HW4

RYSP

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1. Prove that $\sqrt[3]{2}$ is algebraic but irrational. State the minimal polynomial it satisfies.
2. Show that the set of all algebraic numbers is countable.
3. Give an example of a transcendental number and briefly explain why it cannot be the root of any polynomial with integer coefficients.
4. Prove that every rational number is algebraic.
5. Is the number $e + \pi$ necessarily transcendental? Explain why this question remains open.
6. Consider the decimal expansion $x = 0.110001000000000000000001\ldots$. Show that x is a Liouville number and hence transcendental.
7. Define the class of computable numbers. Give an example of a number that is transcendental and non-computable.
8. Let $*$ be a binary operation on $\mathbb{Z}$ defined by $a * b = a + b + 1$. Is $*$ associative? Is it commutative? Does it have an identity element?
9. Determine whether the operation $\max(a, b)$ on $\mathbb{N}$ is associative and commutative.
10. Define a binary relation R on $\mathbb{Z}$ by aRb if and only if $a - b$ is even. Prove that R is an equivalence relation.
11. Define a partial order on the power set of $\{1, 2, 3\}$ using the subset relation $\subseteq$. Draw the corresponding Hasse diagram.
12. Find the reflexive, symmetric, and transitive closures of the relation $R = \{(1, 2), (2, 3)\}$ on the set $\{1, 2, 3\}$.
13. Let $(P(S), \cup, \cap)$ be the power set of a finite set S with union and intersection. Prove that it forms a lattice.
14. Define an algebraic system $(A, \cdot)$ where $\cdot$ is a binary operation on $A = \mathbb{Z}_5$. Describe its identity and inverses, if any.

15. Consider the Boolean algebra on $P(\{a, b\})$. List its elements, and verify that each element has a unique complement.

16. Let $G = S_3$ act on $\mathbb{R}^3$ by permuting coordinates. Show that the sum $x_1 + x_2 + x_3$ is a G-invariant.

17. Let $f(x, y) = (x \vee y) \wedge (\neg x \vee y)$. Write f in conjunctive normal form (CNF).

18. Find a complete system of invariants for the action of $G = \mathbb{Z}_2$ on $\mathbb{R}^2$ by swapping coordinates.

19. What is the canonical form of the set $\{4, 1, 3, 4, 2, 1\}$?

20. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

21. Show that if x is algebraic and $x \neq 0$, then $1/x$ is also algebraic.

22. Prove that the sum of two algebraic numbers is again algebraic.

23. Let $f(x) = x^2 - 2$ and $g(x) = x^2 - 3$. Compute $\operatorname{lcm}(f(x), g(x))$ and use it to determine a minimal polynomial for $\sqrt{2} + \sqrt{3}$.

24. Prove that the set of real numbers with finitely many non-zero digits in base 10 is countable.

25. Explain why the set of constructible numbers is a subset of the algebraic numbers.

26. Let $x = \sum_{n=1}^{\infty} 10^{-n!}$. Prove that x is a Liouville number.

27. Prove that the set $A = \mathbb{Z}$ with the binary operation $a \circ b = ab + 1$ is not a group.

28. Define a binary operation $\diamond$ on $\mathbb{N}$ by $a \diamond b = \min(a, b)$. Is this operation associative? Does it have an identity?

29. Let $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \pmod{3}\}$. Prove that R is an equivalence relation and describe its equivalence classes.

30. Define a relation R on $\mathbb{Z}$ by aRb if and only if $a \leq b^2$. Is R reflexive? Symmetric? Transitive?

31. Let $A = \{1, 2, 3, 4\}$. Define a relation $R = \{(1, 2), (2, 3), (3, 4)\}$. Find its transitive closure explicitly.

32. Show that the structure $(\mathbb{Z}_6, +)$ is a finite group. Determine its identity and inverses.

33. Let $(A, \wedge, \vee)$ be a lattice. Prove that $a \leq b$ if and only if $a \vee b = b$ and $a \wedge b = a$.

34. Let $G = \mathbb{Z}_3$ act on $\mathbb{C}$ by multiplication: $g{\cdot}z = e^{2\pi i g/3} z$. Find all G-invariant polynomials in z and $\bar{z}$.

35. Find a complete system of invariants for the action of S_2 on the set of 2-element subsets of $\{1, 2, 3\}$.

36. Consider the equivalence relation on $\mathbb{R}^2$ given by $(x, y) \sim (u, v)$ if $x^2 + y^2 = u^2 + v^2$. Find a canonical form for each equivalence class.

37. Let $f(x, y) = (x \wedge y) \vee (\neg x \wedge \neg y)$. Express f in disjunctive normal form (DNF).

38. Define a function $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = xyz$. Let $G = S_3$ act on $\mathbb{R}^3$ by permuting coordinates. Is f a G-invariant? Justify.

39. Let $f(x, y) = x^2 + y^2$ and define an equivalence relation $x \sim y$ if $f(x) = f(y)$. Describe the equivalence classes and give a canonical representative for each.

40. Develop linear algebra and abstract algebra methods to find the minimal polynomial of the sum of two algebraic numbers.