§9.3 Infinite Ramsey & Compactness Principle

$\boxed{\text{Thm}}$ (Infinite Ramsey)

In $\forall$ c-coloring of all 2-subsets of $\mathbb{N}$, $\exists$ infinite $X \subseteq \mathbb{N}$ whose all 2-subsets have the same color.

In other words, $\forall$ c-edge-coloring of K_ω, the infinite complete graph on $\mathbb{N}$ has a monochromatic copy of K_ω.

Pf/ Let $S_1 = \mathbb{N}$. Suppose $S_i, x_1, \ldots, x_{i-1}$ are given, where $S_i \subseteq \mathbb{N}$ is infinite. Now, we choose $x_i \in S_i - \{x_1, \ldots, x_{i-1}\}$. Choose a color C_i s.t. $\exists$ infinitely many 2-subsets of color C_i containing x_i and another element of S_i. We let

$S_{i+1} = \{x_1, \cdots, x_i\} \cup \{y : y \in S_i, \{x_i, y\} \text{ has color } c_i\}$

Repeat this, then we have a infinite sequence,

$x_1, x_2, \cdots$. $\Rightarrow$ Among all $\{c_i\}$, $\exists$ color k

that appears infinitely many times, i.e.,

$c_{i_1} = c_{i_2} = \cdots = k$, for some $i_1 < i_2 < \cdots$ (infinitely)

Then we take $X = \{x_{i_1}, x_{i_2}, \cdots\}$ and the

desired conditions are satisfied. $\blacksquare$

H: hypergraph; $\chi(H) := \min c$ s.t. $\exists$ function

$f : V(H) \to [c]$ w/ no edge of H is monochromatic

$R(k,k) \sim$ Consider $H = (V, E)$ w/

$V(H) =$ edges of K_n

$E(H) =$ edges of a copy of K_k in K_n

$R(k,k) > n \Leftarrow \chi(H) > 2$

$\boxed{\text{Thm}}$ (Compactness principle)

$H=(V,E)$: hypergraph w/ all the edges finite

r: Const.

$\chi(H[W]) \le r$, $\forall$ finite set W
$\overset{?}{\Rightarrow} \chi(H) \le r$

(Eq.: $\chi(H) > r \overset{?}{\Rightarrow} \exists$ finite $H' \subseteq H$ s.t. $\chi(H') > r$)

Pf 1/ (assuming that r is countably infinite)

Suppose $V=\mathbb{N}$. $\forall n \in \mathbb{Z}^+$, $\exists$ coloring $f_n : [n] \to [r]$

s.t. no edge of $H[[n]]$ is monochromatic.

We define $f: \mathbb{N} \to [r]$ as follows:

suppose $f(1), f(2), \dots, f(j-1)$ are already defined,

s.t. $S_{j-1} = \{n : n \ge j-1, f(i) = f_n(i)\ \forall 1 \le i \le j-1\}$ is infinite.

Choose $f(j)$ s.t. $\exists$ infinitely many n w/

Possible!

$f_n(i) = f(i), \forall\, 1 \leq i \leq j$. (By Pigeonhole)

$\Rightarrow \forall\, i, \exists$ infinitely many $n \geq i$ s.t.

$f_n(k) = f(k), \forall\, k \in [i]$

$\boxed{\text{Claim}}$ f is a proper r-coloring

If $\exists$ monochromatic edge X in H,

then $\exists\, i$ s.t. $X \subseteq [i]$

$\Rightarrow \exists\, n > i$ s.t. $f_n(k) = f(k), \forall k \in [i]$

So, in f_n, X is a monochromatic edge

of $H[[n]]$, contradiction! $\blacksquare$

Pf/2 (Assuming the axiom of choice)

Consider $T :=$ set of all functions $f: V \to [n]$

$T \Leftrightarrow \prod_{v \in V} [r]$

We give a discrete topology to $[r]$, then we have a product topology, where $\prod_{v \in V} X_v$ is a basis if $X_v = [r]$ for all but finitely many v. By Tychoff thm, T is compact as $[r]$ is compact. For finite $W \subseteq V$, let F_W be the set of functions $f \in T$ s.t. no subset W is a monochromatic edge in f $\Rightarrow F_W$ is closed because its complement is a union of finitely many open sets. Then $\chi(H[W]) \leq r \Rightarrow F_W \neq \phi$.

If $W_1, \dots, W_t$ are finite subsets of V, then $\bigcap F_{W_i} \supseteq F_{\cup W_i} \neq \phi$. Therefore,

$\{F_W : \text{finite } W \subseteq V\}$ has the property that $\forall$ finite intersection $\neq \phi$. As T is compact, $\bigcap_{\text{finite } W \subseteq V} \overline{F_W} \neq \phi$. Let $f \in \bigcap_{\text{finite } W \subseteq V} \overline{F_W}$, then no edge is monochromatic in f. $\square$

$\boxed{\text{Cor}}$ (finite version)

$\forall t, c, \exists n_0$ s.t. [($n \geq n_0$ and all 2-subsets of $[n]$ are colored by c colors) $\Rightarrow$ $\exists$ t-subset of $[n]$ whose all 2-SSs have the same color]. ($R(t, t, \cdots, t) \leq n_0$)

pf/ Let H be a hypergraph whose vtxs are 2-subsets of $\mathbb{N}$ and edges are 2-subsets of a t-subset of $\mathbb{N}$. By the infinite

Ramsey, $\forall$ c-coloring of 2-subsets of $\mathbb{N}$ has an infinite monochromatic subset $\Rightarrow \chi(H) > c$. By the compactness, $\exists$ finite W s.t. $\chi(H[W]) > c$. WMA W is the set of all 2-subsets of a set $X \subseteq [n_0] \Rightarrow \forall$ c-coloring of 2-subsets of $[n_0]$ creates a SS of size t that is monochromatic. $\square$

§ 9.4 Finding a convex n-gon

$\boxed{\text{Thm}}$ (Erdős, Szekeres 1935)

Let $m \geq 3$. $\exists N$ s.t. $\forall$ set of N PTs on $\mathbb{R}^2$ in general position must have m PTs forming a convex m-gon.

Pf/ Let $N = R(m, 5; 4)$

We color a set of 4 PTs by red if they form a convex 4-gon, blue otherwise.

By the def of Ramsey #, either $\exists$ Red m points OR BLUE 5 PTs.

$\boxed{\text{Lem}}$ $\forall$ 5 PTs in GP, $\exists$ 4 of them forming a convex 4-gon.

$\Rightarrow$ "BLUE 5 PTs" is impossible

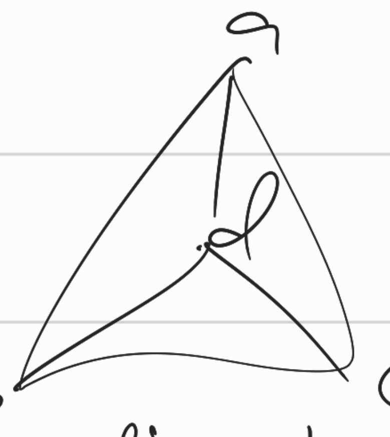
$\Rightarrow \exists$ a set of m PTs whose 4-SSs are all Red $\Rightarrow$ they form a Convex m-gon. $\boxtimes$

Pf/2 (Johnson 80')

Take $N = R(m, m; 3)$

Say for $\{a, b, c\}$, we color Red if $\triangle abc$ contain even # other PTs, Blue o.w.

By the Ramsey Thm, $\exists$ m PTs s.t. all triples have the same color. If these are not convex, then $\exists$ 4 PTs s.t. one is contained in $\triangle$(other 3).



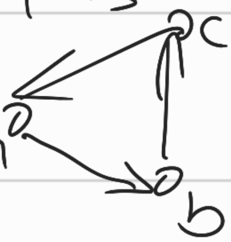
Consider the # other PTs in $\triangle$,

$\#abc = \#abd + \#acd + \#bcd + 1$, Contradiction! $\boxtimes$

Pf/ 3 [M. Tarsy] Set $N = R(m, m; 3)$

Put an ordering $<$ for these N PTs.

We color $a<b<c$ by Red if Path a



is counter-clockwise; by Blue otherwise.

By Ramsey, $\exists$ m PTs s.t. $\forall$ triple has the same color $\Rightarrow$ they must form a convex m-gon. ∎

$\boxed{\text{Conj.}}$ (Erdös-Szekeres 35')

$$N = 2^{m-2} + 1$$

True for $m = 3, 4, 5$

Before 2017, all bounds were $\sim 4^n$

In 2017, Andrew Suk, $N \le 2^{m + 6m^{2/3}\log m}$, $m \to \infty$

In 2020, Holmsen, Mojarrad, Pach, Tardos

$N \leq 2^{m + c\sqrt{m \log m}}$

$\boxed{\text{Ch 10}}$ Progressions

§10.1 van der Waerden's Thm

$\boxed{\text{Thm}}$ (Van der Waerden 1927)

$\forall k, l, \exists N \in \mathbb{Z}^+$ s.t. $\forall$ k-coloring of $[n]$, there is a monochromatic l-AP.

(originally asked by I. Schur)

Let $W(l, k)$ be the min. N.

(van der Waerden #)

By Compactness principle, the above thm. $\Leftrightarrow$ $\forall$ partition of $\mathbb{N}$ into finitely many sets, one of the parts contains $\forall$ long AP.

Pf/ Induction

Let $X_{l,m}$ be the set of all seqs.

$x \in \{0, \cdots, l\}^m$, $x = (x(1), x(2), \cdots, x(m))$

s.t. if $x(i) = l$ then $x(j) = l$, $\forall j \geq i$.

We say 2 seqs. $x, y \in X_{l,m}$ are l-equiv.

if $\exists i \in \{0, \cdots, m\}$ s.t.

$$\begin{cases} x(j) \neq l, \ y(j) \neq l, \ \forall j < i \\ x(j) = y(j) = l, \ \forall j \geq i \end{cases}$$

$\Rightarrow X_{l,m}$ is partitioned into $m+1$ l-equiv. classes (according to the position of the first l)

$\boxed{\text{Claim}}$ For $l, m \geq 1$ and k, $\exists N = N(l, m, k)$

s.t. $\forall C : [N] \to \{0, 1, \cdots, k-1\}$, $\exists$

$a, d_1, d_2, \cdots, d_m \in \mathbb{Z}^+$ s.t.

$C(a + \sum_{i=1}^{m} x(i) d(i))$ is constant on

all seqs. in each ℓ-equiv. class.

$\boxed{\text{Claim}} \Rightarrow$ vdW's Thm.

$m=1$: $C(a + x(1) d(1))$ is constant.

There are 2 ℓ-equiv. classes on $X_{\ell,1}$

#1 : $(0), (1), \cdots, (\ell-1)$; #2 : (ℓ)

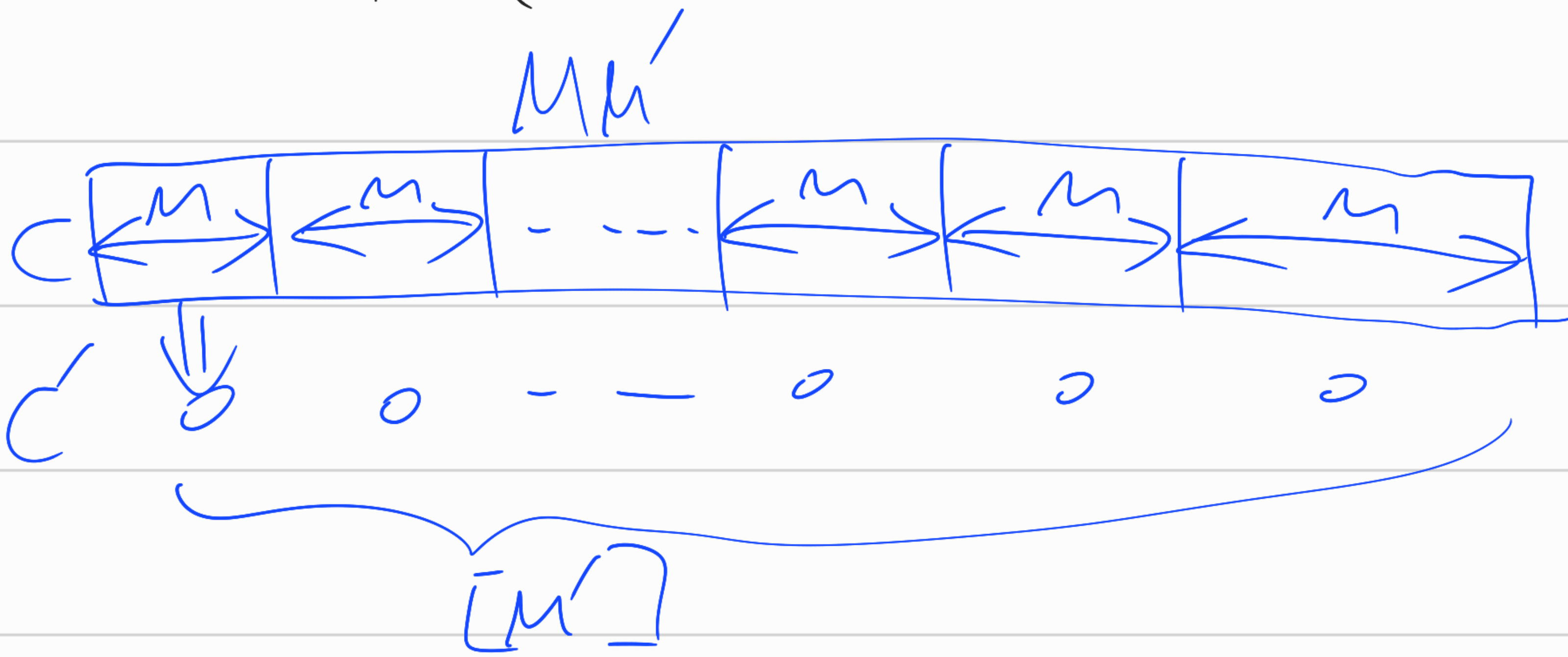
$\Rightarrow c(a) = c(a+d_1) = \cdots = c(a+(\ell-1)d_1)$

Suppose the claim is false. Choose a counterexample w/ min. ℓ. Among all of them w/ same ℓ, we choose the one w/ min. m. If $\ell=1$, it's trivial as each class contains only one seq. Now, WMA

$l > 1$. Suppose $m \neq 1$, let $M = N(l, m-1, k)$ and let $M' = N(l, 1, k^m)$, $N = MM'$.

Suppose $C : [N] \to \{0, \sim, k-1\}$. define $C' : [M'] \to \{0, \sim, k^M - 1\}$ s.t.

$$C'(x) = \sum_{i=1}^{m} C\big((x-1)M + i\big) k^{i-1}$$



$\boxed{\text{Obs}}$ $C'(x) = C'(y)$

$\Leftrightarrow C\big((x-1)M + i\big) = C\big((y-1)M + i\big), \forall i \in [M]$

By the assumption on $N(l, 1, k^M)$, there exist $a', d' > 0$ s.t.

$c'(a') = c'(a'+d') = \cdots = c'(a'+(\ell-1)d')$

Now consider the interval of length M $\{(a'-1)M+1, \cdots, a'M\}$. By the assumption on $M = N(\ell, m-1, k)$, $\exists\, a, d_2, d_3, \cdots, d_m$ s.t. $a + \sum_{i=2}^{m} x(i) d_i \in \{(a'-1)M+1, \cdots, a'M\}$ AND $C(a + \sum_{i=2}^{m} x(i) d_i)$ is constant for all x in each $(\ell-1)$-equiv. class of $\{0, \cdots, \ell\}^{m-1}$ $\Longrightarrow$

$C(a + \sum_{i=1}^{m} x(i) d_i)$ is constant on every ℓ-equiv. class of $\{0, 1, \cdots, \ell\}^m$, contradiction!

Thus, WMA $m=1$. Let $N = N(\ell-1, k, k)$.

Suppose $C: [N] \to \{0, \cdots, k-1\}$. Then $\exists\, a, d_1, \cdots, d_k$ s.t. $a + \sum_{i=1}^{k} x(i) d_i \leq N$

AND $C\left(a+\sum_i x(i) d_i\right)$ is constant for x in each $(\ell-1)$-equiv. class.

$(x \in \{0, \dots, \ell-1\}^K)$. Consider

$$\left.\begin{array}{l} C\left(a+(\ell-1)(d_1+d_2+\cdots+d_K)\right) \\ C\left(a+(\ell-1)(d_2+d_3+\cdots+d_K)\right) \\ \quad\vdots \\ C\left(a+(\ell-1)d_K\right) \\ C(a) \end{array}\right\} \begin{array}{l} K+1 \\ \text{values} \end{array}$$

$\xRightarrow{\text{Pigeonhole}}$ $\exists$ Two of them are equal, i.e.,

$\exists\ 1 \le u < v \le K+1$ s.t.

$$C\left(a+\sum_{i=u}^{K}(\ell-1)d_i\right)=C\left(a+\sum_{i=v}^{K}(\ell-1)d_i\right)$$

Consider $C\left(\left(a+\sum_{i=v}^{K}(\ell-1)d_i\right)+x(i)\sum_{j=u}^{v-1}d_j\right)$

We'll take $a'=a+\sum_{j=v}^{K}(\ell-1)d_j$, $d_1'=\sum_{j=u}^{v-1}d_j$

$\boxed{\text{Claim}}$ $C(a' + x(1)d_1')$ is constant for every l-equiv class.

Note again that when m=1

$X_{l,1}$ → (0), (1), (2), ..., (l-1)
→ (l)

Consider

$C(a + \quad (l-1)d_v + \cdots + (l-1)d_k)$

$C(a + d_u + d_2 + \cdots + d_{v-1} + \cdots \quad \text{———} \quad)$

$C(a + 2d_u + 2d_2 + \cdots + 2d_{v-1} + \cdots \quad \text{———} \quad \text{———} \quad)$

$\vdots$

$C(a + (l-2)d_u + \cdots + (l-2)d_{v-1} + \quad \text{———} \quad)$

They are equal by the induction. [(l-1)-equiv.]

Also $C\left(a + (l-1)\sum_{j=u}^{k} d_j\right) = C\left(a + (l-1)\sum_{j=v}^{k} d_j\right)$. ▨

$\boxed{\text{Conj.}}$ (Erdős, Turan 36')

Every set of positive upper density has a 3-AP.

Proved by Roth, 1953.

$\boxed{\text{Conj.}}$ (Stronger, Erdős, 57')

Every set of positive upper density has an l-AP. Proved by Szemeredi, 1975.

Also by Fürstenberg, 1977 using Ergodic.

$\boxed{\text{Thm}}$ (Bergelson, Leibman 96')

$P_1, \dots, P_k$: Polynomials in $\mathbb{Z}[x]$ w/ $P_i(0) = 0$

If A is a set of positive upper density, then $\exists\, a, d > 0$ s.t.

$a, a+P_1(d), \dots, a+P_k(d) \in A$

Current best known bound for $W(\ell, c)$

$W(\ell, c) \le 2^{2^{c^{2^{2^{\ell+9}}}}}$ [Gowers, 2001]

[Graham, Solymosi, 2006]

$W(3, c) \le 2^{2^{2^{O(c)}}}$

$\boxed{\text{Conj.}}$ (Erdös)

If $\sum_{a \in A} 1/a = \infty$, then $A \ni \ell$-AP, $\forall \ell$

$\boxed{\text{Thm}}$ (Green, Tao, 2004)

True for $A = \{\text{primes}\}$