Covering cubes by hyperplanes

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Joint work with Alexander Clifton (Emory).

My collaborator

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(Ph.D student at Emory)

A naive question

The n-dimensional cube Q^n consists of the binary vectors $\{0,1\}^n$.

An affine hyperplane is:

$$\{\vec{x} : a_1x_1 + \cdots + a_nx_n = b\}.$$

Question

What is the minimum number of affine hyperplanes that cover all the vertices of Q^n?

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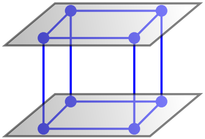
Answer: 2.



$x_i = 1$

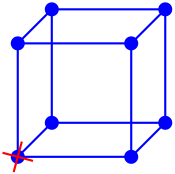
$x_i = 0$

The Alon-Füredi Theorem

A new question

Suppose we would like to avoid exactly one vertex of the cube, how many affine hyperplanes are needed?

For Q^3, 3 planes are needed.



Theorem (Alon, Füredi 1993)

Any set of affine hyperplanes that covers all the vertices of the n-cube Q^n but one contains at least n affine hyperplanes.

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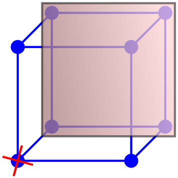

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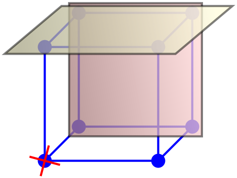

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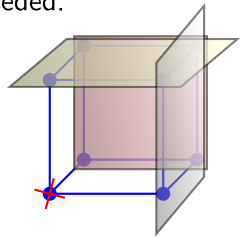

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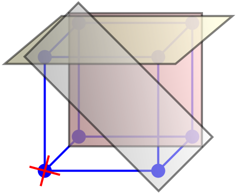

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An outline of the proof of Alon-Füredi Theorem

Proof. Suppose $H_1 \cdots\cdots H_m$ affine hyperplanes covers $Q^n - \{0\}$

$0 \notin H_i$

$H_i: \quad \langle \vec{x}, \underset{\in \mathbb{R}^n}{\vec{a}_i} \rangle = \underset{\in \mathbb{R}}{b_i} \qquad b_i \neq 0$

$$p(x_1 \cdots\cdots x_n) = \prod_{i=1}^{m} \left(\langle \vec{x}, \vec{a}_i \rangle - b_i \right)$$

① $p(\vec{x}) = 0$ for $x \in Q^n - \{0\}$

② $p(\vec{0}) \neq 0$.

$\deg p = m$

p is in n vars.

$\Longrightarrow \quad m \geq n.$

$p \neq 0$ at $\vec{0}$

$= 0 \quad Q^n \setminus \{0\}$

$p = (x_1 - 1) \cdots\cdots (x_n - 1)$

$x_1 = 1$

$\vdots$

$x_{n-1} = 1$

$x_1 + \cdots + x_n = 1$

p

Question (Bukh's homework assignment at CMU)

What happens if we would like to cover the vertices of Q^n at least twice, with one vertex uncovered?

$\boxed{2n}$

$x_1 = 1$
$\vdots$
$x_n = 1$

twice.

$\geq n+1$

Remove $\boxed{\text{one}}$ hyperplane.

$Q^n - \{0\}$ ≥ 1.

$\geq n$

$\underline{n+1}$.

$x_1 = 1$
$\vdots$
$x_n = 1$

e_i $x_1 + x_2 + \cdots + x_n = 1$

$(, \ldots, 1, \ldots, 1 \cdots)$

i-th j-th

$x_i = 1$
$x_j = 1$

$\geq n+1$

Covering the cube k times

Denote by $f(n,k)$ the minimum number of affine hyperplanes needed to cover every vertex of Q^n at least k times (except for $\vec{0}$ which is not covered at all).

We call such a cover an almost k-cover of the n-cube.

$$f(n,1) = n.$$

$$f(n,2) = n+1.$$

What is the next?

Upper and lower bounds

$$\boxed{f(n,k) \le n + \binom{k}{2}}$$

If $\vec{v}$ has t coordinates $= 1$,
then $\vec{v}$ has already been covered t times

Take

$$x_1 = 1, \cdots, x_n = 1,$$

$$x_1 + \cdots + x_n = 1 \quad \text{for } k-1 \text{ times},$$

$$\vdots$$

$x_1 + \cdots + x_n = k-2$ for twice.

$$x_1 + \cdots + x_n = k-1 \quad \text{for } 1 \text{ time}.$$

n

$k-1$

$+k-2$

$+ \vdots$

$+1$

$= \binom{k}{2} + n$

$$\boxed{f(n,k) \ge n + k - 1}$$

Note that removing $k-1$ planes from an almost k-cover still gives an almost 1-cover.

$$\boxed{k = 3: \quad n + 2 \le f(n,3) \le n + 3.}$$

The $k = 3$ case and a natural conjecture

$x_i = 1$
$\Sigma x_i = 1$ twice
$\Sigma x_i = 2$ once

Theorem (H., Clifton 2019)

For $n \geq 2$,

$$f(n,3) = n + 3.$$

$n+k-1 = n+3$

For $n \geq 3$,

$$f(n,4) \in \{n+5, n+6\}.$$

$n+\binom{k}{2} = n+6$

Conjecture (H., Clifton 2019)

For fixed integer $k \geq 1$ and sufficiently large n,

$$f(n,k) = n + \binom{k}{2}.$$

$\geq k$ times $\boxed{kn} \leq$ if n small

THE NULLSTELLENSATZ

If $\mathbb{F}$ is an algebraically closed field, and $f, g_1, \cdots, g_m \in \mathbb{F}[x_1, \cdots, x_n]$, where f vanishes over all common zeros of $g_1, \cdots, g_m$, then there exists an integer k, and polynomials $h_1, \cdots, h_m \in \mathbb{F}[x_1, \cdots, x_n]$, such that

$$f^k = \sum_{i=1}^{m} h_i g_i.$$

When $m = n$, and $g_i = \prod_{s \in S_i} (x_i - s)$, for some $S_1, \cdots, S_n \subset \mathbb{F}$, a stronger result holds: there are polynomials $h_1, \cdots, h_n$ with $\deg h_i \le \deg f - \deg g_i$, such that

$$f = \sum_{i=1}^{n} h_i g_i.$$

We say $\vec{a} = (a_1, \cdots, a_n)$ is a zero of multiplicity t of $f \in \mathbb{F}[x_1, \cdots, x_n]$, if t is the minimum degree of the terms in $f(x_1 + a_1, \cdots, x_n + a_n)$.

$f = (x-1)(y-1)$ at $(1,1)$ it has multiplicity 2

For $i = 1, \cdots, n$, let

$$\underset{\{0\}}{D_i} \subset \underset{\{0,1\}}{S_i} \subset \mathbb{F}. \quad g_i = \prod_{s \in S_i}(x_i - s). \quad \ell_i = \prod_{d \in D_i}(x_i - d).$$

$g_i = (x_i - 1)x_i$ $\qquad$ $\ell_i = x_i$

Theorem (Ball, Serra 2009)

If f has a zero of multiplicity at least t at all the common zeros of $g_1, \cdots, g_n$, except at least one point of $D_1 \times \cdots \times D_n$ where it has a zero of multiplicity less than t, then there are polynomials h_τ satisfying $\deg(h_\tau) \le \deg(f) - \sum_{i \in \tau} \deg(g_i)$, and a non-zero polynomial u satisfying $\deg(u) \le \deg(f) - \sum_{i=1}^{n}(\deg(g_i) - \deg(\ell_i))$, such that

$$f = \boxed{\sum_{\tau \in T(n,t)} g_{\tau(1)} \cdots g_{\tau(t)} h_\tau} + \boxed{u \prod_{i=1}^{n} \frac{g_i}{\ell_i}}. \rightarrow u \cdot \prod (x_i - 1)$$

$T(n,t)$ consists of all non-decreasing sequences of length t on $[n]$.

Outline of our proof using the PCN $(k = 3)$

Goal. $f(n,3) \geq n+3$

Suppose not $H_1 \cdots H_{n+2}$ $H_i: \langle \vec{x}, \vec{a_i} \rangle = 1.$

Let $p_i = \langle \vec{x}, \vec{a_i} \rangle - 1.$

$\deg = n+2$

$f = p_1 \cdots p_{n+2}$

PCN $\Rightarrow$

$(-1)^n \cdot u(\vec{0})$

$$f = \cancel{\sum_{1 \le i \le j \le k \le n} x_i(x_i-1)x_j(x_j-1)x_k(x_k-1)} + \left[\prod_{i=1}^{n}(x_i-1)\right] u$$

$\deg u \le \deg f - n$

≤ 2

$f = 0$ on $Q^n \setminus \{0\}$

$\frac{\partial f}{\partial x_i} \,/\!/\, \frac{\partial^2 f}{\partial x_i \partial x_j}$

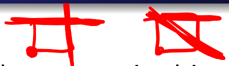
$\Downarrow$

$u \equiv 0$

$f(\vec{0}) = (-1)^n u(\vec{0}) = 0$

Follow-up work

$x_1=1 \cdots x_{n-1}=1$ $x_1+\cdots+x_n=1$



The essence of this proof can be summarized in one sentence:

If f has zeroes of multiplicity at least 3 at $\{0,1\}^n \setminus \{0\}$ and $f(0) \neq 0$, then $\deg(f) \geq n+3$.

$f = p_1 \cdots p_m$

Theorem (Sauermann, Wigderson 2020)

For $k \geq 2$, the minimum possible degree of a polynomial $f(x_1, \cdots, x_n)$ sucht that it has zeroes of multiplicity at least k at $\{0,1\}^n \setminus \{0\}$ and $f(0) \neq 0$, is $n+2k-3$.

$n+\binom{k}{2} \geq f(n,k) \geq n+2k-3$

Corollary (Sauermann, Wigderson 2020)

For $k \geq 2$, an almost k-cover of Q^n has at least $n+2k-3$ hyperplanes.

$f(n,k)$ for fixed n and large k

For small n, $f(n,k) \neq n + \binom{k}{2}$. Actually,

Theorem (H., Clifton 2019)

For fixed n, and k tends to infinity,

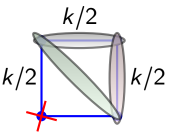
$$f(n,k) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + o(1)\right) k.$$

- Upper bound: use every hyperplane

$$x_{i_1} + \cdots + x_{i_j} = 1$$

a total of $\frac{k}{j\binom{n}{j}}$ times. e.g.

$\frac{3k}{2}$



$x_1 = 1$ $k/2$

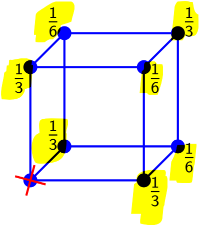
$x_2 = 1$ $k/2$

$x_1 + x_2$ $k/2$

$\frac{3}{2} = 1 + \frac{1}{2}$

$f(n,k)$ for fixed n and large k (ctd.)

- Lower bound: (e.g. $n=3$) assign weights to vertices:



Every affine plane covers vertices of total weight at most 1. Therefore one needs at least

$$k \cdot \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{3} \right) = \frac{11}{6} k$$

hyperplanes.

For general n, assign weight $1/(j\binom{n}{j})$ to vertices whose sum of coordinate is j.

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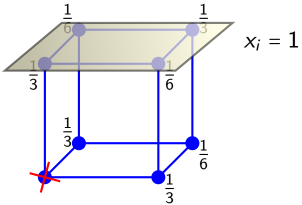

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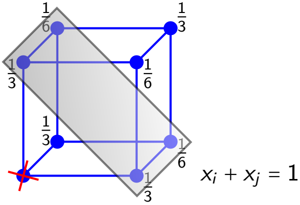

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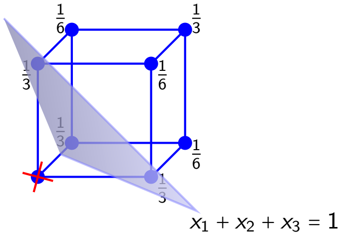

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$= 1 + \frac{1}{2} + \frac{1}{3}$

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An LYM-like inenquality

The Lubell-Yamato-Meshalkin inequality

Let $\mathcal{F}$ be a family of subsets in which no set contains another, then

$$\sum_{S\in\mathcal{F}} \frac{1}{\binom{n}{|S|}} \le 1.$$

Lemma (H., Clifton 2019)

Given n real numbers $a_1, \cdots, a_n$, let

If $a_i > 0$

$$\mathcal{F} = \{S : \emptyset \neq S \subset [n], \boxed{\sum_{i\in S} a_i = 1}\},$$

$\frac{1}{j\binom{n}{j}}$

$\boxed{a_1x_1 + \cdots + a_nx_n = 1}$

then

$$\boxed{\sum_{S\in\mathcal{F}} \frac{1}{|S|\binom{n}{|S|}} \overset{=}{\le} 1.}$$

The inequality is tight for all non-zero binary $(a_1, \cdots, a_n)$.

Proof of the Lemma

We associate every $S \in \mathcal{F}$ (binary vector covered by the plane) with some permutations in $\mathcal{P}_S \subset S_n$.

e.g. When $n = 5$, $S = \{1, 3, 4\}$, it means $a_1 + a_3 + a_4 = 1$, take all permutations in S_5 with prefix (i_1, i_2, i_3) satisfying

$$\{i_1, i_2, i_3\} = \{1, 3, 4\}, \quad a_{i_1} < 1, \quad a_{i_1} + a_{i_2} < 1.$$

We can show:

- $\mathcal{P}_S$ are pairwise disjoint.
- $|\mathcal{P}_S| \geq (|S| - 1)!(n - |S|)!$ (the proof uses the *lorry driver puzzle*.)
- Therefore

$$n! \geq \sum_{S \in \mathcal{F}} |\mathcal{P}_S| = \sum_{S \in \mathcal{F}} (|S| - 1)!(n - |S|)!,$$

which simplifies to our desired result.

Future research problems (I)

$n+\binom{k}{2}$ $\boxed{n+2k-3}$

Problem 1

Prove $f(n,k) = n + \binom{k}{2}$ for large n.

Alon (private communication): for large n, if the almost k-cover contains $\cancel{x_1 = 1, \ldots, x_n = 1}$, then it contains at least $n + \binom{k}{2}$ affine hyperplanes in total.

n

$N_0 = [n]$ → color by hyperplane

$(k-1)$-subset ≥ 1 $\cdots \subseteq N_2 \subseteq N_1 \subseteq N_0.$

Problem 2

Let $g(n,m,k)$ be the minimum number of vertices covered less than k times by m affine hyperplanes not passing through $\vec{0}$. Determine $g(n,m,k)$.

Alon, Füredi 1993: $g(n,m,1) = 2^{n-m}$.

Future research problems (II)

Question: Is it true that for all n, m, k:

$$g(n, m, k) = \boxed{2^{n-d}},$$

where d is the maximum integer such that $f(d, k) \leq m$?

Problem 3

Does there exist an absolute constant $C > 0$, which does not depend on n, such that for a fixed integer n, there exists M_n, so that whenever $k \geq M_n$,

$$f(n, k) \leq \underline{\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k + C}?$$

$n = 2, 3, 4$

Thank you!