Please type your homework solution into a PDF file and submit to the gradescope.com website by 11:59pm KST of the due date. A point will be deducted for each problem if a handwritten solution is submitted. We recommend using LaTeX. You can find useful hints at our KLMS website. It is recommended to use the sample template. (At least, make sure that each problem has a solution in separate pages.) Unprofessional proofs may get a deduction of points, even if the solution is mathematically correct or can be made correct.

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2021 SPRING MAS575 *Combinatorics*
HOMEWORK 4

DUE: MAY 11, 2021

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4.1. Let A and B be two nonempty subsets of $\mathbb{Z}_p$. Let

$$X = \{a + b : a \in A, b \in B, ab \neq 1\}.$$

Show that $|X| \geq \min\{|A| + |B| - 3, p\}$.

4.2. A graph is k-regular if every vertex has degree k. Let p be a prime. Let G be a graph with no loops. Prove that if the average degree of G is greater than $2p - 2$ and the maximum degree is at most $2p - 1$, then G contains a p-regular subgraph.

4.3. Suppose that there exist m affine hyperplanes covering each point in $\{0, 1\}^n - \{0\}$ at least twice but not covering 0. What is the minimum m in terms of n?

4.4. Let p be a prime and $\mathbb{F}_p = GF(p)$ be the field of size p. Let $f_1, f_2, \ldots, f_m$ be polynomials in $\mathbb{F}_p[x_1, x_2, \ldots, x_n]$ with no constant terms. Let $Q_1, Q_2, \ldots, Q_m$ be subsets of $\mathbb{F}_p$ such that $0 \in Q_i$ for all i. If $\sum_{i=1}^m \deg(f_i)|\mathbb{F}_p \setminus Q_i| < n$, then there exists a vector $x \in \{0, 1\}^n$ such that $f_i(x) \in Q_i$ for all i and $x \neq 0$.

4.5. In a party, n couples are invited. They decided to sit around a round table with $2n + 1$ chairs such that the i-th couple are seated from each other by distance d_i (meaning that they are separated by $d_i - 1$ chairs). Prove that if $2n + 1$ is a prime and $d_1, d_2, \ldots, d_n \leq n$, then this is possible.