11.4. Proof of Rado's theorem.

Thm (Rado)

The system $Ax=0$ is regular on $\mathbb{N}$
if and only if
A satisfies the columns condition.

Let $\mathcal{A}$ be a family of finite subsets of $\mathbb{N}$.
We say $\mathcal{A}$ is homogeneous
if $nX \in \mathcal{A}$ for all $n \in \mathbb{N}$, $X \in \mathcal{A}$.

We say $\mathcal{A}$ is regular
if for every r-coloring of $\mathbb{N}$ for any r
there exists a monochromatic member in $\mathcal{A}$.

<u>Lemma</u>. Let $\mathcal{A}$ be a homogeneous regular family of finite subsets of $\mathbb{N}$. Let $M>0$ be fixed.
If χ is an r-coloring of $\mathbb{N}$,
then there exist $A \in \mathcal{A}$ and $d>0$
such that

$$a+\lambda d \quad \text{for all } |\lambda| \leq M, \text{ all } a \in A$$

have the same color.

Proof. By the compactness principle, there exists $R>0$
such that

any r-coloring of $[R]$ induces a monochromatic set $A \in \mathcal{A}$.

Let χ be an r-coloring of $\mathbb{N}$.

Let χ^* be an r^R-coloring of $\mathbb{N}$ such that

$$\chi^*(\alpha) = \chi^*(\beta) \iff \begin{cases} \chi(\alpha) = \chi(\beta) \\ \chi(2\alpha) = \chi(2\beta) \\ \chi(3\alpha) = \chi(3\beta) \\ \vdots \\ \chi(R\alpha) = \chi(R\beta) \end{cases}$$

Let $T = MR!$.

By the van der Waerden theorem, there exists a monochromatic arithmetic progression of length $2T+1$ under χ^*. Thus, there exist c and $e > 0$ such that

$$\chi^*(c + \mu e) \text{ is constant for all } |\mu| \leq T.$$

Now consider χ as an r-coloring of

$$c, 2c, 3c, 4c, \ldots, Rc.$$

Since $\mathcal{A}$ is regular ~~and homogenous~~, there exists a set $B \in \mathcal{A}$ such that

$B \subseteq [R]$ and cB is monochromatic under χ.

Now let $y = \operatorname{lcm}(B)$

$A = cB \in \mathcal{A}$ (because $\mathcal{A}$ is homogeneous)

$$d = ey$$

Then for each $a \in A$, $|\lambda| \leq M$

$$a = cb, \quad b \in B$$

$$\Rightarrow a + \lambda d = cb + \lambda ey$$
$$= b\left(c + \lambda e\left(\frac{y}{b}\right)\right)$$

$$\left|\frac{y}{b}\right| \leq R! \quad \Rightarrow \quad \left|\frac{\lambda y}{b}\right| \leq MR! = T.$$

By the choice of c and e

$$\chi^*\left(c + \lambda e\left(\frac{y}{b}\right)\right) = \chi^*(c)$$

$$\Rightarrow \chi(a + \lambda d) = \chi(a)$$

for each $a \in A$

Since A is monochromatic in χ

all of $a + \lambda d$ for all $a \in A$, $|\lambda| \leq M$

have the same color. $\square$

Lemma. Let $\mathcal{A}$ be a homogeneous regular family of finite subsets of $\mathbb{N}$. For every positive integer M and c, and every r-coloring χ of $\mathbb{N}$ there exist $A \in \mathcal{A}$ and $d > 0$ such that

$$a + \lambda d \quad \text{for all } a \in A, \ |\lambda| \le M$$

and

$$cd$$

have the same color.

Proof. Induction on r. We may assume $r > 1$.
By the compactness principle from the induction hypothesis, for all M and c, there is $T > 0$ such that every $(r-1)$-coloring of T induces a pair $A \in \mathcal{A}$ and $d > 0$ so that all of

$$a + \lambda d \quad \text{for } a \in A, \ |\lambda| \le M$$

and

$$cd$$

have the same color.

Let χ be an r-coloring of $\mathbb{N}$.
By the previous lemma, there exist $A \in \mathcal{A}$ and $d' > 0$ such that all of $a + \lambda d'$ with $a \in A$ and $|\lambda| \le TM$ have the same color.

If for some $|\mu| \leq T$,
$\mu cd'$ have the same color as well,
then we can set $d = \mu d'$

$\Rightarrow$ $a + \lambda d$ for all $a \in A$, $|\lambda| \leq M$
and cd
will have the same color.

Therefore we may assume
$\mu cd'$ for all $|\mu| \leq T$
avoid this particular color.

Let χ' be an $(r-1)$-coloring of $[T]$
defined by $\chi'(\alpha) = \chi(cd'\alpha)$.

By the induction hypothesis and the assumption on T,
there exist $B \in \mathcal{A}$ and $d'' > 0$
such that $b + \lambda d'' \in [T]$ for all $|\lambda| \leq M$, $b \in B$
and
all of $b + \lambda d''$,
cd''
have the same color under χ'.

Let $A = cd'B$, $d = cd'd''$.

Then for all $a \in A$,

$a = cd'b$ for some $b \in B$.

for all $|\lambda| \le m$,

$$\begin{aligned}
\chi(a+\lambda d) &= \chi(\underline{cd'}b + \lambda \underline{cd'}d'') \\
&= \chi'(b+\lambda d'') \\
&= \chi'(b) \\
&= \chi(a)
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
&= \chi'(cd'') \\
&= \chi(c^2 d' d'') \\
&= \chi(cd).
\end{aligned}$$

$\square$

For positive integers m, p, c,
a set S of positive integers is called
an (m, p, c)-set
if there exist a sequence $y = (y_1, y_2, \ldots, y_{m+1}) \in \mathbb{N}^{m+1}$
of positive integers such that

$\color{red}{\overline{}\ \ \overline{}\ \ \overline{}}$ — $\color{red}{J_0\quad J_0\quad J_0}$

$$S = D(m, p, c; y)$$

$$:= \bigcup_{j=1}^{m+1} \left\{ cy_j + \sum_{i=j+1}^{m+1} \lambda_i y_i : |\lambda_i| \le p \text{ for all } i \right\}$$

$$\begin{aligned}
= &\{ cy_1 + \lambda_2 y_2 + \lambda_3 y_3 + \cdots + \lambda_{m+1} y_{m+1} : |\lambda| \le p \} \\
\cup &\{ cy_2 + \lambda_3 y_3 + \cdots + \lambda_{m+1} y_{m+1} : |\lambda_i| \le p \} \\
\cup &\{ cy_3 + \cdots + \lambda_{m+1} y_{m+1} : |\lambda| \le p \} \\
\cdots \cup &\{ cy_{m+1} \}
\end{aligned}$$

$D(1, 10, 5; (1, 10))$

$= \{5 \cdot 1 + 10 x_2 : |x_2| \le 10\}$

$\cup \{10 \cdot 5\}$

Prop. For all positive integers m, p, c, r,

if $\mathbb{N}$ is r-colored, then there exists a monochromatic (m, p, c)-set S.

Proof Induction on m.

If $m = 1$, then an (m, p, c)-set is

$$D(1, p, c; y_1, y_2) = \{c y_1 + \lambda_2 y_2 : |\lambda_2| \le p\} \cup \{c y_2\}$$

for some y_1, y_2.

By the lemma based on the van der Waerden thm in Section 11.2 there is a monochromatic $(1, p, c)$-set.

(Apply the lemma to a new coloring χ',

$$\chi'(\alpha) = \chi(c\,\alpha)$$

Take $s = 1$)

Thus we may assume $m > 1$.

Let $\mathcal{A}$ be the set of all $(m-1, p, c)$-sets.

By the induction hypothesis, $\mathcal{L}$ is regular.
Clearly $\mathcal{A}$ is homogeneous.
By the previous lemma, there exist $A \in \mathcal{A}$, $d > 0$
such that $a + \lambda d$ for all $a \in A$
and $|\lambda| \leq p$
cd
have the same color.
Take $y_{m+1} := d$,
$\Rightarrow$ The union of all these numbers
is an (m, p, c)-set. $\square$

Proof of Rado's theorem, backward direction:
Let A be a matrix satisfying the columns condition.
$c_1, c_2, \ldots, c_n$: column vectors of A.
$(I_1, I_2, \ldots, I_t)$: partition of $[n]$
satisfying the columns condition.

$$\sum_{i \in I_1} c_i = 0$$

$$\sum_{i \in I_2} c_i \in \langle c_j : j \in I_1 \rangle$$

$$\vdots$$

$\sum_{i \in I_2} c_i$ = linear combination of vectors in $\{c_j : j \in I_1\}$

$$\Rightarrow \quad \begin{array}{c} I_1 \quad I_2 \\ \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \end{array} \begin{pmatrix} \boxed{\begin{matrix} * \\ * \end{matrix}} \\ \boxed{\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}} \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \qquad \begin{matrix} \text{rows for } I_1 \\ \text{rows for } I_2 \end{matrix}$$

We have rational vectors

$$z_1, z_2, \dots, z_t \in \mathbb{Q}^n$$

such that

$$A z_j = 0 \quad j \in [t]$$

and

$$z_{ji} = \begin{cases} 1 & \text{if } i \in I_j \\ 0 & \text{if } i \in I_{j'},\ j' > j \\ \text{any} & \text{if } i \in I_{j'},\ j' < j. \end{cases}$$

Let us choose a large integer c such that

$$c z_i \in \mathbb{Z}^n \quad \text{for all } i.$$

Take $Z_i = c z_i$ for each i.

Then now we have a parametric solution of $Ax = 0$

Take $x = \sum_{i=1}^{t} y_i Z_i$

$\left(Az_i = 0 \text{ for all } i \right) \Rightarrow Ax = 0$,

Let ρ be the largest absolute value of the entries in $z_1, z_2, \ldots, z_t$.

Then there is a monochromatic $(t-1, \rho, 1)$-set S,
which will give the monochromatic solution

$$y_1 + \lambda_2 y_2 + \cdots + \lambda_t y_t$$

$$y_2 + \lambda_2 y_3 + \cdots + \lambda_t y_t$$

$$\vdots$$

$$+ y_t$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = y_i z_i$$

$$z_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad z_2 = \begin{pmatrix} * \\ * \\ * \\ \hline 1 \\ 1 \\ 1 \\ \hline \vdots \end{pmatrix} \quad z_3 = \begin{pmatrix} * \\ * \\ * \\ \hline * \\ * \\ * \\ \hline 1 \\ \vdots \\ 0 \end{pmatrix} \cdots$$

$$I_1 \begin{cases} x_1 = y_1 + \underline{*}\, y_2^{\le p} + \underline{*}\, y_3 + \cdots + \underline{*}\, y_t . \\ x_2 = y_1 + * y_2 + * y_3 + \cdots + * y_t . \end{cases}$$

$$I_2 \begin{cases} x_\square = y_2 + * y_3 + \cdots + * y_t \\ \vdots \end{cases}$$

$$I_3 \quad x_\square = y_3 + * y_4 + \cdots + * y_t .$$

$\square$.

11.5. Folkman's theorem

Thm (Folkman) For every r-coloring of $\mathbb{N}$ and a positive integer k, there exists a set S of k positive integers such that all of $\sum_{a \in T} a$ for all $T \subseteq S$, $T \neq \emptyset$ have the same color.

Finite version:

For all integers r and k, there exist $n = n(r,k)$ such that for every r-coloring of $[n]$ there exist $a_1 < a_2 < \cdots < a_k$ with all $\sum a_i \leq n$ such that all of $\sum_{i \in I} a_i$ for $\emptyset \neq I \subseteq [n]$ have the same color.

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Lemma

For all integers r and k,
there exist $n = n(r,k)$
such that for every r-coloring of $[n]$
there exist $a_1 < a_2 < \cdots < a_k$
with
all $\sum a_i \leq n$
such that
the color of $\sum_{i \in I} a_i$ for $\emptyset \neq I \subseteq [n]$
depends only on $\max(I)$.

<u>Proof</u> Induction on k.
Trivial if $k=1$. Assume $k>1$.

Let $n = n(r,k) = 2W(r, n(r,k-1)+1)$

Let χ be an r-coloring of $[n]$.
By considering $\{\frac{n}{2}+1, \frac{n}{2}+2, \ldots, \frac{n}{2}+n\}$
we find a monochromatic arithmetic progression of length $n(r,k-1)+1$ in this set.
So, there is a_k, $d>0$ such that

$$a_k, a_k+d, a_k+2d, \ldots, a_k+(n(r,k-1))d$$

have the same color and

$$\frac{n}{2} < a_k, \quad a_k+(n(r,k-1))d \leq n$$

$$\Longrightarrow n(r,k-1)d \leq \frac{n}{2}$$

We identify the multiples of d with $\mathbb{Z}$

Apply the induction hypothesis (coloring $[n(r,k-1)]$)

$\Rightarrow$ We find $a_1 < a_2 < \cdots < a_{k-1}$

$d \mid a_1, a_2, \ldots, a_{k-1}$,

$\underline{a_1 + a_2 + \cdots + a_{k-1}} \leq n(r,k-1)d$

such that

the color of $\sum_{i \in I} a_i$, for all $I \subseteq [n-1]$, $I \neq \emptyset$

depends only on $\max(I)$.

$$\left(\sum_{i \in I} a_i\right) + a_k \leq a_k + n(r,k-1)d,$$

So, $a_1 + a_2 + \cdots + a_k \leq a_k + n(r,k-1)d$.

$\sum a_i$ is a multiple of d

$0 \leq \qquad \leq n(r,k-1)d$.

Thus, if $k \in I$, then for all $I \subseteq [n]$, $I \neq \emptyset$

$\sum_{i \in I} a_i$ have the same color.

$\square$

Proof of Folkman's theorem:

Find $a_1 < a_2 < \cdots < a_{(r-1)k+1}$

so that $\sum_{i \in I} a_i$ depends only on $\max(I)$.

for each $\phi \neq I \subseteq [(r-1)k+1]$

By the pigeonhole principle,

k of them induces the same color

Pick those k. $\square$