§10.2 Hales-Jewett Thm

Def/ A Combinatorial line of $[k]^n$ is a set L of k points described by a ~~root~~ $\in ([k] \cup \{*\})^n$.

E.g., $L = (1,2,*) = \{(1,2,x),\ x \in [k]\}$

$L = (*,1,*) = \{(x,1,x),\ x \in [k]\}$

$\boxed{\text{Thm}}$ (H-J, 1963)

$\forall k, r, \exists n$ s.t. $\forall$ r-coloring of $[k]^n$ induces a monochromatic CL.

$HJ(k,r) :=$ min. such n

$HJ(1,r) = 1$ ($[1]^n$. line $= 1$ point)

$\boxed{\text{Claim}}$ $HJ(2,r) = r$

$\chi : [2]^r \to [r] \Rightarrow \exists$ monochromatic line

① $HJ(2,r) \geq r$

Color $x \in [2]^n$ by # 1's

$\Rightarrow$ # colors $= n+1$

$\Rightarrow$ If $n+1 \leq r$, then in this $(n+1)$-coloring,

$\nexists$ monochromatic CL

$\Rightarrow HJ(2,r) \geq r$

② $HJ(2,r) \leq r$

Consider the following seq. of length $r+1$

$$\left\{\begin{array}{l} 111\text{---}111 \\ 111\text{---}112 \\ 111\text{---}122 \\ \vdots \\ 122\text{---}222 \\ 222\text{---}222 \end{array}\right.$$

Fix a $\chi : [2]^r \to [r]$

Pigeonhole $\Rightarrow \exists$ Two of them have the same color, and they form a CL

$\Rightarrow HJ(2,r) \leq r$

$\boxed{\mathbb{Z}}$

Pf/ (of vdW via HJ)

Let $n = HJ(\underbrace{l}_{\text{length of the AP}}, \underbrace{r}_{\#\text{ colors}})$

We color $[l]^n$ by coloring

$(x_1, -, x_n)$ w/ the color of $\sum_i x_i$

By HJ, $\exists$ monochromatic CL

$\Rightarrow \exists$ monochromatic $\{a + id : i \in [l]\}$

$\blacksquare$

Pf/ (of HJ, by Shelah 88')

Fix r, induction on K.

$HJ(1,r) = 1$, $HJ(2,r) = r$

Assume that $HJ(K-1, r) < \infty$

we'll select a sequence of "image" Z

$t_1, t_2, \dots, t_m$ and m later.

Let $n = t_1 + t_2 + \dots + t_m$. Now, let's consider an r-coloring of

$[k]^n = [k]^{t_1} \times [k]^{t_2} \times \dots \times [k]^{t_m}$

$\boxed{\text{Claim}}$ $\exists$ root $\tau = \tau_1 \tau_2 \dots \tau_m$, where $\tau_i \in ([k] \cup \{x\})^{t_i}$ such that the CL represented by τ is monochromatic.

(After selecting $t_1, \dots, t_m$ "nicely")

(How to select t_m)

For each $w \in [k]^{t_m}$, we consider an $(r^{k^{n-t_m}})$-coloring of $[k]^{t_m}$, where the color of $w \in [k]^{t_m}$ $\Longleftrightarrow$ the color of $(x, w) \in [k]^n$, for all $x \in [k]^{n-t_m}$

Key Idea: If $t_m \geq HJ(2, r^{k^{n-t_m}})$, then $\exists$ root $\tau_m \in ([k] \cup \{x\})^{t_m}$ s.t.

assigning "x" by $\underset{\tilde{\tau}_m(k)}{\underline{k}}$ or $\underset{\tilde{\tau}_m(k-1)}{\underline{k-1}}$ gives the same color.

$\Rightarrow (x, \tilde{\tau}_m(k-1))$ & $(x, \tilde{\tau}_m(k))$ have the same color, $\forall x \in [k]^{n-t_m}$.

Suppose we already have $\tau_{i+1}, \tau_{i+2}, \ldots, \tau_m$, where $\tau_j \in ([k] \cup \{x\})^{t_j}$. We consider a coloring of $w \in [k]^{t_i}$ by the color

$$(\underbrace{x}_{t_1+\cdots t_{i-1}}, \underbrace{w}_{t_i}, \underbrace{\tau_{i+1}(u_{i+1})}_{t_{i+1}}, \underbrace{\tau_{i+2}(u_{i+2})}_{t_{i+2}}, \ldots, \underbrace{\tau_m(u_m)}_{t_m})$$

for all $x \in [k]^{t_1+\cdots+t_{i-1}}$, $u_{i+1}, \cdots u_m \in [k]$.

$\Rightarrow$ # possible colors $= r^{k^{t_1+\cdots+t_{i-1}+m-i}} := R_i$

$\Rightarrow$ If $t_i \geq HJ(2, R_i)$, then $\exists$ root

$\tilde{\tau}_i \in ([k] \cup \{*\})^{t_i}$ s.t. $\tilde{\tau}_i(k-1)$ and $\tilde{\tau}_i(k)$

have the same color

$\Rightarrow \forall\, x \in [k]^{t_1+\cdots+t_{i-1}}, u_{i+1}, \cdots, u_m \in [k]$,

$(x, \tilde{\tau}_i(\tilde{u}_i), \tilde{\tau}_{i+1}(u_{i+1}), \cdots, \tilde{\tau}_m(u_m))$

has the same color for $\tilde{u}_i \in \{k-1, k\}$

$\tilde{\tau}_m \Rightarrow \tilde{\tau}_{m-1} \Rightarrow \cdots \Rightarrow \tilde{\tau}_1$, in the end,

let $m = HJ(k-1, r)$, and let

$t_i = k^{r^{t_1+\cdots t_{i-1}+m-i}}$ for all $1 \leq i \leq m$.

Consider $\tilde{\tau}_1, \tilde{\tau}_2, \cdots, \tilde{\tau}_m$,

$(\tilde{\tau}_1(u_1), \tilde{\tau}_2(u_2), \cdots, \tilde{\tau}_m(u_m))$, $u_i \in [k]$

$\Leftrightarrow$ coloring of $[k]^m$

By the def of HJ, $\exists$ monochromatic CL described by $\sigma \in ([k-1] \cup \{*\})^m$, also note that $\sigma(k-1)$ and $\sigma(k)$ have the same color $\Rightarrow \{\sigma(1), \ldots, \sigma(k)\}$ is a monochromatic CL. $\blacksquare$

§10.3 Affine Ramsey Thm.

$\mathbb{F}$: finite field

Def/ $X \subseteq \mathbb{F}^n$ is a <u>t-space</u> of $\mathbb{F}^n$ if it's an affine subspace of dimension t.

t-space = translate of a linear space of dim. t

$W \subseteq \mathbb{F}^n$: linear space, $a \in \mathbb{F}^n \Rightarrow a+W$: affine space

Example: 0-space $\Longleftrightarrow$ a point

1-space $\Longleftrightarrow$ a line

2-space $\Longleftrightarrow$ a plane

$\begin{bmatrix} V \\ t \end{bmatrix}$:= set of all t-spaces of V.

$\boxed{\text{Thm}}$ (Graham, Leeb, Rothschild 72') ["Affine Ramsey Thm"] $\mathbb{F}$: finite field

$\forall k, r, \exists N$ s.t. $\forall n \geq N$,

$\forall$ r-coloring of t-space of $\mathbb{F}^n$ induces a k-space W whose all t-spaces have the same color.

In other words, if $\dim V = n \geq N$, let $\chi : \begin{bmatrix} V \\ t \end{bmatrix} \to [r]$, then $\exists W \in \begin{bmatrix} V \\ k \end{bmatrix}$ s.t. χ is constant on $\begin{bmatrix} W \\ t \end{bmatrix}$.

$\boxed{\text{Lem}}$ (True when $t=0$.) $\mathbb{F}$: finite field

$\forall r, k$, $\exists N$ s.t. $\forall n \geq N$,

$\forall$ r-coloring of $\mathbb{F}^n$ induces a monochromatic k-space.

Pf/ (Simple application of H-J)

Take $m = HJ(|\mathbb{F}|^k, r)$, $N = mk$,

WMA $n = N$. There is a natural bijection from $\mathbb{F}^n$ to $(\mathbb{F}^k)^m$. By the def/ of HJ,

$\exists$ monochromatic CL described by a root

$\tau \in (\mathbb{F}^k \cup \{*\})^m$. And points in this line

$\Rightarrow a + x_1 b_1 + x_2 b_2 + \cdots + x_k b_k,\ b_i \in \{0,1\}^n,\ x_i \in \mathbb{F}$

$\Rightarrow$ Affine k-space. $\square$

Cor (of HJ)

$\mathbb{F}$: finite field, let $m = HJ(|\mathbb{F}|^{u+1}, c)$,

and let χ: c-coloring of the ordered

$(u+1)$-tuples $(x_0, x_1, \cdots, x_u)$, where $x_i \in \mathbb{F}^m$.

Then $\exists$ parallel affine lines $L_0, L_1, \cdots, L_u$

in $\mathbb{F}^m$ s.t. $L_0 \times L_1 \times \cdots \times L_u$ is monochromatic.

Pf/ Let us regard each $(u+1)$-tuple

$(x_0, x_1, \dots, x_u)$, $x_i \in \mathbb{F}^m$ as an $m \times (u+1)$ matrix. Consider rows, note that

possible distinct rows $\le |\mathbb{F}|^{u+1}$

$m = HJ(|\mathbb{F}|^{u+1}, c) \Rightarrow$

$\exists$ monochromatic CL L described by a column vector $\tau \in (\mathbb{F}^{u+1} \cup \{*\})^m$

$\Rightarrow L_i = \{a_i + x_i \underset{\sim}{b} : x_i \in \mathbb{F}\}$, $i \in \{0, 1, \dots u\}$

position of $*$'s

they're monochromatic & parallel.

$\Rightarrow L_0 \times L_1 \times \dots \times L_u$ is monochromatic.

$\boxtimes$

$V = \mathbb{F}^n$, $\chi : \begin{bmatrix} V \\ t \end{bmatrix} \to [r]$.

Let B be an $(n+1)$-space of V,

$p : B \to \mathbb{F}^n$, surjective projection.

Then for each $T \in \begin{bmatrix} B \\ t \end{bmatrix}$, we say

T is <u>transversal</u> (w.r.t. p) if

$\dim p(T) = t$; <u>vertical</u> otherwise.

i.e., $\dim p(T) = t-1$

& $p^{-1}(p(T)) = T$

We say $B \in \begin{bmatrix} V \\ n+1 \end{bmatrix}$ is <u>special</u> w.r.t.

χ and p if $\forall$ transversal $T_1, T_2 \in \begin{bmatrix} B \\ t \end{bmatrix}$,

$p(T_1) = p(T_2) \Rightarrow \chi(T_1) = \chi(T_2)$.

$\mathbb{F}$: finite field

Lem $\forall t, u, r$, $\exists\, m = M^{(t)}(u; r)$ s.t. $\forall$ r-coloring $\chi : \begin{bmatrix} \mathbb{F}^{u+m} \\ t \end{bmatrix} \to [r]$ and the projection $p : \mathbb{F}^{u+m} \to \mathbb{F}^{u}$ by taking the first u coordinates, $\exists$ special $(u+1)$-space B w.r.t. χ and p.

Pf/ Let $\nu = \nu(t, u) :=$ # t-spaces in a u-space

($\mathbb{F}$ is finite $\Rightarrow$ ν is finite)

Let $m = HJ(\mathbb{F}^{u+1}, r^{\nu})$. Fix an r-coloring $\chi : \begin{bmatrix} \mathbb{F}^{u+m} \\ t \end{bmatrix} \to [r]$. Let $0 = e_0, e_1, \ldots, e_u$ be vectors in $\mathbb{F}^{u}$ s.t. $e_i = (0, 0, \ldots, \underset{\substack{\uparrow \\ i\text{-th position}}}{1}, 0, \ldots, 0)$, for $i \in [u]$.

$\{e_1, \ldots, e_u\}$ is the standard basis of $\mathbb{F}^{u}$.

Let $A_i = p^{-1}(\{e_i\})$ for $0 \le i \le m$

$\Rightarrow A_i$ is a copy of $\mathbb{F}^m$.

Let $(x_0, x_1, \dots, x_u)$ be a $(u+1)$-tuple of vectors in $\mathbb{F}^m$, and let $y_i = (e_i, x_i) \in A_i$.

Then $\{y_0, y_1, \dots, y_u\}$ generates a unique u-space $X \subseteq \mathbb{F}^m$. $\left(X = \left\{\sum_{i=0}^{u} c_i y_i : \sum_{i=0}^{u} c_i = 1\right\}\right)$

Consider $P|_X$, we can check that it's bijective.

$\Rightarrow X$ is transversal. Let $(T_i)_{i\in[v]}$ be the list of all t-space of $\mathbb{F}^u$ in some preassigned ordering. We define a c-coloring χ' of $(\mathbb{F}^m)^{u+1}$ by:

$\chi'(x_0, x_1, \dots, x_u) = (\chi(T_1'), \chi(T_2'), \dots, \chi(T_v'))$,

where T_i' is the unique t-space in X s.t. $P(T_i') = T_i$ ($\because P|_X$ is bijective)

In X', two $(u+1)$-tuples have the same color

$\Longleftrightarrow$ In X, their corresponding t-spaces have the same color

By the previous Corollary, $\exists$ parallel affine lines $L_0, L_1, \dots, L_u$ s.t. $L_0 \times L_1 \times \dots \times L_u$ is monochromatic in X'. Also, $\{L_0, L_1, \dots, L_u\}$ generates a $(u+1)$-space B.

$\boxed{\text{Claim}}$ B is special w.r.t. X and P.

Pf/ (of claim) Let $T \in \begin{bmatrix} B \\ t \end{bmatrix}$ be transversal, then $\exists\, j \in [v]$ s.t. $P(T) = T_j$. We can extend T to a transversal u-space $X \subseteq B$, $P(X) = H^u$.

Since $A_i = P^{-1}(\{e_i\})$, $\exists\, Y_i \in X \cap A_i$, $\forall\, 0 \leq i \leq u$

$\Rightarrow X$ is generated by $Y_0, Y_1, \dots, Y_u$ and T is contained in X.

$\Rightarrow \chi(T)$ is the j-th coordinate of

$\chi'(y_0, y_1, \dots, y_n)$, where $y_i = (e_i, x_i)$.

As χ' is constant, $\chi(T)$ is determined by j

$\Leftrightarrow P(T)$ $\blacksquare$

$\boxed{\text{Claim}}$ $\forall\, t, r, k_1, k_2, \dots, k_r, \exists\, n = N^{(t)}(k_1, \dots, k_r)$

satisfying the following:

$\forall V$: n-space; $\chi : \begin{bmatrix} V \\ t \end{bmatrix} \to [r]$, there exists

$W \in \begin{bmatrix} V \\ k_i \end{bmatrix}$ for some $i \in [r]$ s.t.

$\chi(T) = i$, $\forall\, T \in \begin{bmatrix} W \\ t \end{bmatrix}$ (Stronger than Thm.)

Pf/ (of the Claim) [J. Spencer 79'] Suppose NOT, among all

counterexamples, choose ones w/ min. t and

among them, choose one w/ min. $\sum_{i=1}^{r} k_i$.

We've already known that it's true when $t=0$.

So $t>0$. Let $s = \max_{1\le i\le r} N^{(t)}(k_1, \dots, k_i - 1, k_{i+1}, \dots, k_r)$

$u = N^{(t-1)}(s; r) = N^{(t-1)}(\underbrace{s, s, s, \dots, s}_{r \text{ of them}})$,

$m = M^{(t)}(u; r)$ [from the previous Lem.]

$n = u + m$. We'll show that n satisfies the conditions.

Let $V = \mathbb{F}^n$; $\chi : \begin{bmatrix} \mathbb{F}^n \\ t \end{bmatrix} \to [r]$; $P : \mathbb{F}^n \to \mathbb{F}^u$ taking

the first u coordinates. By the previous lemma,

$\exists$ special $(u+1)$-space B w.r.t. χ and P.

Let $\chi' : \begin{bmatrix} \mathbb{F}^u \\ t-1 \end{bmatrix} \to [r]$ s.t. $\chi'(T) = \chi(\underbrace{P^{-1}(T) \cap B}_{(P|_B)^{-1}(T)})$

$\dim(B) = u+1$, $\dim P(B) = u$

$\Downarrow$

$\dim (P|_B)^{-1}(T) = \dim T + 1 = t$

In other words, χ' gives an r-coloring to

each $(t-1)$-space of $\mathbb{F}^u$ from the color of a

vertical t-space. By the minimality of t,

$\exists X \in \begin{bmatrix} A^u \\ s \end{bmatrix}$ s.t. all $(t-1)$-spaces of X have the same color, say 1, under X'. Now,

$s \geq N^{(t)}(k_1-1, k_2, \ldots, k_r) \Rightarrow \exists W_1 \subseteq X$ s.t. either (1) $\dim W_1 = k_1 - 1$ and all t-spaces of W_1 have color 1;

OR (2) $\dim W_i = k_i$ and all t-spaces of W_i have color i, for some i. If it's (2), then we're done. So WMA it's (1). Take $W = P|_B^{-1}(W_1)$

$\Rightarrow \dim W = k_1$ and W is a vertical k_1-space of B.

Let T be a t-space of W. If T is transversal, then by (1), the color of T is 1; elif T is vertical, $X'(P|_B(T)) = 1 \Rightarrow X(T) = 1$

$\Rightarrow \begin{bmatrix} W \\ t \end{bmatrix}$ is monochromatic w/ color 1.

$\boxed{\text{Thm}}$ (Vector Space Ramsey Theorem)

$\forall r, t > 0, k, \exists n$ s.t.

$\forall r$-coloring of the t-dim. linear subspaces of an n-dim. vector space V over a fixed finite field $\mathbb{F}$, $\exists$ k-dim. linear subspaces W whose all t-dim. linear subspaces have the same color.

t-dim. subspace $\Leftrightarrow$ $(t-1)$-space