7.8. Covering the cube by affine hyperplanes

An affine hyperplane is a set of vectors
$$H = \{x \in \mathbb{R}^n : a \cdot x = b\}$$
for some $a, b \in \mathbb{R}^n$.

Thm (Alon, Füredi 1993)
If m affine hyperplanes in $\mathbb{R}^n$ avoids 0 but cover all other 2^n-1 points in $\{0,1\}^n$, then $m \geq n$.

$$\underbrace{x_1 = 1, \quad x_2 = 1, \quad \ldots, x_n = 1}_{n \text{ affine hyperplanes}}$$

Lemma: Let P be a polynomial in $\mathbb{R}[x_1, x_2, \ldots, x_n]$ with $P(0, 0, \ldots, 0) \neq 0$.
If $P(x_1, x_2, \ldots, x_n) = 0$ for all $(x_1, x_2, \ldots, x_n) \in \{0,1\}^n - \{(0, \ldots, 0)\}$
then the degree of P is at least n.

Proof of Alon–Füredi assuming the lemma.
Let $H_1, H_2, \ldots, H_m$ be the affine hyperplanes covering all of $\{0,1\}^n$ but 0.
let $a_i, b_i \in \mathbb{R}^n$ such that
$$H_i = \{x \in \mathbb{R}^n : a_i \cdot x = b_i\}$$

Define

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{m} (a_i \cdot x - b_i)$$

$\Rightarrow m = \deg(p) \geq n$.

Proof of Lemma. Suppose not.

Define $f(x_1, x_2, \ldots, x_n) = p(x_1, \ldots, x_n) - c \cdot \prod_{i=1}^{n} (x_i - 1)$

where c is chosen so that $f(0, 0, \ldots, 0) = 0$.

$c \neq 0$.

$\Rightarrow$ The coefficient of $\prod x_i$ in f is $c \neq 0$.

$\Rightarrow$ There exist $x_i \in \{0, 1\}^n$ such that $f(x_1, x_2, \ldots, x_n) \neq 0$. (by the Combinatorial Nullstellensatz).

Contradiction $\square$

7.9. Partitioning into pairs with prescribed differences

Thm (Dyson's conjecture 1962; proved by Wilson 1962, Gunson 1962)

The constant term in the expansion of
$$\prod_{1\le i\ne j\le n}\left(1-\frac{x_i}{x_j}\right)^{a_i}$$
is equal to
$$\frac{(a_1+a_2+\cdots+a_n)!}{a_1!\,a_2!\cdots a_n!}.$$

In other words, if $a=\sum_{i=1}^{n} a_i$,

in $\prod_{1\le i<j\le n}(-1)^{a_j}(x_j-x_i)^{a_i+a_j}$,

the coefficient of $\prod_{i=1}^{n} x_i^{a-a_i}$ is equal to
$$\frac{(a_1+a_2+\cdots+a_n)!}{a_1!\,a_2!\cdots a_n!}.$$

Proof (Good 1970).

Induction on $n+\sum_{i=1}^{n} a_i$.

Let $F(x;a_1,a_2,\ldots,a_n)=$ constant term of $\prod_{i\ne j}\left(1-\frac{x_i}{x_j}\right)^{a_i}$

We may assume $n>1$.

If $a_i=0$ for some i, then

$$F(x; a_1, a_2, \dots a_n) = F(x_1, \dots x_{j-1}, x_{j+1}, \dots, x_n; a_1, a_2, \dots a_{j-1}, a_{j+1}, \dots, a_n)$$

$$= \frac{(a_1 + \dots + a_{j-1} + a_{j+1} + \dots + a_n)!}{a_1! \dots a_{j-1}!\, a_{j+1}! \dots a_n!} = \frac{(a_1 + \dots + a_n)!}{a_1! \dots a_n!}$$ by induction.

So, we may assume $a_i \neq 0$ for all i.

Assume x_i's are distinct.

Let $$f_j(x) = \frac{\prod_{i \neq j} (x - x_i)}{\prod_{i \neq j} (x_j - x_i)}$$

$$\Rightarrow f_j(x) = \begin{cases} 1 & \text{if } x = x_j \\ 0 & \text{if } x = x_i \quad i \neq j \end{cases}$$

f_j is a polynomial of degree $n-1$.

Let $f(x) = f_1(x) + f_2(x) + \dots + f_n(x)$.

$f(x_i) = 1$ for all i

f is a polynomial of degree $\leq n-1$.

$$\Rightarrow \boxed{f(x) \equiv 1.}$$

$$\Rightarrow \sum_{j=1}^{n} \frac{\prod_{i \neq j} (-x_i)}{\prod_{i \neq j} (x_j - x_i)} = 1.$$

$$\Rightarrow \sum_{j=1}^{n} \prod_{i \neq j} \left(1 - \frac{x_j}{x_i}\right)^{-1} = 1$$

$$F(x;a)\cdot\sum_{j=1}^{n}\prod_{i\neq j}\left(1-\frac{x_j}{x_i}\right)^{-1}$$

$$=\sum_{j}\prod F(x;a_1,\dots,a_{j-1},a_j-1,a_{j+1},\dots,a_n)$$

$$=\sum\frac{(a_1+\dots+a_n-1)!}{a_1!\,a_2!\dots a_{j-1}!\,(a_j-1)!\,a_{j+1}!\dots a_n!}$$

$$=\sum\frac{(a_1+\dots+a_n)!}{a_1!\dots a_n!}\;\frac{a_j}{a_1+\dots+a_n}$$

$$=\frac{(a_1+\dots+a_n)!}{a_1!\dots a_n!}$$

$\square$

Thm (Karasev, Petrov 2012)

Let p be an odd prime. $m=\frac{p-1}{2}$

Let $d_1,d_2,\dots,d_m$ be integers not divisible by p.

Then there exists a permutation $\{x_1,x_2,\dots,x_m,\ y_1,y_2,\dots,y_m\}$ of $\{1,2,3,\dots,p-1\}$ such that

$$y_i-x_i\equiv d_i \pmod p$$

for all $i=1,2\dots m$.

Proof.

Let

$$f(x_1, x_2, \cdots, x_m) = (x_1+d_1)(x_2+d_2)\cdots(x_m+d_m) \times \prod_{1\le i<j\le m}(x_i-x_j)(x_i+d_i-x_j)(x_i-x_j-d_j)(x_i+d_i-x_j-d_j)$$

be a polynomial over $GF(p)$.

The degree of $f = m+4\binom{m}{2} = m+2m(m-1)$

$$= 2m^2 - m$$
$$= (p-1)m - m$$
$$= (p-2)m.$$

Let us consider the coefficient of $\prod_{i=1}^{m} x_i^{(p-2)}$

$= $ the coeff of $\prod x_i^{p-2}$ in $x_1x_2\cdots x_m \prod_{i<j}(x_i-x_j)^4$

$= $ the coeff of $\prod x_i^{p-3}$ in $\prod_{i<j}(x_i-x_j)^4$

$\boxed{\text{Take } a_i = 2}$ $\left(\sum a_j\right) - a_i = 2m-2$

$$= (p-1)-2 = p-3.$$

$$\Rightarrow = \frac{(2+2+\cdots+2)!}{2!2!\cdots 2!} = \frac{(2m)!}{2^m} \not\equiv 0 \pmod p$$

By the Combinatorial Nullstellensatz, there exist $x_i \in \{1, \ldots, p-1\}$ for all i such that

$$f(x_1, x_2, \ldots, x_n) \not\equiv 0 \pmod{p}$$

$\Rightarrow$ x_i's are distinct

$x_i + d_i$'s are distinct.

$$\left.\begin{array}{l} x_i \not\equiv x_j \pmod{p} \\ x_i + d_i \not\equiv x_j \pmod{p} \\ x_i + d_i \not\equiv x_j + d_j \pmod{p} \\ x_i \not\equiv x_j + d_j \pmod{p} \end{array}\right\}$$

Take $y_i \equiv x_i + d_i \pmod{p}$.

$\square$

7.11. Regular subgraphs

Thm (Alon, Friedland, Kalai 1984)

Every (multi)graph with average degree > 4 and maximum degree ≤ 5 contains a 3-regular subgraph.

Proof (Alon 1999)

Let $M = V \times E$ incidence matrix of G over $GF(3)$.

$$M = \begin{array}{c} \\ \\ \\ \end{array} \begin{pmatrix} & e & \\ & 1 & \\ & 1 & \\ & & \end{pmatrix} \updownarrow \text{vertices}$$

$$\longleftrightarrow$$

edges $\qquad E = E(G)$.

$m = |E(G)|$, $\quad n = |V(G)|$.

For $x \in (GF(3))^{E(G)}$,

$$f(x) = \prod_{v \in V} \left(1 - \left(\sum_{e \sim v} x_e\right)^2\right) - \prod_{e \in E} (1 - x_e)$$

$\begin{cases} 1 & \text{if } \sum_{e \sim v} x_e = 0 \pmod 3 \ \forall v \\ 0 & \text{otherwise} \end{cases}$

We will set $L_e = \{0,1\}$ for $e \in E$,

$$z^2 \equiv \begin{cases} 1 & \text{if } z \not\equiv 0 \pmod 3 \\ 0 & \text{if } z \equiv 0 \pmod 3 \end{cases}$$

- $f(0,0,\dots,0) = 1 - 1 = 0$.
- $f(X) \neq 0 \iff$ for each vertex v $\sum_{e \sim v} x_e \equiv 0 \pmod 3$

$x_i \in L_e$

$\Rightarrow \sum x_e = 0$ or 3,

H = subgraph of G consisting of edges such that $x_e = 1$

$\Rightarrow$ H has at least one edge & H is 3-regular.

So, enough to find $x_i \in L_i$ so that $f(X) \neq 0$.

The coefficient of $\prod_e x_e =$

$$\deg\left(\prod_v \left(1 - \left(\sum_{e \sim v} x_e\right)^2\right)\right) = \sum_v 2 = 2|V|.$$

$$\deg \prod (1 - x_e) = |E|$$

Since the average degree $> k$

$$\frac{2|E|}{|V|} > 4 \Rightarrow 4|V| < |E|.$$

$\therefore \deg(f) = |E|.$

The coeff of $\prod X_e = -(-1)^{|E|} \not\equiv 0 \pmod 3$

$\Rightarrow$ There exist $x_i \in L_i$

so that $f(x) \not\equiv 0 \pmod 3$

by Combinatorial Nullstellensatz $\square$

7.11. List Coloring

Let L be a mapping that maps a vertex of G to a set $L(v)$ of colors available to v.
list assignment

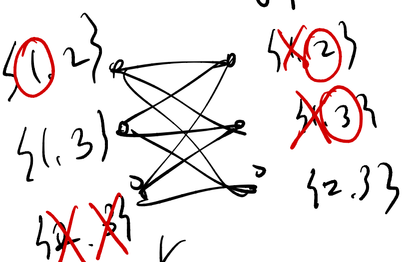
A coloring $c: V(G) \to \bigcup_{v \in V} L(v)$ is an <u>L-list-coloring</u> if whenever v and w are adjacent $c(v) \neq c(w)$.

We say G is L-list-colorable (L-choosable) if it has an L-list-coloring.

We say G is k-list-colorable (or k-choosable) if for any list assignment L of G with $|L(v)| \geq k$, G is L-list-colorable.

The list chromatic number $\chi_\ell(G)$ or choice number $ch(G)$ of a graph G is the minimum k such that G is k-list-colorable.

$$\underbrace{\chi(G) \leq \chi_\ell(G)}_{\text{gap could be big.}}$$



$K_{3,3}$

$\chi(K_{3,3}) = 2$

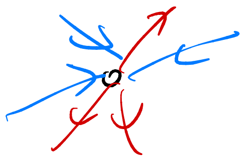
$\chi_\ell(K_{3,3}) > 2$

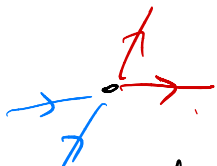
Thm (Alon, Tarsi)

Every simple bipartite planar graph is 3-choosable.

(planar + $\chi(G) \leq 2$ $\Rightarrow$ $\chi_\ell(G) \leq 3$)

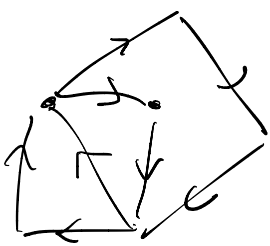
We say a directed graph D is Eulerian if at each vertex, the in-degree is equal to the out-degree.



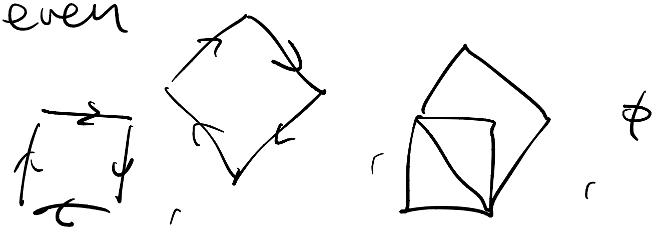


let $EE(D)$ = # Eulerian subgraphs of D with even # edges on the vertex set $V(D)$.

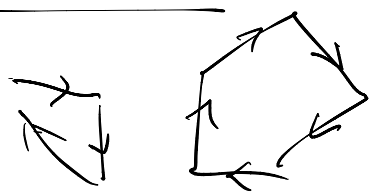
$EO(D)$ = # Eulerian subgraphs of D on the vertex set $V(D)$ with odd # edges.



even



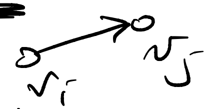
odd



Thm. Let D be a loopless directed graph on the vertex set $V = \{v_1, v_2, \dots, v_n\}$ and let d_i be the out-degree of v_i. Let L be a list assignment such that $|L(v_i)| > d_i$. If $EE(D) \neq EO(D)$, then D is L-list-colorable (as an undirected graph).

Proof. We assume $L(v) \subseteq \mathbb{Z}$, $|L(v)| = d_i + 1$.
We define

$$f_G(x_1, x_2, \dots, x_n) = \prod_{\underline{v_i v_j \in E(D)}} (x_i - x_j)$$



Goal: $f_G(c_1, c_2, \dots, c_n) \neq 0$ for some $c_i \in L(v_i)$ for each i

The degree of f_G = # edges $= |E(D)|$

$$= \sum_{i=1}^{n} d_i$$

$\Rightarrow$ Goal: Show that the coefficient of $\prod x_i^{d_i}$ is nonzero

Consider the expansion of f_G.

Let $DE(d_1, d_2, \dots, d_n)$ = # choices of ends of each edge so that x_i is chosen d_i times for each i and the head (-1) is chosen even # times.

$DO(d_1, \dots, d_n)$ = # choices of ends of each edge so that x_i is chosen d_i times for each i and the head (+) is chosen odd # times.

Then the coefficient of $\prod x_i^{d_i} =$

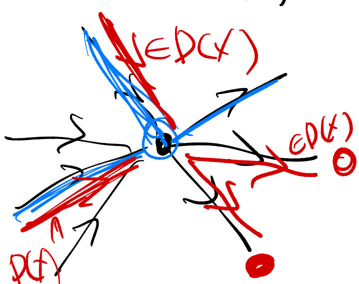
$$DE(d_1, d_2, \dots, d_n) - DO(d_1, d_2, \dots, d_n).$$

Claim: $DE(d_1, d_2, \dots, d_n) - DO(d_1, d_2, \dots, d_n) = EE(D) - EO(D)$

Proof by showing a bijection.

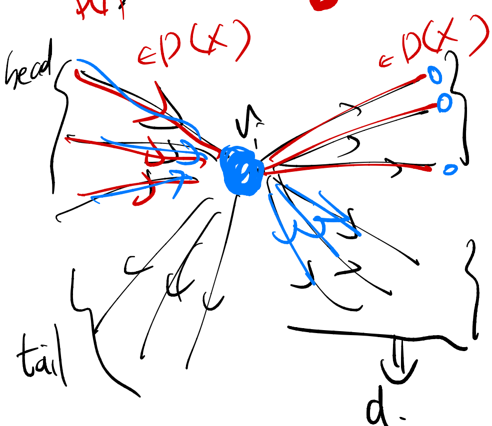
Consider a choice X of ends so that x_i is chosen d_i times at each i.

$D(X)$ = set of edges whose heads are chosen



$d_i = 3$

$\Rightarrow D(X)$ is an Eulerian subgraph of D.



blue edges $= d_i$

red edges $\to$ $D(X)$

$\Downarrow$

out-deg of v_i in $D(X)$

= in-deg of v_i in $D(X)$

D maps X to an Eulerian subgraph

If $X \in DE$, then

the head was chosen even # times

$\Rightarrow |E(D(X))|$ is even

If $X \in DO$, then $|E(D(X))|$ is odd.

$\therefore DE(d_1, \dots, d_n) - DO(d_1, \dots, d_n)$

$= EE(D) - EO(D)$. $\square$

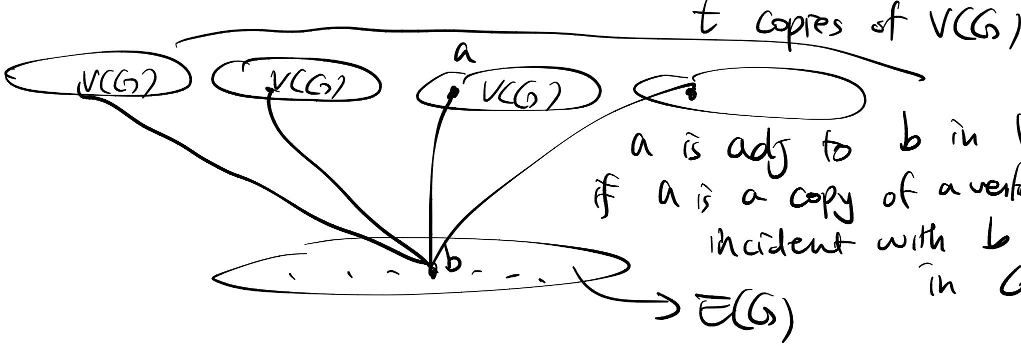
<u>Lemma</u>. Let G be an undirected graph.

Let $d(G) = \max\limits_{\substack{H:\text{subgraph} \\ \text{with} \geq 1 \text{ vertex}}} \frac{|E(H)|}{|V(H)|}$

Then there exists an orientation of G so that the out-degree of each vertex is $\leq \lceil d(G) \rceil$.

<u>Proof</u> Let $t = \lceil d(G) \rceil$

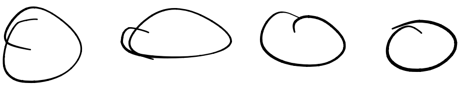
We construct a bipartite graph $H = (V', E')$



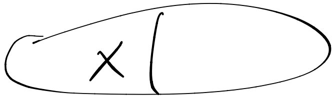
a is adj to b in H if a is a copy of a vertex incident with b in G.

We claim that H has a matching covering $E(G)$.

We use Hall's theorem; Suppose not.



Let $X \subseteq E(G)$, $X \neq \phi$



$$|N_H(X)| = t \cdot \left(\begin{array}{l}\#\text{vertices} \\ \text{incident with} \\ \text{some edges in } X\end{array}\right)$$

$$\frac{|X|}{|\#\text{vertices incident with } X|} \leq d(G) \leq t.$$

$$\Rightarrow \quad |X| \leq t \cdot (\#\text{vertices incident with } X)$$
$$= |N_H(X)|.$$

$$\therefore \quad |N_H(X)| \geq |X|$$

$\Rightarrow$ H has a matching covering $E(G)$.

$\Rightarrow$ G has an orientation so that each vertex is a tail of at most t edges. $\square$

Thm (Alon, Tarsi)
Every simple bipartite planar graph is 3-choosable

Proof.
By Euler's formula

$$|E(G)| \leq 2|V(G)| - \cancel{4}$$

$$\boxed{|E(G)| \leq 2|V(G)|}$$

$\Longrightarrow$ $d(G) \leq 2$

$\Rightarrow$ There is an orientation D of G such that each vertex has out-~~degree~~ degree ≤ 2.

~~EE~~

$\boxed{EO(D) = 0}$ (Since G is bipartite every cycle has even length and so every Eulerian subgraph of D has even # edges)

$EE(D) \neq 0$

(There is an Even Eulerian subgraph with $\emptyset$ as its set of edges)

For every L with $|L(v)| \geq 3 = 2+1$
G is L-list-colorable.

$\Longrightarrow$ G is 3-choosable. $\square$