2020 FALL MAS583 HW6

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1 Problem 1

1.1 Question

Let G be a graph and let P denote the probability that a random subgraph of G obtained by picking each edge of G with probability $1/2$, independently, is connected (and spanning). Let Q denote the probability that in a random two-coloring of G, where each edge is chosen, randomly and independently, to be either red or blue, the red graph and the blue graph are both connected (and spanning). Is $Q \leq P^2$?

1.2 Answer

Definition 1.1 *Given a simple graph $G = (V, E)$, and a spanning subgraph of G, $G_1 = (V, E_1)$, we say $G_2 = (V, E_2)$ is the **inverse graph** of G_1 on G, if $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$.*

Definition 1.2 *Given a simple graph G, and a subgraph G_1 of G, we say G_1 has property $\mathcal{Q}_1(G)$ if G_1 is connected (and spanning), and we say G_1 has property $\mathcal{Q}_2(G)$ if G_2, the inverse graph of G_1 on G, is connected (and spanning).*

Clearly, with the above definitions, $Q \leq P^2$ is equivalent to the following proposition.

Proposition 1.3 *Given a simple graph G, we obtain G_1, a random subgraph of G by picking each edge of G with probability $1/2$, then we have*

$$\Pr[G_1 \in \mathcal{Q}_1(G) \cap \mathcal{Q}_2(G)] \leq \Pr[G_1 \in \mathcal{Q}_1(G)] \cdot \Pr[G_1 \in \mathcal{Q}_2(G)].$$

With FKG Inequality, suffice it to show that $\mathcal{Q}_1(G)$ is monotonically increasing and $\mathcal{Q}_2(G)$ is monotonically decreasing, for any given G. Actually, this is obvious as given any graph G, if a subgraph of G, G_1 has property $\mathcal{Q}_1(G)$, i.e., G_1 is connected (and spanning), then after adding any edges on G_1, it's still connected (and spanning); and if G_1 has property $\mathcal{Q}_2(G)$, i.e., the inverse graph of G_1 on G, G_2, is connected and (and spanning), then after deleting any edges of G_1, which means adding these edges to G_2, G_1 still has property $\mathcal{Q}_2(G)$, completing the proof.

2 Problem 2

2.1 Question

A family of subsets $\mathcal{G}$ is called *intersecting* if $G_1 \cap G_2 \neq \emptyset$ for all $G_1, G_2 \in \mathcal{G}$. Let $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_k$ be k intersecting families of subsets of $\{1, 2, ..., n\}$. Prove that

$$\left| \bigcup_{i=1}^{k} \mathcal{F}_i \right| \leq 2^n - 2^{n-k}.$$

2.2 Answer

We will prove the proposition by induction. When $k = 1$, the proposition is true as for each pair $(S, 2^{[n]} \setminus S)$, where $S \subset 2^n$, at most one of them can be in a intersecting family of subsets. Now suppose the proposition is true for $k-1$. Given $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_k$ be k intersecting families of subsets of $\{1, 2, ..., n\}$, we add some elements into each $\mathcal{F}_i$, if necessary, such that each of them becomes maximal with $|\mathcal{F}_i| = 2^{n-1}$. As we need to find an upper bound for $\left| \bigcup_{i=1}^{k} \mathcal{F}_i \right|$, our operation is acceptable.

Lemma 2.1 (Proposition 6.3.1 on the textbook) *Let $\mathcal{A}$ be a monotone increasing families of subsets of $[n]$ and let $\mathcal{B}$ be a monotone decreasing families of subsets of $[n]$. Then*

$$2^n |\mathcal{A} \cap \mathcal{B}| \leq |\mathcal{A}| \cdot |\mathcal{B}|.$$

Let $\mathcal{A} = \bigcup_{i=1}^{k-1} \mathcal{F}_i$ and let $\mathcal{B} = 2^{[n]} \setminus \mathcal{F}_k$, by the induction hypothesis, we have

$$|\mathcal{A}| \leq 2^n - 2^{n-k+1},$$

and as each $\mathcal{F}_i$ is monotone increasing because that for any $F' \supset F \in \mathcal{F}_i$, suppose $F' \notin \mathcal{F}_i$, as $\mathcal{F}_i$ is maximal, $2^{[n]} \setminus F' \in \mathcal{F}_i$, but $F \cap (2^{[n]} \setminus F') = \emptyset$, which is a contradiction, we have $\mathcal{A}$ is monotone increasing and $\mathcal{B}$ is monotone decreasing, by the lemma, we have

$$|\mathcal{A} \cap \mathcal{B}| \leq 2^{-n}(2^n - 2^{n-k+1})2^{n-1} = 2^{n-1} - 2^{n-k}.$$

Therefore,

$$\left| \bigcup_{i=1}^{k} \mathcal{F}_i \right| = |\mathcal{A}| + |\mathcal{F}_k| - |\mathcal{A} \cap \mathcal{F}_k| = |\mathcal{A}| + |\mathcal{F}_k| - (|\mathcal{A}| - |\mathcal{A} \cap \mathcal{B}|) = |\mathcal{F}_k| + |\mathcal{A} \cap \mathcal{B}| \leq 2^n - 2^{n-k},$$

completing the proof.

3 Problem 3

3.1 Question

Show that the probability that in the random graph $G(2k, 1/2)$ the maximum degree is at most $k-1$ is at least $1/4^k$.

3.2 Answer

For any vertex $v \in [2k]$, let A_v be the event that the degree of v is at most $k-1$. Clearly,

$$\Pr[A_v] = \sum_{i=0}^{k-1} \binom{2k-1}{i} / 2^{2k-1} = 1/2.$$

Note that A_v is monotone decreasing, in the sense that if A_v happens in a graph $G = (2k, E)$, then it happens in any graph $G' = (2k, E' \subset E)$, thus,

$$\Pr[\Delta(G(2k, 1/2)) \leq k-1] = \Pr[\bigcap_v A_v] \geq \prod_v \Pr[A_v] = (1/2)^{2k} = 1/4^k,$$

completing the proof.