HW2

RYSP

July 11, 2025

1. Verify $\neg P \equiv P \Rightarrow 0$.

2. According to the number of input of propositional variables, we call the operation defined by the logical connectives unary, binary, ..., n-nary. What is unary, binary basic logical connectives? From the above problem 1, we can use binary logical connectives uniformly define logical connectives.

3. Connect the logical operation and electric circuit(series, parallel).

4. If x and y are any real numbers, show that

$$||x| - |y|| \leq |x - y|.$$

5. If $A = \{\varnothing, \{\varnothing\}\}$ and f is the empty set, determine which of the following eight statements are true and which are false:

   (a) A has exactly 2 elements.

   (b) $f \in A$

   (c) $f \subset A$

   (d) $\{\} \in A$

   (e) $\{\} \subset A$

   (f) $\{\} \in P(A)$

   (g) $\{\} \subset P(A)$

   (h) $P(A)$ has exactly 4 elements.

6. If $A = \{x \mid x = 4k + 3 \text{ for some } k \in \mathbb{N}\}$ and $C = \{x \mid x = 20q + 19 \text{ for some } q \in \mathbb{N}\}$, show that $C \subset A$.

7. What can you conclude if you know that $A \subset P(B)$ and that $P(X)$ denotes the power set of X (i.e., the set of all subsets of X)?

8. Is the following "proof" correct that for any three sets A, B, C, if $A \subset B$, $B \subset C$, then $A \subset C$?
   Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$, $C = \{1, 2, 3, 4, 5, 6, 7\}$. Then $A \subset B$, $B \subset C$, and $A \subset C$.
   If $A \subset B$ and $B \subset C$ then $A \subset C$.

9. Prove that if $A \subset B$ and $A \neq \varnothing$, then $A - B = \varnothing$.

10. Prove that if A and B are sets, then $A \cup B = B \cup A$.

11. Which numbers leave a remainder of 1 when divided by 3 and a remainder of 3 when divided by 4?

12. What numbers leave a remainder of 2 when divided by 3 and a remainder of 3 when divided by 4?

13. If $A = [1, 5]$, $B = [2, 4]$, $C = [3, 6]$ (intervals of real numbers), then what is $A-(B-C)$? Recall $X - Y = \{a \mid a \in X, a \notin Y\}$.

14. Prove that $A - (B \cup C) = (A - B) \cap (A - C)$.

15. Give an example of three sets whose intersection is $\varnothing$ but no two of them are disjoint.

16. List all elements of $P(\{a, b, c, d\})$.

17. Let $p_1, p_2, \ldots, p_k$ be the first k primes, and $M = (p_1 p_2 \cdots p_k) + 1$. Show that M has a prime factor larger than p_k.

18. (a) Use the Multiplication Principle of Counting to determine the number of factors of $2^3 \times 3^5 \times 7^7$.
    (b) What number from 1 to 1000 has the greatest number of factors?

19. Suppose there are 15 items in A and 13 items in B, and the universe has 30 items.
    (a) If there are 3 items in both A and B, how many items are neither in A nor B?
    (b) What is the smallest number of items that could be neither in A nor B?
    (c) What is the largest number of items that could be neither in A nor B?

20. There are 100 items in the universe. There are 33 items in A, 12 in A alone, 12 in A and B, 12 in A and C. There are 36 in B, of which 13 are also in C.
    (a) If there are 3 items in A, B, C and 30 in neither, how many are in C alone?
    (b) If there are 5 items in A, B, C and 30 in neither, how many are in C alone?
    (c) If there are 4 items in A, B, C and 30 in neither, how many are in C alone?

21. If $P(A) \subset P(B)$, must $A \subset B$?

22. Prove by induction or otherwise: $1 + 3 + 5 + \cdots + (2n - 1) = n!$.

23. A chessboard has 2^n squares on each side. One corner square is removed. Prove the resulting board can be evenly covered with L-shaped tiles, each having three squares of the same size.

24. Using 2-cent and 5-cent stamps, show that any postage ≥ 4 can be formed.

(a) If stamps are 3 and 7 cents, what is the smallest n so that all n or more can be formed?

(b) What if stamps are 5 and 8 cents?

25. Prove by induction or otherwise:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$

26. Show by induction: the sum of three consecutive positive cubes is divisible by 9.

27. Find and prove a formula for $1^3 + 2^3 + \cdots + n^3$.

28. If n is a positive integer, prove $n \leq 2^n$.

29. If $n \geq 4$, prove $n! \geq 2^n$.

30. Prove by induction: for every $n \in \mathbb{N}$,

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

31. Prove by induction: for every $n \in \mathbb{N}$,

$$4^{n+1} > (n+4)^2$$

32. (Tower of Hanoi) Three posts, n disks of different sizes, each move consists of moving one disk, never place a disk on top of a smaller disk. How many moves does it take? [Hint: recursive formula]

33. Use the Division Algorithm to express $\gcd(117, 31)$ as a linear combination of 117 and 31.

34. Determine the number of $2 \times 2 \times 2$ triangles in an $n \times n \times n$ triangle.

35. Determine the number of triangles of any size in an $8 \times 8 \times 8$ triangle.