§10.4 Gallai's Thm.

$V = \{v_1, v_2, \dots, v_t\} \subseteq \mathbb{R}^n$

Def/ We say W is homothetic to V if

$\exists$ ordering of vectors in W, $(w_1, \dots, w_t)$,
$a \in \mathbb{R}^m$, $b \in \mathbb{R}$ s.t. $w_i = a + b v_i$, $\forall i$

$\boxed{\text{Thm}}$ (Gallai)

$\forall$ coloring of $\mathbb{R}^m$ into finitely many colors

and $\forall$ finite set $V \subseteq \mathbb{R}^m$, $\exists$ $W \subseteq \mathbb{R}^m$

s.t. W is monochromatic and homothetic to V.

Pf/ r: # colors; $V = \{v_1, \dots, v_k\}$; $N = HJ(k, r)$

Pick a function $\psi : [k]^N \to \mathbb{R}^m$,

$\psi(x_1, \dots, x_N) = \sum c_i v_{x_i}$

$\{c_i\}_{i \in [N]}$ to be determined later

If we can choose c_i's s.t. ψ is injective, then from the coloring of $\mathbb{R}^m$ we have an r-coloring of $[k]^N$ s.t. there is a monochromatic comb. line. $\Rightarrow \underbrace{\boxed{A} + \boxed{B}\, v_i}_{\text{monochromatic}}$ $i \in [k]$

For example, $(3, 2, *) \to$ monochromatic: $\left\{ \begin{array}{l} \underbrace{c_1 v_3 + c_2 v_2}_{A} + \underbrace{c_3}_{B} v_1 \\ c_1 v_3 + c_2 v_2 + c_3 v_2 \\ c_1 v_3 + c_2 v_2 + c_3 v_3 \end{array} \right.$

$\Rightarrow w_i = A + B v_i$ is monochromatic

Goal: choose $c_1, \sim, c_N$ s.t. ψ is injective, i.e., $\sum_{i=1}^{N} c_i (v_{x_i} - v_{x_i'}) \neq 0, \ \forall x \neq x' \in [k]^N$

equations are finite $\Rightarrow \exists$ such choices $\blacksquare$

$\boxed{\text{Cor}}$ If $\mathbb{N}^2$ is colored by finitely many colors, then $\forall t$, $\exists$ x_0, y_0, d s.t. (x_0+id, y_0+jd) have the same color for all $0 \le i, j < t$.

$\boxed{\text{Ch 11}}$ Monochromatic solutions

§11.1 Schur's Thm.

$\boxed{\text{Thm}}$ (Schur 1916)

$\forall r>0, \exists N$ s.t. in $\forall$ r-coloring of $[N]$,

$\exists\, x, y \in [N]$ s.t. $x, y, x+y$ have the same color.

Pf/ $N = R(3;r)-1$

Let $\chi : [N] \to [r]$. We construct K_{N+1} and

Color edge ij in $\chi(|i-j|)$. By Ramsey's,

$\exists\, a<b<c \in [N+1]$ s.t.

$\chi(b-a) = \chi(c-b) = \chi(c-a)$

Take $x = c-b$, $y = b-a$ $\blacksquare$

$\boxed{\text{Rmk}}$ $[N]$ has $\geq \frac{1}{22}N^2 + O(N)$ monochromatic

triples of the form $\{x, y, x+y\}$. (Tight!) A Robertson 98', Schoen 99'

$\boxed{\text{Cor}}$ (Schur 1916)

$n \in \mathbb{Z}^+$. p: suff. large prime. $\overset{?}{\Longrightarrow}$

$x^n + y^n \equiv z^n \pmod{p}$ has nonzero solutions.

Pf/ Let p be suff. large s.t. $\forall$ n-coloring of $[p-1]$ includes a monochromatic triple of the form $\{x, y, x+y\}$.

Let g be the primitive root modulo p, i.e.,

$\{1, 2, \dots, p-1\} = \{1, g, g^2, \dots, g^{p-2}\} \pmod{p}$

For each $a \in [p-1]$, if $a \equiv g^i \pmod{p}$, $0 \le i \le p-2$,

then we color a by $i \bmod n$.

Suppose $a+b=c$ w/ $a, b, c \in [p-1]$ having the same color.

Then $a \equiv g^{nx+r}$, $b \equiv g^{ny+r}$, $c \equiv g^{nz+r} \pmod{p}$,

where r is the color of a, b, c.

$\Rightarrow g^{nx+r} + g^{ny+r} \equiv g^{nz+r} \pmod{p}$

$\Rightarrow g^{nx} + g^{ny} \equiv g^{nz} \pmod{p}$

$\Rightarrow (g^x)^n + (g^y)^n \equiv (g^z)^n$ ▨

$\boxed{\text{Thm}}$ (Rado) Let $r \geq 1$.

For integers $k_1, \dots, k_r \geq 3$, $\exists N \in \mathbb{Z}^+$ s.t.

$\forall$ r-coloring of $[N]$, $\exists j \in [r]$ and

$x_1, x_2, \dots, x_{k_j} \in [N]$ of color j s.t.

$x_1 + x_2 + \dots + x_{k_j - 1} = x_{k_j}$

(it implies Schur's by taking $k_i = 3, \forall i \in [r]$)

Pf/ $N = R(k_1, \dots, k_r) - 1$. —

▨

§ 11.2 Regular Linear Homogeneous Eqs.

Def/ A set S of eqs. w/ variables $x_1, x_2, \ldots, x_n$ is <u>r-regular</u> on a set A if $\forall$ r-coloring of A induces a monochromatic solution $x_1, x_2, \ldots, x_n \in A$.

For instance, $S = (x_1 + x_2 = x_3)$, then by Schur's, S is r-regular on $\mathbb{Z}^+$, $\forall r$.

We say S is <u>regular</u> if S is r-regular for all $r \in \mathbb{Z}^+$.

Example: $\{x_1 + x_2 + \cdots + x_{100} = x_{101}\}$ is regular on $\mathbb{N}$.

Q: How to characterize a regular set of eqs.?

$\boxed{\text{Thm}}$ (Rado). Let $S = S(x_1, \ldots, x_n)$ be a set of a single linear homogeneous eq.

$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ w/ $c_i \in \mathbb{Z}, \forall i \in [n]$

Then S is regular $\Longleftrightarrow \sum_{i\in I} c_i = 0$ for some $\phi \neq I \subseteq [n]$.

Ex. $\begin{cases} x+y=z & (\checkmark,\ 1+(-1)=0) \\ x+y=2z & (\checkmark,\ 1+1-2=0 \\ x+y=3z & (\text{NOT regular}) \end{cases}$

Pf/ $\Rightarrow$) Suppose S is regular.

Let $c_1, c_2, \ldots, c_n$ be fixed $\mathbb{Z}$. Suppose that $\sum_{i\in I} c_i \neq 0$, $\forall \phi \neq I \subseteq [n]$. Choose a prime p s.t. $\sum c_i \not\equiv 0 \pmod{p}$ [possible since $n < \infty$]

Define $\chi : \mathbb{N} \to [p-1]$ as $\chi(m) \equiv a \pmod{p}$, where $m = p^l \cdot a$, $p \nmid a$, $l, a \in \mathbb{Z}$. We call such a coloring the <u>base p</u> coloring.

$\boxed{\text{Claim}}$ $\exists$ non-monochromatic sol. w/ χ.

Suppose $x_1, -, x_n$ is a sol. of S w/ $\chi(x_i)=u$, $\forall i \in [n]$. For each i, let l_i be the max. $\mathbb{Z}$ s.t. $p^{l_i} \mid x_i$ and let $l = \min_i l_i$.

We have $\frac{x_i}{p^{l_i}} \equiv u \pmod p$

$\Rightarrow \frac{x_i}{p^l} \equiv u$ (if $l = l_i$) or 0 (otherwise) $\pmod p$

Let $I = \{i \in [n] : l = l_i\}$. We have

$\sum_{i=1}^n c_i x_i \equiv 0 \pmod{p^{l+1}}$, but meanwhile

$\sum_{i=1}^n c_i x_i \equiv \sum_{i \in I} p^l \cdot u \cdot c_i \not\equiv 0 \pmod{p^{l+1}}$,

Contradiction! $\blacksquare$

Pf/($\Leftarrow$) Suppose $\exists\, \phi \neq I \subseteq [n]$ s.t. $\sum_{i \in I} c_i = 0$.

WLOG, WMA $c_1 + c_2 + \cdots + c_k = 0$, for some $k \in [n]$.

Moreover, WMA $c_i \neq 0$, $\forall i$. Let $\chi : \mathbb{N} \to [r]$.

If $k = n$, we can take $x_i = 1$, $\forall i \in [n]$ and we have a monochromatic solution. $\Rightarrow$

WMA $k<n$. And we choose the max. k,

in particular, $c_{k+1}+c_{k+2}+\cdots+c_n \neq 0$.

Let $A = \gcd(c_1, \ldots, c_k)$ and $B = c_{k+1}+\cdots+c_n$,

$s = A/\gcd(A,B)$, $t = -B/\gcd(A,B)$ $(At+Bs=0)$

Fact: $\exists\, \lambda_1, \ldots, \lambda_k \in \mathbb{Z}$ s.t. (Since A is gcd)

$c_1\lambda_1 + c_2\lambda_2 + \cdots + c_k\lambda_k = At$. Now let

$$x_i = \begin{cases} a+\lambda_i d & \text{if } i \in [k] \\ sd & \text{if } i > k \end{cases}$$

$$\Rightarrow \sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{k} c_i(a+\lambda_i d) + \sum_{i=k+1}^{n} c_i sd$$

$$= (At+Bs)d = 0$$

$\Rightarrow (x_1, x_2, \ldots, x_n)$ is a sol. of S.

Now it suffices to show that:

(HW)

$\boxed{\text{Lem}}$ $\forall k, r, s \geq 1$, $\exists\, n = n(k,r,s)$ s.t.

$\forall$ r-coloring of $[n]$, $\exists\, a, d > 0$ s.t.

$\{a, a+d, \dots, a+(k-1)d\} \cup \{sd\}$ is monochromatic.

(Take $k' = 2k$, we have monochromatic

$\{a, a+d, a+2d, \dots, a+(2k-1)d, sd\}$. Set

$a' = a + kd \Rightarrow a' + \lambda d$ have the same color

for all λ w/ $|\lambda| < k$.)

$\blacksquare$

§11.3 Radó's Thm

Q: when is $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$ regular?

integer-valued matrix w/ n columns (C_i's)

(A has a single row $\Rightarrow$ §11.2)

Def/ ("Columns Condition")

We say A satisfies the column condition

if $\exists$ partition $(I_1, I_2, \dots, I_t)$ of $[n]$

w/ $I_k \neq \emptyset, \forall k \in [t]$ s.t.

$\forall i \in [t], \sum_{j \in I_i} C_j \in \mathrm{Span}_{\mathbb{Q}}(\{C_k : k \in I_{i'} \text{ for some } i' < i\})$

$\boxed{\text{Example}}$ $\sum_{j \in I_1} C_j = 0,$

$\sum_{j \in I_2} C_j \in \langle C_j : j \in I_1 \rangle$

$\sum_{j \in I_3} C_j \in \langle C_j : j \in I_1 \cup I_2 \rangle$

$\sum_{j \in I_t} C_j \in \langle C_j : j \in \bigcup_{k<t} I_k \rangle$

$\boxed{\text{Thm}}$ (Rado)

$Ax=0$ is regular on $\mathbb{Z}^+ \Longleftrightarrow A$ satisfies "CD"

Pf/ ($\Rightarrow$) [we'll use the "base p" coloring, cf. §11.2]

$\boxed{\text{Lem}}$ Let $j, k \in \mathbb{Z}^+$ and $c_1, c_2, \ldots, c_j \in \mathbb{Z}^k$

s.t. c_1 is NOT a linear combination over $\mathbb{Q}$

of the (possibly empty) set $\{c_2, c_3, \ldots, c_j\}$.

Then $\exists$ finite set F of primes s.t.

$\forall$ prime $p \notin F$, $m \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the

vector $p^m c_1$ is NOT a linear comb. of

$c_2, c_3, \ldots, c_j$ modulo p^{m+1}.

Pf/ (of Lem)

Since c_1 is NOT in the span over $\mathbb{Q}$

of $\{c_2, c_3, \ldots, c_j\}$, $\exists\, u \in \mathbb{Q}^k$ s.t.

$c_i \cdot u \neq 0 \Leftrightarrow i=1$. WMA $u \in \mathbb{Z}^k$ by multiplying a large $\mathbb{Z}$. Let $F = \{p : p$ is prime, $p \mid u \cdot c_1\}$

Suppose p is prime and $m \in \mathbb{N}$ s.t.

$p^m c_1$ is a linear comb. of $c_2, c_3 \ldots c_j$ mod p^{m+1}.

$\Rightarrow p^m \cdot c_1 \equiv a_2 c_2 + a_3 c_3 + \cdots + a_j c_j \pmod{p^{m+1}}$

$\Rightarrow p^m \cdot c_1 u \equiv 0 \pmod{p^{m+1}}$

$\Rightarrow c_1 \cdot u \equiv 0 \pmod{p} \Rightarrow p \in F$

$\Rightarrow F = \{\text{primes}\}$, contradiction!

Suppose $Ax=0$ is regular on $\mathbb{Z}^+$.

For disjoint $I, J \subseteq [n]$, if $I \neq \phi$, then

either $\sum_{i \in I} c_i$ is a linear comb. aer $\mathbb{Q}$ of $\{c_j : j \in J\}$

or $\overset{\text{Lem.}}{\Longrightarrow} \exists$ finite $F_{I,J}$ of primes s.t. $\left[(p \notin F_{I,J}, m \in \mathbb{N})\right.$

$\Rightarrow p^m \sum_{i \in I} c_i$ is NOT a linear comb. of $\{c_j : j \in J\}$ mod $\left.p^{m+1}\right]$.

Let $F = \bigcup_{I,J} F_{I,J}$ over all disj. pairs $I, J \subseteq [n]$

(F is still finite) Choose $p \notin F$ and let

$\chi : \mathbb{Z}^+ \to [p-1]$ be a base-p coloring.

Suppose $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is a monochromatic solution

of $Ax = 0$. Let l_i be the max. $\mathbb{Z}$ s.t. $p^{l_i} \mid x_i$.

Let $a = \chi(x_1) = \chi(x_2) = \cdots = \chi(x_n)$, we have

$x_i \equiv p^{l_i} \cdot a \pmod{p^{l_i+1}}$, $\forall i \in [n]$. WMA

$l_1 \le l_2 \le \cdots \le l_n$, and let $b_1 < b_2 < \cdots < b_t$ s.t.

$\{b_1, \ldots, b_t\} = \{l_1, \ldots, l_n\}$, and let

$I_i = \{ j \in [n] : l_j = b_i \}$.

$$\sum_{i=1}^{n} x_i c_i = \sum_{i=1}^{t} \sum_{j \in I_i} x_j c_j = 0 \Rightarrow \forall m \in [t]$$

$$\sum_{i=1}^{m-1} \sum_{j \in I_i} x_j c_j + \sum_{i=m}^{t} \sum_{j \in I_i} x_j c_j = 0$$

$$\Rightarrow \sum_{i=1}^{m-1} \sum_{j \in I_i} x_j c_j + \sum_{j \in I_m} x_j c_j \equiv 0 \pmod{p^{b_m+1}}$$

$$\Rightarrow \sum_{i=1}^{m-1} \sum_{j \in I_i} x_j c_j + p^{b_m} \cdot a \cdot \sum_{j \in I_m} c_j \equiv 0 \pmod{p^{b_m+1}}$$

$\left(\because\ x_j \equiv p^{b_m} \cdot a \pmod{p^{b_m+1}},\ \forall j \in I_m \right)$

Let $\tilde{a} \in \mathbb{Z}$ s.t. $a\tilde{a} \equiv 1 \pmod{p}$

$$\Rightarrow p^{b_m} \sum_{j \in I_m} c_j \equiv -\tilde{a} \sum_{i=1}^{m-1} \sum_{j \in I_i} x_j c_j \pmod{p^{b_m+1}}$$

Since $p \notin T$, by the previous Lemma,

$\sum_{j \in I_m} c_j$ is a linear comb. over $\mathbb{Q}$ of

$\{c_j : j \in I_i \text{ for some } i < m\} \Rightarrow A$ satisfies "(*)" $\blacksquare$