9.3. Infinite Ramsey & Compactness Principle.

$\mathbb{N}=\{1,2,3,\dots\}$

Thm (Infinite Ramsey) $c \geq 1$

In any c-coloring of all 2-subsets of $\mathbb{N}$, there exists an infinite subset X whose all 2-subsets have the same color.

In other wors, $\omega \to \aleph_0$

any c-edge-coloring of K_ω has a monochromatic copy of K_ω.

Proof. Let $S_1=\mathbb{N}$. $S_i \subseteq \mathbb{N}$, S_i infinite.

Suppose S_i, $x_1, x_2, \dots, x_{i-1}$ are given.

Choose $x_i \in S_i - \{x_1, \dots, x_{i-1}\}$

Choose a color c_i such that there are infinitely many 2-subsets of color c_i containing x_i and another element of S_i

$x_1\ x_2\ \dots\ x_i$

$$S_{i+1}=\{x_1,\dots,x_i\} \cup \{y : y \in S_i,\ \{x_i, y\} \text{ has color } c_i\}$$

Repeat this
$\Rightarrow$ We have an infinite sequence
$\{x_1, x_2, \ldots\ldots \}$
Among all $\{c_i\}$, there is one choice k that appears infinitely many times.

$$c_{i_1} = c_{i_2} = c_{i_3} = \cdots = k$$
$$i_1 < i_2 < i_3 \cdots$$

Then take
$$X = \{x_{i_1}, x_{i_2}, x_{i_3}, \ldots\}$$
$\{x_{i_a}, x_{i_b}\} \Rightarrow$ color k.
$a<b$ $\qquad c_{i_a} = k$

$\square$

H: <u>hypergraph</u> if
$H = (V, E)$
$V :=$ set
$E :=$ set of subsets of V.

$\chi(H) =$ min c such that
there exists a function
$f: V(H) \longrightarrow \{1, \dots, c\}$
with the property that
no edge of H is monochromatic.

chromatic number

$R(k,k) \Longleftrightarrow$ Consider a hypergraph H

$V(H) =$ edges of K_n.

$E(H) =$ edges of a copy of K_k in K_n

$R(k,k) > n \Longleftarrow \chi(H) > 2$

$R(k,k) \leq n \Longrightarrow \chi(H) \leq 2$

Thm (compactness principle)

Let $H=(V,E)$ be a hypergraph such that all edges are finite.
Let r be a constant.
If $\chi(H[W]) \leq r$ for any finite set W
then $\chi(H) \leq r$.

(If $\chi(H) > r$, then there exists a finite subgraph H' such that $\chi(H') > r$.)

Proof 1 (Assume that V is countably infinite)

Suppose $V = \mathbb{N}$.
For all positive integers n
there is a coloring
$$f_n : \{1, \ldots, n\} \longrightarrow \{1, \ldots, r\}$$
such that no edge of $H[\{1, \ldots, n\}]$ is monochromatic.
We define $f : \mathbb{N} \longrightarrow \{1, \ldots, r\}$ as follows.
Suppose $f(1), f(2), \ldots, f(j-1)$ are already defined

so that

$S_{j-1} = \{ n : n \geq j-1, \quad f(i) = f_n(i)$ for all $1 \leq i \leq j-1 \}$

is infinite.

Choose $f(j)$ such that

$\left(\# n \text{ with } f_n(i) = f(i) \text{ for all } 1 \leq i \leq j \right)$

is infinite.

$\boxed{f(1)}$ $\boxed{f(2)}$ $\boxed{f(3)}$ - - -

$\Rightarrow$ For any i, there are infinitely many n such that $n \geq i$

$$\begin{cases} f_n(1) = f(1) \\ f_n(2) = f(2) \\ \vdots \\ f_n(i) = f(i) \end{cases}$$

Claim: f is a proper r-coloring

If there is a monochromatic edge X in H, then there is i such that $X \subseteq \{1, \dots, i\}$

$\Rightarrow$ there is $n > i$ such that $f_n(1) = f(1)$, $f_n(2) = f(2)$, ..., $f_n(i) = f(i)$

So, in f_n, X is a monochromatic edge of $H[\{1, 2, \ldots, n\}]$.
Contradiction. $\square$

Proof 2. (Assuming the axiom of choice).

Consider $T :=$ set of all functions $f : V \to \{1, \ldots, r\}$.

$$T \hookrightarrow \left(\prod_{v \in V} \underline{\{1, \ldots, r\}}. \right)$$

We give a discrete topology to $\{1, \ldots, r\}$

→ We have a product topology.

$\left(\prod_{v \in X} X_v \text{ is a basis if } X_v = \{1, \ldots, r\} \text{ for all but finitely many } v. \right)$

By the Tychonoff theorem,

T is compact, because $\{1, \ldots, r\}$ is compact.

For a finite subset W of V, let F_W be the set of functions $f \in T$ such that no subset of W is a monochromatic edge in f.

F_W is closed because its complement is a union of finitely many open sets.

Since $\chi(H[W]) \le r$

$\Rightarrow F_W \neq \phi$.

If $W_1, W_2, \dots W_t$ are finite subsets of V then

$$F_{W_1} \cap F_{W_2} \cap \cdots \cap F_{W_t} \supseteq F_{W_1 \cup W_2 \cup \cdots \cup W_t} \neq \phi.$$

$\therefore \{F_W : W \subseteq V,\ W \text{ finite}\}$ has the property that every finite intersection is nonempty

T is compact

$$\Rightarrow \bigcap_{\substack{W \subseteq V \\ W:\text{finite}}} F_W \neq \phi.$$

Let $f \in \bigcap_{\substack{W \subseteq V \\ W:\text{finite}}} F_W$

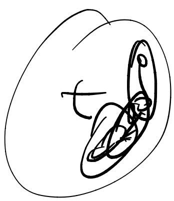
No edge is monochromatic in f. $\square$

Gr of the infinite Ramsey :

For all t, c, there exists n_0 such that
if $n \geq n_0$ and
all 2-subsets of $\{1 \ldots n\}$
are colored by c colors.
then there exists a t-subset of $\{1 \ldots n\}$
whose all 2-subsets
have the same color.

$(R(t, t, \ldots, t) \leq n_0)$

Proof. Let H be a hypergraph
whose vertices are 2-subsets of $\mathbb{N}$
and edges are 2-subsets
of a t-subset of $\mathbb{N}$.



By the infinite Ramsey
any c-coloring of
2-subsets of $\mathbb{N}$
has an infinite monochromatic
subset.

$\Longrightarrow \chi(H) > c$

By the compactness there is a finite
set W such that $\chi(H[W]) > c$.

We may assume that W is the set of all 2-subsets of a set $X \subseteq \{1, \dots, n_0\}$.

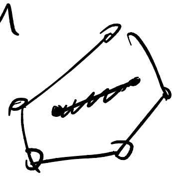
$\Rightarrow$ $\chi(H[W]) > c$

$\rightarrow$ Any c-coloring of 2-subsets of $\{1, \dots, n_0\}$ creates a subset of size t that is monochromatic. $\square$

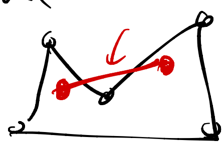
9.4. Finding a convex n-gon

<u>Thm</u> (Erdős, Szekeres 1935)
Let $m \geq 3$.
There exists N such that
any set of N points on $\mathbb{R}^2$
in general position
must have m points
forming a convex m-gon.

Convex 5-gon



Not convex



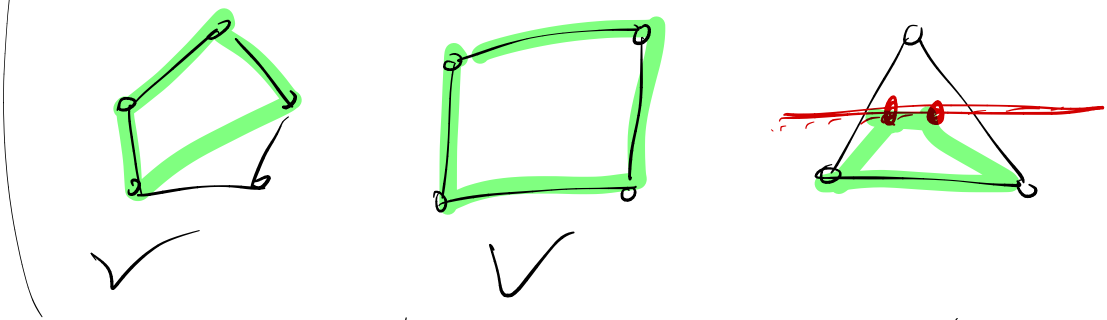
<u>Proof</u> Let $N = R(m, 5 ; 4)$

We color a set of 4 points
by red if they form a convex 4-gon
blue otherwise.

By the definition of the Ramsey number,
either there are red m points
or there are blue 5 points

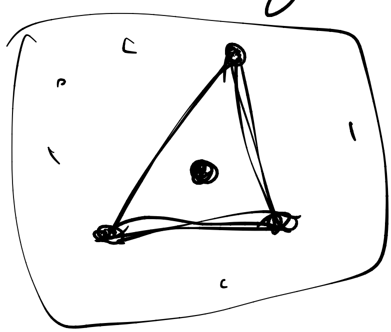
<u>Lemma</u>. In any 5 points in general position
there are 4 points forming a convex 4-gon.

Consider the convex hull of the 5 points



Thus there exists a set of m points whose 4-subsets are all red.

$\rightarrow$ They form a convex m-gon.



$\square$

Proof 2. (Johnson 1986)

Take $N = R(m, m; 3)$

Say for $\{a, b, c\}$

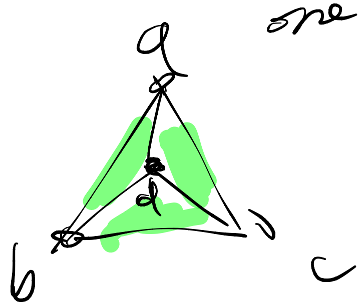
we color red if

$\triangle abc$ contains even # other points

blue otherwise.

By the Ramsey theorem
there are m points such that
all triples have the same color.

If these are not convex, then
there are 4 points
such that
one is contained in the triangle
formed by the other three.



$$\#abc = \#abd + \#acd + \#bcd + 1 \pmod 2$$

$$\Downarrow$$

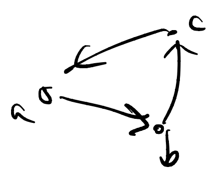
$$0 \equiv 1 \pmod 2$$

Contradiction.

Proof 3 $\quad N = R(m, m; 3)$

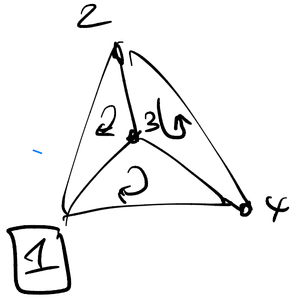
Put an ordering $<$ for these N points.

We color $a<b<c$ by red
if the path $a \to b \to c \to a$
is counterclockwise



by blue otherwise.

By Ramsey, there are m points such that any triple has the same color



$\therefore$ They form a convex m-gon. $\square$

M. Tarsy

Conj by Erdős–Szekeres 1935

$$N = 2^{m-2}+1$$

True for $m = 3, 4, 5$.

Before 2017, all bounds were $\sim 4^n$

In 2017, Andrew Suk

$$N \le 2^{m + 6 m^{\frac{2}{3}} \log m}$$

for large m.

In 2020, Holmsen, Mojarrad, Pach, Tardos

$$N \le 2^{m + c\sqrt{m}\log m}$$

10. Progressions

10.1. van der Waerden's theorem

An ℓ-term arithmetic progression (ℓ-AP) is a set of numbers of the form
$$a,\ a+d,\ a+2d,\ \ldots,\ a+(\ell-1)d$$
for some a, $d \neq 0$.

A k-coloring of a set A is a function
$$\chi : A \to \{1, \ldots, r\}.$$
(let $[r] = \{1, 2, \ldots, r\}$)

Theorem (van der Waerden 1927)

For every k and ℓ, there exists a positive integer N such that for every k-coloring of $[N]$, there is a monochromatic ℓ-term arithmetic progression.

asked by I. Schur.

Let us write $W(\ell, k)$ for the minimum such integer N.

length

colors

"van der Waerden number"

Compactness principle

van der Waerden theorem $\longrightarrow \{1, 2, 3, \ldots\}$

$\stackrel{\text{Id}}{\Longleftrightarrow}$ Any partition of $\mathbb{N}$ into finitely many sets, one of the parts contains arbitrary long arithmetic progressions.

<u>Proof</u> of van der Waerden's theorem:

Let $X_{\ell,m}$ = set of all sequences

$$x \in \{0, 1, 2, \ldots, \ell\}^m$$

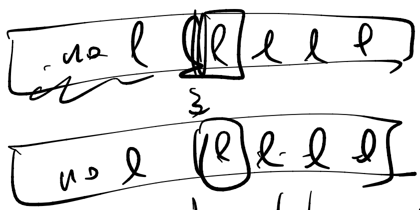
$$x = (x(1), x(2), \ldots, x(m))$$

such that

if $x(i) = \ell$, then $x(j) = \ell$ for all $j \geq i$.

We say 2 sequences $x, y \in X_{\ell,m}$ are ℓ-equivalent if there exists $i \in \{0, \ldots, m\}$ such that

$$\begin{cases} \cdot\ x(j)\neq l,\quad y(j)\neq l & \text{for all } j<i \\ \cdot\ x(j)=y(j)=l & \text{for all } j\geq i. \end{cases}$$



$X_{l,m}$ is partitioned into $\underline{m+1}$ l-equivalence classes

CLAIM: For $l, m \geq 1$ and k, there exists
$N = N(l,m,k)$
such that
for every function $C : [N(l,m,k)] \to \{0,1,\dots,k-1\}$
there exist positive integers
$a, d_1, d_2, \dots, d_m$
such that

$$\underline{C\left(a + \sum_{i=1}^{m} x(i)\, d_i\right)}$$

is constant
on all sequences in
each l-equivalence class of $X_{l,m}$.

CLAIM $\Rightarrow$ vdW theorem

$\boxed{m=1}$: $C(a + x(i)d_1)$ is constant

There are 2 l-equivalence classes on $X_{l,1}$

$X_{l,1} \to$ $\boxed{(0), (1), (2), \ldots, (l-1)}$

$\to (l)$

2 l-equivalence classes

$C(a) = C(a+d_1) = C(a+2d_1) = \cdots = C(a+(l-1)d_1)$

Suppose the claim is false.
Choose a counter-example with minimum l.
Among all of them with same l,
we choose the one with minimum m.

If $l=1$: trivially true / Each equivalence class has only 1 seq.

m

$X_{l,m} \to$
(0 0 0 0 0)
(1 1 1 1 1)
(0 1 1 1 ... 1)
(0 0 1 1 ... 1) ...

We may assume $\ell > 1$.

Suppose $m \neq 1$. Let $M = N(\ell, m-1, k)$

and $M' = N(\ell, 1, k^M)$

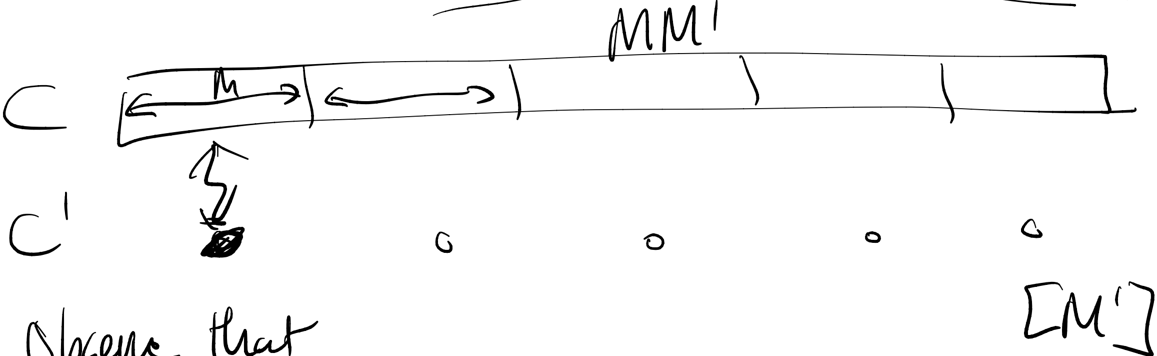
$N = MM'$

Suppose $c : [N] \longrightarrow \{0, 1, \dots, k-1\}$.

Define $c' : [M'] \longrightarrow \{0, \dots, k^M - 1\}$

So that

$$c'(x) = \sum_{i=1}^{M} c\big((x-1)M + i\big) k^{i-1}$$



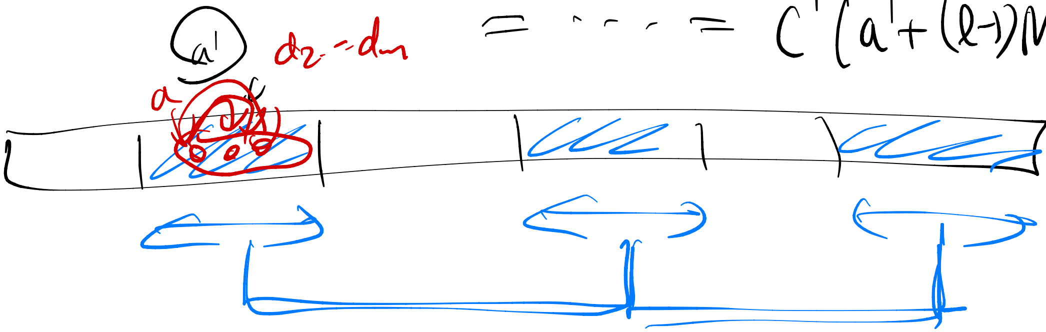
Observe that

$c'(x) = c'(y) \iff$

$c\big((x-1)M + i\big) = c\big((y-1)M + i\big)$

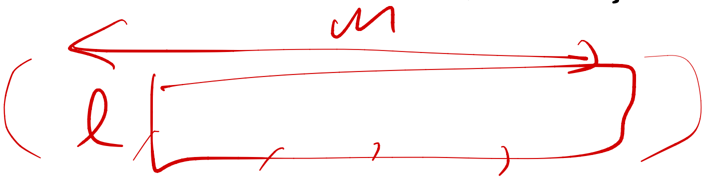
for all $i \in \{1, \dots, M\}$.

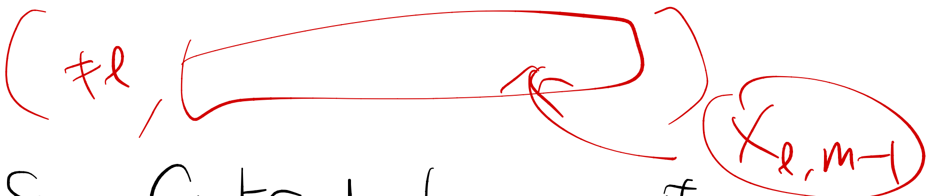
By the assumption on $N(\ell, 1, k^M)$,
there exist a', $d' > 0$ such that
$$C'(a') = C'(a'+d') = C'(a'+2d')$$
$$= \cdots = C'(a'+(\ell-1)M')$$



Consider the interval $\{(a'-1)M+1, \dots, (a'-1)M+M\}$.
By the assumption on $M = N(\ell, m-1, k)$
there exist a, $d_2, d_3, \dots, d_m$
such that
$$a + \sum_{i=2}^{m} x(i) d_i \in \{(a'-1)M+1, \dots, (a'-1)M+M\}$$
and
$$C\left(a + \sum_{i=2}^{m} x(i) d_i\right) \text{ is constant}$$
for all x in each $(\ell-1)$-equivalence class of $\{0, \dots, \ell\}^{m-1}$

$\Rightarrow$ $C\left(a+\sum_{i=1}^{m} x(i)\, d_i\right)$ is constant on every ℓ-equivalence class of $\{0,1,\dots,\ell\}^{m}$





So, Contradiction if $m \neq 1$.

Thus, we may assume $m=1$.

Let $N = N(\ell-1, k, k)$

Suppose $C : [N] \to \{0,\dots,k-1\}$

Then there exist $a, d_1, \dots, d_k$ such that

$$a+\sum_{i=1}^{k} x(i)\, d_i \le N$$

and $C\left(a+\sum x(i)\, d_i\right)$ is constant for x in each $(l-1)$-equivalence class.

$$x \in \{0, 1, \dots, l-1\}^k.$$

$$\left.\begin{array}{l} C(a+(l-1)(d_1+d_2+\cdots+d_k)) \\ C(a+(l-1)(d_2+d_3+\cdots+d_k)) \\ C(a+(l-1)(d_3+d_4+\cdots+d_k)) \\ \quad\vdots \\ C(a+(l-1)d_k) \\ C(a) \end{array}\right\} k+1 \text{ values}$$

By the pigeonhole principle, 2 of them are equal.

There are $1 \le u < v \le k+1$ such that

$$C\left(a+\sum_{i=u}^{k}(l-1)d_i\right) = C\left(a+\sum_{i=v}^{k}(l-1)d_i\right)$$

Consider

$$C\left(\left(a+\sum_{i=v}^{k}(l-1)d_i\right)+x(i)\boxed{\sum_{i=u}^{v-1}d_i}\right)$$

We will take

$$\begin{cases} a' = a + \sum_{i=v}^{k} (l-1) d_i \\ d_1' = \sum_{i=u}^{v-1} d_i \end{cases}$$

We claim that

$C(a' + x(1)\, d_1')$ is constant for every l-equivalence class, $X_{l,m}$

$m=1$

$$X_{l,m} \to (0)\,(1)\,(2) \ldots, (l-1)$$

$$X_{l,m} \to \boxed{(l)}$$

$C(a + \quad (l-1)d_v + (l-1)d_{v+1} + \cdots + (l-1)d_k)$

$(0, 0, \ldots\ 0, (l-1), (l-1), \ldots, l-1)$

$C(a + \quad d_u + d_{u+1} + \cdots + d_{v-1} + \quad " \quad " \quad)$

$(0, 0, 1, 1, \cdots, l-1, l-1, \ldots, l-1)$

$C(a + \quad 2d_u + 2d_{u+1} + \cdots + 2d_{v-1} + \quad " \quad " \quad)$

$(0, 0, 2, 2, \cdots 2, l-1, l-1, \ldots, l-1)$

$C(a + (l-1)d_u + (l-1)d_{u+1} + \cdots + (l-1)d_{v-1} + (l-1)d_v + \cdots + (l-1)d_k)$

$(0, 0, (l-1), \ldots, (l-1), l-1, \ldots, l-1)$

$\quad\ \ \uparrow$
$\quad\ \ u$

$\|$

$$\overset{\|}{C}\left(a+(\ell-1)\sum_{i=r}^{k} d_i\right)$$

$\square$

Conj (Erdős, Turán 1936)

Every set of positive upper density has a 3-AP.

Proved by Roth 1953

1957 Erdős: conjectured:

Every set of positive upper density has a ℓ-AP.

Proved by Szemerédi 1975,

1977 proved by Fürstenberg using ergodic methods.

Thm. [Bergelson, Leibman 1996)

$p_1, p_2, \dots, p_k$: polynomials in $\mathbb{Z}[x]$

$p_i(0) = 0$

If A is a set of positive upper density then

there exist $a > 0$, $d > 0$ such that

$$\left.\begin{array}{l} a \\ a + p_1(d) \\ a + p_2(d) \\ \vdots \\ a + p_k(d) \end{array}\right\} \in A.$$

Current best bound for $W(l, c)$

$$W(l, c) \le 2^{2^{c^{2^{2^{l+9}}}}} \qquad \text{Gowers 2001.}$$

Graham, Solymosi 2006.

$$W(3, c) \le 2^{2^{2^{O(c)}}}$$

Erdős conj ($5000)

If $$\sum_{a \in A} \frac{1}{a} = \infty$$

then A contains an l-AP for every l.

Thm (Green, Tao (2004))

True for $A = \{\text{primes}\}$