HW3

RYSP

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1. **Function Counting:**
   (a) How many functions $f : \{1,2,3,4\} \to \{a,b\}$ are there? [2^4]
   (b) How many *injective* functions $g : \{1,2,3\} \to \{a,b,c,d\}$? [4!/1!]
   (c) How many surjective function $\phi : \{1,2,3,4\} \to \{a,b,c\}$? [4!/1!]
   (d) How many *bijective* functions $h : \{1,2,3\} \to \{a,b,c\}$? [3!]
   (e) Let A, B be finite sets, $|A| = n$, $|B| = m$. For which values of n, m can there exist:
      (i) An injective function $A \to B$?
      (ii) A surjective function $A \to B$?
      (iii) A bijective function $A \to B$?

      Explain.

2. **Images and Preimages:**
   (a) Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 3n + 2$. Is f injective? Surjective? Bijective? Prove your answers.
   (b) Give an explicit example of a function $g : \mathbb{N} \to \mathbb{N}$ that is injective but not surjective.
   (c) Give an explicit example of a function $h : \mathbb{N} \to \mathbb{N}$ that is surjective but not injective.

3. **Composition and Inverses:**
   (a) Let $f(x) = 2x + 1$ and $g(x) = x - 3$ for $x \in \mathbb{R}$. Find $f \circ g$ and $g \circ f$. Are these compositions invertible?
   (b) Prove or disprove: If $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f$ is injective.
   (c) Suppose $f : A \to B$ and $g : B \to A$ satisfy $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Show that f and g are inverses and both bijective.

4. **Function Operations:**
   (a) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Write the restriction $f|_{[0,2]}$ and describe its domain and range.

(b) Give an example of a function $f : A \to B$ that can be extended to a larger domain $Y \supset A$.

(c) Suppose $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$, $g(x) = 2x$. Write the comma product $(f, g) : \mathbb{R} \to \mathbb{R}^2$ and the cross product $f \times g : \mathbb{R}^2 \to \mathbb{R}^2$.

5. **Relations and Equivalence:**

   (a) Prove that "congruence modulo n" ($a \sim b$ if $a \equiv b \pmod{n}$) is an equivalence relation on $\mathbb{Z}$.

   (b) Describe the partition of $\mathbb{Z}$ induced by congruence mod 4. What is a canonical form for each class?

   (c) Let $A = \{1, 2, 3, 4\}$ and R be the relation aRb if a divides b. Is R a partial order? Draw the Hasse diagram (lattice) for (A, R).

6. **Countable and Uncountable Sets:**

   (a) Prove that the set $E = \{2, 4, 6, \ldots\}$ of positive even numbers is countably infinite by constructing an explicit bijection with $\mathbb{N}$.

   (b) Show that the set of all finite subsets of $\mathbb{N}$ is countable.

   (c) Is the set of real numbers in $(0, 1)$ with only finitely many nonzero decimal digits countable or uncountable? Prove your answer.

7. **Cantor's Diagonal Argument:**

   (a) Prove that the set of all infinite binary sequences is uncountable using Cantor's diagonal argument.

   (b) Briefly explain why the set of all polynomials with rational coefficients is countable.

   (c) Give an example of an uncountable subset of $\mathbb{R}$ that is not an interval.

8. Prove or disprove: There is a bijection between $\mathbb{R}$ and $\mathbb{R}^2$.

9. The set of algebraic numbers (roots of integer polynomials) is countable. Briefly outline why.

10. Let S be the set of all functions $f : \mathbb{N} \to \mathbb{N}$ which are not eventually constant (that is, for all n there exists $m > n$ with $f(m) \neq f(n)$). Is S countable or uncountable? Prove your answer.

11. Prove that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$ (i.e., the power set of the naturals has the same cardinality as the real numbers).

12. Suppose $f : A \to B$ is a function and $S \subseteq B$. Prove that $f(f^{-1}(S)) \subseteq S$. Give an example where equality does not hold.

13. Let A and B be infinite sets. Is it always true that $|A \cup B| = \max\{|A|, |B|\}$? Explain and give examples.

14. Fix positive integers n, m. Prove there are m^n n-letter words in an m-letter alphabet, and relate this to the number of functions $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$.

15. Describe all possible partitions of the set $A = \{1, 2, 3\}$. For each, write an equivalence relation corresponding to the partition.

16. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$. Compute $f^{-1}((1, e^2))$ and explain your reasoning.

17. True or False? If $f : A \to B$ and $g : B \to C$ are surjective, then $g \circ f$ is surjective. Prove or provide a counterexample.

18. Explain (with justification) why $|\mathbb{Q}| < |\mathbb{R}|$. (Hint: you may cite Cantor's argument or use density.)

19. The continuum hypothesis posits that there is no set whose cardinality is strictly between that of the integers and the real numbers. Research: Is this statement provable from the usual axioms of set theory? (Briefly summarize in your own words.)

20. Show there is a bijection between $(0, 1)$ and $\mathbb{R}^2$ (hint: try "interleaving decimals" or construct a step-by-step map).

21. Is the set of all infinite sequences of rational numbers countable or uncountable? Prove your answer.

22. Let $f : A \to B$ be bijective. Prove that the inverse function f^{-1} is unique.

23. Draw the Hasse diagram for the power set of $\{a, b, c\}$, ordered by inclusion.

24. Let $f : \mathbb{Z} \to \mathbb{Z}_6$ be $f(n) = n \bmod 6$. Describe all fibers (preimages of points) and the induced equivalence classes.

25. How many equivalence relations are there on a 4-element set? [Hint: Bell numbers, partition of integers, Young diagram]

26. Give an explicit example of a function $f : A \to B$ that cannot be extended to any larger domain $A' \supset A$ in a way that preserves surjectivity. Explain.