RELATIVE EXPANDERS OR WEAKLY RELATIVELY RAMANUJAN GRAPHS

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Abstract

Let G be a fixed graph with largest (adjacency matrix) eigenvalue λ_0 *and with its universal cover having spectral radius* ρ. *We show that a random cover of large degree over G has its "new" eigenvalues bounded in absolute value by roughly* $\sqrt{\lambda_0 \rho}$.

This gives a positive result about finite quotients of certain trees having "small" eigenvalues, provided we ignore the "old" eigenvalues. This positive result contrasts with the negative result of A. Lubotzky and T. Nagnibeda which showed that there is a tree all of whose finite quotients are not "Ramanujan" in the sense of Lubotzky, R. Phillips, and P. Sarnak and of Y. Greenberg.

Our main result is a "relative version" of the Broder-Shamir bound on eigenvalues of random regular graphs. Some of their combinatorial techniques are replaced by spectral techniques on the universal cover of G. For the choice of G that specializes our main theorem to the Broder-Shamir setting, our result slightly improves theirs.

1. Introduction

The term *Ramanujan* has arisen in connection with the eigenvalues or spectrum of a graph, or more precisely the graph's adjacency matrix.* In [G] a finite graph, X, is called *Ramanujan* if $\mathrm{Spec}(X) \subset [-\rho, \rho] \cup \{-\lambda_0, \lambda_0\}$, where ρ is the spectral radius of X's universal cover (i.e., of the adjacency matrix thereof), and λ_0 is the Perron-Frobenius (or largest) eigenvalue of X. If X is k-regular, then this means that $\lambda = \pm k$ or $|\lambda| \le 2\sqrt{k-1}$ for each eigenvalue, λ, of X; this agrees with the definition in [LPS].

Lubotzky and Nagnibeda (see [LN]) have shown that there are trees, T, with finite quotients where none of these quotients are Ramanujan in the above sense. We soon explain why this negative result may be considered surprising. The main goal

*It arose because the proof that certain graphs' eigenvalues (in [LPS]) were small relied upon known parts of the Ramanujan conjectures.

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of this paper is to show that there is a positive result for “most” finite quotients of a tree, provided that one weakens the notion of being Ramanujan and provided that one considers a “relative” notion of being weakly Ramanujan. (We also conjecture that “weakly Ramanujan” can be replaced by “Ramanujan.”) To do so, we “relativize” the Broder-Shamir method for bounding the second eigenvalue (in [BS]), generalizing their result and slightly improving it in the original setting (and, to us, the special setting) of regular graphs.

It is known that for certain k there are infinitely many k-regular graphs that are Ramanujan (see [LPS], [M], [Mo]). Furthermore, it is known that “most” k-regular graphs* with k even are “weakly Ramanujan” in the following sense. Say that X is *ν-weakly Ramanujan* if $\mathrm{Spec}(X) \subset [-\nu, \nu] \cup \{-\lambda_0, \lambda_0\}$. (Also, we usually insist that $\nu < \lambda_0$ to prevent a trivial situation.) Then building a k-regular graph from $k/2$ permutations (assuming that k is even), a number of papers have shown that most graphs are ν-weakly Ramanujan (see [BS], [FKS], [F1]) for certain values of ν; for example, in [F1] it is shown that there is a constant C such that most k-regular graphs on a sufficiently large number of vertices are ν-weakly Ramanujan with $\nu = 2\sqrt{k} + 2\log k + C$. Furthermore, numerical experiments (like those in [F2]) suggest that most random regular graphs on a large number of vertices are Ramanujan.

It therefore seems plausible to conjecture that most k-regular graphs are Ramanujan, that is, most finite quotients of the k-regular tree, T, are Ramanujan (where the word “most” is given any “reasonable” interpretation). This makes the negative result of Lubotzky and Nagnibeda surprising: the notion of Ramanujan seems highly dependent on the tree.

For what follows, we recall the notion of a *covering map*. If G, H are undirected graphs without multiple edges or self-loops, a morphism (i.e., graph homomorphism) $\pi\colon H \to G$ is called a *covering map* if for every vertex, h, of H, π gives a bijection from the edges incident upon h with those incident upon $\pi(h)$. Also, G is called the *base graph* and H the *covering graph*. If G is connected, then the size of π^{-1} of a vertex or edge is constant and is called the *degree* of the covering map. We can also define *covering maps* for graphs that are directed and/or have multiple edges and/or self-loops (see Sec. 5).

If A_H, A_G are the adjacency matrices of finite graphs H, G with a covering map $\pi\colon H \to G$, then any A_G eigenfunction, f, pulls back to an eigenfunction $\pi^* f = f \circ \pi$ of A_H. Such an eigenfunction is called an *old eigenfunction (for π)*, and the resulting eigenvalue of A_H from A_G is an *old eigenvalue*. Since A_G is symmetric, the linear span of the old eigenfunctions is the space of functions that are pullbacks,

*Here “most” means in the sense of the random k-regular graph used by A. Broder and E. Shamir, to be described later in this paper. This is not the same as the “uniform regular graph” model, but the two models are contiguous (see [GJKW]).

$\pi^* f = f \circ \pi$, of an arbitrary f on G; this space is called the space of *old functions*. Its orthogonal complement is called the space of *new functions*, which are just those functions that sum to zero on each "vertex fiber," $\pi^{-1}(v)$, for all vertices, v, of G. A *new eigenfunction/value* is an eigenfunction/value coming from a new function. Since A_H is symmetric, the new and old eigenpairs give a complete set of eigenpairs of A_H.

The result of Lubotzky and Nagnibeda uses the fact that there are many graphs, G, such that any finite quotient of G's universal cover admits a covering map to G. If such a G's eigenvalues are outside $[-\rho, \rho] \cup \{-\lambda_0, \lambda_0\}$ as above, none of T's finite quotients will be Ramanujan. In this paper we show that in this situation the new eigenvalues, that is, those not coming from G, are weakly Ramanujan.

More generally, in this paper we study the following notion.

Definition 1.1
A covering map of graphs, $\pi : H \to G$, is called ν-*weakly Ramanujan* if the new spectrum of the cover lies in $[-\nu, \nu]$, and it is called *Ramanujan* if we may take ν to be the spectral radius of the universal cover of G.

We prove a generalization of the Broder-Shamir result (the expected eigenvalue result in [BS]). For any graph, $G = (V, E)$, we form a probability space of degree n covers of G, denoted $\mathscr{C}_n(G)$, as follows: our random graph has vertex set $V_n = V \times \{1, \ldots, n\}$, and for each $e \in E$ we choose an arbitrary orientation of e, (u, v), and choose uniformly a random permutation, σ_e, on $\{1, \ldots, n\}$ (permutations of different edges are chosen independently); we form edges from (u, i) to $(v, \sigma_e(i))$ for all i. This model of random cover (sometimes "random lift") has also been studied in [AL1], [AL2], [ALM], and [LR].

THEOREM 1.2
Let G be a fixed graph, let λ_0 denote the largest eigenvalue of G, and let ρ denote the spectral radius of the universal cover of ρ. There is a function $\alpha(n)$ such that $\alpha(n) \to 0$ as $n \to \infty$, and there are positive constants C_1, C_2 such that the expected value is

$$\mathrm{E}_{\mathscr{C}_n(G)}\Big(\sum_{\lambda \text{ new}} \lambda^t\Big) \le C_2 \nu^t,$$

where

$$\nu = \sqrt{\lambda_0 \rho} + \alpha(n)$$

and $0 < t \le 2\lfloor C_1 \log n \rfloor$. *This theorem holds for G containing multiple edges and self-loops, with $\mathscr{C}_n(G)$ replaced by any Broder-Shamir family of models of a random cover of degree n (as in Sec. 5).*

In particular, the probability of a graph in $\mathscr{C}_n(G)$ being ν-weakly Ramanujan with $\nu = \sqrt{\lambda_0\rho} + \alpha(n)$ goes to 1 as $n \to \infty$ for some function, $\alpha(n)$, with $\alpha(n) \to 0$ as $n \to \infty$.

A more precise form of Theorem 1.2 and some of its implications (including a precise description of the $\alpha(n)$ above) are given in Sections 2 and 5.

We claim that Theorem 1.2 gives a positive result as mentioned earlier. Indeed, it is not hard to see that there are many trees, T (including those occurring in [LN]), such that for some graph, G, any finite quotient of T occurs in $\mathscr{C}_n(G)$ for the appropriate n; for example, from [LN] there are graphs, G (without half-loops), such that every finite quotient of G's universal cover admits a covering map to G. It is easy to see that all covers of G (without half-loops) occur in $\mathscr{C}_n(G)$, and that the probability of a cover, H, of $G = (V, E)$ occurring is

$$(n!)^{|V|-|E|}/|\operatorname{Aut}(H/G)|,$$

where $\operatorname{Aut}(H/G)$ is the group of automorphisms of H over G (see [F2]). $\mathscr{C}_n(G)$ becomes a seemingly reasonable model of a probabilistic space of finite quotients of T of a given number of vertices. Our generalization of the Broder-Shamir result says that most of the resulting covering maps are weakly Ramanujan.

We remark that there are trees, T, that admit a finite quotient (and therefore infinitely many finite quotients) such that there is no "minimal" finite quotient, G, covered by all finite quotients. However, according to [F2], there is a minimal *pregraph* (in the sense of [F2]) that is covered by all finite quotients. It is therefore important to generalize the results of this paper to pregraphs, for example, to generalize Theorem 1.2 to allow G to be a pregraph. This is the subject of a work of the author in progress.

If in Theorem 1.2 we take G to have one vertex with $d/2$ whole-loops (see Sec. 5), then we are in the setting of d-regular graphs generated by $d/2$ permutations as in [BS]; however, our result slightly improves upon that in [BS]. One key point in the Broder-Shamir trace method is to estimate the number of closed walks from a given vertex on a tree of a given length; their estimate (see [BS, Lem. 5]) involves a weaker estimate of this number than the estimate we use; we use the beautiful (and simple) estimate based on the spectral radius of the tree, as done in [B].

We mention that our strengthening of the Broder-Shamir result is interesting for the following reason. The eigenvalue estimates for random graphs proven by the author and by J. Kahn and E. Szemerédi (in [FKS], [F1]) involve undetermined constants; hence there is no known *fixed* value of the degree, k, for which their estimates are nontrivial; it is known only that as $k \to \infty$ their results become interesting (and ultimately improve upon those of Broder and Shamir). However, the original Broder-Shamir result yields $(\alpha(n) + 2^{1/2}k^{3/4})$-weakly Ramanujan (for "most" graphs) where $\alpha(n) \to 0$ as $n \to \infty$ (for k even); this result is interesting for every even $k > 4$. So our strengthening of the Broder and Shamir result gives new interesting bounds for random k-regular graphs for small k and any particular fixed even value of $k > 2$. In our result the $2^{1/2}k^{3/4}$ is improved to $\sqrt{2k}(k-1)^{1/4}$.

Another interesting note is that our version of Broder and Shamir gives the first direct* results for the k-regular random graph model based on k perfect matchings (when the degree of the cover is even). Thus we obtain the first direct results for odd degree random graphs (by taking G to be one vertex with half-loops; see Sec. 5).

The rest of this paper is organized as follows. In Sections 2 and 3 we prove Theorem 1.2 in the case where the base graph, G, has no self-loops or multiple edges; this gives us the essential ideas to prove Theorem 1.2 in any case. In Section 2 we also give a more precise form of Theorem 1.2 (Th. 2.7) and a number of interesting consequences. In Section 4 we give a relative version of the Alon-Boppana bound, which is a new eigenvalue lower bound (for any graph cover) to complement the Broder-Shamir theorems of Section 2; namely, we show that any cover of G of degree n has a new eigenvalue as large as $\rho - \alpha(n)$ with $\alpha(n) \to 0$ as $n \to \infty$. In Section 5 we describe some generalizations of the Broder-Shamir and Alon-Boppana theorems for a general base graph, and we give some directions for future work.

2. A simple case

Our main theorems (Ths. 1.2, 2.7) are less awkward to prove when the base is a graph with no self-loops or multiple edges. We first deal with this case, assuming the model $\mathscr{C}_n(G)$ in the introduction; this case illustrates all the main ideas. The more general situation follows the same ideas and is described in Section 5.

We wish to use the trace method to bound the eigenvalues of H, a random element in $\mathscr{C}_n(G)$. This means we bound the expected value of the trace of the adjacency matrix of H; that is, we bound the probability that a walk of a given length from a given vertex results in a cycle.

Throughout this section, if e is oriented as (u, v) (for the purpose of forming our random graph cover, H, in $\mathscr{C}_n(G)$ from the σ_e's as in the introduction), we may write $\sigma_{u,v}$ for σ_e and $\sigma_{v,u}$ for σ_e^{-1}.

So given a vertex in H, $u_0 = (v_0, i_0)$, a walk in H starting from u_0 is determined by its projection in G. The walk in H will be a cycle precisely when the following two conditions hold:

(1) the corresponding walk in G is a cycle, $v_0, v_1, \ldots, v_k = v_0$, and

(2) we return to the original vertex over v_0 in H, that is,

$$i_0 = \sigma_{v_{k-1},v_k} \circ \sigma_{v_{k-2},v_{k-1}} \circ \cdots \circ \sigma_{v_0,v_1}(i_0). \quad (1)$$

*One can get "indirect" results on odd degree random graphs by starting with an even degree random graph (with the Broder-Shamir model) and adding a perfect matching (assuming an even number of vertices).

With the cycle $v_0, v_1, \ldots, v_k = v_0$ we associate the *cyclic word*

$$w = \sigma_{v_{k-1},v_k}\sigma_{v_{k-2},v_{k-1}}\cdots\sigma_{v_0,v_1},$$

and we write $P(w)$ for the probability that equation (1) holds for a fixed i_0. (Clearly, this probability is independent of i_0.)

More generally, by a *word* we mean a string

$$\sigma_{v_{k-1},v_k}\sigma_{v_{k-2},v_{k-1}}\cdots\sigma_{v_0,v_1},$$

where $\{v_i, v_{i+1}\}$ is an edge in G for all i, and where v_k need not equal v_0.

If A_H is the adjacency matrix of H, then clearly,

$$\mathrm{E}\big(\mathrm{Tr}(A_H^k)\big) = \sum_{w\in W_k} P(w)n,$$

where W_k is the collection of all cyclic words of length k in G. The problem is reduced to estimating this sum involving the $P(w)$'s.

First we notice that $\sigma_{v,v'}\sigma_{v',v}$ is always the identity. Thus to evaluate $P(w)$, we may cancel all consecutive pairs of inverses in w, potentially reducing the size of w. We call the new word obtained the *reduction* of w (which is easily seen to be independent of the order in which the reductions are made). If Irred_m denotes the irreducible cyclical words of length m, we have

$$\sum_{w\in W_k} P(w) = \sum_{m=0}^{k} \sum_{w\in \mathrm{Irred}_m} P(w)n_k(w),$$

where $n_k(w)$ denotes the number of cyclical words of length k that reduce to w. Of course, $n_k(w) = 0$ if k and $|w|$, the length of w, have different parity.

LEMMA 2.1 (M. Buck)
Let e be the empty word. Then $n_k(e) \le |V_G|\rho^k$, where ρ is the spectral radius of the adjacency matrix of the universal cover of G, and V_G is the set of vertices of G.

Proof
We repeat the proof from [B] (part of [B, Prop. 3.1]) since we use the same idea for bounding the number of other types of walks. Let x be a vertex of the universal cover, T, of G, and let A_T be the adjacency matrix of T. By spectral theory, we know that the bounded operator A_T is self-adjoint, and hence $\|A_T\| = \rho$. Then if δ_x is the function that is 1 on x and 0 on other vertices,

$$(\delta_x, A_T^k\delta_x) \le \|A_T\|^k\|\delta_x\|^2 = \rho^k.$$

But the left-hand side of the above equation corresponds to those walks on x's image in G whose corresponding cyclical word reduces to e. So applying this to one x for each vertex in G yields the lemma. □

Clearly, $P(e) = 1$ when e is the empty word. Hence

$$\sum_{w \in W_k} P(w) \leq n|V_G|\rho^k + \sum_{m=1}^{k} \sum_{w \in \mathrm{Irred}_m} P(w) n_k(w).$$

Next we relativize two of the key lemmas in the Broder-Shamir analysis.

LEMMA 2.2
Let w be an irreducible cyclic word of length $k > 0$ that is not of the form $w = w_a^{-1} w_b^j w_a$ for any words w_a, w_b with $w_b \neq e$ and $j \geq 2$. Then

$$P(w) \leq \frac{1}{n-k} + \binom{k}{2} \frac{k^2}{(n-k)^2}.$$

Proof
The proof is essentially the same as in [BS]. We explain this approach in our context in Section 3; the lemma is an immediate consequence of Lemmas 3.1, 3.2, and 3.5. □

LEMMA 2.3
Let w be any irreducible cyclic word of length k. Then

$$P(w) \leq \frac{k}{n-k} + \binom{k}{2} \frac{k^2}{(n-k)^2}.$$

Proof
Similarly, this lemma is an immediate consequence of Lemmas 3.1, 3.2, and 3.6, and it is essentially the same as in [BS]. □

We now need another counting lemma, using spectral techniques as in [B].

LEMMA 2.4
The number of cyclic words of length k that reduce to a word of the form $w_a^{-1} w_b^j w_a$ with $w_b \neq e$ and $j \geq 2$ is at most

$$|V_G| k(k-1) \binom{k}{2} \rho^k.$$

Proof
If $w = w_a^{-1} w_b^j w_a$ with $w_b \neq e$ and $j \geq 2$, then there is a "cyclic shift," $\widetilde{w}$, of w,

$$\widetilde{w} = \sigma_{v_t,v_{t+1}} \sigma_{v_{t-1},v_t} \cdots \sigma_{v_0,v_1} \sigma_{v_{k-1},v_k} \cdots \sigma_{v_{t+1},v_{t+2}},$$

such that $\widetilde{w}$ reduces to w_b^j. Since there are k cyclic shifts of w, it suffices to show that the number of words of length k reducing to a word of the form w_b^j with $w_b \neq e$ and $j \geq 2$ is at most $|V_G|(k-1)\binom{k}{2}\rho^k$.

For any vertex $v_0 \in V$, fix a vertex x of the universal cover, T, of G, lying over v_0. Each irreducible word w_b beginning with σ_{v_0,v_1} for some vertex v_1 corresponds uniquely to a vertex, y, of T. A word reduces to w_b^j with $j \geq 2$ precisely when its corresponding walk starting at x (in T) does the following:

(1) for some $\ell_1 > 0$ its first ℓ_1's σ's reach y, thereby "tracing out" w_b,

(2) for some $\ell_2 > 0$ its next ℓ_2's σ's again trace w_b, and

(3) the rest of its σ's trace w_b^i for some $i \geq 0$.

It follows that the number of such words is bounded by

$$\sum_{i=0}^{k-2} \sum_{\ell_1+\ell_2 \leq k} (A_G^{\ell_1}\delta_x)(y)(A_G^{\ell_2}\delta_x)(y)(A_G^{k-\ell_1-\ell_2}\delta_x)(y^i),$$

where y^i is the vertex corresponding to w_b^i, and where A_G is the adjacency matrix of G. Summing over all $y \neq x$ yields a bound for the words with reduction to w_b^j, $j \geq 2$, and any $w_b \neq e$. We now estimate as follows:

$$(A_G^{k-\ell_1-\ell_2}\delta_x)(y^i) \leq \|A_G^{k-\ell_1-\ell_2}\delta_x\|_2 \leq \rho^{k-\ell_1-\ell_2}\|\delta_x\|_2 = \rho^{k-\ell_1-\ell_2}.$$

Hence

$$\begin{aligned}
&\sum_{y\neq x}(A_G^{\ell_1}\delta_x)(y)(A_G^{\ell_2}\delta_x)(y)(A_G^{k-\ell_1-\ell_2}\delta_x)(y^i) \\
&\qquad \leq \rho^{k-\ell_1-\ell_2}\sum_{y\neq x}(A_G^{\ell_1}\delta_x)(y)(A_G^{\ell_2}\delta_x)(y) \\
&\qquad \leq \rho^{k-\ell_1-\ell_2}(A_G^{\ell_1}\delta_x, A_G^{\ell_2}\delta_x) \leq \rho^{k-\ell_1-\ell_2}\rho^{\ell_1}\rho^{\ell_2} = \rho^k.
\end{aligned}$$

It follows that the total number of words of length k that reduce to a word of the form w_b^j with $j \geq 2$ and w_b beginning at a fixed vertex, v_0, is at most

$$\sum_{i=0}^{k-2} \sum_{\ell_1+\ell_2 \leq k} \rho^k = (k-1)\sum_{\ell_1+\ell_2 \leq k} \rho^k = (k-1)\binom{k}{2}\rho^k,$$

recalling that the ℓ_i are positive integers.

Hence the total number of words of length k that reduce to a word of the form w_b^j with $j \geq 2$ with $w_b \neq e$ is at most

$$|V_G|(k-1)\binom{k}{2}\rho^k. \qquad \square$$

Combining Lemmas 2.1 – 2.4 yields the following lemma.

LEMMA 2.5
We have

$$\mathrm{E}\big(\mathrm{Tr}(A_H^k)\big) \leq |V_G|\rho^k n + |V_G|k(k-1)\binom{k}{2}\rho^k\left(\frac{kn}{n-k} + \binom{k}{2}\frac{k^2 n}{(n-k)^2}\right)$$
$$+ \mathrm{Tr}(A_G^k)\left(\frac{n}{n-k} + \binom{k}{2}\frac{k^2 n}{(n-k)^2}\right).$$

In particular, if $k \leq n/2$, we have

$$\mathrm{E}\big(\mathrm{Tr}(A_H^k)\big) \leq |V_G|\rho^k(n + 2k^8) + \mathrm{Tr}(A_G^k) + \frac{|V_G|\lambda_0^k 4k^4}{n}.$$

Proof
There are $\mathrm{Tr}(A_G^k)$ cyclic walks of length k in G. Each walk either
(1) reduces to e,
(2) reduces to $w_a^{-1} w_b^j w_a$ with $w_b \neq e$ and $j \geq 2$, or
(3) does neither (1) nor (2).
In case (1) we have $P(w) = 1$, and in the other cases we use one of Lemmas 2.1 – 2.4 to bound $P(w)$. The first statement follows, and the second statement follows from the first, using the bound $\mathrm{Tr}(A_G^k) \leq |V_G|\lambda_0^k$. $\square$

Finally, we arrive at the essential eigenvalue estimate.

THEOREM 2.6
If $k \leq n/2$, then we have

$$\mathrm{E}\Big(\sum_{\lambda \text{ new}} \lambda^k\Big) \leq |V_G|\rho^k(n + 2k^8) + \frac{|V_G|\lambda_0^k 4k^4}{n}. \tag{2}$$

Proof
We have

$$\mathrm{Tr}(A_H^k) = \Big(\sum_{\lambda \text{ old}} \lambda^k\Big) + \Big(\sum_{\lambda \text{ new}} \lambda^k\Big) = \mathrm{Tr}(A_G^k) + \Big(\sum_{\lambda \text{ new}} \lambda^k\Big),$$

so the theorem follows from Lemma 2.5. □

We apply Theorem 2.6 with $k = 2\lfloor \log n / \log(\lambda_0/\rho)\rfloor$, assuming $\lambda_0 > \rho$. For this value of k there are positive constants c_1, c_2 for which

$$c_1\left(\frac{\lambda_0}{\rho}\right)^{k/2} \le n \le c_2\left(\frac{\lambda_0}{\rho}\right)^{k/2}.$$

(Actually, one can take $c_2 = 1/c_1 = \lambda_0/\rho$.) The next theorem follows almost at once.

THEOREM 2.7
Let G be fixed. There is a C such that for any n, setting $k_0 = 2\lfloor \log n / \log(\lambda_0/\rho)\rfloor$, we have for any $k \le k_0$,

$$\mathrm{E}_{\mathscr{C}_n(G)}(\rho_{\mathrm{new}}^k) \le (Ck_0)^{4k/k_0}(\lambda_0\rho)^{k/2}. \tag{3}$$

Proof
The case $k = k_0$ follows easily from Theorem 2.6. That k can be taken smaller follows from Jensen's inequality. □

We now state a number of consequences.

COROLLARY 2.8
For fixed G we have

$$\mathrm{E}_{\mathscr{C}_n(G)}(\rho_{\mathrm{new}}) \le \sqrt{\lambda_0\rho} + O\left(\frac{\log\log n}{\log n}\right).$$

Proof
We take $k = 1$ in Theorem 2.7, whereupon

$$(Ck_0)^{4k/k_0} \le e^{C'\log(k_0)/k_0} \le 1 + \frac{C''\log(k_0)}{k_0} \le 1 + \frac{C'''\log\log n}{\log n}.$$

Applying this to equation (3) yields the following corollary. □

COROLLARY 2.9
For any fixed G and $B > 0$, there are positive constants C_1, C_2 such that

$$\rho_{new} \ge \sqrt{\lambda_0\rho}\,\bigl(1 + \alpha(n)\bigr)$$

in $\mathscr{C}_n(G)$ with probability at most

$$C_1(\log n)^4 n^{-C_2\alpha(n)}$$

for any $\alpha(n) \le B$.

Proof
If P is the aforementioned probability, then

$$\mathrm{E}_{\mathscr{C}_n(G)}(\rho_{\mathrm{new}}^k) \geq P\left(\sqrt{\lambda_0 \rho}\ (1+\alpha(n))\right)^k$$

for any k. Now take $k = k_0$ as in Theorem 2.7; equation (3) implies that

$$P\left(1+\alpha(n)\right)^{k_0} \leq (Ck_0)^4.$$

Since k_0 is proportional to $\log n$, the corollary follows. □

Corollary 2.9, in turn, has various corollaries depending on which function $\alpha(n)$ we choose. If we take $\alpha(n)$ to be constant, we conclude the following theorem.

THEOREM 2.10
For any fixed G and $\epsilon > 0$, there are $C, \delta > 0$, such that the largest new eigenvalue is at least $(1+\epsilon)\sqrt{\lambda_0\rho}$ with probability at most $Cn^{-\delta}$.

We also conclude another theorem by taking $\alpha(n) = C \log\log n / \log n$ with C sufficiently large.

THEOREM 2.11
For a fixed G there is a C such that the probability that ρ_{new} is at most $\sqrt{\lambda_0\rho} + C \log\log n/\log n$ goes to 1 as $n \to \infty$.

3. The Broder-Shamir approach

In this section we describe the remarkable and beautiful approach of Broder and Shamir [BS] to analyzing the $P(w)$'s of Section 2 and proving Lemmas 2.2 and 2.3.

Fix a word, w, of length k. (We may later insist that w be irreducible.) To study $P(w)$, let

$$w = \sigma_{v_{k-1},v_k}\sigma_{v_{k-2},v_{k-1}} \cdots \sigma_{v_0,v_1},$$

and fix an $i_0 \in \{1,\ldots,n\}$. We determine where w takes (v_0, i_0) by determining the steps of the walk in order, that is, first determining $i_1 = \sigma_{v_0,v_1}(i_0)$, then $i_2 = \sigma_{v_1,v_2}(i_1)$, and so on. Initially, we view all $\sigma_{u,v}$'s as "completely random" or "completely undetermined," each taking on any one of the $n!$ permutations on $\{1,\ldots,n\}$ with the same probability. Then we determine $i_1 = \sigma_{v_0,v_1}(i_0)$ as being chosen from $\{1,\ldots,n\}$, each with probability $1/n$. This determining of i_1 conditions the $\sigma_{u,v}$'s in that $\sigma_{v_0,v_1}(i_0)$ is fixed (as is $\sigma_{v_1,v_0}(i_1)$) and σ_{v_0,v_1} can now take on only $(n-1)!$ possible permutations. Assume that for some s we have determined $i_j = \sigma_{v_{j-1},v_j}(i_{j-1})$ for $j = 1,\ldots,s-1$, and now we wish to determine $i_s = \sigma_{v_{s-1},v_s}(i_{s-1})$. There are two possibilities:

(1) a *forced choice*, where $\sigma_{v_{s-1},v_s}(i_{s-1})$ has already been determined (previously in the walk), and
(2) a *free choice*, where $\sigma_{v_{s-1},v_s}(i_{s-1})$ has not been determined.

For a free choice, i_s takes on one of possibly $n-t$ values from 1 to n with equal probability, where t is the number of values of σ_{v_{s-1},v_s} which have been determined up to that point; clearly, $t \le s-1$.

For a free choice, we say that *a coincidence* has occurred if (v_s, i_s) has been previously visited in the walk; that is, $(v_s, i_s) = (v_j, i_j)$ for some $j < s$ (with $j = 0$ possible). A coincidence occurs with probability at most $(s-1)/(n-s+1)$.

We record the following two simple but important observations.

LEMMA 3.1
Fix a word, $w = \sigma_{v_{k-1},v_k} \cdots \sigma_{v_0,v_1}$, of length k, and fix an i_0. The probability that the walk determined by w and i_0 has two or more coincidences is at most

$$\binom{k}{2} \frac{k-1}{n-k+1} \, \frac{k-2}{n-k+2}.$$

Proof
There are $\binom{k}{2}$ ways of choosing two of the choices of $i_1, \ldots, i_k$ to be both coincidences; the first coincidence occurs with probability at most $(k-1)/(n-k+1)$, and the second at most $(k-2)/(n-k+2)$. □

LEMMA 3.2
If w is irreducible, $k > 0$, and there are no coincidences, then $i_k \neq i_0$. Moreover, $(v_s, i_s) \neq (v_t, i_t)$ for any $s \neq t$.

Proof
Assume, to the contrary, that there are s, t with $0 \le s < t \le k$ with $(v_s, i_s) = (v_t, i_t)$. Let s, t be as such with t as small as possible. The minimality of t implies that $(v_s, i_s) \neq (v_r, i_r)$ for any $0 \le s < r \le t-1$.

Since $(v_s, i_s) = (v_t, i_t)$ and since i_t was not a coincidence, $\sigma_{v_{t-1},v_t}(i_{t-1})$ was already determined. But this can happen only in the case where for some $j < t$ we have either

(1) $(v_t, i_t) = (v_j, i_j)$ and $(v_{t-1}, i_{t-1}) = (v_{j-1}, i_{j-1})$, or
(2) $(v_t, i_t) = (v_{j-1}, i_{j-1})$ and $(v_{t-1}, i_{t-1}) = (v_j, i_j)$.

Case (1) is impossible since $(v_{t-1}, i_{t-1}) = (v_{j-1}, i_{j-1})$ contradicts the minimality of t. Case (2) requires $j = t-1$ to avoid having $(v_{t-1}, i_{t-1}) = (v_j, i_j)$ contradict the minimality of t; but then $v_t = v_{j-1} = v_{t-2}$, and w is reducible (since it contains the subword $\sigma_{v_{t-1},v_t}\sigma_{v_{t-2},v_{t-1}} = \sigma_{v_{t-1},v_t}\sigma_{v_t,v_{t-1}}$). Hence both cases (1) and (2) lead to contradictions, and so we derive a contradiction by our assumption that $(v_s, i_s) = (v_t, i_t)$ for some $s \neq t$. $\square$

Essentially, the same proof yields the following stronger lemma.

LEMMA 3.3

Let w be an irreducible cyclic word of length k, and assume that i_p (as above) is a free choice for some p between 1 and k. Let none of $i_p, i_{p+1}, \ldots i_k$ be a coincidence (i.e., each is either a forced choice or a free choice that is not a coincidence). Then the vertices (v_t, i_t) for $t \geq p$ will all be distinct and will not coincide with any vertex (v_r, i_r) for $r < p$.

Proof

We are claiming that $(v_s, i_s) \neq (v_t, i_t)$ for any $s < t$ and $t \geq p$. If not, again fix an s and t with t minimal; clearly, $t > p$ since i_p is a free choice and not a coincidence. The same two-case analysis as in the proof of Lemma 3.2 yields a contradiction. $\square$

LEMMA 3.4

Let w be an irreducible cylcic word such that $i_k = i_0$ in which only one coincidence occurs. Then for some $j \geq 1$ we may write $w = w_a^{-1} w_b^j w_a$ where

(1) *$w_b w_a$ is irreducible, and*

(2) *if $|w_a| = s$ and $|w_b| = t$, then the coincidence occurs at i_{t+s}, the coincidence being $(v_{t+s}, i_{t+s}) = (v_s, i_s)$.*

Proof

Clearly, there are unique s, t such that the coincidence is $(v_{t+s}, i_{t+s}) = (v_s, i_s)$. Let w_a be the word from i_0 to i_s, and let w_b be that from i_{s+1} to i_{s+t}. After i_{t+s}, all other choices must be forced, in view of Lemma 3.3 and the fact that $i_k = i_0$ and that there is exactly one coincidence occurring. At i_{s+t+1}, we must either

(1) begin to follow w_b, or

(2) begin to follow w_a^{-1}.

Since w is irreducible, if we begin to follow w_b we must traverse it in its entirety, returning to (v_s, i_s) again. Eventually we follow w_a^{-1}, whereupon the irreducibility of w implies that we end and reach (v_k, i_k) when we finish traversing w_a^{-1}. This implies the lemma. $\square$

LEMMA 3.5

Let w be irreducible of length $k > 0$. Assume that $w \neq w_a^{-1} w_b^j w_a$ for any irreducible words w_a, w_b with $j \geq 2$. Then the probability that $i_k = i_0$ and exactly one *coincidence occurs is at most*

$$\frac{1}{n-k+1}.$$

Proof

Let w_a be the longest irreducible subword of w such that $w = w_a^{-1} w_b w_a$ (with $w_b w_a$ irreducible). If $|w_a| = s$, then $i_k = i_0$ if and only if $i_{k-s} = i_s$. By Lemma 3.4, $i_1, \ldots, i_{k-s-1}$ are free choices, and i_{k-s} is a coincidence and must take on the value i_s. This coincidence occurs with probability at most

$$\frac{1}{n-k+s+1} \leq \frac{1}{n-k+1}. \qquad \square$$

Similarly, we have the following useful lemma.

LEMMA 3.6

Let w be any irreducible word of length $k > 0$. Then the probability that $i_k = i_0$ and exactly one coincidence occurs is at most

$$\frac{k}{n-k+1}.$$

Proof

Let w_a be the longest irreducible subword of w such that $w = w_a^{-1} w_c w_a$ (with $w_c w_a$ irreducible). There are at most k positive integers, j, such that $w_c = w_b^j$. Lemma 3.4 shows that $i_k = i_0$ requires there to be such a j, and for each j there is one specific coincidence (of the form $i_{s+t} = i_s$ for a given s and t) that must occur. For each j-value, the associated event occurs with probability at most $1/(n-k+1)$. $\square$

4. Alon-Boppana bounds

Fix a graph, G, whose universal cover has spectral radius ρ. In this section we explain that the largest new eigenvalue of a cover, H, of G of degree n must be at least $\rho - \alpha(n)$, where $\alpha(n)$ is a function of n tending to zero as $n \to \infty$. The case of d-regular graphs, where G is a bouquet of loops of total degree d, was first claimed in [A] (as due to N. Alon and R. Boppana) and appears in [N].

THEOREM 4.1

Let G be a fixed graph. There exists a function $\alpha = \alpha(n)$ defined for a positive integer n such that

(1) $\alpha(n) \to 0$ *as* $n \to \infty$, *and*

(2) *for any covering map* $\pi : H \to G$ *of degree* n, *there is a new eigenvalue of absolute value at least* $\rho - \alpha(n)$.

In [N], where G is a bouquet of loops of total degree d, $\alpha(n)$ was shown to be, at most, proportional to $1/\log n$. In the independent works of J. Friedman and N. Kahale (see [F2]), $\alpha(n)$ was shown to be, at most, proportional to $1/\log^2 n$. We use a weaker technique to prove Theorem 4.1 which does not estimate $\alpha(n)$.

Proof
We make use of the following lemma, which is a special case of part of [B, Prop. 3.1].

LEMMA 4.2 (Buck)
Let G be a connected graph, and fix a vertex $v \in V_G$. Then for any $\epsilon > 0$, there is an r_0 such that the number of walks of length $2r$ from v to itself is at least $(\rho - \epsilon)^{2r}$, provided that $r \geq r_0$.

By Lemma 2.1, this number of walks is bounded above by ρ^{2r} for all $r > 0$.

Now fix an $\epsilon > 0$, and let r_0 be as in Lemma 4.2. Let G's maximum degree be D. Let

$$n_0 = 1 + D + D(D-1) + D(D-1)^2 + \cdots + D(D-1)^{2r_0}.$$

Then in any subset of greater than n_0 vertices of a graph of maximum degree less than or equal to D, there are two vertices of distance greater than $2r_0$.

Now consider a covering map $\pi : H \to G$ of degree $n > n_0$; we can fix $u, v \in V_H$ of distance greater than $2r_0$ such that $\pi(u) = \pi(v)$. Let $f = \chi_u - \chi_v$ be the function that is 1 on u, -1 on v, and 0 elsewhere. Then $(A_H^{2r_0} f, f)$ is the sum of the number of walks of length $2r_0$ from, respectively, u and v, which return to their starting vertex. So

$$(A_H^{2r_0} f, f) \geq 2(\rho - \epsilon)^{2r_0}.$$

But f is a new function of the L^2-norm $\sqrt{2}$, and so the norm of $A_H^{2r_0}$ restricted to the new functions is at least $(\rho - \epsilon)^{2r_0}$. Hence the largest eigenvalue of $A_H^{2r_0}$ restricted to L^2_{new} is at least $(\rho - \epsilon)^{2r_0}$, and so that of A_H is at least $\rho - \epsilon$. This proves the theorem. □

5. Generalizations and concluding remarks

Up to now we have developed Broder-Shamir theorems (i.e., Th. 2.7 and its consequences) and the Alon-Boppana theorem (Th. 4.1) for only one model, $\mathscr{C}_n(G)$, of a random cover of G, and we have assumed that G has no multiple edges or self-loops. It is easy to generalize these theorems to

(1) graphs with multiple edges,
(2) graphs with self-loops (either half-loops or whole-loops in the terminology of [F2]), and

(3) graphs with weighted edges (where the adjacency matrix entries are sums of the appropriate edge weights).

Furthermore, define a *C-Broder-Shamir permutation model* to be a probability space of permutation on n-elements for each n (or some collection of n) such that for any n, k with $k \le n/4$ we have that if k-values of the permutation, σ, are fixed, any undetermined value, $\sigma(i)$, of the permutation has $\sigma(i) = j$ with probability at most $(1/n)+(Ck/n^2)$ (for all j). Then the Broder-Shamir theorems generalize to a random cover of the G-model given by any independent permutations, $\{\sigma_e\}_{e\in E}$, which are C-Broder-Shamir for some C (independent of n). The details and examples can be found in [F4].

We now give some directions for further work.

It would be nice to generalize the theorems here to allow the base graph to be a "pregraph" (in the sense of [F2]). Then there would be a relative Broder-Shamir theorem for quotients of every fixed tree, T.

Given a graph (or pregraph), G, with a "reasonable" (we remain vague here) model of a random degree n cover of G, one can conjecture that $\rho_{\text{new}} \le \rho$ with probability tending to 1 (or even, less ambitiously, nonzero probability). One could weaken this "Ramanujan" condition to having $\rho_{\text{new}} \le \rho + \omega(n)$, where ω is some suitable function of n. One could also ask a similar question about Galois covers (see [F2]). [LPS] gives examples of Galois covers where the base graph has one or two vertices.

Another interesting direction would be to fix a cover $\pi : G_0 \to G$ with G_0 infinite. Then one could ask about the above conjectures, as well as the theorems in this paper, where we take a "random" finite quotient of G_0 that covers G and take ρ to be the spectral radius of G_0. (The Alon-Boppana theorem easily generalizes to this situation.)

We remark that there are some very interesting covers with a small new spectral radius in certain cases. For example, it is not hard to see that the Boolean n-cube*, B^n, has one degree 2 cover all of whose eigenvalues are $\pm\sqrt{n}$ (see [F3]).

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*This is the graph with vertices $\{0, 1\}^n$ and edge between two vertices of Hamming distance 1, that is, two vertices that differ in exactly one coordinate.

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