2020 FALL MAS583 HW3

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1 Problem 1

1.1 Question

As shown in Section 3.1, the Ramsey number $R(k,k)$ satisfies

$$R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$$

for every integer n. Conclude that

$$R(k,k) \geq (1 - o(1)) \frac{k}{e} 2^{k/2}.$$

1.2 Answer

From the first inequality, we can get

$$R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}} \geq n - (\frac{ne}{k})^k 2^{1-\binom{k}{2}}.$$

By letting $n = \frac{k}{e} 2^{k/2}$, we get

$$R(k,k) \geq n - (\frac{ne}{k})^k 2^{1-\binom{k}{2}} \geq \frac{k}{e} 2^{k/2} - 2^{1+k/2} = (1 - 2e/k) \frac{k}{e} 2^{k/2} = (1 - o(1)) \frac{k}{e} 2^{k/2},$$

completing the proof.

2 Problem 2

2.1 Question

Prove that the Ramsey number $R(4,k)$ satisfies

$$R(4,k) \geq \Omega((k/\ln k)^2).$$

2.2 Answer

First, the following theorem (Theorem 3.1.3 in the textbook) should be introduced.

Theorem 2.1 *For all integers n and $p \in [0,1]$,*

$$R(k,l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{l}(1-p)^{\binom{l}{2}}.$$

Proof. Considering a random 2-coloring of K_n obtained by coloring each edge independently either red or blue, where each edge is red with probability p. Let X be the number of red k-sets plus the number of blue l-sets. Linearity of expectation gives

$$\mathbb{E}[X] = \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}}.$$

Thus, there exists a 2-coloring with $s \leq \mathbb{E}[X]$ "bad" sets. Removing one point from each bad set gives a coloring of at least $n - s$ points with no bad sets. ■
Using the above theorem, we have for all integers n and $p \in [0,1]$,

$$\begin{aligned} R(4,k) &> n - \binom{n}{4} p^6 - \binom{n}{k}(1-p)^{\binom{k}{2}} \\ &\geq n - n^4 p^6/24 - (ne/k)^k (1-p)^{k^2/2} \\ &\geq n - n^4 p^6/24 - (ne/k)^k e^{-pk^2/2}. \end{aligned}$$

Letting $n = (k/\ln k)^2$ and $p = n^{-1/2} = \ln k/k$, we have $n^4 p^6/24 = n/24$ and

$$\begin{aligned} (ne/k)^k e^{-pk^2/2} &\leq (k/(\ln k)^2)^k / e^{k(\ln k/2 - 1)} \\ &\leq (k/\ln k)^2/2 \\ &\leq n/2, \end{aligned}$$

when k is large enough. Thus, $R(4,k) \geq 11n/24 \geq 11/24(k/\ln k)^2$, completing the proof.

3 Problem 3

3.1 Question

Prove that every three-uniform hypergraph with n vertices and $m \geq n/3$ edges contains an independent set (i.e., a set of vertices containing no edges) of size at least

$$\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}.$$

3.2 Answer

Let G be such a three-uniform hypergraph. Considering a random 2-coloring of G by coloring each vertex independently either red or blue, where each vertex is red with probability p. Let X be the number of red vertices and Y be the number of edges in G whose all three incident vertices are red. Clearly, $\mathbb{E}[X] = np$ and $\mathbb{E}[Y] \leq mp^3$, thus, $\mathbb{E}[X - Y] \geq np - mp^3$, which implies that there exists a coloring where $X - Y \geq np - mp^3$. With any such coloring, for every edge in G whose all three incident vertices are red, we choose one vertex and re-color it to blue, which will result in that we have at least $X - Y$ red vertices that make an independent set. Letting $p = \sqrt{n/3m}$, we have

$$X - Y \geq np - mp^3 = \frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}},$$

completing the proof.

4 Problem 4

4.1 Question

(*) Show that there is a finite n_0 such that any directed graph on $n > n_0$ vertices in which each outdegree is at least $\log_2 n - \frac{1}{10} \log_2 \log_2 n$ contains an even simple directed cycle.

4.2 Answer

Let G be such a directed graph. Define the hypergraph H on vertex set $V = V(G)$ and edge set $F = \{N(u) : u \in V\}$ where $N(u) = \{v \in V : u \to v\} \cup \{u\}$. Clearly, H is a hypergraph where each hyperedge contains at least $\log_2 n - \frac{1}{10} \log_2 \log_2 n + 1$ vertices. If we can prove that there is a 2-coloring of H such that no edge is monochromatic, then we can find an even directed cycle as follows: first, pick any vertex $v_0 \in V$; for each v_i, let v_{i+1} be any vertex in $N(v_i)$ with a different color then v_i; repeatedly do this until we have a cycle, which must happen as the graph is finite and it's easy to see that the cycle is even. So we only need to prove that such 2-coloring exists. It's proved that for r large enough, every r-uniform hypergraph with at most $2^{r-1}\sqrt{r/\ln r}$ edges can be two-colored with no monochromatic edge (Radhakrishnan and Srinivasan, 2000). Thus, we only need to prove that $n \leq 2^{r(n)-1}\sqrt{r(n)/\ln r(n)}$ where $r(n) = \log_2 n - \frac{1}{10} \log_2 \log_2 n + 1$, for sufficiently large n. Actually,

$$\begin{aligned} 2^{r(n)-1}\sqrt{r(n)/\ln r(n)} &\geq 2^{\log_2 n - \frac{1}{10} \log_2 \log_2 n} \sqrt{\frac{\log_2 n - \frac{1}{10} \log_2 \log_2 n}{\ln \log_2 n}} \\ &\geq \frac{n}{(\log_2 n)^{1/10}} \sqrt{\frac{\log_2 n - \frac{1}{10} \log_2 \log_2 n}{\ln \log_2 n}} \\ &\geq n, \end{aligned}$$

as $\log_2 \log_2 n = o(\log_2 n)$ and $\ln \log_2 n = o(\sqrt{\log_2 n})$, completing the proof.

4.3 Reference

https://people.math.sc.edu/lu/teaching/2019spring_778
Radhakrishnan, Jaikumar, and Aravind Srinivasan. "Improved bounds and algorithms for hypergraph 2-coloring." Random Structures & Algorithms 16.1 (2000): 4-32.