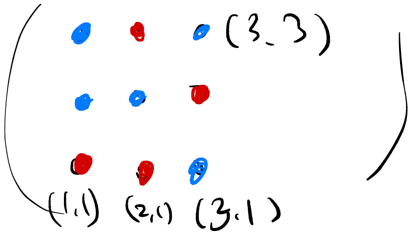
10.2. Hales-Jewett theorem

$[k] = \{1, 2, \dots, k\}$

An r-coloring of $[k]^n$ = function $\chi : [k]^n \to [r]$.

$[3]^2$



A combinatorial line of $[k]^n$ is a set L of k points described by a root $\in ([k] \cup \{*\})^n$

$\boxed{k=3}$

$(1, 2, *) \longrightarrow L = \{(1, 2, 1), (1, 2, 2), (1, 2, 3)\}$

$L = \{(1, 2, x) : x \in [k]\}$

$(*, *, *) \longrightarrow L = \{(x, x, x) : x \in [k]\}$

$(*, 1, *) \longrightarrow L = \{(x, 1, x) : x \in [k]\}$

Thm [Hales-Jewett 1963]
For every k and r, there exists n such that every r-coloring of $[k]^n$ induces a monochromatic combinatorial line.

$HJ(k, r) :=$ minimum such n

$HJ(1, r) = 1$.

$[1]^n$

line = 1 point.

Claim: $HJ(2, r) = r$

($\chi : [2]^r \to [r] \Rightarrow$ there is a mochromatic combinatorial line)

(1) Why $HJ(2,r) \geq r$?

Color each $x \in [2]^n$ by # 1's.

Combinatorial line

$(\underline{*}, \underline{*}, \ldots)$ $* = 1$ or 2

distinct colors $= \boxed{n+1}$

$\Longrightarrow$ If $n+1 \leq r$, then in this coloring there is no monochromatic combinatorial line.

So $\boxed{n \geq r}$.

(2) Why $HJ(2,r) \leq r$

Let $\chi : [2]^r \to [r]$ be a fixed r-coloring.

$$
r+1 \text{ sequences} \left\{ \begin{array}{cccccc}
\multicolumn{6}{c}{\overset{r}{\longleftrightarrow}} \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 \\
1 & 1 & 1 & 1 & 2 & 2 \\
1 & 1 & 1 & 2 & 2 & 2 \\
1 & 1 & 2 & 2 & 2 & 2 \\
1 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2
\end{array} \right.
$$

2 of them have the same color.

Combinatorial line

$(11, ***, 222)$

$\square$

Proof of

Van der Waerden's theorem via Hales-Jewett theorem:

Proof. Let $n = HJ(\ell, r)$.

ℓ — length of the AP. r — # colors

We color $[\ell]^n$ by

$$(x_1, x_2, \dots, x_n) \longmapsto \text{color of } \sum x_i.$$

There is a monochromatic combinatorial line.

$$((1, 2), (*, 2, 3), \dots) \to a$$

$$* \to i$$

$$\Longrightarrow \{a + id : i = 1, 2, \dots, \ell\}$$ is monochromatic

Take $W(\ell, r) := \ell \cdot HJ(\ell, r)$

$(1, 2, *, *, 3)$

$(1, 2, 1, 1, 3)$ $\quad a = 1 + 2 + 3$

$(1, 2, 2, 2, 3)$ $\quad d = 1 + 1$.

$(1, 2, 3, 3, 3)$

$a + 1d$

$a + 2d$

$a + 3d$.

$\square$

Proof of Hales–Jewett theorem
due to Shelah (1988)

Fix r, Induction on k

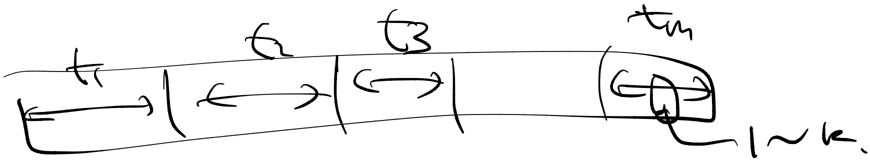
$HJ(1,r)=1$, $HJ(2,r)=r$.

Assume that $HJ(k-1,r)$ is finite.

We will select a sequence of huge integers $t_1, t_2, \dots, t_m$ and m later.

$$n = t_1 + t_2 + \cdots + t_m$$

Now, let us consider an r-coloring of $[k]^n = [k]^{t_1} \times [k]^{t_2} \times \cdots \times [k]^{t_m}$



We claim that there is a root $\tau = \tau_1 \tau_2 \cdots \tau_m$ where $\tau_i \in ([k] \cup \{*\})^{t_i}$ such that the combinatorial line represented by τ is monochromatic.

(after selecting $t_1, t_2 \dots t_m$, m nicely).

How to select t_m?

For each $w \in [k]^{t_m}$, we consider an $\left(r^{k^{t_1+t_2+\cdots+t_{m-1}}}\right)$-coloring of $[k]^{t_m}$

$\left(\text{Color of } w \in [k]^{t_m}\right)$

$\iff$ color of $(x, w) \in [k]^n$ for every $x \in [k]^{t_1+\cdots+t_{m-1}}$

Key Idea: If $t_m \geq HJ\left(2, r^{k^{t_1+t_2+\cdots+t_{m-1}}}\right)$ then there is a root $\tau_m \in \left([k] \cup \{*\}\right)^{t_m}$ such that

$* \to k-1 \quad \tau_m(k-1)$
$* \to k \quad \tau_m(k)$
$\to$ Same color.

Color of $\tau_m(k-1)$ = color of $\tau_m(k)$

$\Downarrow$

For all $x \in [k]^{t_1+\cdots+t_{m-1}}$

$(x, \tau_m(k-1))$ $(x, \tau_m(k))$ have the same color.

$\left(\{k-1, k\}^{t_m}\right)$

Suppose we have

$$\tau_{i+1}, \tau_{i+2}, \dots, \tau_m.$$

$$\left(\tau_j \in ([k] \cup \{*\})^{t_j}\right)$$

We consider a coloring of $w \in [k]^{t_i}$ by the color of

$$\Big(\underbrace{x}_{t_1+\cdots+t_{i-1}}, \underbrace{w}_{t_i}, \tau_{i+1}(u_{i+1}), \tau_{i+2}(u_{i+2}), \dots, \tau_m(u_m)\Big)$$

$$\in [k]^{t_1+t_2+\cdots+t_m}$$

for all $x \in [k]^{t_1+t_2+\cdots+t_{i-1}}$

and $u_{i+1}, u_{i+2}, \dots, u_m \in [k]$.

$$\#\text{possible colors} = r^{k^{t_1+t_2+\cdots+t_{i-1}+m-i}}$$

If $t_i \geq HJ(2, r^{k^{t_1+t_2+\cdots+t_{i-1}+m-i}})$

then there is a root $\tau_i \in ([k] \cup \{*\})^{t_i}$ such that

$\tau_i(k-1)$ and $\tau_i(k)$ have the same color

$\Rightarrow$ for all $x \in [k]^{t_1+\cdots+t_{i-1}}$

and $u_{i+1}, u_{i+2}, \dots, u_m \in [k]$

$$(x, \tau_i(k-1), \tau_{i+1}(u_{i+1}), \dots, \tau_m(u_m))$$

and $(x, \tau_i(k), \tau_{i+1}(u_{i+1}), \dots, \tau_m(u_m))$

have the same color

$\tau_m \Rightarrow \tau_{m-1} \Rightarrow \tau_{m-2} \cdots \longrightarrow \tau_1$

let $m = HJ(k-1, r)$

Let $t_i = k^{r^{t_1+t_2+\cdots+t_{i-1}+m-i}}$ for all $1 \le i \le m$.

Consider $\tau_1, \tau_2, \ldots, \tau_m$

$(\tau_1(u_1), \tau_2(u_2), \ldots, \tau_m(u_m))$

$u_1, u_2, \ldots, u_m \in [k]$.

$\rightarrow$ Coloring of $[k]^m$.

$\Rightarrow$ By the definition of $HJ(k-1, r)$ there is a monochromatic combinatorial line described by $\sigma \in ([k-1] \cup \{*\})^m$.

$(\tau_1, \tau_2, \ldots \tau_m) \leftarrow \underline{\sigma(1)}, \underline{\sigma(2)}, \ldots, \underline{\sigma(k-1)}$.

$u_1, u_2, \ldots, u_m$

$k-1 \Longleftrightarrow k$ $\quad \boxed{\sigma(k)}$?

$\updownarrow$

$\boxed{\sigma(k-1)}$

$\Rightarrow$ $\sigma(k-1)$ and $\sigma(k)$ have the same color

$\therefore$ $\{\sigma(1), \sigma(2), \ldots \sigma(k)\}$ is a monochromatic combinatorial line.

10.3. Affine Ramsey Theorem

$\mathbb{F}$: finite field

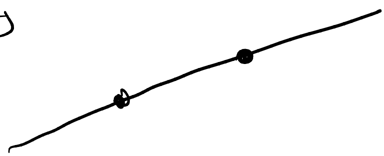
$X \subseteq \mathbb{F}^n$ is a t-space of $\mathbb{F}^n$ if it is an affine subspace of dimension t.

t-space = translate of a linear subspace of dimension t

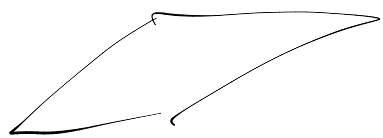
vector subspace $W \subseteq \mathbb{F}^n$, $a \in \mathbb{F}^n$ $\Rightarrow$ $a + W$: affine subspace of dim t.

0-space $\leftrightarrow$ 1 point

1-space $\leftrightarrow$



2-space $\leftrightarrow$



$\begin{bmatrix} V \\ t \end{bmatrix}$ $\Rightarrow$ set of all t-spaces of V.

<u>Thm</u> (Graham, Leeb, Rothschild 1972)
(Affine Ramsey Theorem)
For all k, r, there is N such that for all $n \geq N$,
any r-coloring of t-spaces of $\mathbb{F}^n$
($\mathbb{F}$: finite)

induces a k-space W whose all t-spaces have the same color.

In other words

If $\dim V = n \geq N$

$\chi : \begin{bmatrix} V \\ t \end{bmatrix} \to [r]$

then there is $W \in \begin{bmatrix} V \\ k \end{bmatrix}$ such that χ is constant on $\begin{bmatrix} W \\ t \end{bmatrix}$.

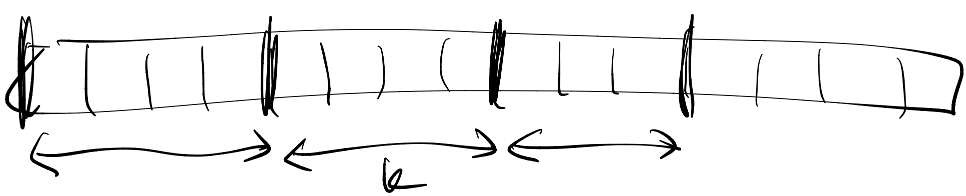
<u>Lemma</u> (true when $t=0$) $\mathbb{F}$: finite

For all r, k, there is N such that for all $n \geq N$, any r-coloring of $\mathbb{F}^n$ induces a monochromatic k-space.

<u>Proof.</u> Take $m = HJ(|\mathbb{F}|^k, r)$ and $N = m \cdot k$. Assume $n = N$.

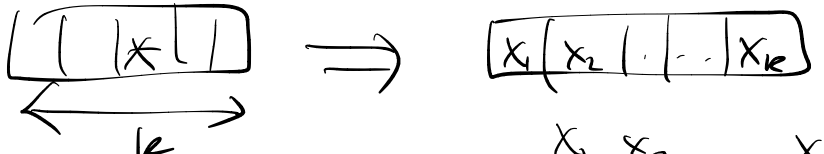
There is a natural bijection from $\mathbb{F}^n$ to $(\mathbb{F}^k)^m$.

$\mathbb{F}^n$



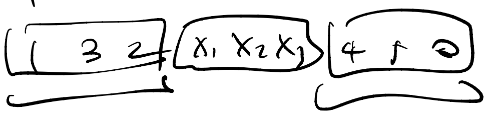
$\in (\mathbb{F}^k)^m$

There is a combinatorial line described by a root $\tau \in (\mathbb{F}^k \cup \{*\})^m$.



$x_1, x_2, \dots, x_k \in \mathbb{F}$.

points in this line



$$\Longrightarrow \quad a + x_1 b_1 + x_2 b_2 + \dots + x_k b_k$$

$$b_i \in \{0,1\}^n$$

$$x_i \in \mathbb{F}.$$

$\Longrightarrow$ affine k-space. $\square$

<u>Corollary of</u> Hales–Jewett theorem:

$\mathbb{F}$: finite field, Let $m = HJ(|\mathbb{F}|^{u+1}, c)$

Let χ: c-coloring of the ordered $(u+1)$-tuples $(x_0, x_1, \dots, x_u)$ where $x_i \in \mathbb{F}^m$.

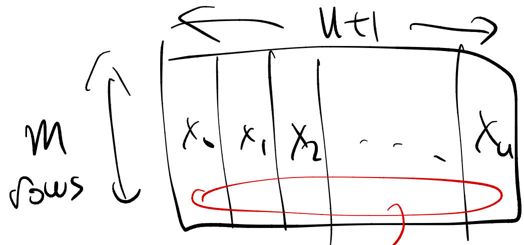
Then there exist parallel affine lines $L_0, L_1, \dots, L_u$ in $\mathbb{F}^m$ such that

$L_0 \times L_1 \times \dots \times L_u$ is monochromatic.

Proof. Let us regard

$(u+1)$-tuples $(x_0, x_1, \ldots, x_u)$ $x_i \in \mathbb{F}^m$

as a $m \times (u+1)$ matrix.



Consider rows,

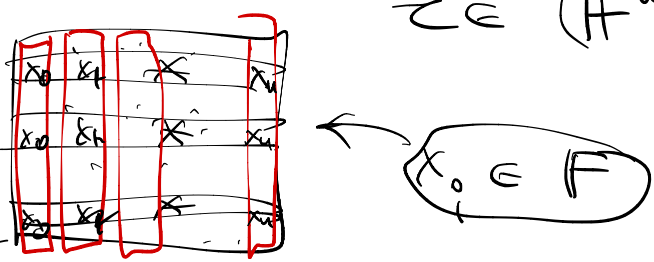
possible distinct rows $\leq |\mathbb{F}|^{u+1}$.

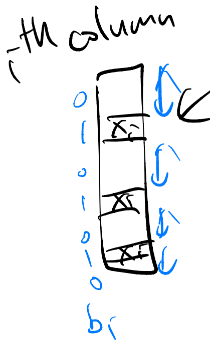
$m = HJ(|\mathbb{F}|^{u+1}, c)$

$\Rightarrow$ There is a combinatorial line L described by a column vector

$\tau \in (\mathbb{F}^{u+1} \cup \{*\})^m$

$\Rightarrow$





$L_i = \{ a_i + x_i b_i : x_i \in \mathbb{F} \}$

$\rightarrow$ affine line.

(1-space)

$L_0 \times L_1 \times \cdots \times L_u$ is monochromatic $\square$

$V = \mathbb{F}^n$

$\chi : \begin{bmatrix} V \\ t \end{bmatrix} \to [r]$

Let B be a $(u+1)$-space of V

$p : B \to \mathbb{F}^u$ be a surjective projection.

Then for each t-space $T \in \begin{bmatrix} B \\ t \end{bmatrix}$,

we say

T is transversal (with respect to p)

if $\dim p(T) = \dim T$.

T is vertical otherwise

(vertical $\Rightarrow$ $\dim p(T) = t-1$

and

$p^{-1}(p(T)) = T$.)

We say $B \in \begin{bmatrix} V \\ u+1 \end{bmatrix}$ is <u>special</u>

with respect to χ and p

if $\chi(T_1) = \chi(T_2)$

whenever $p(T_1) = p(T_2)$

for all transversals

T_1 and T_2

$\in \begin{bmatrix} B \\ t \end{bmatrix}$.

Lemma: For all t, u, r, there exist $m = M^{(t)}(u; r)$ such that for any r-coloring $\chi : \begin{bmatrix} \mathbb{F}^{u+m} \\ t \end{bmatrix} \to [r]$ and the projection $p : \mathbb{F}^{u+m} \to \mathbb{F}^{u}$ by taking the 1st u coordinates, there exists a special $(u+1)$-space B with respect to χ and p.

$\mathbb{F}$: finite.

Proof. Let $v = v(t, u) = \#$ t-spaces in a u-space. (Since $\mathbb{F}$ is finite, v is finite.)

Let $m = HJ(\mathbb{F}^{u+1}, r^{v})$

Let us fix an r-coloring $\chi : \begin{bmatrix} \mathbb{F}^{u+m} \\ t \end{bmatrix} \to [r]$.

Let $e_0 = 0, e_1, e_2, \ldots, e_u$ be vectors in $\mathbb{F}^u$ so that $e_i = (0, 0, \ldots, 0, 1, 0 \ldots \ldots 0)$ (ith position) when $i \geq 1$. $\{e_1, e_2, \ldots, e_u\}$ is the standard basis of $\mathbb{F}^u$.

Let $A_i = p^{-1}(\{e_i\})$ for $0 \leq i \leq u$.

$\Rightarrow$ A_i is a copy of $\mathbb{F}^m$

Let $(x_0, x_1, \dots, x_u)$ be a $(u+1)$-tuple of vectors in $\mathbb{F}^m$

$\Rightarrow$ let $y_i = (e_i, x_i) \in A_i$

$$y_i = (\underbrace{e_i}_{u}\ \underbrace{x_i}_{m})$$

Then $\{y_0, y_1, y_2, \dots, y_u\}$ generates a unique n-space $X \leq \mathbb{F}^m$.

(In affine geometry

$$X = \left\{ \sum_{i=0}^{u} c_i y_i \ : \ \sum_{i=0}^{u} c_i = 1 \right\}$$

projection p, $\Longrightarrow$ $p|_X$ is bijective.

$(\dim X = u = \dim \mathbb{F}^u)$

$\Rightarrow$ X is transversal

Let $T_1, T_2, \dots, T_v$ be the list of all t-spaces of $\mathbb{F}_u$ in some preassigned ordering.

We define a c-coloring of $(\mathbb{F}^m)^{u+1}$ by

$$\chi'(x_0, x_1, \dots, x_u) = (\chi(T_1'), \chi(T_2'), \dots, \chi(T_v'))$$

where T_i' is the unique t-space in X such that $p(T_i') = T_i$

$(T_i' = p^{-1}(T_i))$

In this coloring,

Two $(u+1)$-tuples have the same color

$\Longleftrightarrow$ their corresponding t-spaces have the same color.

By the previous lemma using the Hales-Jewett theorem

there exist affine lines $L_0, L_1, \ldots, L_u$ parallel to each other

so that

$L_0 \times L_1 \times \cdots \times L_u$ is monochromatic in X'.

$\{L_0, L_1, \ldots, L_u\}$ generates a $(u+1)$-space B.

We claim that B is special with respect to χ and p.

Let $T \in \left[{B \atop t}\right]$ be transversal.

$\Rightarrow$ There is $j \in \{1, \ldots, v\}$ such that

$$p(T) = T_j$$

We can extend T to a transversal n-space $X \subseteq B$ by linear algebra

$$p(X) = \mathbb{F}^n.$$

Since $A_i = p^{-1}(\{e_i\})$, there exists $y_i \in X \cap A_i$ for each $i \in \{0, 1, \ldots, n\}$

Then X is generated by $y_0, y_1, \ldots, y_n$.

and T is contained in X.

$\Rightarrow \lambda(T)$ is the j^{th} coordinate of $\lambda'(x_0, x_1, x_2, \ldots, x_n)$

where $y_i = (e_i, x_i)$

As λ' is constant

$\lambda(T)$ is determined by j

$\Rightarrow P(T)$

$\square$

CLAIM: For all $t, r, k_1, k_2, \dots, k_r$, there exists $n = N^{(t)}(k_1, k_2, \dots, k_r)$ satisfying the following.

Let V be an n-space and $\chi : \begin{bmatrix} V \\ t \end{bmatrix} \to [r]$ be an r-coloring of t-spaces of V.

Then there exists $W \in \begin{bmatrix} V \\ k_i \end{bmatrix}$ for some $1 \le i \le r$ such that

$$\chi(T) = i \text{ for all } T \in \begin{bmatrix} W \\ t \end{bmatrix}.$$

Proof of CLAIM:

Among all counterexamples, choose one with minimum t and subject to this, choose $\sum_{i=1}^{r} k_i$ minimum.

If $t=0$, then we are done from the lemma (using the Hales-Jewett)

So, we may assume $t>0$.

Let $s = \max_{1 \le i \le r} N^{(t)}(\underline{k_1, \dots,}\ k_i - 1, \underline{k_{i+1}, \dots, k_r})$

$$u = N^{(t-1)}(s ; r) = N^{(t-1)}(\underbrace{s, s, s, \dots, s}_{r})$$

$m = M^{(t)}(u ; r)$ $\leftarrow$ From the previous lemma

$n = u + m$

We will show that this n satisfies the claim.

Let $V = \mathbb{F}^n$. Let $\chi : \begin{bmatrix} \mathbb{F}^{u+m} \\ t \end{bmatrix} \to [r]$ be an r-coloring.

Let $p : \mathbb{F}^{u+m} \to \mathbb{F}^u$ be the projection taking the 1st u coordinates.

By the definition of m and the previous lemma, there exists a special $(u+1)$-space B with respect to χ and p.

Let $\chi' : \begin{bmatrix} \mathbb{F}^u \\ t-1 \end{bmatrix} \to [r]$ be an r-coloring of $(t-1)$-spaces of $\mathbb{F}^u$ such that

$$\chi'(T) = \chi\left(\left(p|_B\right)^{-1}(T)\right)$$

T has dim $t-1$

$$p|_B^{-1}(T) \subseteq B$$

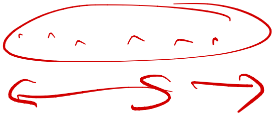
dim $(u+1)$

$$\boxed{\dim p(B) = u}$$

$$\dim p|_B^{-1}(T) = (t-1)+1 = t.$$

In other words χ' gives an r-coloring to $(t-1)$-spaces of $\mathbb{F}^u$ from a color of a vertical t-space.

By the induction hypothesis (assumption that t is minimum), there is $X \in \begin{bmatrix} \mathbb{F}^u \\ s \end{bmatrix}$ such that all $(t-1)$-spaces of X have the same color, say 1, under χ'.



Now, $s \geq N^{(t)}(k_1-1, k_2, k_3, \ldots, k_r)$

$\Rightarrow$ By the assumption, there exists $W_1 \subseteq X$ such that either

(1) $\dim W_1 = k_1 - 1$ and all t-spaces of W_1 have color 1

OR

(2) $\dim W_1 = k_i$ and all t-spaces of W_1 have color i.

~~If~~ we have (2), then we are done.
We may assume we have (1).

Take $W = p|_B^{-1}(W_1)$

$\Rightarrow$ $\dim W = k_1$
W is a vertical k_1-space of B.

T : t-space of W.

If T is transversal, then by (1), the color of T is 1.

If T is vertical,

$$\chi'(p|_B(T)) = 1$$

$$\Rightarrow \chi(T) = 1.$$

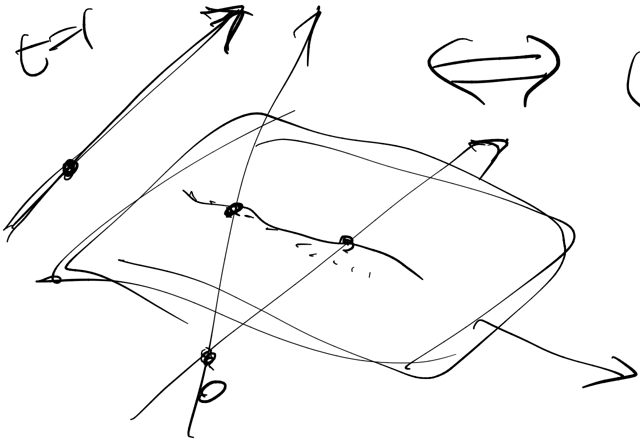
$\therefore$ So all t-spaces of W have the same color 1. $\square$.

This proof was due to J. Spencer 1979.

Thm (Vector Space Ramsey Theorem)

For all $r, t > 0, k$, there exists n such that for any r-coloring of the t-dimensional $\underbrace{}_{\text{(linear)}}$ subspaces of an n-dimensional vector space V over a fixed finite field $\mathbb{F}$, there exists a k-dimensional linear subspace W whose all t-dimensional subspaces have the same color.

χ : coloring of t-dimensional subspaces



$\Longleftrightarrow$ $(t-1)$-space.

$\square$