1. *Forward problem* (initial boundary value problem for wave equation

$$\begin{aligned} \frac{1}{c^2(x)} u_{tt} &= \Delta u \text{ in } \Omega \times [0, 2T], \\ u|_{t=0} &= 0, \ u_t|_{t=0} = 0 \text{ in } \Omega, \\ u_n|_{\partial\Omega} &= f, \end{aligned}$$

where Ω is a disk in $\mathbb{R}^2$

$$\Omega = \{x \in \mathbb{R}^2 | \ |x| < r\},$$

c is the speed of sound, and f is a Neumann boundary source (control). Control f runs over control space F. It is enough assume that $F = C_0^\infty(S \times [0, T])$, where $S = \{x \in \mathbb{R}^2 | \ |x| = r\}$. The solution will be denoted by u^f.

For solving forward problem we use Matlab, and we choose f as a rectangular pulse (depend on time only) or as Ricker pulse.

2. *Inverse problem* is finding c by the inverse data:

$$u^f|_{S \times [0,2T]}, \ f \in F.$$

We will consider a bit another problem: to find eikonal function

$$\tau(x) = dist(x, S),$$

where distance means the travel time from the point x to the boundary S. Notice, that fuction τ determines c since it satisfies eiconal equation

$$|\nabla \tau| = 1/c.$$

3. *Control problem.*

Let u^h be a solution to the forward problem (corresponding to the control h) and v be an arbitrary solution to the wave equation. Then one can get differential equality

$$\frac{1}{c^2}(v u_t^f - u^f v_t)_t = \operatorname{div}(v \nabla u^f - u^f \nabla v).$$

After inegration over $\Omega \times [0, s]$ we have integral equality

$$\int_{\Omega^s} \frac{1}{c^2}(v u_t^f - u^f v_t) dx = \int_S (v f - u^f v_n) dS dt,$$

where $S = \partial\Omega = \{x \in \mathbb{R}^2 | \ |x| = r\}$, v_n is the normal derivative and Ω^s is the set filled with wave u up to the moment s.More precisely

$$\Omega^s = \{x \in \Omega | \ \tau(x) \leq s\},$$

where τ is the eikonal,

$$\tau(x) = dist(x, S),$$

where distance means the travel time from the point x to the boundary S. If we put v the even part of u^g with respect to $t=T$

$$v(x,t) = u^g_+(x,t) = \frac{1}{2}(u^g(x,t) + u^g(x, 2T-t))$$

then

$$\int_{\Omega^s} \frac{1}{c^2} u^h_t u^g(,.T)dx = \int_S [u^g_+ h - u^h (g_n)_+] dSdt. \tag{1}$$

Now, if we put $v=\varphi$,where φ is arbitrary harmonic function (does not depend on time), then we have

$$\int_{\Omega^s} \frac{1}{c^2} u^h_t(,.T)\varphi dx = \int_S [\varphi h - u^h \varphi_n] dSdt. \tag{2}$$

Let

$$f(.,t) = \int_0^t h(,.t')dt' \stackrel{def}{=} Ih(t).$$

Then 1 and 2 and can be rewritten as

$$C(f,g) \stackrel{def}{=} \int_{\Omega^s} \frac{1}{c^2} u^f u^g(.,T)dx = \int_{S\times[0,T]} [u^g_+ If - Iu^f (g_n)_+] dSdt, \tag{3}$$

$$\Phi(f) \stackrel{def}{=} \int_{\Omega^s} \frac{1}{c^2} u^f(,.T)\varphi dx = \int_S [\varphi If - Iu^f \varphi_n] dSdt. \tag{4}$$

Thus we have bilinear functional C and linear functional Φ. Let us consider equation

$$A(f,g) = \Phi(f), \quad \forall f \in F^s \tag{5}$$

with respect to $g \in F^s$. By definition this is equalent to the equation

$$\int_{\Omega^s} \frac{1}{c^2} u^f u^g(,.T)dx = \int_{\Omega^s} \frac{1}{c^2} u^f(,.T)dx.$$

But the last equality holds if anf only if

$$u^g(,.T) = \varphi.$$

It follows from the approximate boundary contollability, that is any funaction from $L^2(\Omega^s)$ can be appoximated by a wave $u^f(,.T)$. Another words the following equality holds

$$\overline{U^s} = L^2(\Omega^s)$$

where

$$U^s = \{u^f(.,T)|\ f \in F^s\} \subset L^2(\Omega^s)$$

is the reachable set and $F^s \subset F$ is the subspace of the control space that consist of controls acting only during period $[T-s,T]$. Such controls generate waves which fill the domain Ω^s up to the moment T. Notice, that equation $A(f,g) = \Phi(f)$ is not solvable in general, but it may be shown it is dense solvable that is there exist a sequence of controls $\{g_n\}$ such that $A(f, g_n) \to \Phi(f)$. Formally one can consruct a wider control space in which the equation $A(f, g) = \Phi(f)$ will be solvable. So, let g is the solution of this equation. It implyes the equality $u^g(, .T) = \varphi$.Let us put

$$\begin{aligned} \varphi(x) &= \frac{1}{|x - x_0|}, \quad n = 3, \\ \varphi(x) &= \ln|x - x_0|, \quad n = 2. \end{aligned} \tag{6}$$

These functions are fundamental solutions to the Laplace equations (up to some constants) and if c then they are harmonic in Ω^s if $x_0 \notin \Omega^s$.

How we are going to find $\tau(x_0)$? Let us fix a point x_0 and choose φ as in 6. If $x_0 \notin \Omega^s$ then after solving 5 we get control g that generates

$$u^g(., T) \approx \varphi \text{ in } \Omega^s.$$

Then the value of $\Phi(g)$ equals

$$\Phi(g)(s) = \int_{\Omega^s} \frac{1}{c^2} \varphi^2 dx.$$

This value tends to ∞ when s tends to $\tau(x_0)$. At this moment we get blow-up

$$\Phi(g)(\tau(x_0)) = \infty.$$