3.8. Threshold version of Lovász's Two Families Theorem

Thm (Füredi 1984)

Let $U_1, U_2, \dots, U_m$
$V_1, V_2, \dots, V_m$ be subspaces of a vector space W over $\mathbb{F}$.

$\dim(U_i) = r$, $\dim(V_i) = s$

(1) $\dim(U_i \cap V_i) \le t$ for all $i = 1, 2, \dots, m$

(2) $\dim(U_i \cap V_j) > t$ for all $1 \le i < j \le m$.

Then

$$m \le \binom{r+s-2t}{r-t}$$

<u>Proof</u>.

We may assume that $|\mathbb{F}|$ is sufficiently large
$n = \dim W$

There exists a subspace V of W of dimension $n-t$ in general position with U_i, V_i, $U_i \cap V_j$ for all $i \le j$.

$$\dim(U_i \cap V) = \dim U_i - t = r - t$$

$$\dim V_i \cap V = \dim V_i - t = s - t$$

$$\dim (U_i \cap V) \cap (V_i \cap V) = \dim(U_i \cap V_i \cap V) = \max(\dim(U_i \cap V_i) - t, 0) = 0$$

$$\dim (U_i \cap V) \cap (V_j \cap V) = \max(\dim(U_i \cap V_j) - t, 0) > 0.$$

$$\left.\begin{array}{l} U_i' = U_i \cap V \\ V_i' = V_i \cap V \end{array}\right\} \text{Apply Lovász's theorem}$$

$$\Rightarrow m \le \binom{r+s-2t}{r-t}.$$

Corollary.

Let $A_1, A_2, \ldots, A_m$

$B_1, B_2, \ldots, B_m$ be subsets of $\{1, 2, \ldots, n\}$

$|A_i| = r, \quad |B_i| = s$

$|A_i \cap B_i| \le t$ for all i

$|A_i \cap B_j| > t$ for all $i < j$.

$$\Rightarrow \quad m \le \binom{r+s-2t}{r-t}.$$

Proof. Map $i \to e_i$

$$1 \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad 2 \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \ldots$$

$$A_i \to U_i \le \mathbb{R}^n$$

$$B_i \to V_i \le \mathbb{R}^n.$$

$$|A_i \cap B_i| = \dim(U_i \cap V_i)$$

$$|A_i \cap B_j| = \dim(U_i \cap V_j)$$

$\square$

4. Two geometric applications

4.1. Chromatic number of $\mathbb{R}^n$

Let G_n be the graph on $\mathbb{R}^n$ such that two points (vertices) x, y are adjacent in G_n if and only if $d(x, y) = 1$.

(unit distance graph of $\mathbb{R}^n$)

Q: What is the chromatic number of G_n? $\chi(G_n)$

($\chi(G) =$ min k such that G has a k-coloring

k-coloring := a function from $V(G) \to \{1, \ldots, k\}$ such that adjacent vertices get distinct color.)

Thm. (de Bruijn, Erdős 1951)
G: graph, k: integer
If every finite subgraph of G has a k-coloring then G has a k-coloring.

("compactness" of the chromatic number)

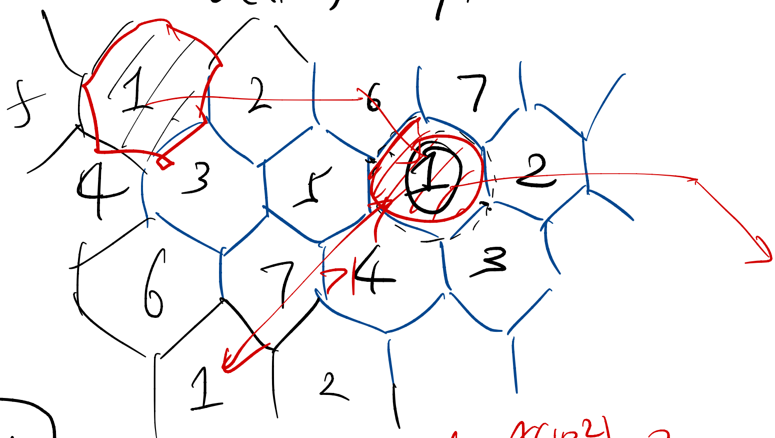
Question: What is $\chi(\mathbb{R}^2)$?

$\chi(\mathbb{R}^n) = \chi(G_n)$

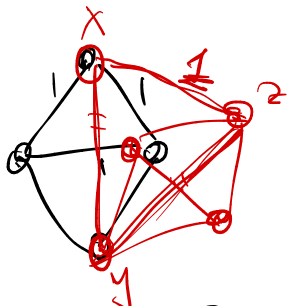
Proposed by Hadwiger (1944), Nelson (1950)

Known for $\chi(\mathbb{R}^2)$

$\chi(\mathbb{R}^2) \le 7$.



$\boxed{\chi(\mathbb{R}^2) \ge 4}$



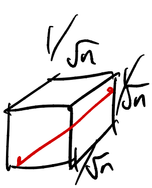
If $\chi(\mathbb{R}^2)=3$,
color of x
= color of y
= color of z

2018. Aubrey de Grey : $\chi(\mathbb{R}^2) \ge 5$.

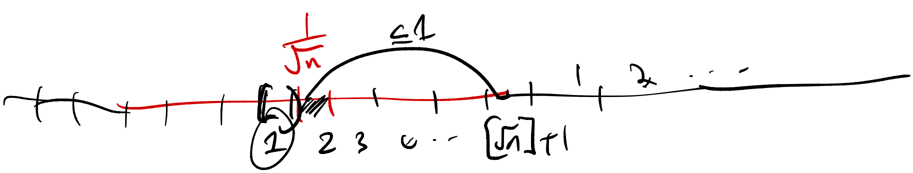
$\chi(\mathbb{R}^n)$?

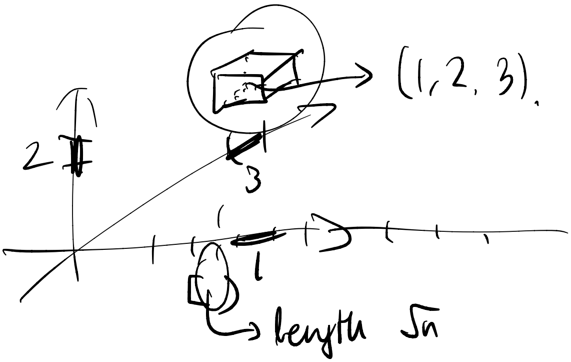
Why is it bounded?

Partition $\mathbb{R}^n$ into cubes whose sides have $\sim 1/\sqrt{n}$



$\Rightarrow$





$\boxed{[x,\ x+\frac{1}{\sqrt{n}})}$

$$\Rightarrow \quad \chi(\mathbb{R}^n) \le \left([\sqrt{n}]+1\right)^n \sim e^{\frac{1}{2}n\log n}$$

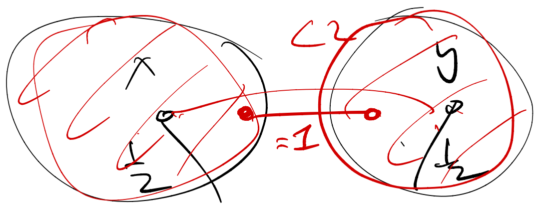
Prop. $\chi(\mathbb{R}^n) \le 9^n$.

Proof. Let X be a maximal set of points in $\mathbb{R}^n$ such that $d(x,y) \ge \frac{1}{2}$ for every distinct point x, y in X.

$$\Rightarrow \quad \bigcup_{x\in X} \underbrace{B(x,\tfrac{1}{2})}_{\text{open ball of radius } \frac{1}{2}} = \mathbb{R}^n.$$

Moreover $B(x,\frac{1}{4})$ for all $x \in X$ are disjoint.

Goal: Color $B(x,\frac{1}{2})$ by the same color so that $B(x,\frac{1}{2})$, $B(y,\frac{1}{2})$ have distinct color if $d(x,y) < 2$



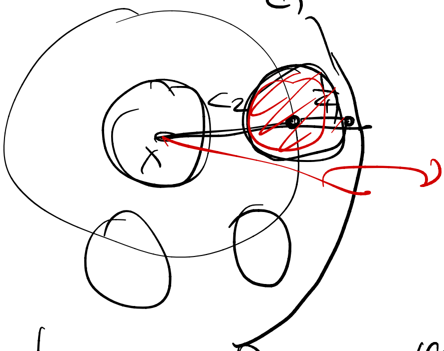
What is the max degree of this graph?

$\to$ what is an upper bound on # y's such that

$y \in X \qquad d(x,y) < 2,$

$B(x, \frac{1}{4})$ are disjoint.

$$\Rightarrow \deg_G(x) + 1 \leq \frac{\mathrm{Vol}(B(x, \frac{9}{4}))}{\mathrm{Vol}(B(x, \frac{1}{4}))}$$



$2 + \frac{1}{4}$

$$= \frac{(\frac{9}{4})^n}{(\frac{1}{4})^n} = 9^n.$$

Larman, Rogers 1972: $\chi(\mathbb{R}^n) \leq (3 + o(1))^n$.

<u>Thm</u> (Frankl, Wilson 1981)
For large n, $\chi(\mathbb{R}^n) > 1.2^n$.

Proof.

$$\boxed{\chi(G)\,\alpha(G) \geq |V(G)|}$$

$\alpha(G)$ = max size of an independent set in G

An independent set is a set of vertices such that no 2 vertices are adjacent.

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)} \geq \frac{|V(G)|}{\bigcirc}.$$

<u>Goal</u>: Construct some finite set of points in $\mathbb{R}^n$ such that any subset whose pair of points have distance $\neq$ fixed constant has small size.

Choose a prime $p < \frac{n}{2}$.

A subset of $\{1, 2, \dots, n\}$ $\longrightarrow$ $\mathbb{R}^n$

$A \longmapsto$ characteristic vector $\varphi(A) = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}$

$A, B \subseteq \{1, \dots, n\}$

$$d(\varphi(A), \varphi(B)) = \sqrt{\sum (a_i - b_i)^2}$$
$$= \sqrt{|A \triangle B|}$$

If both A and B have size $2p-1$, then

$$d(\varphi(A), \varphi(B)) = \sqrt{|A \triangle B|}$$
$$= \sqrt{|A| + |B| - 2|A \cap B|}$$
$$= \sqrt{4p - 2 - 2|A \cap B|}.$$

$|A \cap B| \neq p-1 \iff d(\varphi(A), \varphi(B)) \neq \sqrt{2p}$

Let G_p be a graph on subsets of size $2p-1$ of $\{1, \cdots, n\}$

such that two vertices A, B are adjacent if $\underline{|A \cap B|} = p-1$

$$\chi(G_p) \leq \chi(\mathbb{R}^n)$$

<u>Goal</u>: Upper bound on $\alpha(G_p)$?

$\alpha(G_p)$ = independent set

= $\mathcal{F}$ of subsets of size $2p-1$ in $\{1, \cdots, n\}$

such that for all $X, Y \in \mathcal{F}$, $|X \cap Y| \neq p-1$

Use Alon-Babai-Suzuki.

$L = \{0, 1, \cdots, p-2\}$.

$|X \cap Y| \in L + p\mathbb{Z}$ for all $X \neq Y \in \mathcal{F}$

$$\Rightarrow \quad |\mathcal{F}| \leq \binom{n}{p-1}.$$

$$\alpha(G_p) \leq \binom{n}{p-1}$$

$$\therefore \chi(G_p) \geq \frac{\binom{n}{2p-1}}{\binom{n}{p-1}} = \frac{\cancel{\frac{2p}{n+1}}\binom{n+1}{2p}}{\cancel{\frac{p}{n+1}}\binom{n+1}{p}}$$

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

$$= 2\,\frac{\binom{n+1}{2p}}{\binom{n+1}{p}}.$$

Stirling's formula.

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n(1+o(1))$$

$$\Rightarrow \binom{n}{\alpha n} = \frac{n!}{(\alpha n)!\,(n-\alpha n)!}$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi}\sqrt{\alpha n}\sqrt{(1-\alpha)n}}\;\frac{\left(\frac{n}{e}\right)^n(1+o(1))}{\left(\frac{\alpha n}{e}\right)^{\alpha n}\left(\frac{(1-\alpha)n}{e}\right)^{(1-\alpha)n}}$$

$$= \frac{1}{\sqrt{2\pi}}\;\frac{1+o(1)}{\sqrt{\alpha(1-\alpha)}}\;\frac{1}{\sqrt{n}}\;\frac{1}{\alpha^{\alpha n}(1-\alpha)^{(1-\alpha)n}}$$

$$= \frac{1}{\sqrt{2\pi}}\;\frac{1+o(1)}{\sqrt{\alpha(1-\alpha)}}\;\frac{1}{\sqrt{n}}\;e^{\underline{-\alpha n\log\alpha-(1-\alpha)n\log(1-\alpha)}}$$

$$= \frac{1+o(1)}{\sqrt{2\pi}\sqrt{\alpha(1-\alpha)}}\;\frac{1}{\sqrt{n}}\;e^{\boxed{H(\alpha)}n}$$

$$H(\alpha) = -\alpha n\log\alpha-(1-\alpha)n\log(1-\alpha)$$

$$\binom{n+1}{2p}\Big/\binom{n+1}{p} \Rightarrow \text{maximize } H(2\alpha)-H(\alpha)$$

where $\alpha = \frac{p}{n+1}$

Find α such that
$H(2\alpha) - H(\alpha)$ is maximized.

$\left(H(2\alpha) - H(\alpha) \right)' = 0$

$\Rightarrow \alpha = \frac{2-\sqrt{2}}{4} = 0.1464\cdots$

For large n, choose α such that

$\frac{p}{n+1} \sim 0.146\cdots$

(There is a prime p between k and $(1+\varepsilon)k$ if k is large)

$$\Rightarrow \boxed{2\frac{\binom{n+1}{2p}}{\binom{n+1}{p}} > (1.2)^n}$$

$e^{H(\alpha)n} > (1.2071)^n$

$\left(c\frac{1}{\sqrt{n}}(1.2071)^n > 1.2^n \right.$
$\left. \text{for large } n \right)$

$\Rightarrow \chi(\mathbb{R}^n) > (1.2)^n.$
for large n.

$\square$

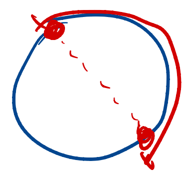
4.2. Borsuk's conjecture — disproof

Borsuk-Ulam Theorem: (1933)

Let S^n be the n-dimensional sphere.
If $n+1$ closed sets $X_1, X_2, \dots, X_{n+1}$ cover S^n then
there exists $i \in \{1, 2, \dots, n+1\}$ such that
X_i contains a pair of antipodal points.

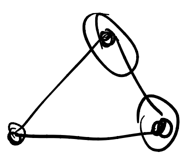
S^0

S^1



Conjecture (Borsuk 1933)
Every set of finite diameter in $\mathbb{R}^n$ can be partitioned into
$n+1$ sets of smaller diameter.

(Consider an n-dimensional simplex



If true, then $n+1$ is tight)

Let $f(n)$ be the minimum integer such that every set of finite diameter in $\mathbb{R}^n$ can be partitioned into $f(n)$ sets of smaller diameter.

$f(n) \geq n+1$.

Borsuk's conjecture: $f(n) = n+1$

Schramm 88: $f(n) < \left(\sqrt{\frac{3}{2}} + o(1)\right)^n = (1.224\cdots + o(1))^n$.

Known cases: true if
- $n = 2, 3$
- the set is centrally symmetric and convex
- convex with smooth boundary

Thm (Kahn, Kalai 93)

$$f(d) > 1.2^{\sqrt{d}}$$

for all sufficiently large d.

Proof. Let $n = 4p-1$ for a prime p. Choose n as big as possible and yet $\binom{n}{2} \le d$.

Let $k = 2p-1$. $\Rightarrow$ $\boxed{n = 2k+1}$

A: subset of $\{1, \ldots, n\}$ $\longrightarrow$ $\Phi(A)$ vector in $\mathbb{R}^{\binom{n}{2}}$

$$\Phi(A)_{ij} = \begin{cases} 1 & \text{if } i \in A,\ j \notin A \\ 0 & \text{otherwise} \end{cases} \quad (1 \le i < j \le n)$$

We may assume that $\Phi(A)$ is a vector in $\mathbb{R}^d$.

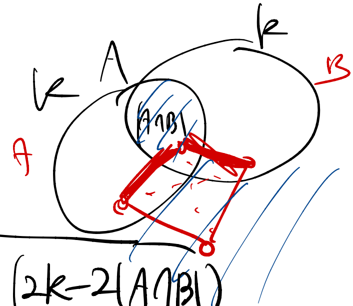
$$\|\Phi(A)\| = \sqrt{k(n-k)}$$

if $|A| = k$.

For 2 sets A, B of size k in $\{1, \dots, n\}$

$$d(\Phi(A), \Phi(B))$$

$$= \sqrt{\sum_{1 \le i < j \le n} (x_{ij} - y_{ij})^2}$$



$$= \sqrt{(n - (2k - |A \cap B|) + |A \cap B|)(2k - 2|A \cap B|)}$$

$$= \sqrt{(n - 2k + 2|A \cap B|)(2k - 2|A \cap B|)}$$

$(n - x)x$ is maximized when $x = \frac{n}{2}$

$d(\Phi(A), \Phi(B))$ is maximized

when $2k - 2|A \cap B| = \frac{n}{2} = \frac{2k+1}{2} = k + \frac{1}{2}$

$$\Rightarrow \quad 2|A \cap B| = k - \frac{1}{2}$$

$$= 2p - \frac{3}{2}$$

$$|A \cap B| = p - \frac{3}{4}$$

$$\boxed{|A \cap B| = p - 1}$$

let G_p be the graph on subsets of size k of $\{1, \dots, n\}$

such that

A is adjacent to $B \iff |A \cap B| = p - 1$.

$\alpha(G_p)$ = max size of an independent set of G_p

= max size of a set of k-subsets of $\{1, \dots, n\}$ such that no 2 of them have intersection size $= p-1$

$\begin{cases} k = 2p-1 \\ L = \{0, 1, \dots, p-2\} \end{cases}$

Alon-Babai-Suzuki' theorem

$$\Longrightarrow \alpha(G_p) \le \binom{n}{p-1} = \binom{4p-1}{p-1}.$$

$$f(d) \ge \chi(G_p) \ge \frac{|V(G_p)|}{\alpha(G_p)} = \frac{\binom{4p-1}{2p-1}}{\binom{4p-1}{p-1}}$$

$$= \frac{\frac{2p}{4p}\binom{4p}{2p}}{\frac{p}{4p}\binom{4p}{p}}$$

$$= 2 \frac{\frac{(4p)!}{(2p)!(2p)!}}{\frac{(4p)!}{p!\,(3p)!}}$$

$$= 2 \cdot \frac{(3p)!\,p!}{((2p)!)^2}$$

Stirling

$$\Rightarrow \quad \geq 2 \cdot \frac{\sqrt{2\pi(3p)}\left(\frac{3p}{e}\right)^{3p} \sqrt{2\pi p}\left(\frac{p}{e}\right)^{p}}{\left(\sqrt{2\pi(2p)}\left(\frac{2p}{e}\right)^{2p}\right)^{2}}(1+o(1))$$

$$= 2\frac{\sqrt{3 \cdot 2}}{2}\left(\frac{3^3}{16}\right)^{p}(1+o(1))$$

$$= \sqrt{6}\left(\frac{27}{16}\right)^{p}(1+o(1))$$

$$= c \cdot \left(\frac{27}{16}\right)^{\frac{n}{4}}(1+o(1))$$

$$= c \cdot (1.13975\cdots)^{n}(1+o(1))$$

$$\left(d+ < \binom{n}{2} \leq d \Rightarrow \frac{n(n-1)}{2} > d+ \Rightarrow \frac{n^2}{2} > d \Rightarrow n > \sqrt{2d} \right)$$

$$= c \cdot \left(\left(\frac{27}{16}\right)^{\sqrt{2}}\right)^{\sqrt{d}}(1+o(1))$$

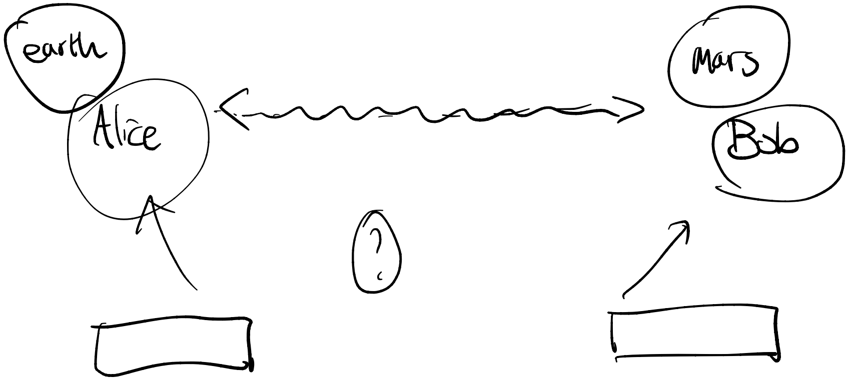
$$= c\,(1.2032\cdots)^{\sqrt{d}}(1+o(1))$$

$$> 1.2^{\sqrt{d}}$$

for all sufficiently large d.

5. Applications of Schwartz-Zippel Lemma

5.1. COMMUNICATION COMPLEXITY: INTRODUCTION



"Communication protocol": Set of rules describing how Alice and Bob would exchange information in what order.

The communication complexity of a problem
= #bits transmitted until Alice or Bob can determine the answer

Introduced by Yao 1979.

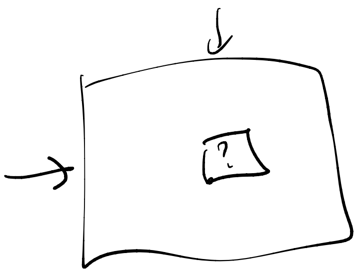
$a_1, a_2, \ldots, a_n$: list of possible inputs to Alice
$b_1, b_2, \ldots, b_m$: list of possible inputs to Bob

$c_{ij} = 0$ or 1 ...

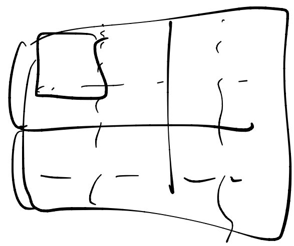
The goal is to determine c_{ij} when Alice has a_i, Bob has b_j.

The communication complexity
= min # bits to be exchanged in order for Alice or Bob to determine c_{ij} in the worst case

Let $C = (c_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le m}}$ "Communication matrix".

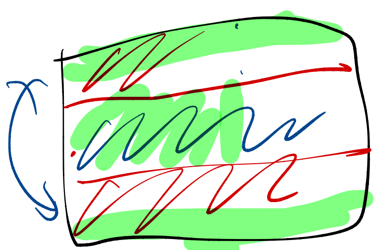


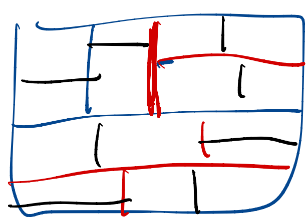
$K(C)$



Combinatorially,

$K(C)$ = min # rounds in order to partition C into almost homogeneous matrices
where in each round,
we can split each of the current submatrices into 2 submatrices vertically or horizontally





A matrix is almost homogeneous

if each column vector is an all-1 or all-0 vector

OR

each row vector is an all-1 or all-0 vector.

Trivial Protocol

Alice sends n to Bob

$\sim \lceil \log_2 n \rceil$ bits

$$\Rightarrow \begin{cases} K(C) \leq \lceil \log_2 n \rceil \\ K(C) \leq \lceil \log_2 m \rceil. \end{cases}$$