KAIST
2021 MAS575 Combinatorics
Homework 5

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May 24, 2021

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HW 5.1

Let G be a simple graph on the vertex set $\{v_1, \ldots, v_n\}$. let $A = (a_{ij})$ be an $n \times n$ real symmetric matrix such that for all $i, j \in [n]$, $a_{ij} \neq 0$ iff v_i is adjacent to v_j. Let $\alpha(G)$ be the maximum size of an *independent* set in G. Among n eigenvalues of A, let N^+ be the number of positive eigenvalues of A and N^- be the number of negative eigenvalues of A. Prove that $\alpha(G) \leq \min(n - N^+, n - N^-)$.

Proof. Recall Cauchy's interlacing theorem.

Theorem 1.1 (Corollary in the lecture). *Let A be an $n \times n$ real symmetric matrix and let B be a $k \times k$ principal submatrix of A. Let $r_1 \geq \cdots \geq r_n$ be the eigenvalues of A and let $s_1 \geq \cdots \geq s_k$ be the eigenvalues of B. Then $r_{i+n-k} \leq s_i \leq r_i$ for each $i \in [k]$.*

Set $k = \alpha(G)$ and fix an independent set $K \subset \{v_1, \ldots, v_n\}$ with $|K| = k$. We now let B be the principal submatrix of A induced by the subgraph of G on K. Let $r_1 \geq \cdots \geq r_n$ be the eigenvalues of A and let $s_1 \geq \cdots \geq s_k$ be the eigenvalues of B. By the above theorem, we have $r_{i+n-k} \leq s_i \leq r_i$ for each $i \in [k]$. Note that B is a zero-matrix and thus $s_i = 0$ for each $i \in [k]$, which implies that $r_{i+n-k} \leq 0$ and $r_i \geq 0$ for each $i \in [k]$. This immediately gives that both N^+ and N^- are at most $n - k$ and thus provides the desired bound for $k = \alpha(G)$, completing the proof. $\square$

HW 5.2

Assume that n is not too small. Prove that if A is a subset of $[n]$ with $|A| > 2n/3$, then A has an arithmetic progression of length 3.

Remark 2.1. *When $n = 4$ or $n = 5$ the result does not hold. When $n = 4$, we can choose $A = \{1, 2, 4\}$ with $|A| = 3 > 8/3 = 2n/3$ but A does not contain any arithmetic progression of length 3. When $n = 5$, we can choose $A = \{1, 2, 4, 5\}$ with $|A| = 4 > 10/3 = 2n/3$ but A does not contain any arithmetic progression of length 3.*

Proof. First consider the case when 3 divides n. We partition $[n]$ into $n/3$ consecutive triples

$$\{1, 2, 3\}, \{4, 5, 6\}, \ldots, \{n - 2, n - 1, n\}.$$

The key fact is that if $A \subset [n]$ contains no arithmetic progression, then A can have at most 2 elements of each triple, which implies that $|A| \leq 2n/3$. This is a contradiction with the given condition $|A| > 2n/3$, completing the proof.

Now we assume 3 does not divide n and $n \geq 7$. First consider the case when $n \equiv 1 \pmod 3$, we partition $[n]$ into $(n + 2)/3$ groups

$$\{1, 2, 3\}, \{4, 5, 6\}, \ldots, \{n - 3, n - 2, n - 1\}, \{n\}.$$

Suppose $A \subset [n]$ contains no arithmetic progression. Together with the condition that $|A| > 2n/3$, A contains exactly two elements in each triple and n must be in A. Moreover, to avoid 3-AP, $n - 2$ and $n - 1$ cannot be in A together, so $n - 3 \in A$, then $n - 6 \notin A$, which implies that both $n - 5$ and $n - 4$ are in A. However, this forms a 3-AP $n - 5, n - 4, n - 3$, contradicting with the assumption that A contains no arithmetic progression.

Now consider the case when $n \equiv 2 \pmod 3$. In this case, $|A| > 2n/3$ means that $|A| \geq (2n + 2)/3$. Suppose $A \subset [n]$ contains no arithmetic progression. We partition $[n]$ into $(n + 1)/3$ groups

$$\{1, 2, 3\}, \{4, 5, 6\}, \ldots, \{n - 4, n - 3, n - 2\}, \{n - 1, n\}.$$

Then to avoid arithmetic progressions, A must contain exactly two elements in each triple and both $n - 1$ and n must be in A. Then the above argument for the case when $n \equiv 1 \pmod 3$ can be applied and this completes the proof. □

HW 5.3

Prove that for all n, there exists N satisfying the following: for every bipartite graph G with the bipartition (A, B) such that no two vertices in A have the same set of neighbors and $|A| \geq N$, there exist distinct vertices $a_1, a_2, \ldots, a_n \in A$ and $b_1, b_2, \ldots, b_n \in B$ such that one of the following hold:

1. For all $1 \leq i, j \leq n, a_i$ is adjacent to b_j iff $i = j$.
2. For all $1 \leq i, j \leq n, a_i$ is adjacent to b_j iff $i \leq j$.
3. For all $1 \leq i, j \leq n, a_i$ is adjacent to b_j iff $i \neq j$.

Remark 3.1. *We have $|A|$ different subsets of $[|B|]$. Equivalently, we have a $\{0,1\}$-matrix with $|A|$ columns and $|B|$ rows such that no two columns are identical.*

Remark 3.2. *Equivalently, we need to show that with the given conditions, when $|A|$ is sufficiently large, we cannot avoid all the three induced subgraphs.*

Proof. Consider a $\{0,1\}$-matrix M with $|A|$ columns and $|B|$ rows, representing the adjacency matrix of G. The given conditions is equivalent to that no two columns of M are identical. We need to show that when $|A|$ is sufficiently large, we cannot avoid all the three submatrices I_n, $J_n - I_n$, and U_n when the permutations of rows and columns are allowed, where I_n is the identity $n \times n$ matrix, J_n is the all-one $n \times n$ matrix, and $U_n = (u_{i,j})$ is the $n \times n$ matrix with $u_{i,j} = 1$ iff $1 \leq j \leq i \leq n$. We follow the paper *Unavoidable Minors of Large 3-Connected Binary Matroids.*

Definition 3.1. *Let $n, p \geq 0$ with $n + p > 0$. A $\{0,1\}$-matrix $M = (m_{i,j})$ is $[n,p]$-**semidiagonal** if M has exactly $n + p$ columns and at least n rows, and, for every row $i \in [n] \setminus \{n + p\}$, we have $m_{i,i} \neq m_{i,i+1}$ and $m_{i,i+1} = m_{i,j}$ for all $j \in [n + p] \setminus [i]$.*

Lemma 3.1. *For all $n \geq 2$, let $g_1(n) = 3^{n+1}$. Let C be a $\{0,1\}$-matrix with at least $g_1(n)$ columns with no two columns identical. Then there is an $[n,0]$-semidiagonal matrix D obtained from C by deleting columns and permuting rows.*

Proof of Lemma 3.1. We may assume that C has exactly $g_1(n)$ columns. We will inductively construct a sequence of matrices $C = C_0, C_1, \ldots, C_n = D$ where for each $m \in \{0, 1, \ldots, n\}$, C_m is $[m, g_1(n-m)]$-semidiagonal and has been obtained from C by deleting columns and permuting rows. It it easy to check that $C_0 = C$ is $[0, g_1(n)]$-semidiagonal. Now suppose $m \in [n]$ and $C_{m-1} = (c_{i,j})$ is $[m-1, g_1(n-m+1)]$-semidiagonal and has been obtained from C by deleting columns and permuting rows. Note that C_{m-1} has $m - 1, g_1(n - m + 1) \geq m + 2$ columns. By the definition of semidiagonality and the property of C, the m-th and $(m+1)$-th columns of C_{m-1} are not identical but agree in the first $m-1$ rows. Therefore, there exists $i \geq m$ such that $c_{i,m} \neq c_{i,m+1}$. Let $J = [m-1+g_1(n-m+1)] \setminus [m+1]$ and consider the entries $c_{i,t}$ for $t \in J$, where $|J| = m - 1 + g_1(n-m+1) - (m+1) > 2g_1(n-m)$. By pigeonhole, there exist $J' \subset J$ with $|J'| = g_1(n-m)$ and $\alpha \in \{0,1\}$ such that $c_{i,t} = \alpha$ for all $t \in J'$. Moreover, since $c_{i,m} \neq c_{i,m+1}$, there is an $m' \in \{m, m+1\}$ such that $c_{i,m'} \neq \alpha$. We construct C_m from C_{m-1} by only keeping columns with index in $[m-1] \cup \{m'\} \cup J'$ and swapping rows i and m. It is easy to check that the constructed C_m is $[m, g_1(n-m)]$-semidiagonal and has been obtained from C by deleting columns and permuting rows. By induction, we complete the proof. $\square$

Definition 3.2. *We say that a $k \times k$ $\{0,1\}$-matrix $M = (m_{i,j})$ is **good** if $M \in \{I_k, J_k - I_k, U_k, J_k - U_k\}$.*

Lemma 3.2. *For all $n \geq 2$, let $g_2(n) = 2R(n,n)$, where $R(\cdot,\cdot)$ is the Ramsey number. Suppose $D = (d_{i,j})$ is a $[g_2(n), p]$-semidiagonal matrix. Then D has a good principal submatrix E that has n columns.*

Proof. Let $m = g_2(n) = 2R(n,n)$ and consider $d_{1,1}, d_{2,2}, \ldots, d_{m,m}$. By pigeonhole, there exist $J \subset [m]$ with $|J| = R(n,n)$ such that all $d_{i,i}$ are identical for $i \in J$. We construct a principal submatrix $F = (f_{i,j})$ of D by only keeping the columns and rows with index in J. Note that all elements on the main diagonal of F are identical. Also note that by the definition of semidiagonality, all $f_{i,j}$ with $j > i$ are identical and not equal to the elements on the main diagonal. Consider a 2-edge-coloring of $K = K_{R(n,n)}$ with vertex set $[R(n,n)]$ where for all $i > j$, the edge (i,j) is colored by $f_{i,j}$. By the definition of Ramsey numbers, there exists $J' \subset [R(n,n)]$ with $|J'| = n$ such that the induced subgraph of K on J' is monochromatic. Construct E from F by only keeping the rows and columns with index in J'. It is easy to check that E is a principal submatrix of D and E is good, which completes the proof. $\square$

Combining the above to lemmas, we let $N = g_1(g_2(n+1))$ and require $|A| \geq N$, and then we conclude that there exists a $(n+1)\times(n+1)$ submatrix of M that is in $\{I_{n+1}, J_{n+1} - I_{n+1}, U_{n+1}, J_{n+1} - U_{n+1}\}$ when the permutations of rows and columns are allowed. By choosing n rows and n columns properly with some permutations if needed, we can always have a $n \times n$ submatrix that is I_n or $J_n - I_n$ or U_n, completing the proof. $\square$

HW 5.4

Let t, r be positive integers. Prove that there exists a number N such that any r-coloring of numbers in $[N]$ contains an arithmetic progression $a, a+d, \ldots, a+(t-1)d \in [N]$ of length t $(d \neq 0)$ such that $a, a+d, \ldots, a+(t-1)d$ and d have the same color.

Remark 4.1. *Maybe we can leave the first D numbers $[D]$ representing the possible distances uncolored and apply Van der Waerden to show that no matter how we color the first D numbers, we will always have a monochromatic l-AP with step size $d \leq D$ in a specific color. However, I cannot figure out a proper argument currently.*

Proof. Recall Van der Waerden's Theorem.

Theorem 4.1 (Van der Waerden's Theorem)**.** *For any positive integers k and l, there exists a positive integer $N_0 = N_0(k, l)$ such that any k-coloring of $[N]$ with $N \geq N_0$ creates a monochromatic l-AP.*

□

HW 5.5

Prove that for all k and q, there exists N such that every sequence $a_1, a_2, \ldots, a_N$ of positive integers with $a_1 < a_2 < \cdots < a_N$ and $a_{i+1} - a_i \le q$ for all $i \in [N-1]$ has a subsequence that is an arithmetic progression of length k.

Remark 5.1. *This result directly follows from Szemerédi's Theorem because for every sequence $a_1, a_2, \ldots, a_N$ of positive integers with $a_1 < a_2 < \cdots < a_N$ and $a_{i+1} - a_i \le q$ for all $i \in [N-1]$, we may assume that $a_1 = 1$ and then it has a upper density at least $1/(q+1)$ in $[N(q+1)]$.*

Proof. Recall Van der Waerden's Theorem again.

Theorem 5.1 (Van der Waerden's Theorem)**.** *For any positive integers k and l, there exists a positive integer $N_0 = N_0(k, l)$ such that any k-coloring of $[N]$ with $N \ge N_0$ creates a monochromatic l-AP.*

We may assume that $a_1 = 1$. Fix a sequence $A = (1 = a_1, a_2, \ldots, a_M)$ of positive integers with $a_1 < a_2 < \cdots < a_M$ and $a_{i+1} - a_i \le q$ for all $i \in [M-1]$, where M is a positive integer to be determined later. Clearly, $a_M \le (M-1)q + 1$. We apply a q-coloring of $[Mq]$ as follows:

1. We color $V_0 = A$ with color 0;
2. We color $V_1 = \{a + 1 : a \in A\} \setminus V_0$ with color 1;
3. We color $V_2 = \{a + 2 : a \in A\} \setminus (V_0 \cup V_1)$ with color 2;
4. $\ldots$
5. We color $V_{q-1} = \{a + q - 1 : a \in A\} \setminus \bigcup_{i=0}^{q-2} V_i$ with color $q - 1$.

By the above Van der Waerden's Theorem, there exists a positive integer $N_0 = N_0(q, k)$ such that any q-coloring of $[N]$ with $N \ge N_0$ creates a monochromatic k-AP. Therefore, we can choose M such that $Mq \ge N_0$ and then there exists $j \in \{0, 1, \ldots, q-1\}$ such that V_j contains a k-AP, say $(b, b+d, b+2d, \ldots, b+(k-1)d)$ with $d \ne 0$, which gives a k-AP $(b-j, b-j+d, b-j+2d, \ldots, b-j+(k-1)d)$ in A, completing the proof.

□