

3.4. Wedge products - 2nd exterior power.

Let V be a vector space over \mathbb{F} .

Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V .

For a given vector space V ,

we define the dual space V^*

as the vector space consisting of

all linear functions from V to \mathbb{F} .

Observation: $\dim(V^*) = \dim(V)$

An alternating bilinear form on V over \mathbb{F} is a function $B: V \times V \rightarrow \mathbb{F}$ satisfying

$$(1) \quad B(v, v) = 0 \quad \text{for all } v \in V.$$

$$(2) \quad B(u, v) = -B(v, u) \quad \text{for all } u, v \in V.$$

$$(3) \quad B(c_1 u_1 + c_2 u_2, v) = c_1 B(u_1, v) + c_2 B(u_2, v).$$

The 2nd exterior power of V , denoted by $\Lambda^2 V$ is the dual space of the vector space of all alternating bilinear forms on V over \mathbb{F} .

For an alternating bilinear form $B: V \times V \rightarrow \mathbb{F}$ and $u, v \in V$, we define $\underline{u \wedge v} \in \Lambda^2 V$ so that

$$\underline{u \wedge v} \in \Lambda^2 V$$

u wedge v

$$(u \wedge v)(B) = B(u, v) \in \mathbb{F}.$$

Lemma. $\{v_i \wedge v_j : i < j \leq n\}$

Proof. For $a = \sum_{i=1}^n a_i v_i$ $b = \sum_{j=1}^n b_j v_j$ is a basis of $\Lambda^2 V$.

We define an alternating bilinear form

$$E_{ij}(a, b) = a_i b_j - a_j b_i \in F.$$

$$\begin{cases} E_{ij}(v_i, v_j) = 1 & \text{if } i = i', j = j' \\ 0 & \text{otherwise} \end{cases}$$

Claim 1: $\{v_i \wedge v_j : i < j\}$ is linearly indep.

$$\sum c_{ij} (v_i \wedge v_j) = 0$$

$$0 = \sum_{i < j} c_{ij} (v_i \wedge v_j) (E_{i', j'})$$

$$= \sum_{i < j} c_{ij} E_{i', j'}(v_i, v_j)$$

$$= c_{i' j'}$$

The space of all alternating bilinear forms

~ the space of skew-symmetric $n \times n$ matrices



$$\dim \leq \binom{n}{2}$$

$$\binom{n}{2}$$

$\Rightarrow \{v_i \wedge v_j : i < j\}$ is a basis
of $\Lambda^2 V$.

* $v \wedge v = 0$

$$\begin{aligned} (\sum c_i v_i) \wedge (\sum c_j v_j) \\ = \sum_{i,j} (c_i v_i \wedge v_j + c_j v_j \wedge v_i) \\ + \underline{c_i v_i \wedge v_i} \end{aligned}$$

* $v \wedge u = -u \wedge v = 0.$

$$\begin{aligned} ((v \wedge u)(B)) \\ = B(v, u) \\ = -B(u, v) \\ = -(u \wedge v)(B). \end{aligned}$$

*

$$(c_1 u_1 + c_2 u_2) \wedge v = c_1 (u_1 \wedge v) + c_2 (u_2 \wedge v)$$

$$\begin{aligned} ((c_1 u_1 + c_2 u_2) \wedge v)(B) \\ = B(c_1 u_1 + c_2 u_2, v) \\ = c_1 B(u_1, v) + c_2 B(u_2, v) \\ = c_1 (u_1 \wedge v)(B) + c_2 (u_2 \wedge v)(B) \\ = (c_1 u_1 \wedge v + c_2 u_2 \wedge v)(B). \end{aligned}$$

* $u \wedge v = 0 \Leftrightarrow \{u, v\}$ is linearly dependent.

(\Leftarrow) $u = k v$, easy

(\Rightarrow) If $\{u, v\}$ is independent, then we can extend to a basis.
 $u \wedge v \neq 0$.

3.5. Wedge products - k^{th} exterior power

V : vector space over a field \mathbb{F} , finite dimensional
 An alternating multilinear form of degree k on V
 is a function $M: V^k \rightarrow \mathbb{F}$ with the following properties:

- (1) $M(w_1, w_2, \dots, w_k) = 0$ if $w_i = w_j$ for some $i \neq j$.
- (2) $M(w_1, w_2, \dots, \cancel{w_i}, \dots, \cancel{w_j}, \dots, w_k)$
 $= -M(w_1, w_2, \dots, \overset{\leftarrow}{w_j}, \dots, \overset{\rightarrow}{w_i}, \dots, w_k)$
- (3) $M(c_1 a_1 + c_2 a_2, w_2, w_3, \dots, w_k)$
 $= c_1 M(a_1, w_2, w_3, \dots, w_k)$
 $+ c_2 M(a_2, w_2, w_3, \dots, w_k)$.
 for $c_1, c_2 \in \mathbb{F}$
 $w_2, w_3, \dots, w_k \in V$
 $a_1, a_2 \in V$.

Lemma. Let M be an alternating multilinear form of degree k on V .

Let $u_1, u_2, \dots, u_k, w_1, w_2, \dots, w_k \in V$.

If u_1, u_2, \dots, u_k are in the subspace spanned by w_1, w_2, \dots, w_k , then

$$M(u_1, u_2, \dots, u_k) = \lambda M(w_1, w_2, \dots, w_k)$$

for some $\lambda \in \mathbb{F}$.

easy

The k^{th} exterior power of V

$= \Lambda^k V =$ dual space of the vector space of all alternating multilinear forms of degree k on V .

For $w_1, w_2, \dots, w_k \in V$,

we write $w_1 \wedge w_2 \wedge \dots \wedge w_k \in \Lambda^k V$

defined as

$$(w_1 \wedge w_2 \wedge \dots \wedge w_k)(M) = M(w_1, \dots, w_k)$$

If $\{v_1, v_2, \dots, v_n\}$ is a basis of V , then M is determined by

$$M(v_{i_1}, v_{i_2}, \dots, v_{i_k}) \quad (1 \leq i_1 < i_2 < \dots < i_k \leq n)$$

There are $\binom{n}{k}$ values to determine M .

\Rightarrow dimension of the vector space
of the alternating multilinear form
of degree k

$$\leq \binom{n}{k}.$$

$$\Rightarrow \dim \Lambda^k V \leq \binom{n}{k}.$$

Lemma: $\{v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k} : 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$
is a basis of $\Lambda^k V$

If $\{v_1, v_2, \dots, v_n\}$ is a basis of V .

Proof. For $(1 \leq i_1 < i_2 < \dots < i_k \leq n)$, let

us define $M(w_1, w_2, \dots, w_k)$
 w_i

so that

$$\text{If } w_i = \sum w_{ij} v_j \text{ then}$$

$$M_{i_1 i_2 \dots i_k}(w_1, \dots, w_k) = \det \begin{pmatrix} w_{1i_1} & w_{2i_1} & \dots & w_{ki_1} \\ w_{1i_2} & w_{2i_2} & \dots & w_{ki_2} \\ \vdots & \vdots & \ddots & \vdots \\ w_{1i_k} & w_{2i_k} & \dots & w_{ki_k} \end{pmatrix}$$

$M_{i_1 i_2 \dots i_k}$ is an alternating multilinear form of degree k .

$$\left\{ \begin{array}{l} M_{i_1 i_2 \dots i_k}(v_{i_1}, v_{i_2}, \dots, v_{i_k}) = \det \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = 1 \\ M_{i_1 i_2 \dots i_k}(v_j, v_{j_2}, \dots, v_{j_k}) = 0 \\ \quad \quad \quad \{i_1, \dots, i_k\} \neq \{j, \dots, j_k\} \end{array} \right.$$

$$\text{If } \sum \alpha_{j_1 \dots j_k} v_{j_1} \wedge v_{j_2} \wedge \dots \wedge v_{j_k} = 0$$

$$\text{then } \sum \alpha_{j_1 \dots j_k} (v_{j_1} \wedge \dots \wedge v_{j_k}) (M_{i_1 \dots i_k}) = 0$$

$$\sum \alpha_{j_1 \dots j_k} \underbrace{M_{i_1 i_2 \dots i_k}(v_{j_1}, \dots, v_{j_k})}_{\alpha_{i_1 \dots i_k} = 0} = 0$$

$$\left\{ v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k} \right\} \text{ is linearly indep. } \square$$

Key properties

(1) $v_1 \wedge v_2 \wedge \dots \wedge v_k = 0$ if $v_i = v_j$ for some $i \neq j$.

(2) $v_1 \wedge v_2 \wedge \dots \wedge \underbrace{v_i \wedge \dots \wedge v_j}_{+} \wedge \dots \wedge v_k$
 $= - v_1 \wedge v_2 \wedge \dots \wedge v_j \wedge \dots \wedge \underbrace{v_i \wedge \dots \wedge v_k}_{+}$

Lemma. $v_1 \wedge v_2 \wedge \dots \wedge v_k \neq 0$
 $\Leftrightarrow \{v_1, v_2, \dots, v_k\}$ is linearly independent

Proof. If $\{v_1, \dots, v_k\}$ is linearly dependent
then $v_1 \wedge \dots \wedge v_k = 0$ (\because easy)

If $\{v_1, \dots, v_k\}$ is linearly independent,
then extend it to a base
 $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$

$\Rightarrow v_1 \wedge \dots \wedge v_k$ is an element of a basis
 $\Rightarrow v_1 \wedge \dots \wedge v_k \neq 0$. \square

① \leftrightarrow \det

$$w_i = \sum_{j=1}^n w_{ij} v_j$$

$$w_1 \wedge w_2 \wedge \dots \wedge w_n$$

$$= (\sum w_{1j} v_j) \wedge (\sum w_{2j} v_j) \wedge \dots \wedge (\sum w_{nj} v_j)$$

$$= \sum w_{1i_1} w_{2i_2} \dots w_{ni_n} v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_n}$$

$$\text{Let } \{i_1, i_2, \dots, i_n\} \subset \{1, 2, \dots, m\}$$

$$\begin{aligned}
 &= \sum_{\sigma} \underbrace{\omega_{1\sigma(1)} \omega_{2\sigma(2)} \dots \omega_{n\sigma(n)}}_{\sigma: \text{permutation}} \text{Sign}(\sigma) \underbrace{v_1 \wedge v_2 \wedge \dots \wedge v_n}_{\text{where } W = \begin{pmatrix} w_{11} & w_{12} & \dots \\ w_{21} & w_{22} & \dots \\ \vdots & \vdots & \ddots \\ w_{n1} & w_{n2} & \dots \end{pmatrix}}
 \end{aligned}$$

For $a \in \wedge^k V$, $b \in \wedge^l V$, we define $a \wedge b$

$a \wedge b \in \wedge^{k+l} V$
 $a \wedge b : M \mapsto \mathbb{F}$; M: alternating multilinear form of degree $k+l$.

For $w_1, \dots, w_e \in V$,

$$M|_{w_1, \dots, w_e} = M \left(\underbrace{\dots}_{k}, \underbrace{w_1, \dots, w_e}_{l} \right)$$

$M \mapsto \underbrace{M|_{w_1, \dots, w_e}}_{\text{is an alternating}} : V^k \rightarrow \mathbb{F}$
multilinear form of degree k.

$$a(M|w_1 \dots w_e) \in F.$$

We define

$$(a \wedge b)(M) = b(a(M|w_1 \dots w_e))$$

(let $N(w_1, w_2, \dots, w_e) = a(M|w_1 \dots w_e)$)
 $\Rightarrow N$ is an alternating multilinear form
of degree e .
 $\Rightarrow b(N)$ is defined.

$$\star \quad \underbrace{(u_1 \wedge \dots \wedge u_k)}_{\in \Lambda^k V} \wedge \underbrace{(w_1 \wedge \dots \wedge w_e)}_{\in \Lambda^e V} = u_1 \wedge \dots \wedge u_k \wedge w_1 \wedge \dots \wedge w_e \quad \in \Lambda^{k+e} V.$$

Lemma.

$$(1) \quad \text{If } a \in \Lambda^k V, \quad b \in \Lambda^e V \quad \text{then}$$

$$b \wedge a = (-)^{ke} a \wedge b$$



$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{matrix}$$

$$(2) \quad (c, a_1 + c_2 a_2) \wedge b = c(a_1 \wedge b) + c_2 (a_2 \wedge b)$$

$$(3) \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c \quad \in \Lambda^{k+k+m} V.$$

$$a \in \Lambda^k V, \quad b \in \Lambda^e V, \quad c \in \Lambda^m V.$$

□

Skew Version of

3.6 Bollobás Two Families Theorem.

Thm. (Skew version of Bollobás' Two Families Theorem)

$$\begin{array}{c} A_1, A_2, \dots, A_m \\ B_1, B_2, \dots, B_n \end{array} \left. \begin{array}{l} \text{subsets of } \{1, \dots, n\} \\ |A_i|=r \\ |B_i|=s \end{array} \right\} \text{for all } i$$

(1) $A_i \cap B_j = \emptyset$

(2) $A_i \cap B_j \neq \emptyset \quad \text{for all } i < j$

$$\Rightarrow M \leq \binom{r+s}{r}$$

Lemma. In \mathbb{R}^n , we can pick a set X of infinitely many points such that every subset of at most n points is linearly independent.

Proof. Let $\alpha_k = \begin{pmatrix} 1 \\ k \\ k^2 \\ \vdots \\ k^{n-1} \end{pmatrix} \in \mathbb{R}^n$

$\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is linearly independent.

$$\Leftrightarrow \det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 4 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n^n \end{pmatrix} \neq 0$$

$$\det \begin{pmatrix} (1 & 1 & \dots & 1) \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n+1} & x_2^{n+1} & \dots & x_n^{n+1} \end{pmatrix} = \prod_{i < j} (x_i - x_j)$$

Proof of Skew version of Babilos' Two Families theorem
(Horasz 1977)

Let X be an infinite subset of \mathbb{R}^{n+s}
such that every subset of size $\leq n+s$
is linearly independent.

For each $x \in \bigcup_{i=1}^m (A_i \cup B_i)$,

We associate x with some element of X .
(In other words)
we may assume $A_i, B_i \subseteq X$

For a subset $I \subseteq X$, let

$$w_I = \bigwedge_{i \in I} w_i \in \Lambda^{|I|} \mathbb{R}^{n+s}$$

$$a_i = w_{A_i} \in \Lambda^n \mathbb{R}^{n+s}$$

$$b_i = w_{B_i} \in \Lambda^s \mathbb{R}^{n+s}$$

$$a_i \wedge b_i \neq 0 \quad (\text{because } A_i \cap B_i = \emptyset)$$

& only res vectors in X
are linearly indep.

If $i \neq j$ then
 $a_i \wedge b_j = 0 \quad (A_i \cap B_j \neq \emptyset)$

Claim: $\{a_1, a_2, \dots, a_m\}$ is linearly independent
 in $V^r \mathbb{R}^{r+s}$

Suppose $\sum c_j a_j = 0$

$$\sum c_j a_j \wedge b_i = 0$$

$$\underbrace{c_j a_j \wedge b_i}_{c_j = 0} = 0$$

$$\therefore m \leq \dim V^r \mathbb{R}^{r+s} = \binom{r+s}{r}.$$

Thus (Lovasz)
 Let W be a vector space over a field F .

Let U_1, U_2, \dots, U_m

V_1, V_2, \dots, V_m be subspaces of W
 such that

$$\dim U_i = r \quad \dim V_i = s$$

$U_i \cap V_i = \{0\}$ for all i

$U_i \cap V_j \neq \{0\}$ for all $i < j$

Then $m \leq \binom{r+s}{r}$

Partial Proof (When $\dim W = r+s$)

For a subspace U of W with a basis
 $\{x_1, \dots, x_p\}$

We write

$$U = x_1 \wedge x_2 \wedge \dots \wedge x_p$$

let $a_i = \wedge U_i \in \wedge^r W$

$$b_i = \wedge V_i \in \wedge^s W$$

$$U_i \cap V_i = \{0\} \Rightarrow a_i \wedge b_i = 0$$

If $i < j$ $U_i \cap V_j \neq \{0\} \Rightarrow a_i \wedge b_j = 0$.

$\{a_1, a_2, \dots, a_m\}$ is linearly independent

$$\Rightarrow m \leq \dim \wedge^r W = \binom{r+s}{r}$$

□

3.7. Subspaces in general position

For subspaces U_1, U_2, \dots, U_m of a vector space W over a field \mathbb{F} , $\dim W = n$,

we say a subspace V of W is in general position
with respect to $\{U_1, U_2, \dots, U_m\}$

$$\text{If } \dim U_i \cap V = \max(\dim(U_i) - t, 0) \text{ for all } i=1, 2, \dots, m.$$

Goal : If $|\mathbb{F}|$ is big, then there is one.

Thm (Existence of a subspace in general position)

$$\text{Let } s \in \{0, 1, \dots, n\}$$

Given a list of subspaces U_1, U_2, \dots, U_m of a vector space W of dimension n over a field \mathbb{F} ,

there is an s -dimensional subspace V in general position with respect to U_1, \dots, U_m

$$\text{if } |\mathbb{F}| > sm.$$

1880

Lemma (DeMillo-Lipton-Schwartz-Zippel ⁽¹⁸⁸⁰⁾)

(sparse zero lemma)

Let $f \in \mathbb{F}[X_1, X_2, \dots, X_n]$ be a nonzero polynomial of degree d .

Let $S \subseteq \mathbb{F}$ be a finite set with $|S| = N$.

Then the number of tuples $(\alpha_1, \alpha_2, \dots, \alpha_n) \in S^n$ with $f(\alpha_1, \alpha_2, \dots, \alpha_n) = 0$

is at most dN^{d-1} .

In other words, if we take $\alpha_i \in \Omega$ uniformly at random, then

$$P(f(\alpha_1, \alpha_2, \dots, \alpha_n) = 0) \leq \frac{d}{N}.$$

Proof. Induction on n .

If $n=1$, then f has at most d roots.

Now let us assume $n > 1$.

Write

$$f(\alpha_1, \dots, \alpha_n) = g_0 + g_1 \alpha_n + g_2 \alpha_n^2 + \dots + g_k \alpha_n^k$$

where $g_i \in F[x_1, \dots, x_{n-1}]$, $\deg g_i \leq d-i$.

roots of $f(x_1, \dots, x_n)$

$$\begin{aligned} & \#(\alpha_1, \dots, \alpha_{n-1}) \text{ with } g_k(\alpha_1, \dots, \alpha_{n-1}) = 0 \\ & \quad (\text{by induction}) \leq (d-k) N^{n-2} \\ & \Rightarrow \#(\alpha_1, \dots, \alpha_n) \text{ with } g_k(\alpha_1, \dots, \alpha_{n-1}) = 0 \\ & \quad \& f(\alpha_1, \dots, \alpha_n) = 0 \\ & \leq (d-k) (N^{n-2} \cdot N = (d-k) N^{n-1}) \end{aligned}$$

$$\#(\alpha_1, \dots, \alpha_{n-1}) \text{ with } g_k(\alpha_1, \dots, \alpha_{n-1}) \neq 0 \leq N^{n-1}$$

\Rightarrow There are $\leq k$ choices of α_n for each of them
so that $f(\alpha_1, \dots, \alpha_n) = 0$.

$$\begin{aligned} & \Rightarrow \#(\alpha_1, \dots, \alpha_n) \text{ with } f(\alpha_1, \dots, \alpha_n) = 0 \\ & \quad g_k(\alpha_1, \dots, \alpha_{n-1}) \neq 0 \\ & \leq k \cdot N^{n-1} \end{aligned}$$

$$\Rightarrow \#(\alpha_1, \dots, \alpha_n) \text{ such that } f(\alpha_1, \dots, \alpha_n) = 0 \\ \leq k \cdot N^{n-1} + (d-k)N^{n-1} = dN^{n-1} \quad \square$$

Proof of the theorem on
the existence of a subspace
in general position.

- Let $t = n - s$
 We may assume $\dim(U_i) \geq t$
 by enlarging U_i whenever $\dim U_i < t$.

Goal: Find a subspace V such that
 $\left\{ \begin{array}{l} \dim(U_i \cap V) = \dim U_i - t \\ \dim V = n - t \end{array} \right.$

We fix a basis W .
 Each vector of W is represented by
 this basis.

Let

$$\text{Let } P_i(x_1, x_2, \dots, x_{n-t}) = \det \begin{bmatrix} n & w_1^i & w_2^i & \cdots & w_t^i & | & x_1 & x_2 & \cdots & x_{n-t} \end{bmatrix}$$

First t vectors
of a basis of U_i

where $x_1, x_2, \dots, x_{n-t} \in F^n$

$\Rightarrow P_i \in F[x_{11}, x_{12}, \dots, x_{1n}, x_{21}, x_{22}, \dots, x_{2n}, \dots, x_{n-t,1}, x_{n-t,2}, \dots, x_{n-t,n}]$

$$\deg P_i = n-t$$

Let X be a vector space spanned by x_1, x_2, \dots, x_{n-t} .

If $P_i \neq 0$, then $\det(U_i + X) = n$

$$\begin{aligned}\dim(U_i \cap X) &= \dim U_i + \dim X \\ &\quad - \dim(U_i + X) \\ &= \dim U_i + (n-t) - n \\ &= \dim U_i - t\end{aligned}$$

Goal: Find x_1, x_2, \dots, x_{n-t} so that

$P_i(x_1, x_2, \dots, x_{n-t}) \neq 0$ for all i .

Let $g = P_1 P_2 \cdots P_m$.

$$\deg g = m(n-t) = ms.$$

roots of $g \leq (ms) |F|^{(n-t)n-1}$

\Rightarrow If $|F|^{(n-t)n} > (ms) |F|^{(n-t)n-1}$

then there is an assignment
 $x_1, x_2, \dots, x_{n-t} \in F^n$ such that
 $g(x_1, \dots, x_{n-t}) \neq 0.$

$$\hookrightarrow |F| > MS$$



Corollary Let t be a nonnegative integer.
 Let U_1, U_2, \dots, U_m be subspaces of
 a vector space W over a field F
 $\dim W = n.$

There is a linear transformation

$$g : W \rightarrow W'$$

such that $\dim W' = t$

$$\dim g(U_i) = \min(\dim(U_i), t)$$

for each i .

$$\text{If } |F| > m(n-t)$$

We say such a linear transformation
 is in general position
 with respect to $U_1, U_2, \dots, U_m.$

Proof. Let V be a subspace of $\dim n-t$
 of W
 in general position with $U_1, \dots, U_m.$

Take any linear transformation $\varphi: W \rightarrow W$
such that $\text{Ker } \varphi = V$.

$$\begin{aligned}\dim(\varphi(U_i)) &= \dim U_i - \dim(U_i \cap \text{Ker } \varphi) \\ &= \dim U_i - \max(\dim U_i - t, 0) \\ &= \min(t, \dim U_i)\end{aligned}$$

Proof of Lovasz Two Families theorem:

- We may assume $|F|$ is big by taking an extension field of F

Enough to consider the case that $\dim W > r+s$.

There is a linear transformation $\varphi: W \rightarrow W'$ in general position with

$$U_i + U_j \quad \text{for all } i \leq j$$

such that

$$W' = \varphi(W)$$

$$\dim W' = \dim \varphi(W) = r+s$$

$$\begin{aligned}\dim \underline{\varphi(U_i + U_j)} &= \min(\dim(U_i + U_j), r+s) \\ &= \dim(U_i + U_j)\end{aligned}$$

$$\Rightarrow \dim \underline{\varphi(U_i)} = \dim U_i$$

For $i \leq j$

$$\begin{aligned} \dim(\varphi(U_i) \cap \varphi(V_j)) &= \dim \varphi(U_i) + \dim \varphi(V_j) \\ &\quad - \dim(\varphi(U_i) + \varphi(V_j)) \\ &= \dim U_i + \dim V_j \\ &\quad - \dim(U_i + V_j) \\ &= \dim U_i \cap V_j \end{aligned}$$

So, $\varphi(U_1), \dots, \varphi(U_m)$
 $\varphi(V_1), \dots, \varphi(V_n)$ are subspaces
of $\varphi(W)$

where $\left\{ \begin{array}{l} \varphi(U_i) \cap \varphi(V_j) \neq \{0\} \text{ for } i < j \\ \varphi(U_i) \cap \varphi(V_i) = \{0\} \text{ for all } i \end{array} \right.$

$$\dim \varphi(W) = r+s$$

$$\dim \varphi(U_i) = r \quad \dim \varphi(V_j) = s$$

$$\Rightarrow m \leq \dim \overline{\varphi}(W) = \binom{r+s}{r}$$
□