Please type your homework solution into a PDF file and submit to the gradescope.com website by 11:59pm KST of the due date. A point will be deducted for each problem if a handwritten solution is submitted. We recommend using LaTeX. You can find useful hints at our KLMS website. It is recommended to use the sample template. (At least, make sure that each problem has a solution in separate pages.) Unprofessional proofs may get a deduction of points, even if the solution is mathematically correct or can be made correct.

2021 SPRING MAS575 *Combinatorics*
HOMEWORK 6

DUE: JUNE 8, 2021

Note that $\mathbb{N}$ is the set of positive integers.

6.1. Prove that for every positive integer r and every r-coloring of $\mathbb{N}$, there exist three positive integers x, y, and z such that x, $x+y$, z, and $x+yz$ have the same color.

6.2. Prove that for every positive integer r and every r-coloring of $\mathbb{N}$, there exist three distinct positive integers x, y, and z of the same color such that $xy^2 = z^3$.

6.3. Prove that for every positive integer k, there exists a prime p such that there are k consecutive quadratic residue modulo p.

(An integer q is a quadratic residue modulo p if $q \equiv x^2 \pmod{p}$ for some integer x.)

6.4. Let n be a positive integer. Prove that there is a $(2n)$-coloring χ of all rational numbers such that

$$\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i = 1$$

has no rational solutions such that $\chi(x_i) = \chi(y_i)$ for all $i = 1, 2, \ldots, n$.

6.5. Prove that the following are equivalent.

1. For every positive integer r and every r-coloring of $\mathbb{N}$, there exist distinct positive integers x_1, x_2, …, x_n of the same color such that $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$.
2. For every positive integer r and every r-coloring of $\mathbb{N}$, there exist positive integers x_1, x_2, …, x_n of the same color such that
$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$$
and there are distinct integers λ_1, λ_2, …, λ_n such that $c_1\lambda_1 + c_2\lambda_2 + \cdots + c_n\lambda_n = 0$.