KAIST
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Homework 3

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HW 3.1

The $(n-1)$-*dimensional unit sphere* is defined as $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Let $m(n)$ be the maximum number of points in a set X of points in S^{n-1} such that $\|x - y\| \in \{a, b\}, \forall x \neq y \in X$, for some fixed a and b. Prove that

$$n(n+1)/2 \leq m(n) \leq n(n+3)/2.$$

Proof. WLOG, we assume that $0 < a < b < 2$, otherwise the cases are very special and clearly not maximal. Let $X = \{x_i\}_{i\in[m]}$ be a maximal set of m points in S^{n-1} satisfying the conditions. Note that for all $x \neq y \in S^{n-1}$, $\|x - y\| = \sqrt{2 - 2\langle x, y\rangle}$, therefore the condition can be rephrased as $\langle x, y\rangle \in \{c, d\}, \forall x \neq y \in X$, where $c = 1 - a^2/2$ and $d = 1 - b^2/2$ with $-1 < d < c < 1$. We define functions $f_i : S^{n-1} \to \mathbb{R}$ as $f_i(y) = (\langle x_i, y\rangle - c)(\langle x_i, y\rangle - d)$. Note that each f_i is a polynomial on S^{n-1} with degree ≤ 2. We have $f_i(x_j) = (1-c)(1-d) > 0$ if $i = j$ and 0 otherwise. We claim that all these f_i are linearly independent. If the claim is true, we will have m is at most the dimensional of the space of all polynomials from S^{n-1} to $\mathbb{R}$ with degree ≤ 2, which is $\binom{n}{2}+(n-1)+n+1 = n(n+3)/2$, providing the desired upper bound, where $\binom{n}{2}$ represents all the y_iy_j terms with $i \neq j$, $(n-1)$ represents all the y_i^2 terms noting that $y_n^2 = 1 - \sum_{j\in[n-1]} y_j^2$, and the last n and 1 represent the linear terms and the constant, respectively. To prove the claim, assume $\sum_{i\in[m]} \alpha_i f_i \equiv 0$, let $y = x_j$ for some $j \in [m]$ we have $\alpha_j f_j(x_j) = 0$, which implies that $\alpha_j = 0$ for all j as j can be arbitrarily chosen. Regarding the lower bound, consider the orthogonal basis of $\mathbb{R}^{n+1}$, $(e_i)_{i\in[n+1]}$, where e_i is the vector in $\mathbb{R}^n$ with only the i-th entry to be 1 and all the other entries are 0. Consider the set $\{e_i + e_j\}_{(i,j)\in\binom{[n+1]}{2}}$, all points in which are in a subspace $\{z \in \mathbb{R}^{n+1} : \sum_{k\in[n+1]} z_k = 2, \|z\| = \sqrt{2}\} \cong S^{n-1}$ and it clearly satisfies the condition as the inner product of two distinct points in it must be 0 or 1, completing the proof. $\square$

HW 3.2

Let k be a positive integer. Let $(r_i)_{i\in[k]}$ be positive integers. For each $i \in [m]$, let $(A_{i,j})_{j\in[k]}$ be a k-tuple of pairwise disjoint sets such that $|A_{i,j}| = r_j, \forall j \in [k]$. Suppose that for each $i \neq i'$, there exist $j_1 < j_2$ and $j_1' < j_2'$ such that

$$A_{i,j_1} \cap A_{i',j_2'} \neq \emptyset \text{ and } A_{i,j_2} \cap A_{i',j_1'} \neq \emptyset.$$

Prove that

$$m \leq \frac{(\sum_{i\in[k]} r_i)!}{\prod_{i\in[k]} r_i!}.$$

Remark 2.1. *The RHS intuitively represents the case when we use the same $\sum_{i\in[k]} r_i$ numbers to construct all k-tuples, and the inequality says that is the best possible.*

Proof. Let us consider a reduced problem first, equivalently when all $r_i = 1$.

Lemma 2.1. *Let k be a positive integer. For each $i \in [m]$, let A_i be an ordered list of length k with pairwise disjoint elements in $[n]$. If for each $i \neq i'$, there exist $a, b \in [n]$ with $a \neq b$ such that a appears before b in A_i and a appears after b in $A_{i'}$, then $m \leq k!$.*

Proof of Lemma 2.1. Consider all the permutations of $[n]$ and choose π among them uniformly at random. For each A_i, the probability that A_i is a subsequence of π in the sense that if x appears before y in A_i then $\pi^{-1}(x) < \pi^{-1}(y)$, is clearly $1/k!$. Note that if A_i is a subsequence of π, then no other $A_{i'}$ could be, otherwise there will be no pair (a, b) appearing in different orders. Therefore, we have m disjoint events, each of which happens with probability $1/k!$, providing the desired inequality. □

Now, it remains to restore the original setting. WLOG, we assume all numbers appearing in all these sets are in $[n]$ for some n. First, let us give each k-tuple all possible $r_i!$ orders for its corresponding r_i, and for each i, we join all $A_{i,j}$ together (with all possible $S = \prod_{j\in[k]} r_j!$ ordered instances) to be ordered lists $\mathcal{A}_{i,s}$ of length $R = \sum_{j\in[k]} r_j$ for all $s \in [S]$, where each s is associated with an order among all S possible ones. Now we have mS many ordered lists. It is easy to check that the condition that for each $i \neq i'$, there exist $j_1 < j_2$ and $j_1' < j_2'$ such that $A_{i,j_1} \cap A_{i',j_2'} \neq \emptyset$ and $A_{i,j_2} \cap A_{i',j_1'} \neq \emptyset$ is a sufficient condition of that there exist $a, b \in [n]$ with $a \neq b$ such that a appears before b in $\mathcal{A}_{i,s}$ and b appears after a in $\mathcal{A}_{i',s'}$, for all $s, s' \in [S]$. And for two distinct ordered lists generated from the same original k-tuple, the latter condition can always be satisfied in any set that is given different orders. Therefore, we now have mS many ordered lists of length R. By Lemma 2.1, we have $mS \leq R!$, giving us the desired inequality.

□

HW 3.3

Let $(X_i)_{i\in[n]}$ be disjoint sets. Let $(r_i)_{i\in[n]}$ and $(s_i)_{i\in[n]}$ be positive integers. Suppose that A_{ij} and B_{ij} are subsets of X_i for $(i,j) \in [n] \times [m]$ such that $|A_{ij}| = r_i$ and $|B_{ij}| = s_i$. In addition,

$$(\bigcup_i A_{ij}) \cap (\bigcup_i B_{ij}) = \emptyset, \forall j \in [m].$$

$$(\bigcup_i A_{ij}) \cap (\bigcup_i B_{ik}) \neq \emptyset, \forall 1 \leq j < k \leq m.$$

Prove that

$$m \leq \prod_{i\in[n]} \binom{r_i + s_i}{r_i}.$$

Remark 3.1. *It is a generalization of the skew version of Bollobás Two Families Theorem. Besides, because all X_i are disjoint, the conditions can be rephrased as $A_{ij} \cap B_{i'j'} = \emptyset$ if $i \neq i'$ or $j = j'$, and $\forall j < k, \exists i$ s.t. $A_{ij} \cap B_{ik} \neq \emptyset$.*

Proof. First, we state the lemma that is used in the proof of the original skew version of Bollobás Two Families Theorem as shown in the lecture.

Lemma 3.1. *For any fixed $n \in \mathbb{N}$, we can pick a set Y of infinitely many points in $\mathbb{R}^n$ such that every subset of at most n points is linearly independent.*

WLOG, we may see each X_i as an infinite subset of $\mathbb{R}^{r_i+s_i}$ such that every subset of size $r_i + s_i$ is linearly independent, where with each $k \in X_i$ we associate a vector $w_k \in \mathbb{R}^{r_i+s_i}$ such that $\{w_k\}_{k\in X_i}$ is in general position. Besides, as all X_i are disjoint and we can see all these $\mathbb{R}^{r_i+s_i}$ as disjoint subspace of a larger space. For any $K \subset X_i$, we let

$$w_K = \bigwedge_{k\in K} w_k \in \wedge^{|K|}\mathbb{R}^{r_i+s_i}.$$

Let

$$a_{ij} = w_{A_{ij}} \in \wedge^{r_i}\mathbb{R}^{r_i+s_i}, b_{ij} = w_{B_{ij}} \in \wedge^{s_i}\mathbb{R}^{r_i+s_i},$$

and we let

$$a_j = \bigwedge_{i\in[n]} a_{ij}, b_j = \bigwedge_{i\in[n]} b_{ij}.$$

By the property of the wedge product and Lemma 3.1, we have

$$a_j \wedge b_j \neq 0, \forall j$$

because $a_{ij} \wedge b_{i'j} \neq 0, \forall i, i', j$ as $A_{ij} \cap B_{i'j} = \emptyset, \forall i, i', j$ and

$$a_j \wedge b_k = 0, \forall j < k$$

because likewise $\forall 1 \leq j < k \leq m, \exists i$ *s.t.* $a_{ij} \wedge b_{ik} = 0$.

We claim that $\{a_j\}_{j\in[m]}$ is linearly independent in $\bigwedge_{i\in[n]} \wedge^{r_i}\mathbb{R}^{r_i+s_i}$ which has dimension $\prod_{i\in[n]} \binom{r_i+s_i}{r_i}$ and thus gives the desired inequality. To see this, suppose $\sum_j c_j a_j = 0$, then for any fixed k, we have

$$0 = \sum_j c_j a_j \wedge b_k = c_k a_k \wedge b_k,$$

which gives $c_k = 0$, completing the proof. $\square$

HW 3.4

Let a, b, c be positive integers. Let $A = (A_{ij})_{(i,j)\in[m]\times[3]}$ be a matrix of finite sets such that

1. $|A_{i,1}| = a, |A_{i,2}| = b, |A_{i,3}| = c, \forall i$;
2. $A_{i,j} \cap A_{i,j'} = \emptyset, \forall i, j, j'$ with $j \neq j'$;
3. $\forall 1 \leq i < j \leq m, A_{i,1} \cap A_{j,2} \neq \emptyset \vee A_{i,1} \cap A_{j,3} \neq \emptyset \vee A_{i,2} \cap A_{j,3} \neq \emptyset$.

Prove that

$$m \leq \frac{(a+b+c)!}{a!b!c!}.$$

Proof. We first state a Lemma about general position.

Lemma 4.1. *Let U and V be linear spaces over an infinite field $\mathbb{F}$ and let $(A_i, B_i)_i$ be pairs of subspaces of U such that $\dim(A_i + B_i) \leq \dim(V)$ for each i. Then there exists a linear map $T : U \to V$ such that for each i, $\dim(T(A_i)) = \dim(A_i), \dim(T(B_i)) = \dim(B_i), \dim(T(A_i) \cap T(B_i)) = \dim(A_i \cap B_i)$.*

Let $\mathbb{F}$ be an infinite field and let U denote the full linear space containing all these A_{ij} over $\mathbb{F}$ with $\dim(U) \geq a + b + c$. Consider $V = V_1 \oplus V_2$ over $\mathbb{F}$ for some vector spaces V_1 and V_2 with $\dim(V_1) = a + b + c$ and $\dim(V_2) = b + c$. By Lemma 4.1, we can find linear transformations $T_1 : U \to V_1$ and $T_2 : U \to V_2$ such that for $i \in \{1, 2\}$, $\dim(T_i(A_{j,k})) = \dim(A_{j,k}), \forall j, \forall i \leq k \leq 3$ and $\dim(T_i(A_{j,k}) \cap T_i(A_{j',k'})) = \dim(A_{j,k} \cap A_{j',k'}), \forall j, j', \forall i \leq k, k' \leq 3$. We let $v_i = (\wedge T_1(A_{i,1}) \wedge (\wedge T_2(A_{i,2}))$ and let $w_i = (\wedge T_1(A_{i,2}) \wedge (\wedge T_1(A_{i,3}) \wedge (\wedge T_2(A_{i,3}))$, where $\wedge T_k(A_{i,j}) = \bigwedge_{x\in A_{i,j}} T_k(x) \in \wedge^{|A_{i,j}|} V_k$. We claim that $v_i \wedge w_j = 0, \forall i < j$ and $v_i \wedge w_i \neq 0, \forall i$. To see this, if $i < j$, then $A_{i,1} \cap A_{j,2} \neq \emptyset \vee A_{i,1} \cap A_{j,3} \neq \emptyset \vee A_{i,2} \cap A_{j,3} \neq \emptyset$, say $A_{i,k} \cap A_{j,k'} \neq \emptyset$ for some $1 \leq k < k' \leq 3$, especially $k \in \{1, 2\}$, then $T_k(A_{i,k}) \cap T_k(A_{j,k'})$ has positive dimension while $T_k(A_{i,k})$ is a factor in v_i and $T_k(A_{j,k'})$ is a factor in w_j, so $v_i \wedge w_j = 0$. On the other hand, for each i, $v_i \wedge w_i \neq 0$ because it is a wedge product of disjoint subspaces. Now we claim that $\{v_j\}_{j\in[m]}$ is linearly independent in $(\wedge^a V_1) \wedge (\wedge^b V_2)$ which has dimension $\binom{a+b+c}{a}\binom{b+c}{b} = \frac{(a+b+c)!}{a!b!c!}$ and thus gives the desired inequality. To see this, suppose $\sum_{j\in[m]} c_j a_j = 0$, then we enumerate k from m decreasing to 1, each time we have

$$0 = \sum_{j\leq k} c_j a_j \wedge b_k = c_k a_k \wedge b_k,$$

which gives $c_k = 0$, completing the proof. $\square$

HW 3.5

Let L be a vector space of functions from $\mathbb{F}^n$ to $\mathbb{F}$ such that if $f \in L$ and $f(x) = 0$ for more than d points x on a line in $\mathbb{F}^n$, then $f(x) = 0$ for all points on the line. Prove that the dimension of L is at most $(d+1)^n$.

Remark 5.1. *Equivalently, we can prove that if* $\dim(L) > (d+1)^n$*, then* $\exists f \in L$ *such that there are* $d+2$ *points* $\{x_i\}_{i\in[d+2]}$ *on a line in* $\mathbb{F}^n$ *such that* $f(x_i) = 0, \forall i \in [d+1]$ *but* $f(x_{d+2}) \neq 0$*. Besides,* $(d+1)^n$ *is the dimension of the space of polynomials from* $\mathbb{F}^n$ *to* $\mathbb{F}$ *with the order of each variable* $\leq d$*.*

Remark 5.2. *For each* f*, and* $v, w \in \mathbb{F}^n$*,* $w \neq 0$*, we let* $\tilde{f}_{v,w} : \mathbb{F} \to \mathbb{F}$ *be defined as* $\tilde{f}_{v,w}(t) = f(v+tw)$*. The condition says that if* $f \in L$*, then for each* v, w*,* $\tilde{f}_{v,w}$ *has* $\leq d$ *zeros or* $\tilde{f}_{v,w} \equiv 0$*.*

Remark 5.3. *When* $n = 1$*, assume* L *has dimension* $m > d+1$*, let* $\{f_i\}_{i\in[m]}$ *be a basis of* L*. Choose* $d+1$ *distinct points (on a line which is the whole* $\mathbb{F}$*)* $\{y_i\}_{i\in[d+1]} \subset \mathbb{F}$ *and consider the system of* $d+1$ *linear equations* $\sum_{i\in[m]} c_i f_i(y_j) = 0, j \in [d+1]$ *on* $m > d+1$ *variables* $(c_i)_{i\in[m]}$*. Therefore, there must exist a nontrivial solution* $(c_i)_{i\in[m]}$*. However, because* L *is a vector space,* $g := \sum_{i\in[m]} c_i f_i \in L$ *and now we have* $g \not\equiv 0$ *but* g *vanishes on* $d+1$ *distinct points on a line, contradicting with the given condition.*

Proof. We already prove the case when $n = 1$ above. Now we assume $n > 1$ and suppose the opposite, i.e., assume that L has dimension $m > (d+1)^n$, let $\{f_i\}_{i\in[m]}$ be a basis of L. Choose $d+1$ distinct points $Y = \{y_i\}_{i\in[d+1]} \subset \mathbb{F}$ and consider the system of $(d+1)^n$ linear equations $\sum_{i\in[m]} c_i f_i(y_J) = 0, J = (j_k)_{k\in[n]} \in [d+1]^n$, where $y_J = (y_{j_k})_{k\in[n]}$. Note that the number of variable m is strictly more than the number of linear equations $(d+1)^n$, therefore there must exist a nontrivial solution $(c_i)_{i\in[m]}$. On the other hand, because L is a vector space and $\{f_i\}_{i\in[m]}$ is a basis, $g := \sum_{i\in[m]} c_i f_i \in L$ and $g \not\equiv 0$. Now we have $g(y_J) = 0$ for all $J \in [d+1]^n$. We fix $n-1$ variables, e.g., we fix $x_2, .., x_n$ with each $x_k \in Y$ and we have $g(y, x_2, ..., x_n) = 0$ for each $y \in Y$. Note that all points $(y, x_2, ..., x_n)_{y\in Y}$ are $d+1$ points on a line $\{(v, x_2, ..., x_n)\}_{v\in\mathbb{F}}$ in $\mathbb{F}^n$, thus we have $g(v, x_2, ..., x_n) = 0$ for all $v \in \mathbb{F}$. Also note that this holds for any $x_2, .., x_n \in Y$ as they can be arbitrarily chosen in Y. Now we fix $v_1 \in \mathbb{F}$ and fix $x_3, ..., x_n \in Y$, and let x_2 enumerate in Y, similarly, we can conclude that $g(v_1, v_2, x_3, ..., x_n) = 0$ for all $v_1, v_2 \in \mathbb{F}$. Again, note that this holds for any fixed $x_3, .., x_n \in Y$. We repeat this process and finally we have $g(x) = 0$ for all $x \in \mathbb{F}^n$, i.e., $g \equiv 0$, which is a contradiction, completing the proof. $\square$