KAIST
2021 MAS575 Combinatorics
Homework 1

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HW 1.1

Suppose that there are m red clubs $(R_i)_{i\in[m]}$, and m blue clubs $(B_i)_{i\in[m]}$ in a university of n students. Suppose that the following rules are satisfied:

1. $|R_i \cap B_i|$ is odd for every i.
2. $|R_i \cap B_j|$ is even for every $i \neq j$.

Prove that $m \leq n$.

Proof. Let $[n]$ denote the students and let $M, N \in \mathbb{F}_2^{n\times m}$ such that $M_{ij} = 1$ iff $i \in R_j$ and $N_{ij} = 1$ iff $i \in B_j$, for all $(i,j) \in [n] \times [m]$. Consider the matrix multiplication on $\mathbb{F}_2$ and we have $(M^T N)_{kl} = \sum_p M_{pk}N_{pl} = |R_k \cap B_l| = 1$ iff $k = l$, for all $(k,l) \in [m]^2$, i.e., $M^T N = I_m$. Particularly, $n \geq \text{rank}(M) \geq \text{rank}(M^T N) = m$, completing the proof. $\square$

HW 1.2

Let us consider the following variation of the odd rule in some university with n students:

1. Every club has an even number of members.
2. Every pair of clubs shares an odd number of members.

Prove that there are at most n clubs if n is odd, and at most $n-1$ clubs if n is even.

Proof. Let $[n]$ denote the students. Suppose we have maximal m clubs satisfying the above conditions. Now we add a new student $(n+1)$ who joins all the m existing clubs and also opens a new club as its only member. Consider now the $m+1$ clubs $(C_i)_{i\in[m+1]}$ and $n+1$ students $[n+1]$, it is easy to check that

1. Every club has an odd number of members.
2. Every pair of clubs shares an even number of members.

Let $A \in \mathbb{F}_2^{(n+1)\times(m+1)}$ such that $A_{ij} = 1$ iff $i \in C_j$, for all $(i,j) \in [n+1] \times [m+1]$. Consider the matrix multiplication on $\mathbb{F}_2$ and we have $A^T A = I_{m+1}$ with full rank $m+1$. Therefore, we have $n+1 \geq \operatorname{rank}(A) \geq \operatorname{rank}(A^T A) = m+1$. It remains to show that $m \leq n-1$ when n is even. Suppose the opposite, i.e., $m = n$ are even. This time we delete the new student we just added and consider back the original situation. We have clubs $(C_i)_{i\in[m]}$ and student $[n]$. Let $A \in \mathbb{F}_2^{n\times m}$ such that $A_{ij} = 1$ iff $i \in C_j$, for all $(i,j) \in [n] \times [m]$. We have $A^T A = J_m - I_m$ where J_m is the all-1 $m \times m$ matrix. $A^T A$ has full rank as its determinant is $(-1)^{m-1}(m-1) = 1$ with even m. By the similar inequality $n \geq \operatorname{rank}(A) \geq \operatorname{rank}(A^T A) = m$, we now know that it should be equality, which gives that A has full rank. However, the summation of all rows of A is zero vector as each column has even number of 1s which is equal to the number of members in the corresponding club. This contradicts with the full rankness, completing the proof.

□

HW 1.3

Let $\mathcal{A}, \mathcal{B}$ be families of subsets of $[n]$ such that $|A \cap B|$ is odd for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that $|\mathcal{A}||\mathcal{B}| \leq 2^{n-1}$.

Proof. We may assume that both $\mathcal{A}$ and $\mathcal{B}$ are nonempty, otherwise there is nothing to prove. Suppose $|\mathcal{A}| = \{A_i\}_{i\in[a]}$ and $|\mathcal{B}| = \{B_j\}_{j\in[b]}$, and let $U = \{u_i\}_{i\in[a]}$ where $u_i \in \mathbb{F}_2^n$ is the characteristic vector of A_i, likewise we construct $V = \{v_j\}_{j\in[b]}$ for $\mathcal{B}$. It is easy to see that the given condition is equivalent to $u_i \cdot v_j = 1$ (on $\mathbb{F}_2$) for all $(i, j) \in [a] \times [b]$. Let $U' = u_1 + U = \{u_1 + u : u \in U\}$ and let $V' = v_1 + V = \{v_1 + v : v \in V\}$. Furthermore, we let $\tilde{U} = U \cup U'$. Clearly, $|U| = |U'| = a$, $|V| = |V'| = b$, and $U \cap U' = \emptyset$ as fix any $v \in V$, $u \cdot v = 1$ for any $u \in U$ and $u' \cdot v = 0$ for any $u' \in U'$. Therefore, $|\tilde{U}| = 2a$ and suffices it to show that $|\tilde{U}||V'| \leq 2^n$. Let $\mathcal{U} = \mathrm{span}(\tilde{U})$ and let $\mathcal{V} = \mathrm{span}(V')$. We claim that $\mathcal{U}$ and $\mathcal{V}$ are orthogonal to each other as subspaces of $\mathbb{F}_2^n$. Indeed, for any $x \in \tilde{U}$ and $y \in V'$, we have $x \cdot y = 0$ $(1 + 1$ or $0 + 0)$. So $\mathcal{U}$ and $\mathcal{V}$ are also orthogonal to each other as their bases from which they span are orthogonal to each other as shown above. Thus, $|\tilde{U}||V'| \leq |\mathcal{U}||\mathcal{V}| \leq 2^{\dim(\mathcal{U})+\dim(\mathcal{V})} \leq 2^n$, completing the proof. $\square$

HW 1.4

Let $\{A_i\}_{i\in[m]}$ be an intersecting antichain of subsets of $[n]$ such that $|A_i| \le n/2$ for each i. Prove that

$$\sum_{i\in[m]} \binom{n-1}{|A_i|-1}^{-1} \le 1.$$

(A family $\{A_i\}_{i\in[m]}$ is an *antichain* iff $A_i \nsubseteq A_j$ for all $i \neq j$.)

Proof. This problem can be seen as a combination of the LYM inequality and the Erdös-Ko-Rado Theorem. Let $\mathcal{F}$ denote $\{A_i\}_{i\in[m]}$, the intersecting antichain we are studying. We follow Bollobás' paper *Sperner Systems Consisting of Pairs of Complementary Subsets.*

Lemma 4.1. *Put $\epsilon = e^{2\pi i/n}$ and $T = \{1, \epsilon, \epsilon^2, ..., \epsilon^{n-1}\}$. Let $\mathcal{B} = \{B_i\}_{i\in[l]}$ be an intersecting antichain of T such that*

1. *B_i has the form $\{\epsilon^k\}_{p_i \le k \le q_i}, \forall i$, for some $q_i - p_i \le n$;*
2. *$|B_i| \le n/2, \forall i$.*

Then

$$v(\mathcal{B}) = \sum_{i\in[l]} |B_i|^{-1} \le 1.$$

Remark 4.1. *This setting is equivalent to using consecutive numbers with module.*

Proof of Lemma 4.1. Let $m = \min\{|B_i| : B_i \in \mathcal{B}\}$ and let $\mathcal{C} = \{B_i \in \mathcal{B} : |B_i| = m\}$. We call $C_1, ..., C_s \in \mathcal{C}$ a *sequence* if $C_{i+1} = \epsilon C_i, \forall i \in [s-1]$. Clearly, $s \le m$ for any sequence, otherwise we will have two disjoint member in $\mathcal{B}$.

Decompose $\mathcal{C}$ into maximal sequences $\mathcal{C}_1, ..., \mathcal{C}_p$. Let $\mathcal{C}'_j$ consist of all $(m+1)$-subset of T, satisfying the first condition in the statement of the lemma and containing $\mathcal{C}_j$. Then we have

1. $|\mathcal{C}'_j| = |\mathcal{C}_j| + 1, \forall j$,
2. $\mathcal{C}'_j \cap \mathcal{B} = \emptyset, \forall j$,
3. $\mathcal{C}'_j \cap \mathcal{C}'_k = \emptyset, \forall j \neq k$.

Furthermore, $\mathcal{B}' = (\mathcal{B} - \mathcal{C}) \cup \bigcup_{j\in[p]} \mathcal{C}'_j$ also satisfies the two conditions in the statement and is also intersecting. As $|\mathcal{C}_j| \le m, \forall j$, we have

$$v(\mathcal{C}_j) = |\mathcal{C}_j|/m \le (|\mathcal{C}_j| + 1)/(m+1) = v(\mathcal{C}'_j),$$

which gives $v(\mathcal{B}) \le v(\mathcal{B}')$. By repeating this process, we can see that we may assume without loss of generality $|B_i| = \lfloor n/2 \rfloor, \forall i$, which make the conclusion trivial as again otherwise we will have two disjoint members. $\square$

Now, let $\mathcal{S}_k = \{S \subset [n] : |S| = k, 1 \in S\}$. Let α be the number of bijections from $[n]$ to T (equivalent to permutations) and let $\beta(A)$ be the number of bijections $\phi : [n] \to T$ such that $\phi(A)$ satisfies the first condition in Lemma 4.1. Let $\phi(\mathcal{F}) = \{\phi(A)\}_{A \in \mathcal{F}}$ and it is easy to check that $\phi(\mathcal{F})$ satisfies the two conditions in Lemma 4.1 and it's intersecting. Therefore, by the lemma, we have

$$\sum_{A \in \mathcal{F}} \frac{\beta(A)}{\alpha |A|} \leq 1.$$

The conclusion in Lemma 4.1 holds with equality for $\mathcal{B} = \phi(\mathcal{S}_k)$ for any bijection ϕ from $[n]$ to T, and the conclusion

$$\sum_{A \in \mathcal{F}} \binom{n-1}{|A|-1}^{-1} \leq 1$$

also holds with inequality for $\mathcal{F} = \mathcal{S}_k$. Therefore,

$$\frac{\beta(A)}{\alpha |A|} = \binom{n-1}{|A|-1}^{-1}$$

and the desired result follows. □

4.1 A previous attempt

Below is my previous proof where the red part is not true. I hoped to find a way to prove this using an argument similar to that for the Bollobás theorem but failed.

Proof. This problem can be seen as a combination of the LYM inequality and the Erdös-Ko-Rado Theorem. Let $\mathcal{F}$ denote $\{A_i\}_{i \in [m]}$, the intersecting antichain we are studying.

Claim 4.1. *For $s \in [n]$ and $l \in [\lfloor n/2 \rfloor]$, let $B_{s,l}$ denote the set of l consecutive numbers $\{s-1+i\}_{i \in [l]}$ (module n so that each element is in $[n]$). We claim that for each s, there exists at most one $l = l(s)$ such that $B_{s,l} \in \mathcal{F}$.*

The claim is trivial, if there are $l_1 < l_2$ such that both B_{s,l_1} and B_{s,l_2} are in $\mathcal{F}$, then $B_{s,l_1} \subset B_{s,l_2}$, which contradicts with the condition that $\mathcal{F}$ is an antichain. Now we count the number L of tuples (π, s, l) such that $\pi(B_{s,l}) \in \mathcal{F}$ and $\pi(s)$ is the minimum in $\pi(B_{s,l})$. For each $A_i \in \mathcal{F}$, we have

$$\frac{n}{|A_i|} |A_i|!(n - |A_i|)!$$

tuples (π, s, l) satisfying the conditions and specifically $\pi(B_{s,l}) = A_i$, where the denominator $|A_i|$ comes from that $\pi(s)$ is the minimum in $\pi(B_{s,l})$. Therefore,

$$L = \sum_{i \in [m]} \frac{n}{|A_i|} |A_i|!(n - |A_i|)!.$$

Now we claim that $L \leq n!$ which completes the proof. Specifically, we prove that for each fixed permutation π, at most one pair (s, l) can make (π, s, l) satisfy the conditions. Suppose the opposite, i.e., there are $(s, l) \neq (s', l')$ such that both $\pi(s, l)$ and $\pi(s', l')$ satisfy the conditions. By the claim, we know $s \neq s'$, otherwise we must have $l = l'$. But as $\pi(s)$ is the minimum in $\pi(B_{s,l})$ and $\pi(s')$ is the minimum in $\pi(B_{s',l'})$, and all the elements are distinct, we must have $\pi(B_{s,l}) \cap \pi(B_{s',l'}) = \emptyset$, which contradicts with that $\mathcal{F}$ is intersecting, completing the proof. □

HW 1.5

Let $1/2 \le p \le 1$. Let $(X_i)_{i\in[n]}$ be independent random variables such that $\Pr[X_i = 1] = p$ and $\Pr[X_i = 0] = 1 - p$. Let $(\alpha_i)_{i\in[n]}$ be nonnegative real numbers such that $\sum_{i\in[n]} \alpha_i = 1$. Prove that

$$\Pr[\sum_{i\in[n]} \alpha_i X_i \ge 1/2] \ge p.$$

Hint: Erdös-Ko-Rado Theorem.

Proof. We may assume that $n \ge 2$, otherwise there is nothing to prove. Besides, we may assume that each $\alpha_i > 0$ as $\alpha_i = 0$ makes the corresponding random variable essentially meaningless in terms of the summation. Let $\mathcal{F}$ be the maximal family of subsets of $[n]$ such that $\sum_{i\in A} \alpha_i \ge 1/2$ for all $A \in \mathcal{F}$. We have

$$\Pr[\sum_{i\in[n]} \alpha_i X_i \ge 1/2] = \sum_{A\in\mathcal{F}} p^{|A|}(1-p)^{n-|A|}.$$

Observe that for any $A \subset [n]$, at least one of A and $\overline{A} = [n] \setminus A$ should be in $\mathcal{F}$ as $\sum_{i\in[n]} \alpha_i = \sum_{i\in A} \alpha_i + \sum_{i\in\overline{A}} \alpha_i = 1$. Furthermore, if both A and $\overline{A}$ are in $\mathcal{F}$, i.e., $\sum_{i\in A} \alpha_i = \sum_{i\in\overline{A}} \alpha_i = 1/2$, we discard one of them. By doing this, we have $\mathcal{F}'$ with $|\mathcal{F}'| = 2^{n-1}$ and $\mathcal{F}'$ is intersecting because if $A \cap B = \emptyset$ and $\sum_{i\in A} \alpha_i, \sum_{i\in B} \alpha_i \ge 1/2$, then $\sum_{i\in A\cup B} \alpha_i = 1$ with $A \cup B \subsetneq [n]$, which contradicts with the assumption that each $\alpha_i > 0$. Now suffices it to show that

$$\sum_{A\in\mathcal{F}'} p^{|A|}(1-p)^{n-|A|} \ge p.$$

For $k \in [n]$, let $a_k = |\{A \in \mathcal{F}' : |A| = k\}|$ $(a_0 = 0)$. We use the following Erdös-Ko-Rado Theorem.

Theorem 5.1. *If $2k \le n$ then every intersecting family of k-element subsets of $[n]$ has at most $\binom{n-1}{k-1}$ members.*

Therefore, $a_k \le \binom{n-1}{k-1}$ for all $k \le \lfloor n/2 \rfloor$. And we observe that $a_k + a_{n-k} = \binom{n}{k}$ for all $k \in [n]$, and $p = \sum_{k\in[n]} p^k(1-p)^{n-k}\binom{n-1}{k-1}$. Thus, we have

$$\begin{aligned}&(\sum_{A\in\mathcal{F}'} p^{|A|}(1-p)^{n-|A|}) - p\\ &= \sum_{k\in[n]} p^k(1-p)^{n-k}(a_k - \binom{n-1}{k-1})\end{aligned}$$

Let f_k denote $p^k(1-p)^{n-k}(a_k - \binom{n-1}{k-1})$, we observe $a_k - \binom{n-1}{k-1} = \binom{n-1}{k} - a_{n-k}$ and thus have

$$f_k + f_{n-k} = (p^k(1-p)^{n-k} - p^{n-k}(1-p)^k)(a_k - \binom{n-1}{k-1}).$$

Therefore, if n is odd, as $f_n = 0$, we have

$$\begin{aligned}\sum_{k\in[n]} f_k &= \sum_{k\in[\lfloor n/2\rfloor]} f_k + f_{n-k}\\ &= \sum_{k\in[\lfloor n/2\rfloor]} (p^k(1-p)^{n-k} - p^{n-k}(1-p)^k)(a_k - \binom{n-1}{k-1})) \ge 0,\end{aligned}$$

as each term is nonnegative with $1/2 \leq p \leq 1$; similarly when n is even, we have

$$\sum_{k\in[n]} f_k = (f_{n/2} + f_{n/2})/2 + \sum_{k\in[\frac{n}{2}-1]} f_k + f_{n-k} \geq 0,$$

completing the proof. $\square$