6.2. SLICE RANK

May 18 Terrence Tao

$f : A^k \to \mathbb{F}$ is called a slice if

$$f(x_1, x_2, \ldots, x_k) = h(x_i)\, g(x_1, x_2, \ldots, x_{i-1}, x_i, \ldots, x_k)$$

for some functions $h : A \to \mathbb{F}$, $g : A^{k-1} \to \mathbb{F}$, some i

For a function $f : A^k \to \mathbb{F}$, the slice rank of f is the minimum integer m such that f is a linear combination of m slices.

If $k=2$, then (A: finite)
the slice rank of $f : A^2 \to \mathbb{F}$ is equal to the rank of a matrix

$$\big(f(i,j)\big)_{i \in A,\, j \in A}.$$

(cf. Every rank-r matrix can be written as a linear combination of r rank-1 matrices.)

(Rank of diagonal hypermatrices)

Lemma. Let $k \geq 2$, A: finite set, $\mathbb{F}$: field.
Let $f : A^k \to \mathbb{F}$ be a function such that

$$f(x_1, x_2, \ldots, x_k) \neq 0 \implies x_1 = x_2 = \cdots = x_k$$

Then the slice rank of f is equal to

$$|\{x \in A : f(x, x, \ldots, x) \neq 0\}|.$$

Proof. Induction on k.

If $k=2$, trivial (diagonal matrices)

We can assume $k > 2$.

We may also assume $f(a, a, \ldots, a) \neq 0$ for all $a \in A$.

For each $a \in A$,

let $f_a(x_1, x_2, \ldots, x_k) = \begin{cases} 1 & \text{if } x_1 = \cdots = x_k = a \\ 0 & \text{otherwise} \end{cases}$

Then f_a is a slice.

Furthermore $\underline{f(x)} = \sum_{a \in A} \underline{\underline{f(a, a, \ldots, a)}\, f_a(x)}$

$\Rightarrow$ (The slice rank of $f \leq |A|$.

Suppose that the slice rank of $f \le |A|-1$.

$$f(x_1, x_2, \dots, x_k) = \sum_{i \in 1}^{k} \sum_{\alpha \in I_i} f_{i,\alpha}(x_i)\, g_{i,\alpha}(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$$

for some sets $I_1, I_2, \dots, I_k$

and functions $f_{i,\alpha} : A \to \mathbb{F}$

$g_{i,\alpha} : A^{k-1} \to \mathbb{F}$

$\alpha \in I_i \quad 1 \le i \le k$

$|I_1| + |I_2| + \dots + |I_k| \le |A| - 1$

Let W be the vector space of functions $h : A \to \mathbb{F}$ that are "orthogonal" to all $f_{k,\alpha}$ for all $\alpha \in I_k$, in the sence that

$$\sum_{a \in A} f_{k,\alpha}(a)\, h(a) = 0$$

(For 2 functions $h_1, h_2 : A \to \mathbb{F}$

$$\langle h_1, h_2 \rangle = \sum_{a \in A} h_1(a)\, h_2(a) \dots)$$

Let $d = \dim W = \dim\left(\begin{array}{c}\text{vector space of}\\ \text{all functions}\\ h : A \to \mathbb{F}\end{array}\right) - \dim\left(\begin{array}{c}\text{vector space of}\\ \text{spans of all } f_{k,\alpha}\end{array}\right)$

$$\left(W = \left(\begin{array}{c}\text{vector space of}\\ \text{spans of all}\\ f_{k,\alpha}\end{array}\right)^{\perp} \right)$$

$$\dim\begin{pmatrix}\text{vector space}\\ \text{of}\\ \text{spans of}\\ \text{all } f_{k,\alpha}\end{pmatrix} \le |I_k|.$$

So, $d = \dim W \ge |A| - |I_k|$

Choose a function $h \in W$ such that

$A' = \{x \in A : h(x) \ne 0\}$ is maximal.

Then $|A'| \ge d$

(Why? If not, consider a linear transformation

$$W \longrightarrow \mathbb{F}^{A'}$$
$$f \longmapsto (f(x) : x \in A')$$

$\dim(\mathbb{F}^{A'}) = |A'|$

$\therefore \dim \ker(\downarrow) = d - |A'| > 0$

there is a function $h' \in W$, $h' \not\equiv 0$

such that $h(a) = 0$ for all $a \in A'$

Then consider $h'' = h + h'$

$h''(a) \ne 0$ for all $a \in A'$

$h''(b) \ne 0$ for some $b \notin A'$.

Contradiction to the maximality.)

So, h is a function in W so that

$h(a) \neq 0$ for all $a \in A'$. $|A'| \geq d$

Now consider

$$\sum_{x_k \in A} f(x_1, x_2, \ldots, x_k) h(x_k)$$

$$= \sum_{i=1}^{k-1} \sum_{\alpha \in I_i} f_{i,\alpha}(x_i) \sum_{x_k \in A} g_{i,\alpha}(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) h(x_k)$$

$$+ \sum_{\alpha \in I_k} \sum_{x_k \in A} \underline{f_{k,\alpha}(x_k)}\, g_{k,\alpha}(x_1, \ldots, x_{k-1})\, \underline{h(x_k)}$$

$$= \sum_{i=1}^{k-1} \sum_{\alpha \in I_i} f_{i,\alpha}(x_i) \sum_{x_k \in A} g_{i,\alpha}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) h(x_k)$$

$$+ \sum_{\alpha \in I_k} g_{k,\alpha}(x_1, \ldots, x_{k-1}) \underbrace{\sum_{x_k \in A} f_{k\alpha}(x_k) h(x_k)}_{\substack{\| \\ 0}}$$

because $h \in W$

$$= \sum_{i=1}^{k-1} \sum_{\alpha \in I_i} \underbrace{f_{i\alpha}(x_i)}_{A \to \mathbb{F}} \underbrace{\left(\sum_{x_k \in A} g_{i\alpha}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) h(x_k) \right)}_{: A^{k-2} \to \mathbb{F}}$$

Let $f' : A^{k-1} \to \mathbb{F}$

$$f'(x_1, \ldots, x_{k-1}) = \sum_{x_k \in A} f(x_1, \ldots, x_k) h(x_k).$$

f' is written as a sum of

$|I_1|+|I_2|+\cdots+|I_{k-1}|$ slices,

$\Rightarrow$ The slice rank of $f' \le |I_1|+\cdots+|I_{k-1}|$.

If $f'(x_1, x_2, \ldots x_{k-1}) \neq 0$ then

$f(x_1, x_2 \ldots, x_k) \neq 0$ for some $x_k \in A$

$\Longrightarrow x_1 = x_2 = \cdots = x_{k-1}$

By the induction hypothesis

the slice rank of $f' = |\{x \in A : f'(x, x, \ldots, x) \neq 0\}|$

$$\left(f'(x, x, \ldots x) = \sum_{x_k \in A} f(x, x, \ldots x, x_k)\, h(x_k) \right.$$

$$= \underbrace{f(x, x, \ldots, x)}_{\neq 0}\ \underbrace{h(x)}_{\neq 0 \text{ if } x \in A'}$$

$(\text{slice rank of } f') \ge |A'| \ge d \ge |A| - |I_k|$

$\therefore\ |I_1| + \cdots + |I_{k-1}| \ge |A| - |I_k|$

$\therefore\ |I_1| + \cdots + |I_{k-1}| + |I_k| \ge |A|$. $\square$

6.3. CAP SET PROBLEM — Revisited

For $A \subseteq \mathbb{F}_3^n$, let

$$f(x,y,z) = \begin{cases} 1 & \text{if } x+y+z=0 \\ 0 & \text{otherwise} \end{cases}$$

for $x, y, z \in A$.

Observe: if x, y, z is a 3-term arithmetic progression in $\mathbb{F}_3^n$,

then $x+z = 2y = -y$

$\Rightarrow x+y+z=0$

$\Rightarrow$ If A has no 3-term arithmetic progression,

then $f(x,y,z) = \begin{cases} 1 & \text{if } x=y=z \\ 0 & \text{otherwise} \end{cases}$.

$\Rightarrow$ (The slice rank of f $= |A|$.)

__Lemma.__ The slice rank of f is at most $3N$ where

$$N = \sum_{\substack{a,b,c \geq 0 \\ a+b+c=n \\ b+2c \leq 2n/3}} \frac{n!}{a!\,b!\,c!}$$

Proof. $f(x,y,z) = \prod_{i=1}^{n} \left(1 - \underline{(x_i + y_i + z_i)^2}\right)$

The right-hand side is a linear combination of monomials of the form

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$$

where

$$i_1, i_2, \ldots, i_n, j_1, \ldots, j_n, k_1, \ldots, k_n \geq 0, \leq 2$$

$$i_1 + i_2 + \cdots + i_n + j_1 + \cdots + j_n + k_1 + \cdots + k_n \leq 2n$$

$$\sum_{l=1}^{n} i_l + \sum_{l=1}^{n} j_l + \sum_{l=1}^{n} k_l \leq 2n$$

$$\Rightarrow \sum i_l \leq \frac{2n}{3} \text{ or } \sum j_l \leq \frac{2n}{3} \text{ or } \sum k_l \leq \frac{2n}{3}$$

Among all monomials with $\sum i_l \leq \frac{2n}{3}$ we can rearrange and regroup these monomials as

$$\sum_{\alpha} f_\alpha(x)\, g_\alpha(y,z)$$

where $\alpha = (i_1, i_2, \ldots, i_n)$

$$0 \leq i_1, \ldots, i_n \leq 2$$

$$i_1 + i_2 + \cdots + i_n \leq \frac{2n}{3}$$

$\Rightarrow$ (# choices of α) $= N$

Thus

$$f(x,y,z) = \overset{N}{\sum_{\alpha}} f_\alpha(x)\, g_\alpha(y,z) + \overset{N}{\sum_{\beta}} \bar{f}_\beta(y)\, g_\beta(x,z) + \overset{N}{\sum_{\gamma}} \bar{\bar{f}}_\gamma(z)\, \bar{\bar{g}}_\gamma(x,y)$$

$\therefore$ f is a sum of $3N$ slices.

$\Rightarrow$ The slice rank of $f \leq 3N$. $\square$

Therefore

$$|A| \leq 3N = o(2.756^n)$$

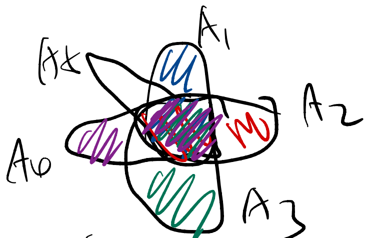
6.4. SUNFLOWER LEMMA

Sunflower of size r

= family of sets $A_1, A_2, \ldots, A_r$

such that $A_i \cap A_j = A_1 \cap A_2 \cap \cdots \cap A_r$

for all $1 \le i < j \le r$.



Thm (Erdős – Rado 1960)

There exists a function $f(k, r)$ such that

every family of distinct k-sets with more than $f(k,r)$ members contains a sunflower of size r.

Proof We proceed by induction on k.

If $k=1$, then $f(1, r) = r-1$.

$\underbrace{\circ \quad \circ \quad \circ \quad \circ \quad \circ}_{r \text{ sets}}$

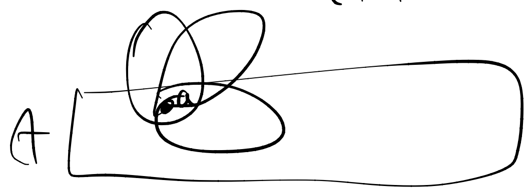
If $k>1$, then : let $\mathcal{F}$ be the family

Let $A_1, A_2, \ldots, A_t$ be a maximal subfamily of pairwise disjoint members.

$A_1, A_2, \dots A_t$: $\Longrightarrow$ sunflower

So, we may assume $t < r$.

Let $A = \bigcup_{i=1}^{t} A_i$, $|A| \leq (r-1)k$



For each $a \in A$, consider

$$\mathcal{F}_a = \{ X - \{a\} : X \in \mathcal{F}, a \in X \}$$

If $\mathcal{F}_a$ has a sunflower of size r then $\mathcal{F}$ has a sunflower of size r.

By induction we may assume $|\mathcal{F}_a| \leq f(k-1, r)$

$$\Rightarrow |\mathcal{F}| \leq |A| \, f(k-1,r) \leq (r-1)k \, f(k-1,r)$$

We take $f(k,r) = (r-1)k \, f(k-1,r)$ □

$$\boxed{f(k,r) \leq (r-1)^k \, k!}$$

Bound by Erdős-Rado

Erdős-Rado: $f(k,r) \geq (r-1)^k$.

Conj (Erdős-Rado) For every r, there exists c such that

$$f(k,r) \leq c^k$$

for all k.

OPEN, even for $r=3$. (Erdős \$1000 for $r=3$)

1996 Kostochka $f(k,r) \leq D(r,\alpha)\, k! \left(\frac{(\log\log\log k)^2}{\alpha \log\log k}\right)^k$

for every $0<\alpha<1$

D: function

2020 Alweiss, Lovett, Wu, Zhang

$$f(k,r) \leq (\log k)^k \, (r \log\log k)^{O(k)}$$

2020 Rao

$$f(k,r) \leq (\alpha\, r \log(rk))^k$$

for a universal constant $\alpha>1$.

WEAKENING DUE TO FÜREDI:

Thm (FÜREDI 80)

Let $\mathcal{F}$ be a family of sets of size $\leq k$.

If $|\mathcal{F}| > (r-1)^k$, then there exist r members $A_1, A_2, \ldots, A_r$ in $\mathcal{F}$ such that

$$\left| \bigcup_{i<j} (A_i \cap A_j) \right| < k.$$

Proof. Induction on $r+k$

If all $X \in \mathcal{F}$ have size $<k$, then trivial by induction.

So, we may assume that

there is $S \in \mathcal{F}$ such that $|S| = k$.

For every $X \subseteq S$, let

$$\mathcal{F}_X = \{ A - X : A \in \mathcal{F}, A \cap S = X \}$$

Then each member of $\mathcal{F}_X$ has

$\leq k - |X|$ elements.

By the induction hypothesis,

if $|\mathcal{F}_X| > (r-2)^{k-|X|}$, then

$\mathcal{F}_X$ contains $r-1$ members $A_1', A_2', \ldots, A_{r-1}'$

such that $\left| \bigcup_{i<j} (A_i' \cap A_j') \right| < k - |X|$

$A_i' = A_i - X$ for some $A_i \in \mathcal{F}$

let $A_r := S$.

Then $A_1, A_2, \ldots, A_r$ satisfy

$$\left| \bigcup_{i<j} (A_i \cap A_j) \right| < k$$

(If $X = S$, then $|\mathcal{F}_X| = 1$)

Thus, we may assume

$$|\mathcal{F}_X| \leq (r-2)^{k-|X|}$$

$$|\mathcal{F}| = \sum_{X \subseteq S} |\mathcal{F}_X| \leq \sum_{X \subseteq S} (r-2)^{k-|X|}$$

$$= \sum_{i=0}^{k} \binom{k}{i} (r-2)^{k-i}$$

$$= \sum_{j=0}^{k} \binom{k}{j} (k-2)^{j}$$

$$= \left(1 + (k-2)\right)^{k} = (r-1)^{k}$$

$$\square$$

6.5. Erdős-Szemerédi sunflower conjecture.

Conj (Erdős-Szemerédi 1978)
There exists $\varepsilon > 0$ such that
for all $n \geq 2$,
every family of subsets of $\{1, 2, \ldots, n\}$
with $|\mathcal{F}| > 2^{(1-\varepsilon)n}$
contains a sunflower of size 3.

↳ $A, B, C \in \mathcal{F}$
so that $A \cap B = B \cap C = C \cap A$.

Known: Erdős-Rado conjecture implies Erdős-Szemerédi.

Thm (Erdős-Szemerédi 78)
If Erdős-Rado conjecture holds with c,
then every family of subsets of $\{1, 2, \ldots, n\}$
with $|\mathcal{F}| > 2^{(1-\frac{c}{4})n}$
contains a sunflower of size 3.

Alon, Shpilka, Umans 2013:
Cap set conjecture $\Rightarrow$ Erdős-Szemerédi Conjecture.

$\Uparrow$

Ellenberg, Gijswijt

Thm (Naslund, Sawin 2017)

Let $\mathcal{F}$ be a family of subsets of $\{1,2,\dots,n\}$ with no sunflower of size 3.

Then
$$|\mathcal{F}| \le 3(n+1)\sum_{k\le \frac{n}{3}} \binom{n}{k}.$$

Proof. Map a subset of $\{1,2,\dots,n\}$ to a 0-1 vector.

Let S be the set of characteristic vectors of members of $\mathcal{F}$.

If $\mathcal{F}$ has no sunflower of size 3, then for all distinct $x,y,z\in S$ there is i such that exactly 2 of x_i, y_i, z_i is 1.

$$\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

Let S_ℓ be the set of vectors in S with ℓ 1's.

$$S = S_0 \cup S_1 \cup \dots \cup S_n$$

CLAIM: For $x,y,z \in S_\ell$

$x=y=z$ OR $\{x_i, y_i, z_i\} = \{0,1,1\}$ for some i.

If $x=y\neq z$ then trivially

then as all of x, y, z have exactly l 1's there is i such that

$$\{x_i, y_i, z_i\} = \{0, 1, 1\}.$$

If $x\neq y\neq z\neq x$ then Since $\mathcal{F}$ has no sunflowers of size 3

there is i such that

$$\{x_i, y_i, z_i\} = \{0, 1, 1\}$$

For $x, y, z \in S_l$, define

$$f_l(x,y,z) = \prod_{i=1}^{n} \left(2-(x_i+y_i+z_i)\right)$$

Then $f_l(x,y,z)\neq 0 \iff x_i+y_i+z_i\neq 2$ for all i

$\iff x=y=z.$

By the lemma on the slice rank of a diagonal hypermatrix, the slice rank of f_l is equal to $|S_l|$.

By expanding f_{e_1} we have a linear combination of monomials $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} y_1^{j_1} \cdots y_n^{j_n} z_1^{k_1} \cdots z_n^{k_n}$

where

$$0 \le i_1, i_2, \ldots, i_n \le 1,$$
$$j_1, j_2, \ldots, j_n$$
$$k_1, k_2, \ldots, k_n$$

$$0 \le \sum i_e + \sum j_e + \sum k_e \le n$$

By the pigeonhole principle

$$\sum i_e \le \frac{n}{3} \text{ or } \sum j_e \le \frac{n}{3} \text{ or } \sum k_e \le \frac{n}{3}$$

By the same idea from "Cap set problem revisited"

we deduce that

the slice rank of f_e

$$\le 3 \cdot \sum_{k \le \frac{n}{3}} \binom{n}{k}$$

$$\therefore |S_e| \le 3 \sum_{0 \le k \le \frac{n}{3}} \binom{n}{k}$$

$$\Rightarrow |S| = |S_0| + |S_1| + \cdots + |S_n|$$

$$\le 3(n+1) \underline{\sum_{0 \le k \le \frac{n}{3}}} \binom{n}{k}. \qquad \square$$

How to estimate?

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

If $0<x<1$

$$\frac{(1+x)^n}{x^k} = \sum_{i=0}^{k} \binom{n}{i} x^{i-k} + \sum_{i=k+1}^{n} \binom{n}{i} x^{i-k}$$

$$\geq \sum_{i=0}^{k} \binom{n}{i}$$

$$\therefore \sum_{i=0}^{k} \binom{n}{i} \leq \frac{(1+x)^n}{x^k}$$

Assume $k < \frac{1}{2}n$. $\quad x = \frac{k}{n-k}$

$$\sum_{i=0}^{k} \binom{n}{i} \leq \frac{\left(1+\frac{k}{n-k}\right)^n}{\left(\frac{k}{n-k}\right)^k} = \frac{n^n}{k^k\,(n-k)^{n-k}}$$

(If $k = \alpha n$ then)

$$\leq \frac{n^n}{(\alpha n)^{\alpha n}\,((1-\alpha)n)^{(1-\alpha)n}} = \frac{1}{\alpha^{\alpha n}(1-\alpha)^{(1-\alpha)n}}$$

If $\alpha = \frac{1}{3}$, then

$$\sum_{i=0}^{\frac{n}{3}} \binom{n}{i} \leq \frac{1}{\left(\frac{1}{3}\right)^{\frac{1}{3}n}\left(\frac{2}{3}\right)^{\frac{2}{3}n}} = \frac{3^n}{2^{\frac{2}{3}n}} = (1.8899\ldots)^n$$

This proves the conjecture of Erdős and Szemerédi