Calculus V

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Contents

1 Integral Transforms

Integral transforms spread everywhere. Here is a summary to the frequently used integral transforms. Integral transform consists of **kernel** and **domain**.

1.1 Laplace Transform

Use $\mathscr{L}$ to denote Laplace transform, by definition,

$$\mathscr{L}[f(t)](s) := \int_{[0,\infty)} e^{-st} f(t) dt = F(s) = \tilde{f}(s).$$

By convention, Laplace transform $\mathscr{L}$ transforms real variable t (time) to complex variable s (frequency, $s = \sigma + i\omega$), then

kernel: e^{-st}

domain: $[0, \infty)$

1.2 Inverse Laplace Transform

The **inverse Laplace transform** recovers $f(t)$ from $F(s)$, denoted by

$$f(t) = \mathscr{L}^{-1}[F(s)](t).$$

Key points:

- The inverse transform is generally given by the **complex Bromwich integral** (the **inverse Mellin transform**):

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds,$$

  where the contour path in the complex plane is chosen such that all singularities of $F(s)$ lie to the left of the line $\mathrm{Re}(s) = \gamma$.

- In practice, inverse Laplace transforms are often computed using **tables** of transforms and properties such as linearity, shifting, convolution, and partial fraction decomposition.

- Common properties useful for inversion:

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a),$$
$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s),$$
$$\mathcal{L}\{f * g\} = F(s)G(s),$$

  where $*$ denotes convolution.

- The inverse transform is unique under suitable growth conditions on $f(t)$.

Remark 1.1. The inverse Laplace transform is a powerful tool for solving differential and integral equations by transforming to the **complex frequency domain** (s), manipulating algebraically, then returning to the **time domain** (t).

1.3 Fourier Transform

Suppose $f(x)$ is a function defined on $\mathbb{R}$ that satisfies suitable **integrability** and **smoothness** conditions.

The *Fourier transform* $\hat{f}(\omega)$ of f is defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx.$$

The *Fourier integral representation* (or inverse Fourier transform) allows us to recover $f(x)$ from $\hat{f}(\omega)$:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\, d\omega.$$

This integral expresses f as a continuous superposition of complex exponentials $e^{i\omega x}$, weighted by $\hat{f}(\omega)$.

Remark 1.2.

- The Fourier integral representation generalizes the Fourier series representation to non-periodic functions.
- It plays a fundamental role in signal processing, differential equations, and quantum mechanics.

Remark 1.3. The Fourier integral representation of function $f(x)$ is essentially the same as inverse Fourier transform.

Use $\mathscr{F}$ to denote Fourier transform, by definition,

$$\mathscr{F}[f(x)](\omega) := \int_{\mathbb{R}} e^{-i\omega x} f(x)dx = F(\omega) = \hat{f}(\omega).$$

By convention, Fourier transform $\mathscr{F}$ transforms real variable x (space) to real variable ω (real frequency), then

kernel: $e^{-i\omega x}$

domain: $\mathbb{R}$

Remark 1.4 (real Fourier transform)**.**

1.4 Fourier Cosine Transform

Use $\mathscr{C}$ to denote cosine transform, by defintion,

$$\mathscr{C}[f(x)](\omega) := \int_{[0,\infty)} \cos(\omega x) f(x)dx = F_c(\omega) = \hat{f}_c(\omega).$$

Suppose cosine transform $\mathscr{C}$ transforms x to ω, then

kernel: $\cos \omega x$

domain: $[0, \infty)$

1.5 Sine Transform

Use $\mathscr{S}$ to denote sine transform, by definition,

$$\mathscr{S}[f(x)](\omega) := \int_{[0,\infty)} \sin(\omega x) f(x) dx = F_s(\omega) = \hat{f}_s(\omega).$$

kernel: $\sin \omega x$

domain: $[0, \infty)$

1.6 Inverse Fourier Cosine Transform

Given the cosine transform

$$F_c(\omega) = \mathscr{C}[f(x)](\omega) = \int_0^\infty \cos(\omega x) f(x)\, dx,$$

the inverse cosine transform recovers $f(x)$ by

$$\boxed{f(x) = \frac{2}{\pi} \int_0^\infty \cos(\omega x) F_c(\omega)\, d\omega.}$$

kernel: $\frac{2}{\pi} \cos \omega x$ **domain:** $[0, \infty)$

1.7 Inverse Fourier Sine Transform

Given the sine transform

$$F_s(\omega) = \mathscr{S}[f(x)](\omega) = \int_0^\infty \sin(\omega x) f(x)\, dx,$$

the inverse sine transform recovers $f(x)$ by

$$\boxed{f(x) = \frac{2}{\pi} \int_0^\infty \sin(\omega x) F_s(\omega)\, d\omega.}$$

kernel: $\frac{2}{\pi} \sin \omega x$ **domain:** $[0, \infty)$

1.8 Connection Between Laplace and Fourier Transforms

The **Laplace transform** $\mathscr{L}$ and the **Fourier transform** $\mathscr{F}$ are closely related integral transforms widely used to analyze differential equations and boundary value problems.

Recall the Laplace transform definition:

$$\mathscr{L}[f(t)](s) := \int_0^\infty e^{-st} f(t)\, dt, \quad s = \sigma + i\omega,$$

with kernel e^{-st} and domain $[0, \infty)$.

The Fourier transform is defined by

$$\mathscr{F}[f(x)](\omega) := \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx,$$

with kernel $e^{-i\omega x}$ and domain $\mathbb{R}$.

Remark 1.5 (key connection)**.** When the Laplace transform parameter s is restricted to the imaginary axis, i.e., $s = i\omega$ with $\sigma = 0$, the Laplace transform reduces to the Fourier transform of a function defined on $[0, \infty)$:

$$\mathscr{L}[f(t)](i\omega) = \int_{0}^{\infty} e^{-i\omega t} f(t)\, dt,$$

which resembles the Fourier transform but on a half-line domain rather than $\mathbb{R}$.

1.9 Operational Properties for Laplace and Fourier Transforms

Integral transforms simplify the analysis of differential and integral equations by converting differentiation, translation, and other operations into algebraic manipulations in the transform domain.

- **Linearity:**

$$\mathscr{L}[af(t) + bg(t)] = aF(s) + bG(s), \quad \mathscr{F}[af(x) + bg(x)] = a\hat{F}(\omega) + b\hat{G}(\omega).$$

- **Translation (Time/Space Shift):**

$$\mathscr{L}[f(t-a)u(t-a)](s) = e^{-as}F(s), \quad a \geq 0,$$

  where $u(t)$ is the Heaviside step function.

$$\mathscr{F}[f(x-a)](\omega) = e^{-i\omega a}\hat{F}(\omega).$$

- **Scaling:**

$$\mathscr{L}[f(at)](s) = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0,$$

$$\mathscr{F}[f(ax)](\omega) = \frac{1}{|a|}\hat{F}\left(\frac{\omega}{a}\right).$$

- **Differentiation:**

$$\mathscr{L}[f'(t)](s) = sF(s) - f(0),$$

  and more generally,

$$\mathscr{L}[f^{(n)}(t)](s) = s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0).$$

  For the Fourier transform,

$$\mathscr{F}\left[\frac{d^n}{dx^n}f(x)\right](\omega) = (i\omega)^n \hat{F}(\omega).$$

- **Integration/Convolution:**
  For functions f, g, their convolution is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

  The Laplace transform of the convolution satisfies

$$\mathscr{L}[f * g](s) = F(s) \cdot G(s),$$

  where $F(s) = \mathscr{L}[f](s)$ and $G(s) = \mathscr{L}[g](s)$.
  Similarly, the Fourier transform converts convolution into multiplication:

$$\mathscr{F}[f * g](\omega) = \hat{F}(\omega) \cdot \hat{G}(\omega),$$

  where $\hat{F}(\omega) = \mathscr{F}[f](\omega)$ and $\hat{G}(\omega) = \mathscr{F}[g](\omega)$.

- **Multiplication by t^n or x^n:**

$$\mathscr{L}[t^n f(t)](s) = (-1)^n \frac{d^n}{ds^n} F(s),$$

$$\mathscr{F}[x^n f(x)](\omega) = i^n \frac{d^n}{d\omega^n} \hat{F}(\omega).$$

Remark 1.6. These properties transform differential equations into algebraic equations in the complex domain, greatly simplifying the solution of ODEs and PDEs such as heat, wave, and Laplace equations. Initial and boundary conditions translate into conditions on transformed functions, making the Laplace and Fourier transforms essential tools in mathematical physics and engineering.

1.10 Integration and Convolution

Convolution is a fundamental operation that combines two functions into a third function expressing how the shape of one is modified by the other. It appears naturally in systems characterized by linear time-**invariant** behavior, integral equations, and many PDEs.

Definition 1.1 (Convolution)**.**

$$(f * g)(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau,$$

for functions f, g defined on $[0, \infty)$.

Definition 1.2 (Convolution on $\mathbb{R}$)**.** For functions $f, g : \mathbb{R} \to \mathbb{C}$ (or $\mathbb{R}$) integrable on $\mathbb{R}$, the *convolution* $f * g$ is defined by

$$(f * g)(t) := \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\,d\tau,$$

provided the integral converges.

Remark 1.7 (Laplace Transform of Convolution)**.** The Laplace transform converts convolution in the time domain into multiplication in the complex frequency domain:

$$\mathscr{L}[f * g](s) = \int_0^\infty e^{-st} \left(\int_0^t f(\tau) g(t-\tau) d\tau \right) dt.$$

Interchanging the order of integration (justified by Fubini's theorem under suitable conditions), and using the substitution $u = t - \tau$, this evaluates to

$$\mathscr{L}[f * g](s) = F(s) \cdot G(s),$$

where

$$F(s) = \mathscr{L}[f](s) = \int_0^\infty e^{-st} f(t) dt, \quad G(s) = \mathscr{L}[g](s) = \int_0^\infty e^{-st} g(t) dt.$$

This property allows solving integral equations by transforming convolution integrals into algebraic products.

Remark 1.8 (Fourier Transform of Convolution)**.** For functions f, g defined on $\mathbb{R}$, the convolution is defined by

$$(f * g)(x) := \int_{-\infty}^\infty f(\tau) g(x-\tau) d\tau.$$

The Fourier transform turns convolution into pointwise multiplication:

$$\mathscr{F}[f * g](\omega) = \int_{-\infty}^\infty e^{-i\omega x} \left(\int_{-\infty}^\infty f(\tau) g(x-\tau) d\tau \right) dx = \hat{F}(\omega) \hat{G}(\omega),$$

where

$$\hat{F}(\omega) = \mathscr{F}[f](\omega), \quad \hat{G}(\omega) = \mathscr{F}[g](\omega).$$

This transforms integral equations and PDEs with convolution kernels into algebraic equations easier to manipulate and invert.

Remark 1.9 (more results)**.**

- Convolution is commutative: $f * g = g * f$.
- It is associative and distributive over addition.
- Convolution with the delta function $\delta(t)$ leaves the function unchanged.
- Convolution plays a key role in the solutions of linear systems, Green's functions, and impulse responses.

Proposition 1 (Convolution with the Delta Function)**.** Let $\delta(t)$ be the Dirac delta distribution. For any suitable function $f(t)$, the convolution satisfies

$$(f * \delta)(t) = f(t).$$

Explanation. By definition of convolution on $\mathbb{R}$,

$$(f * \delta)(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\, d\tau.$$

Using the sifting property of the delta function,

$$\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\, d\tau = f(t).$$

Thus, convolution with δ acts as the identity operation. $\square$

2 Special Functions

2.1 Exponential Integral

$$\mathrm{Ei}(x)$$

The **Exponential Integral** $\mathrm{Ei}(x)$ is defined by the Cauchy principal value integral

$$\mathrm{Ei}(x) = \mathrm{P.V.} \int_{-\infty}^{x} \frac{e^t}{t}\, dt,$$

where the integrand has a singularity at $t = 0$. It arises in applications involving integrals of the form $\int \frac{e^t}{t} dt$.

- For $x > 0$, $\mathrm{Ei}(x)$ grows rapidly.
- For $x < 0$, it relates to $E_1(|x|)$.
- Can be expanded in series and numerically computed.

2.2 Generalized Exponential Integral

$$E_n(z)$$

The generalized functions $E_n(z)$, for integer $n \geq 1$, are defined by

$$E_n(z) = \int_1^{\infty} \frac{e^{-zt}}{t^n}\, dt.$$

- For $n = 1$,

  $$E_1(z) = \int_z^{\infty} \frac{e^{-t}}{t} dt,$$

  valid for $\mathrm{Re}(z) > 0$.
- They satisfy the recurrence relation

  $$nE_{n+1}(z) = e^{-z} - zE_n(z).$$
- Used in differential equations and integral transforms.

Remark 2.1. The classical exponential integral functions Ei and E_1 are closely related as follows:

$$\begin{cases} E_1(x) = \displaystyle\int_x^{\infty} \frac{e^{-t}}{t}\, dt, & x > 0, \\ \boxed{\mathrm{Ei}(x) = -E_1(-x)}, & x < 0, \\ E_n(z) = \displaystyle\int_1^{\infty} \frac{e^{-zt}}{t^n}\, dt, & n \in \mathbb{N}, \\ nE_{n+1}(z) = e^{-z} - zE_n(z). \end{cases}$$

Here, $E_n(z)$ generalizes the exponential integral for $n \geq 1$, and the recurrence relation relates different orders.

2.3 Logarithmic Integral

$$\mathrm{li}(x)$$

The **Logarithmic Integral** $\mathrm{li}(x)$ is defined as

$$\mathrm{li}(x) = \mathrm{P.V.} \int_0^x \frac{dt}{\ln t},$$

with a singularity at $t = 1$ handled by the principal value.

- Important in number theory, especially the Prime Number Theorem.
- Related to the exponential integral by

$$\mathrm{li}(x) = \mathrm{Ei}(\ln x), \quad x > 0.$$

2.4 Summary Table

Function	Definition	Domain / Parameters	Key Properties
$\mathrm{Ei}(x)$	$\mathrm{P.V.} \int_{-\infty}^x \frac{e^t}{t} dt$	$x \in \mathbb{R} \setminus \{0\}$	Singularity at 0; rapid growth for $x > 0$
$E_n(z)$	$\int_1^\infty \frac{e^{-zt}}{t^n} dt$	$n \in \mathbb{N},\ z \in \mathbb{C}$	Recurrence relation; generalizes E_1
$\mathrm{li}(x)$	$\mathrm{P.V.} \int_0^x \frac{dt}{\ln t}$	$x > 0$	Approximates prime counting; $\mathrm{li}(x) = \mathrm{Ei}(\ln x)$

2.5 More Special Functions

$$\begin{aligned}
\ln x &= \int_1^x \frac{dt}{t}, && x > 0, \\
\mathrm{Si}(x) &= \int_0^x \frac{\sin t}{t}\, dt, && \text{(Sine integral, with integrand 1 at } t = 0\text{)}, \\
C(x) &= \int_0^x \cos\!\Big(\tfrac{\pi t^2}{2}\Big)\, dt, && \text{(Fresnel cosine integral)}, \\
S(x) &= \int_0^x \sin\!\Big(\tfrac{\pi t^2}{2}\Big)\, dt, && \text{(Fresnel sine integral)}, \\
\mathrm{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du, && \text{(Error function)}, \\
\mathrm{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du, && \mathrm{erf}(x) + \mathrm{erfc}(x) = 1, \\
\Gamma(z) &= \int_0^\infty e^{-t} t^{z-1}\, dt, && \Re(z) > 0 \quad \text{(Gamma function)}.
\end{aligned}$$

Remark 2.2. These integral definitions not only generalize elementary functions (e.g. the logarithm and factorial via $\Gamma(n + 1) = n!$) but also play central roles in applications across differential equations, signal processing, and probability theory.

2.6 Legendre Functions and Polynomials

The *Legendre functions* are solutions to the Legendre differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0,$$

where n can be a non-negative integer or, more generally, a complex number, and the variable x is taken in the interval $[-1, 1]$ for polynomials or in the complex plane $\mathbb{C}$ with appropriate branch cuts for general functions.

Legendre Polynomials $P_n(x)$: When n is a non-negative integer, the solutions that are regular on $[-1, 1]$ are called *Legendre polynomials*. These polynomials are widely used in physics and numerical analysis.

- **Polynomial nature:** $P_n(x)$ is a polynomial of degree n.
- **Orthogonality:** They satisfy the orthogonality relation

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{nm},$$

where δ_{nm} is the Kronecker delta.

- **Generating function:**

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |t| < 1.$$

- **Rodrigues' formula:**

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n.$$

Remark 2.3. Legendre polynomials frequently appear in solving Laplace's equation in spherical coordinates and in expansions of functions defined on the sphere.

General Legendre Functions: For non-integer (possibly complex) degree ν, the Legendre equation admits two linearly independent solutions often denoted $P_\nu(x)$ (the first kind) and $Q_\nu(x)$ (the second kind).

- $P_\nu(x)$ generalizes Legendre polynomials beyond integer degrees.
- $Q_\nu(x)$ is generally singular on the interval $[-1, 1]$.
- These functions are defined as analytic functions on $\mathbb{C}$ with branch cuts typically along $[-1, 1]$.
- They play a key role in boundary value problems where parameters are non-integral or complex.

2.7 Bessel Functions and Bessel Polynomials

The *Bessel functions of the first kind* $J_\nu(x)$ are solutions to Bessel's differential equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0,$$

where $\nu \in \mathbb{R}$ or $\mathbb{C}$ is the order of the function.

Key properties of Bessel functions $J_\nu(x)$:

- **Series representation:**

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}.$$

- **Integral representation:**

$$J_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos(\nu\tau - x\sin\tau)\,d\tau.$$

- **Orthogonality relations** hold under appropriate conditions on the order and interval.

- **Asymptotic behavior for large x:**

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

Remark 2.4. Bessel functions naturally arise in problems with cylindrical symmetry such as vibrations of circular membranes, heat conduction in cylinders, and wave propagation.

Bessel Polynomials Distinct from Bessel functions, the *Bessel polynomials* $\{y_n(x)\}$ form a sequence of polynomials defined by the generating function

$$\sum_{n=0}^{\infty} y_n(x)\frac{t^n}{n!} = \frac{1}{1-2xt+t^2}.$$

Explicitly, they are given by

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!\,k!} \left(\frac{x}{2}\right)^k.$$

They satisfy the differential equation

$$x^2 y_n''(x) + (2x+2)y_n'(x) - n(n+1)y_n(x) = 0,$$

which differs from the classical Bessel differential equation.

Remark 2.5. Bessel polynomials find applications in approximation theory and signal processing, differing from classical Bessel functions in being finite-degree polynomials rather than transcendental functions.

3 Classification of Differential Equations

According to the number of dependent variables:

$$\text{Differential Equations}\begin{cases}\text{Differential Equation}\\ \text{System of Differential Equations}\end{cases}$$

According to the number of independent variables:

$$\text{Differential Equations}\begin{cases}\text{Ordinary Differential Equations}\\ \text{Partial Differential Equations}\end{cases}$$

Therefore, totally, there are four types of differential equations:

(i) Ordinary differential equation

(ii) Partial differential equation

(iii) System of ODEs

(iv) System of PDEs

3.1 Terminologies

(i) interval of validity: the domain where the solution of differential equations is valid, also, it is part of the solution of differential equations

(ii) solution curve: the graph of solution of differential equations is called a solution curve, which is different from the graph of corresponding function since the domain of validity may differ

(iii) implicit solution / representation: not the solution of every DE has the explicit expression, so the **relation** of the solution satisfies is called an implicit solution (of the DE)

(iv) particular solution: A solution of a differential equation that is free of arbitrary parameters is called a particular solution

(v) family of solutions: the solution containing parameters

(vi) singular solution: a solution of DEs which cannot be obtained by evaluating the family of solutions

(vii) integral-defined function: some integral cannot be represented by elementary functions (including their operations under a finite sequence), so to define such an integral-defined function, the integral representation cannot be simplified further (non-elementary)

(viii) transient term: vanishing as times goes to infinity (complementary function / homogeneous solution)

(ix) steady-state solution: particular solution

(x) fundamental period: If T is the smallest positive value for which $f(x+T) = f(x)$ holds, then T is called the fundamental period of f. (For period functions the fundamental period not necessarily exists)

(xi) fundamental domain: the region where multi valued function restricted to makes the principal function well defined

3.2 Initial-Value Problems (IVPs)

Definition 3.1 (Initial-Value Problem)**.** Let I be an interval containing x_0. The problem of finding y satisfying

$$\frac{d^n y}{dx^n} = f\big(x,\, y,\, y',\, y'',\, \ldots,\, y^{(n-1)}\big)$$

together with

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

is called an *initial-value problem* (IVP). The tuple $(y_0, y_1, \ldots, y_{n-1})$ gives the *initial conditions* (IC).

Remark 3.1. All initial conditions are specified at the same point $x_0 \in I$.

3.2.1 Fixed-Point Background

Definition 3.2 (Contraction)**.** A map $T : (X, d) \to (X, d)$ is a *contraction* if there exists $L \in [0, 1)$ such that

$$d(Tx, Ty) \leq L\, d(x, y), \quad \forall x, y \in X.$$

Theorem 3.1 (Banach Fixed-Point)**.** If (X, d) is complete and T is a contraction with constant $L < 1$, then:

(a) T has a unique fixed point $x^* \in X$;

(b) Iteration $x_{n+1} = T(x_n)$ converges to x^* with

$$d(x_n, x^*) \leq \frac{L^n}{1 - L} d(x_1, x_0).$$

Idea. The contraction property shows $\{x_n\}$ is Cauchy; completeness gives convergence to some x^*. Continuity of T ensures $T(x^*) = x^*$. Uniqueness follows from $d(x^*, y^*) \leq Ld(x^*, y^*)$. The error bound comes from iterating the contraction estimate. □

3.2.2 Picard–Lindelöf for First-Order Systems

Definition 3.3 (Lipschitz in x)**.** Let $D \subset \mathbb{R} \times \mathbb{R}^n$. A function $f : D \to \mathbb{R}^n$ is *Lipschitz in the second variable* if

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|, \quad \forall (t,x),(t,y) \in D.$$

Theorem 3.2 (Picard–Lindelöf)**.** Let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and

$$D = \{(t,x) : |t - t_0| \leq a,\ \|x - x_0\| \leq b\}.$$

If f is continuous and Lipschitz in x on D with constant L, and

$$M = \max_{(t,x)\in D} \|f(t,x)\|, \quad h = \min\left\{a, \frac{b}{M}\right\},$$

then the IVP

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0$$

has a unique solution on $[t_0 - h, t_0 + h]$.

Idea. Work in the Banach space of continuous functions $\varphi : [t_0 - h, t_0 + h] \to \mathbb{R}^n$ with sup norm. The Picard operator

$$(T\varphi)(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds$$

is a contraction for h small enough. Apply Banach's theorem. □

3.2.3 General nth-Order ODEs

Definition 3.4 (Locally Lipschitz)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is *locally Lipschitz* at $x_0 \in X$ if there exist $r > 0$ and $L \geq 0$ such that

$$d_Y\big(f(x), f(y)\big) \leq L\, d_X(x,y) \quad \text{for all } x, y \in B_X(x_0, r).$$

We say f is *locally Lipschitz on* X if it is locally Lipschitz at every $x_0 \in X$.

Definition 3.5 (Locally Lipschitz in the second variable)**.** Let $D \subset \mathbb{R} \times \mathbb{R}^n$. A function $f : D \to \mathbb{R}^m$ is *locally Lipschitz in* x (the second variable) if for every $(t_0, x_0) \in D$ there exist a neighborhood $U \subset D$ of (t_0, x_0) and a constant $L \geq 0$ such that

$$\|f(t,x) - f(t,y)\| \leq L\,\|x - y\| \quad \text{for all } (t,x),(t,y) \in U \text{ with the same } t.$$

Proposition 2. If $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is C^1 on the open set Ω, then f is locally Lipschitz on Ω. More precisely, for any compact $K \subset \Omega$,

$$\|f(x) - f(y)\| \leq \Big(\sup_{z\in K} \|Df(z)\|\Big)\, \|x - y\| \qquad (x, y \in K).$$

Remark 3.2.

- Lipschitz ⇒ locally Lipschitz ⇒ continuous.
- In ODE theory, local Lipschitz in x (the state variable) is the key hypothesis for uniqueness (Picard–Lindelöf).

Theorem 3.3 (Existence–Uniqueness)**.** If f in

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \quad y^{(j)}(x_0) = \eta_j,$$

is continuous and locally Lipschitz in $(y, \dots, y^{(n-1)})$, then there exists $h > 0$ such that the IVP has a unique solution on $[x_0 - h, x_0 + h]$.

Sketch. Introduce $y_0 = y$, $y_1 = y'$, $\dots, y_{n-1} = y^{(n-1)}$, set $Y = (y_0, \dots, y_{n-1})^T$, rewrite as $Y' = G(x, Y)$, and apply Picard–Lindelöf for systems. □

3.2.4 Linear Case

Corollary 1. If $a_n(x) \neq 0$ and all $a_k(x), g(x)$ are continuous on I, then the linear ODE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

has a unique solution on I for any initial data $y^{(j)}(x_0) = y_j$.

Idea. Rewrite as a first-order linear system and apply Picard–Lindelöf globally. □

Remark 3.3.

$$\boxed{\text{Banach FPT} \;\Rightarrow\; \text{Picard–Lindelöf} \;\Rightarrow\; \text{Existence–Uniqueness for IVPs}}$$

3.3 Boundary-Value Problems (BVPs)

Definition 3.6 (Boundary-Value Problem)**.** Let L be a linear differential operator acting on a suitable function space $\mathcal{V}$, and let f be a given function on a domain $\Omega \subset \mathbb{R}^n$. A *boundary-value problem* consists of:

$$\begin{cases} Lu = f & \text{in } \Omega, \\ Bu = g & \text{on } \partial\Omega, \end{cases}$$

where B is a *boundary operator* encoding the boundary conditions on $\partial\Omega$.

3.3.1 Types of Boundary Conditions

(i) **Dirichlet:**

$$u|_{\partial\Omega} = g.$$

In vector space terms, $u - g \in \mathcal{V}_0 := \{v \in \mathcal{V} : v|_{\partial\Omega} = 0\}$.

(ii) **Neumann:**

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g.$$

In vector space terms, this is a linear constraint given by a functional $\ell(v) = \frac{\partial v}{\partial n}|_{\partial\Omega} - g$ vanishing.

(iii) **Robin (mixed):**

$$\alpha\, u + \beta\, \frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega,$$

with given $\alpha, \beta \in \mathbb{R}$, not both zero. Admissible functions form the affine subspace

$$\mathcal{V}_{\alpha,\beta,g} = \left\{ v \in \mathcal{V} \;\middle|\; \alpha\, v + \beta\, \frac{\partial v}{\partial n} = g \text{ on } \partial\Omega \right\}.$$

Remark 3.4 (Normal Derivative and Complex Analogue)**.** Let $\Omega \subset \mathbb{R}^n$ have smooth boundary $\partial\Omega$, and let $\mathbf{n}(x)$ denote the outward unit normal vector at $x \in \partial\Omega$. For a scalar field $u : \Omega \to \mathbb{R}$, the *normal derivative* is defined by

$$\frac{\partial u}{\partial n}(x) := \nabla u(x) \cdot \mathbf{n}(x),$$

that is, the directional derivative of u in the normal direction. This quantity measures the rate of change of u as one moves outward from $\partial\Omega$.

In **Neumann boundary conditions**, one prescribes

$$\frac{\partial u}{\partial n}(x) = g(x), \quad x \in \partial\Omega,$$

where g is a given function. In **Robin boundary conditions**, one prescribes a linear combination

$$\alpha(x)\, u(x) + \beta(x)\, \frac{\partial u}{\partial n}(x) = g(x), \quad x \in \partial\Omega.$$

Complex Analogue: If u is harmonic in $\Omega \subset \mathbb{C}$ and $f = u + iv$ is analytic in Ω, then

$$\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n} = \frac{\partial u}{\partial x} \cos\theta + \frac{\partial u}{\partial y} \sin\theta,$$

where θ is the angle the outward normal $\mathbf{n}$ makes with the positive x-axis. By the Cauchy–Riemann equations,

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial \tau},$$

where $\frac{\partial}{\partial \tau}$ denotes the derivative along the tangent direction. This duality is often used in potential theory and conformal mapping to switch between normal derivatives of u and tangential derivatives of its harmonic conjugate v.

Remark 3.5. In all three cases, the boundary condition is a *linear constraint* on $\mathcal{V}$, and the set of admissible functions is an affine subspace determined by the homogeneous BC space and a particular function matching g.

3.3.2 Existence and Uniqueness for BVPs

Theorem 3.4 (General Principle for Linear BVPs)**.** Let $L : \mathcal{V} \to \mathcal{V}^*$ be a bounded, coercive, linear operator on a Hilbert space $\mathcal{V}_B$ of functions satisfying the homogeneous boundary conditions $Bu = 0$. Then, for any $f \in \mathcal{V}^*$, there exists a unique $u \in \mathcal{V}_B$ such that

$$Lu = f \quad \text{in the weak sense.}$$

Idea. Use the Lax–Milgram theorem: coercivity ensures invertibility, boundedness ensures continuity of the bilinear form $a(u, v) = \langle Lu, v\rangle$, and f is a bounded functional. □

Example 3.1 (Poisson Equation)**.** Let $\Omega \subset \mathbb{R}^n$ be bounded with smooth boundary. The Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega)$ for each $f \in L^2(\Omega)$. This follows from Lax–Milgram with $a(u, v) = \int_\Omega \nabla u \cdot \nabla v$.

3.4 Eigenvalue Problems (EPs)

Definition 3.7 (Eigenvalue Problem)**.** Let $L : \mathcal{V} \to \mathcal{V}$ be a linear operator, and let B encode homogeneous boundary conditions. An *eigenvalue problem* is:

$$\begin{cases} Lu = \lambda\, u & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\lambda \in \mathbb{C}$ and $u \in \mathcal{V}_B \setminus \{0\}$.

Remark 3.6. In the vector space framework:

- $\mathcal{V}_B = \{v \in \mathcal{V} : Bv = 0\}$ is the BC subspace.
- The eigenproblem is: find (λ, u) with $u \in \mathcal{V}_B$, $u \neq 0$, such that $Lu = \lambda u$.
- If L is self-adjoint on a Hilbert space, λ is real, and eigenfunctions for distinct eigenvalues are orthogonal.

Example 3.2 (Sturm–Liouville Problem)**.** Find (λ, u) such that

$$-(p(x)u')' + q(x)u = \lambda\, w(x)u, \quad a < x < b,$$

with Dirichlet BC $u(a) = u(b) = 0$. Here L is self-adjoint on $L^2_w(a, b)$, the spectrum is discrete, and eigenfunctions form an orthogonal basis.

3.5 Periodic Problems

Definition 3.8 (Periodic Boundary–Value Problem)**.** Let L be a differential operator on (a, b), and let $T = b - a > 0$. A *periodic boundary–value problem* (PBVP) seeks a function u such that

$$\begin{cases} Lu = f & \text{on } (a, b), \\ u(a) = u(b), \quad u'(a) = u'(b), \quad \ldots, \quad u^{(m-1)}(a) = u^{(m-1)}(b), \end{cases}$$

where m is the order of the differential equation.

Remark 3.7. The equalities at a and b enforce that u and its derivatives up to order $m - 1$ match at the two endpoints. For smooth solutions, this implies that u extends to a T–periodic function on $\mathbb{R}$.

Example 3.3 (Second–Order Periodic Problem)**.** Consider

$$-u''(x) + q(x)u(x) = \lambda u(x), \quad a < x < b,$$

with

$$u(a) = u(b), \quad u'(a) = u'(b).$$

If q is continuous and $T = b - a$, solutions correspond to T–periodic eigenfunctions of the associated Sturm–Liouville operator.

Remark 3.8 (Self–Adjointness and Spectrum)**.** If L is formally self–adjoint and q, w, p satisfy the regularity and positivity assumptions of Sturm–Liouville theory, then the periodic BCs make the boundary term vanish:

$$p(x)\left(u'(x)\overline{v(x)} - u(x)\overline{v'(x)}\right)\Big|_a^b = 0$$

because the values at a and b cancel. Hence L is symmetric on the periodic domain, and in the regular case it is self–adjoint. The spectrum consists of real eigenvalues

$$\lambda_1 \le \lambda_2 \le \cdots \to +\infty,$$

and eigenfunctions form an orthogonal basis in $L^2_w(a, b)$.

Remark 3.9 (Floquet Theory Connection)**.** For a second–order linear ODE

$$y'' + p(x)\,y' + q(x)\,y = 0, \quad p, q \ T\text{–periodic},$$

Floquet theory states that solutions can be written as

$$y(x) = e^{\mu x} P(x),$$

where P is T–periodic and $\mu \in \mathbb{C}$ is the *Floquet exponent*. The periodic boundary–value problem corresponds to the special case $\mu = 0$, i.e. purely periodic solutions. Spectral analysis of the associated operator then yields *bands* of allowed λ values (Bloch waves) and *gaps* where no bounded periodic solutions exist.

Outline of Floquet's Theorem. Consider the first–order system

$$Y'(x) = A(x)Y(x), \quad A(x+T) = A(x), \quad A \in C(\mathbb{R}).$$

Let $\Phi(x)$ be the *fundamental matrix* with $\Phi(0) = I$. Then $\Phi(x+T)$ is also a fundamental matrix, so there exists a constant matrix C such that

$$\Phi(x+T) = \Phi(x)C.$$

Setting $x = 0$ gives $C = \Phi(T)$, the *monodromy matrix*. The eigenvalues ρ_j of C are called *Floquet multipliers*, and we may write $\rho_j = e^{\mu_j T}$, where μ_j are the Floquet exponents (possibly complex).

Now define

$$P(x) := \Phi(x)e^{-xB}, \quad B := \frac{1}{T}\log C,$$

where log denotes a chosen matrix logarithm. Then $P(x+T) = P(x)$, and every solution can be expressed as

$$Y(x) = P(x)e^{xB}Y(0),$$

which in the scalar second–order case reduces to

$$y(x) = e^{\mu x}P(x),$$

with P T–periodic and μ a Floquet exponent. □

Remark 3.10 (Applications)**.** Periodic problems and Floquet theory appear in:

- Fourier series and harmonic analysis on $[0, T]$.
- Bloch wave theory and band structure in solid state physics.
- Stability analysis of periodic solutions in dynamical systems.
- Mathieu's equation and Hill's equation in parametric resonance.

Remark 3.11 (Finite-Dimensional Analogy)**.** The relationship between IVPs, BVPs, EPs, and periodic problems can be understood by comparing to linear algebra:

- **IVP:** Given a matrix $A \in \mathbb{R}^{n\times n}$ and an initial vector x_0, the system

  $$x'(t) = Ax(t), \quad x(0) = x_0,$$

  has a unique solution $x(t) = e^{At}x_0$ for all t.

- **BVP:** Instead of specifying $x(0)$, suppose we require

  $$B_1x(0) = g_1, \quad B_2x(T) = g_2,$$

  where B_1 and B_2 are matrices enforcing boundary constraints. Now, solvability depends on whether the combined constraints are compatible with the dynamics given by A.

- **EP:** The eigenproblem

  $$Au = \lambda u$$

  corresponds to finding nonzero u satisfying a *homogeneous BVP* for a discrete system. The vector space $\mathbb{R}^n$ is restricted to the subspace $\mathcal{V}_B := \{v : Bv = 0\}$ by the boundary conditions.

- **Periodic Problem:** A periodic BVP corresponds to

  $$x(0) = x(T), \quad x'(0) = x'(T),$$

  which, for $x'(t) = Ax(t)$, means

  $$e^{AT}x(0) = x(0).$$

  This is equivalent to $x(0)$ being an eigenvector of e^{AT} with eigenvalue 1, i.e. a Floquet multiplier equal to 1.

In this analogy:

- A plays the role of the differential operator L.
- B is the boundary operator.
- Solving $Ax = b$ with boundary constraints is the discrete analogue of a BVP.
- Solving $Au = \lambda u$ with boundary constraints is the discrete analogue of an EP.
- Periodic problems correspond to finding eigenvectors of the monodromy operator e^{AT} with eigenvalue 1.

4 Euler–Cauchy Equations

Definition 4.1 (Euler–Cauchy Equation)**.** An *Euler–Cauchy* (or *equipotential*) differential equation of order n has the form

$$a_n\, t^n \frac{d^n y}{dt^n} + a_{n-1}\, t^{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\, t\, \frac{dy}{dt} + a_0\, y = f(t),$$

where $a_0, \ldots, a_n$ are constants and $t > 0$.

4.1 Homogeneous Case (Order 2)

We focus on the second-order equation

$$t^2 y'' + \alpha\, t\, y' + \beta\, y = 0, \quad t > 0.$$

Step 1: Trial Solution (Ansatz). Seek a solution of the form

$$y(t) = t^r,$$

which gives

$$y' = r\, t^{r-1}, \quad y'' = r(r-1)\, t^{r-2}.$$

Step 2: Indicial Equation. Substituting into the ODE:

$$t^2 \big[r(r-1)t^{r-2}\big] + \alpha\, t \big[r\, t^{r-1}\big] + \beta\, t^r = \big[r(r-1) + \alpha\, r + \beta\big]\, t^r.$$

Thus, the characteristic (indicial) equation is

$$r^2 + (\alpha - 1)\, r + \beta = 0.$$

Step 3: Roots and General Solution. Let r_1, r_2 be the roots of $r^2 + (\alpha - 1)r + \beta = 0$:

- **Distinct real roots:** $r_1 \neq r_2 \in \mathbb{R}$

$$y(t) = C_1\, t^{r_1} + C_2\, t^{r_2}.$$

- **Repeated root:** $r_1 = r_2 = r$

$$y(t) = \big(C_1 + C_2 \ln t\big)\, t^r.$$

- **Complex roots:** $r_{1,2} = \mu \pm i\nu$

$$y(t) = t^\mu \left[C_1 \cos(\nu \ln t) + C_2 \sin(\nu \ln t)\right].$$

4.2 Nonhomogeneous Case

For

$$t^2y'' + \alpha\, t\, y' + \beta\, y = g(t),$$

use the substitution

$$t = e^u, \quad Y(u) = y(e^u),$$

which yields

$$\frac{dy}{dt} = \frac{1}{t}Y'(u), \quad \frac{d^2y}{dt^2} = \frac{1}{t^2}\left[Y''(u) - Y'(u)\right].$$

The equation becomes a constant-coefficient equation in u:

$$Y'' + (\alpha - 1)\, Y' + \beta\, Y = G(u), \quad G(u) = g(e^u).$$

Solve for $Y(u)$ by standard methods (e.g., undetermined coefficients, variation of parameters), then transform back:

$$y(t) = Y(\ln t).$$

5 Techniques in Matrix Theory

5.1 Powers of Matrices

Theorem 5.1 (Cayley–Hamilton Theorem)**.** Let $A \in \mathbb{F}^{n\times n}$ and let

$$\chi_A(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

be its characteristic polynomial. Then

$$\chi_A(A) \;=\; A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I \;=\; 0.$$

Proof. Consider $p(\lambda) = \lambda I - A$ and its adjugate $\operatorname{adj}(p(\lambda))$, which satisfies

$$\operatorname{adj}(p(\lambda))\, p(\lambda) = \det(p(\lambda))\, I = \chi_A(\lambda)\, I.$$

That is,

$$\operatorname{adj}(\lambda I - A)\, (\lambda I - A) = \chi_A(\lambda)\, I.$$

This is an identity in $\mathbb{F}[\lambda]$ with matrix coefficients. Substitute $\lambda = A$: since $(AI - A) = 0$, the left-hand side vanishes,

$$\operatorname{adj}(AI - A)\, (AI - A) = 0,$$

and hence

$$0 = \chi_A(A)\, I = \chi_A(A),$$

which proves the theorem. $\square$

Proposition 3 (Polynomial Representation of A^m)**.** Let $A \in \mathbb{F}^{n\times n}$ and $m \geq 0$. Suppose there are scalars $c_0, \dots, c_{n-1}$ such that

$$A^m = c_0 I + c_1 A + \cdots + c_{n-1}A^{n-1}.$$

Then this holds if and only if every eigenvalue λ of A satisfies

$$\lambda^m = c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1}.$$

Proof. $\Rightarrow$ Let λ be an eigenvalue of A with eigenvector $v \neq 0$. Then

$$A^k v = \lambda^k v, \quad k = 0, 1, \dots, m.$$

Applying the given matrix identity to v gives

$$\lambda^m v = (c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1})v.$$

Since $v \neq 0$, the scalar identity follows.

$\Leftarrow$ Suppose $\lambda^m - (c_0 + \cdots + c_{n-1}\lambda^{n-1}) = 0$ for every eigenvalue λ of A. Then

$$p(x) = x^m - (c_0 + c_1 x + \cdots + c_{n-1}x^{n-1})$$

vanishes on the spectrum of A. By the Cayley–Hamilton theorem, the minimal polynomial of A divides $p(x)$, hence $p(A) = 0$, i.e.

$$A^m = c_0 I + c_1 A + \cdots + c_{n-1}A^{n-1}.$$

$\square$

5.2 Power Method and Rayleigh Quotient

Definition 5.1 (Dominant Eigenvalue)**.** Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A \in \mathbb{F}^{n\times n}$. An eigenvalue λ_k is *dominant* if

$$|\lambda_k| > |\lambda_i|, \quad \forall\, i \neq k.$$

Any eigenvector for λ_k is called a *dominant eigenvector*.

Proposition 4 (Convergence of the Power Method)**.** Let A have a dominant eigenvalue λ_k with eigenvector v_k. For any $v_0 \neq 0$ not orthogonal to v_k, define

$$v_m = A^m v_0, \quad m = 0, 1, \ldots$$

Then

$$\frac{v_m}{\|v_m\|} \;\to\; \pm v_k$$

as $m \to \infty$. Moreover, the Rayleigh quotient

$$R(v_m) = \frac{\langle A v_m,\, v_m\rangle}{\langle v_m,\, v_m\rangle}$$

converges to λ_k.

5.3 Method of Deflation

Let A be symmetric with eigenpairs $(\lambda_1, v_1), \ldots, (\lambda_n, v_n)$ ordered so that $|\lambda_1| > |\lambda_2| \geq \cdots$. To find (λ_2, v_2) after (λ_1, v_1):

(i) **Deflate:**

$$A^{(1)} = A - \lambda_1 v_1 v_1^\top$$

removes the component along v_1.

(ii) **Apply Power Method** to $A^{(1)}$ to approximate v_2.

(iii) **Estimate** λ_2 via the Rayleigh quotient:

$$\lambda_2 \approx \frac{v_2^\top A v_2}{v_2^\top v_2}.$$

Remark 5.1. For diagonalizable but non-symmetric A, use bi-orthogonal deflation:

$$A^{(1)} = A - \lambda_1 v_1 w_1^\top,$$

where w_1 is a left eigenvector for λ_1 with $w_1^\top v_1 = 1$.

5.4 Congruent Matrices

Definition 5.2. Two square matrices A, B over $\mathbb{F}$ are *congruent* if $\exists$ invertible P such that

$$B = P^\top A P.$$

5.4.1 Properties

- Congruence is an equivalence relation (reflexive, symmetric, transitive).
- Preserves symmetry and rank.
- Preserves bilinear forms $\langle x, y\rangle = x^\top A y$.

5.4.2 Special Cases

- **Real symmetric** case: preserves the numbers of positive, negative, zero eigenvalues (Sylvester's Law of Inertia).
- **Complex Hermitian** case: replace $P^\top$ with P^*.

5.4.3 Canonical Forms for Quadratic Forms

- Over $\mathbb{R}$: there exists P s.t.

$$P^\top A P = \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  with inertia (p, q).

- Positive definite $\Rightarrow$ congruent to I_n.
- Nondegenerate 2×2: canonical elliptic and hyperbolic forms.

5.4.4 Applications

- Diagonalization/classification of quadratic forms and bilinear forms.
- Classification of conic sections via quadratic part matrix.
- Inertia tensor transformations in physics.
- Classification of PDEs by principal part.

5.4.5 Example

Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$B = P^\top A P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$$

so A and B are congruent.

5.5 Matrices as Variables

If $Av = \lambda v$ for nonzero vector v, then (v, λ) is called a eigen pair of A. Here the linear operator defined by A is between column spaces. In fact, A induces $X \mapsto AX$ between matrix spaces. If there is $AX = \lambda X$, and $X \neq 0$, then X is called a eigen matrix of A, where (λ, X) is called a generalized eigen pair of A.

Remark 5.2. Depending on the domain of our induced linear operator, the eigen vector / matrix / function denote the same mathematical entity.

Remark 5.3. Not every linear operator can be directly extracted as a matrix multiplication. For examples, $X \mapsto AX + XA, X^T, [A, X]$.

Remark 5.4 (Eigen system, spectrum)**.** All eigen pairs form the eigen system, and specturm refers to the set of eigenvalues.

6 Series Representation and Integral Representation

6.1 Orthogonal Expansions and Orthogonal Series

Let $\Phi = \{\phi_0, \phi_1, \dots\}$ be a sequence of orthonormal functions on a domain D with respect to the inner product

$$\langle f, g \rangle = \int_D f(x)\, \overline{g(x)}\, w(x)\, dx.$$

Definition 6.1 (Orthogonal Series)**.** Given f defined on D, the *associated orthogonal series* of f with respect to Φ is

$$\Phi(f)(x) = \sum_{n=0}^{\infty} \langle f, \phi_n \rangle\, \phi_n(x).$$

If $\Phi(f)$ converges to f, it is called the *orthogonal expansion / orthogonal series representation* of f.

Remark 6.1. A sequence Φ is *complete* if the only continuous function orthogonal to every ϕ_n is the zero function.

6.2 Fourier Series (Periodic)

The Fourier series is a special case of an orthogonal series.

Definition 6.2 (Real Fourier Series)**.** On $[-p, p]$, take

$$\Phi = \left\{ 1, \cos\frac{n\pi x}{p}, \sin\frac{n\pi x}{p} : n \geq 1 \right\}.$$

Then

$$\mathrm{FS}(f)(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p} \right],$$

where

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(t)\, dt, \quad a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\frac{n\pi t}{p}\, dt, \quad b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\frac{n\pi t}{p}\, dt.$$

Definition 6.3 (Complex Fourier Series)**.** On $[-p, p]$, take $\Phi = \{e^{in\pi x/p} : n \in \mathbb{Z}\}$. Then

$$\Phi(f)(x) = \sum_{n\in\mathbb{Z}} c_n e^{in\pi x/p}, \quad c_n = \frac{1}{2p}\int_{-p}^{p} f(t) e^{-in\pi t/p}\, dt.$$

Definition 6.4 (Fundamental Frequency and Spectrum)**.** The $2p$-periodic extension of f has fundamental angular frequency $\omega = \frac{\pi}{p}$. The *frequency spectrum* of f is

$$\{(n\omega, |c_n|) : n \in \mathbb{Z}\}.$$

6.3 Other Classical Orthogonal Series

Many Sturm–Liouville problems yield orthogonal expansions:

- **Legendre Series** ($[-1, 1]$, weight 1):

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x), \quad a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx.$$

- **Bessel Series** ($[0, 1]$, weight x):

$$f(x) \sim \sum_{n=1}^{\infty} a_n J_\nu(\alpha_{\nu,n} x), \quad a_n = \frac{2}{[J_{\nu+1}(\alpha_{\nu,n})]^2} \int_0^1 x f(x) J_\nu(\alpha_{\nu,n} x)\, dx,$$

  where $\alpha_{\nu,n}$ is the n-th positive zero of J_ν.

- **Chebyshev Series** ($[-1, 1]$, weight $(1 - x^2)^{-1/2}$):

$$f(x) \sim \sum_{n=0}^{\infty} a_n T_n(x), \quad a_n = \frac{2 - \delta_{n0}}{\pi} \int_{-1}^{1} \frac{f(x) T_n(x)}{\sqrt{1 - x^2}}\, dx.$$

6.4 Fourier Series in Two Variables

For $[0, b] \times [0, c]$,

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}(t) \sin \frac{m\pi x}{b} \sin \frac{n\pi y}{c}$$

or other bases depending on boundary conditions.

Example 6.1 (Heat Equation)**.**

$$u_t = k(u_{xx} + u_{yy}), \quad u|_{\partial\Omega} = 0.$$

Separation of variables yields

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t} \sin \frac{m\pi x}{b} \sin \frac{n\pi y}{c},$$

with

$$A_{mn} = \frac{4}{bc} \int_0^b \int_0^c f(x, y) \sin \frac{m\pi x}{b} \sin \frac{n\pi y}{c}\, dy\, dx.$$

6.5 Double Cosine Series

For $f(x, y)$ on $[0, b] \times [0, c]$,

$$f(x, y) = A_{00} + \sum_{m=1}^{\infty} A_{m0} \cos \frac{m\pi x}{b} + \sum_{n=1}^{\infty} A_{0n} \cos \frac{n\pi y}{c} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos \frac{m\pi x}{b} \cos \frac{n\pi y}{c}.$$

Coefficients follow from cosine orthogonality.

6.6 Integral Representations (Non-periodic)

Theorem 6.1 (Real Fourier integral (inversion))**.** Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f \in L^1(\mathbb{R})$ and f has bounded variation on every finite interval (e.g. f is piecewise C^1 with $f, f' \in L^1$). Define

$$A(\omega) = \int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx, \qquad B(\omega) = \int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx \quad (\omega \in \mathbb{R}).$$

Then for every $x \in \mathbb{R}$ at which the one-sided limits $f(x\pm)$ exist and are finite, the integral

$$\frac{1}{\pi}\int_0^{\infty}\big[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\big]\,d\omega$$

converges and equals the midpoint value

$$\frac{f(x+) + f(x-)}{2}.$$

In particular, if f is continuous at x, the integral equals $f(x)$.

Proof. Let the (nonunitary) Fourier transform be $F(\omega) = \int_{\mathbb{R}} f(x)e^{-i\omega x}\,dx$. For real f, one has $F(\omega) = A(\omega) - iB(\omega)$ with A even and B odd. The complex inversion formula gives $f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} F(\omega)e^{i\omega x}\,d\omega$ at every Lebesgue point of f. Expanding $F(\omega)e^{i\omega x}$ and using the parity of A, B kills the imaginary part and halves the domain, yielding

$$\begin{aligned} f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\big[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\big]\,d\omega \\ = \frac{1}{\pi}\int_0^{\infty}\big[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\big]\,d\omega. \end{aligned}$$

□

Remark 6.2.

(i) For even f one has $B \equiv 0$, giving the cosine transform pair; for odd f, $A \equiv 0$, giving the sine transform pair.

(ii) Other normalizations (e.g. 2π split as $(2\pi)^{-1/2}$ factors) are equivalent up to constant changes in A, B.

(iii) If $f \in L^2(\mathbb{R})$, the same formula holds in L^2 (Plancherel) sense. For functions not in L^1 (e.g. $f(x) = 1/x^2$ near 0), one needs a distributional/regularized interpretation.

Make a summary, the definition follows:

Definition 6.5 (Real Fourier Integral Representation)**.**

$$f(x) = \frac{1}{\pi}\int_0^{\infty}[A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega,$$

where

$$A(\omega) = \int_{-\infty}^{\infty} f(x)\cos\omega x dx, B(\omega) = \int_{\mathbb{R}} f(x)\sin\omega x dx.$$

Remark 6.3. Analogous to real Fourier series, there are cosine (and sine) integral representation of even (and odd) functions:

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\omega)\cos\omega x]d\omega,$$

where

$$A(\omega) = 2\int_0^\infty f(x)\cos\omega x dx.$$

Many special functions have both series and integral representations:

- **Fourier Transform: (complex Fourier integral = inverse Fourier transform)**

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,d\omega, \quad \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt.$$

- **Laplace Transform:**

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,ds, \quad F(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

- **Bessel Function:**

$$J_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos(\nu\tau - x\sin\tau)\,d\tau.$$

- **Legendre Polynomial:**

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \left(x + \sqrt{x^2-1}\cos\theta\right)^n\,d\theta.$$

- **Factorial:**

  1) Euler's (Gamma) integral on $(0,\infty)$. For $\Re s > 0$ the Gamma function is

$$\Gamma(s) = \int_0^\infty x^{\,s-1}e^{-x}\,dx.$$

  Setting $s = n+1$ gives, for $n \in \mathbb{N}$,

$$n! \;=\; \Gamma(n+1) \;=\; \int_0^\infty x^{\,n}e^{-x}\,dx.$$

  Proof (by parts): Let $I_n = \int_0^\infty x^n e^{-x}dx$. Then $I_n = [-x^n e^{-x}]_0^\infty + n\int_0^\infty x^{n-1}e^{-x}dx = n\,I_{n-1}$, with $I_0 = 1$, hence $I_n = n!$.

2) Beta–Gamma relation (integral on $(0,1)$). The Beta function

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

yields, for integers $m, n \geq 0$,

$$\int_0^1 t^n(1-t)^m\,dt = \frac{n!\,m!}{(n+m+1)!}.$$

Equivalently, $n! = \dfrac{(n+m+1)!}{m!}\displaystyle\int_0^1 t^n(1-t)^m\,dt$ (any fixed $m \geq 0$).

3) Cauchy's integral (coefficient extractor). Using $e^z = \sum_{k\geq 0}\frac{z^k}{k!}$ and Cauchy's formula, for any positively oriented circle C around 0,

$$n! \;=\; \frac{1}{2\pi i}\oint_C \frac{e^z}{z^{\,n+1}}\,dz.$$

4) Hankel's contour for the Gamma function. For $\Re s > 0$,

$$\Gamma(s) = \frac{1}{2\pi i}\int_H (-t)^{-s}e^{-t}\,dt,$$

where H is the Hankel contour encircling the nonpositive real axis. Setting $s = n+1$ gives

$$n! \;=\; \frac{1}{2\pi i}\int_H (-t)^{-(n+1)}e^{-t}\,dt.$$

Remark 6.4 (Wallis integrals, Wallis product, and Stirling's formula)**.**

1. Wallis integrals and closed forms Define

$$I_n := \int_0^{\pi/2} \sin^n x\,dx \qquad (n \in \mathbb{N}_0).$$

Integration by parts with $u = \sin^{n-1} x,\ dv = \sin x\,dx$ gives the two–step recurrence

$$I_n = \frac{n-1}{n}\,I_{n-2}, \qquad I_0 = \frac{\pi}{2},\ I_1 = 1.$$

Hence

$$I_{2n} = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{\pi}{2},$$

$$I_{2n+1} = \frac{(2n)!!}{(2n+1)!!} = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)}.$$

2. Wallis product for π Set $A_n := \dfrac{I_{2n}}{I_{2n+1}}$. Using the recurrence twice,

$$\frac{A_{n+1}}{A_n} = \frac{I_{2n+2}I_{2n+1}}{I_{2n}I_{2n+3}} = \frac{2n+1}{2n+2} \cdot \frac{2n+3}{2n+2} = \frac{(2n+1)(2n+3)}{(2n+2)^2} < 1,$$

so (A_n) is strictly decreasing. Since $\sin^{2n} x > \sin^{2n+1} x$ on $(0, \pi/2)$, we also have $A_n > 1$; thus $A_n \downarrow A$ for some $A \geq 1$.

From the closed forms,

$$A_n = \frac{I_{2n}}{I_{2n+1}} = \frac{\pi}{2} \prod_{k=1}^{n} \frac{(2k-1)(2k+1)}{(2k)^2}.$$

Taking limits and using $A = \lim A_n$ gives Wallis's product:

$$\boxed{\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{(2k)^2}{(2k-1)(2k+1)}} \qquad \Big(\text{equivalently } \lim_{n\to\infty} A_n = 1\Big).$$

3. Double factorial ratio and the central binomial coefficient From the explicit formulas,

$$\frac{(2n)!!}{(2n-1)!!} = \frac{I_{2n}}{I_{2n+1}} \cdot \sqrt{\frac{(2n)^2}{(2n+1)(2n-1)}} \sim \sqrt{\pi n} \qquad (n \to \infty),$$

so, using $(2n)!! = 2^n n!$ and $(2n-1)!! = \dfrac{(2n)!}{2^n n!}$,

$$\frac{2^{2n}(n!)^2}{(2n)!} \sim \sqrt{\pi n} \quad \Longleftrightarrow \quad \boxed{\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}}.$$

4. Stirling's formula from Wallis Define

$$a_n := \frac{n!\, e^n}{n^{n+\frac{1}{2}}} \quad (n \geq 1).$$

Then

$$\frac{a_{2n}}{a_n^2} = \frac{(2n)!}{(n!)^2} \cdot \frac{n^{2n+1}}{(2n)^{2n+\frac{1}{2}}} = \frac{\binom{2n}{n}}{2^{2n}} \cdot \frac{\sqrt{n}}{\sqrt{2}} \xrightarrow[n\to\infty]{} \frac{1}{\sqrt{2\pi}}$$

by the central–binomial asymptotic above. Since (a_n) is bounded and slowly varying (e.g. by log–convexity of Γ or standard elementary bounds), the limit $\lim_{n\to\infty} a_n = A$ exists, and the functional equation $A/A^2 = 1/\sqrt{2\pi}$ forces

$$A = \sqrt{2\pi}.$$

Hence the Stirling asymptotic

$$\boxed{n! \sim \sqrt{2\pi n} \left(\tfrac{n}{e}\right)^n} \qquad (n \to \infty).$$

The argument also yields the sharp central–binomial estimate

$$\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}}\Big(1 + O\big(\tfrac{1}{n}\big)\Big),$$

from which one can refine Stirling with error bounds if desired.

Remark 6.5. Integral representations are powerful for asymptotics, analytic continuation, and numerics; series are better for spectral/PDE computation.

6.7 Summary Table: Series vs. Integral

Function	**Domain / Weight**	**Series Form**	**Integral Form**
Fourier (Real)	$[-p, p]$, $w = 1$	$\frac{a_0}{2} + \sum a_n \cos + b_n \sin$	$\frac{1}{2\pi} \int \hat{f} e^{i\omega x}$
Fourier (Complex)	$[-p, p]$, $w = 1$	$\sum c_n e^{in\pi x/p}$	Same as above
Legendre P_n	$[-1, 1]$, $w = 1$	$\sum a_n P_n$	$\frac{1}{\pi} \int (x + \sqrt{x^2 - 1} \cos\theta)^n$
Bessel J_ν	$[0, 1]$, $w = x$	$\sum a_n J_\nu(\alpha_{\nu,n} x)$	$\frac{1}{\pi} \int \cos(\nu\tau - x \sin\tau)$
Chebyshev T_n	$[-1, 1]$, $(1 - x^2)^{-1/2}$	$\sum a_n T_n$	$T_n(\cos\theta) = \cos(n\theta)$
Laplace	$t \geq 0$	Expansion of $F(s)$ in $1/s$	$\frac{1}{2\pi i} \int F(s) e^{st}$

Remark 6.6. The phrase "series $\leftrightarrow$ integral conversions use orthogonality inversion, generating functions, or transform limits" refers to the main mechanisms for moving between discrete and continuous representations:

(i) **Orthogonality inversion.** If

$$f(x) = \sum_{n=0}^{\infty} a_n \, \phi_n(x),$$

where $\{\phi_n\}$ is orthogonal, the coefficients are obtained from

$$a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

This yields an *integral representation* for a_n. Example: Legendre expansion

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) \, dx.$$

(ii) **Generating functions.** Many orthogonal systems have a generating function $G(t, x)$ whose series in t yields the basis:

$$G(t, x) = \sum_{n=0}^{\infty} \phi_n(x) t^n.$$

Summing the series symbolically or using contour integration can lead to an integral form. Example: Legendre polynomials

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

$\Rightarrow$

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left(x+\sqrt{x^2-1}\cos\theta\right)^n d\theta.$$

(iii) **Transform limits.** Some series are discretized versions of an integral transform; taking a suitable limit converts the sum into an integral. Example: Fourier series $\to$ Fourier integral:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p} \quad \xrightarrow{p\to\infty} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\, d\omega.$$

7 Introduction to Sturm–Liouville Theory

Sturm–Liouville (S–L) theory provides a systematic framework for analyzing linear second–order differential equations, especially those arising from boundary–value problems (BVPs) in mathematical physics.

7.1 General Form

The **general Sturm–Liouville equation** is

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y=\lambda\, w(x)y, \tag{1}$$

where:

- $p(x) > 0$, $q(x)$, $w(x)$ are given real–valued coefficient functions on the interval of interest,
- $w(x) > 0$ is the *weight function*,
- $\lambda \in \mathbb{R}$ is the *spectral parameter.*

Equation (1) can be expressed using the Sturm–Liouville operator $\mathscr{L}$:

$$\mathscr{L}y \equiv -\frac{d}{dx}[p(x)y'(x)]+q(x)\,y(x),$$

so that the eigenvalue problem is

$$\mathscr{L}y=\lambda\, w(x)\, y,$$

together with suitable boundary conditions.

Remark 7.1 (Positivity, Ambient Space, and Self–Adjointness)**.** In the classical *regular* Sturm–Liouville setting on $[a, b]$, one assumes

$$p(x)>0, \quad w(x)>0, \quad p,q,w\in C[a,b].$$

Positivity of p ensures that the divergence-form operator

$$\tau[y] := -\frac{d}{dx}(p(x)\,y'(x))+q(x)\,y(x)$$

is (formally) self–adjoint with respect to the weight $w(x)$ in the Hilbert space

$$L^2_w([a,b]) \;:=\; \left\{ f:[a,b]\to\mathbb{C} \;\middle|\; \int_a^b |f(x)|^2 w(x)\,dx<\infty \right\},$$

equipped with the inner product

$$\langle f,g\rangle_w=\int_a^b f(x)\,\overline{g(x)}\,w(x)\,dx.$$

In this space:

- Eigenfunctions $\{\phi_n\}$ corresponding to distinct eigenvalues are orthogonal in $\langle\cdot,\cdot\rangle_w$.
- Under regularity conditions and separated homogeneous boundary conditions, $\{\phi_n\}$ form a complete orthogonal basis of $L^2_w([a,b])$.

Let

$$Ay := w(x)^{-1}\,\tau[y], \quad \mathcal{D}(A) = \left\{y \in H^1[a,b] \;\middle|\; \tau[y] \in L^2_w,\ \alpha_1 y(a) + \alpha_2 y'(a) = 0,\ \beta_1 y(b) + \beta_2 y'(b) = 0\right\}.$$

For $y, z \in \mathcal{D}(A)$, Green's identity yields

$$\langle Ay, z\rangle_w - \langle y, Az\rangle_w = \Big[p(x)\big(y'(x)\,\overline{z(x)} - y(x)\,\overline{z'(x)}\big)\Big]_{x=a}^{x=b}.$$

The separated homogeneous boundary conditions make this boundary term vanish, hence A is symmetric. For regular endpoints, the symmetric operator A is in fact self–adjoint. This implies:

- All eigenvalues are real.
- Eigenfunctions for distinct eigenvalues are orthogonal in L^2_w.
- The spectrum is discrete with $\lambda_n \to +\infty$.

Proof that $\lambda_n \to +\infty$ for a regular Sturm–Liouville problem. Let

$$Ay := -\frac{d}{dx}\big(p(x)y'(x)\big) + q(x)y(x)$$

on $[a,b]$ with p, q, w continuous, $p > 0$, $w > 0$ on $[a,b]$, and separated homogeneous boundary conditions that make A self–adjoint in $L^2_w([a,b])$.

The associated closed, semibounded quadratic form is

$$\mathfrak{a}[y] = \int_a^b \big(p|y'|^2 + q\,|y|^2\big)\,dx, \qquad \|y\|^2_{L^2_w} = \int_a^b w\,|y|^2\,dx,$$

with form domain $\mathcal{D}$ determined by the boundary conditions. The Rayleigh quotient is

$$R[y] = \frac{\mathfrak{a}[y]}{\|y\|^2_{L^2_w}}, \qquad y \in \mathcal{D}\setminus\{0\}.$$

By the Courant–Fischer min–max principle,

$$\lambda_n = \min_{\substack{V\subset\mathcal{D} \\ \dim V = n}} \ \max_{0\neq y\in V} R[y],$$

where the eigenvalues are ordered nondecreasingly and repeated by multiplicity.
Step 1: Neumann bracketing. Partition $[a,b]$ into N equal subintervals of length

$$\ell = \frac{b-a}{N}$$

with points $a = x_0 < x_1 < \cdots < x_N = b$. Define a comparison operator C_N by adding *Neumann* boundary conditions

$$y'(x_j) = 0, \qquad 1 \le j \le N-1,$$

at the interior cut points, while keeping the original separated conditions at a and b. This enlarges the form domain, so by the min–max monotonicity,

$$\nu_k(C_N) \;\le\; \lambda_k(A), \quad \forall\, k,$$

where $\{\nu_k(C_N)\}$ are the eigenvalues of C_N.

Step 2: Spectrum of C_N. Because of the Neumann conditions at interior points, C_N is the orthogonal direct sum of N Sturm–Liouville operators on intervals $[x_{j-1}, x_j]$ of length ℓ with Neumann boundary conditions at both ends (except possibly modified at the outermost ends; this only strengthens the bound below). Thus the spectrum of C_N is the multiset union of the spectra on the subintervals.

On each subinterval, the first Neumann eigenvalue is 0 (constants). Let

$$p_{\min} := \min_{[a,b]} p > 0, \quad w_{\max} := \max_{[a,b]} w < \infty, \quad q_{\min} := \min_{[a,b]} q.$$

By the one–dimensional Neumann Poincaré–Wirtinger inequality, for functions orthogonal to constants we have

$$\int_{x_{j-1}}^{x_j} p|y'|^2\, dx \;\ge\; p_{\min} \frac{\pi^2}{\ell^2} \int_{x_{j-1}}^{x_j} |y - \overline{y}|^2\, dx,$$

where $\overline{y}$ is the average of y over the subinterval. Moreover, $\int w|y|^2 \le w_{\max} \int |y|^2$, hence

$$R[y] \;\ge\; \frac{p_{\min}\,\pi^2}{w_{\max}\,\ell^2} + \frac{q_{\min}}{w_{\max}} \;\;\gtrsim\;\; \frac{c_1}{\ell^2} - c_0,$$

with constants c_1, c_0 depending only on $p_{\min}, w_{\max}, q_{\min}$.

Thus, on each subinterval:

- the first eigenvalue is 0,
- the second eigenvalue is at least $c_1/\ell^2 - c_0 = \dfrac{c_1 N^2}{(b-a)^2} - c_0$.

Step 3: Ordering of eigenvalues. Since C_N is the direct sum of N copies, we have

$$\nu_1(C_N) = \cdots = \nu_N(C_N) = 0, \qquad \nu_{N+1}(C_N) \;\ge\; c\,N^2 - C,$$

for positive constants c, C independent of N.

Step 4: Transfer to A. From $\nu_k(C_N) \le \lambda_k(A)$, we get

$$\lambda_{N+1}(A) \;\ge\; \nu_{N+1}(C_N) \;\ge\; c\,N^2 - C.$$

Letting $N \to \infty$ yields $\lambda_{N+1}(A) \to +\infty$, and hence

$$\boxed{\lambda_n \to +\infty \quad \text{as} \quad n \to \infty.}$$

$\square$

7.2 Regular and Singular Problems

Definition 7.1 (Regular S–L Problem)**.** The S–L problem on $[a, b]$ is *regular* if:

(i) p, p', q, w are continuous on $[a, b]$,

(ii) $p(x) > 0$, $w(x) > 0$ for all $x \in [a, b]$,

(iii) Boundary conditions are homogeneous linear:

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0.$$

Definition 7.2 (Singular S–L Problem)**.** The S–L problem is *singular* if it fails to be regular due to one or more of:

- $p(x)$ vanishes at an endpoint,
- $q(x)$ or $w(x)$ become infinite at an endpoint,
- The interval of definition is infinite.

7.3 Eigenvalues and Eigenfunctions

We write the family of problems as

$$L_\lambda\{y\} = 0,$$

where L_λ is a differential operator containing the spectral parameter λ. An *eigenvalue* λ is one for which there exists a nontrivial solution y (an *eigenfunction*) satisfying the boundary conditions.

7.4 Fundamental Properties (Regular Case)

Theorem 7.1 (Properties of Regular S–L Problems)**.** Let the S–L problem on $[a, b]$ be regular. Then:

(a) There exists an infinite sequence of real eigenvalues:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lambda_n \to +\infty \text{ as } n \to \infty.$$

(b) Each eigenvalue is *simple* (its eigenspace is one–dimensional).

(c) Eigenfunctions corresponding to distinct eigenvalues are linearly independent.

(d) The eigenfunctions are orthogonal with respect to the weight $w(x)$:

$$\int_a^b \phi_m(x)\phi_n(x)w(x)\,dx = 0, \quad m \neq n.$$

(e) The set $\{\phi_n\}$ forms a complete orthogonal basis for $L^2_w[a, b]$.

7.5 Separation of Variables and Eigenfunction Expansions

Many PDEs reduce to S–L problems via separation of variables:

1. Assume $u(x,t) = X(x)T(t)$.
2. The spatial part yields a Sturm–Liouville eigenproblem for $X(x)$.
3. The temporal part gives ODEs for $T(t)$ with λ from the spatial problem.

Solutions can then be written as

$$u(x,t) = \sum_{n=1}^{\infty} c_n \, \phi_n(x) \, T_n(t),$$

where $\{\phi_n\}$ are S–L eigenfunctions.

7.6 Classical Examples

Vibrating String (Wave Equation):

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0$$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$, $X_n(x) = \sin\frac{n\pi x}{L}$.

Legendre's Equation:

$$-\frac{d}{dx}\left[(1-x^2)y'(x)\right] = \lambda y(x), \quad x \in [-1,1]$$

$\Rightarrow \lambda_n = n(n+1)$, $\phi_n = P_n(x)$.

Bessel's Equation:

$$-\frac{d}{dx}[x\,y'(x)] + \frac{\nu^2}{x}y(x) = \lambda\,x\,y(x), \quad x > 0$$

$\Rightarrow$ Solutions are $J_\nu(\sqrt{\lambda}x)$, $Y_\nu(\sqrt{\lambda}x)$; eigenvalues from boundary conditions.

7.7 PDE Applications

Heat Equation: For $u_t = ku_{xx}$ with homogeneous Dirichlet BCs:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k(n\pi/L)^2 t} \sin\frac{n\pi x}{L}.$$

Wave Equation: For $u_{tt} = c^2 u_{xx}$ with homogeneous Dirichlet BCs:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right] \sin\frac{n\pi x}{L}.$$

7.8 Orthogonality and Completeness

The eigenfunctions of a regular S–L problem satisfy

$$\int_a^b \phi_m(x)\phi_n(x)w(x)\,dx = 0, \quad m \neq n,$$

and form a complete basis in $L^2_w[a, b]$, allowing expansions of suitable functions in the eigenbasis.

8 Introduction to PDE

Partial differential equations (PDEs), like ordinary differential equations (ODEs), are classified as linear or nonlinear. Analogous to a linear ODE, the dependent variable and its partial derivatives appear only to the first power in a linear PDE. In this and the chapters that follow, we are concerned only with linear partial differential equations.

Let u denote the dependent variable and x, y the independent variables. Then the general form of a linear second-order PDE is

$$A\,\frac{\partial^2 u}{\partial x^2} \;+\; B\,\frac{\partial^2 u}{\partial x \partial y} \;+\; C\,\frac{\partial^2 u}{\partial y^2} \;+\; D\,\frac{\partial u}{\partial x} \;+\; E\,\frac{\partial u}{\partial y} \;+\; F\,u \;=\; G(x,y),$$

where the coefficients $A, B, C, \ldots, F$ are constants or functions of x, y. When $G(x, y) \equiv 0$, equation is said to be *homogeneous*; otherwise it is *nonhomogeneous*.

Remark 8.1. If $u : \mathbb{R}^2 \to \mathbb{R}$, we write

$$Du = \Big\{ \frac{\partial u}{\partial x_i} : i = 1, 2 \Big\}, \qquad D^2 u = \Big\{ \frac{\partial^2 u}{\partial x_i \partial x_j} : i, j = 1, 2 \Big\}.$$

More generally, for $u : \mathbb{R}^n \to \mathbb{R}$ and any integer $k \geq 1$,

$$D^k u = \{ D^\alpha u : |\alpha| = k \},$$

so that

$$D^k u : \mathbb{R}^n \;\longrightarrow\; \mathbb{R}^{\binom{n+k-1}{k}}.$$

Remark 8.2 (Conic Sections, Quadratic Forms, and Canonical Forms)**.** We recall that a *conic section* in the plane is defined by a general second-degree equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

The quadratic part can be represented using a symmetric matrix and quadratic form:

$$\mathbf{x}^\top M \mathbf{x} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + bxy + cy^2$$

Two quadratic forms are *congruent* if their matrices satisfy $M' = P^\top M P$ for some invertible P. Under an appropriate congruence transformation (rotation and translation), each nondegenerate conic may be brought to one of the canonical forms:

$$\begin{aligned} \text{Ellipse: } & \frac{x'^2}{\alpha^2} + \frac{y'^2}{\beta^2} = 1, && (\Delta < 0, \text{ positive definite}) \\ \text{Parabola: } & y' = \gamma\, x'^2, && (\Delta = 0, \text{ singular}) \\ \text{Hyperbola: } & \frac{x'^2}{\alpha^2} - \frac{y'^2}{\beta^2} = 1. && (\Delta > 0, \text{ indefinite}) \end{aligned}$$

The classification corresponds to the *congruence canonical forms* of the matrix M:

$$\begin{cases} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} & \text{(ellipse/hyperbola)} \\ \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} & \text{(parabola)} \end{cases}$$

where λ_1, λ_2 have same sign for ellipses, opposite for hyperbolas.

Observe the formal similarity between:

- The quadratic form $\mathbf{x}^\top M\mathbf{x} = ax^2 + bxy + cy^2$ for conics
- The principal part $A\,u_{xx} + B\,u_{xy} + C\,u_{yy}$ of second-order PDEs

In both cases, the classification depends on the discriminant

$$\Delta = B^2 - 4AC = -4\det M$$

with identical criteria:

- $\Delta < 0$: Elliptic PDE/Ellipse (positive definite)
- $\Delta = 0$: Parabolic PDE/Parabola (singular)
- $\Delta > 0$: Hyperbolic PDE/Hyperbola (indefinite)

The congruence transformation $M' = P^\top MP$ preserves the sign of Δ (by Sylvester's Law of Inertia), maintaining the classification under coordinate changes.

8.1 Method of Separation of Variables

It is not our intention to examine procedures for finding general solutions of linear partial differential equations. Not only is it often difficult to obtain a general solution of a linear second-order PDE, but a general solution is usually not all that useful in applications. Thus our focus throughout will be on finding particular solutions of some of the important linear PDEs, that is, equations that appear in many applications.

Although there are several methods that can be tried to find particular solutions of a linear PDE, the one we are interested in at the moment is called the *method of separation of variables*. In this method, if we are seeking a particular solution of, say, a linear second-order PDE in which the independent variables are x and y, then we seek to find a **particular solution** in the form of a product of a function of x and a function of y:

$$u(x, y) = X(x)\,Y(y).$$

With this assumption, it is sometimes possible to reduce a linear PDE in two variables to two ordinary differential equations. To this end we observe that

$$u_x = X'(x)\,Y(y), \quad u_y = X(x)\,Y'(y), \quad u_{xx} = X''(x)\,Y(y), \quad u_{yy} = X(x)\,Y''(y),$$

where the primes denote ordinary differentiation.

Find product solutions of

$$u_{xx} + 4\,u_y = 0.$$

Solution. Substituting $u(x,y) = X(x)\,Y(y)$ into the PDE yields

$$X''(x)\,Y(y) + 4\,X(x)\,Y'(y) \;=\; 0.$$

Divide both sides by $4\,X(x)\,Y(y)$ to separate variables:

$$\frac{X''(x)}{4\,X(x)} + \frac{Y'(y)}{Y(y)} = 0 \quad\Longrightarrow\quad \frac{X''(x)}{4\,X(x)} = -\,\frac{Y'(y)}{Y(y)}.$$

Since the left-hand side depends only on x and the right-hand side only on y, each must be equal to a constant. It is convenient to write the separation constant as λ. Thus

$$\frac{X''(x)}{4\,X(x)} = \lambda, \quad -\frac{Y'(y)}{Y(y)} = \lambda,$$

which gives the two ODEs

$$X''(x) - 4\,\lambda\,X(x) = 0, \qquad Y'(y) + \lambda\,Y(y) = 0.$$

We consider three cases for λ:

$$\lambda = 0, \quad \lambda = -a^2 < 0, \quad \lambda = +a^2 > 0, \quad a > 0.$$

Case I. $\lambda = 0$:

$$X'' = 0, \quad Y' = 0.$$

Integrating gives

$$X(x) = c_1 + c_2\,x, \qquad Y(y) = c_3,$$

so a particular product solution is

$$u(x,y) = X(x)\,Y(y) = (c_1 + c_2\,x)\,c_3 = A_1 + B_1\,x,$$

where $A_1 = c_1c_3$ and $B_1 = c_2c_3$.

Case II. $\lambda = -a^2$:

$$X'' - 4(-a^2)\,X = X'' + 4a^2X = 0, \quad Y' + (-a^2)\,Y = Y' - a^2Y = 0.$$

The general solutions are

$$X(x) = c_4\cosh(2ax) + c_5\sinh(2ax), \qquad Y(y) = c_6e^{\,a^2y}.$$

Hence another particular solution is

$$u(x,y) = \big(c_4\cosh(2ax) + c_5\sinh(2ax)\big)\,c_6e^{\,a^2y} = A_2\,e^{\,a^2y}\cosh(2ax) \;+\; B_2\,e^{\,a^2y}\sinh(2ax),$$

where $A_2 = c_4c_6$ and $B_2 = c_5c_6$.

Case III. $\lambda = +a^2$:

$$X'' - 4a^2 X = 0, \quad Y' + a^2 Y = 0.$$

The general solutions are

$$X(x) = c_7 \cos(2ax) + c_8 \sin(2ax), \qquad Y(y) = c_9 e^{-a^2 y}.$$

Thus a further particular solution is

$$u(x,y) = \big(c_7 \cos(2ax) + c_8 \sin(2ax)\big)\, c_9 e^{-a^2 y} = A_3\, e^{-a^2 y} \cos(2ax) \;+\; B_3\, e^{-a^2 y} \sin(2ax),$$

where $A_3 = c_7 c_9$ and $B_3 = c_8 c_9$.

8.2 Classification of equations

A linear second-order partial differential equation in two independent variables with constant coefficients can be classified as one of three types. This classification depends only on the coefficients of the second-order derivatives. Of course, we assume that at least one of the coefficients A, B, and C is not zero.

Definition 8.1 (Classification of equations)**.** The linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

where A, B, C, D, E, F, and G are real constants, is said to be

- **hyperbolic** if $B^2 - 4AC > 0$,
- **parabolic** if $B^2 - 4AC = 0$,
- **elliptic** if $B^2 - 4AC < 0$.

Remark 8.3. The classification can be understood using matrix notation. Let the quadratic form associated with the second-order derivatives be written as:

$$\mathbf{x}^\top M \mathbf{x}, \quad \text{where} \quad M = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}.$$

By congruently transforming this quadratic form into its canonical form, the discriminant $\Delta = B^2 - 4AC$ determines the type of the equation, analogous to the classification of conic sections.

8.3 Classical PDEs and Boundary-Value Problems

The following are classical partial differential equations (PDEs) of mathematical physics:

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k > 0, \quad \text{(Heat equation)}$$

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad \text{(Wave equation)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \text{(Laplace's equation)}$$

These equations are known, respectively, as the **one-dimensional heat equation**, the **one-dimensional wave equation**, and **Laplace's equation** in two dimensions. Here, "one-dimensional" refers to the fact that x denotes a spatial dimension while t represents time; "two-dimensional" in means that both x and y are spatial dimensions. Laplace's equation is often written as $\nabla^2 u = 0$, where

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

8.4 Heat equation

Equation occurs in the theory of heat flow—that is, heat transferred by conduction in a rod or thin wire. The function $u(x,t)$ represents temperature. Problems in mechanical vibrations often lead to the wave equation . For purposes of discussion, a solution $u(x,t)$ of will represent the displacement of an idealized string. Finally, a solution $u(x,y)$ of Laplace's equation can be interpreted as the steady-state (time-independent) temperature distribution throughout a thin, two-dimensional plate.

Even though we have to make many simplifying assumptions, it is worthwhile to see how equations such as and arise.

Suppose a thin circular rod of length L has a cross-sectional area A and coincides with the x-axis on the interval $[0,L]$. Let us assume:

- The flow of heat within the rod takes place only in the x-direction.
- The lateral surface of the rod is insulated (no heat escapes).
- No heat is generated within the rod (no chemical or electrical sources).
- The rod is homogeneous (constant mass per unit volume $\mathbb{R}ho$).
- The specific heat capacity γ and thermal conductivity K are constants.

To derive the PDE for the temperature $u(x,t)$, we use two empirical laws of heat conduction:

(i) The quantity of heat Q in an element of mass m is

$$Q = \gamma m u, \quad (4)$$

where u is the temperature of the element.

(ii) The rate of heat flow Q_t through a cross-section is proportional to the area A and the temperature gradient:

$$Q_t = -KAu_x. \quad (5)$$

The minus sign ensures $Q_t > 0$ when $u_x < 0$ (heat flows from hot to cold).

For a thin slice between x and $x + \Delta x$, the mass is

$$m = \rho A \Delta x,$$

and the heat content is

$$Q = \gamma \rho A \Delta x \, u. \tag{6}$$

The net rate of heat flow into the slice is

$$-KAu_x(x,t) - \big[-KAu_x(x+\Delta x,t) \big] = KA\big[u_x(x+\Delta x,t) - u_x(x,t) \big]. \tag{7}$$

Differentiating the heat content with respect to time yields

$$Q_t = \gamma \rho A \Delta x \, u_t. \tag{8}$$

By conservation of energy, the net heat flow into the slice equals the rate of change of heat content:

$$KA \frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x} = \gamma \rho A \, u_t.$$

Dividing both sides by $\gamma \rho A$,

$$\frac{K}{\gamma \rho} \frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x} = u_t. \tag{9}$$

Taking the limit as $\Delta x \to 0$, this becomes

$$\frac{K}{\gamma \rho} u_{xx} = u_t.$$

Defining the thermal diffusivity

$$k = \frac{K}{\gamma \rho},$$

we obtain the classical heat equation:

$$u_t = k u_{xx}.$$

8.5 Wave Equation

Consider a string of length L (e.g., a guitar string) stretched between $x = 0$ and $x = L$. Let $u(x,t)$ denote the vertical displacement of the string at position x and time t. Assume:

- The string is perfectly flexible and homogeneous, with constant mass density ρ (mass per unit length).
- Displacements are small: $|u| \ll L$ and slopes are small: $|u_x| \ll 1$.
- The tension T is constant and large compared to other forces like gravity.
- No external forces act on the string.

For a small segment of length Δx, the net vertical force due to tension is:

$$T \sin\theta_2 - T \sin\theta_1 \approx T(\tan\theta_2 - \tan\theta_1) = T\big[u_x(x + \Delta x, t) - u_x(x, t)\big].$$

By Newton's second law, this net force equals the mass times the acceleration:

$$T\big[u_x(x + \Delta x, t) - u_x(x, t)\big] = \rho \Delta x \, u_{tt}(x, t).$$

Dividing both sides by Δx and taking the limit as $\Delta x \to 0$ gives the PDE:

$$T u_{xx} = \rho u_{tt}.$$

Rearranging, define the wave speed squared:

$$a^2 = \frac{T}{\rho}.$$

Hence, the wave equation governing the string's displacement is

$$u_{tt} = a^2 u_{xx}.$$

8.6 Laplace's equation

Laplace's equation in 2D and 3D arises in time-independent problems involving potentials (electrostatic, gravitational, fluid flow). A solution $u(x, y)$ of can represent the steady-state temperature distribution in a plate.

8.7 Boundary and Initial Conditions

Solutions of and may require initial conditions (IC) and boundary conditions (BC):

- For the heat equation: $u(x, 0) = f(x)$ (initial temperature).
- For the wave equation:

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x).$$

- Common BC types:
  - Dirichlet: $u =$ constant at boundary.
  - Neumann: $\frac{\partial u}{\partial n} = 0$ (insulated).
  - Robin: $\frac{\partial u}{\partial n} + hu = 0$ (heat loss to surroundings).

8.8 Boundary-Value Problems

Problems such as

$$\text{Solve: } a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 \le x \le L,\ t \ge 0$$

$$\text{Subject to: } \begin{cases} \text{(BC) } u(0,t) = 0, & u(L,t) = 0,\ t \ge 0 \\ \text{(IC) } u(x,0) = f(x), & \frac{\partial u}{\partial t}\Big|_{t=0} = g(x),\ 0 \le x \le L \end{cases}$$

and

$$\text{Solve: } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le a,\ 0 \le y \le b$$

$$\text{Subject to: } \begin{cases} \text{(BC) } \frac{\partial u}{\partial x}\Big|_{x=0} = 0, & \frac{\partial u}{\partial x}\Big|_{x=a} = 0,\ 0 < y < b \\ u(x,0) = 0, & u(x,b) = f(x),\ 0 < x < a \end{cases}$$

are called **boundary-value problems**. The problem is classified as a **homogeneous BVP** since the partial differential equation and the boundary conditions are homogeneous.

8.9 Variations

The partial differential equations must be modified to take into consideration internal or external influences acting on the physical system. More general forms of the one-dimensional heat and wave equations are, respectively,

$$k \frac{\partial^2 u}{\partial x^2} + F(x,t,u,u_x) = \frac{\partial u}{\partial t}$$

and

$$a^2 \frac{\partial^2 u}{\partial x^2} + F(x,t,u,u_t) = \frac{\partial^2 u}{\partial t^2}.$$

For example, if there is heat transfer from the lateral surface of a rod into a surrounding medium that is held at a constant temperature u_m, then the heat equation becomes

$$k \frac{\partial^2 u}{\partial x^2} - h(u - u_m) = \frac{\partial u}{\partial t},$$

where h is a constant.

In the function F could represent the various forces acting on the string. For example, when external, damping, and elastic restoring forces are taken into account, assumes the form

$$a^2 \frac{\partial^2 u}{\partial x^2} + \underbrace{f(x,t)}_{\text{external force}} - \underbrace{c\frac{\partial u}{\partial t}}_{\text{damping}} - \underbrace{ku}_{\text{restoring force}} = \frac{\partial^2 u}{\partial t^2}.$$

8.10 Heat equation

Consider a thin rod of length L with an initial temperature $f(x)$ throughout and whose ends are held at temperature zero for all time $t > 0$. If the rod satisfies the assumptions, then the temperature $u(x,t)$ in the rod is determined from the boundary-value problem:

$$
\begin{aligned}
k\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0 \\
u(0,t) &= 0, \quad u(L,t) = 0,\ t > 0 \\
u(x,0) &= f(x), \quad 0 < x < L
\end{aligned}
$$

Using the product $u(x,t) = X(x)T(t)$, and λ as the separation constant, leads to:

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

This gives two ODEs:

$$
\begin{aligned}
X'' + \lambda X &= 0 \\
T' + k\lambda T &= 0
\end{aligned}
$$

The boundary conditions in become $X(0)T(t) = 0$ and $X(L)T(t) = 0$, which imply:

$$X(0) = 0, \quad X(L) = 0$$

This constitutes a Sturm-Liouville problem:

$$
\begin{aligned}
X'' + \lambda X = 0,& \\
X(0) = 0, \quad X(L) = 0&
\end{aligned}
$$

The general solutions for different cases of λ are:

- $\lambda = 0$: $X(x) = c_1 + c_2 x$
- $\lambda = -\alpha^2 < 0$: $X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$
- $\lambda = \alpha^2 > 0$: $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$

Only the case $\lambda = \alpha^2 > 0$ yields non-trivial solutions. Applying the boundary conditions gives:

$$X(x) = c_2 \sin \alpha x$$

with $\alpha L = n\pi$ or $\alpha = n\pi/L$ for $n = 1, 2, 3, \ldots$

The eigenvalues and eigenfunctions are:

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

The solution for $T(t)$ is:

$$T(t) = c_3 e^{-k(n^2\pi^2/L^2)t}$$

Thus, the product solutions are:

$$u_n(x,t) = A_n e^{-k(n^2\pi^2/L^2)t} \sin\left(\frac{n\pi}{L}x\right)$$

The general solution is the infinite series:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k(n^2\pi^2/L^2)t} \sin\left(\frac{n\pi}{L}x\right)$$

The coefficients A_n are determined from the initial condition:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

8.11 Wave equation

The vertical displacement $u(x,t)$ of a vibrating string of length L is determined from:

$$\begin{aligned} a^2\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L,\ t > 0 \\ u(0,t) = 0,& \quad u(L,t) = 0,\ t > 0 \\ u(x,0) = f(x),& \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x),\ 0 < x < L \end{aligned}$$

Using separation of variables $u(x,t) = X(x)T(t)$ gives:

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

The resulting ODEs are:

$$\begin{aligned} X'' + \lambda X &= 0 \\ T'' + a^2\lambda T &= 0 \end{aligned}$$

The boundary conditions lead to the same Sturm-Liouville problem as before, with solutions:

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad \lambda_n = \frac{n^2\pi^2}{L^2}$$

The solution for $T(t)$ is:

$$T(t) = A_n\cos\left(\frac{n\pi a}{L}t\right) + B_n\sin\left(\frac{n\pi a}{L}t\right)$$

Thus, the general solution is:

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi a}{L}t\right) + B_n\sin\left(\frac{n\pi a}{L}t\right)\right]\sin\left(\frac{n\pi}{L}x\right)$$

The coefficients are determined by:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

D'Alembert's Solution For an infinite string, the wave equation can be solved using the transformation $\xi = x + at$, $\eta = x - at$, leading to:

$$u(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds \quad (14)$$

8.12 Laplace's equation

Suppose we wish to find the steady-state temperature $u(x, y)$ in a rectangular plate whose vertical edges $x = 0$ and $x = a$ are insulated, and whose upper and lower edges $y = b$ and $y = 0$ are maintained at temperatures $f(x)$ and 0, respectively. When no heat escapes from the lateral faces of the plate, we solve the following boundary-value problem:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \quad 0 < x < a,\ 0 < y < b \\
\left.\frac{\partial u}{\partial x}\right|_{x=0} &= 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0,\ 0 < y < b \\
u(x,0) &= 0, \quad u(x,b) = f(x),\ 0 \le x \le a
\end{aligned}$$

Using separation of variables $u(x, y) = X(x)Y(y)$, we obtain:

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

This leads to two ODEs:

$$\begin{aligned}
X'' + \lambda X &= 0 \\
Y'' - \lambda Y &= 0
\end{aligned}$$

The boundary conditions become:

$$X'(0) = 0, \quad X'(a) = 0, \quad Y(0) = 0$$

The Sturm-Liouville problem for $X(x)$ is:

$$\begin{aligned}
X'' + \lambda X &= 0, \\
X'(0) = 0, &\quad X'(a) = 0
\end{aligned}$$

- For $\lambda = 0$: $X(x) = c_1 + c_2x$ with $X'(0) = c_2 = 0$ and $X'(a) = 0$ implies $X(x) = c_1$

- For $\lambda = -\alpha^2 < 0$: No non-trivial solutions
- For $\lambda = \alpha^2 > 0$: $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$ with $X'(0) = \alpha c_2 = 0$ and $X'(a) = -\alpha c_1 \sin \alpha a = 0$

The eigenvalues and eigenfunctions are:

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \quad X_n(x) = \cos\left(\frac{n\pi}{a}x\right), \quad n = 0, 1, 2, \ldots$$

For $Y(y)$, we have:

- For $n = 0$: $Y(y) = c_3 + c_4 y$ with $Y(0) = c_3 = 0$ gives $Y(y) = c_4 y$
- For $n \geq 1$: $Y(y) = c_3 \cosh\left(\frac{n\pi}{a}y\right) + c_4 \sinh\left(\frac{n\pi}{a}y\right)$ with $Y(0) = c_3 = 0$ gives $Y(y) = c_4 \sinh\left(\frac{n\pi}{a}y\right)$

The general solution is:

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}y\right) \cos\left(\frac{n\pi}{a}x\right) \quad (7)$$

The coefficients are determined by:

$$A_0 = \frac{1}{ab} \int_0^a f(x) dx \quad (8)$$

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \cos\left(\frac{n\pi}{a}x\right) dx \quad (9)$$

The solution for a Dirichlet problem with different boundary conditions:

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}y\right) \sin\left(\frac{n\pi}{a}x\right)$$

where

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right) dx \quad (10)$$

For nonhomogeneous boundary conditions on all sides, we can split the problem into two subproblems with homogeneous conditions on parallel sides. The solution is the sum of the solutions to these subproblems.

Problem 1 Problem 2

$$\begin{cases} \nabla^2 u_1 = 0 \\ u_1(0, y) = 0, u_1(a, y) = 0 \\ u_1(x, 0) = f(x), u_1(x, b) = g(x) \end{cases} \qquad \begin{cases} \nabla^2 u_2 = 0 \\ u_2(0, y) = F(y), u_2(a, y) = G(y) \\ u_2(x, 0) = 0, u_2(x, b) = 0 \end{cases}$$

The complete solution is $u(x, y) = u_1(x, y) + u_2(x, y)$.

8.13 Nonhomogeneous Boundary-Value Problems

A boundary-value problem is said to be **nonhomogeneous** if either the partial differential equation or the boundary conditions are nonhomogeneous. The method of separation of variables may not be directly applicable to such problems. We examine two techniques:

(i) Change of dependent variable $u = v + \psi$ that transforms the problem into:

- An ODE problem for ψ
- A homogeneous PDE problem for v solvable by separation of variables

(ii) Orthogonal series expansions approach

8.14 Time Independent PDE and BCs

Consider the problem:

$$k\frac{\partial^2 u}{\partial x^2} + F(x) = \frac{\partial u}{\partial t}, \quad 0 < x < L, t > 0$$
$$u(0,t) = u_0, \quad u(L,t) = u_1, t > 0$$
$$u(x,0) = f(x), \quad 0 < x < L$$

This models heat distribution in a rod with internal heat generation $F(x)$ and fixed boundary temperatures.

8.15 Wave Equation

Consider a string of length L (e.g., a guitar string) stretched between $x = 0$ and $x = L$. Let $u(x,t)$ denote the vertical displacement of the string at position x and time t. Assume:

- The string is perfectly flexible and homogeneous, with constant mass density ρ (mass per unit length).
- Displacements are small: $|u| \ll L$ and slopes are small: $|u_x| \ll 1$.
- The tension T is constant and large compared to other forces like gravity.
- No external forces act on the string.

For a small segment of length Δx, the net vertical force due to tension is:

$$T\sin\theta_2 - T\sin\theta_1 \approx T(\tan\theta_2 - \tan\theta_1) = T\big[u_x(x+\Delta x, t) - u_x(x,t)\big].$$

By Newton's second law, this net force equals the mass times the acceleration:

$$T\big[u_x(x+\Delta x, t) - u_x(x,t)\big] = \rho\Delta x\, u_{tt}(x,t).$$

Dividing both sides by Δx and taking the limit as $\Delta x \to 0$ gives the PDE:

$$Tu_{xx} = \rho u_{tt}.$$

Rearranging, define the wave speed squared:

$$a^2 = \frac{T}{\rho}.$$

Hence, the wave equation governing the string's displacement is

$$u_{tt} = a^2 u_{xx}.$$

Let $u(x,t) = v(x,t) + \psi(x)$, which transforms the problem into:
Problem A (ODE for ψ):

$$k\psi'' + F(x) = 0$$
$$\psi(0) = u_0, \quad \psi(L) = u_1$$

Problem B (Homogeneous PDE for v):

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$$
$$v(0,t) = 0, \quad v(L,t) = 0$$
$$v(x,0) = f(x) - \psi(x)$$

The solution is $u(x,t) = \psi(x) + v(x,t)$.
Solve:

$$k\frac{\partial^2 u}{\partial x^2} + r = \frac{\partial u}{\partial t}, \quad 0 < x < 1, t > 0$$
$$u(0,t) = 0, \quad u(1,t) = u_1, t > 0$$
$$u(x,0) = f(x), \quad 0 < x < 1$$

Solution:

1. Let $u(x,t) = v(x,t) + \psi(x)$
2. Solve $\psi'' = -\frac{r}{k}$ with $\psi(0) = 0$, $\psi(1) = u_1$:

$$\psi(x) = -\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_1\right)x$$

3. Solve the homogeneous problem for $v(x,t)$:

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x$$

where

$$A_n = 2\int_0^1 [f(x) - \psi(x)] \sin n\pi x \, dx$$

4. Final solution:

$$u(x,t) = \psi(x) + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x$$

For problems like:

$$\begin{aligned} k\frac{\partial^2 u}{\partial x^2} + F(x,t) = \frac{\partial u}{\partial t}, \quad 0 < x < L, t > 0 \\ u(0,t) = u_0(t), \quad u(L,t) = u_1(t), t > 0 \\ u(x,0) = f(x), \quad 0 < x < L \end{aligned}$$

we use a different approach with $\psi(x,t) = u_0(t) + \frac{x}{L}[u_1(t) - u_0(t)]$.
Solve:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, t > 0 \\ u(0,t) = \cos t, \quad u(1,t) = 0, t > 0 \\ u(x,0) = 0, \quad 0 < x < 1 \end{aligned}$$

Solution:

1. Let $u(x,t) = v(x,t) + (1-x)\cos t$
2. The transformed problem for $v(x,t)$:

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} + (1-x)\sin t = \frac{\partial v}{\partial t} \\ v(0,t) = 0, \quad v(1,t) = 0 \\ v(x,0) = x - 1 \end{aligned}$$

3. Expand in eigenfunctions $\sin n\pi x$:

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin n\pi x$$

$$(1-x)\sin t = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin t \sin n\pi x$$

4. Solve ODEs for coefficients $v_n(t)$
5. Final solution combines $\psi(x,t)$ and series solution

For problems with homogeneous BCs, we can directly assume:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi}{L} x$$

and expand $F(x,t)$ similarly.

8.16 Using Orthogonal Series Expansions

The temperature in a rod of unit length with heat transfer at the right boundary is determined from:

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, t > 0$$
$$u(0,t) = 0$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=1} = -hu(1,t), \quad h > 0, t > 0$$
$$u(x,0) = 1, \quad 0 \le x \le 1$$

Using separation of variables $u(x,t) = X(x)T(t)$ with $-\lambda$ as the separation constant:

$$X'' + \lambda X = 0$$
$$T' + k\lambda T = 0$$
$$X(0) = 0 \quad \text{and} \quad X'(1) = -hX(1)$$

The Sturm-Liouville problem is:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(1) + hX(1) = 0$$

For $\lambda = \alpha^2 > 0$, the general solution is $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$.

- From $X(0) = 0$: $c_1 = 0$, so $X(x) = c_2 \sin \alpha x$
- From $X'(1) = -hX(1)$: $\alpha \cos \alpha = -h \sin \alpha$ or $\tan \alpha = -\alpha/h$

The eigenvalues $\lambda_n = \alpha_n^2$ are solutions to $\tan \alpha_n = -\alpha_n/h$, with eigenfunctions $X_n(x) = \sin \alpha_n x$.

The solution takes the form:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin \alpha_n x$$

Initial condition at $t = 0$:

$$1 = \sum_{n=1}^{\infty} A_n \sin \alpha_n x$$

The coefficients A_n are found using orthogonality:

$$A_n = \frac{\int_0^1 \sin \alpha_n x \, dx}{\int_0^1 \sin^2 \alpha_n x \, dx}$$

Evaluating the integrals:

$$\int_0^1 \sin \alpha_n x \, dx = \frac{1 - \cos \alpha_n}{\alpha_n}$$
$$\int_0^1 \sin^2 \alpha_n x \, dx = \frac{1}{2} - \frac{\sin 2\alpha_n}{4\alpha_n} = \frac{h + \cos^2 \alpha_n}{2h}$$

Thus:

$$A_n = \frac{2h(1-\cos\alpha_n)}{\alpha_n(h+\cos^2\alpha_n)}$$

$$u(x,t) = 2h\sum_{n=1}^{\infty} \frac{1-\cos\alpha_n}{\alpha_n(h+\cos^2\alpha_n)} e^{-k\alpha_n^2 t}\sin\alpha_n x$$

The twist angle $\theta(x,t)$ of a vibrating shaft is determined from:

$$\begin{aligned}
a^2\frac{\partial^2\theta}{\partial x^2} &= \frac{\partial^2\theta}{\partial t^2}, \quad 0<x<1, t>0 \\
\theta(0,t) &= 0 \\
\left.\frac{\partial\theta}{\partial x}\right|_{x=1} &= 0, \quad t>0 \\
\theta(x,0) &= x, \quad \left.\frac{\partial\theta}{\partial t}\right|_{t=0} = 0, \quad 0<x<1
\end{aligned}$$

The Sturm-Liouville problem is:

$$X'' + \lambda X = 0, \quad X(0)=0, \quad X'(1)=0$$

- For $\lambda = \alpha^2 > 0$: $X(x) = c_1\cos\alpha x + c_2\sin\alpha x$
- Boundary conditions give $\alpha_n = \frac{(2n-1)\pi}{2}$ and $X_n(x) = \sin\left(\frac{(2n-1)\pi}{2}x\right)$

The solution form:

$$\theta(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(a\frac{2n-1}{2}\pi t\right)\sin\left(\frac{2n-1}{2}\pi x\right)$$

Initial condition:

$$x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2}\pi x\right)$$

Coefficients:

$$A_n = \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2}$$

$$\theta(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}\cos\left(a\frac{2n-1}{2}\pi t\right)\sin\left(\frac{2n-1}{2}\pi x\right)$$

9 Integral Evaluations

9.1 Improper Integrals on Symmetric Intervals and Cauchy Principal Value

Suppose f is continuous on the open interval $(-q, q)$. The improper integral

$$\int_{-q}^{q} f(x)\,dx$$

is defined by two distinct limits:

$$\int_{-q}^{q} f(x)\,dx = \lim_{r\to q^-} \int_{-r}^{0} f(x)\,dx + \lim_{R\to q^-} \int_{0}^{R} f(x)\,dx. \tag{5}$$

If both limits in (5) exist finitely, the integral is said to be *convergent*. If one or both limits fail to exist, the integral is *divergent*.

When we know *a priori* that the integral converges, we can evaluate it by the *symmetric* limit:

$$\int_{-q}^{q} f(x)\,dx = \lim_{R\to q^-} \int_{-R}^{R} f(x)\,dx. \tag{6}$$

The symmetric limit in (6) may exist even if the improper integral is divergent in the usual sense.

Example: Consider

$$\int_{-q}^{q} x\,dx.$$

The improper integral is divergent since

$$\lim_{R\to q^-} \int_{-R}^{0} x\,dx = \lim_{R\to q^-} \left(\frac{-R^2}{2}\right) = -\infty,$$

and similarly,

$$\lim_{R\to q^-} \int_{0}^{R} x\,dx = \lim_{R\to q^-} \left(\frac{R^2}{2}\right) = +\infty.$$

However, using the symmetric limit (6), we obtain

$$\lim_{R\to q^-} \int_{-R}^{R} x\,dx = \lim_{R\to q^-} \left[\frac{x^2}{2}\right]_{-R}^{R} = \lim_{R\to q^-} \left(\frac{R^2}{2}\frac{(-R)^2}{2}\right) = 0. \tag{7}$$

This symmetric limit is called the *Cauchy principal value* of the integral and is denoted by

$$\text{P.V.} \int_{-q}^{q} f(x)\,dx = \lim_{R\to q^-} \int_{-R}^{R} f(x)\,dx.$$

Remark 9.1. The Cauchy principal value (P.V.) can be understood as a form of *weak integral* or *distributional integral* in many contexts. It essentially extends the notion of integration to handle certain singularities or divergences by interpreting the integral as a limit against test functions, i.e., in a weak (distributional) sense rather than a classical point-wise sense.

Theorem 9.1 (Behavior of Integral as $R \to \infty$)**.** Suppose

$$f(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials with $\deg P = n$ and $\deg Q = m$, and

$$m \geq n + 2.$$

Let C_R be the semicircular contour defined by

$$z = Re^{i\theta}, \quad \theta \in [0, \pi].$$

Then

$$\int_{C_R} f(z)\, dz \to 0 \quad \text{as } R \to \infty.$$

Theorem 9.2. Suppose

$$f(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials with degrees

$$\deg P = n, \quad \deg Q = m,$$

and

$$m \geq n + 1.$$

Let C_R be the semicircular contour

$$z = Re^{i\theta}, \quad \theta \in [0, \pi],$$

and let $a \geq 0$ be a real parameter.

Then

$$\lim_{R \to \infty} \int_{C_R} \frac{P(z)}{Q(z)} e^{iaz}\, dz = 0.$$

Theorem 9.3. The *Fourier integral* of a function $f(x)$ is an integral representation expressing f as a continuous superposition of complex exponentials. It serves as the *inverse Fourier transform.*

Given the Fourier transform $\hat{f}(\omega)$ defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx,$$

the function $f(x)$ can be recovered by the inverse Fourier transform:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega.$$

This integral representation

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \right) e^{i\omega x} d\omega$$

is called the *Fourier integral* and generalizes the Fourier series to functions defined on the entire real line.

Theorem 9.4. Suppose f has a simple pole at $z = c$, where $c \in \mathbb{R}$.

Define the semicircular contour

$$C_r = \{z = c + re^{i\theta} : \theta \in [0, \pi]\},$$

a semicircle of radius r centered at c in the upper half-plane.

Then,

$$\lim_{r \to 0} \int_{C_r} f(z)\, dz = \pi i \ \mathrm{Res}(f, c).$$

Remark 9.2 (Classical integral vs. weak integral)**.** A classical (Lebesgue or Riemann) integral of a function f over some domain is a limit of sums of function values times small intervals, requiring certain regularity and absolute integrability.

A weak integral or distributional integral involves interpreting integrals in terms of how f acts on a space of test functions φ (usually smooth functions with compact support).

Cauchy Principal Value as a distribution:

For functions with singularities or integrals that diverge normally (like

$$\int_{-\infty}^{\infty} \frac{1}{x}\, dx),$$

the principal value integral can be seen as a distribution called the principal value distribution p.v.$\frac{1}{x}$.

This distribution is defined by how it acts on test functions φ:

$$\langle \mathrm{p.v.}\frac{1}{x}, \varphi \rangle = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\varphi(x)}{x}\, dx.$$

Here the limit avoids the singularity symmetrically, much like the symmetric limit in the Cauchy principal value.

Interpretation:

The P.V. integral effectively renormalizes or regularizes an otherwise divergent integral by symmetric limiting.

This regularization matches the way distributions extend classical functions, allowing us to assign finite values to otherwise divergent integrals, but only when tested against smooth functions.

In this sense, the P.V. is a weak notion of integration.

Connections in analysis and physics:

The Cauchy principal value is fundamental in the theory of singular integral operators, Hilbert transforms, and boundary values of analytic functions.

It often arises in solving PDEs, harmonic analysis, and quantum field theory as a way to handle divergent integrals meaningfully.

Summary:

The Cauchy principal value is indeed a prototype example of a weak integral or distributional integral.

It replaces classical integration by a limit process symmetric around singularities, making it possible to define integrals for functions not integrable in the usual sense.

Formally, it corresponds to a distribution acting on test functions, fitting perfectly into the framework of weak (distributional) integration.

If you want, I can provide a more formal treatment or examples showing how P.V. integrals appear as distributions!

10 Conformal Mappings

Let C_1 and C_2 be two curves intersecting at a point, with tangent vectors

$$z_1' \quad \text{and} \quad z_2',$$

respectively.

The angle α between C_1 and C_2 at the point of intersection is given by

$$\cos\alpha = \frac{\operatorname{Re}\left(z_1'\overline{z_2'}\right)}{|z_1'|\,|z_2'|}.$$

Here, z_1' and z_2' are complex numbers representing the tangent vectors to the curves at the intersection point. The numerator $\operatorname{Re}(z_1'\overline{z_2'})$ corresponds to the dot product of the two vectors when viewed as vectors in $\mathbb{R}^2$. The denominator $|z_1'|\,|z_2'|$ normalizes by the magnitudes of the tangent vectors. Thus, this formula gives the cosine of the angle between the two tangent vectors, and hence the angle α between the curves.

11 Harmonic Functions and the Dirichlet Problem

Definition 11.1. A twice continuously differentiable function $u = u(x, y)$ defined on a region $R \subset \mathbb{R}^2$ is called *harmonic* if it satisfies Laplace's equation:

$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{in } R.$$

Dirichlet Problem. Given a bounded domain R with boundary ∂R, and a continuous function g defined on ∂R, the *Dirichlet problem* is to find a function u such that

$$\begin{cases} \Delta u = 0 & \text{in } R, \\ u = g & \text{on } \partial R. \end{cases}$$

A function u that satisfies the above is called a *solution to the Dirichlet problem* on R.

Remarks:

- The solution u is harmonic and bounded in R.
- The boundary condition $u = g$ prescribes the values of u on the entire boundary ∂R.
- Under suitable conditions on R and g, the Dirichlet problem has a unique solution.

12 Complex Viewpoints in Engineering

12.1 Vector Fields and Analyticity

(i) Suppose that

$$\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$$

is a vector field in a domain D, and $P(x,y)$ and $Q(x,y)$ are continuous and have continuous first partial derivatives in D. If

$$\operatorname{div}\mathbf{F} = 0 \quad \text{and} \quad \operatorname{curl}\mathbf{F} = 0,$$

then the complex function

$$g(z) = P(x,y) + iQ(x,y)$$

is analytic in D.

(ii) Conversely, if $g(z)$ is analytic in D, then

$$\mathbf{F}(x,y) = g(z)$$

defines a vector field in D for which

$$\operatorname{div}\mathbf{F} = 0 \quad \text{and} \quad \operatorname{curl}\mathbf{F} = 0.$$

12.2 Potential Functions

Suppose that $F(x,y)$ is a vector field in a simply connected domain D with both $\operatorname{div} F = 0$ and $\operatorname{curl} F = 0$. By Theorem 18.3.3, the analytic function

$$g(z) = P(x,y) + iQ(x,y)$$

has an antiderivative

$$G(z) = f(x,y) + ic(x,y) \tag{4}$$

in D, which is called a *complex potential* for the vector field F.

Note that

$$g(z) = G'(z) = \frac{\partial f}{\partial x} + i\frac{\partial c}{\partial x} = \frac{\partial f}{\partial x} + 2i\frac{\partial f}{\partial y}$$

and so

$$\frac{\partial f}{\partial x} = P \quad \text{and} \quad \frac{\partial f}{\partial y} = Q. \tag{5}$$

Therefore, $F = \nabla f$ and the harmonic function f is called a (real) *potential function* for F.[1]

If F is an electric field, the electric potential function φ is defined to be $-f$ and

$$F = -\nabla\varphi.$$

[1] When the potential f is specified on the boundary of a region R, we can use conformal mapping techniques to solve the resulting Dirichlet problem. The equipotential lines $f(x,y) = c$ can be sketched and the vector field F can be determined using (5).

12.3 Circulation and Flux

Let $g(z) = P(x,y) + i\,Q(x,y)$ be a complex-valued function representing the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$.

Circulation: The circulation of $\mathbf{F}$ around a closed curve C is given by the line integral

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \mathrm{Re}\oint_C g(z)\,dz,$$

where Re denotes the real part.

Net Flux: The net outward flux of $\mathbf{F}$ across C can be expressed as

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C P\,dy - Q\,dx = \mathrm{Im}\oint_C g(z)\,dz,$$

where Im denotes the imaginary part.

12.4 Steady-State Fluid Flow

The vector $V(x,y) = P(x,y) + iQ(x,y)$ may also be interpreted as the velocity vector of a two-dimensional steady-state fluid flow at a point (x,y) in a domain D. The velocity at all points in the domain is therefore independent of time, and all movement takes place in planes that are parallel to a z-plane.

The physical interpretation of the conditions $\operatorname{div} V = 0$ and $\operatorname{curl} V = 0$ was discussed in Section 9.7. Recall that if $\operatorname{curl} V = 0$ in D, the flow is called *irrotational.* If a small circular paddle wheel is placed in the fluid, the net angular velocity on the boundary of the wheel is zero, and so the wheel will not rotate. If $\operatorname{div} V = 0$ in D, the flow is called *incompressible.* In a simply connected domain D, an incompressible flow has the special property that the amount of fluid in the interior of any simple closed contour C is independent of time. The rate at which fluid enters the interior of C matches the rate at which it leaves, and consequently there can be no fluid sources or sinks at points in D.

If $\operatorname{div} V = 0$ and $\operatorname{curl} V = 0$, then V has a complex velocity potential

$$G(z) = f(x,y) + ic(x,y)$$

that satisfies $G'(z) = V$. In this setting, special importance is placed on the level curves $c(x,y) = c$.

If $z(t) = x(t) + iy(t)$ is the path of a particle (such as a small cork) that has been placed in the fluid, then

$$\frac{dx}{dt} = P(x,y) \tag{6}$$
$$\frac{dy}{dt} = Q(x,y).$$

Hence,

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \quad \text{or} \quad Q(x,y)\,dx - P(x,y)\,dy = 0.$$

This differential equation is exact, since $\operatorname{div} V = 0$ implies

$$\frac{\partial Q}{\partial y} = \frac{\partial P}{\partial x}.$$

By the Cauchy–Riemann equations,

$$\frac{\partial c}{\partial x} = -\frac{\partial f}{\partial y} = -Q \quad \text{and} \quad \frac{\partial c}{\partial y} = \frac{\partial f}{\partial x} = P,$$

and therefore all solutions of (6) satisfy

$$c(x, y) = \text{constant}.$$

The function $c(x, y)$ is therefore called a *stream function* and the level curves $c(x, y) = c$ are *streamlines* for the flow.

12.5 Area Integrals and Contour Integrals of Complex Functions (Cauchy-Green Formula)

The Cauchy-Green formula serves as a bridge connecting area integrals inside a domain with contour integrals along its boundary:

For a complex differentiable function $f = u + iv$,

$$\oint_C f(z)\, dz = 2i \iint_D \frac{\partial f}{\partial \bar{z}}\, dx\, dy,$$

where $C = \partial D$ is the boundary of the domain D, and

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

is the **Wirtinger derivative**.

When f is analytic, $\frac{\partial f}{\partial \bar{z}} = 0$, so

$$\oint_C f(z)\, dz = 0,$$

which is the Cauchy integral theorem.

The complex derivative $f'(z)$ relates to area integrals as follows:

Cauchy integral formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz,$$

Differentiating with respect to z_0 gives

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz,$$

which shows the complex derivative can be expressed via contour integrals.

The Wirtinger derivative is used for area integrals of non-analytic functions:

For a complex function that is not analytic, $\frac{\partial f}{\partial \bar{z}} \neq 0$. The Cauchy-Green formula expresses the contour integral in terms of an area integral, explicitly revealing the non-analytic part of the function.

12.6 Summary of the Approach

Use the Cauchy-Green formula to convert contour integrals into area integrals, where the integrand involves $\frac{\partial f}{\partial \bar{z}}$, thus combining area integrals with complex derivatives (or Wirtinger derivatives).

Use contour integral expressions involving derivatives to compute function values and derivatives.

Both numerically and theoretically, area integrals (over domains) and contour integrals (over boundaries) can be combined and converted via derivative relations.

12.7 Applications

- Compute values and derivatives of analytic functions using the Cauchy integral formula and its derivative formula.
- Compute contour integrals of non-analytic functions by converting them into area integrals involving $\frac{\partial f}{\partial \bar{z}}$ via the Cauchy-Green formula.
- Analyze non-analytic functions through decomposition by Wirtinger derivatives and study their properties via area integrals.

Content	**Explanation**
Contour integral $\leftrightarrow$ area integral	Linked by the Cauchy–Green formula; the area integral uses the Wirtinger derivative $\partial_{\bar{z}}$.
Complex derivative expression	Complex derivatives can be represented by contour integrals with second–order poles, connecting derivatives and integrals.
Wirtinger derivative	Used to analyze non-analytic functions; the area integral explicitly reflects the non-analytic part.
Combined approach	Integrates contour integrals, area integrals, and derivatives to study function values, derivatives, and structural properties.

Remark 12.1 (Constructing Special Flows)**.** The process of constructing an irrotational and incompressible flow that remains inside a given region R is called *streamlining.* Since the streamlines are described by $c(x, y) = c$, two distinct streamlines do not intersect. Therefore, if the boundary is itself a streamline, a particle that starts inside R cannot leave R. This is the content of the following theorem:

Theorem 12.1 (Streamlining)**.** Suppose that

$$G(z) = f(x, y) + ic(x, y)$$

is analytic in a region R and $c(x, y)$ is constant on the boundary of R. Then

$$V(x, y) = G'(z)$$

defines an irrotational and incompressible fluid flow in R. Moreover, if a particle is placed inside R, its path $z = z(t)$ remains in R.

Complex Functions as Flows We also may interpret a complex function $w = f(z)$ as a two-dimensional fluid flow by considering the complex number $f(z)$ as a vector based at the point z. The vector $f(z)$ specifies the speed and direction of the flow at a given point z.

If $x(t) + iy(t)$ is a parametric representation for the path of a particle in the flow, the tangent vector $T = x'(t) + iy'(t)$ must coincide with $f(x(t) + iy(t))$. When $f(z) = u(x, y) + iv(x, y)$, it follows that the path of the particle must satisfy the system of differential equations

$$\frac{dx}{dt} = u(x, y)\frac{dy}{dt} = v(x, y).$$

We call the family of solutions of this system the *streamlines* of the flow associated with $f(z)$.