

Please type your homework solution into a PDF file and submit to the gradescope.com website by 11:59pm KST of the due date. A point will be deducted for each problem if a handwritten solution is submitted. We recommend using \LaTeX . You can find useful hints at our KLMS website. It is recommended to use the sample template. (At least, make sure that each problem has a solution in separate pages.) Unprofessional proofs may get a deduction of points, even if the solution is mathematically correct or can be made correct.

2021 SPRING MAS575 Combinatorics
HOMEWORK 6

DUE: JUNE 8, 2021

Note that \mathbb{N} is the set of positive integers.

6.1. Prove that for every positive integer r and every r -coloring of \mathbb{N} , there exist three positive integers x , y , and z such that x , $x + y$, z , and $x + yz$ have the same color.

6.2. Prove that for every positive integer r and every r -coloring of \mathbb{N} , there exist three positive integers x , y , and z of the same color such that $xy^2 = z^3$.

6.3. Prove that for every positive integer k , there exists a prime p such that there are k consecutive quadratic residue modulo p .

(An integer q is a quadratic residue modulo p if $q \equiv x^2 \pmod{p}$ for some integer x .)

6.4. Let n be a positive integer. Prove that there is a $(2n)$ -coloring χ of all rational numbers such that

$$\sum_{i=1}^n x_i - \sum_{i=1}^n y_i = 1$$

has no rational solutions such that $\chi(x_i) = \chi(y_i)$ for all $i = 1, 2, \dots, n$.

6.5. Prove that the following are equivalent.

1. For every positive integer r and every r -coloring of \mathbb{N} , there exist distinct positive integers x_1, x_2, \dots, x_n of the same color such that $c_1 x_1 + c_2 x_2 + \dots + c_n x_n = 0$.
2. For every positive integer r and every r -coloring of \mathbb{N} , there exist positive integers x_1, x_2, \dots, x_n of the same color such that

$$c_1 x_1 + c_2 x_2 + \dots + c_n x_n = 0$$

and there are distinct integers $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $c_1 \lambda_1 + c_2 \lambda_2 + \dots + c_n \lambda_n = 0$.