Please type your homework solution into a PDF file and submit to the gradescope.com website by 11:59pm KST of the due date. A point will be deducted for each problem if a handwritten solution is submitted. We recommend using LaTeX. You can find useful hints at our KLMS website. It is recommended to use the sample template. (At least, make sure that each problem has a solution in separate pages.) Unprofessional proofs may get a deduction of points, even if the solution is mathematically correct or can be made correct.

2021 Spring MAS575 *Combinatorics*
Homework 5

Due: May 25, 2021

5.1. Let G be a simple graph G on the vertex set $\{v_1, v_2, \dots, v_n\}$. Let $A = (a_{ij})$ be an $n \times n$ real symmetric matrix with zero diagonal entries such that for all $i \neq j$, $a_{ij} \neq 0$ if and only if v_i is adjacent to v_j. Let $\alpha(G)$ be the maximum size of an *independent* set in G. (A set of vertices is *independent* if no two vertices in the set are adjacent.)

Among n eigenvalues of A, let N^+ be the number of positive eigenvalues of A and N^- be the number of negative eigenvalues of A. Prove that $\alpha(G) \leq \min(n - N^+, n - N^-)$.

5.2. Assume that n is not too small. Prove that if A is a subset of $\{1, 2, \dots, n\}$ with $|A| > 2n/3$, then A has an arithmetic progression of length 3.

5.3. Prove that for all n, there exists N satisfying the following; For every bipartite graph G with the bipartition (A, B) such that no two vertices in A have the same set of neighbors and $|A| \geq N$, there exist distinct vertices $a_1, a_2, \dots, a_n \in A$ and $b_1, b_2, \dots, b_n \in B$ such that one of the following hold:

1. For all $1 \leq i, j \leq n$, a_i is adjacent to b_j if and only if $i = j$.
2. For all $1 \leq i, j \leq n$, a_i is adjacent to b_j if and only if $i \leq j$.
3. For all $1 \leq i, j \leq n$, a_i is adjacent to b_j if and only if $i \neq j$.

5.4. Let t, r be a positive integer. Prove that there exists a number N such that any r-coloring of numbers in $\{1, 2, \dots, N\}$ contains an arithmetic progression a, $a + d$, $a + 2d$, $\dots$, $a + (t-1)d \in [N]$ of length t ($d \neq 0$) such that a, $a + d$, $a + 2d$, $\dots$, $a + (t-1)d$ and d have the same color.

5.5. Prove that for all k and q, there exists N such that every sequence a_1, a_2, $\dots$, a_N of positive intgers with

$$a_1 < a_2 < \cdots < a_N \text{ and } a_{i+1} - a_i \leq q \text{ for all } i \in \{1, 2, \dots, N-1\}$$

has a subsequence that is an arithmetic progression of length k.