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Graphs which Contain all Small Graphs

BÉLA BOLLOBÁS AND ANDREW THOMASON

Given a graph H, is there a strongly regular graph G containing H as an induced subgraph? This question, posed by M. Rosenfeld, is answered here in the affirmative. In fact, given r, we shall be looking for graphs of small order which contain *every* graph of order r as an induced subgraph. In order to abbreviate this rather clumsy description, we shall say that we are looking for *r-full graphs* of small order. Note that if G is an r-full graph of order n then $\binom{n}{r}$ is at least the number of non-isomorphic graphs of order r, so

$$\binom{n}{r} \geq 2^{\binom{r}{2}}/r!$$

and thus

$$n \geq 2^{(r-1)/2}.$$

We shall show that if n is slightly greater than $2^{(r-1)/2}$, not only does there exist an r-full graph of order n but almost every graph of order n is r-full. Rather curiously, we are unable to give a concrete example of an r-full graph of such small order. We do, however, exhibit an r-full graph of order 2^r which is rather close to being strongly regular. Finally, we display a strongly regular r-full graph of order roughly $r^2 2^{2r}$.

We shall consider the set $\mathcal{G}_n$ of all $2^{\binom{n}{2}}$ graphs with vertex set $\{1, 2, \ldots, n\}$. As usual, $\mathcal{G}_n$ is regarded as a probability space in which any two graphs have the same probability; equivalently, the edges are chosen independently and with probability $\frac{1}{2}$. A subset $\mathcal{H}_n \subset \mathcal{G}_n$ is said to contain *almost every* graph of order n if $|\mathcal{H}_n|/|\mathcal{G}_n| \to 1$ as $n \to \infty$.

THEOREM 1. *Almost every graph of order $n = \lceil r^2 2^{r/2} \rceil$ is r-full.*

PROOF. In order to simplify the notation we shall assume that r is even. Put $m = r2^{r/2}$ and consider the probability space $\mathcal{G}_m$ of all graphs with vertex set $V^* = \{1, 2, \ldots, m\}$. Let H be a fixed graph with vertex set $\{x_1, x_2, \ldots, x_r\}$. Given $G \in \mathcal{G}_m$ denote by $Y_r = Y_r(G)$ the number of r-subsets $\{y_1, y_2, \ldots, y_r\}$ of V^*, $y_1 < y_2 < \cdots < y_r$, for which $x_i \to y_i$ gives an isomorphism between H and the subgraph of G induced by $\{y_1, y_2, \ldots, y_r\}$. The expectation of the random variable Y_r is clearly

$$E_r = E(Y_r) \geq \binom{m}{r} 2^{-\binom{r}{2}} \geq \frac{1}{2r}\left(\frac{em}{r}\right)^r 2^{-\binom{r}{2}} > e^r,$$

provided $r > 8$. It is easily seen that inequality (7) of [1, p. 421] holds for Y_r (which is slightly different from the random variable Y_r in [1]), that is

$$P(Y=0) < 2r^4 m^{-2} + 2E_r^{-1} < 3r^2 2^{-r}, \tag{1}$$

provided r is sufficiently large.

Now let us consider the set $\mathcal{G}_n$ of all graphs with vertex set $V = \{1, 2, \ldots, n\}$, $n = rm$. Partition V into r sets of m vertices each: $V = \bigcup_{i=1}^r V_i^*$. By (1) the probability that V_i^* contains no spanned subgraph isomorphic to a fixed graph H of order r is at most $3r^2 2^{-r}$. Since the sets V_i^* are disjoint, the probability that a graph $G \in \mathcal{G}_n$ contains no subgraph isomorphic to H is at most $(3r^2 2^{-r})^r$. Since we have at most $2^{\binom{r}{2}}$ choices for H, the

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probability that a graph $G \in \mathscr{G}_n$ does not contain *every* graph of order r as a spanned subgraph is at most

$$2^{\binom{r}{2}}(3r^2 2^{-r})^r,$$

which tends to 0 as $r \to \infty$.

The r-full graph of order 2^r we referred to in the introduction is the following graph P. Its vertex set is the power set $\mathscr{P}(X)$ of $X = \{1, 2, \ldots, r\}$. Let $A, B \in \mathscr{P}(X)$ be two distinct vertices of P. If A and B are non-empty we join A to B iff $|A \cap B|$ is even; if A is empty we join A to B iff $|B|$ is even. The graph P is then close to being strongly regular in the following sense. Each vertex has degree $2^{r-1} - 1$, every edge is in $2^{r-2} - 1$ or $2^{r-2} - 2$ triangles and in the complement every edge is in $2^{r-2} - 1$ or 2^{r-2} triangles.

THEOREM 2. *The graph P is r-full.*

PROOF. Let H be a graph with vertex set $\{v_1, v_2, \ldots, v_r\}$. We claim that there are sets $A_1, A_2, \ldots, A_r$ uniquely determined by H, such that

$$A_i \subset \{1, 2, \ldots, i\}, \qquad i \in A_i,$$

and, for $i \neq j$,

$$|A_i \cap A_j| \text{ is even iff } v_i v_j \text{ is an edge of } H.$$

Indeed, having chosen $A_1, A_2, \ldots, A_{j-1}$ and also $A_j \cap \{1, 2, \ldots, i-1\}$, our choice of whether i is in A_j will affect $A_j \cap A_i$ (since $i \in A_i$) but none of $A_j \cap A_k$, $k < i$ (since $A_k \subset \{1, 2, \ldots, k\}$). Hence with the unique proper choice of $A_j \cap \{i\}$ we can make sure that all the numbers $|A_k \cap A_l|$, $1 \leq k < l \leq j-1$, and $|A_k \cap A_j|$, $1 \leq k \leq i$, have the required parities. This shows the existence and uniqueness of the sets $A_1, A_2, \ldots, A_r$.

Clearly the map $v_i \to A_i$ is an isomorphism between H and the subgraph of P induced by $\{A_1, A_2, \ldots, A_r\}$.

We remark that for $r > 5$ the graph P is in fact $(r+1)$-full. For P contains a complete graph of order $2^{\lfloor r/2 \rfloor}$ whose vertices are unions of the sets $\{1, 2\}, \{3, 4\}, \ldots, \{2\lfloor r/2 \rfloor - 1, 2\lfloor r/2 \rfloor\}$. On the other hand if H of order $r+1$ is not complete we may order its vertices $v_1, v_2, \ldots, v_{r+1}$ so that $v_1 v_{r+1}$ is not an edge of H. The sets $A_1, A_2, \ldots, A_r$ are then chosen as in the proof of Theorem 2, and a set A_{r+1} can then be chosen similarly so that $|A_{r+1} \cap A_j|$ is even iff $v_{r+1} v_j$ is an edge of H, $j \leq r$. Since $1 \in A_{r+1}$, $A_{r+1} \neq \emptyset$ so $v_i \to A_i$ is an imbedding of H in P.

Finally we turn to strongly regular graphs. Let $q \equiv 1 \pmod 4$ be a prime power so that -1 is a square in the field $F = \mathbb{F}_q$ of order q. Denote by χ the quadratic residue character on F. The *Paley graph* Q_q has vertex set F and edge set $\{xy : \chi(x - y) = 1\}$. It is well known that this graph is strongly regular (see [2]). We shall show that if q is large enough then Q_q is r-full. The proof is based on Weil's theorem proving the Riemann hypothesis for algebraic curves over finite fields (see, for example, Schmidt [3]). The result we need is that if $f(X)$ is a polynomial over F of degree m and it is not a constant multiple of the square of another polynomial then

$$\left| \sum_{x \in F} \chi(f(x)) \right| \leq m q^{\frac{1}{2}}. \tag{2}$$

THEOREM 3. *If $q \geq (2^{r-2}(r-1)+1)^2$ then Q_q is r-full.*

PROOF. We fix q and apply induction on r. Assume that $r \geq 2$ and Q_q is $(r-1)$-full. Let H be a graph of order r. Pick a vertex u of H and choose an induced subgraph H^* of Q_q isomorphic to $H-u$. Let $V^* = V(H^*) \subset F$, denote by A the set of vertices in V^* corresponding to $\Gamma(u)$ in $H-u$ and put $B = V^* - A$.

Given a subset W of V^*, define the polynomial $f_W(x)$ by

$$f_W(X) = \prod_{w \in W} (X - w).$$

Then for $x \in F$ we have

$$\pi(x) = \prod_{a \in A} (1 + \chi(x-a)) \prod_{b \in B} (1 - \chi(x-b)) = \sum_{W \subset V^*} (-1)^{|B \cap W|} \chi(f_W(x)).$$

If $x \notin V^*$ then $\pi(x)$ is 0 unless $\Gamma(x) \cap V^* = A$ when $\pi(x) = 2^{r-1}$. Therefore the subgraph of Q_q induced by $\{X\} \cup V^*$ is isomorphic to H if $\pi(x) \neq 0$. Thus the theorem will be proved if we show that $\sum' \pi(x)$ is positive, where the prime indicates that the sum is over $F - V^*$. Note that if $x \in V^*$ then $\pi(x) = 0$ or 2^{r-2}, so

$$\sum{}' \pi(x) \geq \sum_{x \in F} \pi(x) - (r-1)2^{r-2}.$$

Furthermore,

$$\begin{aligned} \sum_{x \in F} \pi(x) &= \sum_{x \in F} \sum_{W \subset V^*} (-1)^{|B \cap W|} \chi(f_W(x)) \\ &= \sum_{W \subset V^*} (-1)^{|B \cap W|} \sum_{x \in F} \chi(f_W(x)). \end{aligned}$$

Since $f_\varnothing(x) = 1$, $\sum_{x \in F} \chi(f_\varnothing(x)) = q$, whereas, if $W \neq \varnothing$, inequality (2) gives

$$\left| \sum_{x \in F} \chi(f_W(x)) \right| \leq |W| q^{\frac{1}{2}}.$$

Consequently, if $\varnothing \neq W \neq V^*$, then

$$\left| \sum_{x \in F} \chi(f_W(x)) \right| + \left| \sum_{x \in F} \chi(f_{V^* - W}(x)) \right| \leq (r-1) q^{\frac{1}{2}}.$$

Therefore

$$\sum{}' \pi(x) \geq \sum_{x \in F} \pi(x) - (r-1)2^{r-2} \geq q - 2^{r-2}(r-1) q^{\frac{1}{2}} - (r-1)2^{r-2} > 0,$$

completing the proof.

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B. BOLLOBÁS AND A. THOMASON
Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, U.K.