Singularities of Normal Functions

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*Based on joint work with Mark Green, building in part on earlier joint work with Rodolfo Aguilar.

Outline

I. Introduction

Normal functions are basic Hodge theoretic invariants of a pair (X, Z) where X is a smooth projective variety and $Z \subset X$ is an algebraic subvariety or an algebraic cycle. Attached to a normal function ν is an infinitesimal invariant $\delta\nu$. Of particular interest is the analysis of ν and $\delta\nu$ when the pair (X, Z) degenerates to a singular pair (X_0, Z_0). Attached to ν_{lim} and $(\delta\nu)_{\mathsf{lim}}$ are the singularities $\mathrm{sing}(\nu)$ and $\mathrm{sing}(\delta\nu)$, and a basic structral result in a natural identification

$$\boxed{\mathrm{sing}(\nu) = \mathrm{sing}(\delta\nu)}$$

After recalling background material, in this talk we will sketch a proof of the above result. A basic ingredient in the argument is a nilpotent orbit theorem for normal functions. Since the singularities of a normal function occur in codimension ≥ 2 there are subtleties in this construction that are not present in the classical nilpotent orbit theorem.

Following the proof of the main result we will discuss a method for computing $\delta\nu_Z$ in practice and will show how this can be used to resolve an old question regarding lines on quintic 3-folds.

II. Background

- A basic invariant of a smooth projective variety X is the polarized Hodge structure (PHS) on its cohomology groups $H^n(X, \mathbb{Q})$. In the classical case when the period domain $\mathbb{D}$ of all PHS's of the same type is Hermitian symmetric one may attach algebro-geometric objects, such as theta-divisors, to $H^n(X, \mathbb{Q})$. In the non-classical case, since a general PHS of this type does not arise from the cohomology of an algebraic variety this is not possible, and for a variety X_t varying in a family the "surrogate" object of an infinitesimal variation of Hodge structure (IVHS) has been of use in extracting geometric information from the family.

- For a pair (X, Z) one has the functorial mixed Hodge structure (MHS) defined on $H^n(X, Z; \mathbb{Q})$. There is a corresponding notion of variation of mixed Hodge structure (VMHS) where the pair varies in a family. Although there is the definition of a period mapping and its differential, so far as we are aware in this case a cohomological formula for the differential has only recently been derived and used. There is also the concept of an infinitesimal variation of mixed Hodge structrue (IVMHS), and again so far as we are aware this has also only recently been formalized.

- A particular type of pair is when X is smooth projective and $Z \subset X$ is an algebraic cycle whose fundamental class $[Z]$ is zero in integral homology. In this talk, for simplicity of notation we will take the case

$$\dim X = 2m - 1$$

$$\dim Z = m - 1 \quad (\text{thus } \operatorname{codim}_X Z = m)$$

Denoting by $|Z|$ the support of the cycle Z, there are invariants of the MHS on $H^{2m-1}(X, |Z|)$, one of which is the Abel-Jacobi image $AJ_X(Z) \in J(X)$ (notation explained below). When we have a family $\{(X_t, Z_t), t \in B\}$ the corresponding $AJ_{X_t}(Z_t) \in J(X_t)$ give a <u>normal function</u>

$\nu : B \to \mathcal{J}$ where $\mathcal{J}$ is the family of $J(X_t)$'s. One may extract from the differential of the period mapping arising from the $H^{2m-1}(X_t, |Z_t|)$'s the infinitesimal invariant $\delta\nu$, alluded to above and defined below.

- For the study of singularities we will be interested in the case when B is smooth with smooth relative completion $\overline{B}$ when $B = \overline{B} \setminus D$ with D a normal crossing divisor. Thus $D = \bigcup D_I$ with strata $D_I = \bigcap_{i \in I} D_i$. Usually, we shall take

$$B = (\Delta^*)^r \subset \Delta^r = \overline{B} \quad \text{with coordinates } t = (t_1, ..., t_r).$$

III. Notations

- Hodge structure (HS) is $(H, F^\bullet)$ where $H = H_{\mathbb{Z}} \otimes \mathbb{C}$, $F^n \subset F^{n-1} \subset \cdots F^0 = H$ with $n =$ weight; opposite condition $F^p \oplus \bar{F}^{n-p+1} \xrightarrow{\sim} H$ for $0 \leq p \leq n$ satisfied, assumed polarized by $Q : H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q}$;
- Unless otherwise mentioned $n = 2m - 1$ and

$$
\begin{tikzcd}[column sep=small]
 & F^m H^{2m-1}(X) \backslash H^{2m-1}/H_{\mathbb{Z}}^{2m-1} \\
J(X) \arrow[ur, phantom, sloped, "="] \arrow[dr, phantom, sloped, "="] & \\
 & F^m \overset{\vee}{H^{2m-1}}(X)/H_{2m-1}(X, \mathbb{Z})
\end{tikzcd}
$$

- Mixed Hodge structure (MHS) is $(H, W_\bullet, F^\bullet)$ with $W_0 \subset W_1 \subset \cdots \subset W_k/\mathbb{Q}$, and where $F^\bullet$ induces a HS of weight n on $\mathrm{Gr}^n_W(H)$.
- $AJ_X : \mathrm{CH}^m(X)_{\mathrm{hom}} \to J(X)$

$$\begin{array}{ll} & \begin{cases} 0 \to H^{2m-2}(|Z|)_0 \to H^{2m-1}(X, |Z|) \to H^{2m-1}(X) \to 0, \\ \text{and extension class } e_{X,Z} \in \mathrm{Ext}^1_{MHS}(H^{2m-1}(X), H^{2m-2}(Z)_0) \end{cases} \\ AJ_X(Z) & \\ & \begin{cases} \langle AJ_X(Z), \omega \rangle = \int_\Gamma \omega, \quad \omega \in F^m H^{2m-1}(X), \\ \text{and } \Gamma \in H_{2m-1}(X, |Z|) \text{ with } \partial\Gamma = Z \end{cases} \end{array}$$

- Higher AJ maps are defined on algebraic cycles in $X \times (\mathbb{P}^1 \setminus \{0, \infty\})^n$

$$AJ_X^n : CH^m(X, n)_{\text{hom}} \to J(X, n) \begin{array}{l} \nearrow F^m \setminus H^{2m-n-1} / H_{\mathbb{Z}}^{2m-n-1} \\ \qquad\quad \vee \\ \searrow F^m H^{2m-n-1} / H_{\mathbb{Z}, 2m-n-1} \end{array}$$

where the top term is a class in $\mathrm{Ext}^1_{MHS}(\cdot, \cdot)$, and the second term is $\int_\Gamma \omega + T_Z(\omega)$ where T_z is the current given by a differential form in L^1 involving $\log z_i$'s and dz_i/z_i's.

- MHS's appearing in Abel-Jacobi maps have special properties
  - length 2
  - one term is a $\mathbb{Z}(-(m-1)) \cong H_{\mathbb{Z}}^{m-1,m-1}$ (n=0 case)
- Variation of Hodge structure (VHS) is $\{\mathcal{H}, \mathcal{F}^{\bullet}, \nabla; B\}$ where Gauss-Manin connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_B$ has
  - $\nabla^2 = 0$
  - $\mathcal{H}_{\nabla} \cong \mathbb{H}_{\mathbb{C}}$ where $\mathbb{H}_{\mathbb{Z}} \to B$ a local system
  - $\mathcal{F}^p \xrightarrow{\nabla} \mathcal{F}^{p-1} \otimes \Omega^1_B$

  There is also the notation of a variation of mixed Hodge structure (VMHS); weight filtration preserved by ∇.

IV. Normal Functions and their Infinitesimal Invariants

- $\mathcal{J} = \mathcal{F}^m \setminus \mathcal{H}/\mathbb{H}_{\mathbb{Z}}$; for $\nu : B \to \mathcal{J}$ and for any local lift $\hat{\nu} : B \to \mathcal{H}$ the quasi-horizontality condition

$$\nabla \hat{v} \in \mathcal{F}^{m-1} \otimes \Omega^1_B$$

  is well defined , non-trivial for $m \geq 2$, and gives the sheaf $\mathcal{J}_h \subset \mathcal{J}$ of normal functions.

- Normal functions are a particular type of VMHS

- Koszul type complex

$$\mathcal{K}^\bullet = \{\mathcal{F}^m \xrightarrow{\nabla} \mathcal{F}^{m-1} \otimes \Omega^1_B \xrightarrow{\nabla} \mathcal{F}^{m-2} \otimes \Omega^2_B \to \cdots\}$$

and

$$\delta\nu = [\nabla\hat{\nu}] \in H^1(\mathcal{K}^\bullet)$$

defines the infinitesimal invariant of ν.

$$\boxed{\delta\nu = 0 \Leftrightarrow \text{may choose } \hat{\nu} \text{ to be a section of } \mathcal{H}_\nabla = \mathbb{H}_\mathbb{C}}$$

- In the geometric case $\mathcal{X} \xrightarrow{f} B, f^{-1}(t) = X_t$ and
    - $\mathcal{H} = R_f^{2m-1}\mathbb{C}_{\mathcal{X}} \otimes \mathcal{O}_B = \{H^{2m-1}(X_t)\text{'s}\}$
    - $\mathcal{J} = \{J(X_t)\text{'s}\}$

  Given $\mathcal{Z} \subset \mathcal{X}$ with $\mathcal{Z} \cdot X_t = Z_t \in CH^m(X_t)_{\text{hom}}$ we have

$$\nu_{\mathcal{Z}} : B \to \mathcal{J}_h.$$

$$\boxed{\delta\nu_Z = 0 \Leftrightarrow \nu_Z \text{ lifts to a section of } R_f^{2m-1}\mathbb{C}_{\mathcal{X}}}$$

V. Singularities of Period Mappings

- For a VHS over $B = \Delta^{*n} \subset \Delta^n = \overline{B}$ with $B = \overline{B} \setminus D$ where D is a normal crossing divisor (NCD), there is at $\{0\} \in \overline{B}$ a limiting mixed Hodge structure (LMHS).

  There are also VLMHS's along the open strata D_I^* of D
- there are quasi-unipotent monodromies T_i corresponding to the generators of $\pi_1(B)$; will assume T_i unipotent with logarithm N_i

- (How to define the LMHS?)
  - $U = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$
  - $U \to \Delta^*$ by $t = \exp(2\pi iz)$
  - $U^r \to B$ the universal cover
  - local system $\mathbb{H}_{\mathbb{Z}}$ trivial up on $\tilde{B}$; $\mathbb{H}_{\mathbb{C}} \cong U^r \times H$
- Period domain $\mathbb{D} =$ PHS's $\{H, F^\bullet, Q\}$ on H satisfying Hodge-Riemman I,II

- Compact dual $\check{\mathbb{D}} = \{H, F^\bullet, Q\}$'s satisfying HR I but not opposite condition and HR II

$$
\begin{array}{ccc}
\mathbb{D} & \subset & \check{\mathbb{D}} \\
\| & & \| \\
G_{\mathbb{R}}/G_0 & & G_{\mathbb{C}}/\mathbb{D}
\end{array}
\qquad \mathbb{D} = \text{open } G_{\mathbb{R}}\text{-orbit in } \check{\mathbb{D}}
$$

  - $\tilde{\Phi}(z + e_i) = T_i \tilde{\Phi}(z)$;
  - $\tilde{\Psi}(z) := \exp\left(-\sum z_i N_i\right)\tilde{\Phi}(z) \in \check{\mathbb{D}}$
  - $\tilde{\Psi}(z + e_i) = \tilde{\Psi}(z) = \Psi(t), t \in B$
  - $\Psi(t)$ extends to $\overline{B} \to \check{\mathbb{D}}$ (Gauss-Manin has regular singular points)
  - $\Psi(0) = F^\bullet_{\mathrm{lim}}$

$$\Longrightarrow \Phi(t) \sim \exp\big(\sum_i l(t_i) N_i\big) \cdot F^{\bullet}_{\mathrm{lim}}, \quad l(t) = \frac{\log t}{2\pi\sqrt{-1}}$$

$\boxed{\text{singular part of } \Phi(t)}$

- Alternatively there are canonical extensions $\mathcal{H}_e \to \overline{B}$ of $\mathcal{H} \to B$ where

$$\mathcal{H}_e \xrightarrow{\nabla} \mathcal{H}_e \otimes \Omega^1_{\overline{B}}(\log D)$$

and an identification

$$F^\bullet_{\mathrm{lim}} = \mathcal{F}^\bullet_{e,\{0\}}$$

- Any nilpotent operator

$$N : H \to H$$

$$N^{k+1} = 0, \quad N^k \neq 0$$

defines a unique weight filtration $W_\bullet(N)$

$$W_0 \subset W_1 \subset \cdots \subset W_{2k} = H$$

with

$$\begin{cases} N : W_i \to W_{i-2} \\ N^l : \mathrm{Gr}_{k+l} \xrightarrow{\sim} \mathrm{Gr}_{k-l} \end{cases}$$

For $N = \sum \lambda_i N_i, \lambda_i > 0$, $W(N)$ is independent of the λ_i (proof uses Hodge theory) and

$$\{H, W_\bullet(N), F^\bullet_{\mathrm{lim}}\}$$

gives a LMHS

- Thus have an identification

$$H = \mathcal{H}_{e,\{0\}} = H_{\lim}$$

VI.

Singularities of Normal Functions and of their Infinitesimal Invariants

- Everything is now modulo torsion (think of $\otimes\mathbb{Q}$); $\mathcal{J} \to B$ extends to $\mathcal{J}_e \to \overline{B}$ where

$$0 \to \mathcal{J}_e^0 \to \mathcal{J}_e \xrightarrow{\sigma} \mathcal{G}$$

  - $\mathcal{J}_e^0$ is a fiber space of connected, abelian complex Lie groups
  - $\mathcal{G}$ is a constructible sheaf supported on D which is a local system on the D_I^*'s

- admissable normal function (ANF) ν extends to

$$\nu_e : \overline{B} \to \mathcal{J}_e$$

- in the geometric case where we have $\overline{\mathcal{Z}} \subset \overline{\mathcal{X}}$ any normal function is admissible

Definition: $\mathrm{sing}(\nu) = \sigma(\nu_e)$

- How to describe $\mathrm{sing}(\nu)$? In $B = (\Delta^*)^r$ with monodromies T_i and $N_i = \log T_i$, $\mathrm{sing}(\nu) :=$ value of $\sigma(\nu)$ at $\mathcal{G}_{\{0\}}$ is described by monodromy of ν.

monodromy of ν: think of $\int_\Gamma \omega$ and monodromy on $H_{2m-1}(X, Z; \mathbb{Q}) \cong H^{2m-1}(X \setminus |Z|, \mathbb{Q})$; first take $r = 1$

- $\tilde{T}_\gamma = \begin{pmatrix} T_\gamma & E_\gamma \\ 0 & 1 \end{pmatrix}$, $\gamma \in \pi_1(\Delta^*)$
- $0 \to H^{2m-1}(X, \mathbb{Q}) \to H^{2m-1}(X \setminus |Z|, \mathbb{Q}) \to \mathbb{Z}(-m) \to 0$
- split sequence over $\mathbb{Q}$, $\hat{\nu} = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$ gives

$$(\tilde{T}_\gamma - I)\hat{\nu} = 0 \Longleftrightarrow E_\gamma(1) = (T_\gamma - I)\lambda$$

- $(\tilde{T}_\gamma - I)\hat{\nu} \longleftrightarrow (\tilde{T}_\gamma - I)\Gamma =$ vanishing cycle in $H_{2m-1}(X, \mathbb{Q})$

and by the local invariant cycle theorem (Clemens-Schimid sequence)

$$(\tilde{T}_\gamma - I)\Gamma = (T_\gamma - I)\lambda \quad \text{(over } \mathbb{Q}\text{)}$$

(monodromy of ν cont.'d) For $r \geq 2$ we have for each i

- $\begin{cases} (T_i - I)\hat{\nu} = (T_i - I)\lambda_i \\ \lambda_i \to \lambda_i + \eta, \quad \eta \in H = H^{2m-1}(X;\mathbb{Q}) \end{cases}$
- $[T_i - I, T_j - I] = 0$

suggests Koszul complex

$$(K_M^\bullet) \quad 0 \to H \to \bigoplus_i (T_i - I)H \to \bigoplus_{i<j} (T_i - I)(T_j - I)H \to \cdots$$

$$\Longrightarrow \mathrm{sing}(\nu) \in H^1(K_M^\bullet)$$

- How to describe $\mathrm{sing}(\delta\nu)$? For this will use the

Residues of $\delta\nu$: Here $H_{\lim} \equiv \mathcal{H}_{e,\{0\}}$ is a MHS with $W_\bullet(N)$ and $F^\bullet_{\lim}$ with $N_i : F^p_{\lim} \to F^{p-1}_{\lim}$ and $N_i : W_k(N) \to W_{k-2}(N)$

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{H}_e & \overset{\nabla}{\to} & \mathcal{H}_e \otimes \Omega^1_{\bar{B}}(\log D) & \overset{\nabla}{\to} & \mathcal{H}_e \otimes \Omega^2_{\bar{B}}(\log D) & \to \cdots \\
 & & \cup & & \cup & & \cup & \\
0 & \to & \mathfrak{F}^m_e & \to & \mathfrak{F}^{m-1}_e \otimes \Omega^1_{\bar{B}}(\log D) & \to & \mathfrak{F}^{m-2}_e \otimes \Omega^2_{\bar{B}}(\log D) & \to \cdots \\
 & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} & \\
0 & \to & F^m_{\lim} & \to & \bigoplus N_i F^m_{\lim} & \to & \bigoplus_{i<j} N_i N_j F^m_{\lim} & \to \cdots
\end{array}$$

$(K^\bullet_R)$

$$\boxed{\Longrightarrow \quad \mathrm{sing}(\delta\nu) \in H^1(K^\bullet_R)}$$

- Also have $F^p_{\lim} \subset H_{\lim}$ and $N_i = H_{\lim} \to \mathcal{H}_{\lim}$ give $K^\bullet_{\lim}$ based on $H_{\lim}, N_i$ with $K^\bullet_R \subset K^\bullet_{\lim}$

Punch line: $T_i - I = N_i A_i$ where $A_i : H \xrightarrow{\sim} H_{\lim}$ gives

$$
\begin{array}{ccc}
K^\bullet_M & \xrightarrow{\sim} & K^\bullet_{\lim} \\
 & & \cup \\
 & & K^\bullet_R
\end{array}
$$

and under the induced maps on H^1

$$\operatorname{sing}(\nu) = \operatorname{sing}(\delta\nu)$$

CKS $H^1(K^\bullet_{\mathrm{lim}})$ has a MHS with highest weight $2m-2$,
$H^1(K^\bullet_{\mathrm{lim}}) \cong \mathrm{Gr}_W^{2m-2}(H^1(K^\bullet_{\mathrm{lim}}))$ and

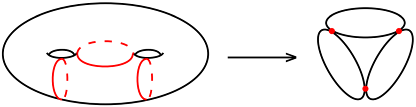
$$\boxed{\mathrm{sing}(\nu) = \mathrm{sing}(\delta\nu) \in Hg^{m-1}(H^1(K^\bullet_{\mathrm{lim}}))}$$

Here using that H_{lim} has a $\mathbb{Q}$-structure and A_i is defined $/\mathbb{Q}$ (but not $/\mathbb{Z}$ unless all $(T_i - I)^2 = 0$)

$$\begin{cases} \mathrm{sing}(\nu) \leftrightarrow \mathbb{Q}\text{-structure} \\ \mathrm{sing}(\delta\nu) \leftrightarrow F^\bullet \end{cases}$$ which together gives a class in Hg^{m-1}

Example: X_0 has nodes p_i which are the limits of vanishing cycles δ_i ($\cong S^d, \dim_{\mathbb{C}} X = d$)

$$H^1(K_M^\bullet) \cong \{\text{relations } \sum r_i\delta_i \sim 0, r_i \in \mathbb{Q}\}$$



$$\delta_1 + \delta_2 + \delta_3 \sim 0$$

VII. Sketch of the proof of the main theorem

As in the proof of the usual nilpotent orbit theorem given a locally defined lift $\hat{\nu}(t) \in \mathcal{H}^{2m+1}$ of an ANF $\nu(t)$ it is convenient to pull $\hat{\nu}(t)$ up to U^r to have $\hat{\nu}(z) \in \mathcal{H}_z^{2m+1}$ with

$$\boxed{\hat{\nu}(z + e_i) = \tilde{T}_i \hat{\nu}(z)}$$

Then we have over U^r the equations

$$\begin{cases} \hat{\nu}(z + e_i) - \tilde{T}_i \hat{\nu}(z) = (T_i - I)\lambda_i, \quad \lambda_i \in H_{\mathbb{Q}} \\ (T_i - I)(T_j - I)\lambda_i = (T_j - I)(T_i - I)\lambda_j \end{cases}$$

and critically

$$(T_i - I)(T_j - I)\lambda_i = (T_j - I)(T_i - I)\lambda_j;$$

note symmetry here in i, j

The main step in the argument is the

Lemma

There exists a unique H-valued polynomial $P(z)$ such that

$$P(z+e_i) - T_i P(z) = (T_i - I)\lambda_i$$

Proof. We will do the first non-trivial case when $r = 2$. The method used below to derive it can, with some effort, be extended to the general case.

Setting

$$\begin{cases} g(z) = \exp(z_1 N_1 + z_2 N_2) \\ g_i(z) = \exp(z_i N_i), \quad i = 1, 2 \end{cases}$$

the formula for $P(z)$ is

$$P(z_1, z_2) = g(z)\lambda + (g_1(z_1) - I)(g_2(z_2) - I)\lambda_i + (g_1(z_1) - I)\lambda_1 + (g_2(z_2) - I)\lambda_2$$

Here $\lambda \in H$ and, reflecting the critical symmetry above, in the second term i may be taken to be 1 or 2. One then checks that the above functional equation for $P(z)$ holds. The argument uses especially the third of the above relations.

To derive the formula for $P(z_1, z_2)$ one sets

$$q_k(z) = \begin{cases} 1 & k = 0 \\ \frac{z(z-1)\cdots(z-(k-1))}{k!} & k \geq 1 \end{cases}$$

Then

$$q_{k+1}(z+1) - q_k(z) = q_{k-1}(z).$$

Setting

$$P(z_1, z_2) = \sum a_{k,l} q_k(z_1) q_l(z_2)$$

one then uses the relations above to recursively determine the $a_{k,l}, k + l > 0$, in terms of a_{00}. The result is

$$\begin{aligned} P(z_2, z_2) = & a_{00} + \sum_{k>0} q_k(z_1)(T_1 - I)^k (a_{00} + \lambda_1) \\ & + \sum_{l>0} q_l(z_2)(T_2 - I)^l (a_{00} + \lambda_2) \\ & + \sum_{\substack{k>0 \\ l>0}} q_k(z_1) q_l(z_2)(T_1 - I)^k (T_2 - I)^l (a_{00} + \lambda_i) \end{aligned}$$

where, again reflecting the third relation above, i may be 1 or 2 in the last term. The a_{00} reflects that we may add to each λ_i a fixed λ.

We note that $P(z_1, z_2)$ depends only on a_{00} and the $(T_i - I)\lambda_i$ for $i = 1, 2$. Having determined the a_{kl} in terms of a_{00} one then checks that the equation just above is summarized in the equation for $P(z_1, z_2)$.

From the above it follows that

$$(P - \hat{\nu})(z + e_i) = (P - \hat{\nu})(z).$$

Thus $P(z) - \hat{\nu}(z)$ descends to $B = (\Delta^*)^r$; we write

$$P(z) = \hat{\nu}(z) + Q(t).$$

Since ν is an admissible normal function it follows that $Q(t)$ is holomorphic on $\bar{B} = \Delta^r$. We will write $\equiv$ to mean congruence modulo holomorphic terms on $\bar{B}$.

From the formula for P we obtain

$$g(z)^{-1}dP(z) \equiv \sum_i N_i \lambda_i dz_i$$

It follows that

$$g(z)^{-1}d\hat{\nu}(z) \equiv \sum_j N_j \lambda_j dz_j = (\frac{1}{2\pi i}) \sum (N_j \lambda_j) \frac{dt_j}{t_j}$$

Since

$$\hat{\nu}(z + e_i) - \hat{\nu}(z) = (T_i - I)\hat{\nu}(z) = (T_i - I)\lambda_i$$

it follows that $d\hat{\nu}(z)$ descends to $\nabla\hat{\nu}(t)$. Taking $g(z)^{-1}d\hat{\nu}(z)$ and letting $\mathrm{Im} z \to 0$ picks out the residue term in $\nabla\hat{\nu}$. This gives

$$\mathrm{Res}_{\{o\}}(\delta\nu) = \textstyle\sum_i N_i \lambda_i$$

$\square$

As remarked above, a subtlety is that in the usual construction of nilpotent orbits one uses $g(z)$ in the above. For the construction of $P(z)$ it is necessary to have a power series in the $(T_i - I)$'s, not one in the N_i's. The reason for this is the different λ_i's that appear on the right hand side of the expression for $P(z_1, z_2)$.

Another subtlety is that for $F^\bullet(z)$ the filtration on H given by the VHS on B pulled back to U^r, the approximating nilpotent orbit is obtained from

$$G^\bullet(z) = g(z)^{-1} F^\bullet(z) \in \check{\mathbb{D}}$$

Then $G^\bullet(z)$ descends to B and the resulting $G^\bullet(t)$ extends across $t = 0$ and setting $F_\infty^\bullet = G^\bullet(0)$ the nilpotent orbit is $g(z) \cdot F_\infty^\bullet$. Using that

$$\nabla \hat{\nu}(z) \in F^{m-1}(z) \otimes \Omega^1_{U^r}$$

we may infer that all

$$N_i \lambda_i \in F^{m-1}_{\lim}$$

Finally the $\lambda_i \in H_{\mathbb{Q}}$ so that

$$N_i \lambda_i \in Hg_{\mathrm{lim}}^{m-1}$$

as has been noted above.

VIII. Computation of $\delta\nu$

- Have seen that $\delta\nu$ has a structure and there are a number of examples showing that $\delta\nu \neq 0$ for *generic* X; no method yet for computing $\delta\nu(Z)$ for *specific* X, Z and which involves the geometry of $Z \subset X$.
- Will discuss one here; first listing existing computational techniques for $\delta\nu$
  - relevant Koszul group is zero
  - argument using global monodromy
  - for generic X's

- family of Z's branches over family of X's along normal to the branch locus $\delta\nu(Z)$ reduces to $dAJ_X(Z)$ which can be computed classically
- for 1-cycle Z in 3-folds X use surfaces $Y \subset X$ with $Z \subset Y \subset X$ so that computation involves co-dimension 1 subvarieties; here Y varies in a Noether-Lefschetz locus for the X's
- combination of geometric and cohomological methods along subspaces $T' \subset T$'s where differential of period mapping for X's have small rank (adjunction method of Collino et al)
- degeneration argument

- Needed is a cohomological formula for $\delta\nu(Z)$; will illustrate such a general method in the case $\dim X = 3$, $\mathrm{codim}_X Z = 2$, applies to give

$$\delta\nu(Z) \neq 0$$

for *any* non-singular X in a family, including long-standing case $X =$ quintic in $\mathbb{P}^4$ and $Z = L' - L''$ where L', L'' are rigid lines in X (normal bundles are $\bigoplus^2 \mathcal{O}(-1)$); long known to be true for generic X
- Notations:
  - $X =$ smooth 3-fold
  - $Z =$ algebraic 1-cycle with support $|Z|$ a disjoint of smooth curves

- $H^2(|Z|)_0 =$ cokernel of $\{H^2(X) \to H^2(|Z|)\}$

then have

$$0 \to H^2(|Z|)_0 \to H^3(X, |Z|) \to H^3(X) \to 0$$

- first step is cohomological formula for the differential of period mapping

$$(X, Z) \to H^3(X, |Z|)$$

For the 2-step complex $T_X \to N_{Z/X}$ well known that

$$\mathrm{TDef}(X, Z) \cong \mathbb{H}^1(T_X \to N_{Z/X})$$

and standard methods give

$$\mathbb{H}^1(T_X \to N_{Z/X}) \to \mathrm{End}_F^{-1}\{H^3(X, |Z|)\}$$

for the differential of period mapping

- For application will make simplifying assumptions

$$H^0(N_{Z/X}) = H^1(N_{Z/X}) = 0$$

geometrically means Z deforms uniquely and freely with X (to 1st-order); then for

$$0 \to T_{X,Z} \to T_X \to N_{Z/X} \to 0$$

we have

$$\mathbb{H}^1(T_X \to N_{Z/X}) \cong H^1(T_{X,Z}) \cong H^1(T_X)$$

- For the differential of the period mapping pass to $F^\bullet$-associated graded to have the basic diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] \\
 & H^{1,1}(|Z|)_0 \cong \mathbb{C} \arrow[d] \\
H^1(T_X) \otimes H^{2,1}(X) \arrow[r] \arrow[ur, dashed, "\beta"] \arrow[dr, "\alpha"'] & \mathrm{Gr}^1_F H^3(X, |Z|) \arrow[d] \\
 & H^{1,2}(X) \arrow[d] \\
 & 0
\end{tikzcd}
$$

Then have

$$\boxed{\beta|_{\ker(\alpha)} \neq 0 \Longrightarrow \delta\nu(Z) \neq 0.}$$

- Reason for restricting to $\ker\alpha$: if $t = (t_1, ..., t_n)$ coordinates in $H^1(T_X)$ with $\partial_i = \frac{\partial}{\partial t_i}$ and

$$\partial_i \to \eta_i \in H^{2,1}(X)$$

then we want to compute the

$$\partial/\partial t_i(\int_{\Gamma_t} \omega_t), \quad \omega_t \in H^{2,1}(X_t)$$

This is roughly

$$\int_{\Gamma_t} \nabla_{\partial/\partial t_i}(\omega_t) + \int_{\nabla_{\partial/\partial t_i}(\Gamma_t)} \omega_t$$

where the ∇'s are the Gauss-Manin connections (for different local systems); thus if

$$\alpha(\sum \eta_i \otimes \omega_i) = 0$$

first term in

$$\sum_i \partial/\partial t_i(\int_{\Gamma_t} \omega_{i,t})|_{t=\{0\}}$$

drops out and $\sum_i \eta_i \otimes \omega_i \in \ker \alpha$ turns out to be localized along Z

- if as differential forms

$$\alpha(\sum_i \eta_i \otimes \omega_i) = \bar{\partial}\gamma$$

where $\eta_i \in A^{0,1}(T_X), \omega_i \in A^{2,1}(X), \gamma \in A^{1,1}(X)$

then we have the basic formula

$$\boxed{\beta(\sum_i \eta_i \otimes \omega_i) = \int_Z \gamma}$$

One must check that the RHS is independent of the choices of the η_i, ω_i, and γ. For γ this follows from $[Z] = 0$ in $H_2(X)$. For the η_i and ω_i the issue is more subtle; needs to use the assumption that Z uniquely and freely deforms with X expressed cohomologically by $H^q(N_{Z/X}) = 0$ for $q = 0, 1$.

- Applying the basic formula requires of course a specific cohomological computation. A first result is
  - $X \subset \mathbb{P}V^*$ smooth quintic 3-fold ($V \cong \mathbb{C}^5$)
  - $L \subset X$ a line with $N_{L/X} \cong \overset{2}{\bigoplus} \mathcal{O}_L(-1)$
  - $Z = L - (\frac{1}{5})H_1 \cap H_2 \cap X$ where H_1, H_2 are general hyperplanes

  Then for ν_Z the corresponding normal function

$$\delta\nu_Z(X) \neq 0$$

  For X generic the result $\nu_Z(X) \neq 0$ in $\mathcal{J}(X)$ is classical, as is also the result for $\delta\nu_Z(X)$, the point here is to have the above for <u>any</u> smooth X.

- For the proof of $\delta\nu_Z(X) \neq 0$ one needs a polynomial description of the cohomology groups involved.
  In the case at hand may assume L, H_1, H_2 are fixed in $\mathbb{P}V^*$ with X varying.
  - $X = \{F(x) = 0\}, \quad F \in V^4$
  - $L = \{x_3 = x_4 = x_5 = 0\} \implies F = x_3F_3 + x_4F_4 + x_5F_5$ where $F_\alpha \in V^4$
  - since L deforms with X a general deformation is $F + \varepsilon G = 0$ where
    $$G = x_3G_3 + x_4G_4 + x_5G_5, G_\alpha \in V^4/\mathcal{J}_{F,4}$$
  - $\eta_i \leftrightarrow G_i$ as above
  - $\omega_i \leftrightarrow P_i \in V^5/\mathcal{J}_{\mathcal{F},\triangledown}$
  - $\alpha(\sum \eta_i \otimes \omega_i) = 0 \Leftrightarrow \sum G_iP_i \in \mathcal{J}_{F,10}$

- $L \subset X \implies F_{x_1}, F_{x_2} \in \{x_3, x_4, x_5\}$

With these ingredients a computation using Koszul complexes and that $F_{x_1}, ..., F_{x_5}$ form a regular sequence leads to the result.

The computation is delicate in that the relevant Koszul cohomology groups just barely vanish. The method and result have been extended to several other cases where polynomial descriptions of cohomology are available