KAIST
2021 MAS575 Combinatorics
Homework 2

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Contents

1 HW 2.1 **2**

2 HW 2.2 **3**

3 HW 2.3 **5**

4 HW 2.4 **6**

5 HW 2.5 **7**

HW 2.1

Let p be a prime and let L be a subset of $\{0, 1, 2, ..., p-1\}$ of size s. Suppose that $\{A_i\}_{i\in[m]}$ and $\{B_i\}_{i\in[m]}$ are two families of subsets of $[n]$ satisfying the following:

1. $|A_i \cap B_i| \notin L + p\mathbb{Z}, \forall i$;

2. $|A_i \cap B_j| \in L + p\mathbb{Z}, \forall i < j$.

Prove that $m \leq \sum_{i=0}^{s} \binom{n}{i}$.

Proof. Let $a_i = (a_{ij})_{j\in[n]}, b_i = (b_{ij})_{j\in[n]} \in \mathbb{F}_p^n$ be the characteristic vectors of A_i and B_i, respectively, i.e., $a_{ij} = 1$ if $j \in A_i$ and $a_{ij} = 0$ otherwise, likewise $b_{ij} = 1$ if $j \in B_i$ and $b_{ij} = 0$ otherwise. Let $L = \{l_i\}_{i\in[s]}$, and define f_i for $i \in [m]$ on $\mathbb{F}_p^n$ as

$$f_i(x) = \prod_{j\in[s]} (x \cdot a_i - l_j), x \in \mathbb{F}_p^n.$$

We obtain $\tilde{f}_i$ from f_i by repeatedly replacing x_j^2 to x_j for all j after expansion, until each $\tilde{f}_i$ becomes multilinear of degree $\leq s$ with variables $x_1, x_2, ..., x_n$. Clearly, we have $\tilde{f}_i(b_i) = f_i(b_i) \neq 0, \forall i$ and $\tilde{f}_i(b_j) = f_i(b_j) = 0, \forall i < j$.

Claim 1.1. *$\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_m$ are linearly independent.*

To see this, suppose $\sum_{i\in[m]} c_i \tilde{f}_i(x) \equiv 0, \forall x \in \mathbb{F}_p^n$, then let $x = b_m$ we have

$$0 = \sum_{i\in[m]} c_i \tilde{f}_i(b_m) = \sum_{i\in[m]} c_i f_i(b_m) = c_m f_m(b_m),$$

which implies $c_m = 0$. Therefore, we get $\sum_{i\in[m-1]} c_i \tilde{f}_i \equiv 0$ and this time we let $x = b_{m-1}$ and have

$$0 = \sum_{i\in[m-1]} c_i \tilde{f}_i(b_{m-1}) = \sum_{i\in[m-1]} c_i f_i(b_{m-1}) = c_{m-1} f_{m-1}(b_{m-1}),$$

which implies $c_{m-1} = 0$. By repeating this process, we can see

$$c_m = c_{m-1} = ... = c_1 = 0,$$

completing the proof of the claim. Therefore, m is at most the dimension of the span of all multilinear polynomials of degree $\leq s$ with variables $x_1, x_2, ..., x_n$, i.e.,

$$m \leq \sum_{i=0}^{s} \binom{n}{i},$$

completing the proof. □

HW 2.2

Let $K = \{k_i\}_{i\in[r]}, L = \{l_i\}_{i\in[s]} \subset \mathbb{N}$ and assume that $k_i > s - r, \forall i$. Let $\mathcal{F}$ be a family of subsets of $[n]$. Prove that if

1. $|A| \in K, \forall A \in \mathcal{F}$,

2. $|A \cap B| \in L, \forall A \neq B \in \mathcal{F}$,

then

$$|\mathcal{F}| \leq \sum_{i=0}^{r-1} \binom{n}{s-i}$$

Proof. WLOG, we may assume that L is minimal in the sense that for each $l_i \in L$, there exist $A \neq B \in \mathcal{F}$ such that $|A \cap B| = l_i$, because otherwise we can remove these "unused" l_i from L and s will not increase, thus $k_i > s - r$ still holds. For such L, we have $l_i < |A|$ for each $i \in [s]$ and each $A \in \mathcal{F}$. Let $\mathcal{F} = \{A_i\}_{i\in[m]}$ and let $a_i \in \mathbb{R}^n$ be the characteristic vector of A_i for each i. Define f_i for each $i \in [m]$ on $\mathbb{R}^n$ as

$$f_i(x) = \prod_{j\in[s]} (x \cdot a_i - l_j), x \in \mathbb{R}^n.$$

Besides, for each $I \subset [n]$ with $|I| \leq s - r$, we define g_I on $\mathbb{R}^n$ as

$$g_I(x) = \prod_{i\in[r]} (\sum_{j\in[n]} x_j - k_i) \prod_{v\in I} x_v, x \in \mathbb{R}^n.$$

We obtain $\tilde{f}_i$ from f_i by repeatedly replacing x_j^2 to x_j for all j after expansion, until each $\tilde{f}_i$ becomes multilinear of degree $\leq s$ with variables $x_1, x_2, ..., x_n$. Similarly, we have multilinear $\tilde{g}_I$ for each I. It is easy to check that $\tilde{f}_i(a_j) = f_i(a_j) \neq 0$ iff $i = j$, and $\tilde{g}_I(a_i) = g_I(a_i) = 0$ for each I and each i .

Claim 2.1. *All $\tilde{f}_i$ for $i \in [m]$ and $\tilde{g}_I$ for $I \subset [n]$ with $|I| \leq s - r$ are linearly independent.*

If the claim is true, as all these functions are multilinear of degree $\leq s$ with n variables, we have

$$m + \sum_{i=0}^{s-r} \binom{n}{i} \leq \sum_{i=0}^{s} \binom{n}{i}$$

and the desired result follows. To prove the claim, suppose $\sum_i \alpha_i \tilde{f}_i + \sum_I \beta_I \tilde{g}_I \equiv 0$. By letting $x = a_j$ for some j, we have

$$0 = \sum_i \alpha_i \tilde{f}_i(a_j) + \sum_I \beta_I \tilde{g}_I(a_j) = \sum_i \alpha_i f_i(a_j) + \sum_I \beta_I g_I(a_j) = \alpha_j f_j(a_j),$$

implying that $\alpha_j = 0$, for any j, as j can be arbitrarily chosen. Now it remains to show all $\beta_I = 0$. Suppose there exists I such that $\beta_I \neq 0$, then we choose a minimal I_0 among them in the sense that $\beta_I = 0, \forall I \subsetneq I_0$. Let x^* be the characteristic vector of I_0, and we have

$$\tilde{g}_I(x^*) = g_I(x^*) = \prod_{i\in[r]} (|I_0| - k_i) \prod_{v\in I} x_v^*.$$

As $|I_0| \leq s - r < k_i, \forall i$, the above is 0 iff $\prod_{v \in I} x_v^* = 0$, which holds for each I such that $I \setminus I_0 \neq \emptyset$. Therefore, by the minimality of I_0,

$$0 = \sum_I \beta_I \tilde{g}_I(x^*) = \sum_I \beta_I g_I(x^*) = \sum_{I : I \setminus I_0 = \emptyset} \beta_I g_I(x^*) = \beta_{I_0} g_{I_0}(x^*).$$

However, clearly, $g_{I_0}(x^*) \neq 0$, contradicting with the assumption that $\beta_I \neq 0$, completing the proof.

$\square$

HW 2.3

Let $n > 1$. Construct two families $\{A_i\}_{i\in[m]}$ and $\{B_i\}_{i\in[m]}$ of sets such that

1. $A_i \cap B_i = \emptyset, \forall i \in [m]$,
2. $A_i \cap B_j \neq \emptyset, \forall 1 \le i < j \le m$,

but

$$\sum_{i\in[m]} \binom{|A_i| + |B_i|}{|A_i|}^{-1} \ge n.$$

Proof. WLOG, we may assume that n is integer, otherwise we use $\lceil n \rceil$ as n. Let $\{A_i\}_{i\in[m]} = 2^{[n]}$ with $m = 2^n$ such that $|A_i| \ge |A_j|, \forall i \le j$ and let $B_i = [n] \setminus A_i, \forall i$. It is easy to check that the two conditions are satisfied and for each i,

$$\binom{|A_i| + |B_i|}{|A_i|} = \binom{n}{|A_i|}.$$

Therefore,

$$\sum_{i\in[m]} \binom{|A_i| + |B_i|}{|A_i|}^{-1} = \sum_{A\in[n]} \binom{n}{|A|}^{-1} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{i}^{-1} = n + 1 \ge n,$$

giving a desired construction. □

HW 2.4

Let $\mathcal{F}$ be a k-uniform family of subsets of $[n]$ such that for every member F of $\mathcal{F}$, there is a coloring of $[n]$ by k colors so that F is the only member of $\mathcal{F}$ with all k colors appearing. Prove that $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Proof. Let $\mathcal{F} = \{A_i\}_{i\in[m]}$. For each $i \in [m]$, let $c_i : [n] \to [k]$ be any coloring such that A_i is the only member of $\mathcal{F}$ with all k colors appearing and $c_i(n) = k$. Further for each $j \in [k]$ we let $c_{ij} \in \mathbb{F}_2^{n-1}$ be the characteristic vector of $c_i^{-1}(j) \setminus \{n\}$, the subset of $[n-1]$ whose members are colored by color j. For each $i \in [m]$, we let $a_i \in \mathbb{F}_2^{n-1}$ be the characteristic vector of $A_i \setminus \{n\}$, and we define for each $i \in [m]$

$$f_i(x) = \prod_{j=1}^{k-1} c_{ij} \cdot x, x \in \mathbb{F}_2^{n-1}.$$

Note that the computation is over $\mathbb{F}_2$ and we can check that $f_i(a_v) = 1$ iff $i = v$. To see this, for each $i \in [m]$ and each $j \in [k-1]$, $c_{ij} \cdot a_i = 1$, counting the number in $A_i \setminus \{n\}$ colored by c_i with color j, as we already required $c_i(n) = k$. While when $i \neq v$, under coloring c_i, either some color $j \in [k-1]$ does not appear in A_v so that $c_{ij} \cdot a_v = 0$, or only color k is missing in A_v so that for some $j \in [k-1]$, $c_{ij} \cdot a_v = 2 = 0$. For each $i \in [m]$, f_i is multilinear of degree **exactly** $k-1$ with variables $x_1, x_2, ..., x_{n-1}$. To see this, each $x_v, v \in [n-1]$ survives in at most one $c_{ij} \cdot x$, and as c_i colors A_i with all k colors appearing, each $c_{ij} \cdot x$ contains some x_v. Now we claim that all these $(f_i)_{i\in[m]}$ are linearly independent. Suppose $\sum_{i\in[m]} c_i f_i(x) \equiv 0$, let $x = a_j$ for some $j \in [m]$, we have

$$0 = \sum_{i\in[m]} c_i f_i(a_j) = c_j f_j(a_j),$$

implying that $c_j = 0$ for each j, as j can be arbitrarily chosen. Therefore, m is at most the dimension of the span of all multilinear polynomials of degree exactly $k-1$ with variables $x_1, x_2, ..., x_{n-1}$, i.e.,

$$m \leq \binom{n-1}{k-1},$$

completing the proof. $\square$

HW 2.5

Let $\mathcal{F}_1, \mathcal{F}_2$ be k-uniform families of subsets of $[n]$. Let L_1, L_2 be disjoint subsets of integers. Prove that if $\mathcal{F}_1$ is L_1-intersecting and $\mathcal{F}_2$ is L_2-intersecting, then

$$|\mathcal{F}_1||\mathcal{F}_2| \leq \binom{n}{k}.$$

Proof. Hint is found in the following book. Babai, László, and Péter Frankl. *Linear algebra methods in combinatorics.* University of Chicago, 1988.

Lemma 5.1. *Let $\mathcal{F}_1, \mathcal{F}_2$ be k-uniform families of subsets of $[n]$ and let $\mathcal{F}_2^\sigma$ denote the image of $\mathcal{F}_2$ under the permutation σ of the universe $[n]$, then $\mathbb{E}_\sigma|\mathcal{F}_1 \cap \mathcal{F}_2^\sigma| = |\mathcal{F}_1||\mathcal{F}_2|/\binom{n}{k}$, where σ is chosen uniformly at random among all the permutations of $[n]$.*

Proof of Lemma. Let $A \in \mathcal{F}_2$ and consider the probability p that $A^\sigma \in \mathcal{F}_1$ when choosing a permutation σ uniformly at random. For each $B \in \mathcal{F}_1$, it is easy to see that the probability that $A^\sigma = B$ is $k!(n-k)!/n! = 1/\binom{n}{k}$. Therefore, $p = |\mathcal{F}_1|/\binom{n}{k}$, which is the contribution of A to $\mathbb{E}_\sigma|\mathcal{F}_1 \cap \mathcal{F}_2^\sigma|$, implying the desired result. $\square$

With the above lemma, it suffices to show that $\mathbb{E}_\sigma|\mathcal{F}_1 \cap \mathcal{F}_2^\sigma| \leq 1$ with the given conditions. To see this, fix a permutation σ, suppose there exist $A_1 \neq A_2$ such that $\{A_1, A_2\} \subset \mathcal{F}_1$ and $\{A_1, A_2\} \subset \mathcal{F}_2^\sigma$. Let B_i be the member in $\mathcal{F}_2$ such that $B_i^\sigma = A_i$ for $i \in [2]$, then $|A_1 \cap A_2| = |B_1 \cap B_2|$, contradicting with the condition that L_1 and L_2 are disjoint. Therefore, $|\mathcal{F}_1 \cap \mathcal{F}_2^\sigma| \leq 1$ for any permutation σ, completing the proof. $\square$