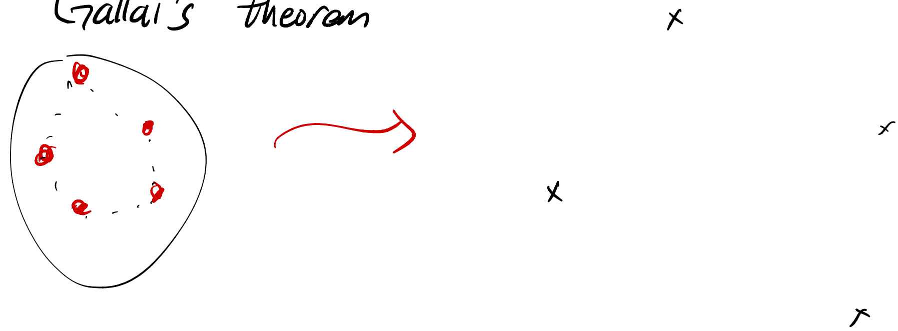
10.4. Gallai's theorem



$V = \{v_1, v_2, \ldots, v_t\} \subseteq \mathbb{R}^m$

We say W is homothetic to V if there are an ordering of vectors in W $w_1, w_2, \ldots, w_t$ and $a \in \mathbb{R}^m$, $b \in \mathbb{R}$ such that $w_i = a + b v_i$ for all i.

Thm (Gallai) For every coloring of $\mathbb{R}^m$ into finitely many colors and for every finite set $V \subseteq \mathbb{R}^m$, there exists a monochromatic set $W \subseteq \mathbb{R}^m$ homothetic to V.

Proof. Let r be the # colors.

Let $k = |V|$ $\quad V = \{v_1, v_2, \dots, v_k\}$

Let $N = HJ(k, r)$

Pick a function $\psi : [k]^N \to \mathbb{R}^m$

$$\psi(x_1, x_2, \dots, x_N) = \sum_{i=1}^{N} c_i v_{x_i}$$

for some $c_1, c_2, \dots, c_N$ fixed.

$\left((1,1,1) \to c_1v_1 + c_2v_1 + c_3v_1 \right)$

If we can choose $c_1, c_2, \dots, c_N$ so that ψ is injective, then

from the coloring of $\mathbb{R}^m$ we have an r-coloring of $[k]^N$.

$\Rightarrow$ there is a monochromatic combinatorial line.

$\Rightarrow$ all of $\boxed{a} + \boxed{b}\, v_i$ have the same color $\quad i = 1, 2, \dots, k$

For instance,

$$(3, 2, *) \to \left.\begin{array}{l} \overbrace{\boxed{c_1v_3 + c_2v_2}}^{a} + \overbrace{\boxed{c_3}}^{b} v_1 \\ c_1v_3 + c_2v_2 + c_3v_2 \\ c_1v_3 + c_2v_2 + c_3v_3 \end{array}\right\} \text{same color}$$

$w_i = a + bv_i$

$\Rightarrow W = \{a + b v_i : i \in \{1, 2, \dots, k\}\}$ is monochromatic.

How do we choose $c_1, \dots, c_k$ so that ψ is injective?

$$\sum_{i=1}^{N} c_i (v_{x_i} - v_{x_i'}) \neq 0$$

for all $x, x' \in [k]^N$

equations is finite

$\Rightarrow$ (There is a choice $c_1 \dots c_N$ to avoid all of them.)

$\square$

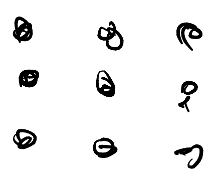
Corollary

$\mathbb{N} = \{1, 2, \dots\}$

If $\mathbb{N}^2$ is colored by finitely many colors then for each t, there exist x_0, y_0 and d such that

$$(x_0 + id,\ y_0 + jd)$$

for all $0 \le i, j < t$ have the same color.



11. Monochromatic Solutions

11.1. Schur's theorem

Thm (Schur 1916)

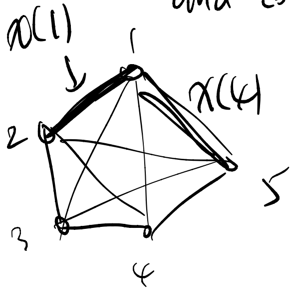
For any $r>0$, there exists N such that in every r-coloring of $[N]=\{1,2,\dots,N\}$ there exist $x, y \in [N]$ such that $x, y, x+y$ have the same color.

<u>Proof</u>. $N = R(\underbrace{3,3,3,\dots,3}_{r}) - 1$

Let $\chi : [N] \to [r]$ be an r-coloring.

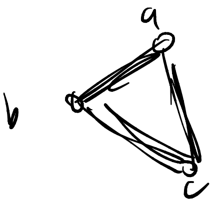
We construct K_{N+1} on $\{1,2,\dots,N+1\}$

and color an edge ij with $\chi(|i-j|)$.



By the Ramsey theorem, there exist $\underbrace{a, b, c}_{\text{distinct}} \in [N+1]$ such that

$$\chi(|a-b|) = \chi(|b-c|) = \chi(|a-c|)$$



Assume $a<b<c$.

$$\chi(b-a) = \chi(c-b) = \chi(c-a)$$

$$c-a = (c-b)+(b-a)$$

Take $x := c-b$, $y = b-a$

$\Rightarrow x+y, x, y$ have the same color. $\square$

Remark: Known that

$[N]$ has $\geq \frac{1}{22}N^2 + O(N)$ monochromatic triples of the form $\{x, y, x+y\}$

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Cor (Schur 1916)

Let n be a positive integer. If p is a sufficiently large prime, then

$$x^n + y^n \equiv z^n \pmod{p}$$

has ~~no~~ nonzero solutions.

> FLT $n>2$
> $x^n + y^n = z^n$
> has no nontrivial integer solutions

Proof: Let p be a prime. Assume p is large so that any n-coloring of $[p-1]$ induces a monochromatic triple of the form $\{x, y, x+y\}$

Let g be the primitive root modulo p.
(there is g so that $\{1, 2, \ldots, p-1\} = \{1, g, g^2, \ldots, g^{p-2}\} \pmod{p}$)

For each $a \in [p-1]$ if $a \equiv g^i \pmod{p}$

then we color a by $\boxed{i \bmod n}$

$\boxed{\#\text{colors} = n}$

Suppose $a+b=c$, a, b, c have the same color.

$a, b, c \in [p-1]$

$$a \equiv g^{nx+r} \pmod{p}$$
$$b \equiv g^{ny+r} \pmod{p}$$
$$c \equiv g^{nz+r} \pmod{p}$$

(r : the color of a, b, c)

$$g^{nx+r} + g^{ny+r} \equiv g^{nz+r} \pmod{p}$$

$$\Rightarrow g^{nx} + g^{ny} \equiv g^{nz} \pmod{p}$$

$$\Rightarrow (g^x)^n + (g^y)^n \equiv (g^z)^n. \qquad \square$$

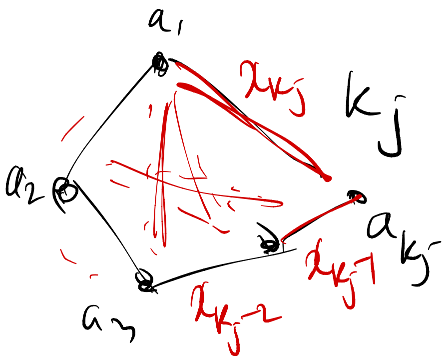
Thm (Rado?)

Let $r \geq 1$. For integers $k_1, k_2, \ldots, k_r \geq 3$, there exists a positive integer N such that for every r-coloring of $[N]$, there exist $j \in [r]$ and $x_1, x_2, \ldots, x_{k_j} \in [N]$ of color j such that

$$x_1 + x_2 + \cdots + x_{k_j - 1} = x_{k_j}.$$

(Schur's thm : $k_1 = k_2 = \cdots = k_r = 3$)

Proof $N = R(k_1, k_2, \ldots, k_r) - 1$



$$
\begin{aligned}
& x(a_2 - a_1) \\
&= x(a_3 - a_2) \\
&\ \vdots \\
&= x(a_{k_j} - a_{k_j-1}) \\
&= x(a_{k_j} - a_1)
\end{aligned}
$$

11.2. Regular Linear Homogeneous Equations

A set S of equations with variables $x_1, x_2, \dots, x_n$ is r-regular on a set A if any r-coloring of A induces a monochromatic solution $x_1, x_2, \dots, x_n \in A$.

For instance
$S = (x_1 + x_2 = x_3)$ on $\mathbb{N} = \{1, 2, \dots\}$
Then by Schur's theorem
S is r-regular for any r.

We say S is regular if S is r-regular for any $r > 0$.

Ex. $\{x_1 + x_2 + \dots + x_{100} = x_{101}\}$ is regular on $\mathbb{N}$.

Q: Characterize a regular set of equations.

Thm (Rado) Let $S = S(x_1, x_2, \dots, x_n)$ be a set of a single linear homogeneous equation
$$c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots + c_n x_n = 0$$
with integer coefficients $c_1, c_2, \dots, c_n$.

Then S is regular on $\mathbb{N}$
if and only if

$$\sum_{i\in I} c_i = 0 \quad \text{for some nonempty } I \subseteq \{1, \ldots, n\}.$$

$\boxed{Ex}$

$$\begin{cases} x+y-z=0 & \boxed{1+(-1)=0} \\ x+y=2z & 1+1-2=0 \\ x+y=3z & \text{Not regular} \end{cases}$$

<u>Proof</u> of the forward direction:
Let $c_1, c_2, \ldots c_n$ be fixed integers
Suppose that $\sum_{i\in I} c_i \neq 0$
for all nonempty subsets $I \subseteq \{1, 2, \ldots n\}$.
Choose a prime p such that

$$\sum c_i \not\equiv 0 \pmod p$$

(Since n is finite, such p exists.)

Now we are going to define a $(p-1)$-coloring

$$\chi : \mathbb{N} \to \{1, 2, 3, \ldots, p-1\}$$

as follows.

$$\chi(m) = a \pmod p \quad \text{if} \quad \left(\begin{array}{l} m = p^l \cdot a \\ p \nmid a, \ a \in \mathbb{Z} \end{array} \right.$$

($\hookrightarrow$ reemainder in a/p.)

$$\boxed{0 \le \chi \le p-1}$$

We call such a coloring the <u>base p</u> coloring.

We claim that there is no monochromatic solution with χ.

Suppose that $x_1, x_2, \dots, x_n$ is a solution of S with $\chi(x_1) = \chi(x_2) = \cdots = \chi(x_n) = u$.

For each i, let l_i be the maximum integer so that $p^{l_i} \mid x_i$.

Let $l = \min(l_1, l_2, \dots, l_n)$.

$$\frac{x_i}{p^{l_i}} \equiv u \pmod p$$

$$\frac{x_i}{p^{l}} \equiv \underset{l = l_i}{\underbrace{u}^{(\neq 0)}} \text{ or } \underset{l < l_i}{0} \pmod p$$

Let $I = \{ i \in [n] : l = l_i \}$

$$\sum_{i=1}^{n} c_i x_i \equiv 0 \pmod{p^{l+1}}$$

$$\Rightarrow \quad \sum c_i \frac{x_i}{p^l} \equiv 0 \pmod p$$

$$\sum_{i\in I} c_i u \equiv 0 \pmod p$$

$$p \nmid u \Rightarrow \quad \sum_{i \in I} c_i \equiv 0 \pmod p$$

Contradiction.

Proof of the backward direction:

Suppose $c_1 + c_2 + \cdots + c_k = 0$ for some $1 \leq k \leq n$.

$c_i \neq 0$ for all i.

Let $\Delta : \mathbb{N} \to [r]$ be an r-coloring of $\mathbb{N}$.

If $k = n$, then we can take $x_1 = x_2 = \cdots = x_n = 1$.

$$c_1 1 + c_2 \cdot 1 + \cdots + c_n \cdot 1 = 0$$

We have a monochromatic solution.

Therefore we may assume $k < n$.

Choose k maximum

$\Rightarrow$ We may assume

$$c_{k+1} + c_{k+2} + \cdots + c_n \neq 0.$$

Let $A = \gcd(c_1, c_2, \ldots, c_k)$

$B = c_{k+1} + c_{k+2} + \cdots + c_n$

$$s = \frac{A}{\gcd(A,B)}, \qquad t = -\frac{B}{\gcd(A,B)},$$

$$(At + Bs = 0)$$

By a theorem in elementary number theory there are integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$$c_1\lambda_1 + c_2\lambda_2 + \cdots + c_k\lambda_k = At.$$

Now let

$$x_i = \begin{cases} a + \lambda_i d & \text{if } 1 \le i \le k \\ sd & \text{if } i > k, \end{cases}$$

$$\sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{k} c_i(a + \lambda_i d) + \sum_{i=k+1}^{n} c_i sd$$

$$= \cancel{\overset{0}{\sum_{i=1}^{k} c_i a}} + \Big(\sum_{i=1}^{k} c_i\lambda_i\Big) d + sdB$$

$$= (At + Bs)d = 0.$$

So, $(x_1, x_2, \ldots, x_n)$ is a solution of S.

Now it is enough to prove the following (with different parameters)

Take $k' = 2k$

$$a,\ a+d,\ a+2d,\ \ldots,\ a+(2k-1)d,\qquad sd.$$

$a' = a + kd \Rightarrow a' + \lambda d$ for any $|\lambda| < k$ have the same color

Lemma For all $k, r, s \geq 1$, there exists $n = n(k,r,s)$ such that for every r-coloring of $[n]$, there exist $a, d > 0$ so that
$$\{a, a+d, a+2d, \dots, a+(k-1)d\} \cup \{sd\}$$
is monochromatic.

Proof. Induction on r. We may assume $r > 1$.

Let $n = s \cdot W(\underline{kn(k, r-1, s)}, r)$
[van der Waerden no.

Let A be an r-coloring of $[n]$.

Then there exists a monochromatic arithmetic progression (say red)
$$\{a + id' : 0 \leq i < kn(k, r-1, s)\}$$
for some a and $d' > 0$ in $[W(kn(k, r-1, s), r)]$.

If $sd'j$ is red for some $j \in [n(k, r-1, s)]$ then let $d = d'j$

$\Rightarrow$ sd and
$a + id$ for all $0 \leq i < k$
are all red.

Therefore we may assume that $sd'j$ is not red for all $j \in [n(k, r-1, s)]$.

$$(sd')\cdot 1,\quad (sd')\cdot 2,\quad \ldots,\quad (sd')\cdot n(k,r-1,s)$$

$$\downarrow \qquad\quad \downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$1 \qquad\quad 2 \qquad\qquad\qquad\qquad n(k,r-1,s)$$

Let χ' be a coloring of $[n(k,r-1,s)]$ such that $\chi'(i) = \chi(sd'i)$.

By the induction hypothesis there are $a', \bar d' > 0$ such that

$$a',\ a'+\bar d',\ a'+2\bar d',\ \ldots,\ a'+(k-1)\bar d'$$

and $s\bar d'$

have the same color under χ'.

(χ' is an $(r-1)$-coloring!)

$$\Rightarrow \underbrace{\boxed{sd'a'}}_{=a},\ sd'a' + \overbrace{\boxed{sd'\bar d'}}^{=d},\ sd'a'+2sd'\bar d',\ \ldots,\ sd'a'+(k-1)sd'\bar d'$$

and $s\underbrace{\boxed{d's\bar d'}}_{\to d}$

have the same color under χ.

$a = sd'a'$, $d = sd'\bar d'$ work.

11.3. Rado's theorem

When is $A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0$ regular?

A: integer-valued matrix with n columns

"Columns condition" c_i = i^{th} column vector

A satisfies the columns condition if there exists a partition $(I_1, I_2, \dots I_t)$ of $[n]$ into t nonempty subsets such that

for each $i = 1, 2, \dots, t$

$\sum_{j \in I_i} c_j$ is in the span (over $\mathbb{Q}$) of the set $\{c_j : j \in I_{i'} \text{ for } i' < i\}$

$$\sum_{j \in I_1} c_j = 0$$

$$\sum_{j \in I_2} c_j \in \langle c_j : j \in I_1 \rangle$$

$$\sum_{j \in I_3} c_j \in \langle c_j : j \in I_1 \cup I_2 \rangle$$

$$\vdots$$

$$\sum_{j \in I_t} c_j \in \langle c_j : j \in I_1 \cup I_2 \cup \dots \cup I_{t-1} \rangle$$

Thm (Rado)

The system $Ax=0$ is regular on $\mathbb{N}$
if and only if
A satisfies the columns condition.

Proof of the forward direction:
(We will use the base p coloring for some large prime p.)

<u>Lemma</u>: Let j, k be positive integers
and $c_1, c_2, \dots, c_j \in \mathbb{Z}^k$ be vectors
such that c_1 is not a linear combination (over $\mathbb{Q}$)
of the (possibly empty) set $\{c_2, c_3, \dots, c_j\}$.
Then there exists a finite set F of primes
such that for any prime $p \notin F$
and any nonnegative integer m,
the vector
$$p^m c_1$$
is not a linear combination of $c_2, \dots, c_j$
<u>modulo p^{m+1}</u>.

Proof. Since c_1 is not in the span (over $\mathbb{Q}$)
of $\{c_2, c_3, \dots, c_j\}$,
there exists a vector $u \in \mathbb{Q}^k$
such that
$$c_2 \cdot u = c_3 \cdot u = \cdots = c_j \cdot u = 0$$
$$c_1 \cdot u \neq 0.$$
By multiplying a big integer,
we may assume $u \in \mathbb{Z}^k$.

Let F be the set of all primes p such that $p \mid \underline{u \cdot c_1}$.

Suppose p is a prime and m is a non-negative integer such that
$p^m c_1$
is a linear combination of $c_2, c_3, \dots, c_j \bmod p^{m+1}$.

$$p^m c_1 \equiv a_2 c_2 + a_3 c_3 + \cdots + a_j c_j \pmod{p^{m+1}}.$$

$$p^m c_1 \cdot u \equiv a_2 \underbrace{c_2 \cdot u}_{=0} + a_3 \underbrace{c_3 \cdot u}_{=0} + \cdots + a_j \underbrace{c_j \cdot u}_{=0} \pmod{p^{m+1}}$$

$$\Rightarrow p^m c_1 \cdot u \equiv 0 \pmod{p^{m+1}}$$

$$c_1 \cdot u \equiv 0 \pmod{p}$$

$$p \mid c_1 \cdot u \Rightarrow p \in F. \quad \square$$

Now let's prove the forward direction of Rado's theorem.

Let A be a matrix with n columns so that $Ax = 0$ is regular on $\mathbb{N}$.

For 2 disjoint subsets I, J of $[n]$, if $I \neq \varnothing$, then either

(by the previous lemma)

- $\sum_{i \in I} c_i$ is a linear combination (over $\mathbb{Q}$) of vectors in $\{c_j : j \in J\}$

OR

- there exists a finite set $F_{I,J}$ of primes such that for any prime $p \notin F_{I,J}$ and any nonnegative integer m, the vector $p^m \sum_{i \in I} c_i$ is not a linear combination of $\{c_j : j \in J\}$ modulo p^{m+1}.

Let F be the union of all $F_{I,J}$ over all pairs I, J where $F_{I,J}$ is defined.

F is finite, $p \notin F$.

Let $\chi : \mathbb{N} \to [p-1]$ be a base-p coloring

$$\left(\chi(m) = a \bmod p \quad \text{when} \quad m = p^l \cdot a, \ p \nmid a \right)$$

Suppose that $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a monochromatic solution of $Ax = 0$.

$$\chi(x_1) = \chi(x_2) = \cdots = \chi(x_n) = a$$

Let l_i be the integer such that

$$p^{l_i} \mid x_i, \quad p^{l_i+1} \nmid x_i.$$

Since $\chi(x_i) = a$, we have

$$x_i \equiv p^{l_i} \cdot a \pmod{p^{l_i+1}}$$

Now, we may assume $l_1 \le l_2 \le \cdots \le l_n$.

Let $b_1 < b_2 < \cdots < b_t$ be the integers such that

$$\{b_1, b_2, \ldots, b_t\} = \{l_1, l_2, \ldots, l_n\}$$

Let $I_i = \{ j \in \{1, 2, \ldots, n\} : l_j = b_i \}$

$$\left[\begin{array}{l} l : \ 1, \ 1, \ 2, \ 2, \ 2, \ 7 \\ I_1 = \{1, 2\} \quad I_2 = \{3, 4, 5\} \quad I_3 = \{6\}. \end{array} \right.$$

$$\sum_{i=1}^{n} x_i c_i = 0 \implies \sum_{i=1}^{t} \sum_{j \in I_i} x_j c_j = 0$$

For each $m \in [t]$,

$$\sum_{i=1}^{m-1} \sum_{j \in I_i} x_j c_j + \underbrace{\sum_{i=m}^{t} \sum_{j \in I_i} x_j c_j}_{\text{divisible by } p^{b_m}} = 0$$

$\boxed{\text{In mod } p^{b_m+1}}$

$$\sum_{i=1}^{m-1} \sum_{j \in I_i} x_j c_j + \sum_{j \in I_m} x_j c_j \equiv 0 \pmod{p^{b_m+1}}$$

$$x_j \equiv p^{b_m} a \pmod{p^{b_m+1}}$$

$$\implies \sum_{i=1}^{m-1} \sum_{j \in I_i} x_j c_j + a p^{b_m} \sum_{j \in I_m} c_j \equiv 0 \pmod{p^{b_m+1}}$$

There is an integer $\tilde{a}$ such that

$$a\tilde{a} \equiv 1 \pmod{p}$$

$$\implies p^{b_m} \sum_{j \in I_m} c_j = -\tilde{a} \sum_{i=1}^{m-1} \sum_{j \in I_i} x_j c_j \pmod{p^{b_m+1}}$$

$p \notin F$

By the previous lemma,

$\sum_{j \in I_m} c_j$ is a linear combination (over $\mathbb{Q}$) of $\{c_j : j \in I_i,\ i < m\}$

So, A satisfies the columns condetion.