6.4 Sunflower Lemma

Def/ **Sunflower of size r** is a family of sets $(A_i)_{i\in[r]}$ s.t. $A_i \cap A_j = \bigcap_{k\in[r]} A_k$, $\forall i\neq j \in [r]$

Thm (Erdös-Rado 60')
$\exists f(k,r)$ s.t. $\forall$ family of k-sets w/ more than $f(k,r)$ members contains a SF of size r.

Pf/ 1° $k=1$, $f(1,r)=r-1$ suffices

2° $k>1$, Let $\mathcal{F}$ be the family, and Let $(A_i)_{i\in[t]}$ be a maximal subfamily of pairwise disjoint members.

Note that $(A_i)_{i\in[t]}$ forms a SF.

So we may assume $t<r$

Let $A = \bigcup_{i=1}^{t} A_i$ w/ $|A| \leq (r-1)\cdot k$

For $a\in A$, Consider

$\tilde{\mathcal{F}}_a = \{x-\{a\} : x\in \mathcal{F}, a\in x\}$

$\boxed{\text{Obs}}$ $\tilde{\mathcal{F}}_a$ has a SF of size r

$\Rightarrow \mathcal{F}$ has a SF of size r

And $\tilde{\mathcal{F}}_a$ is a family of $(k-1)$-sets

By induction hypothesis, if $\mathcal{F}$ contains no r-SF, then $\tilde{\mathcal{F}}_a$ contains no r-SF.

thus $|\tilde{\mathcal{F}}_a| \leq f(k-1,r)$

$\Rightarrow |\mathcal{F}| \leq |A|\cdot f(k-1,r) \leq (r-1)k\cdot f(k-1,r)$

Let $f(k,r) \geq (r-1)k\cdot f(k-1,r)$ $\boxed{\;}$

We take $f(k,r) = (r-1)^k k!$

$\boxed{\text{LB}}$ (Erdős–Rado)

$f(k,r) \geq (r-1)^k$

$\boxed{\text{Conj}}$ (E-R) [open even for $r=3$]

$\forall r, \exists C$ s.t. $f(k,r) \leq C^k, \forall k$

Recent development

1990 Kostochka

2020 Alweiss, Lovett, Wu, Zhang

2020, Rao $f(k,r) \leq (\alpha r \log(rk))^k$,

for some universal const. $\alpha < 1$

Thm (Füredi 80')

$\mathcal{F}$: family of sets of size $\leq k$

if $|\mathcal{F}| > (r-1)^k$, then $\exists$ r members

$(A_i)_{i\in[r]}$ in $\mathcal{F}$ s.t.

$$\left|\bigcup_{i<j}(A_i \cap A_j)\right| < k$$

Pf/ Induction on $r+k$

And we may assume that $\exists S \in \mathcal{F}$ w/ $|S| = k$

For $X \subseteq S$, let

$$\mathcal{F}_X = \{A - X : A \in \mathcal{F}, A \cap S = X\}$$

Then $\forall$ member of $\mathcal{F}_X$ has size

$\leq k - |X|$

By the IH, if $|\mathcal{F}_X| > (r-2)^{k-|X|}$,

then $\exists (r-1)\ (A_i^X)_{i\in[r-1]}$ in $\mathcal{F}_X$ s.t.

$$\left|\bigcup_{i<j}(A_i^X \cap A_j^X)\right| < k - |X|$$

Note that each $A_i^X = A_i - X$ for some $A_i \in \tilde{F}$. Let $A_r := S$, then $(A_i)_{i\in[r]}$ satisfies that

$$\left|\bigcup_{i<j}(A_i \cap A_j)\right| < k$$

So we may assume that $|\tilde{F}_X| \le (r-2)^{k-|X|}$

$$\Rightarrow |\tilde{F}| = \sum_{X\subseteq S}|\tilde{F}_X| \le \sum_{X\subseteq S}(r-2)^{k-|X|}$$

$$= \sum_{i=0}^{k}\binom{k}{i}(k-2)^{i}$$

$$= (r-1)^k, \text{ contradiction!}$$

$\square$

§6.5 Erdős–Szemerédi SF Conj.

Conj (E-S, 78')

$\exists \varepsilon > 0$ s.t. $\forall n \geq 2$, every family $\mathcal{F}$ of SSs of $[n]$ w/ $|\mathcal{F}| > 2^{(1-\varepsilon)n}$ contains a 3-SF, i.e., $\exists A, B, C \in \mathcal{F}$ s.t. $A \cap B = A \cap C = B \cap C$

Rmk. E-R Conj. $\Rightarrow$ E-S Conj.

Thm (E-S, 78')

If E-R Conj. holds w/ c, then $\forall$ family $\mathcal{F}$ of SSs of $[n]$ w/ $|\mathcal{F}| > 2^{(1-c/4)n}$ contains a 3-SF.

Rmk (Alon, Shpilka, Umans 2013)

Cap set conj. $\Rightarrow$ E-S Conj.

it's true, proven by

Ellenberg & Gijswijt

Thm (Naslund, Sawin 2017)

$\mathcal{F}$: family of SSs of $[n]$ w/o 3-SF

$\stackrel{?}{\Rightarrow} |\mathcal{F}| \le 3(n+1) \sum_{k \le n/3} \binom{n}{k}$

Pf/ Let $S = S_{\mathcal{F}}$ be the set of characteristic vectors of members of $\mathcal{F}$.

Note $\mathcal{F}$ contains no 3-SF implies that $\forall$ distinct $x, y, z \in S$, $\exists\, i$ s.t. exactly two of x_i, y_i, z_i are 1.

Let $S_\ell \subseteq S$ of vectors w/ ℓ 1's.

$\boxed{\text{Claim}}$ For $x, y, z \in S_l$, $x = y = z$ OR $\{x_i, y_i, z_i\} = \{0, 1, 1\}$ (multiset)

For $x, y, z \in S_l$, define

$$f_l(x, y, z) = \prod_{i=1}^{n} \left(2 - (x_i + y_i + z_i)\right)$$

Then $f_l(x, y, z) \neq 0 \iff x = y = z$

$\Rightarrow sl(f_l) = |S_l|$

We expand f_l into linear combination of monomials $x_1^{i_1} \cdots x_n^{i_n} y_1^{j_1} \cdots y_n^{j_n} z_1^{k_1} \cdots z_n^{k_n}$ s.t.

$0 \leq x_t, y_t, z_t \leq 1, \forall t \in [n]$

$0 \leq \sum x_t + \sum y_t + \sum z_t \leq n$

$\Rightarrow \sum x_t \leq n/3$ OR $\sum y_t \leq n/3$ OR $\sum z_t \leq n/3$

$\Rightarrow sr(f_l) \leq 3 \cdot \sum_{k \leq n/3} \binom{n}{k}$ (cf. §6.3)

$\Rightarrow |S| = \sum_{l=0}^{n} |S_l| \leq 3(n+1) \sum_{k \leq n/3} \binom{n}{k}$ $\boxtimes$

Estimation? Let $0<x<1$.

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

$$(1+x)^n / x^k = \sum_{i=0}^{k} \binom{n}{i} x^{i-k} + \sum_{i=k+1}^{n} \binom{n}{i} x^{i-k}$$

$$\geq \sum_{i=0}^{k} \binom{n}{i}$$

Assume $k < n/2$ and set $x = k/(n-k)$.

$$\sum_{i=0}^{k} \binom{n}{i} \leq \frac{n^n}{k^k (n-k)^{n-k}}$$

Esp., if $k = \alpha n$, RHS $= \left(\alpha^{\alpha} (1-\alpha)^{1-\alpha}\right)^{-n}$

$\Rightarrow |S| \leq (1.889\ldots)^n \Rightarrow$ E-S Conj.