§7.8 Covering the Cube by affine hyperplanes

Def/ (Affine hyperplanes)

An AH is a set of vectors $\{x \in \mathbb{R}^n : a x = b\}$ for some $a, b \in \mathbb{R}^n$.

$\boxed{\text{Thm}}$ (Alon Füredi 93')

If m AHs in $\mathbb{R}^n$ avoid 0 but cover all other $2^n - 1$ pts in $\{0,1\}^n \overset{?}{\Longrightarrow} m \geq n$

Tight. Consider $(\{x_i = 1\})_{i \in [n]}$. $(m = n)$

$\boxed{\text{Lem}}$ P: polynomial in $\mathbb{R}[x_1, \dots, x_n]$ w/ $P(0) \neq 0$

If $P(x_1, \dots, x_n) = 0$, $\forall (x_1, \dots, x_n) \in \{0,1\}^n - \{0\}$

$\overset{?}{\Longrightarrow} \deg(P) \geq n$ [This Lemma implies the above Theorem]

Let $(H_i)_{i \in [m]}$ be the AHs covering $\{0,1\}^n - \{0\}$

Let $a_i, b_i \in \mathbb{R}^n$ s.t. $H_i = \{x \in \mathbb{R}^n : a_i x = b_i\}$, $i \in [m]$

Def $P(x_1,\dots,x_n) := \prod_{i=1}^{m} (a_i x - b_i)$ w/ $\deg(P) = m$

$\boxed{\text{Obs}}$ $x \in H_i$ for some $i \in [m] \Rightarrow P(x) = 0$

$\Rightarrow P(x) = 0, \forall x \in \{0,1\}^n - \{0\} \overset{\text{Lem}}{\Longrightarrow} \deg(P) \geq n$

$\blacksquare$

Pf/ (of the Lemma)

Suppose NOT, i.e., $\deg(P) < n$ (esp. $\prod_i x_i \notin P$)

Def. $f(x_1,\dots,x_n) := P(x_1,\dots,x_n) - c\prod_{i=1}^{n}(x_i - 1)$,

where we choose $c \neq 0$ s.t. $f(0) = 0$.

$\Rightarrow \mathrm{coef}_f(\prod_i x_i) = -c \neq 0$

$\overset{\text{CN}}{\Longrightarrow} \exists x \in \{0,1\}^n$ s.t. $f(x) \neq 0$, contradiction!

(But $f(0) = 0$, and $f(x) = 0 - 0 = 0$ if $x \in \{0,1\}^n - \{0\}$)

$\blacksquare$

§7.9 Partitioning into pairs w/ prescribed differences

$\boxed{\text{Thm}}$ (Dyson's conj. 62'; Proved by $\begin{cases} \text{Wilson } 62' \\ \text{Gunson } 62' \end{cases}$, indep.)

The constant term in the expansion of $\prod_{1\le i\neq j\le n}\left(1-\frac{x_i}{x_j}\right)^{a_i}$ $= \prod_{i<j} \frac{(-1)^{a_j}(x_j-x_i)^{a_i+a_j}}{x_j^{a_i}x_i^{a_j}}$

is $(a_1+\cdots+a_n)!/(a_1!\cdots a_n!)$.

In other words, if $a=\sum_{i=1}^{n} a_i$, then in $\prod_{1\le i<j\le n}(-1)^{a_j}(x_j-x_i)^{a_i+a_j}$, $\text{Coef.}\left(\prod_{i=1}^{n} x_i^{a-a_i}\right)$ is

$a!/(a_1!\cdots a_n!)$

Pf/ (Good 70') $a=\sum_{i=1}^{n} a_i$. Induction on $n+a$.

Let $F(x;a_1,\dots,a_n)$ be the const. term we're interested in.

If $a_i=0$ for some i, then x_i makes no difference

on $F \xRightarrow{\text{Induction}} F=\frac{(a-a_i)!}{a_1!\cdots a_{i-1}!a_{i+1}!\cdots a_n!}=\frac{a!}{a_1!\cdots a_n!}$

(Also, trivial when $n=1$)

So, we may assume $n>1$, $a_i \neq 0, \forall i \in [n]$

When x_i's are pairwise distinct, Let

$$f_j(x) := \prod_{i \neq j} (x - x_i)/(x_j - x_i), \text{ for } j \in [n]$$

$\boxed{\text{obs}}$ $f_j(x) = \begin{cases} 1, & \text{if } x = x_j \\ 0, & \text{if } x = x_i \text{ for some } i \neq j \end{cases}$

AND, $\deg(f_j) = n-1$

Let $f(x) := \sum_{j \in [n]} f_j(x)$, then $f(x_i) = 1, \forall i \in [n]$.

AND $\deg(f) \leq n-1 \Rightarrow f \equiv 1$, esp.

$$f(0) = \sum_j \prod_{i \neq j} {-x_i}/{x_j - x_i} = 1$$

$$\Rightarrow \sum_j \prod_{i \neq j} \left(1 - \frac{x_j}{x_i}\right)^{-1} = 1$$

Note that $F(x; a_1, \dots, a_n) \cdot \sum_j \prod_{i \neq j} \left(1 - \frac{x_j}{x_i}\right)^{-1}$

$$= \sum_j F(x; a_1, \dots, a_{j-1}, a_j - 1, a_{j+1}, \dots, a_n)$$

$$\underset{\text{(Induction)}}{=} \sum_j \frac{(a-1)!\, a_j}{\prod_k a_k!} = \frac{(a-1)!}{\prod_k a_k!} \cdot a = \frac{a!}{\prod_k a_k!}$$

$\boxed{\text{Thm}}$ (Karasev, Petrov 2012)

p: odd prime, set $m = p-1/2$, let $(d_i)_{i\in[m]} \in \mathbb{Z}$ NOT divisible by p $\Rightarrow \exists \{x_1, \cdots, x_m, y_1, \cdots, y_m\} \in S_{p-1}$

s.t. $y_i - x_i \equiv d_i \pmod p$, $\forall i \in [m]$

Find pairs in $[p-1]$ w/ prescribed differences

Pf/ (By CN)

$f(x_1, \cdots, x_m) :=$
$$\prod_{k\in[m]} (x_k + d_k) \cdot \prod_{i<j} (x_i - x_j)(x_i + d_i - x_j)(x_i - x_j - d_j)(x_i + d_i - x_j - d_j)$$

w/ $\deg(f) = m + 4\binom{m}{2} = 2m^2 - m = (p-2)m$

Consider coef $\left(\prod_i x_i^{p-2}\right)$ in f, which is

coef $\left(\prod_i x_i^{p-2}\right)$ in $x_1 \cdots x_m \prod_{i<j} (x_i - x_j)^4$

$= \text{Coef}\left(\prod_i x_i^{p-3}\right)$ in $\prod_{i<j} (x_i - x_j)^4$

Dyson's w/ $a_i = 2, \forall i$ $\quad 2m!/(2!)^m \not\equiv 0 \pmod p$ ($\because 2m < p$)

$\overset{CN}{\Longrightarrow} \exists\, x_i \in [p-1], \forall i$ s.t. $f(x_1, \dots, x_m) \not\equiv 0 \pmod{p}$

$\Longrightarrow$ What we want.

$$\begin{cases} x_i \neq -d_i, \ \forall i & (y_i \neq 0) \\ x_i \neq x_j, \ \forall i \neq j & (x_i\text{'s are distinct}) \\ x_i \neq x_j + d_j, \ \forall i \neq j & (y_j \neq x_i) \\ x_i + d_i \neq x_j + d_j, \ \forall i \neq j & (y_i\text{'s are distinct}) \end{cases} \pmod{p}$$

Take $y_i = x_i + d_i$

§7.10 Regular Subgraphs

$\boxed{\text{Thm}}$ (Alon, Friedland, Kalai 84')

$\forall$ (multi)graph w/ average deg > 4, max. deg ≤ 5, contains a 3-reg. subgraph.

Pf/ (Alon 98')

Let $M = V \times E$ be the incidence matrix of G (modulo 3) on $\mathbb{F}_3$ $\left(M = \begin{pmatrix} & 1 & \\ & 1 & \end{pmatrix}\right) \updownarrow V$, $\overleftrightarrow{\quad E \quad}$

the 2 vtxs incident to the edge

Let $m = |E|$ and $n = |V|$

For $x : E \to \mathbb{F}_3$ (OR $\mathbb{F}_3^E$), let

$$f(x) := \prod_{v \in V}\left(1 - \left(\sum_{e \sim v} x_e\right)^2\right) - \underbrace{\prod_{e \in E}(1 - x_e)}_{= \mathbb{1}_{x \equiv 0}}$$

$\boxed{\text{Obs}}$ $z \in \mathbb{F}_3$, $z^2 = 1$ if $z \neq 0$ and $z^2 = 0$ otherwise

$$\Rightarrow \prod_{v \in V}\left(1 - \left(\sum_{e \sim v} x_e\right)^2\right) = \mathbb{1}_{\sum_{e \sim v} x_e \equiv 0 \ (\text{mod } 3),\ \forall v \in V}$$

$f(0) = 1 - 1 = 0$, AND set $L_e = \{0, 1\}, \forall e \in E$

$f(x) \neq 0$, $x_e \in L_e \Leftrightarrow \sum_{e \sim v} x_e \equiv 0 \pmod 3, \forall v \in V$

$\xRightarrow{\text{Max. deg} \leq 5} \sum_{e \sim v} x_e \in \{0, 3\}, \forall v \in V$

Then we take H be the subgraph of G consisting of $e \in E$ s.t. $x_e = 1$, then H is non-empty and 3-regular.

Now NTS $\exists x_e \in L_e$ s.t. $f(x) \neq 0$, to see this

$\text{Coef.}\left(\prod x_e\right) = -(-1)^m \not\equiv 0 \pmod 3$

$\deg\left(\prod\left(1 - \left(\sum x_e\right)^2\right)\right) = 2|V| = 2n$

$\deg\left(\prod\left(1 - x_e\right)\right) = |E| = m$

average deg $:= 2m/n > 4 \Leftrightarrow m > 2n$

$\Rightarrow \deg(f) = m$

$\xRightarrow{\text{CN}} \exists x_i \in L_i$ s.t. $f(x) \neq 0 \pmod 3$

§7.1 | List coloring

L is called a list assignment.

For $v \in V$, $L(v)$ is the set of colors available for v. An L-list-coloring (L-LC) is a coloring c s.t. $c(v) \in L(v), \forall v$, AND, $c(v) \neq c(w), \forall v \sim w$.

A graph G is L-list-colorable if it admits an L-LC. (AKA L-choosable)

We say G is k-list-colorable (k-LC) (AKA k-choosable) if $\forall$ list assignment L of G w/ $|L(v)| \geq k, \forall v$, G is L-list-colorable.

The list chromatic # of G is (AKA choice #, $ch(G)$)

$$\chi_L(G) := \min\{k : G \text{ is } k\text{-LC}\}$$

$\chi(G) \leq \chi_L(G)$ $[L(v) = [k], \forall v]$

$\boxed{\text{Rmk}}$ The gap could be big, consider $K_{n,n}$

$\chi(K_{n,n}) = 2, \forall n$, But $\chi_L(K_{n,n}) \to \infty$, as $n \to \infty$

$\boxed{\text{Thm}}$ (Alon Tarsi 90's)

$\forall$ simple bipartite planar graph is 3-LC

(planar + $\chi(G) \le 2 \Rightarrow \chi_L(G) \le 3$)

Def/ A directed graph D is Eulerian if the in-deg. is equal to the out-deg. for each vtx.

$EE(D) :=$ # Eulerian spanning subgraphs of D w/ even edges

(keeping the whole $V(D)$)

$EO(D) :=$ # Eulerian spanning subgraphs of D w/ odd edges

($EE \ne EO \Rightarrow$ L-LC) (Loopless)

$\boxed{\text{Thm}}$ D: digraph on $V(D) = \{v_1, \dots, v_n\}$ and let $d_i = \text{out-deg}(v_i)$, L: list assignment $|L(v_i)| > d_i$

$EE(D) \ne EO(D) \overset{?}{\Longrightarrow}$ The underlying graph of D is L-LC

Pf/ (of EE $\neq$ EO $\Rightarrow$ L-LC)

We may assume $L(v_i) \subseteq \mathbb{Z}$ and $|L(v_i)| = d_i + 1$, $\forall i$

Let $f_D(x_1, \cdots, x_n) := \prod_{v_i v_j \in E(D)} (x_i - x_j)$

$\boxed{\text{Goal}}$ $f_D(c_1, \cdots, c_n) \neq 0$ ~~for~~ for some $c_i \in L(v_i)$, $\forall i$

$\boxed{\text{Obs}}$ $\deg(f_D) = |E(D)| = \sum_{i \in [n]} d_i$

It suffices to show $\mathrm{Coef}\left(\prod x_i^{d_i}\right) \neq 0$

Expanding f_D,

$DE(d_1, \cdots, d_n) :=$ # choices of ends

of each edge s.t. x_i is chosen d_i times, $\forall i$ and

the head (-1) is chosen even times; Similarly,

$DO(d_1, \cdots, d_n) :=$ # choices of ends

of each edge s.t. x_i is chosen d_i times, $\forall i$ and

the head (-1) is chosen odd times

Then $\operatorname{coef}\left(\prod x_i^{d_i}\right) = DE(d_1, \dots, d_n) - DO(d_1, \dots, d_n)$

$\boxed{\text{Claim}}$ $DE - DO = EE - EO$

Pf/ (of the claim) By showing a bijection

Consider a choice X of ends s.t. x_i is

chosen d_i times, $\forall i$, Let

$D(X) :=$ set of edges whose heads are chosen

Fix v_i, 4 types of its incident edges

In/Out-edges, head/tail chosen

So $IH + OT = OH + OT = d_i \Rightarrow IH = OH$

$\Rightarrow D(X)$ is an Eulerian subgraph of D.

D maps X to an Eulerian subgraph, in particular,

if $X \in DE$, i.e., # heads chosen is even

$\Rightarrow |E(D(X))|$ is even; Similarly,

$X \in DO \Rightarrow |E(D(X))|$ is odd

$\Rightarrow DE - DO = EE - EO$ $\blacksquare$

$\boxed{\text{Lem}}$ G: undirected graph, $d(G) := \max_{\emptyset \neq H \subseteq G} \frac{|E(H)|}{|V(H)|}$

(?) $\Rightarrow \exists$ orientation of G s.t. out-deg of $\forall$ vtx $\leq \lceil d(G) \rceil$

Pf/ (of Lem) Set $t := \lceil d(G) \rceil$

We construct a bipartite graph $H = (V', E')$

$$\overbrace{V(G) \quad \cdots \quad V(G)}^{t \text{ copies of } V(G)} \qquad \left(V' = \bigcup_{i \in [t]} V_i\right)$$

$$E(G) \qquad (V_E)$$

$a \in V'$ is adjacent to $b \in V_E$ in H if

a is a copy of a vtx incident w/ b in G

$\boxed{\text{Claim}}$ H has a matching covering $E(G)$

Pf/ (of Claim, by Hall's) Suppose NOT.

Let $X \subseteq E(G)$ $(X \neq \emptyset)$, $|N_H(X)| = t \cdot$ (# vtxs incident w/ some edge in X)

By the def. of $d(G)$, we have

$|X| /$ # vtxs incident w/ some edge in $X \leq d(G) \leq t$

$\Rightarrow |X| \leq |N_H(X)|$, by Hall's $\blacksquare$

With this claim $\Rightarrow$ G has an orientation s.t.

$\forall$ vtx is a tail of $\leq t$ edges. $\blacksquare$

$\boxed{\text{Recall}}$

$\boxed{\text{Thm}}$ (Alon Tarsi 90's)

$\forall$ simple bipartite planar graph is 3-LC

(planar + $\chi(G) \leq 2 \Rightarrow \chi_l(G) \leq 3$)

Pf/ By Euler's formula,

$|E(G)| \leq 2|V(G)| - 4$ (bipartite, planar, $|V| \geq 3$)

More generally $|E| \leq 2|V|$ ($\forall$ bipartite, planar)

$\Rightarrow d(G) \leq 2$

$\Rightarrow \exists$ orientation D of G s.t. each vex has out-deg. ≤ 2, Consider $EE(D)$ and $EO(D)$

$\boxed{\text{Claim}}$ $EO(D) = 0$

To see this, G bipartite $\Rightarrow$ cycles all have even length $\Rightarrow \forall$ Eulerian subgraph of D has even edges

$\boxed{\text{Claim}}$ $EE(D) \neq 0$

Because there is an Even Eulerian subgraph w/ ϕ as its edge set.

$EO \neq EE \Rightarrow L$-LC w/ $|L(v)| = 3 > 2 \geq d(G)$

$\Rightarrow G$ is 3-LC $\blacksquare$