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Note

A note on acyclic edge coloring of complete bipartite graphs

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ABSTRACT

An *acyclic edge coloring* of a graph is a proper edge coloring such that there are no bichromatic (2-*colored*) cycles. The *acyclic chromatic index* of a graph is the minimum number k such that there is an acyclic edge coloring using k colors and is denoted by $a'(G)$. Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in a graph G. A complete bipartite graph with n vertices on each side is denoted by $K_{n,n}$. Alon, McDiarmid and Reed observed that $a'(K_{p-1,p-1}) = p$ for every prime p. In this paper we prove that $a'(K_{p,p}) \le p+2 = \Delta+2$ when p is prime. Basavaraju, Chandran and Kummini proved that $a'(K_{n,n}) \ge n+2 = \Delta+2$ when n is odd, which combined with our result implies that $a'(K_{p,p}) = p+2 = \Delta+2$ when p is an odd prime. Moreover we show that if we remove any edge from $K_{p,p}$, the resulting graph is acyclically $\Delta + 1 = p + 1$-edge-colorable.

All graphs considered in this paper are finite and simple. A *proper edge coloring* of $G = (V, E)$ is a map $c : E \to C$ (where C is the set of available *colors*) with $c(e) \neq c(f)$ for any adjacent edges e,f. The minimum number of colors needed to properly color the edges of G is the *chromatic index* of G and is denoted by $\chi'(G)$. A proper edge coloring c is *acyclic* if there are no bichromatic cycles in the graph. In other words, an edge coloring is acyclic if the union of any two color classes is a disjoint union of paths (i.e., a linear forest) in G. The *acyclic edge chromatic number* (also called the *acyclic chromatic index*), denoted by $a'(G)$, is the minimum number of colors required in an acyclic edge coloring of G. The concept of acyclic coloring of a graph was introduced by Grünbaum [7]. Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in a graph G. By Vizing's theorem $\Delta \le \chi'(G) \le \Delta + 1$ (see [6] for the proof). Since any acyclic edge coloring is also proper, $a'(G) \ge \chi'(G) \ge \Delta$.

It has been conjectured by Alon, Sudakov, and Zaks [2] that $a'(G) \le \Delta + 2$ for any G. Using probabilistic arguments Alon, McDiarmid, and Reed [1] proved that $a'(G) \le 60\Delta$. The best known result up to now for arbitrary graphs is that of Molloy and Reed [9], who showed that $a'(G) \le 16\Delta$.

Though the best known upper bound for the general case is far from the conjectured $\Delta + 2$, the conjecture has been shown to be true for some special classes of graphs. Alon, Sudakov, and Zaks [2] proved that there exists a constant k such that $a'(G) \le \Delta + 2$ for any graph G whose girth is at least $k\Delta \log \Delta$. They also proved that $a'(G) \le \Delta + 2$ for almost all Δ-regular graphs. This result was improved by Nešetřil and Wormald [12], who showed that random regular graphs almost always have $a'(G) \le \Delta + 1$. Muthu, Narayanan, and Subramanian proved the conjecture for grid-like graphs [10] and outerplanar graphs [11]. In fact they gave a better bound of $\Delta + 1$ for these classes of graphs. From Burnstein's [5] result it follows that the conjecture is true for subcubic graphs. Skulrattankulchai [13] gave a polynomial time algorithm for coloring a subcubic graph using $\Delta + 2 = 5$ colors.

A complete bipartite graph with n vertices on each side is denoted by $K_{n,n}$. We denote the sides by A and B. Thus $V(K_{n,n}) = A \cup B$.

Our result: In this paper, we prove the following theorem:

Theorem 1. *$a'(K_{p,p}) \le p + 2 = \Delta + 2$, when p is an odd prime*

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Remarks. (1) Basavaraju, Chandran and Kummini [3] proved that $a'(K_{n,n}) \geq n+2 = \Delta+2$, when n is odd. From Theorem 1, we can infer that $a'(K_{p,p}) = p+2 = \Delta+2$.
(2) The complete bipartite graph, $K_{n,n}$ is said to have a *perfect 1-factorization* if the edges of $K_{n,n}$ can be decomposed into n disjoint perfect matchings such that the union of any two perfect matchings forms a hamiltonian cycle. Alon, McDiarmid and Reed [1] observed that $a'(K_{p-1,p-1}) = p$. It is easy to see that if $K_{n+2,n+2}$ has a perfect 1-factorization, then $a'(K_{n,n}) \leq a'(K_{n+1,n+1}) \leq n+2$. It is known that (see [4]), if $n+2 \in \{p, 2p-1, p^2\}$, where p is an odd prime or when $n+2 < 50$ and odd, then $K_{n+2,n+2}$ has a perfect 1-factorization. Combining with the result that $a'(K_{n,n}) \geq n+2 = \Delta+2$ when n is odd, one gets $a'(K_{n,n}) = n+2 = \Delta+2$ for the above mentioned values of $n+2$. As of now, these are the only values of n for which we know the exact value for $a'(K_{n,n})$. Note that we cannot apply the simple argument mentioned here when $n = p$.
(3) To get an upper bound for $a'(K_{n,n})$, the best method we can think of is to look for the smallest prime number p such that $p \geq n+2$. Then $a'(K_{n,n}) \leq p$. A weakening of the result of Iwaniec and Pintz [8] gives that for every sufficiently large integer x, there exists a prime number in the range $[x, x + x^{0.6}]$.

1. Proof of Theorem 1

Let $A = \{0, 1, \ldots, p-1\}$ and $B = \{0, 1, \ldots, p-1\}$. Let $\pi_0, \pi_1, \ldots, \pi_{p-1}$ be the permutation defined by $\pi_i : a \longmapsto (a+i) \pmod p$. Let M_i be the perfect matching corresponding to the permutation π_i. It is easy to verify that if $i \neq j$, then $M_i \cap M_j = \emptyset$. Now we claim the following:

Claim 1. *If $i \neq j$, then $M_i \cup M_j$ forms a Hamiltonian cycle (i.e., $M_0, M_1, \ldots M_{p-1}$ form a perfect 1-factorization).*

Proof. First note that the union of any two perfect matchings forms a collection of disjoint cycles. Suppose two matchings M_i and M_j $(i > j)$ are such that a cycle of length $2k < 2p$ gets formed by the edges of $M_i \cup M_j$ (recall that all cycles are of even length in $K_{p,p}$). Without loss of generality let this cycle contain the vertex $a \in A$. It is easy to see that $(\pi_j^{-1}\pi_i)^k(a) = a$. Noting that $(\pi_j^{-1}\pi_i)(a) = a + i - j \pmod p$, we have $(\pi_j^{-1}\pi_i)^k(a) = a + ki - kj = a + k(i-j) \pmod p = a \pmod p$, which implies that $k(i-j) = 0 \pmod p$. Since $i - j \neq 0 \pmod p$, we have $k = 0 \pmod p$, a contradiction since $k < p$. Thus $M_i \cup M_j$ forms a cycle of length $2p$ (a Hamiltonian cycle) when i and j are distinct. □

Now consider the multiplicative group Z_p^*, and let x be a generator of this group. Define a permutation π of $\{1, 2, \ldots, p-1\}$ by $\pi : a \longmapsto ax \pmod p$. Let M be the matching corresponding to the permutation π.

Claim 2. $|M \cap M_i| = 1$, *for each M_i, $1 \leq i \leq p-1$ and $M_0 \cap M = \emptyset$ (i.e., for each M_i, $1 \leq i \leq p-1$, the matchings M and M_i have exactly one edge in common; also the matchings M and M_0 do not have any edge in common).*

Proof. By the definition of M, we infer that $M_0 \cap M = \emptyset$. Now let $a = i(x-1)^{-1} \pmod p$. Note that since $i \neq 0$, $a \neq 0$. We have $\pi_i(a) = a + i = i(x-1)^{-1} + i = i(x-1)^{-1}(1 + x - 1) = i(x-1)^{-1}(x) = ax \pmod p = \pi(a)$. Thus it follows that the edge $(a, ax) \in M \cap M_i$ for $a = i(x-1)^{-1} \pmod p$. Therefore $|M \cap M_i| \geq 1$ for $1 \leq i \leq p-1$. Since $|M| = p-1$, we can also infer that $|M \cap M_i| = 1$. □

Now color the edges of $K_{p,p}$ as follows to get a coloring f with $p+2$ colors:

(1) if $e \in M_i \setminus M$ (where $0 \leq i \leq p-1$), then it is colored with color c_i;
(2) if $e \in M - (1, x)$, then it is colored with color c_p;
(3) edge $e = (1, x)$ is colored with color c_{p+1}.

Claim 3. *The coloring f is acyclic.*

Proof. Obviously f is a proper coloring. Let c_i and c_j be two colors. We consider different values for i and j with $i > j$ and show that a (c_i, c_j) bichromatic cycle cannot exist.

Case 1: $i = p+1$

Since there is only one edge colored c_{p+1}, there cannot be any bichromatic cycle involving the color c_{p+1}.

Case 2: $i, j < p$

Note that $M_i \cup M_j$ forms a Hamiltonian cycle by Claim 1. Now at least one edge of M_i belongs to M (by Claim 2) and is colored c_p or c_{p+1} with respect to the coloring f, breaking the possible (c_i, c_j) bichromatic cycle. Therefore there cannot be any (c_i, c_j) bichromatic cycle when $i, j < p$.

Case 3: $i = p$

Suppose M_j is a matching such that a cycle of length $2k < 2p$ (no cycles of length $2p$ can be formed as there are only $p-2$ edges of color c_p) gets formed by the edges of $M \cup M_j$ (recall that all cycles are of even length in $K_{p,p}$). Thus $(\pi_j^{-1}\pi)^k(a) = a \pmod p$. Noting that $(\pi_j^{-1}\pi)(a) = ax - j \pmod p$, we have $(\pi_j^{-1}\pi)^2(a) = (ax-j)x - j = ax^2 - j(x+1) \pmod p$. Similarly $(\pi_j^{-1}\pi)^k(a) = ax^k - j(x^{k-1} + \cdots + x + 1) = ax^k - j(x^k - 1)(x-1)^{-1} = a \pmod p$. We have $a(x^k - 1) - j(x^k-1)(x-1)^{-1} = 0 \pmod p$ and thus $(x^k - 1)(a - j(x-1)^{-1}) = 0 \pmod p$. If $(a - j(x-1)^{-1}) = 0 \pmod p$, then $a = j(x-1)^{-1} \pmod{p}$. But according to Claim 2, we have edge $(a, ax) \in M \cap M_j$. Therefore this edge and thus vertex a cannot be in the cycle formed by $M \cup M_j$, a contradiction. Thus we infer that $(x^k - 1) = 0 \pmod{p}$. This implies that $x^k = 1 \pmod{p}$ and hence $k = p - 1$, since x is a generator. Thus there are $2(p-1)$ edges in the cycle, out of which $p - 1$ are colored c_p, a contradiction since only $p - 2$ edges are colored c_p. □

Theorem 2. *For a prime $p > 2$, if G is a graph obtained by removing just one edge from $K_{p,p}$, then $a'(G) = \Delta + 1 = p + 1$ (the above statement is true even if we delete any number of edges between 1 and $p - 2$).*

Proof. It is easy to infer from the proof of Theorem 1 that $a'(G) \leq p + 1$. The lower bound comes from a simple counting argument: At most one color class can have p edges, since otherwise there will be bichromatic cycles. Thus if $a'(G) \leq p$, then there can be at most $p + (p-1)^2 < p^2 - 1$ edges in G, a contradiction. □

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