Chapter 8. Cauchy's interlacing theorem

8.1. Cauchy's interlacing theorem

A sequence $s_1 \geq s_2 \geq \cdots \geq s_{n-1}$ of length $n-1$ <u>interlaces</u> a sequence $r_1 \geq r_2 \geq \cdots \geq r_n$

if $r_1 \geq s_1 \geq r_2 \geq s_2 \geq r_3 \geq s_3 \cdots \geq s_{n-1} \geq r_n$.

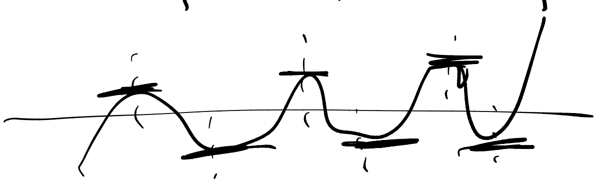
We say that if f is a polynomial of degree n with all roots real

and g is a polynomial of degree $n-1$ with all roots real

and the roots of g interlaces the roots of f

then g interlaces f.

Observe that: if f is a poly with all roots real, then f' interlaces f.



<u>Lemma</u> Let f, g be polynomials with all roots real such that

f has degree n

g has degree $n-1$.

Then the roots of g interlace the roots of f

$\iff$ all roots of $f+\alpha g$ are real for all real α.

Proof

$\Rightarrow$ Let $r_1 \geq r_2 \geq \cdots \geq r_n$ be the root of f

$s_1 \geq s_2 \geq \cdots \geq s_{n-1}$ be the root of g.

First, we claim that we may assume r_i, s_j are distinct.

Otherwise

$$\begin{cases} f = c_f(x-r_1)(x-r_2)\cdots(x-r_n) \\ g = c_g(x-s_1)(x-s_2)\cdots(x-s_{n-1}) \end{cases}$$

Let $f_\varepsilon = c_f(x-r_1)(x-r_2+2\varepsilon)(x-r_3+4\varepsilon)\cdots(x-r_n+(2n-2)\varepsilon)$

$\to$ roots of f_ε

$r_1,\ r_2-2\varepsilon,\ r_3-4\varepsilon, \cdots, r_n-(2n-2)\varepsilon$.

Let $g_\varepsilon = c_g(x-s_1+\varepsilon)(x-s_2+3\varepsilon)\cdots(x-s_{n-1}+(2n-3)\varepsilon)$

roots of $g_\varepsilon =$

$s_1-\varepsilon,\ s_2-3\varepsilon,\ s_3-5\varepsilon, \cdots s_{n-1}-(2n-3)\varepsilon$.

For $0 < \varepsilon <$ small,

the roots of f_ε, roots of g_ε are all distinct,

As $\varepsilon \to 0+$,

$f_\varepsilon \to f$, $g_\varepsilon \to g$

If x_ε is a root of $f_\varepsilon + \alpha g_\varepsilon = 0$

then $x_\varepsilon \to$ a root of $f + \alpha g = 0$.

$\Rightarrow$ all roots of $f + \alpha g = 0$ are real.

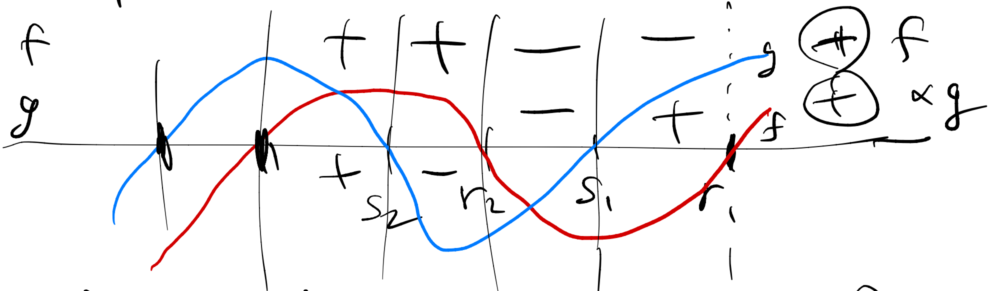
So, we may now assume

$$r_n < s_{n-1} < r_{n-1} < s_{n-2} < r_{n-2} < \cdots < s_1 < r_1.$$

We may assume $\alpha \neq 0$.

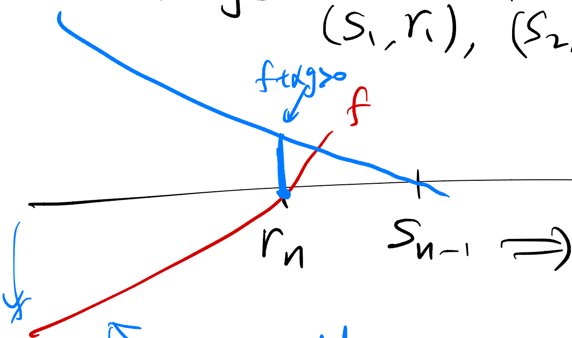
If $f(x) \to \infty$ and $\alpha g(x) \to \infty$ as $x \to \infty$

then



$f + \alpha g = 0$ has no root in (r_1, ∞)
$(r_2, s_1), (r_3, s_2), \ldots, (r_n, s_{n-1})$.

$\Rightarrow$ By the intermediate value theorem,
$f + \alpha g = 0$ has at least one root in
$(s_1, r_1), (s_2, r_2), (s_3, r_3), \ldots, (s_{n-1}, r_{n-1})$.



$\Rightarrow$ $f + \alpha g = 0$ has at least one root in $(-\infty, r_n)$

because $\left(\lim_{x \to -\infty} (f + \alpha g)(x) = -\infty \text{ and } (f + \alpha g)(r_n) > 0 \right)$

or

$\left(\lim_{x \to -\infty} (f + \alpha g)(x) = \infty \quad (f + \alpha g)(r_n) < 0 \right.$

$\Longrightarrow$) We found n real roots of $f+\alpha g=0$.) DONE.

By the similar argument the same conclusion hold

if $f(x) \to \infty$, $\alpha g(x) \to -\infty$

or $f(x) \to -\infty$, $\alpha g(x) \to \infty$

or $f(x) \to -\infty$, $\alpha g(x) \to -\infty$

as $x \to \infty$

($\Longleftarrow$) We may assume f, g have no common root.

Why? If not let us write

$$\begin{cases} f = q f_1 \\ g = q g_1 \end{cases}$$

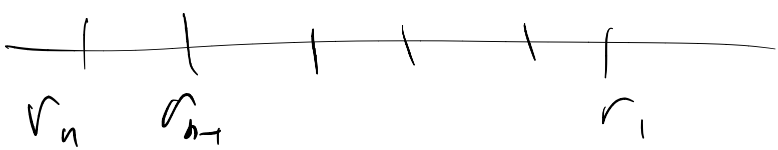
f_1, g_1 relative prime

Then the roots of g_1 interlace the roots of f_1.

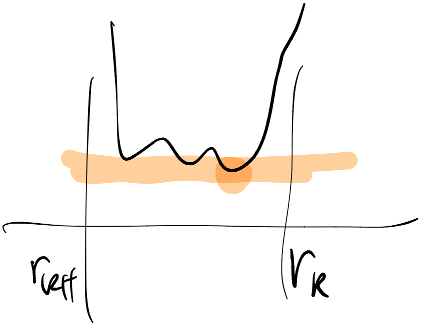
$\Longrightarrow$ the roots of g interlace the roots of f.

So, there is no common root.

If $[r_{k+1}, r_k]$ has no root of g.

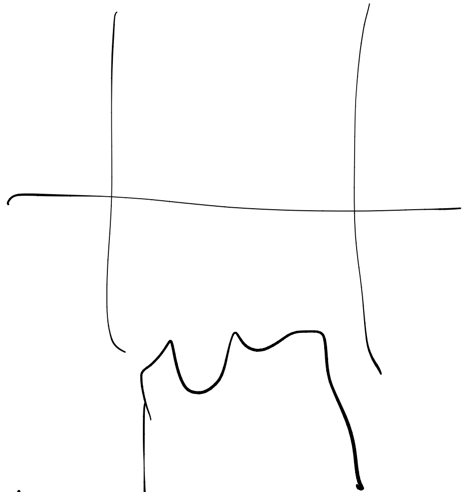


then $|g(x)| > \varepsilon$ for all $x \in [r_{k+1}, r_k$



$$\frac{g(x)}{f(x)}$$

or



Pick $\beta \neq 0$ so that

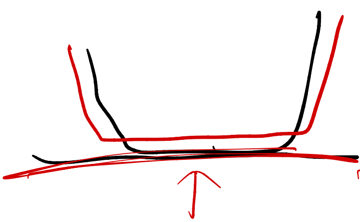
$$\frac{g(x)}{f(x)} = \beta$$ has a root of multiplicity > 1.

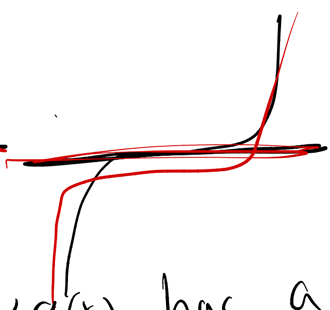
$$g(x) = \beta f(x)$$

$$f(x) - \frac{1}{\beta} g(x) = 0.$$ Let $\alpha = -\frac{1}{\beta}$.

$f(x) + \alpha g(x) = 0$ has a multiple root at x

$x \in [r_{k+1}, r_k]$

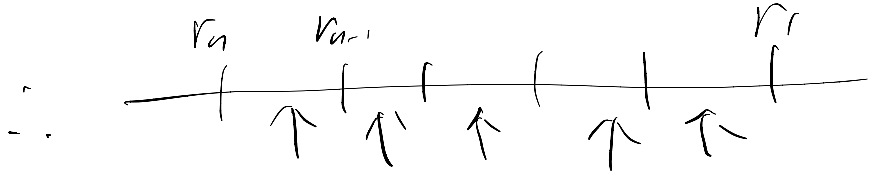




We change α by adding or subtracting a small amount

$\Longrightarrow$ $f(x) + \alpha g(x) = 0$ has a complex root

contradiction



$\therefore$ Since g has degree $n-1$,
for each k, $[r_{k+1}, r_k]$ has exactly one root s_k of g

$$r_n < s_{n-1} < r_{n-1} < s_{n-2} < r_{n-2} < \cdots < r_1.$$

Thm. If A is $\underbrace{\text{a}}_{n\times n}$ real symmetric matrix and B is a principal submatrix of A obtained by deleting the k^{th} row and the k^{th} column, then the eigenvalues of B interlace the eigenvalues of A.

$$\left(\begin{array}{ll} Ax = \lambda x & \lambda = \text{eigenvalues} \\ x \neq 0 & x = \text{eigenvectors} \end{array} \right)$$

Proof We may assume $k = n$.

$$(\det(xI - A) = 0)$$

Let
$$A = \left(\begin{array}{c|c} B & c \\ \hline c^T & d \end{array} \right)$$

$$\underbrace{\det(xI - A)}_{\text{characteristic poly of } A} = \det\left(\begin{array}{c|c} xI - B & -c \\ \hline -c^T & x - d \end{array}\right)$$

$$\det\left(\begin{array}{c|c} xI - B & -c \\ \hline -c^T & x - d + \alpha \end{array}\right)$$

real root

char poly of $\left(\begin{array}{c|c} B & c \\ \hline c^T & d - \alpha \end{array}\right)$

symmetric

$$= \underbrace{\det\left(\begin{array}{c|c} xI - B & -c \\ \hline -c^T & x - d \end{array}\right)}_{\text{char poly of } A} + \underbrace{\det\left(\begin{array}{c|c} xI - B & -c \\ \hline 0 & \alpha \end{array}\right)}_{\text{char poly of } B}$$

$$= \underbrace{\det(xI - A) + \alpha \det(xI - B)}_{\text{has real roots } \forall \alpha}$$

$\therefore$

$\Longrightarrow$ the roots of $\det(xI - B) = 0$ interlace the roots of $\det(xI - A) = 0$

$\square$

Cor

If A : $n \times n$ real symmetric matrix

B : principal $k \times k$ submatrix

r_i : i^{th} largest eigenvalue of A

s_i : i^{th} largest eigenvalue of B

$$\Rightarrow \quad r_{i+(n-k)} \leq s_i \leq r_i$$

8.2. SENSITIVITY CONJECTURE

A boolean function is a function $f: \{0,1\}^n \longrightarrow \{0,1\}$.

For any subset S of $\{1,2,\ldots,n\}$, $x \in \{0,1\}^n$

let x^S be the 0-1 vector obtained by flipping i^{th} coordinate for all $i \in S$

$$\begin{array}{c} 1\,2\,3 \\ \downarrow\,\downarrow\,\downarrow \end{array}$$

$$x = 101 \qquad S = \{1,2\}$$

$$\Longrightarrow \quad x^S = 011$$

For a boolean function $f: \{0,1\}^n \to \{0,1\}$

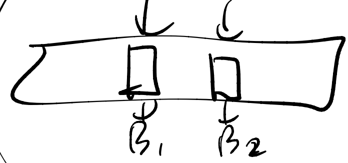
the local sensitivity $s(f,x)$ is the number of indices $i \in \{1,2,\ldots,n\}$ such that

$$f(x) \neq f(x^{\{i\}}).$$

Then the sensitivity of f, $s(f)$ is the $\max_x s(f,x)$.

The local block sensitivity $bs(f,x)$ is the maximum number of blocks $B_1, B_2, \ldots, B_k$ of $\{1,\ldots,n\}$ such that for each B_i,

$$f(x) \neq f(x^{B_i})$$



$\underline{bs(f,x)} \geq \underline{s(f,x)}$

Then the block sensitivity $bs(f)$ of f is $\max\limits_x bs(f,x)$.

Conj (Nisan, Szegedy 1994; sensitivity conjecture)

for some c, $bs(f) \leq (s(f))^c$

for any boolean function f.

The best upper bound was due to Kenyon, Kutin 04 $bs(f) \leq O(e^{s(f)}\sqrt{s(f)})$

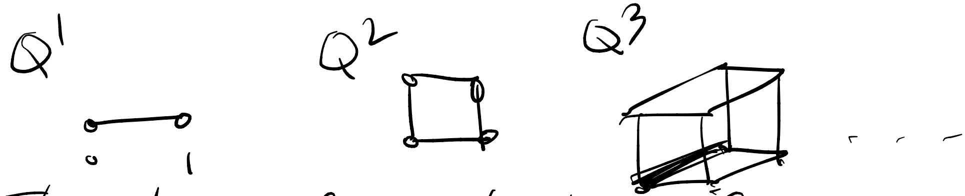
lower bound: $bs(f) = \Omega(s(f)^2)$

Q^n = graph on $\{0,1\}^n$ such that

x, y are adjacent iff they are different only on 1 coordinate,

$y = x^{\{i\}}$ for some i.

"hypercube" cube.



The degree of a boolean function is the degree of the unique multilinear real polynomial representing f. $\deg(f)$.

$$f(x_1, x_2, \dots, x_n) = x_1 x_2 x_3 + x_4 + 1 - x_5$$

$$f(\quad) = x_1 x_2 \cdots x_n$$

$$(1-x_1) - (1-x_a) \cdots$$

For a graph H
$\Delta(H)$ = max degree of vertices in H.

Thm (Nisan Szegedy 94)

$$bs(f) \leq 2\deg^2(f).$$

Thm (Gotsman Linial 92)
The following are equivalent for any increasing function.

(1) For every induced subgraph H of Q^n with $|V(H)| \neq 2^{n-1}$,

$$\Delta(H) \geq h(n) \text{ or } \Delta(Q^n - V(H)) \geq h(n)$$

(2) For every boolean function f,

$$s(f) \geq h(\deg(f))$$

If we have such a function h, then $\underline{bs(f)} \leq 2\deg^2(f) \leq \underline{2\left(h^{-1}(s(f))\right)^2}$

Thm (Huang 2019)

$\left(\begin{array}{l} H : \text{induced subgraph of } Q^n \\ |V(H)| = 2^{n-1}+1 \\ \Rightarrow \quad \Delta(H) \geq \sqrt{n}. \end{array} \right)$

Lemma. $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$A_n = \left(\begin{array}{c|c} A_{n-1} & I \\ \hline I & -A_{n-1} \end{array} \right)$$

Then A_n is a $2^n \times 2^n$ real symmetric matrix whose eigenvalues are $\sqrt{n}$ of multiplicity 2^{n-1} and $-\sqrt{n}$ of multiplicity 2^{n-1}.

Proof. Claim: $A_n^2 = nI$.

$A_1^2 = I$.

$$A_n^2 = \begin{pmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{pmatrix}^2 = \begin{pmatrix} A_{n-1}^2 + I & A_{n-1} - A_{n-1} \\ A_{n-1} - A_{n-1} & I + A_{n-1}^2 \end{pmatrix}$$

$$= \begin{pmatrix} (n-1)I+I & 0 \\ 0 & (n-1)I+I \end{pmatrix} = nI.$$

If λ is an eigenvalue of A_n, $A_n v = \lambda v$ for the corresponding eigenvector $v \neq 0$.

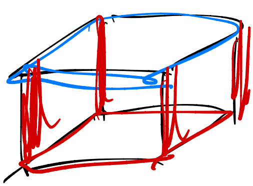
$$nv = A_n^2 v = A_n(\lambda v) = \lambda^2 v$$

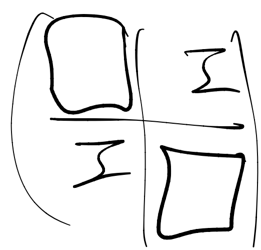
$$\Rightarrow \boxed{\lambda^2 = n} \qquad \lambda = \pm\sqrt{n}$$

$$\operatorname{tr}(A_n) = 2\operatorname{tr}(A_{n-1}) = 2^{n-1}\operatorname{tr}(A_1) = 0.$$

$\Rightarrow$ half of the eigenvalues are $\sqrt{n}$, another half are $-\sqrt{n}$. $\square$

What is A_n?





Lemma. Let H be an m-vertex graph on $\{v_1, \cdots, v_m\}$

Let $A = (a_{ij})$ be a symmetric $m \times m$ matrix such that $a_{ij} \in \{\pm 1, 0\}$ and

$a_{ij} \neq 0 \iff v_i$ is adjacent to v_j.

Let λ_1 be the largest eigenvalue of A.

Then $\lambda_1 \leq \Delta(H)$.

Proof. Let $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}$ be an eigenvector corresponding to λ_1.

$$Av = \lambda_1 v$$

Let us assume that $|v_1| \geq |v_i|$ for all i.

$$|\lambda_1||v_1| = |\lambda_1 v_1| = |(\lambda_1 v)_1| = |(Av)_1|$$

$$= \left|\sum a_{ij} v_j\right|$$

$$\leq \sum_{j \in N(v_1)} |a_{ij}||v_j|$$

$$\leq \deg(v_1) \cdot |v_1|.$$

$$\Rightarrow |\lambda_1| \leq \underline{\deg(v_1)} \leq \underline{\Delta(H)}. \qquad \square$$

Proof of the sensitivity conj.

Let H be an induced subgraph of Q^n with $2^{n-1}+1$ vertices

$$\underline{\lambda_{1+(2^n-(2^{n-1}+1))}(A_n) \leq \underbrace{\lambda_1(A_n[V(H)])}_{\text{largest eigenvalue of } H}} \leq \lambda_1(A_n)$$

by Cauchy's interlacing theorem

$\lambda_\ell(A_n) = \ell^{th}$ largest eigenvalue of A_n.

$\lambda_1(A_n[V(H)]) \geq \underline{2^{n-1}}$th largest eigenvalue of A_n

$$= \sqrt{n}.$$

$$\therefore \lambda_1(A_n[V(H)]) = \sqrt{n}$$

$$\Rightarrow \Delta(H) \geq \sqrt{n}. \qquad \square$$

As a corollary we have

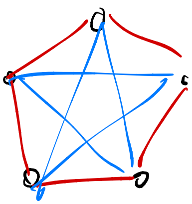
$$\boxed{bs(f) \leq 2 \cdot (s(f))^4.}$$

Chapter 9. Ramsey Theory

9.1. RAMSEY'S THEOREM

$R(k,l)$ = min N such that
for all $n \geq N$, any 2-edge-coloring
of K_n into red or blue creates
either a copy of a red K_k
or a blue K_l.

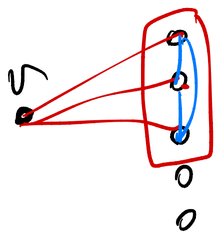
$R(3,3)$



K_5

$R(3,3) > 5$.

$R(3,3) \leq 6$

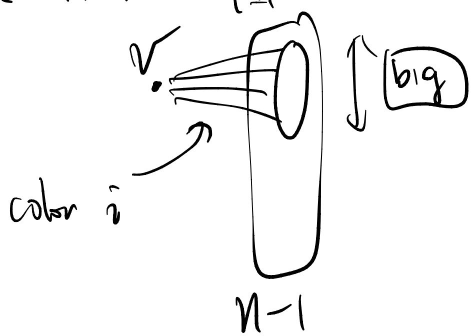


$R(n_1, n_2, n_3, \ldots, n_k)$ = min N such that
for all $n \geq N$,
any coloring of the edges of K_n
into k colors
creates a copy of K_{n_i}
whose all edges are colored by i
for some i.

Thm. $R(n_1, n_2, \ldots, n_k)$ exists.

Claim: $R(n_1, n_2, \ldots, n_k) \le \sum R(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k) - k + 2$

Suppose $n \ge N = \sum_{i=1}^{k} R(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k) - k + 2$



$R(n_1, n_2, \ldots, n_{i-1}, n_i - 1, n_{i+1}, n_{i+2}, \ldots, n_k)$

If there are $\ge R(\qquad)$ neighbors of v joined by edges of color i with v
then we have a copy of K_{n_j} of color j $\rightarrow$ DONE

Thus, we may assume
#such nbrs $\le R(\qquad) - 1$.

$\therefore\ n - 1 \le \sum_{i=1}^{k} \left(R(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k) - 1 \right)$

Contradiction.

1930 RAMSEY.

- Induction on $\sum_{i=1}^{k} n_i$
- $R(1, n_2, \ldots, n_k) = 1$, $R(2, n_2, \ldots, n_k) = R(n_2, \ldots, n_k)$.

Easy exercise.

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

$$R(1,1) = \binom{0}{0} = 1.$$

$$\begin{aligned} R(k,l) &\leq R(k-1,l) + R(k,l-1) \\ &\leq \binom{k+l-3}{k-2} + \binom{k+l-3}{k-1} \\ &= \binom{k+l-2}{k-1}. \end{aligned}$$

$\square$

$$\rightsquigarrow \quad R(k,l) \leq 2^{k+l-2}.$$

9.2. Ramsey's Theorem for sets.

Instead of coloring $\underline{\text{edges}}$ of K_n

$\Leftrightarrow$ $\underline{\text{2-subsets}}$ of $\{1, \dots, n\}$,

What happens if we color m-subsets of $\{1, \dots, n\}$?

Let $R(n_1, n_2, \dots, n_k ; m) =$ min N such that for all n,

if we color all m-subsets of $\{1, \dots, n\}$ into one of k colors,

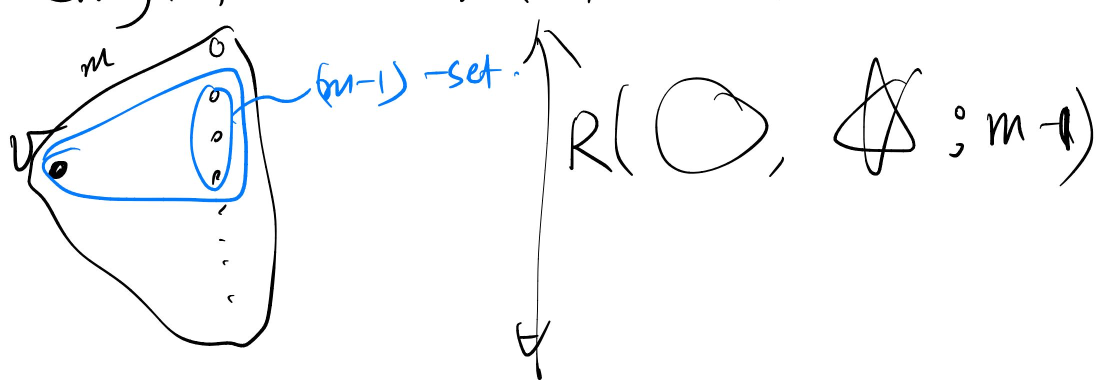
then there exist i and an n_i-subset X such that all m-subsets of X have color i.

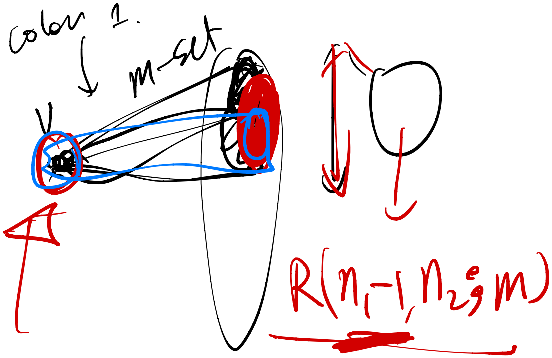
$\underline{\text{Thm}}$. $R(n_1, n_2, \dots, n_k ; m)$ exists.

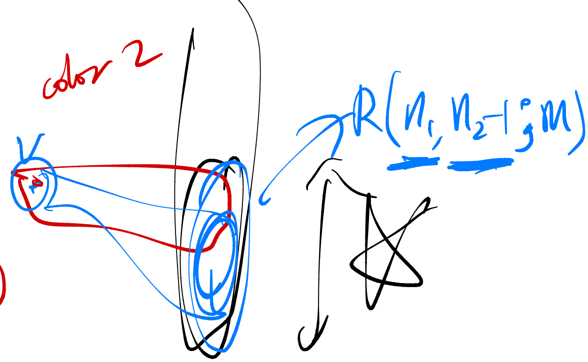
$\underline{\text{Proof}}$ We may assume $k=2$, because

$$R(n_1, n_2, \dots, n_k ; m) \le R\big(R(n_1, n_2, \dots, n_{k-1}; m), n_k ; m\big)$$

Enough to show $R(n_1, n_2 ; m)$ exists.







$$R(n_1, n_2; m) \le R\big(R(n_1-1, n_2; m)\ R(n_1, n_2; m); m-1\big) + 1$$

Consider the
Min Counterexample with
min m,
and then min $n_1 + n_2$.

"hypergraph".