

Please type your homework solution into a PDF file and submit to the gradescope.com website by 11:59pm KST of the due date. A point will be deducted for each problem if a handwritten solution is submitted. We recommend using \LaTeX . You can find useful hints at our KLMS website. It is recommended to use the sample template. (At least, make sure that each problem has a solution in separate pages.) Unprofessional proofs may get a deduction of points, even if the solution is mathematically correct or can be made correct.

2021 SPRING MAS575 Combinatorics
HOMEWORK 5

DUE: MAY 25, 2021

5.1. Let G be a simple graph G on the vertex set $\{v_1, v_2, \dots, v_n\}$. Let $A = (a_{ij})$ be an $n \times n$ real symmetric matrix such that for all $i \neq j$, $a_{ij} \neq 0$ if and only if v_i is adjacent to v_j . Let $\alpha(G)$ be the maximum size of an *independent* set in G . (A set of vertices is *independent* if no two vertices in the set are adjacent.)

Among n eigenvalues of A , let N^+ be the number of positive eigenvalues of A and N^- be the number of negative eigenvalues of A . Prove that $\alpha(G) \leq \min(n - N^+, n - N^-)$.

5.2. Prove that if A is a subset of $\{1, 2, \dots, n\}$ with $|A| > 2n/3$, then A has an arithmetic progression of length 3.

5.3. Prove that for all n , there exists N satisfying the following; For every bipartite graph G with the bipartition (A, B) such that no two vertices in A have the same set of neighbors and $|S| \geq N$, there exist distinct vertices $a_1, a_2, \dots, a_n \in A$ and $b_1, b_2, \dots, b_n \in B$ such that one of the following hold:

1. For all $1 \leq i, j \leq n$, a_i is adjacent to b_j if and only if $i = j$.
2. For all $1 \leq i, j \leq n$, a_i is adjacent to b_j if and only if $i \leq j$.
3. For all $1 \leq i, j \leq n$, a_i is adjacent to b_j if and only if $i \neq j$.

5.4. Let t, r be a positive integer. Prove that there exists a number N such that any r -coloring of numbers in $\{1, 2, \dots, N\}$ contains an arithmetic progression $a, a + d, a + 2d, \dots, a + (t - 1)d \in [N]$ of length t ($d \neq 0$) such that $a, a + d, a + 2d, \dots, a + (t - 1)d$ and d have the same color.

5.5. Prove that for all k and q , there exists N such that every sequence a_1, a_2, \dots, a_N of positive integers with

$$a_1 < a_2 < \dots < a_N \text{ and } a_{i+1} - a_i \leq q \text{ for all } i \in \{1, 2, \dots, N - 1\}$$

has a subsequence that is an arithmetic progression of length k .