2.5. Alon-Babai-Suzuki theorem.

Thm 4. (Alon, Babai, Suzuki 1991)

Let p be a prime.

Let $L \subseteq \{0, 1, \dots, p-1\}$, $|L| = s$

Assume $s + k \leq n$.

Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \dots, n\}$.

If

(1) $|A| \equiv k \pmod{p}$ for all $A \in \mathcal{F}$

(2) $k \notin L + p\mathbb{Z}$

(3) $|A \cap B| \in L + p\mathbb{Z}$ for all distinct $A, B \in \mathcal{F}$.

then

$$|\mathcal{F}| \leq \binom{n}{s}$$

Proof. Let $\mathcal{F} = \{A_1, \dots, A_m\}$ $|A_i| \equiv k \pmod{p}$

Assume $0 \leq k < p$.

Let $a_i \in \mathbb{F}_p^n$ be the characteristic vector of A_i

$$L = \{l_1, l_2, \dots, l_s\}$$

We define $f_i(x) = \prod_{j=1}^{s} (x \cdot a_i - l_j)$ for $x \in \mathbb{F}_p^n$.

Let $\tilde{f_i}$ be the multilinear polynomial obtained from f_i by repeatedly replacing x_j^2 to x_j.

$$\tilde{f_i}(x) = f_i(x) \quad \text{when } x \in \{0,1\}^n.$$

For every subset I of $\{1, 2, \dots, n\}$ with $|I| < s$,

define $g_I(x) = \left(\sum_{i=1}^{n} x_i - k\right) \prod_{i \in I} x_i$ for $x \in \mathbb{F}_p^n$.

Let $\tilde{g}_I$ be the multilinear polynomial obtained from g_I by repeatedly reducing x_j^2 to x_j.

Again, $\tilde{g}_I(x) = g_I(x)$ when $x \in \{0,1\}^n$.

Claim: $\tilde{f}_1, \dots, \tilde{f}_m$, $\tilde{g}_I$ for $I \subseteq \{1, \dots, n\}$, $|I| < s$ are linearly independent.

If true, then

$$m + \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{s-1} \le \underline{\sum_{i=0}^{s} \binom{n}{i}}$$

$$\Longrightarrow m \le \binom{n}{s}$$

Suppose that

$$\sum_{i=1}^{m} \alpha_i \tilde{f}_i(x) + \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| < s}} \beta_I \tilde{g}_I(x) \equiv 0 \pmod{p}$$

Take $x = a_j \Rightarrow$ $\tilde{g}_I(a_j) = g_I(a_j) \equiv 0 \pmod{p}$

$\Rightarrow$ $\tilde{f}_i(a_j) = f_i(a_j) \begin{cases} \not\equiv 0 \pmod{p} & \text{if } i = j \\ \equiv 0 & \text{otherwise.} \end{cases}$

$$\Rightarrow \alpha_j \tilde{f}_j(a_j) \equiv 0 \pmod{p}$$

$$\Rightarrow \alpha_j \equiv 0 \pmod{p}$$

Suppose that there is I such that $\beta_I \not\equiv 0$

Then choose a minimal subset I_0 such that $\beta_{I_0} \not\equiv 0 \pmod{p}$

$|I_0| < k$

Let x^* be the characteristic vector of I_0.

$x^* \in \mathbb{F}_p^n$ $\qquad x^*_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\cdots i \in I_0$, $\cdots j \notin I_0$.

- $\beta_I \equiv 0 \pmod p$ if $I \subsetneq I_0$
- $\tilde{g}_I(x^*) = 0$ if $I \notin I_0$

$$\Rightarrow \sum_{I \subseteq I_0} \beta_I \tilde{g}_I(x^*) \equiv 0 \pmod p$$

$$\Rightarrow \beta_{I_0} \tilde{g}_{I_0}(x^*) \equiv 0 \pmod p$$

$$\tilde{g}_{I_0}(x^*) = g_{I_0}(x^*) = (|I_0| - k)$$

$$\therefore \beta_{I_0} (|I_0| - k) \equiv 0 \pmod p$$

$$\Rightarrow |I_0| \equiv k \pmod p$$

$k < p$, $|I_0| < s \leq p \Rightarrow |I_0| = k$.

We assume $\beta_I = 0$ if $|I| \geq s$.

Subclaim: If $|J| \not\equiv k \pmod p$,
then
$$\sum_{T \subseteq J} \beta_T \equiv 0 \pmod p$$

Proof Let x^* be the characteristic vector of J in $\mathbb{F}_p^n$.

$$0 \equiv \sum_I \beta_I \tilde{g}_I(x^*) = \sum \beta_I \, g_I(x^*)$$

$$= \sum_{I \subseteq J} \beta_I \left(|J| - k \right) \pmod p$$

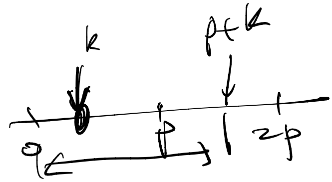
Since $|J| \not\equiv k \pmod p$, $\sum_{I \subseteq J} \beta_I \equiv 0 \pmod p$.

Now, let J_0 be a subset of $\{1, 2, \ldots, n\}$ such that $|J_0| = k+s$

$I_0 \subseteq J_0$

$(k+s \leq n)$

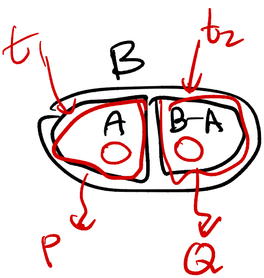
Since $|J_0| < k+p$,



$$\sum_{I \subseteq J} \beta_I \equiv 0 \pmod p \quad \text{for all } \underline{J \subseteq J_0, \ |J| \not\equiv k}$$

Recall Möbius inversion for sets.

- $f : 2^B \to \mathbb{F}$
- $g(X) = \sum_{T \subseteq X} f(T)$.



$$\boxed{\Rightarrow \sum_{A \subseteq T \subseteq B} \underbrace{(-1)^{|B|-|T|}}_{\mu(s_2, t_2)} g(T) = \sum_{B-A \subseteq T \subseteq B} f(T).}$$

Think of P as the set of subsets of A
Q as the set of subsets of $B-A$
$F(X, Y) := f(X \cup Y)$

Take $A = I_0$, $B = J_0$. We obtain

$|J_0| = k + s$

$$\sum_{I_0 \subseteq T \subseteq J_0} (-1)^{|J_0|-|T|} \left(\sum_{U \subseteq T} \beta_U \right) = \sum_{J_0 - I_0 \subseteq T \subseteq J_0} \beta_T$$

$= 0$ unless $|T| = k$

$$(\bmod p) \quad 0 \not\equiv (-1)^{|J_0| - |I_0|} \beta_{I_0} = \sum_{J_0 - I_0 \subseteq T \subseteq J_0} \beta_T$$

$\boxed{|J_0 - I_0| = s}$ $\Longrightarrow$ We already assumed that $\beta_T = 0$ whenever $|T| \geq s$.

$$\Longrightarrow \sum_{J_0 - I_0 \subseteq T \subseteq J_0} \beta_T = 0$$

Contradiction.

$\square$

2.6. k-wise intersection

Thm. (Grolmusz, Sudakov 2002)

p: prime, $L \subseteq \{0, 1, \dots, p-1\}$, $|L| = s$

$k \geq 2$

Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \dots, n\}$ such that

(1) $|A| \notin L + p\mathbb{Z}$ for all $A \in \mathcal{F}$

(2) $|A_1 \cap A_2 \cap \dots \cap A_k| \in L + p\mathbb{Z}$ for all distinct $A_1, \dots, A_k \in \mathcal{F}$.

Then

$$|\mathcal{F}| \leq (k-1) \sum_{i=0}^{s} \binom{n}{i}$$

Proof. $L = \{l_1, \dots, l_s\}$

Let $\mathcal{F}_0 = \mathcal{F}$

We construct $\mathcal{F}_1, \mathcal{F}_2, \dots$

For $i = 1, 2, \dots$, if $\mathcal{F}_{i-1} \neq \phi$, then let $A_1, A_2, \dots, A_d$ be a maximal subfamily of $\mathcal{F}_{i-1}$ such that

$$\left| \bigcap_{j=1}^{d'} A_j \right| \notin L + p\mathbb{Z}$$

for all $\boxed{1 \leq d' \leq d}$

Then such a subfamily exists and $1 \leq d < k$.

let $S_i := A_1$, $T_i = A_1 \cap A_2 \cap \cdots \cap A_d$.

let $\mathcal{F}_i = \mathcal{F}_{i-1} \setminus \{A_1, A_2, \ldots, A_d\}$.

Each time we remove at most $k-1$ members from $\mathcal{F}$

$\Rightarrow$ there is $m \geq \frac{|\mathcal{F}|}{k-1}$ such that $(S_1, T_1), (S_2, T_2), \ldots, (S_m, T_m)$ are defined.

$|S_i \cap T_i| = |T_i| \notin L + p\mathbb{Z}$

If $j > i$ $|S_j \cap T_i| \in L + p\mathbb{Z}$.

by the maximality of $A_1 \ldots A_d$

$|A_1 \cap A_2 \cap \cdots \cap A_d \cap S_j| \in L + p\mathbb{Z}$

let s_i, t_i be the characteristic vectors of S_i, T_i respectively in $\mathbb{F}_p^n$.

Let $f_i(x) = \prod_{j=1}^{s} (x \cdot t_i - l_j)$ for $x \in \mathbb{F}_p^n$.

Define $\tilde{f}_i$ as a multilinear polynomial obtained from f_i by replacing x_j^2 to x_j for each j.

$\Rightarrow$ $\tilde{f}_i(x) = f_i(x)$ for all $x \in \{0,1\}^n$

Claim: $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_m$ are linearly indep.

If not, $\sum_{i=1}^{m} \alpha_i \tilde{f}_i(x) = 0$

Choose ~~minimum~~ maximum j such that $\alpha_j \neq 0$

$$\sum_{i=1}^{m} \alpha_i \tilde{f}_i(s_j)$$

$$= \underbrace{\sum_{i=1}^{j-1} \alpha_i \tilde{f}_i(s_j)}_{} + \alpha_j \tilde{f}_j(s_j) + \sum_{i=j+1}^{m} \alpha_i \hat{f}_i(s_j)$$

0

0

If $j > i \Rightarrow |S_j \cap T_i| \in L + p\mathbb{Z}$

$\Rightarrow f_i(t_j) = 0 \Rightarrow \hat{f}_i(t_j) = 0$

$$= \alpha_j \underbrace{\tilde{f}_j(s_j)}_{\neq 0} = 0$$

$|S_j \cap T_j| \notin L + p\mathbb{Z}$

$\Rightarrow \alpha_j = 0$ Contradiction.

$$\frac{|\mathcal{F}|}{k-1} \le m \le \text{dim of } \overset{\text{subspace of}}{\text{multilinear poly}} \text{ of deg } \le s \text{ with } n \text{ variables}$$

$$= \sum_{i=0}^{s} \binom{n}{i}$$

$$\therefore |\mathcal{F}| \le (k-1) \sum_{i=0}^{s} \binom{n}{i}.$$

<u>Thm</u> (Grolmusz, Sudakov 2002)

$|L| = s$, $k \ge 2$.

$\mathcal{F}$: family of subsets of $\{1, \ldots, n\}$

Such that $|A_1 \cap \cdots \cap A_k| \in L$

for all $\underset{\text{distinct}}{A_1, \ldots, A_k} \in \mathcal{F}$,

Then

$$|\mathcal{F}| \le (k-1) \sum_{i=0}^{s} \binom{n}{i}$$

Proof. $\mathcal{F}_0 = \mathcal{F}$.

For $i = 1, 2, \ldots$, if $\mathcal{F}_{i-1} \ne \phi$, then we construct $\mathcal{F}_i$ as follows.

- If $|A \cap B| \in L$ for all distinct $A, B \in \mathcal{F}_{i-1}$ then take S_i = largest set in $\mathcal{F}_{i-1}$

  $T_i = S_i$

Let $\mathcal{F}_i = \mathcal{F}_{i-1} - \{S_i\}$.

- Otherwise, there exists a maximal subfamily of $\mathcal{F}_{i-1}$,

$A_1, A_2, \dots, A_d$ such that

$$\left|\bigcap_{j=1}^{d'} A_j\right| \notin L$$

for all $2 \le d' \le d$.

$d < k$

Let $\mathcal{F}_i = \mathcal{F}_{i-1} - \{A_1, \dots, A_d\}$

$S_i = A_1$

$T_i = A_1 \cap A_2 \cap \dots \cap A_d$

We obtain $(S_1, T_1)\ (S_2, T_2) \dots (S_m, T_m)$

where $m \ge \dfrac{|\mathcal{F}|}{k-1}$

$|S_i \cap T_i| = |T_i|$ for all i,

$$\left\{\begin{array}{l} |S_j \cap T_i| < |T_i| \ \cancel{\text{or}} \text{ and} \\ |S_j \cap T_i| \in L \end{array}\right\} \text{ when } j > i.$$

let $f_i(x) = \prod_{l_j < |T_i|} (x \cdot t_i - l_j)$

$$f_i(s_i) = \prod_{l_j < |T_i|} (s_i \cdot t_i - l_j)$$

$$= \prod (|S_i \cap T_i| - l_j)$$

$$= \prod_{l_j < |T_i|} (|T_i| - l_j) \neq 0.$$

$j > i$ $\quad f_i(s_j) = \prod_{l_j < |T_i|} (\underline{|S_j \cap T_i|} - l_j) = 0$

Define $\tilde{f}_1 \; \tilde{f}_2 \; \ldots \; \tilde{f}_m$ as before.

$$\tilde{f}_i(s_i) \neq 0 \qquad \tilde{f}_i(s_j) \overset{!}{=} 0 \quad \text{for all } j > i$$

$\Rightarrow \tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_m$ are linearly indep

$$\frac{|\mathcal{F}|}{k-1} \leq m \leq \sum_{i=0}^{s} \binom{n}{i}$$

$$\therefore |\mathcal{F}| \leq (k-1) \sum_{i=0}^{s} \binom{n}{i}. \qquad \square$$

3. Bollobás' Two Families Theorem

3.1. Sperner's theorem and LYM inequality

A family of subsets is an <u>antichain</u> if $X \not\subseteq Y$ for all distinct members X, Y.

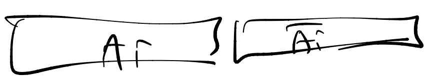
Thm (Sperner 1928)
If $\mathcal{F}$ is an antichain of subsets of $\{1, 2, \ldots, n\}$ then

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Thm (LYM ; Lubell 66, Yamamoto 54, Meschalkin 63)
If $\{A_1, A_2, \ldots, A_m\}$ is an antichain and subsets of $\{1, \ldots, n\}$, then

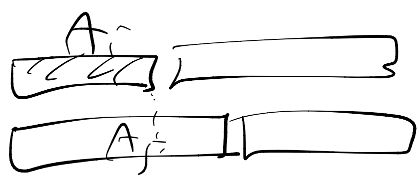
$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \le 1$$

<u>Proof</u>. Consider $n!$ permutations of $\{1, \ldots, n\}$.
For A_i, let P_i be the set of permutations of $\{1, \ldots, n\}$ that starts with A_i.



$|P_i| = |A_i|!\,(n-|A_i|)!$

$P_i \cap P_j = \emptyset$



$$\sum |P_i| \leq n!$$

$$\sum |A_i|!\,(n-|A_i|)! \leq n!$$

$$\sum \frac{1}{\frac{n!}{|A_i|!(n-|A_i|)!}} \leq 1$$

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \leq 1$$

$\square$

Sperner:

$$\binom{n}{|A_i|} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \sum \frac{1}{\binom{n}{|A_i|}} \leq 1$$

$$= m \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

$$\therefore m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

3.2. Bollobás' Two Families Theorem

Thm (Bollobas 1965)

Let $A_1, A_2, \ldots, A_m$

$B_1, B_2, \ldots, B_m$ be subsets of $\{1, 2, \ldots, n\}$

such that

(1) $A_i \cap B_i = \phi$ for all i

(2) $A_i \cap B_j \neq \phi$ for all $i \neq j$

Then

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1 .$$

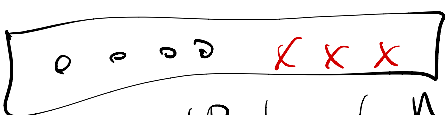
Why does it imply LYM?

If $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ is an antichain

then take $B_i = \{1, 2, \ldots, n\} - A_i$.

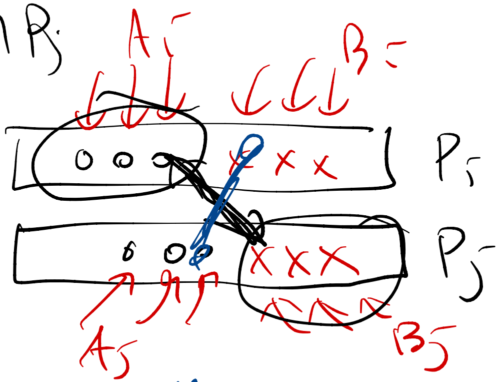
$$\Longrightarrow \sum \frac{1}{\binom{n}{|A_i|}} \leq 1$$

<u>Proof</u>. For each i, let P_i be the set of all permutations of $\{1, 2, \ldots, n\}$ such that all elements of A_i occur before every element of B_i.



$$|P_i| = \binom{n}{|A_i|+|B_i|} |A_i|! \, |B_i|! \, (n - |A_i| - |B_i|)!$$

$P_i \cap P_j$



$P_i \cap P_j = \phi$

$$\sum_{i=1}^{m} |P_i| \le n!$$

$$\sum \binom{n}{|A_i|+|B_i|} |A_i|!\, |B_i|!\, (n-|A_i|-|B_i|)! \le n!$$

$$\sum \frac{1}{\frac{(|A_i|+|B_i|)!}{|A_i|!\,|B_i|!}} \le 1$$

$$\sum \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \le 1 \quad \square$$

Thm. $\left.\begin{matrix} A_1, A_2, \dots, A_m \\ B_1, B_2, \dots, B_m \end{matrix}\right\}$ subsets of $\{1, \dots, n\}$

$|A_i| = r$, $|B_i| = s$ for all i

(1) $A_i \cap B_i = \emptyset$

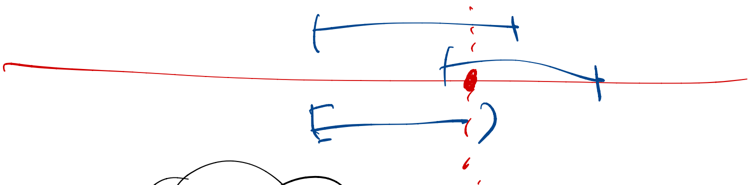
(2) $A_i \cap B_j \neq \phi$ for all $\boxed{i \neq j}$

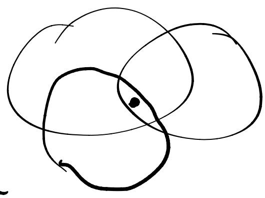
$\Rightarrow \quad m \le \binom{r+s}{r}$.

3.3. Helly-type theorems for finite sets.

Helly's theorem: $\mathcal{F}$: family of convex sets in $\mathbb{R}^n$.
If every set of $\le n+1$ members have a common intersection point
then all members have a common intersection point.

$\boxed{\mathbb{R}^1}$





Lemma. Let $\mathcal{F}$ be an r-uniform family of sets.
If every set of $\le r+1$ members has a common element
then all members have a common element.

Proof. Suppose not.
Choose a counterexample with min $|\mathcal{F}|$.
Let $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$.

$m \ge r+2$. $\quad \left(\bigcap_{i=1}^{m} A_i = \phi\right)$

Since $\mathcal{F} - \{A_i\}$ is not a counterexample,
there is $b_i \in \bigcap_{j \ne i} A_j$.

If $b_1, b_2, \ldots, b_m$ are all distinct

then A_1 contains $b_2, b_3, \ldots, b_m$.

But $|A_1| = r$. Contradiction.

$(m \geq r+2)$

So, there are $i \neq j$ such that $b_i = b_j$.

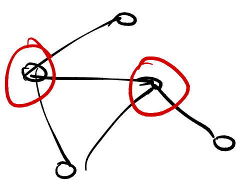
Then all members contain b_i. $\square$

<u>Thm</u> (Erdős, Hajnal, Moon 1964)

Let G be a graph.

If any set of $\leq \binom{s+2}{2}$ edges can be covered by s vertices,

then all edges can be covered by s vertices.



A set S of vertices covers a set X of edges if every edge in X is incident with some vertex in S.

$\boxed{K_{s+2}}$ — $\binom{s+2}{2}$ is tight.

Thm (Bollobás 1965)

Let H be an r-uniform hypergraph.
If any set of $\leq \binom{r+s}{r}$ edges can be covered by s vertices,
then all edges can be covered by s vertices.

Proof. Suppose not. Take H as the minimum counterexample.
Then no set of s vertices can cover all edges of H.
Let $A_1, A_2, \ldots, A_m$ be the edges of H.
For each A_i there is a set B_i of s vertices covering $H - A_i$

$$A_i \cap B_i = \emptyset$$

$$A_i \cap B_j \neq \emptyset$$

Apply Two Families Theorem

$$\Longrightarrow \quad m \leq \binom{r+s}{r}.$$

Then by the assumption,
$A_1, A_2, \ldots A_m$ can be covered by s vertices, contradiction $\square$.