KAIST

2021 MAS575 Combinatorics

Homework 6

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HW 6.1

Prove that for every positive integer r and every r-coloring of $\mathbb{N}$[1], there exist three positive integers x, y, and z such that x, $x+y$, z, and $x+yz$ have the same color.

Proof. We do induction on r. When $r=1$, it is trivial. Now we suppose that for every $(r-1)$-coloring of $\mathbb{N}$, there exist three positive integers x_0, y_0, and z_0 such that x_0, x_0+y_0, z_0, and $x_0+y_0z_0$ have the same color. In particular, we may let $M=M(r-1)$ be a positive integer such that we can always find such $x_0, y_0, z_0 \in [M]$. Recall van der Waerden's Theorem.

Theorem 1.1 (van der Waerden's Theorem)**.** *For any $k, l \in \mathbb{N}$, there exists $N_v = N_v(k,l) \in \mathbb{N}$ such that any k-coloring of $[N]$ with $N \geq N_v$ creates a monochromatic l-AP.*

Fix any r-coloring $c : \mathbb{N} \to [r]$. Apply the above van der Waerden's Theorem with $k=r$ and a sufficiently large l. Then we have a monochromatic l-AP in $\mathbb{N}$: $a, a+d, \ldots, a+(l-1)d$. WLOG, we may assume that they all have color r. If we let $x=a$ and let $y=d$, then it suffices to find $z \in [l-1]$ such that z has color r. So we may assume that for all $t \in [l-1]$, t has a different color with the above AP, i.e., t is colored by some color in $[r-1]$. By the fact that l is sufficiently large ($l \geq M+1$ suffices) and the induction hypothesis, there exists $x_0, y_0, z_0 \in [l-1]$ such that x_0, x_0+y_0, z_0, and $x_0+y_0z_0$ have the same color, completing the proof. $\square$

[1] $\mathbb{N} = \mathbb{Z}^+$

HW 6.2

Prove that for every positive integer r and every r-coloring of $\mathbb{N}$, there exist three distinct positive integers x, y, and z of the same color such that $xy^2 = z^3$.

Remark 2.1. *It suffices to show that for every positive integer r and every r-coloring of $\mathbb{N}$, there exist $x, k \in \mathbb{N}$ with $k > 1$ such that x, xk^2, and xk^3 have the same color (then we let $y = xk^3$ and let $z = xk^2$).*

Proof. Fix an r-coloring of $\mathbb{N}$ and consider the set $T = \{2^t : t \in \mathbb{N}\}$. By van der Waerden's Theorem with $k = r$ and $l = 4$, as well as the isomorphism between $\mathbb{N}$ and T, there exists a monochromatic sequence $2^a, 2^{a+d}, 2^{a+2d}, 2^{a+3d}$ with $a, d \in \mathbb{N}$. Then we let $x = 2^a$, let $y = 2^{a+3d}$, and let $z = 2^{a+2d}$, clearly $xy^2 = z^3$, completing the proof. $\square$

HW 6.3

Prove that for every positive integer k, there exists a prime p such that there are k consecutive quadratic residues modulo p.

Proof. Recall HW 5.4, a generalization of van der Waerden's Theorem.

Lemma 3.1. *Let t, r be positive integers. There exists a number $N_l = N_l(t, r)$ such that any r-coloring of numbers in $[N]$ contains an arithmetic progression $a, a+d, \ldots, a+(t-1)d \in [N]$ of length t $(d \neq 0)$ such that $a, a+d, \ldots, a+(t-1)d$ and d have the same color.*

Also recall Euler's criterion.

Lemma 3.2. *Let p be an odd prime and a be an integer coprime to p. Then*

$$a^{\frac{p-1}{2}} \equiv \begin{cases} 1 & \pmod{p}, \textit{if } a \textit{ is a quadratic residue modulo } p, \\ -1 & \pmod{p}, \textit{otherwise.} \end{cases}$$

Remark 3.1. *Lemma 3.2 implies that gh is a quadratic residue module p if both g and h are quadratic residues or both are quadratic non-residues.*

Let p be a sufficiently large prime to be specified later. Define a 2-coloring $c : [p-1] \to \{1, -1\}$ such that $c(x) \equiv a^{\frac{p-1}{2}} \pmod{p}$. By the definition of N_l in Lemma 3.1, if we require $p > N_l(k, 2)$, then there exists a monochromatic k-AP $a, a+d, \ldots, a+(k-1)d$ with $d \neq 0$ also having the same color. Let $e \in [p-1]$ be the integer such that $de \equiv 1 \pmod{p}$. Note that e is a quadratic residue modulo p if and only if d is a quadratic residue module p. Therefore, by Remark 3.1, we obtain k consecutive quadratic residues modulo p, which are $ae, ae+1, \ldots, ae+k-1$, by multiplying each term in the monochromatic k-AP with e, completing the proof. $\square$

HW 6.4

Let n be a positive integer. Prove that there is a $(2n)$-coloring χ of all rational numbers such that

$$\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i = 1$$

has no rational solutions such that $\chi(x_i) = \chi(y_i)$ for all $i \in [n]$.

Remark 4.1. *When $n = 1$, let $\chi_1 : \mathbb{Q} \to [2]$ be a 2-coloring on all rational numbers. We first fix any 2-coloring for the rational numbers in the interval $[0, 1)$ and for any rational number t outside the interval, we let $m \in [0, 1)$ be the unique number such that $t - m \in \mathbb{Z}$ and set $\chi_1(t) = \chi_1(m)$ if $t - m \equiv 0 \pmod 2$ and set $\chi_1(t) = 3 - \chi_1(m)$ otherwise. It is easy to see that such a coloring satisfies the desired condition. In particular, the "initial" coloring for numbers in $[0, 1)$ could be monochromatic.*

Proof. We define $\chi_n : \mathbb{Q} \to [2n]$ by $\chi_n(t) = i$, where $i \in [2n]$ is the unique number such that

$$2m + \frac{i-1}{n} \le t < 2m + \frac{i}{n}$$

for some $m \in \mathbb{Z}$. Now suppose that there exist $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{Q}$ such that

$$\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i = 1$$

and $\chi_n(x_i) = \chi_n(y_i)$ for all $i \in [n]$, then we have

$$2m_j - \frac{1}{n} < x_j - y_j < 2m_j + \frac{1}{n}$$

for some $m_j \in \mathbb{Z}$, for all $j \in [n]$. Let $M = \sum_{i \in [n]} m_i \in \mathbb{Z}$ and we have

$$2M - 1 < \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i = 1 < 2M + 1,$$

which is impossible for all $M \in Z$, completing the proof. $\square$

HW 6.5

Prove that the following two statements are equivalent, where c_i is a nonzero integer for all $i \in [n]$.

1. For every positive integer r and every r-coloring of $\mathbb{N}$, there exist distinct positive integers $x_1, x_2, \ldots, x_n$ of the same color such that $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$.

2. For every positive integer r and every r-coloring of $\mathbb{N}$, there exist positive integers $x_1, x_2, \ldots, x_n$ of the same color such that $c_1x_1+c_2x_2+\cdots+c_nx_n = 0$ and there are distinct integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $c_1\lambda_1 + c_2\lambda_2 + \cdots + c_n\lambda_n = 0$.

Remark 5.1. *The first part of statement* (2) *exactly says that* $c_1x_1+c_2x_2+\cdots+c_nx_n = 0$ *is **regular**.*

Remark 5.2. *From* (2) *we have a monochromatic solution* $x_1, x_2, \ldots, x_n$*, we may hope that there exists some* d *such that* $x_1 + \lambda_1 d, x_2 + \lambda_2 d, \ldots, x_n + \lambda_n d$ *is also a monochromatic and distinct solution.*

Proof. References:

- `https://www.cs.umd.edu/users/gasarch/COURSES/858/S20/notes/distradotalk.pdf`;
- `https://www.cs.umd.edu/users/gasarch/COURSES/858/S20/notes/distrado.pdf`.

(1) $\Rightarrow$ (2) is trivial by letting $\lambda_i = x_i$ for each $i \in [n]$ in (2). Now suppose that (2) holds, we need to show that (1) holds. Recall van der Waerden's Theorem again.

Theorem 5.1 (van der Waerden's Theorem)**.** *For any* $k, l \in \mathbb{N}$*, there exists* $N_v = N_v(k, l) \in \mathbb{N}$ *such that any* k*-coloring of* $[N]$ *with* $N \geq N_v$ *creates a monochromatic* l*-AP.*

Also recall Rado's theorem.

Theorem 5.2 (Rado's Theorem)**.** *Let* $c_1, \ldots, c_n$ *be nonzero integers. Then the equation*

$$c_1x_1 + \cdots + c_nx_n = 0$$

on variables $x_1, \ldots, x_n$ *is regular if and only if some nonempty subset of the* c_i *sums to zero.*

Lemma 5.1. *Let* s, t, r *be positive integers. For any* r*-coloring* $\chi : \mathbb{N} \to [r]$*, there exist* $a, d \in \mathbb{N}$ *such that for each* $i \in [s]$*,*

$$\chi(i(a - td)) = \chi(i(a - (t-1)d)) = \cdots = \chi(ia) = \cdots = \chi(i(a + (t-1)d)) = \chi(i(a + td)).$$

Proof of Lemma 5.1. Fix an r-coloring $\chi : \mathbb{N} \to [r]$ and we define a new coloring $\chi^* : \mathbb{N} \to [r]^s$ by

$$\chi^*(n) = (\chi(n), \chi(2n), \ldots, \chi(sn)).$$

By van der Waerden's theorem, there exist $a, d \in \mathbb{N}$ such that[2]

$$\chi^*(a - td) = \chi^*(a - (t-1)d) = \cdots = \chi^*(a) = \cdots = \chi^*(a + (t-1)d) = \chi^*(a + td),$$

which immediately completes the proof. □

[2] Although we write $\mathbb{N}$ here, it is notable that $[N]$ for some large yet finite $N = N(t, r)$ suffices.

Lemma 5.2. *Let $b_1, \dots, b_n$ be nonzero integers such that the equation*

$$b_1x_1 + \cdots + b_nx_n = 0$$

on variables $x_1, \dots, x_n$ is regular. Let t, r be positive integers. For any r-coloring $\chi : \mathbb{N} \to [r]$, there exist $e_1, e_2, \dots, e_n, d \in \mathbb{N}$ such that

1. *$b_1e_1 + \cdots + b_ne_n = 0$;*

2. *$(e_i - jd)$ all have the same color for all $i \in [n]$ and $-t \le j \le t$.*

Proof of Lemma 5.2. Let s be a large positive integer to be specified later. Then by Lemma 5.1, there exist $a, D \in \mathbb{N}$ such that for each $i \in [s]$,

$$\chi(i(a - TD)) = \chi(i(a - (T-1)D)) = \cdots = \chi(ia) = \cdots = \chi(i(a + (T-1)D)) = \chi(i(a + TD)),$$

where D and T are large positive integers to be specified later. Consider the coloring $\chi^{**} : [s] \to [r]$ define by $\chi^{**}(i) = \chi(ia)$. By the regularity of $b_1, \dots, b_n$, we may let s be sufficiently large and then there exist $f_1, \dots, f_n \in [s]$ such that

1. $b_1f_1 + \cdots + b_nf_n = 0$, which implies that $b_1(af_1) + \cdots + b_1(af_n) = 0$;

2. $\chi^{**}(f_1) = \chi^{**}(f_2) = \cdots = \chi^{**}(f_n)$, which implies that $\chi(af_1) = \chi(af_2) = \cdots = \chi(af_n)$.

Now, we conclude that $f_i(a - jd)$ all have the same color for all $i \in [n]$ and $-T \le j \le T$. Let $e_i = af_i$ for all $i \in [n]$, we have that $(e_i - f_ijd)$ all have the same color for all $i \in [n]$ and $-T \le j \le T$. Now it suffices to find $d \in \mathbb{N}$ such that

$$\{d, 2d, \dots, td\} \subset \{f_iD, 2f_iD, \dots, Tf_iD\}, \forall i \in [n].$$

Note that we have not chosen d and T yet. We let $d = \prod_{j=1}^{n} f_jD$ and let T be sufficiently large, say, $T = ts^n$, then the desired condition is satisfied, completing the proof. □

Now we are ready to proof the desired result. Let M be a large positive integer to be determined later. Fix a r-coloring $\chi : \mathbb{N} \to [r]$. By Lemma 5.2, there exist $e_1, \dots, e_n, d \in \mathbb{N}$ such that

1. $b_1e_1 + \cdots + b_ne_n = 0$;

2. $(e_i - jd)$ all have the same color for all $i \in [n]$ and $-M \le j \le M$.

Let A be a integer to be determined later. Note that regardless of the value of A, we have

$$\sum_{i=1}^{n} b_i(e_i + Ad\lambda_i) = 0,$$

which gives us a solution $(e_1 + Ad\lambda_1, \dots, e_n + Ad\lambda_n)$. It now suffices to find M such that there exists $A \in \mathbb{Z}$ with

1. $e_i + Ad\lambda_i$ are all distinct for $i \in [n]$;

2. $|A\lambda_i| \le M$ for all $i \in [n]$.

We let $M = 2\binom{n}{2}\max\{|\lambda_i|\}_{i\in[n]}$. The first condition forbidden at most $\binom{n}{2}$ values of A while the second condition allows strictly more than $\binom{n}{2}$ values of A with our choice of M, thus we can find such an A, completing the proof. $\square$

A previous attempt. We do induction on n. When $n = 1$, clearly (2) $\Rightarrow$ (1). Now suppose that (2) $\Rightarrow$ (1) for all $n < k$, we want to show that (2) $\Rightarrow$ (1) for $n = k$. We have that for every positive integer r and every r-coloring of $\mathbb{N}$, there exist positive integers $x_1, x_2, \ldots, x_k$ of the same color such that $c_1x_1 + c_2x_2 + \cdots + c_kx_k = 0$, which exactly says that $c_1x_1 + c_2x_2 + \cdots + c_kx_k = 0$ is regular. By Rado's theorem, there exists a nonempty subset of the c_i summing to zero. We let $\emptyset \neq I \subset [n]$ be such a subset with maximal cardinality $t = |I|$, i.e., $\sum_{i\in I} c_i = 0$, and $\sum_{i\in I'} c_i \neq 0$ for all $I' \subset [n]$ with $|I'| > t$. WLOG, we may further assume that $I = [t]$. Now we have distinct integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$c_1\lambda_1 + c_2\lambda_2 + \cdots + c_n\lambda_n = 0$$

and

$$c_1\lambda_1 + c_2\lambda_1 + \cdots + c_t\lambda_1 = 0.$$

We subtract the second equation from the first one, and we have

$$c_2(\lambda_2 - \lambda_1) + c_3(\lambda_3 - \lambda_1) + \cdots + c_t(\lambda_t - \lambda_1) + c_{t+1}\lambda_{t+1} + \cdots + c_n\lambda_n = 0.$$

We claim that $(\lambda_2 - \lambda_1), (\lambda_3 - \lambda_1), \ldots, (\lambda_t - \lambda_1), \lambda_{t+1}, \ldots, \lambda_n$ are all distinct. The first t ones are clearly distinct and so are the last $n - t$ ones. For the inter-distinctness, suppose there are $i \in [t]$ and $j \in [n] \setminus [t]$ such that $\lambda_i - \lambda_1 = \lambda_j$, ... $\square$