Three proofs of the Cauchy-Davenport Inequality

Following Tao and Vu *Additive Combinatorics*

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Algebaic proof

Combinatorial proof

Fourier Analytic Proof

Additive sets

Definition

- An *additive set* is a pair (A, Z) where Z is a group and $A \subseteq Z$ is a finite, nonempty subset of Z
- Given two additive sets A, B we can define the *sum set* $A + B := \{a + b : a \in A,\ b \in B\}$

Example

$A = \{2, 3\}$, $B = \{4\}$, then $A + B = \{6, 7\}$

Nonexample

$A = \{2, 3\}$, $B = 2\mathbb{Z}$, then $A + B = \mathbb{Z}$, but B not finite.

Cauchy-Davenport theorem

Theorem (Cauchy, 1813 and Davenport, 1935)

Let p be a prime, and A, B are two additive sets in $\mathbb{Z}/p\mathbb{Z}$, then

$$|A + B| \geq \min(|A| + |B| - 1, p)$$

Cauchy-Davenport theorem

Example 1

Let $A = \{1, 3, 5, 9\}$, $B = \{2, 3, 4\}$ be additive sets with ambient group $\mathbb{Z}/11\mathbb{Z}$.
Then $A + B = \{3, 4, 5, 6, 7, 8, 9, 0, 1, 2\}$ so

$$|A + B| = 10 \geq \min(6, 11) \quad \checkmark$$

Cauchy-Davenport theorem

Example 2

Let $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8, 10\}$ in $\mathbb{Z}/11\mathbb{Z}$.

Then $A + B = \{3, 5, 7, 9, 0, 2, 4, 6\}$ so

$$|A + B| = 8 \geq \min(8, 11) \quad \checkmark$$

There are a few things to notice here:

- Compare to last example: $|A|$ is the same, $|B|$ is bigger, but $|A + B|$ is smaller
- "Consecutive" numbers in A, B have common difference of 2.
- Equality is achieved!

Arithmetic progression

- In $\mathbb{Z}$, we call sequence with a common difference of consecutive terms an arithmetic progression.
- In the language of additive sets, if we can write

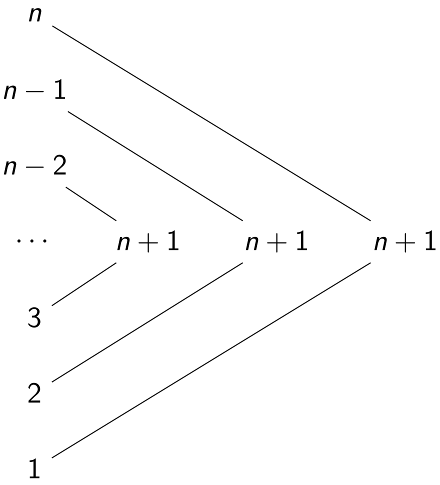
$$A = a + [0, n) \cdot r = \{a, a + r, a + 2r, ..., a + nr\}$$

  then we call A an *arithmetic progression*.
- From last example: $A = 1 + [0, 4) \cdot 2$
- A well known formula which makes me smile is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

  because of this proof:

Arithmetic progression



Each line has a value of $n + 1$.

When n is even, add up for each of $1, 2, \ldots, n$, but that has overcounted by double, so $\frac{n(n+1)}{2}$.

When n is odd, same idea but add up $n + 1$ $(n - 1)$-many times (omitting $\frac{n+1}{2}$) so we get $\frac{(n+1)(n-1)}{2} + \frac{n+1}{2}$.

Arithmetic progression

- It turns out that there is a similar formula for arithmetic progressions. If s_n is the nth partial sum for an arithmetic progression,
$$s_n = \frac{n(a_1 + a_n)}{2}$$
- Same proof
- $\{1, 2, ..., n\} = 1 + [0, n) \cdot 1$ is a special case

Vosper's theorem

Theorem

If $|A|, |B| \geq 2$, $|A + B| \leq p - 2$, the Cauchy-Davenport theorem achieves equality if and only if A, B are both arithmetic progressions with a common difference.

Proof $\emptyset$ of Cauchy-Davenport

- Augstin Louis Cauchy, 1813
- As far as I can tell, *Oevours complete d'Augustin Cauchy* starts in 1815.

Proof 1 of Cauchy-Davenport [Dav35, TV10]

Definition

Let A, B additive sets, $e \in A - B$ define e-transform $A_{(e)}, B_{(e)}$ as

$$A_{(e)} := A \cup (B + e) \supseteq A$$

$$B_{(e)} := B \cap (A - e) \subseteq B$$

Example

Let $A = \{1, 3, 5, 9\}$, $B = \{2, 3, 4\}$, $5 = 9 - 4 \in A - B$

$$A_{(5)} = \{1, 3, 5, 9\} \cup \{7, 8, 9\} = \{1, 3, 5, 7, 8, 9\}$$

$$B_{(5)} = \{2, 3, 4\} \cap \{7, 9, 0, 4\} = \{4\}$$

Proof 1 of Cauchy-Davenport [Dav35, TV10]

- $A_{(5)} + B_{(5)} = \{5, 7, 9, 0, 1, 2\} \subseteq A + B$
- $|A| + |B| = |A_{(5)}| + |B_{(5)}|$.
- These are true in general
- Upshot: e-transformation keeps the total size of the two sets the same, while making the sum set weakly smaller.

Proof 1 of Cauchy-Davenport [Dav35, TV10]

Lemma

If $A, B \subseteq Z$ are additive sets, $n, m \in \mathbb{Z}$ then:

$|A + nB - mB| = |A|$ if and only if there is a subgroup $G \leq Z$ so that $B \subseteq h + G$ and A is a union of cosets of G.

Proof.

Construct such a G. $\square$

Proof 1 of Cauchy-Davenport [Dav35, TV10]

Idea: fix A, see how B acts on A. Induct.

Proof.

Base case: If $|B| = 1$, then $|A + B| = |A| = |A| + |B| - 1 \leq p$. ✓
Induction step: Suppose we know the claim holds for $|B'| < |B|$.
Suppose $\exists e \in A - B$ so $|B_{(e)}| < |B|$. Then

$$\begin{aligned} |A + B| &\geq |A_{(e)} + B_{(e)}| \\ &\geq \min(|A_{(e)}| + |B_{(e)}| - 1, p) \\ &= \min(|A| + |B| - 1, p). \end{aligned}$$

Proof 1 of Cauchy-Davenport [Dav35, TV10]

Now suppose $|B_{(e)}| = |B| \; \forall e$.

- $B \cap (A - e) = B$
- $B \subseteq (A - e) \; \Leftrightarrow B + e \subseteq A$
- Adding *anything* in $A - B$ makes $B + e \subset A$
- So adding *everything* yields $B + (A - B) \subseteq A$
- By lemma, B in coset of subgroup of $\mathbb{Z}/p\mathbb{Z}$, A is union of cosets.
- Only subgroups of $\mathbb{Z}/p\mathbb{Z}$ are $\mathbf{0}$ and itself.

$G = \mathbf{0}$ case:
$|B| = 1$ so back in base case.

$G = \mathbb{Z}/p\mathbb{Z}$ case:
$|A| = p$ so $A + B = \mathbb{Z}/p\mathbb{Z}$, so $|A + B| = p$.

$\square$

A few comments

- This can be phrased nicely because of the *e*-transform. Davenport's proof required lots of keeping track of indices.
- This gives insight into why this is true for a cyclic group of *prime* order.

Warmup question

- Let f be a polynomial over a field. Bound the size of the set $A := \{\alpha : f(\alpha) = 0\}$.

$$|A| \leq \deg f$$

- Said another way, if $|A| > \deg f$, then $\exists a \in A$ with $f(a) \neq 0$
- Question: can we phrase a set (e.g. a sum set $A + B$) as the zero locus of a polynomial? This is the *polynomial method*.

Theorem (The combinatorial Nullstellensatz)

Let F be a field, $p \in F[t_1, t_2, ..., t_n]$ a degree-d polynomial which has nonzero coefficient of $t_1^{d_1} t_2^{d_2} ... t_n^{d_n}$ where $d = d_1 + \cdots d_n$. If $S_i \subset F$ with $|S_i| \geq d_i \ \forall i$, then there exists a tuple $\mathbf{x} = (x_1, x_2, ..., x_n) \in S_1 \times S_2 \times \cdots \times S_n$ for which $p(\mathbf{x}) \neq 0$.

Proof 2 of Cauchy-Davenport [TV10]

Lemma

Let $h \in \mathbb{F}_p[x, y]$. Let $k \geq 0$, and let A, B be additive sets in $\mathbb{F}_p$ with $|A| + |B| = k + 2 + \deg h$. If $(x + y)^k h(x, y)$ has a nonzero coefficient of $x^{|A|-1} y^{|B|-1}$, then

$$|\{\alpha + \beta : \alpha \in A,\ \beta \in B,\ h(\alpha, \beta) \neq 0\}| \geq k + 1$$

Proof.

Contradict combinatorial Nullstellensatz. $\square$

Proof 2 of Cauchy-Davenport [TV10]

Proof.

If $|A| + |B| > p$, then $|A + B| = \mathbb{Z}/p\mathbb{Z}$. If $|A| + |B| \leq p$, Consider the polynomial

$$f = (x + y)^{|A|+|B|-2} = \sum_{n=0}^{|A|+|B|-2} \binom{|A| + |B| - 2}{k} x^n y^{|A|+|B|-2-n}.$$

The coefficient of $x^{|A|-1}y^{|B|-1}$ is $\binom{|A|+|B|-2}{|A|-1}$. Since $|A| + |B| \leq p$, no factors of p appear in the binomial coefficient, so it is nonzero. Previous lemma with $h = 1$ tells us that

$$|\{\alpha + \beta : \alpha \in A,\ \beta \in B,\ h(\alpha, \beta) \neq 0\}| \geq |A| + |B| - 1$$

But since $h \neq 0$ always,

$$\{\alpha + \beta : \alpha \in A,\ \beta \in B,\ h(\alpha, \beta) \neq 0\} = A + B.$$

□

So what?

Define a *restricted sum set* $A \hat{+} B = \{a + b : a \in A, b \in B, a \neq b\}$

Conjecture (Erdos, Heilbronn 1964)

$$|A \hat{+} A| \geq \min(2|A| - 3, p)$$

- Proved 1994 by da Silva, Hamidoune.
- Stronger result for $A \hat{+} B$ proved in 1996 by Alon, Nathanson, Ruzsa

Proof.
Take $h = (x - y)$, and apply lemma. $\square$

Similar technique can be used to give results in particle physics.

Probabilistic method

Philosophy

If you can prove something has the right probability in an appropriate space, that can interpreted as proof.

Example

Assign n balls to m bins at (uniform) random. Let P be the probability that any of the bins contain two or more balls. If $P = 1$, this is the pigeonhole principle.

Example

Existence proof: If I can draw something at random with nonzero probability, it must exist.

Usage

Fourier analysis can be framed as statements about probability and expectation.

Fourier analysis definitions

- Let p be a prime, $f, g : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$
- $\hat{f}(\xi) = \frac{1}{p} \sum_{x \in \mathbb{Z}/p\mathbb{Z}} f(x) e^{-2\pi i x \xi / p}$
- $(f * g)(\xi) = \frac{1}{p} \sum_{x \in \mathbb{Z}/p\mathbb{Z}} f(x - \xi) g(\xi) = \frac{1}{p} \sum_{x \in \mathbb{Z}/p\mathbb{Z}} f(\xi) g(x - \xi)$
- $\operatorname{supp} f = \{x \in \mathbb{Z}/p\mathbb{Z} : f(x) \neq 0\}$

Upshots

- $\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f) + \operatorname{supp}(g)$ as an additive set.
- $\widehat{f * g} = \hat{f} \cdot \hat{g}$ (among other standard identities)

Fourier analysis theorems

Theorem (Tao, '05)

Let p be a prime, $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ a random variable. Then $|\operatorname{supp} f| + |\operatorname{supp} \hat{f}| \geq p + 1$.

Theorem (Tao, '05)

Let A, B, be nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$ with $|A| + |B| \geq p + 1$. Then $\exists$ a function $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ such that $\operatorname{supp} f = A$ and $\operatorname{supp} \hat{f} = B$.

Proofs are not enlightening, so omitted.

Proof 3 of Cauchy-Davenport [TV10]

Let A, B be additive sets in $\mathbb{Z}/p\mathbb{Z}$. We can choose sets X, Y so that

- $|X| = p + 1 - |A|$
- $|Y| = p + 1 - |B|$
- $|X \cap Y| = \max(|X| + |Y| - p, 1)$.

Example

Let $A = \{1, 3, 5, 9\}, B = \{2, 3, 4\} \subseteq \mathbb{Z}/11\mathbb{Z}$ so we want $|X| = 8$ and $|Y| = 9$, and $|X \cap Y| = 6$

$$X = \boxed{0, 1, 2, 3, 4, 5}, \boxed{6, 7}, \cancel{8}, \cancel{9}, \cancel{10}$$
$$Y = \boxed{0, 1, 2, 3, 4, 5}, \cancel{6}, \cancel{7}, \boxed{8, 9, 10}$$

The point is that the only data from A and B which we retain is their size.

Proof 3 of Cauchy-Davenport [TV10]

By previous theorem, since $|A| + |X| = p + 1$, there is:

- A function f with $\operatorname{supp} f = A$ and $\operatorname{supp} \hat{f} = X$.
- g with $\operatorname{supp} g = B$ and $\operatorname{supp} \hat{g} = Y$.

Now convolve $f * g$. We know

- $\operatorname{supp}(f * g) \subseteq \operatorname{supp} f + \operatorname{supp} g = A + B$
- $\operatorname{supp}(\widehat{f * g}) = \operatorname{supp}(\hat{f} \cdot \hat{g}) = X \cap Y$

Then

$$\begin{aligned}
|\operatorname{supp}(f * g)| + |\operatorname{supp}(\widehat{f * g})| &\geq p + 1 \\
|A + B| + |X \cap Y| &\geq p + 1 \\
|A + B| &\geq p + 1 - \max(|X| + |Y| - p, 1) \\
&= p + 1 - \max(p + 2 - |A| - |B|, 1) \\
&= \min(|A| + |B| - 1, p)
\end{aligned}$$

□

Poll

Which was your favorite?

1. Original proof exploiting (sub)group structure of $\mathbb{Z}/p\mathbb{Z}$?
2. Proof counting the zeros of a certain polynomial?
3. **Exploiting the relationship between a Fourier transform and its convolution?**

Thank you!

References

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