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2021 SPRING MAS575 *Combinatorics*
HOMEWORK 1

DUE: MARCH 16, 2021

1.1. Suppose that there are m red clubs $R_1, R_2, \dots, R_m$, and m blue clubs $B_1, B_2, \dots, B_m$ in a university of n students. Suppose that the following rules are satisfied:

(a) $|R_i \cap B_i|$ is odd for every i.

(b) $|R_i \cap B_j|$ is even for every $i \neq j$.

Prove that $m \leq n$.

1.2. Let us consider the following variation of the odd rule in some university with n students:

(i) Every club has an even number of members.

(ii) Every pair of clubs shares an odd number of members.

Prove that there are at most n clubs if n is odd, and at most $n-1$ clubs if n is even.

1.3. Let $\mathcal{A}$, $\mathcal{B}$ be families of subsets of $\{1, 2, \dots, n\}$ such that $|A \cap B|$ is odd for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that $|\mathcal{A}||\mathcal{B}| \leq 2^{n-1}$.

1.4. Let $\{A_1, A_2, \dots, A_m\}$ be an intersecting antichain of subsets of $\{1, 2, \dots, n\}$ such that $|A_i| \leq \frac{1}{2}n$ for each i. Prove that

$$\sum_{i=1}^{m} \frac{1}{\binom{n-1}{|A_i|-1}} \leq 1.$$

(A family $\{A_1, A_2, \dots, A_m\}$ is an *antichain* if and only if $A_i \not\subseteq A_j$ for all $i \neq j$.)

1.5. Let $\frac{1}{2} \leq p < 1$. Let $X_1, X_2, \dots, X_n$ be independent random variables such that $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be nonnegative real numbers such that $\sum_{i=1}^{n} \alpha_i = 1$. Prove that

$$P\left(\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n \geq \frac{1}{2}\right) \geq p.$$

Hint: Erdős-Ko-Rado Theorem.

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2021 SPRING MAS575 *Combinatorics*
HOMEWORK 2

DUE: MARCH 30, 2021

2.1. Let p be a prime and let L be a subset of $\{0, 1, 2, \dots, p-1\}$ of size s. Suppose that $\{A_1, A_2, \dots, A_m\}$ and $\{B_1, B_2, \dots, B_m\}$ are two families of subsets of $\{1, 2, \dots, n\}$ satisfying the following:

(i) $|A_i \cap B_i| \notin L + p\mathbb{Z}$ for all $1 \le i \le m$;

(ii) $|A_i \cap B_j| \in L + p\mathbb{Z}$ for all $1 \le i < j \le m$.

Prove that $m \le \sum_{i=0}^{s} \binom{n}{i}$.

2.2. Let $K = \{k_1, k_2, \dots, k_r\}$ and $L = \{l_1, l_2, \dots, l_s\}$ be two sets of nonnegative integers and assume that $k_i > s - r$. Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \dots, n\}$. Prove that if

(i) $|A| \in K$ for all $A \in \mathcal{F}$,

(ii) $|A \cap B| \in L$ for all $A, B \in \mathcal{F}$, $A \ne B$,

then

$$|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}.$$

2.3. Let $n > 1$. Construct two families $\{A_1, A_2, \dots, A_m\}$ and $\{B_1, B_2, \dots, B_m\}$ of sets such that

(i) $A_i \cap B_i = \emptyset$ for all $i \in \{1, 2, \dots, m\}$, and

(ii) $A_i \cap B_j \ne \emptyset$ for all $1 \le i < j \le m$,

but

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \ge n.$$

2.4. Let $\mathcal{F}$ be a k-uniform family of subsets of $[n]$ such that for every member F of $\mathcal{F}$, there is a coloring of $\{1, 2, \dots, n\}$ by k colors so that F is the only member of $\mathcal{F}$ with all k colors appearing. Prove that $|\mathcal{F}| \le \binom{n-1}{k-1}$.

Hint: Find $|\mathcal{F}|$ linearly independent polynomials in variable $x_2, x_3, \dots, x_n$ over $\mathbb{F}_2$.

2.5. Let $\mathcal{F}_1, \mathcal{F}_2$ be k-uniform families of subsets of $\{1, 2, \dots, n\}$. Let L_1, L_2 be disjoint subsets of integers. Prove that if $\mathcal{F}_1$ is L_1-intersecting and $\mathcal{F}_2$ is L_2-intersecting, then

$$|\mathcal{F}_1||\mathcal{F}_2| \le \binom{n}{k}.$$

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2021 SPRING MAS575 *Combinatorics*
HOMEWORK 3

DUE: APRIL 13, 2021

3.1. The $(n-1)$-*dimensional unit sphere* is defined as $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Let $m(n)$ be the maximum number of points in a set X of points in S^{n-1} such that $\|x-y\| = a$ or b for all $x, y \in X$, $x \neq y$ for some fixed a and b. Prove that

$$n(n+1)/2 \le m(n) \le n(n+3)/2.$$

3.2. Let k be a positive integer. Let $r_1, r_2, \ldots, r_k$ be positive integers. For each $i \in \{1, 2, \ldots, m\}$, let $(A_{i,1}, A_{i,2}, \ldots, A_{i,k})$ be a k-tuple of pairwise disjoint sets such that $|A_{i,j}| = r_j$. Suppose that for each $i \neq i'$, there exist $j_1 < j_2$ and $j'_1 < j'_2$ such that

$$A_{i,j_1} \cap A_{i',j'_2} \neq \emptyset \text{ and } A_{i,j_2} \cap A_{i',j'_1} \neq \emptyset.$$

Prove that $m \le \dfrac{(\sum_{i=1}^k r_i)!}{\prod_{i=1}^k r_i!}$.

3.3. Let $X_1, X_2, \ldots, X_n$ be disjoint sets. Let $r_1, r_2, \ldots, r_n$ and $s_1, s_2, \ldots, s_n$ be positive integers. Suppose that A_{ij} and B_{ij} are subsets of X_i for $i \in [n]$ and $j \in [m]$ such that $|A_{ij}| = r_i$ and $|B_{ij}| = s_i$. In addition,

$$(\bigcup_i A_{ij}) \cap (\bigcup_i B_{ij}) = \emptyset \text{ for all } 1 \le j \le m.$$
$$(\bigcup_i A_{ij}) \cap (\bigcup_i B_{ik}) \neq \emptyset \text{ for all } 1 \le j < k \le m.$$

Prove that $m \le \prod_{i=1}^n \binom{r_i+s_i}{r_i}$.

3.4. Let a, b, c be positive integers. Let

$$\begin{matrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ & \vdots & \\ A_{m,1} & A_{m,2} & A_{m,3} \end{matrix}$$

be a matrix of finite sets such that

(i) $|A_{i,1}| = a$, $|A_{i,2}| = b$, $|A_{i,3}| = c$ for all i, j;

(ii) $A_{i,j} \cap A_{i,j'} = \emptyset$ for all i, j, j' if $j \neq j'$;

(iii) for each $1 \le i < j \le m$, $A_{i,1} \cap A_{j,2} \neq \emptyset$, or $A_{i,1} \cap A_{j,3} \neq \emptyset$, or $A_{i,2} \cap A_{j,3} \neq \emptyset$.

Prove that $m \le \dfrac{(a+b+c)!}{a!b!c!}$.

(Hint: Consider $V = V_1 \oplus V_2$ for some vector space V_1 and V_2 with $\dim(V_1) = a+b+c$ and $\dim(V_2) = b+c$. Find some linear transformations $T_1 : U \to V_1$ and $T_2 : U \to V_2$ with certain properties. Later find linearly independent vectors in $(\bigwedge^a V_1) \wedge (\bigwedge^b V_2)$.)

3.5. Let L be a vector space of functions from $\mathbb{F}^n$ to $\mathbb{F}$ such that if $f(x) = 0$ for more than d points x on a line in $\mathbb{F}^n$, then $f(x) = 0$ for all points on the line. Prove that the dimension of L is at most $(d+1)^n$.

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2021 SPRING MAS575 *Combinatorics*
HOMEWORK 4

DUE: MAY 18, 2017

4.1. Let A and B be two nonempty subsets of $\mathbb{Z}_p$. Let

$$X = \{a + b : a \in A, b \in B, ab \neq 1\}.$$

Show that $|X| \geq \min\{|A| + |B| - 3, p\}$.

4.2. A graph is k-regular if every vertex has degree k. Let p be a prime. Let G be a graph with no loops. Prove that if the average degree of G is greater than $2p - 2$ and the maximum degree is at most $2p - 1$, then G contains a p-regular subgraph.

4.3. Let n be a positive integer. Let $L = \{(x, y, z) \in \mathbb{Z}^3 : |x|, |y|, |z| \leq n\} - \{(0, 0, 0)\}$. Suppose that we want to cover all points in L by planes in $\mathbb{R}^3$ not passing $(0, 0, 0)$. Prove that $6n$ is the minimum number of such planes.

4.4. Suppose that points in $\{0, 1\}^n - \{0\}$ are covered by m hyperplanes without covering 0. Prove that $m \geq n$. Is this tight?

4.5. In a party, n couples are invited. They decided to sit around a round table with $2n + 1$ chairs such that the i-th couple are seated from each other by distance d_i (meaning that they are separated by $d_i - 1$ chairs). Prove that if $2n + 1$ is a prime and $d_1, d_2, \ldots, d_n \leq n$, then this is possible.