1.

Introduction

1.1. Even/Odd Rules for Clubs

$\boxed{\text{Even Rule}}$

(1) Every student club has even # students.

(2) Every pair of clubs must share even # students.

(3) No 2 clubs have identical membership.

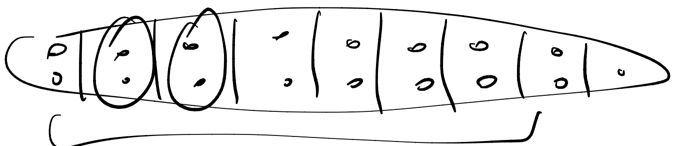
$A_1, A_2, \ldots, A_m$: subsets of $\{1, 2, \ldots, n\}$

(1) $\Rightarrow$ $|A_i|$ is even for all i

(2) $\Rightarrow$ $|A_i \cap A_j|$ is even. for all i, j

(3) $A_1, A_2 \ldots A_m$ are distinct.

Q : How many student clubs can students form?



$\lfloor n/2 \rfloor$

It is possible to create $2^{\lfloor \frac{n}{2} \rfloor}$ clubs following the even rule.

Odd Rule

(1) Every student club has odd # students.

(2) Every pair of clubs must share even # students.

$A_1, A_2, \ldots, A_m \subseteq \{1, 2, \ldots, n\}$

① $|A_i|$ odd

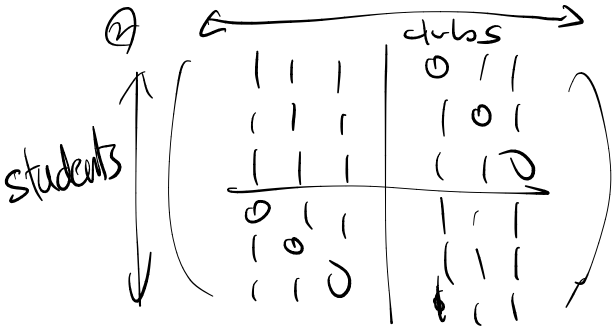
② $|A_i \cap A_j|$ even

How big can m be?

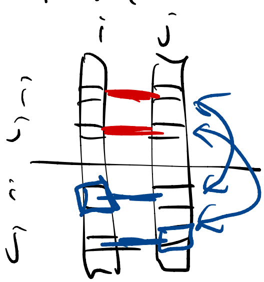
Examples

① n clubs. $A_i = \{ i^{th} \text{ student} \}$.

②



$$\begin{pmatrix} A+I & A \\ A & \boxed{A+I} \end{pmatrix}$$



A: symmetric $\frac{n}{2} \times \frac{n}{2}$ 0-1 matrix

Thm (odd rule)

With n students, no more than n clubs can be formed under the odd rule.

In other words,

if $A_1, A_2, \ldots, A_m \subseteq \{1, 2, \ldots, n\}$

$|A_i|$ odd for all i

$|A_i \cap A_j|$ even for all $i \neq j$

then $m \leq n$.

Proof.

1st proof: For $1 \leq i \leq m$

Let $v_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{pmatrix} \in \mathbb{F}_2^n$

$\boxed{\mathbb{F}_2 \text{ field w/ } \{0, 1\}}$

where

$$v_{ij} = \begin{cases} 1 & \text{if } j \in A_i \\ 0 & \text{otherwise.} \end{cases}$$

$$v_i \cdot v_j = \sum_k v_{ik} v_{jk} = |A_i \cap A_j|$$

$\mathbb{F}_2$

$$= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Claim: $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

Suppose not. $\sum c_i v_i = 0$

$$0 = \left(\sum c_i v_i\right) \cdot v_j = \sum_i c_i v_i \cdot v_j = c_j$$

$\Rightarrow c_j = 0$ for all j.

$v_1, v_2, \ldots, v_m \in \mathbb{F}_2^n$ ← n-dimensional vector space.

$\Rightarrow m \le n$.

2nd proof:

$$M = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \end{pmatrix} = \begin{array}{c|c} & \text{clubs} \\ \hline \text{students} & \begin{pmatrix} m_{ij} \end{pmatrix} \end{array}$$

be an $m \times n$ matrix over $\mathbb{F}_2$.

$$A = M^T M = \begin{array}{c|c} & \text{students} \\ \hline \text{clubs} & \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{pmatrix} \end{array} \begin{array}{c|c} & \text{clubs} \\ \hline \text{st} & \begin{pmatrix} v_1 & \cdots & v_m \end{pmatrix} \end{array}$$

$$= \begin{array}{c|c} & \text{clubs} \\ \hline \text{clubs} & \begin{pmatrix} & & \\ & v_i^T v_j & \\ & & \end{pmatrix} \end{array} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$v_i \cdot v_j$

$m \times m$ identity matrix.

Fact: A, B

$\Rightarrow rk(AB) \le rk(B)$

$m = rk(M^T M) \le rk(M) \le$ # rows of M $= n$

$\therefore m \le n$.

Thm (Even rule)
With n students no more than $2^{\lfloor \frac{n}{2} \rfloor}$ clubs can be formed under the even rule.

If $A_1, A_2, \dots, A_m \subseteq \{1, 2, \dots n\}$

- $|A_i|$ even for all i
- $|A_i \cap A_j|$ even for all i, j
- $A_1, A_2, \dots A_m$ are distinct

then

$$m \le 2^{\lfloor \frac{n}{2} \rfloor}.$$

Proof. let $v_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{pmatrix} \in \mathbb{F}_2^n$. $v_{ij} = \begin{cases} 1 & \text{if } j \in A_i \\ 0 & \text{otherwise} \end{cases}$

Let $U = \langle v_1, v_2, \dots, v_m \rangle \subseteq \mathbb{F}_2^n$

$v_i \cdot v_j = 0$ for all i, j

$\Rightarrow$ For all $x, y \in U$, $x \cdot y = 0$

$\left(\left(\sum c_i v_i\right) \cdot \left(\sum d_i v_i\right) = 0 \right)$

$\Rightarrow \quad U \subseteq U^\perp = \{ y \in \mathbb{F}_2^n : x \cdot y = 0 \text{ for all } x \in U \}$

$$\dim U + \dim(U^\perp) = \dim(\mathbb{F}_2^n)$$

$\le$

$2 \dim U$

$$\Rightarrow \quad \dim U \le \frac{1}{2} \dim(\mathbb{F}_2^n) = \frac{n}{2}$$

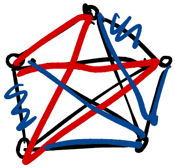
$\dim U \leq \lfloor \frac{n}{2} \rfloor$

\# vectors in $U \leq 2^{\lfloor \frac{n}{2} \rfloor}$

$v_1, v_2 \ldots v_m \in U$

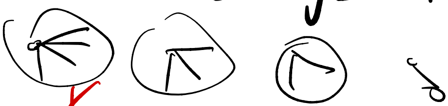
$\Rightarrow m \leq 2^{\lfloor \frac{n}{2} \rfloor}$

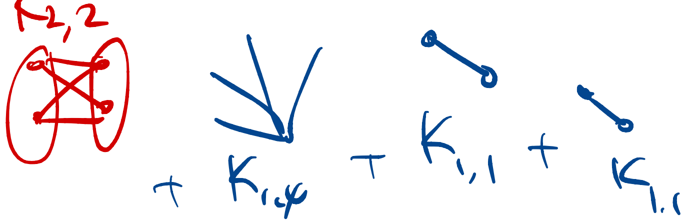
1.2. Partitioning K_n into complete bipartite graphs



K_5

Partition $E(K_n)$ into the edges of complete bipartite graphs?



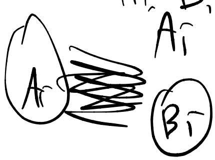


Thm (Graham, Pollack 1971)
The edge set $E(K_n)$ of the complete graph cannot be partitioned into less than $n-1$ copies of the edge sets of complete bipartite graphs.

Proof. (Tverberg 1982)
Suppose that $E(K_n)$ can be partitioned into t edge sets of complete bipartite graphs on (A_i, B_i)

$A_i, B_i \subseteq \{1, 2, \dots, n\}$

$A_i \cap B_i = \emptyset$



$x = (x_1, x_2, \dots, x_n)$

Let $$S(x) = \sum_{1 \le i < j \le n} x_i x_j = \sum_{i=1}^{t} \left(\left(\sum_{j \in A_i} x_j \right) \left(\sum_{k \in B_i} x_k \right) \right)$$

$$\left(\sum x_i\right)^2 = \sum x_i^2 + 2S(x)$$

Consider the following equations:

$$(*)\begin{cases} \sum_{j\in A_i} x_j = 0 & \text{for all } 1\le i\le t \\ \sum_{i=1}^{n} x_i = 0 \end{cases}$$

$t+1$ linear homogeneous equations.

$\boxed{t+1<n}$ $\Rightarrow$ There exists a non-zero solution x satisfying all $(*)$.

$\Rightarrow \left(\sum x_i\right)^2 = 0,\ S(x) = 0$

$\Rightarrow \sum x_i^2 = 0$

$\Rightarrow x_i = 0$ for all i.

Contradiction. $\square$

(Lindström's theorem)

1.3. Finding two collections of the same union / intersections.

<u>Thm.</u> Let $A_1, A_2, \dots, A_{n+1} \subseteq \{1, 2, \dots, n\}$.
$\Rightarrow$ There exist subsets I, J of $\{1, 2, \dots, n+1\}$ such that

$$I \cup J \neq \phi \quad I \cap J = \phi, \quad \bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j.$$

<u>Proof</u> (Lindström, Smet/1970)

Let $v_i = \begin{pmatrix} v_{i1} \\ \vdots \\ v_{in} \end{pmatrix} \in \mathbb{R}^n$ $\quad v_{ij} = \begin{cases} 1 & \text{if } j \in A_i \\ 0 & \text{otherwise.} \end{cases}$

$n+1$ vectors in $\mathbb{R}^n$

$\Longrightarrow$ There exists a <u>non-trivial</u> linear combination

$$\sum_{i=1}^{n+1} \alpha_i v_i = 0.$$

Let $I = \{i : \alpha_i > 0\}$, $J = \{i : \alpha_i < 0\}$.
$I \cap J = \phi$, $I \cup J \neq \phi$.

$$\sum_{i \in I} \alpha_i v_i = \sum_{j \in J} (-\alpha_j) v_j$$

$$\Longrightarrow \bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j.$$

Thm. (Lindström 1993)

Let $A_1, A_2, \dots, A_{n+2} \subseteq \{1,2,\dots,n\}$

$\Rightarrow$ There exist subsets I, J of $\{1,\dots,n+2\}$

$I \cap J = \phi$, $I \cup J \neq \phi$.

$$\bigcup_{i\in I} A_i = \bigcup_{j\in J} A_j, \quad \bigcap_{i\in I} A_i = \bigcap_{j\in J} A_j.$$

Proof.

Let $v_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$

$v_1, v_2, \dots v_{n+2} \in \mathbb{R}^{n+1}$

$\Rightarrow$ There exists a non-trivial linear combination

$$\sum_{i=1}^{n+2} \alpha_i v_i = 0$$

$I = \{i : \alpha_i > 0\}$, $J = \{i : \alpha_i < 0\}$.

$$\bigcup_{i\in I} A_i = \bigcup_{j\in J} A_j \qquad \sum_{i\in I} \alpha_i v_i = \sum_{j\in J} (-\alpha_j) v_j$$

$$\boxed{\sum_{i\in I} \alpha_i = \sum_{j\in J} (-\alpha_j)}$$

$e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$$\boxed{\sum \alpha_i (e - v_i) = \sum (-\alpha_j)(e - v_j)}$$

$$\bigcup_{i\in I} A_i^c = \bigcup_{j\in J} A_j^c \Rightarrow \bigcap_{i\in I} A_i = \bigcap_{j\in J} A_j. \qquad \square$$

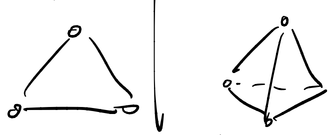
1.4 Two-Distance Sets

$X \subseteq \mathbb{R}^n \qquad |X| = m$

Q1: How large can $|X|$ be, if $\forall x \neq y \in X, \quad \|x-y\| = 1$?

$a_1, a_2, \ldots, a_m \in X \subseteq \mathbb{R}^n$

$\mathbb{R}^2$ | $\mathbb{R}^3$



? $n+1$

We may assume $a_m = 0$.

$\|a_i\| = 1$ for all $1 \le i \le m-1$.

$\|a_i - a_j\| = 1$ for all $1 \le i < j \le m-1$

$\hookrightarrow (a_i - a_j)\cdot(a_i - a_j) = 1$

$\underline{a_i \cdot a_i} + \underline{a_j \cdot a_j} - 2a_i \cdot a_j = 1$

$\therefore \quad a_i \cdot a_j = \frac{1}{2}$

Construct a matrix

$$M = \left(a_1 \middle| a_2 \middle| \cdots \middle| a_{m-1} \right)$$

$$A = M^T M = \underset{m-1}{\begin{pmatrix} 1 & \frac{1}{2} & & \frac{1}{2} \\ \frac{1}{2} & 1 & & \\ \frac{1}{2} & \frac{1}{2} & 1 & \\ \vdots & & & \ddots \\ \frac{1}{2} & & & 1 \end{pmatrix}} \qquad \begin{pmatrix} a_1^T \\ \hline a_2^T \\ \hline \vdots \\ a_{m-1}^T \end{pmatrix} \left(a_1 \middle| a_2 \middle| \cdots \middle| a_{m-1} \right)$$

$\longleftrightarrow$ $m-1$

$\mathrm{rank}(A) = m-1 \le \mathrm{rank}\, M \le \#\text{rows of } M = n$

$\therefore\ m-1 \le n.$ $\boxed{m \le n+1}$

Q2:
what if $\|x-y\|$ takes 2 values?

$x \ne y,\ x, y \in X$

"two-distance set"

Thm (Larman, Rogers, Seidel 1977)

$X = \{a_1, a_2, \cdots, a_m\} \subseteq \mathbb{R}^n$

$\|a_i - a_j\| \in \{d_1, d_2\}$ for all $i \ne j$.

$$\Longrightarrow \quad m \le \binom{n+2}{2} + n + 1$$

Proof. For $x \in \mathbb{R}^n$, define, for all $i \in \{1, \cdots, m\}$

$$f_i(x) = (\|x-a_i\|^2 - d_1^2)(\|x-a_i\|^2 - d_2^2)$$

$f_1, f_2, \cdots, f_m$ are polynomials with n variables $x_1, x_2, \cdots, x_n$.

$$\begin{cases} f_i(a_i) = d_1^2 d_2^2 \\ f_i(a_j) = 0 \end{cases}$$

$\Rightarrow$ $f_1, f_2, \cdots, f_m$ are linearly independent,

$m \le$ dim of a subspace of polynomials containing all of $f_1, \cdots, f_m$.

$$f_i(x) = \left(\sum_{j=1}^{n} (x_j - a_{ij})^2 - d_1^2\right)\left(\sum (x_j - a_{ij})^2 - d_2^2\right)$$

$$= \left(\sum x_j^2 - 2\sum a_{ij} x_j + \sum a_{ij}^2 - d_1^2\right)$$

$$\left(\sum x_j^2 - 2\sum a_{ij} x_j + \sum a_{ij}^2 - d_2^2\right)$$

$$\underbrace{\left(\sum_{j=1}^{n} x_j\right)^2}_{1 \text{ poly}}, \quad \underbrace{\left(\sum_{j=1}^{n} x_j^2\right) x_i}_{n \text{ poly}}, \quad \underbrace{x_i x_j}_{\binom{n}{2}+n}, \quad \underbrace{x_i}_{n}, \quad \underbrace{1}_{1}$$

$f_1, f_2, \dots, f_m$ are contained in the span of the above $1+n+\binom{n}{2}+n+n+1$ polynomials.

$$\Rightarrow m \le 1+n+\binom{n}{2}+n+n+1$$

$$= \boxed{\binom{n}{2}} + n + (n+1) + n + 1$$

$$= \binom{n+2}{2} + n + 1. \qquad \square$$

Not tight.

Thm (Blokhuis 1984)

$X = \{a_1, a_2, \dots, a_m\} \subseteq \mathbb{R}^n$

$\|a_i - a_j\| \in \{d_1, d_2\}$ for all $i \ne j$.

$$\Rightarrow \quad m \le \binom{n+2}{2}$$

Proof. Claim: $f_1, f_2, \ldots, f_m, x_1, x_2, \ldots, x_n, 1$ are linearly independent.

(If true, then
$$m+n+1 \le \binom{n+2}{2} + n + 1 \Rightarrow m \le \binom{n+2}{2}.)$$

Suppose that
$$\sum_{i=1}^{m} c_i f_i(x) + \sum_{j=1}^{n} d_j x_j + b = 0$$

Take $x = y e_k$ for some $y \in \mathbb{R}$

$e_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$ ← k^{th} coordinate

$$\sum_{i=1}^{m} c_i \left(y^2 - 2a_{ik} y + \sum_j a_{ij}^2 - d_1^2 \right)\left(y^2 - 2a_{ik} y + \sum_j a_{ij}^2 - d_2^2 \right)$$
$$+ d_k y + b = 0$$

identity → compare the coefficients of y^3 & y^4.

$\boxed{y^4}$: $\boxed{\sum_{i=1}^{m} c_i = 0}$

y^3 : $\sum_{i=1}^{m} \left(c_i(-2a_{ik}) + c_i(-2a_{ik}) \right) = 0$.

$$\Rightarrow \boxed{\sum_{i=1}^{m} c_i a_{ik} = 0}$$

If we put $x = a_k$:

$$\boxed{c_k d_1^2 d_2^2 + \sum_{j=1}^{n} d_j a_{kj} + b = 0}$$

$$0 = \sum_{k=1}^{m} c_k \left(c_k d_1^2 d_2^2 + \sum_j d_j a_{kj} + b \right)$$

$$= \left(\sum_{k=1}^{m} c_k^2\right)\left(d_1^2 d_2^2\right) + \sum_{j=1}^{n}\left(d_j \sum_{k=1}^{m} c_k a_{kj}\right) + b\left(\sum_{k=1}^{m} c_k\right)$$

$$= \left(\sum_{k=1}^{m} c_k^2\right)\left(d_1^2 d_2^2\right)$$

$$\Longrightarrow \sum_{k=1}^{m} c_k^2 = 0 . \qquad c_1 = c_2 = \cdots = c_m = 0 .$$

$\square$

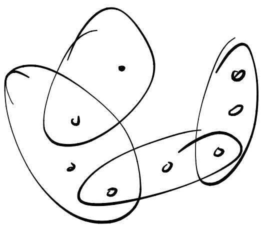
Remark: $\left(\begin{array}{l} X \text{ is an } s\text{-distance set in } \mathbb{R}^n \\ \Longrightarrow |X| \leq \binom{n+s}{s} \\ \text{Bannai, Bannai, Stanton 1983} \end{array} \right)$

1.5. Parallel lectures

KAIST 2 special lectures at the same.

$C_1, C_2, C_3, \dots, C_m$: list of student clubs

Q: Can we assign students to one of 2 lectures so that in each student club, there is at least one student in each lecture hall?



Thm (Erdős 1963)

If each club has $\geq k$ members and $m < 2^{k-1}$, then it is possible to assign students into one of the 2 lecture halls so that no student club is completely in 1 lecture hall.

__Proof__ We randomly assign each student to one of the 2 lecture halls with prob $\frac{1}{2}$ independently.

Let A_i = event that all members of C_i are completely in 1 lecture hall.

$$\Pr(A_i) = 2\left(\frac{1}{2}\right)^{|C_i|} \leq 2 \cdot 2^{-k} = 2^{1-k}.$$

$$\Pr(A_1 \cup A_2 \cup \cdots \cup A_m) \le \sum_{i=1}^{m} \Pr(A_i) = m\, 2^{1-k} < 1$$

"Union bound"

$\Rightarrow$ There is a positive probability that none of A_i occurs. $\square$

<u>Thm</u>. Suppose that each club has ≥ 2 member and $|C_i \cap C_j| \ne 1$ for all i, j.

Then it's possible to assign students in that way.

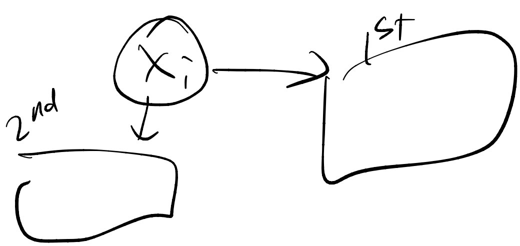
<u>Proof</u> Let $x_1, x_2, \cdots x_n$ be the list of students. We say an assignment of $x_1, \ldots, x_i$ is bad if there is a club whose members are completely in one lecture hall.

Induction on i.

> Claim: There is an assignment of $x_1, \ldots x_i$ that is not bad.

$\boxed{i=1}$: Put x_1 into an arbitrary lecture hall.

$\boxed{i>1}$: Assume that $x_1, x_2, \ldots, x_{i-1}$ are already assigned to one of the 2 lecture halls so that no club is bad.



We may assume there is a club $C \subseteq 1^{st}$ lecture hall

$x_i \in C \subseteq \{x_1, \ldots, x_i\}$

Similarly there is

$x_i \in D \subseteq 2^{nd}$ lecture hall

$D \subseteq \{x_1, \ldots, x_i\}$

$|C \cap D| = 1$. Contradiction. $\square$

1.6. Ramsey Numbers

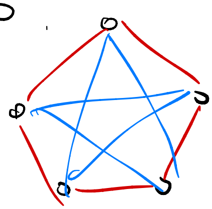
For integers $k, l > 0$, we say

$R(k, l) = \min\{N :$ any coloring of $E(K_N)$ with 2 colors (red or blue) has a red K_k subgraph or a blue K_l subgraph$\}$

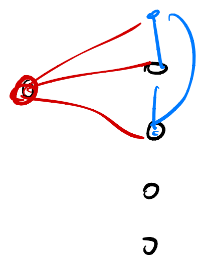
Ramsey number

$R(3,3) = 6$

$R(3.3) > 5$ →



$\begin{cases} \text{no red } K_3 \\ \text{no blue } K_3 \end{cases}$



$R(3.3) \le 6.$

Thm: If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$.

Proof. We color the edges of K_n randomly by red or blue with prob $1/2$ each.

For a set R of k vertices

let A_R = event that R induces a red K_k or a blue K_k.

$$\Pr(A_R) = 2^{-\binom{k}{2}} \times 2 = 2^{1-\binom{k}{2}}$$

$$\Pr\Big(\bigcup_{\substack{R \subseteq V(K_n) \\ |R| = k}} A_R\Big) \le \sum_{\substack{R \subseteq V(K_n) \\ |R| = k}} \Pr(A_R) = \binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

$\Rightarrow$ There is a positive probability that none of A_R occurs

$\Rightarrow R(k,k) > n.$ $\square$

Say $\boxed{n = \lfloor 2^{k/2} \rfloor}$. $k \geq 3$.

$$\binom{n}{k} 2^{1-\binom{k}{2}} < \frac{n^k}{k!} 2^{1-\frac{k(k-1)}{2}} = \frac{n^k}{k!} \frac{2^{1+k/2}}{2^{k^2/2}}$$

$$< \frac{\cancel{2^{\frac{k^2}{2}}}}{k!} \frac{2^{1+k/2}}{\cancel{2^{k^2/2}}}$$

$$= \frac{1}{k!} 2^{1+k/2}.$$

$$< \frac{1}{3!} 2^{1+\frac{3}{2}} = \frac{4\sqrt{2}}{3!} < 1$$

$\therefore R(k,k) > \lfloor 2^{k/2} \rfloor$ if $k \geq 3$.

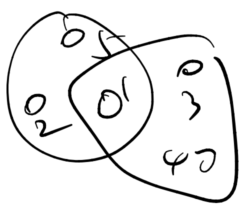
1.7. Family of Sets

A set system (or a family of sets) over X is a set of subsets ~~of~~ X

$\mathcal{F} = \{A_1, A_2, \dots, F_m\}$

A set system $\mathcal{F}$ is k-uniform if $|X| = k$ for all $X \in \mathcal{F}$.

A "hypergraph" is a pair (V, E) of a finite set V and a set E of "edges" (hyperedges) that are subsets of V.

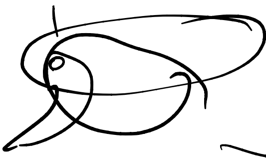


$(\{1,2,3,4,5\}, \{\{1,3,4\}, \{1,2,5\}\})$

2. Set Systems with Restricted Intersection

2.1. Erdős–Ko–Rado Theorem

We say a set system $\mathcal{F}$ is INTERSECTING if
$X \cap Y \neq \phi$ for all $X, Y \in \mathcal{F}$.



A set system over $\{1, 2, \ldots, n\}$ which is intersecting
→ subsets containing 1.
2^{n-1} such sets.

Thm. If $\mathcal{F}$ is an intersecting family of subsets of an n-element set S, then
$$|\mathcal{F}| \leq 2^{n-1}.$$

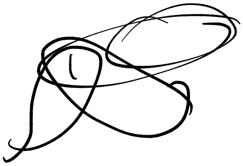
Proof. Suppose $|\mathcal{F}| > 2^{n-1}$.
Then $\mathcal{F}$ contains a pair X, Y
$$Y = S - X$$
$$\Rightarrow X \cap Y = \phi \qquad \square$$

Thm (Erdős–Ko–Rado 1961)
Let $k \leq \frac{n}{2}$.

Let $\mathcal{F}$ be a k-uniform intersecting family of subsets of an n-element set.

Then
$$|\mathcal{F}| \leq \binom{n-1}{k-1}$$

Example :

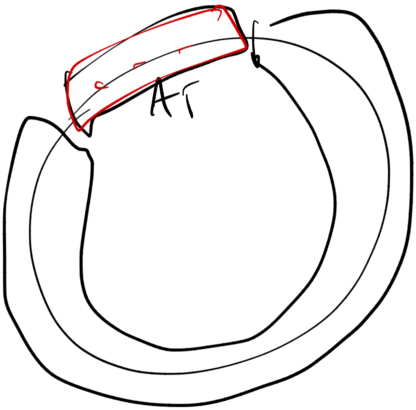


$\binom{n-1}{k-1}$

Proof (Katona 1972)

$S = \{1, \ldots, n\}$

$A_1, A_2, \ldots, A_m \in \mathcal{F}$ $\quad |A_i| = k$

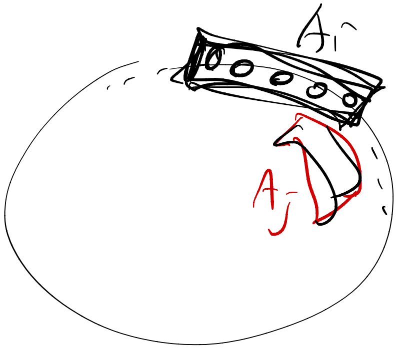


For $i \in \{1, 2, \ldots, m\}$
let P_i be the set of all cyclic permutations of $\{1, 2, \ldots, n\}$
such that all elements of A_i occur consecutively in the cyclic order.

$$|P_i| = k!\,(n-k)!$$

- \# cyclic permutation $= (n-1)!$
- Each cyclic permutation can occur in at most $\underline{k}$ of $P_1, P_2, \ldots, P_m$.

Why?



$A_i \cap A_j = \emptyset$.

$$\sum_{i=1}^{m} |P_i| \leq k \cdot (\#\text{cyclic permutation})$$

$$= k \cdot (n-1)!$$

$$m\, k! \cdot (n-k)! \leq k\,(n-1)!$$

$$m \leq \frac{1}{(k-1)! \cdot (n-k)!} (n-1)! = \binom{n-1}{k-1} \qquad \square$$