Singularities of Normal Functions

Phillip Griffiths*

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I. Introduction

Normal functions are basic Hodge theoretic invariants of a pair (X, Z) where X is a smooth projective variety and $Z \subset X$ is an algebraic subvariety or an algebraic cycle. Attached to a normal function ν is an infinitesimal invariant $\delta\nu$. Of particular interest is the analysis of ν and $\delta\nu$ when the pair (X, Z) degenerates to a singular pair (X_0, Z_0). Attached to ν_{lim} and $(\delta\nu)_{\mathrm{lim}}$ are the singularities $\mathrm{sing}(\nu)$ and $\mathrm{sing}(\delta\nu)$, and a basic structral result in a natural identification

$$\boxed{\mathrm{sing}(\nu) = \mathrm{sing}(\delta\nu)}$$

After recalling background material, in this talk we will sketch a proof of the above result. A basic ingredient in the argument is a nilpotent orbit theorem for normal functions. Since the singularities of a normal function occur in codimension ≥ 2 there are subtleties in this construction that are not present in the classical nilpotent orbit theorem.

Following the proof of the main result we will discuss a method for computing $\delta\nu_Z$ in practice and will show how this can be used to resolve an old question regarding lines on quartic 3-folds.

II. Background

- A basic invariant of a smooth projective variety X is the polarized Hodge structure (PHS) on its cohomology groups $H^n(X,\mathbb{Q})$. In the classical case when the period domain $\mathbb{D}$ of all PHS's of the same type is Hermitian symmetric one may attach algebro-geometric objects, such as theta-divisors, to $H^n(X,\mathbb{Q})$. In the non-classical case, since a general PHS of this type does not arise from the cohomology of an algebraic variety this is not possible, and for a variety X_t varying in a family the "surrogate" object of an infinitesimal variation of Hodge structure (IVHS) has been of use in extracting geometric information from the family.

- For pair (X, Z) one has the functorial mixed Hodge structure (MHS) defined on $H^n(X, Z; \mathbb{Q})$. There is a corresponding notion of variation of mixed Hodge structure (VMHS) where the pair varies in a family. Although there is the definition of a period mapping and its differential, so far as we are aware in this case a cohomological formula for the differential has only recently been derived and used. There is also the concept of an infinitesimal variation of mixed Hodge structrue (IVMHS), and again so far as we are aware this has also only recently been formalized.

- A particular type of pair is when X is smooth projective and $Z \subset X$ is an algebraic cycle whose fundamental class $[Z]$ is zero in integral homology. In this talk, for simplicity of notation we will take the case

$$\begin{aligned} \dim X &= 2m - 1 \\ \dim Z &= m - 1 \quad (\text{thus } \operatorname{codim}_X Z = m) \end{aligned}$$

Denoting by $|Z|$ the support of the cycle Z, there are invariants of the MHS on $H^{2m-1}(X, |Z|)$, one of which is the Abel-Jacobi image $AJ_X(Z) \in J(X)$ (notation explained below). When we have a family $\{(X_t, Z_t), t \in B\}$ the corresponding $AJ_{X_t}(Z_t) \in J(X_t)$ give a <u>normal function</u> $\nu : B \to \mathcal{J}$, where the $\mathcal{J}$ is the family of $J(X_t)$'s. One may extract from the differential of the period mapping arising from the $H^{2m-1}(X_t, |Z_t|)$'s the infinitesimal invariant $\delta\nu$, alluded to above and defined below.

- For the study of singularities we will be interested in the case when B is smooth with smooth relative completion $\overline{B}$ when $B = \overline{B} \setminus D$ with D a normal crossing divisor. Thus $D = \bigcup D_I$ with strata $D_I = \bigcap_{i \in I} D_i$.
  Usually, we shall take

$$B = (\Delta^*)^r \subset \Delta^r = \overline{B}$$

  with coordinates $t = (t_1, ..., t_r)$.

III. Notations

- Hodge structure is $(H, F^\bullet)$ where $F^n \subset F^{n-1} \subset \cdots F^0$ with $n =$ weight;
  $H = H_{\mathbb{Z}} \otimes \mathbb{C}$, assumed polarized by $Q : H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q}$;
- Unless otherwise mentioned $n = 2m-1$ and

$$\begin{aligned} J(X) &= F^m H^{2m-1}(X) \backslash H^{2m-1}/H_{\mathbb{Z}}^{2m-1} \\ &= F^m \overset{\vee}{H^{2m-1}}(X)/H_{2m-1}(X,\mathbb{Z}) \end{aligned}$$

- Mixed Hodge structure (MHS) is $(H, W_\bullet, F^\bullet)$ with $W_0 \subset W_1 \subset \cdots \subset W_k/\mathbb{Q}$, and where $F^\bullet$ induces a HS of weight n on $\mathrm{Gr}_W^n(H)$.

- $AJ_X : \mathrm{CH}^m(X)_{\mathrm{hom}} \to J(X)$

$$\left\{ \begin{array}{c} 0 \to H^{2m-2}(|Z|)_0 \to H^{2m-1}(X \setminus |Z|) \to H^{2m-1}(X) \to 0, \\ \text{and extension class } e_{X,Z} \in \mathrm{Ext}^1_{MHS}(H^{2m-1}(X), H^{2m-2}(Z)_0) \end{array} \right.$$

$$AJ_X(Z)$$

$$\left\{ \begin{array}{c} \langle AJ_X(Z), \omega \rangle = \int_\Gamma \omega, \quad \omega \in F^m H^{2m-1}(X), \\ \text{and } \Gamma \in H_{2m-1}(X, |Z|) \text{ with } \partial\Gamma = Z \end{array} \right.$$

- Higher AJ maps are defined on algebraic cycles in $X \times (\mathbb{P}^1 \setminus \{0, \infty\})^n$

$$AJ_X^n : CH^m(X,n)_{\mathrm{hom}} \to J(X,n) \begin{array}{l} = F^m \backslash H^{2m-1-n}/H_{\mathbb{Z}}^{2m-1-n} \\ = F^m \overset{\vee}{H}{}^{2m-1-n}/H_{\mathbb{Z}}^{2m-1-n} \end{array}$$

where the top term is a class in $\mathrm{Ext}^1_{MHS}(\cdot,\cdot)$, and the second term is $\int \omega + T_z(\omega)$ where T_z is the current given by a differential form in L^1 involving $\log z_i$'s and dz_i/z_i's.

- MHS's appearing in Abel-Jacobi maps have special properties
  - length 2
  - one term is a $\mathbb{Z}(-(m-1)) \cong H_{\mathbb{Z}}^{m-1,m-1}$ (n=0 case)
- Variation of Hodge structure (VHS) is $\{\mathcal{H}, \mathcal{F}^\bullet, \nabla; B\}$ where Gauss-Manin connection ∇ has
  - $\nabla^2 = 0$
  - $\mathcal{H}_\nabla \cong \mathbb{H}_\mathbb{C}$ where $\mathbb{H}_\mathbb{Z} \to B$ a local system
  - $\mathcal{F}^p \xrightarrow{\nabla} \mathcal{F}^{p-1} \otimes \Omega^1_B$

  There is also the notation of a variation of mixed Hodge structure (VMHS).

IV. Normal Functions and their Infinitesimal Invariants

- $\mathcal{J} = \mathcal{F}^m \setminus \mathcal{H}/\mathbb{H}_\mathbb{Z}$; for $\nu : B \to \mathcal{J}$ and for any local lift $\hat{\nu} : B \to \mathcal{H}$ the quasi-horizontality condition

$$\nabla \hat{v} \in \mathcal{F}^{m-1} \otimes \Omega^1_B$$

  is well defined , non-trivial for $m \geq 2$, and gives the sheaf $\mathcal{J}_h \subset \mathcal{J}$ of normal functions.
- Normal functions are a particular type of VMHS
- Koszul complex

$$\mathcal{K}^\bullet = \{\mathcal{F}^m \xrightarrow{\nabla} \mathcal{F}^{m-1} \otimes \Omega^1_B \xrightarrow{\nabla} \mathcal{F}^{m-2} \otimes \Omega^2_B \to \cdots\}$$

  and

$$\delta\nu = [\nabla\hat{\nu}] \in H^1(\mathcal{K}^\bullet)$$

  defines the infinitesimal invariant of ν.

$$\boxed{\delta\nu = 0 \Leftrightarrow \text{may choose } \hat{\nu} \text{ as section of } \mathcal{H}_\nabla = \mathbb{H}_\mathbb{C}}$$

- In the geometric case $\mathcal{X} \xrightarrow{f} B, f^{-1}(t) = X_t$ and
  - $\mathcal{H} = R_f^{2m-1}\mathbb{C}_{\mathcal{X}} \otimes \mathcal{O}_B = \{H^{2m-1}(X_t)\text{'s}\}$
  - $\mathcal{J} = \{J(X_t)\}$

  Given $\mathcal{Z} \subset \mathcal{X}$ with $\mathcal{Z} \cdot X_t \in Z_t \in CH^m(X_t)_{\text{hom}}$ we have

$$\nu_{\mathcal{Z}} : B \to \mathcal{J}_h.$$

$$\boxed{\delta\nu = 0 \Leftrightarrow \nu \text{ lifts to a section of } R_f^{2m-1}\mathbb{C}_{\mathcal{X}}}$$

V. Singularities of Period Mappings

- For a VHS over $B = \Delta^{*n} \subset \Delta^n = \overline{B}$ with $B = \overline{B} \setminus D$ where D is a normal crossing divisor (NCD), there is at $\{0\} \in \overline{B}$ a limiting mixed Hodge structure (LMHS).
  There are also VLMHS's along the open strata D_I^* of D
- there are quasi-unipotent monodromies T_i corresponding to the generators of $\pi_1(B)$; will assume T_i unipotent with logarithm N_i
- How to define the LMHS?
  - $U = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$
  - $U \to \Delta^*$ by $t = \exp(2\pi i z)$
  - $U^r \to B$ the universal cover
  - local system $\mathbb{H}_{\mathbb{Z}}$ trivial up on $\tilde{B}$; $\mathbb{H}_{\mathbb{C}} \cong U^r \times H$

- Period domain $\mathbb{D}$ = PHS's $\{H, F^\bullet, Q\}$ on H satisfying Hodge-Riemman I,II
- Compact dual $\check{\mathbb{D}} = \{H, F^\bullet, Q\}$'s satisfying HR I

$$\begin{array}{ccc} \mathbb{D} & \subset & \check{\mathbb{D}} \\ \| & & \| \\ G_{\mathbb{R}}/G_0 & & G_{\mathbb{C}}/\mathbb{D} \end{array} \quad \text{and } \mathbb{D} = \text{open } G_{\mathbb{R}}\text{-orbit in } \check{\mathbb{D}}$$

  - $\tilde{\Phi}(z + e_i) = T_i \tilde{\Phi}(z)$;
  - $\Psi(z) := \exp\left(-\sum z_i N_i\right)\Phi(z) \in \check{\mathbb{D}}$
  - $\Psi(z + e_i) = \Psi(z) = \Psi(t), t \in B$
  - Ψ extends to $\overline{B} \to \check{\mathbb{D}}$ (Gauss-Manin has regular singular points)
  - $\Psi(0) = F^\bullet_{\lim}$

$$\Longrightarrow \Phi(t) \sim \exp\Big(\sum_i l(t_i)N_i\Big) \cdot F^\bullet_{\lim}, \quad l(t) = \frac{\log t}{2\pi\sqrt{-1}}$$

$$\boxed{\text{singular part of } \Phi(t)}$$

- Alternatively there are canonical extensions $\mathcal{H}_e \to \overline{B}$ of $\mathcal{H} \to B$ where

$$\mathcal{H}_e \xrightarrow{\nabla} \mathcal{H}_e \otimes \Omega^1_{\overline{B}}(\log D)$$

and there is an identification

$$F^\bullet_{\lim} = \mathcal{F}^\bullet_{e,\{0\}}$$

- Any nilpotent operator

$$N : H \to H$$

$$N^{k+1} = 0$$

defines a weight filtration $W_\bullet(N)$

$$W_0 \subset W_1 \subset \cdots \subset W_k = H$$

For $N = \sum \lambda_i N_i, \lambda_i > 0$, $W(N)$ is independent of the λ_i (proof uses Hodge theory) and

$$\{H, W_\bullet(N), F^\bullet_{\text{lim}}\}$$

gives a LMHS

- Thus have an identification

$$H = \mathcal{H}_{e,\{0\}} = H_{\lim}$$

VI.
Singularities of Normal Functions and of their infinitesimal invariants

- Everything is now modulo torsion (think of $\otimes\mathbb{Q}$); $\mathcal{J} \to B$ extends to $\mathcal{J}_e \to \overline{B}$ where

$$0 \to \mathcal{J}_e^0 \to \mathcal{J}_e \xrightarrow{\sigma} \mathcal{G}$$

  - $\mathcal{J}_e^0$ is a fiber space of connected, abelian complex Lie groups
  - $\mathcal{G}$ is a constructible sheaf on D which is a local system on D_I^*'s
  - admissable normal function (ANF) ν extends to

$$\nu_e : \overline{B} \to \mathcal{J}_e$$

  - in the geometric case where we have $\overline{\mathcal{Z}} \subset \overline{\mathcal{X}}$ any normal function is admissible

Definition: $\mathrm{sing}(\nu) = \sigma(\nu_e)$

- What about $\mathrm{sing}(\nu)$ and $\mathrm{sing}(\delta\nu)$? In $B = (\Delta^*)^r$ with monodromies T_i and $N_i = \log T_i$, $\mathrm{sing}(\nu) :=$ value of $\sigma(\nu)$ at $\mathcal{G}_{\{0\}}$ is described by monodromy of ν.
  (i) monodromy of ν; think of $\int_\Gamma \omega$ and monodromy on $H_{2m-1}(X, Z; \mathbb{Q}) \cong H^{2m-1}(X \setminus |Z|, \mathbb{Q})$; first take $r = 1$
    - $\tilde{T}_\gamma = \begin{pmatrix} T_\gamma & E_\gamma \\ 0 & 1 \end{pmatrix}$, $\gamma \in \pi_1(\Delta^*)$
    - $0 \to H^{2m-1}(X, \mathbb{Q}) \to H^{2m-1}(X \setminus |Z|, \mathbb{Q}) \to \mathbb{Z}(-m) \to 0$
    - split sequence over $\mathbb{Q}$, $\hat{\nu} = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$ gives

$$(\tilde{T}_\gamma - I)\hat{\nu} = 0 \Longleftrightarrow E_\gamma(1) = (T_\gamma - I)\lambda$$

    - $(\tilde{T}_\gamma - I)\hat{\nu} \longleftrightarrow (\tilde{T}_\gamma - I)\Gamma =$ vanishing cycle in $H_{2m-1}(X, \mathbb{Q})$

  and by the local invariant cycle theorem (Clemens-Schimid sequence)

$$(\tilde{T}_\gamma - I)\Gamma = (T_\gamma - I)\lambda \quad \text{(over } \mathbb{Q}\text{)}$$

(i) monodromy of ν cont.'d For $r \geq 2$ we have for each i

- $\begin{cases} (T_i - I)\hat{\nu} = (T_i - I)\lambda_i \\ \lambda_i \to \lambda_i + \eta, \quad \eta \in H = H^{2m-1}(X;\mathbb{Q}) \end{cases}$
- $[T_i - I, T_j - I] = 0$

suggests Koszul complex

$$(K_M^\bullet) \quad 0 \to H \to \bigoplus_i (T_i - I)H \to \bigoplus_{i<j} (T_i - I)(T_j - I)H \to \cdots$$

$$\Longrightarrow \operatorname{sing}(\nu) \in H^1(K_M^\bullet)$$

For $\operatorname{sing}(\delta\nu)$ shall use the

- residues of $\delta\nu$: Here $H_{\lim} \equiv \mathcal{H}_{e,\{0\}}$ is a MHS with $W_\bullet(N)$ and $F^\bullet_{\lim}$ with $N_i : F^p_{\lim} \to F^{p-1}_{\lim}$ and $N_i : W_k(N) \to W_{k-2}(N)$

$(K^\bullet_R)$

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{H}_e & \stackrel{\nabla}{\to} & \mathcal{H}_e \otimes \Omega^1_{\bar{B}}(\log D) & \stackrel{\nabla}{\to} & \mathcal{H}_e \otimes \Omega^2_{\bar{B}}(\log D) & \to \cdots \\
 & & \cup & & \cup & & \cup & \\
0 & \to & \mathfrak{F}^m_e & \to & \mathfrak{F}^{m-1}_e \otimes \Omega^1_{\bar{B}}(\log D) & \to & \mathfrak{F}^{m-2}_e \otimes \Omega^2_{\bar{B}}(\log D) & \to \cdots \\
 & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} & \\
0 & \to & F^m_{\lim} & \to & \bigoplus N_i F^m_{\lim} & \to & \bigoplus_{i<j} N_i N_j F^m_{\lim} & \to \cdots
\end{array}$$

$$\boxed{\Longrightarrow \quad \mathrm{sing}(\delta\nu) \in H^1(K^\bullet_R)}$$

- $F^p_{\lim} \subset H_{\lim}$ and $N_i = H_{\lim} \to \mathcal{H}_{\lim}$ give $K^\bullet_{\lim}$ based on $H_{\lim}, N_i$ and $K^\bullet_R \subset K^\bullet_{\lim}$

Punch line: $T_i - I = N_i A_i$ where $A_i : H \xrightarrow{\sim} H_{\lim}$ gives

$$\begin{array}{ccc} K^\bullet_M & \xrightarrow{\sim} & K^\bullet_{\lim} \\ & & \cup \\ & & K^\bullet_R \end{array}$$

and under the induced maps on H^1

$$\operatorname{sing}(\nu) = \operatorname{sing}(\delta\nu)$$

CKS $H^1(K^\bullet_{\lim})$ has a MHS with highest weight $2m-2$, $H^1(K^\bullet_{\lim}) \cong \operatorname{Gr}^{2m-2}_W(H^1(K^\bullet_{\lim}))$ and

$$\boxed{\operatorname{sing}(\nu) = \operatorname{sing}(\delta\nu) \in Hg^{m-1}(H^1(K^\bullet_{\lim}))}$$

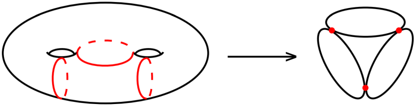
Here using that $H_{\lim}$ has a $\mathbb{Q}$-structure and A_i is defined $/\mathbb{Q}$ (but not $/\mathbb{Z}$ unless all $(T_i - I)^2 = 0$)

$$\begin{cases} \mathrm{sing}(\nu) \leftrightarrow \mathbb{Q}\text{-structure} \\ \mathrm{sing}(\delta\nu) \leftrightarrow F^\bullet \end{cases}$$

which together gives a class in Hg^{m-1}

Example: X_0 has nodes p_i which are the limits of vanishing cycles δ_i ($\cong S^d, \dim_{\mathbb{C}} X = d$)

$$H^1(K_M^\bullet) \cong \{\text{relations} \sum r_i \delta_i \sim 0, r_i \in \mathbb{Q}\}$$



$$\delta_1 + \delta_2 + \delta_3 \sim 0$$

VII. Sketch of the proof of the main theorem

As in the proof of the usual nilpotent orbit theorem given a locally defined lift $\hat{\nu}(t) \in \mathcal{H}^{2m+1}$ of an ANF $\nu(t)$, it is convenient to pull $\hat{\nu}(t)$ up to U^r to have $\hat{\nu}(z) \in \mathcal{H}_z^{2m+1}$ with

$$\hat{\nu}(z + e_i) = \tilde{T}_i \hat{\nu}(z)$$

Then we have over U^r the equations

$$\begin{cases} \hat{\nu}(z + e_i) - \tilde{T}_i \hat{\nu}(z) = (T_i - I)\lambda_i, & \lambda_i \in H_{\mathbb{Q}} \\ (T_i - I)(T_j - I)\lambda_i = (T_j - I)(T_i - I)\lambda_j \end{cases}$$

The main step in the argument is the

Lemma

There exists a unique H-valued polynomial $P(z)$ such that

$$P(z + e_i) - T_i P(z) = (T_i - I)\lambda_i$$

Proof. We will do the first non-trivial case when $r = 2$. The method used below to derive will extend to the general case. Setting

$$\begin{cases} g(z) = \exp(z_1 N_1 + z_2 N_2) \\ g_i(z) = \exp(z_i N_i), \quad i = 1, 2 \end{cases}$$

the formula for $P(z)$ is

$$P(z_1, z_2) = g(z)\lambda + (g_1(z_1) - I)(g_2(z_2) - I)\lambda_i + (g_1(z_1) - I)\lambda_1 + (g_2(z_2) - I)\lambda_2$$

Here $\lambda \in H$ and in the second term i may be taken to be 1 or 2. One then checks that the above functional equation for $P(z)$ holds.

To derive the formula for $P(z_1, z_2)$ one sets

$$q_k(z) = \begin{cases} 1 & k = 0 \\ \frac{z(z-1)\cdots(z-(k-1))}{k!} & k \geq 1 \end{cases}$$

Then

$$q_{k+1}(z+1) - q_k(z) = q_{k-1}(z).$$

Setting

$$P(z_1, z_2) = \sum a_{k,l} q_k(z_1) q_l(z_2)$$

one then uses the relation above to recursively determine the $a_{k,l}, k + l > 0$, in terms of a_{00}. The result is

$$\begin{aligned} P(z_2, z_2) =& a_{00} + \sum_{k>0} q_k(z_1)(T_1 - I)^k (a_{00} + \lambda_1) \\ &+ \sum_{l>0} q_l(z_2)(T_2 - I)^l (a_{00} + \lambda_2) \\ &+ \sum_{\substack{k>0 \\ l>0}} q_k(z_1) q_l(z_2)(T_1 - I)^k (T_2 - I)^l (a_{00} + \lambda_i) \end{aligned}$$

where i may be 1 or 2 in the last term. The a_{00} reflects that we may add to each λ_i a fixed λ.

We note that $P(z_1, z_2)$ depends only on a_{00} and the $(T_i - I)\lambda_i$ for $i = 1, 2$. Having determined the a_{kl} in terms of a_{00} one then checks that the equation just above is summarized in the equation for $P(z_1, z_2)$.

From the above it follows that

$$(P - \hat{\nu})(z + e_i) = (P - \hat{\nu})(z).$$

Thus $P(z) - \hat{\nu}(z)$ descends to $B = (\Delta^*)^r$; we write

$$P(z) = \hat{\nu}(z) + Q(t).$$

Since ν is an admissible normal function it follows that $Q(t)$ is holomorphic on $\bar{B} = \Delta^r$. We will write $\equiv$ to mean congruence modulo holomorphic terms on $\bar{B}$.

From the formula for P we obtain

$$g(z)^{-1}dP(z) \equiv \sum_i N_i \lambda_i dz_i$$

It follows that

$$g(z)^{-1}d\hat{\nu}(z) \equiv \sum_j N_j \lambda_j dz_j = (\frac{1}{2\pi i}) \sum (N_j \lambda_j) \frac{dt_j}{t_j}$$

Since

$$\hat{\nu}(z + e_i) - \hat{\nu}(z) = (T_i - I)\hat{\nu}(z) = (T_i - I)\lambda_i$$

it follows that $d\hat{\nu}(z)$ descends to $\nabla\hat{\nu}(t)$. Taking $g(z)^{-1}d\hat{\nu}(z)$ and letting $\mathrm{Im} z \to 0$ picks out the residue term in $\nabla\hat{\nu}$. This gives

$$\mathrm{Res}_{\{o\}}(\delta\nu) = \sum_i N_i \lambda_i$$

which is what was to be proved. □

As remarked above, a subtlety is that in the usual construction of nilpotent orbits one uses $g(z)$ in the above. For the construction of $P(z)$ it is necessary to have a power series in the $(T_i - I)$'s, not one in the N_i's. The reason for this is the different λ_i's that appear on the right hand side of the expression for $P(z_1, z_2)$.

Another Subtlety is that for $F^\bullet(z)$ the filtration on H given by the VHS on B pulled back to U^r, the approximating nilpotent orbit is obrained from

$$G^\bullet(z) = g(z)^{-1} F^\bullet(z).$$

Then $G^\bullet(z)$ descends to B and the resulting $G^\bullet(t)$ extends across $t = 0$ and setting $F^\bullet_\infty = G^\bullet(0)$ the nilpotent orbit is $g(z) \cdot F^\bullet_\infty$. Using that

$$\nabla \hat{\nu}(z) \in F^{m-1}(z) \otimes \Omega^1_{U^r}$$

we may infer that all

$$N_i \lambda_i \in F^{m-1}_{\mathrm{lim}}$$

Finally the $\lambda_i \in H_{\mathbb{Q}}$ so that

$$N_i \lambda_i \in Hg^{m-1}_{\mathrm{lim}}$$

as has been noted above.

VIII. Computation of $\delta\nu$

- Have seen that $\delta\nu$ has a general structure, and there are a marble of examples showing either $\delta\nu \neq 0$ or $\delta\nu = 0$ for generic X; no method for computing $\delta\nu(Z)$ for specific X, Z which involves the geometry of $Z \subset X$.
- Will discuss one here; first listing existing computational techniques, all for generic X's
    - relevant Koszul group is zero
    - family of Z's branches over family of X's and normal to branch parts $\delta\nu(Z)$ reduces to $dAJ_X(Z)$
    - for 1-cycle Z in 3-folds X uses surfaces $Y \subset X$ with $Z \subset Y \subset X$ so that computation involves co-dimension 1
    - counting of geometric 1 co-homological methods using $T' < T$'s where differential of period mapping for X's have small rank (adjunction method of Callino et al)
    - global argument

- Needed is a cohomological formula for $\delta\nu(Z)$; will illustrate in the case $\dim X = 3$, $\mathrm{codim}_X Z = 2$ such a general method, applies to give
$$\delta\nu(Z) \neq 0$$
for any non-singular X in a variety of example, yielding long-standing case $X =$ quentics in $\mathbb{P}^4$ and $Z = L' - L''$ where L', L'' are rigid lives in X (normal bundles are
$$\bigoplus^2 \mathcal{O}(-1)$$
); long known to be true for generic X
- Notations:
  - $X =$ smooth 3-fold
  - $Z =$ algebraic 1-cycle with support $|Z|$ a disjoint of smooth curves
  - $H^2(|Z|)_0 =$ cokernel of $H^2(X) \to H^2(|Z|)$

then have

$$0 \to H^2(|Z|)_0 \to H^3(X, |Z|) \to H^3(X) \to 0$$

- first step in cohomological formula for the differential of period mapping
$$(X, Z) \to H^3(X, |Z|)$$
For complex $T_X \to N_{Z/X}$ well known that
$$\mathrm{TDef}(X, Z) \cong \mathbb{H}^1(T_X \to N_{Z/X})$$
and standard method give
$$\mathbb{H}^1(T_X \to N_{Z/X}) \to \mathrm{End}_F^{-1}\{H^3(X, |Z|)\}$$
for differential of period mapping

- For application will make simplifying assumption

$$H^0(N_{Z/X}) = H^1(N_{Z/X});$$

means Z deforms uniquely and freely with X (to 1st-order); then for

$$0 \to T_{X,Z} \to T_X \to N_{Z/X} \to 0$$

we have

$$\mathbb{H}^1(T_X \to N_{Z/X}) \cong H^1(T_{X,Z}) \cong H^1(T_X)$$

- For the differential of the period mapping pass to $F^\bullet$-associated guided to have the box diagram:

$$
\begin{tikzcd}
& 0 \arrow[d] \\
& H^{1,1}(|Z|)_0 \cong \mathbb{C} \arrow[d] \\
H^1(T_X) \otimes H^{2,1}(X) \arrow[ur, dashed, "\beta"] \arrow[r] \arrow[dr, "\alpha"'] & \mathrm{Gr}_F^1 H^3(X, |Z|) \arrow[d] \\
& H^{1,2}(X) \arrow[d] \\
& 0
\end{tikzcd}
$$

Then have

$$\boxed{\beta|_{\ker(\alpha)} \neq 0 \Longrightarrow \delta\nu(Z) \neq 0.}$$

- Reason for restricting to have: if $t = (t_1, ..., t_n)$ coordinate in $H^1(T_X)$ with $\partial_i = \frac{\partial}{\partial t_i}$ and

$$\partial_i \to \eta_i \in H^{2,1}(X)$$

then we want to compute the

$$\partial/\partial t_i(\int_{\Gamma_t} \omega_t), \quad \omega_t \in H^{2,1}(X_t)$$

This is sometimes like

$$\int_{\Gamma_t} \nabla_{\partial/\partial t_i}(\omega_t) + \int_{\nabla_{\partial/\partial t_i}(\Gamma_t)} \omega_t$$

where ∇'s Gauss-Manin connection (for differential local system); $\nabla_{\partial/\partial t_i}(\Gamma_t)$ is sometimes like a natural along Z_t; thus if

$$\alpha(\sum \eta_i \otimes \omega_i) = 0$$

first term is

$$\sum_i \partial/\partial t_i(\int_{\Gamma_t} \omega_{i,t})|_{t=\{0\}}$$

drops out over above is localized along Z

- if an differential form

$$\alpha(\sum_i \eta_i \otimes \omega_i) = \bar{\partial}\gamma$$

where $\eta_i \in A^{0,1}(T_X), \omega_i \in A^{2,1}(X), \gamma \in A^{1,1}(X)$
then we have the basic formula

$$\boxed{\beta(\sum_i \eta_i \otimes \omega_i) = \int_Z \gamma}$$

One must check that the RHS is independent of the choices of the η_i, ω_i, and γ. For γ this follows from $[Z] = 0$ in $H_2(X)$.
For the η_i and ω_i the issue is more subtle; needs to use the assumption that Z uniquely and freely deforms with X expressed cohomologically by $H^q(N_{Z/X}) = 0$ for $q = 0, 1$.
These give that

$$\mathbb{H}^1(T_X \to N_{Z/X}) \xrightarrow{\sim} H^1(T_X)$$

is an isomorphism.

- Applying the basic formula of course requires specific cohomological computation. A first result is
  - $X \subset \mathbb{P}V^*$ smooth quartic 3-fold ($V \cong \mathbb{C}^5$)
  - $L \subset X$ a line with $N_{L/X} \cong \bigoplus^2 \mathcal{O}_L(-1)$
  - $Z = L - (\frac{1}{5})H_1 \cap H_2 \cap X$ where H_1, H_2 are general hyperplanes

Then for ν_Z the corresponding normal function

$$\delta\nu_Z(X) \neq 0$$

For X generates the result $\nu_Z(X) \neq 0$ in $\mathcal{J}(X)$ is classical, as is also the result for $\delta\nu_Z(X)$, the point here is to have the above for <u>any</u> smooth X.

- For the proof of $\delta\nu_Z(X) \neq 0$ are never the polynomial description of the cohomology group.
  In the case at hand may assume L, H_1, H_2 are fixed in $\mathbb{P}V^*$ with X varying.
  - $X = \{F(x) = 0\}, \quad F \in V^4$
  - $L = \{x_3 = x_4 = x_5 = 0\} \implies F = x_3F_3 + x_4F_4 + x_5F_5$ where $F_\alpha \in V^4$
  - general deformation $F + \varepsilon G = 0$ where

$$G = x_3G_3 + x_4G_4 + x_5G_5, \, G_\alpha \in V^4/\mathcal{J}_{F,4}$$

  - $\eta_i \leftrightarrow G_i$ as above
  - $\omega_i \leftrightarrow P_i \in V^5/\mathcal{J}_{\mathcal{F},\triangledown}$

- $\alpha(\sum \eta_i \otimes \omega_i) = 0 \Leftrightarrow \sum G_i P_i \in \mathcal{J}_{F,10}$
- $L \subset X \implies F_{x_1}, F_{x_2} \in \{x_3, x_4, x_5\}$

With these ingredients a computation using Koszul complexes and that $F_{x_1}, ..., F_{x_5}$ form a unique sequence leads to the result.
The computation is somewhat delicate in that the relevant Koszul cohomology groups just barely vanish.