3.4. Wedge products – 2^{nd} exterior power.

Let V be a vector space over $\mathbb{F}$.

Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V.

For a given vector space V,
we define the dual space V^*
as the vector space consisting of
all linear functions from V to $\mathbb{F}$.

Observation: $\dim(V^*) = \dim(V)$

An alternating bilinear form on V over $\mathbb{F}$
is a function $B : V \times V \to \mathbb{F}$ satisfying

(1) $B(v, v) = 0$ for all $v \in V$.
(2) $B(u, v) = -B(v, u)$ for all $u, v \in V$.
(3) $B(c_1 u_1 + c_2 u_2, v) = c_1 B(u_1, v) + c_2 B(u_2, v)$.

The 2^{nd} exterior power of V, denoted by $\Lambda^2 V$
is the dual space of the vector space
of all alternating bilinear forms on V over $\mathbb{F}$.

For an alternating bilinear form $B : V \times V \to \mathbb{F}$
and $u, v \in V$,
we define $\underline{u \wedge v} \in \Lambda^2 V$ $\boxed{\underline{u \text{ wedge } v}}$
so that

$$(u \wedge v)(B) = B(u, v) \in \mathbb{F}.$$

Lemma $\{v_i \wedge v_j : 1 \le i < j \le n\}$ is a basis of $\wedge^2 V$.

Proof. For $a = \sum_{i=1}^{n} a_i v_i$ $\quad b = \sum_{i=1}^{n} b_i v_i$

we define an alternating bilinear form

$$E_{ij}(a,b) = a_i b_j - a_j b_i \in \mathbb{F}.$$

$$\begin{pmatrix} i<j \\ i'<j' \end{pmatrix} \quad E_{ij}(v_{i'}, v_{j'}) = \begin{cases} 1 & \text{if } i=i',\ j=j' \\ 0 & \text{otherwise} \end{cases}$$

Claim 1: $\{v_i \wedge v_j : i<j\}$ is linearly indep.

$$\sum c_{ij}(v_i \wedge v_j) = 0$$

$$0 = \sum_{i<j} c_{ij}(v_i \wedge v_j)(E_{i',j'})$$

$$= \sum_{i<j} c_{ij}\, E_{i',j'}(v_i, v_j)$$

$$= c_{i'j'}$$

The space of all alternating bilinear forms

$\sim$ the space of skew-symmetric $n \times n$ matrices

$\binom{n}{2}$

$\dim \le \binom{n}{2}$.

$\Rightarrow \{v_i \wedge v_j : i<j\}$ is a basis of $\Lambda^2 V$.

* $v \wedge v = 0$

$$\left(\sum c_i v_i\right) \wedge \left(\sum c_j v_j\right) = \sum_{i<j} \left(c_i v_i \wedge v_j + c_i v_j \wedge v_i\right) + c_i v_i \wedge v_i = 0.$$

* $v \wedge u = -u \wedge v$

$$(v \wedge u)(B) = B(v,u) = -B(u,v) = -(u \wedge v)(B).$$

* $(c_1 u_1 + c_2 u_2) \wedge v = c_1 (u_1 \wedge v) + c_2 (u_2 \wedge v)$

$$\left((c_1 u_1 + c_2 u_2) \wedge v\right)(B) = B(c_1 u_1 + c_2 u_2, v) = c_1 B(u_1, v) + c_2 B(u_2, v) = c_1\, u_1 \wedge v(B) + c_2 (u_2 \wedge v)(B) = (c_1 u_1 \wedge v + c_2 u_2 \wedge v)(B).$$

* $u \wedge v = 0 \iff \{u, v\}$ is linearly dependent.

($\Leftarrow$) $u = kv$, easy

($\Rightarrow$) If $\{u, v\}$ is independent, then we can extend to a basis.

$u \wedge v \neq 0$.

3.5. Wedge products – k^{th} exterior power

V: vector space over a field $\mathbb{F}$, finite dimensional

An alternating multilinear form of degree k on V is a function $M: V^k \to \mathbb{F}$ with the following properties:

(1) $M(w_1, w_2, \cdots, w_k) = 0$ if $w_i = w_j$ for some $i \neq j$.

(2) $M(w_1, w_2, \cdots, w_i, \cdots, w_j, \cdots, w_k)$

$= -M(w_1, w_2, \cdots, w_j, \cdots, w_i, \cdots, w_k)$

(3) $M(c_1 a_1 + c_2 a_2, w_2, w_3, \cdots, w_k)$

$= c_1 M(a_1, w_2, w_3, \cdots, w_k) + c_2 M(a_2, w_2, w_3, \cdots, w_k)$.

for $c_1, c_2 \in \mathbb{F}$

$w_2, w_3, \cdots, w_k \in V$

$a_1, a_2 \in V$.

<u>Lemma</u>. Let M be an alternating multilinear form of degree k on V.

Let $u_1, u_2, \cdots, u_k, w_1, w_2, \cdots, w_k \in V$.

If $u_1, u_2, \cdots, u_k$ are in the subspace spanned by $w_1, w_2, \cdots, w_k$, then

$M(u_1, u_2, \cdots, u_k) = \lambda M(w_1, w_2, \cdots, w_k)$

for some $\lambda \in \mathbb{F}$.

<u>easy</u>.

The k^{th} exterior power of V

$= \Lambda^k V =$ dual space of the vector space of all alternating multilinear form of degree k on V.

For $\omega_1, \omega_2, \dots, \omega_k \in V$,
we write $w_1 \wedge w_2 \wedge \dots \wedge w_k \in \Lambda^k V$
defined as
$$(\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_k)(M) = M(\omega_1, \dots, \omega_k).$$

If $\{v_1, v_2, \dots, v_n\}$ is a basis of V, then M is determined by
$$M(v_{i_1}, v_{i_2}, \dots, v_{i_k}) \quad 1 \le i_1 < i_2 < \dots < i_k \le n.$$
There are $\binom{n}{k}$ values to determine M.

$\Rightarrow$ dimension of the vector space of the alternating multilinear form of degree k is $\le \binom{n}{k}$.

$\Rightarrow \dim \Lambda^k V \le \binom{n}{k}$.

Lemma: $\{v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k} : 1 \le i_1 < i_2 < \dots < i_k \le n\}$ is a basis of $\Lambda^k V$

if $\{v_1, v_2, \dots, v_n\}$ is a basis of V.

Proof. For $1 \le i_1 < i_2 < \dots < i_k \le n$ let us define $M_{i_1 \dots i_k}(\omega_1, \omega_2, \dots, \omega_k)$
so that

If $w_i = \sum w_{ij} v_j$ then

$$M_{i_1 i_2 \cdots i_k}(w_1, \ldots, w_k) = \det \begin{pmatrix} w_{1i_1} & w_{2i_1} & \cdots & w_{ki_1} \\ w_{1i_2} & w_{2i_2} & \cdots & w_{ki_2} \\ w_{1i_3} & w_{2i_3} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ w_{1i_k} & w_{2i_k} & \cdots & w_{ki_k} \end{pmatrix}$$

$M_{i_1 i_2 \ldots i_k}$ is an alternating multilinear form of degree k.

$$\begin{cases} M_{i_1 i_2 \cdots i_k}(v_{i_1}, v_{i_2}, \ldots, v_{i_k}) = \det \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = 1. \\ M_{i_1 i_2 \cdots i_k}(v_{j_1}, v_{j_2}, \ldots, v_{j_k}) = 0. \end{cases}$$

$\{i_1 \cdots i_k\} \neq \{j_1 \cdots j_k\}$

If $\sum \alpha_{j_1 \cdots j_k} v_{j_1} \wedge v_{j_2} \wedge \cdots \wedge v_{j_k} = 0$

then $\sum \alpha_{j_1 \cdots j_k} (v_{j_1} \wedge \cdots \wedge v_{j_k})(M_{i_1 \cdots i_k}) = 0$

$$\sum \alpha_{j_1 \cdots j_k} \underline{M_{i_1 i_2 \cdots i_k}(v_{j_1}, \ldots, v_{j_k})} = 0$$

$$\alpha_{i_1 \cdots i_k} = 0$$

$\{v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}\}$ is linearly indep. $\square$

Key properties

(1) $v_1 \wedge v_2 \wedge \cdots \wedge v_k = 0$ if $v_i = v_j$ for some $i \neq j$.

(2) $v_1 \wedge v_2 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k$

$= - v_1 \wedge v_2 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k$

Lemma. $v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq 0$

$\iff$ $\{v_1, v_2, \ldots, v_k\}$ is linearly independent

Proof. If $\{v_1, \ldots v_k\}$ is linearly dependent

then $v_1 \wedge \cdots \wedge v_k = 0$ (; easy)

If $\{v_1 \ldots v_k\}$ is linearly independent, then extend it to a base

$\{v_1, \cdots v_k, v_{k+1}, \ldots, v_n\}$

$\Rightarrow$ $v_1 \wedge \cdots \wedge v_k$ is an element of a basis

$\Rightarrow v_1 \wedge \cdots \wedge v_k \neq 0$. $\square$

$\wedge \longleftrightarrow \det$ $\qquad w_i = \sum_{j=1}^{n} w_{ij} v_j$

$w_1 \wedge w_2 \wedge \cdots \wedge w_n$

$= \left(\sum w_{1j} v_j\right) \wedge \left(\sum w_{2j} v_j\right) \wedge \cdots \wedge \left(\sum w_{nj} v_j\right)$

$= \sum w_{1i_1} w_{2i_2} \cdots w_{ni_n} v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_n}$

$\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$

$$= \sum_{\sigma} \boxed{w_{1\sigma(1)} w_{2\sigma(2)} \cdots w_{n\sigma(n)} \operatorname{sign}(\sigma)} \, v_1 \wedge v_2 \wedge \cdots \wedge v_n$$

σ : permutation

$$= (\det W) \, \underline{v_1 \wedge v_2 \wedge \cdots \wedge v_n}$$

where

$$W = \begin{pmatrix} w_{11} & w_{21} & \\ w_{12} & w_{22} & \\ \vdots & \vdots & \cdots \\ w_{1n} & w_{2n} & \end{pmatrix}$$

For $a \in \Lambda^k V$, $b \in \Lambda^l V$, we define $\underline{a \wedge b}$

$a \wedge b \in \Lambda^{k+l} V$

$a \wedge b : M \longmapsto \mathbb{F}$; M: alternating multilinear form of degree $k+l$.

For $w_1, \ldots, w_l \in V$,

$$M|_{w_1 \cdots w_l} := M(\underbrace{-,-,-,-}_{k}, \underbrace{w_1, \ldots, w_l}_{l})$$

$$M \longmapsto \underline{M|_{w_1 \ldots w_l}} : V^k \to \mathbb{F}$$

is an alternating multilinear form of degree k.

$$\underline{a}\left(M|w_1\dots w_\ell\right)\in\mathbb{F}.$$

We define

$$(a\wedge b)(M)=b\left(a\left(M|w_1\dots w_\ell\right)\right)$$

$$\left(\begin{array}{l}\text{let } N(w_1,w_2,\dots w_\ell)=a\left(M|w_1\dots w_\ell\right)\\ \Rightarrow N \text{ is an alternating multilinear form of degree } \ell.\\ \Rightarrow b(N) \text{ is defined.}\end{array}\right)$$

$$\star\quad \Big(\underbrace{(u_1\wedge\dots\wedge u_k)}_{\in\Lambda^k V}\wedge\underbrace{(w_1\wedge\dots\wedge w_\ell)}_{\in\Lambda^\ell V}=\underbrace{u_1\wedge\dots\wedge u_k\wedge w_1\dots\wedge w_\ell}_{\in\Lambda^{k+\ell}V}.$$

Lemma.

(1) If $a\in\Lambda^k V$, $b\in\Lambda^\ell V$, then

$$b\wedge a=(-1)^{k\ell}\,a\wedge b$$

1 2 3 4 5
1 3 2 4 5
3 1 2 4 5

(2) $(c_1a_1+c_2a_2)\wedge b=c_1(a_1\wedge b)+c_2(a_2\wedge b)$

(3) $a\wedge(b\wedge c)=(a\wedge b)\wedge c\quad\in\Lambda^{k+\ell+m}V,$

$a\in\Lambda^k V,\quad b\in\Lambda^\ell V,\quad c\in\Lambda^m V.$

$\square$

3.6 Skew Version of Bollobás Two Families Theorem.

Thm. (Skew version of Bollobás' Two Families Thm)

$\left.\begin{matrix} A_1, A_2, \dots, A_m \\ B_1, B_2, \dots, B_m \end{matrix}\right\}$ subsets of $\{1, \dots, n\}$

$|A_i| = r \qquad |B_i| = s \qquad$ for all i

(1) $A_i \cap B_i = \emptyset$

(2) $A_i \cap B_j \neq \phi$ for all $i < j$

$$\Rightarrow \quad m \le \binom{r+s}{r}.$$

Lemma. In $\mathbb{R}^n$, we can pick a set X of infinitely many points such that every subset of at most n points is linearly independent.

Proof. Let $a_k = \begin{pmatrix} 1 \\ k \\ k^2 \\ \vdots \\ k^{n-1} \end{pmatrix} \in \mathbb{R}^n$

$\{a_{i_1}, a_{i_2}, \dots, a_{i_n}\}$ is linearly independent.

$$\Leftrightarrow \quad \det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ i_1 & i_2 & i_3 & \cdots & i_n \\ i_1^2 & i_2^2 & \cdots & & i_n^2 \\ \vdots & & & & \\ i_1^{n-1} & i_2^{n-1} & \cdots & & i_n^{n-1} \end{pmatrix} \neq 0$$

$$\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{i>j} (x_i - x_j)$$

Proof of skew version of Bollobás Two Families theorem (Lovasz 1977)

Let X be an infinite subset of $\mathbb{R}^{r+s}$ such that every subset of size $\le r+s$ is linearly independent.

For each $x \in \bigcup_{i=1}^{m} (A_i \cup B_i)$,

we associate x with some element of X.

((In other words) we may assume $A_i, B_i \subseteq X$)

For a subset $I \subseteq X$, let

$$w_I = \bigwedge_{i \in I} w_i \in \Lambda^{|I|} \mathbb{R}^{r+s}.$$

Let $a_i = w_{A_i} \in \Lambda^r \mathbb{R}^{r+s}$

$b_i = w_{B_i} \in \Lambda^s \mathbb{R}^{r+s}$

$a_i \wedge b_i \neq 0$ (because $A_i \cap B_i = \phi$) & ~~any~~ r+s vectors in X are linearly indep.

If $i<j$ then

$a_i \wedge b_j = 0 \qquad (A_i \cap B_j \neq \phi)$

Claim: $\{a_1, a_2 \ldots a_m\}$ is linearly independent in $\wedge^r \mathbb{R}^{r+s}$

Suppose $\sum c_j a_j = 0$

$$\sum c_j a_j \wedge b_i = 0$$

$$\overline{c_i a_i \wedge b_i = 0}$$

$$\boxed{c_i = 0}$$

$\therefore \quad m \le \dim V^r \mathbb{R}^{r+s} = \binom{r+s}{r}.$

Thm (Lovasz)

Let W be a vector space over a field $\mathbb{F}$.

Let $U_1, U_2, \ldots, U_m$

$V_1, V_2, \ldots, V_m$ be subspaces of W

Such that $\dim U_i = r, \quad \dim V_i = s$

- $U_i \cap V_i = \{0\}$ for all i
- $U_i \cap V_j \neq \{0\}$ for all $i<j$

Then $\quad m \le \binom{r+s}{r}.$

Partial Proof (When $\underline{\dim W = r+s}$)

For a subspace U of W with a basis $\{x_1, \dots, x_p\}$

We write $\Lambda U = x_1 \wedge x_2 \wedge \cdots \wedge x_p$.

Let
$$a_i = \Lambda U_i \in \Lambda^r W$$
$$b_i = \Lambda V_i \in \Lambda^s W$$

$$U_i \cap V_i = \{0\} \Rightarrow a_i \wedge b_i \neq 0$$

If $i < j$
$$U_i \cap V_j \neq \{0\} \Rightarrow a_i \wedge b_j = 0.$$

$\{a_1, a_2, \dots, a_m\}$ is linearly independent

$$\Rightarrow m \leq \dim \Lambda^r \underline{W} = \binom{r+s}{r}$$

$\square$

3.7. Subspaces in general position

For subspaces $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_m$ of a vector space W over a field $\mathbb{F}$, $\dim W = n$,
we say a subspace V of W is in general position of dim $n-t$ with respect to $\{\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_m\}$
if

$$\dim \mathcal{U}_i \cap V = \max(\dim(\mathcal{U}_i) - t, 0)$$

for all $i = 1, 2, \dots, m$.

<u>Goal</u>: If $|\mathbb{F}|$ is big, then there is one.

<u>Thm</u> (Existence of a subspace in general position)

Let $s \in \{0, 1, \dots, n\}$.
Given a list of subspaces $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_m$ of a vector space W of dimension n over a field $\mathbb{F}$,
there is an s-dimensional subspace V in general position with respect to $\mathcal{U}_1, \dots, \mathcal{U}_m$
if $|\mathbb{F}| > sm$.

<u>Lemma</u> (DeMillo–Lipton–Schwartz–Zippel 1980's)
(sparse zero lemma)

Let $f \in \mathbb{F}[x_1, x_2, \dots, x_n]$ be a nonzero polynomial of degree d.
Let $\Omega \subseteq \mathbb{F}$ be a finite set with $|\Omega| = N$.
Then the number of tuples $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \Omega^n$ with $f(\alpha_1, \alpha_2, \dots, \alpha_n) = 0$
is at most dN^{n-1}.

In other words, if we take $\alpha_i \in \Omega$ uniformly at random, then

$$P(f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0) \le \frac{d}{N}.$$

Proof. Induction on n.

If $n=1$, then f has at most d roots.

Now let us assume $n > 1$.

Write

$$f(x_1, \ldots, x_n) = g_0 + g_1 x_n + g_2 x_n^2 + \ldots + g_k x_n^k$$

where $g_i \in \mathbb{F}[x_1, \ldots, x_{n-1}]$, $\deg g_i \le d - i$.

$\boxed{\# \text{roots of } f(x_1, \ldots x_n)}$

$g_k(\alpha_1 \ldots \alpha_{n-1})$

$(\#(\alpha_1 \ldots \alpha_{n-1})$ with $g_k(\alpha_1 \ldots \alpha_{n-1}) = 0)$

$$\le (d-k) N^{n-2}$$

(by induction)

$\Rightarrow \#(\alpha_1 \ldots \alpha_n)$ with $g_k(\alpha_1 \ldots \alpha_{n-1}) = 0$ & $f(\alpha_1 \ldots \alpha_n) = 0$

$$\le (d-k) N^{n-2} \cdot N = (d-k) N^{n-1}.$$

$\#(\alpha_1 \ldots \alpha_{n-1})$ with $g_k(\alpha_1 \ldots \alpha_{n-1}) \ne 0)$

$$\le N^{n-1}$$

$\Rightarrow$ There are $\le k$ choices of α_n for each of them so that $f(\alpha_1 \ldots \alpha_n) = 0$.

$\Rightarrow (\#(\alpha_1, \ldots, \alpha_n)$ with $f(\alpha_1 \ldots \alpha_n) = 0$, $g_k(\alpha_1 \ldots \alpha_{n-1}) \ne 0)$

$$\le k \cdot N^{n-1}$$

$$\Rightarrow \#\left((\alpha_1, \dots, \alpha_n) \text{ such that } f(\alpha_1, \dots, \alpha_n) = 0\right)$$
$$\leq k \cdot N^{n-1} + (d-k)N^{n-1} = dN^{n-1} \qquad \square$$

Proof of the theorem on the existence of a subspace in general position.

Let $t = n - s$.

We may assume $\dim(U_i) \geq t$ by enlarging U_i whenever $\dim U_i < t$.

<u>Goal</u>: Find a subspace V such that

$$\begin{cases} \dim(U_i \cap V) = \dim U_i - t \\ \dim V = n - t \end{cases}$$

We fix a basis W.
Each vector of W is represented by this basis.

Let

$$P_i(x_1, x_2, \dots, x_{n-t}) = \det \begin{array}{cc} & n \\ n & \begin{bmatrix} w_1^i & w_2^i & \cdots & w_t^i & x_1 & x_2 & \cdots & x_{n-t} \end{bmatrix} \end{array}$$

First t vectors of a basis of U_i

where $x_1, x_2, \ldots, x_{n-t} \in \mathbb{F}^n$

$\Rightarrow$ $p_i \in \mathbb{F}[x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, x_{22}, \ldots, x_{2n}, \ldots, x_{n-t,1}, x_{n-t,2}, \ldots, x_{n-t,n}]$

$\deg p_i = n-t$

Let X be a vector space spanned by $x_1, x_2, \ldots, x_{n-t}$.

If $p_i \neq 0$, then $\dim(U_i + X) = n$

$$\Rightarrow \dim(U_i \cap X) = \dim U_i + \dim X - \dim(U_i + X)$$
$$= \dim U_i + (n-t) - n$$
$$= \dim U_i - t$$

Goal: Find $x_1, x_2, \ldots, x_{n-t}$ so that $p_i(x_1, x_2, \ldots, x_{n-t}) \neq 0$ for all i.

Let $g = p_1 p_2 \cdots p_m$.

$\deg g = m(n-t) = ms$.

$\#\text{roots of } g \leq (ms)\, |\mathbb{F}|^{(n-t)n - 1}$

$\Rightarrow$ If $|\mathbb{F}|^{(n-t)n} > (ms)\, |\mathbb{F}|^{(n-t)n-1}$

then there is an assignment
$x_1, x_2, \ldots x_{n-t} \in \mathbb{F}^n$ such that
$g(x_1, \ldots, x_{n-t}) \neq 0$.

$\Leftarrow |\mathbb{F}| > ms$ $\square$

<u>Corollary</u> Let t be a nonnegative integer.
let $U_1, U_2, \ldots U_m$ be subspaces of
a vector space W over a field $\mathbb{F}$
$\dim W = n$.
There is a linear transformation
$\varphi : W \longrightarrow W'$
such that $\dim W' = t$
$\dim \varphi(U_i) = \min(\dim(U_i), t)$
for each i.
if $|\mathbb{F}| > m(n-t)$.
We say such a linear transformation
is in general position
with respect to $U_1, U_2, \ldots, U_m$.

Proof. Let U be a subspace of dim $n-t$
of W
in general position with $U_1 \ldots U_m$.

Take any linear transformation $\varphi : W \to W$ such that

$$\ker \varphi = V.$$

$$\dim(\varphi(U_i)) = \dim U_i - \dim(U_i \cap \ker(\varphi))$$
$$= \dim U_i - \max(\dim U_i - t, 0)$$
$$= \min(t, \dim U_i)$$

$\square$

Proof of Lovász Two Families theorem:

- We may assume $|\mathbb{F}|$ is big by taking an extension field of $\mathbb{F}$

Enough to consider the case that

$$\dim W > r+s.$$

There is a linear transformation $\varphi : W \to W'$ in general position with $U_i + V_j$ for all $i \le j$

such that

$$W' = \varphi(W)$$

- $\dim W' = \dim \varphi(W) = r+s$
- $\dim \underline{\varphi(U_i+V_j)} = \min(\dim(U_i+V_j), r+s)$
$$= \dim \underline{(U_i+V_j)}$$

$\Longrightarrow \dim \varphi(U_i) = \dim U_i$

For $i \le j$

$$\dim(\varphi(U_i) \cap \varphi(V_j))$$

$$= \dim \varphi(U_i) + \dim \varphi(V_j) - \dim(\varphi(U_i) + \varphi(V_j))$$

$$= \dim U_i + \dim V_j - \dim(U_i + V_j)$$

$$= \dim U_i \cap V_j$$

So, $\varphi(U_1), \ldots \varphi(U_m)$

$\varphi(V_1), \ldots \varphi(V_m)$ are subspaces of $\varphi(W)$

where
$$\begin{cases} \varphi(U_i) \cap \varphi(V_j) \neq \{0\} & \text{for } i < j \\ \varphi(U_i) \cap \varphi(V_i) = \{0\} & \text{for all } i \end{cases}$$

$$\dim \varphi(W) = r + s$$

$\dim \varphi(U_i) = r \quad \dim \varphi(V_j) = s$

$$\Longrightarrow \quad m \le \dim \Lambda^r(W') = \binom{r+s}{r}$$

$\square$