Please type your homework solution into a PDF file and submit to the gradescope.com website by 11:59pm KST of the due date. A point will be deducted for each problem if a handwritten solution is submitted. We recommend using LaTeX. You can find useful hints at our KLMS website. It is recommended to use the sample template. (At least, make sure that each problem has a solution in separate pages.) Unprofessional proofs may get a deduction of points, even if the solution is mathematically correct or can be made correct.

2021 Spring MAS575 *Combinatorics*
Homework 2

Due: March 30, 2021

2.1. Let p be a prime and let L be a subset of $\{0, 1, 2, \dots, p-1\}$ of size s. Suppose that $\{A_1, A_2, \dots, A_m\}$ and $\{B_1, B_2, \dots, B_m\}$ are two families of subsets of $\{1, 2, \dots, n\}$ satisfying the following:

(i) $|A_i \cap B_i| \notin L + p\mathbb{Z}$ for all $1 \le i \le m$;

(ii) $|A_i \cap B_j| \in L + p\mathbb{Z}$ for all $1 \le i < j \le m$.

Prove that $m \le \sum_{i=0}^{s} \binom{n}{i}$.

2.2. Let $K = \{k_1, k_2, \dots, k_r\}$ and $L = \{l_1, l_2, \dots, l_s\}$ be two sets of nonnegative integers and assume that $k_i > s - r$. Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \dots, n\}$. Prove that if

(i) $|A| \in K$ for all $A \in \mathcal{F}$,

(ii) $|A \cap B| \in L$ for all $A, B \in \mathcal{F}$, $A \ne B$,

then

$$|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}.$$

2.3. Let $n > 1$. Construct two families $\{A_1, A_2, \dots, A_m\}$ and $\{B_1, B_2, \dots, B_m\}$ of sets such that

(i) $A_i \cap B_i = \emptyset$ for all $i \in \{1, 2, \dots, m\}$, and

(ii) $A_i \cap B_j \ne \emptyset$ for all $1 \le i < j \le m$,

but

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \ge n.$$

2.4. Let $\mathcal{F}$ be a k-uniform family of subsets of $[n]$ such that for every member F of $\mathcal{F}$, there is a coloring of $\{1, 2, \dots, n\}$ by k colors so that F is the only member of $\mathcal{F}$ with all k colors appearing. Prove that $|\mathcal{F}| \le \binom{n-1}{k-1}$.

Hint: Find $|\mathcal{F}|$ linearly independent polynomials in variable $x_2, x_3, \dots, x_n$ over $\mathbb{F}_2$.

2.5. Let $\mathcal{F}_1, \mathcal{F}_2$ be k-uniform families of subsets of $\{1, 2, \dots, n\}$. Let L_1, L_2 be disjoint subsets of integers. Prove that if $\mathcal{F}_1$ is L_1-intersecting and $\mathcal{F}_2$ is L_2-intersecting, then

$$|\mathcal{F}_1||\mathcal{F}_2| \le \binom{n}{k}.$$