Please type your homework solution into a PDF file and submit to the gradescope.com website by 11:59pm KST of the due date. A point will be deducted for each problem if a handwritten solution is submitted. We recommend using LaTeX. You can find useful hints at our KLMS website. It is recommended to use the sample template. (At least, make sure that each problem has a solution in separate pages.) Unprofessional proofs may get a deduction of points, even if the solution is mathematically correct or can be made correct.

2021 SPRING MAS575 *Combinatorics*
HOMEWORK 1

DUE: MARCH 16, 2021

1.1. Suppose that there are m red clubs $R_1, R_2, \dots, R_m$, and m blue clubs $B_1, B_2, \dots, B_m$ in a university of n students. Suppose that the following rules are satisfied:

(a) $|R_i \cap B_i|$ is odd for every i.

(b) $|R_i \cap B_j|$ is even for every $i \neq j$.

Prove that $m \leq n$.

1.2. Let us consider the following variation of the odd rule in some university with n students:

(i) Every club has an even number of members.

(ii) Every pair of clubs shares an odd number of members.

Prove that there are at most n clubs if n is odd, and at most $n-1$ clubs if n is even.

1.3. Let $\mathcal{A}$, $\mathcal{B}$ be families of subsets of $\{1, 2, \dots, n\}$ such that $|A \cap B|$ is odd for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that $|\mathcal{A}||\mathcal{B}| \leq 2^{n-1}$.

1.4. Let $\{A_1, A_2, \dots, A_m\}$ be an intersecting antichain of subsets of $\{1, 2, \dots, n\}$ such that $|A_i| \leq \frac{1}{2}n$ for each i. Prove that

$$\sum_{i=1}^{m} \frac{1}{\binom{n-1}{|A_i|-1}} \leq 1.$$

(A family $\{A_1, A_2, \dots, A_m\}$ is an *antichain* if and only if $A_i \not\subseteq A_j$ for all $i \neq j$.)

1.5. Let $\frac{1}{2} \leq p < 1$. Let $X_1, X_2, \dots, X_n$ be independent random variables such that $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be nonnegative real numbers such that $\sum_{i=1}^{n} \alpha_i = 1$. Prove that

$$P\left(\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n \geq \frac{1}{2}\right) \geq p.$$

Hint: Erdős-Ko-Rado Theorem.