Ch8 Cauchy's interlacing theorem

§8.1 Cauchy's interlacing theorem

Def/ A sequence $s_1 \geq \cdots \geq s_{n-1}$ of length $n-1$ interlaces a seq. $r_1 \geq \cdots \geq r_n$ if $r_1 \geq s_1 \geq r_2 \geq \cdots \geq s_{n-1} \geq r_n$. For polynomial f w/ $\deg(f) = n$ and all roots real, and polynomial g w/ $\deg(g) = n-1$ and all roots real, we say g interlaces f if the roots of g interlaces the roots of f.

Obs If f is a polynomial w/ all roots real, then f' interlaces f.

Lem f, g: polynomials w/ all roots real, $\deg(f) = n$, $\deg(g) = n-1$ $\overset{\text{①}}{\Longrightarrow}$ (the roots of g interlaces the roots of f) $\Longleftrightarrow$

all roots of $f+\alpha g$ are real, $\forall \alpha \in \mathbb{R}$

Pf/ $\Rightarrow$) Let $r_1 \geqslant \cdots \geqslant r_n$ be the roots of f

and $s_1 \geqslant \cdots \geqslant s_{n-1}$ be the roots of g.

$\boxed{\text{Claim}}$ we may assume that $\nexists\, r_i = s_j$

Otherwise, $f = c_f (x-r_1)(x-r_2)\cdots(x-r_n)$

$g = c_g (x-s_1)\cdots(x-s_{n-1})$. Let

$f_\varepsilon := c_f \big((x-r_1)(x-r_2+2\varepsilon)\cdots(x-r_n+(2n-2)\varepsilon\big)$

w/ roots $r_1, r_2-2\varepsilon, r_3-4\varepsilon, \cdots, r_n-(2n-2)\varepsilon$

$g_\varepsilon := c_g \big((x-s_1+\varepsilon)(x-s_2+3\varepsilon)\cdots(x-s_{n-1}+(2n-3)\varepsilon\big)$

w/ roots $s_1-\varepsilon, s_2-3\varepsilon, \cdots, s_{n-1}-(2n-3)\varepsilon$

For small ε, the roots of f_ε and g_ε are

all distinct. Besides, $f_\varepsilon \to f$ and $g_\varepsilon \to g$ as $\varepsilon \to 0^+$.

Then if x_ε is a root of $f_\varepsilon + \alpha g_\varepsilon = 0$,

then $x_\varepsilon \to x_0$, where x_0 is a root of $f+\alpha g=0$.

So x_ε's are all real $\Rightarrow$ x_0's are all real.

Now, we may assume $r_n < s_{n-1} < r_{n-1} < \cdots < r_2 < s_1 < r_1$,

and $\alpha \neq 0$. If $f(x) \to \infty$ and $\alpha g(x) \to \infty$ as $x \to \infty$

then 1° $f+\alpha g$ has no root in (r_1, ∞), (r_2, s_1),

(r_3, s_2), $\cdots$, (r_n, s_{n-1}); 2° $f+\alpha g$ has ≥ 1

root in each of (s_1, r_1), (s_2, r_2), $\cdots$, (s_{n-1}, r_{n-1}),

$(-\infty, r_n)$ $\Rightarrow$ n real roots. Similarly,

we can find n real roots of $f+\alpha g$ for

$f \to \pm\infty$, $\alpha g \to \mp\infty$ as $x \to \infty$.

$\Leftarrow$) WMA f and g have no common root,

otherwise write $f = q \cdot f_1$, $g = q \cdot f_2$, where

f_1 and f_2 are relatively prime, then (the roots of g_1 interlaces the roots of f_1)
$\Rightarrow$ ($\sim g$ interlaces $\sim f$). Now,
if $[r_{k+1}, r_k]$ contains no root of g, then
$|g(x)| > \varepsilon, \forall x \in [r_{k+1}, r_k]$ for some $\varepsilon > 0$.
Consider $g(x)/f(x)$, we pick $\beta \neq 0$ s.t. $g(x)/f(x) = \beta$
has a root w/ multiplicity > 1, then set $\alpha = -1/\beta$,
and $f(x) + \alpha g(x) = 0$ has a multiple root
in $[r_{k+1}, r_k]$. Then we may change α by a
suff. small value s.t. $f(x) + \tilde{\alpha} g(x) = 0$ "loses"
this multiple root and thus has complex
roots, contradiction! Therefore, $\forall k \in [n-1]$, $[r_{k+1}, r_k]$
contains ≥ 1 roots s_k of g $\Rightarrow$ exactly one, and

$r_n < s_{n-1} < r_{n-1} < \cdots < r_2 < s_1 < r_1$. $\square$

$\boxed{\text{Thm}}$ A: real sym $n \times n$, B: Principal subm. of A, obtained by deleting the k-th row and the k-th col. $\Rightarrow$ the eigenvalues of B interlace those of A.

Pf/ WMA $k=n$ (by permutation if necessary)

Now, $A = \left(\begin{array}{c|c} B & c \\ \hline c^T & d \end{array}\right)$, and

$$\det(xI - A) = \det\left(\begin{array}{c|c} xI-B & -c \\ \hline -c^T & x-d \end{array}\right)$$

$\boxed{\text{Obs}}$ $\det\left(\begin{array}{c|c} xI-B & -c \\ \hline -c^T & x-d+\alpha \end{array}\right) = \det(xI-A) + \alpha \det(xI-B)$ has real roots, $\forall \alpha \in \mathbb{R}$

$\overset{\text{Lem.}}{\Longrightarrow}$ the roots of $\det(xI-B)$ interlace those of $\det(xI-A)$. $\square$

$\boxed{\text{Cor}}$ A: real, Sym, $n \times n$; B: principal subm, $k \times k$.

r_i: eigenvalues of A, $\downarrow$; s_i: eigenvalues of B, $\downarrow$

$\Rightarrow$ $r_{i+(n-k)} \le s_i \le r_i$

§8.2 The sensitivity conjecture

Def/ A <u>boolean function</u> is a func. $f: \{0,1\}^n \to \{0,1\}$

For $\forall S \subseteq [n]$, $\forall x \in \{0,1\}^n$, let x^S be the 0-1 vector obtained by flipping i-th coordinate of x for all $i \in S$. For a boolean function $f: \{0,1\}^n \to \{0,1\}$, the <u>local sensitivity</u> $s(f,x)$ is $|\{i \in [n] : f(x) \neq f(x^{\{i\}})\}|$.

And the <u>sensitivity</u> of f, $s(f) := \max_x s(f,x)$.

The <u>local block sensitivity</u> $bs(f,x)$ is the max. # disjoint blocks $B_1, \ldots, B_k$ of $[n]$ s.t. $f(x) \neq f(x^{B_i})$, $\forall i \in [k]$. $(bs(f,x) \geq s(f,x))$

The <u>block sensitivity</u> $bs(f) := \max_x bs(f,x)$

$\boxed{\text{Conj}}$ (Nisan, Szegedy 94') [Sensitivity Conj.]

$bs(f) \leq (s(f))^c$ for some const. c, $\forall$ boolean f

$\boxed{\text{Best Known}}$ (Kenyon, Kutin 04')

$bs(f) \le \frac{e}{\sqrt{2\pi}} e^{s(f)} \sqrt{s(f)}$, while LB is

$bs(f) = \Omega((s(f))^2)$. HUGE gap!

Consider the hypercube Q^n whose vertex set is $\{0,1\}^n$ and 2 vtxs x, y are adj. if they differ in 1 coordinate, i.e., $y = x^{\{i\}}$ for some i.

Def/ The degree of a boolean function $\deg(f)$ is the degree of the unique multilinear real polynomial representing f. (as $0^2=0, 1^2=1$)

$\Delta(H) :=$ max. degree of H

$\boxed{\text{Thm}}$ (Nisan Szegedy 94')

$bs(f) \le 2\deg^2(f)$

$\boxed{\text{Thm}}$ (Gotsman, Linial 92)

The following are equivalent for $\forall$ increasing function h:

(1) $\forall$ induced $H \subseteq Q^n$ w/ $|V(H)| \neq 2^{n-1}$,

$\max(\Delta(H), \Delta(Q^n - H)) \geq h(n)$

(2) $\forall$ boolean f, $s(f) \geq h(\deg(f))$

$\boxed{\text{RMK}}$ Find a function h satisfying (1),

then we have $bs(f) \leq 2\deg^2(f) \leq 2(h^{-1}(s(f)))^2$

$\boxed{\text{Thm}}$ (Huang 2019)

H: induced subg. of Q^n w/ $|V(H)| = 2^{n-1}+1$

$\overset{?}{\Rightarrow}$ $\Delta(H) \geq \sqrt{n}$

$\boxed{\text{Lem}}$ $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_n = \begin{pmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{pmatrix} \in \mathbb{R}^{2^n \times 2^n}$ sym.,

whose eigenvalues are $\pm\sqrt{n}$, both of multiplicity 2^{n-1}

Pf/ (of Lem)

$\boxed{\text{Claim}}$ $A_n^2 = nI$

$A_1^2 = I$ (✓) _ by induction,

$$A_n^2 = \begin{pmatrix} A_{n-1} & I \\ I & A_{n-1} \end{pmatrix}^2 = \begin{pmatrix} A_{n-1}^2 + I & 0 \\ 0 & A_{n-1}^2 + I \end{pmatrix} = nI$$

Then, if λ is an eigenvalue of A_n,

$\exists v \neq 0$ s.t. $A_n v = \lambda v$ $\Rightarrow$

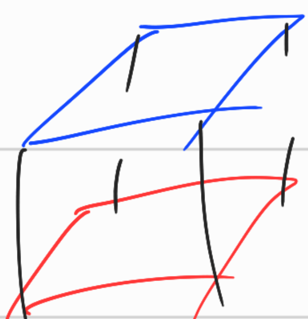
$$nv = A_n^2 v = \lambda^2 v \Rightarrow \lambda^2 = n \Rightarrow \lambda = \pm\sqrt{n}$$

$$\Rightarrow \operatorname{tr}(A_n) = 2\operatorname{tr}(A_{n-1}) = 2^{n-1}\operatorname{tr}(A_1) = 0$$

$\Rightarrow$ half of them are $\sqrt{n}$, half are $-\sqrt{n}$

What is A_n?

(2nd floor)



(1st floor)

$$\text{adj. Matrix} = \begin{pmatrix} A(\text{1st}) & I \\ I & \underbrace{A(\text{2nd})}_{\text{except we have '}-\text{'}} \end{pmatrix}$$

$\boxed{\text{Lem}'}$ H : n-vtx graph; $A = (a_{ij})$: sym, $n \times n$, where $a_{ij} \in \{\pm 1, 0\}$ and $a_{ij} \neq 0 \Leftrightarrow v_i \overset{\text{adj.}}{\sim} v_j$

$\lambda_1 :=$ the largest eigenvalue of A

$\Rightarrow \lambda_1 \leq \Delta(H)$

Pf/ Let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ be an eigenvector of λ_1, $Av = \lambda_1 v$. WLOG, Assume $|v_1| \geq |v_i|, \forall i$

$$\Rightarrow |\lambda_1| \cdot |v_1| = |\lambda_1 v_1| = |(\lambda_1 v)_1| = |(Av)_1| = \left|\sum_j a_{ij} v_j\right|$$

$$\leq \sum_{j \in N(v_1)} |a_{ij}| |v_j| \leq \deg(v_1) |v_1|$$

$$\Rightarrow |\lambda_1| \leq \deg(v_1) \leq \Delta(H)$$

Pf/ (of the Sensitivity Conj.)

Let H be an induced subgraph of Q^n w/ $|V(H)| = 2^{n-1} + 1$. $\Rightarrow$ By Cor. on page 6

$$\lambda_{1+2^n-(2^{n-1}+1)}(A_n) = \lambda_{2^{n-1}}(A_n) \leq \lambda_1(A_n[V(H)]) \leq \lambda_1(A_n)$$

$$\Rightarrow \lambda_1\left(A_n[V(H)]\right) = \sqrt{n} \Rightarrow \Delta(H) \geq \sqrt{n} \qquad \boxtimes$$

$$\Rightarrow bs(f) \leq 2\left(s(f)\right)^4$$

Ch9 Ramsey Theory

§9.1 Ramsey's Thm for graphs

$R(k,l) :=$ min N s.t. $\forall$ 2-edge-coloring of K_n w/ $n \geq N$ has a Red K_k or Blue K_l.

Generalisation

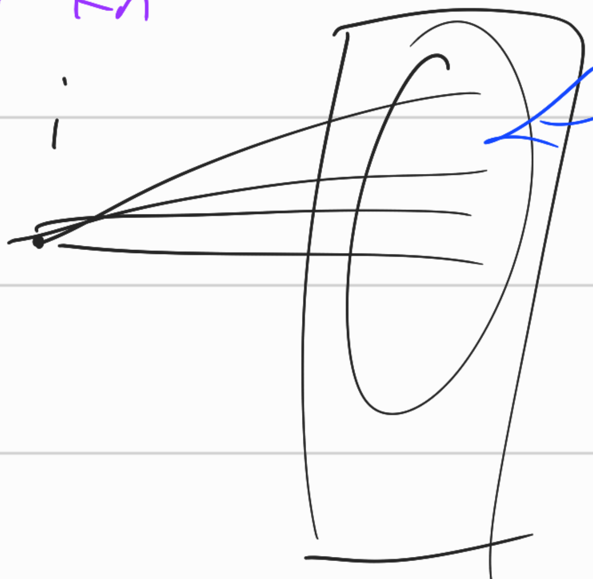
$\Longrightarrow R(n_1, \ldots, n_k) :=$ min N s.t. $\forall$ k-edge-coloring has a monochromatic K_{n_i} in color i.

$\boxed{\text{Thm}}$ (Existance)

$R(n_1, n_2, \cdots, n_k) \leq$

Consider K_n

color i



n-1 vexs

if this has size $\geq R(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k)$, then we must have K_{n_j} in color j for some j

$\Rightarrow$ We need n to be bounded. $\boxed{\text{i}}$

(in the sense of induction)

$\boxed{\text{Thm}}$ (Ramsey 30')

$$R(k,l) \le \binom{k+l-2}{k-1}$$

Pf/ $R(1,1)=1$

$$R(k,l) \le R(k-1,l) + R(k,l-1)$$

$$\le \binom{k+l-3}{k-2} + \binom{k+l-3}{k-1}$$

$$= \binom{k+l-2}{k-1}$$

$$\Rightarrow R(k,l) \le 2^{k+l-2}$$

§9.2 Ramsey Thm for sets

Instead of coloring edges of K_n, i.e., 2-subsets of $[n]$, we consider m-subsets of $[n]$.

Def/ $R(n_1, \cdots, n_k; m) :=$ min N s.t. $\forall$ k-coloring on all m-subsets of $[n]$ w/ $n \geq N$ has $[X \subseteq [n]$ w/ $|X| = n_i$ s.t. all m-subsets of X are in color $i]$ for some $i \in [k]$.

$\boxed{\text{Thm}}$ (Existance)

Pf/ We may assume $k=2$, because

$R(n_1, n_2, \cdots, n_k; m) \leq R(R(n_1, \cdots, n_{k-1}; m), n_k; m)$

NTS $R(n_1, n_2; m)$ exists. Using the similar idea.

$$\Rightarrow R(n_1, n_2; m) \leq R(R(n_1-1, n_2; m), R(n_1, n_2; m); m-1) + 1$$