0.1 Finite Element Method (Galerkin approximation)

Forward problem is reduced to the following ODE system

$$MU''(t) + KU(t) = F(t), \quad t \in [0, T]$$

with zero initial data

$$U(0) = 0, U'(0) = 0.$$

Here

$$M_{ij} = \int_\Omega \frac{1}{c^2} \varphi_i \varphi_j dx$$

is the mass matrix,

$$K_{ij} = \int_\Omega (\nabla \varphi_i, \nabla \varphi_j) dx$$

is the stiffness matrix,

$$F_{i\alpha}(t) = \int_{\partial\Omega} \varphi_i f_\alpha(., t) d\Gamma$$

is the control matrix-function. The solution U is the matrix $N \times N_c$, where N is the number of nodes, N_c is the number of controls. The approximate equality

$$U_{i\alpha}(t) \approx u^{f_\alpha}(x_i, t)$$

holds.

Introduce connection matrix C

$$C = U^*(T) M U(T).$$

Notice, that appoximately

$$C_{\alpha\beta} \approx \int_\Omega \frac{1}{c^2} u^{f_\alpha}(, .T) u^{f_\beta}(, .T) dx.$$

Consider control problem

$$u^f(, .T) = 1$$

with respect to control f. Due to the boundary contollability this equation is equivalent to the following equation w.r.t. vector $d \in \mathbb{R}^{N_c}$

$$Cd = U^*(T) M e,$$

where vector

$$e_i = 1, \ for \ all \ i.$$

Then control

$$f_e = \sum d_\alpha f_\alpha$$

generates the final state

$$U_{i\alpha}(T) = 1, \ for \ all \ i.$$

Now, we change T for any $s \leq T$ and do the same, but for contols which belongs to subspace F_s. Then we will get vectors d_s, controls $f_{e,s}$, such that

$$U_{i\alpha}(T) = 1, \ for \ all \ x_i \in \Omega_s.$$

Now introduce vector

$$E_i = \psi(x_i), \ x_i \neq x_0$$

where ψ is the fundamental solution and consider finction

$$\Phi(s) = U_s^* M E.$$

Then $\Phi(s)$ tends to ∞ as $s \to \tau(x_0)$.