2020 FALL MAS583 HW5

Fanchen Bu [20194185]

October 24, 2020

1 Problem 1

1.1 Question

(*) Prove that for every integer $d > 1$ there is a finite $c(d)$ such that the edges of any bipartite graph with maximum degree d in which every cycle has at least $c(d)$ edges can be colored by $d+1$ colors so that there are no two adjacent edges with the same color and there is no two-colored cycle.

1.2 Answer

First, we will show that for such a bipartite graph, we can find a proper edge-coloring with d colors. Then, we will use the Local Lemma to show that we can recolor some edges with a new color (the $d+1$-th color) and with positive probability, the final edge-coloring satisfies the desired conditions.
Let $\Delta(G)$ denote the maximum degree for a given graph G, and let $\chi'(G)$ denote the chromatic index of a given graph G, which is the minimum number of colors needed for a proper edge-coloring of G.

Theorem 1.1 (Vizing) *For any finite, simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

Proof. The lower bound is trivial, since if G has a vertex v of degree d, then at least d edges share v as a vertex and cannot be colored with less than d colors.
To proof the upper bound, suffice it to show the following lemma.

Lemma 1.2 *For any finite, simple graph G, let v be a vertex such that v and all its neighbors have degree at most k, while at most one neighbor has degree precisely k. Then if $G - v$ is k-edge-colorable, then G is also k-edge-colorable.*

We proof the lemma by induction on k. We can assume that each neighbor u of v has degree $k-1$, except for one of degree k, otherwise, we can add new edges until it's satisfied. Consider any k-edge-coloring of $G - v$. For $i \in [k]$, let X_i be the set of neighbors of v that are missed by color i. So all but one neighbor of v is in precisely two of the X_i, and one neighbor is in precisely one X_i. Hence

$$\sum_{i=1}^{k} |X_i| = 2\deg(v) - 1 < 2k.$$

We can assume that we have chosen the coloring such that $\sum_{i=1}^{k} |X_i|^2$ is minimized. Then for all $i, j \in [k]$,

$$||X_i| - |X_j|| \leq 2.$$

To see this, WLOG, suppose that, say, $|X_1| > |X_2| + 2$, consider the subgraph H made by all edges of color 1 and 2. Each component of H is a part or circuit. At least one component of H contains more vertices in X_1 than in X_2. This component is a path P starting in X_1 and not ending in X_2. Exchanging color 1 and 2 on P reduces $|X_1|^2 + |X_2|^2$, contradicting our assumption on its minimality. This implies that there exists an i with $|X_i| = 1$, since otherwise, each $|X_i|$ is 0 or 2, contradicting the fact that $\sum_{i=1}^{k} |X_i|$ is odd. Now we assume that $|X_k| = 1$, say $X_k = \{u\}$. Let G' be the graph obtained from G by deleting edge vu and all the edges of color k. So $G' - v$ is $(k-1)$-edge-colorable. Moreover, in G', vertex v and all its neighbors have degree at most $k-1$, and at most one neighbor has degree $k-1$. So by the induction hypothesis, G' is $(k-1)$-edge-colorable. Restoring color k, and giving edge vu color k, gives a k-edge-coloring of G. ■

Definition 1.3 *Given a graph G and an edge coloring of G, A color c is* ***absent*** *at a vertex v if no edge that has v as an endpoint receives the color c, otherwise, we say c is* ***present*** *at v.*

Theorem 1.4 (Kőnig) *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

Proof. Suppose, for contradiction, that counterexamples exist. Let G be such a counterexample, i.e., $\chi'(G) = \Delta(G) + 1$, of minimum size, which means we can find an edge of G such that removing it from G will make the remaining graph G' has $\chi'(G') = \Delta(G')$.
Let $e = \{v, w\}$ be such an edge, i.e., $\chi'(G - e) = \Delta(G - e)$. Since $G - e$ can be colored with $\Delta(G) \geq \Delta(G - e)$ colors, each of the $\Delta(G)$ colors must be assigned to an edge incident to either v or w, otherwise, with this color assigned to e, we have a proper $\Delta(G)$-edge-coloring of G.
Now, in $G - e$, $d(v) < \Delta(G)$ and $d(w) < \Delta(G)$, so there is a color c_a absent at v, and a color c_b absent at w, where $c_a \neq c_b$, and c_a is present at w and c_b is present at v. Let P be a path of maximum length having initial vertex w whose edges are alternately colored c_a and c_b. Note that $v \notin P$, since otherwise P would have odd length, implying that the first and last edges of P are both colored with c_a, contradicting that c_a is absent at v. Interchanging the colors c_a and c_b on the edges of P produces a new edge coloring of $G - e$ with $\Delta(G)$ colors in which neither v nor w is incident with an edge colored by c_a. Coloring $e = \{v, w\}$ with c_a makes a proper edge-coloring with $\Delta(G)$ colors of the original graph G, contradicting with the initial assumption, thus completing the proof. ■
Now, let $G = (V, E)$ be a bipartite graph with maximum degree d with girth $c = c(d)$. By theorem 1.4, let $f : E \to [d]$ be a proper edge-coloring of G. Then each edge of G is recolored with a new color $d + 1$ randomly and independently with probability $\frac{1}{32d}$. It remains to show that with positive probability, the following two things hold:

1. the coloring remains proper, i.e., no pair of incident edges are colored $d + 1$

2. the coloring is acyclic, i.e., every cycle of G contains at least 3 different colors

Definition 1.5 *A simple cycle D with even length is called* ***half-monochromatic*** *if half its edges (every other edge) are colored the same by the first coloring f.*

Note that it also contains the only possibility that a cycle is two-colored after f with the assumption that f is a proper edge-coloring. Now, bad events are defined as follows:

1. For each pair of incident edges $B = \{e_1, e_2\}$, let E_B be the event that both e_1 and e_2 are recolored with color $d + 1$

2. For each cycle C which two-colored by the first coloring f, let E_C be the event that no edge of C was recolored with color $d+1$

3. For each half-monochromatic cycle D, let E_D be the event that half the edges of D are recolored with color $d+1$ such that D becomes (or stays) two-colored.

We can easily check that if none of the above events happens, then we have a desired $(d+1)$-edge-coloring. Let H be the dependency graph whose vertices are all these bad events, where E_X and E_Y are adjacent if and only if X and Y intersect, since the occurrence of each E_X depends only on the edges in X. About the probability of the occurrences of the bad events, we have the following lemmas.

Lemma 1.6 *For each event E_B of type 1,*

$$\Pr[E_B] = (\frac{1}{32d})^2 = \frac{1}{1024d^2};$$

for each event E_C, where C has length x, of type 2,

$$\Pr[E_C] = (1 - \frac{1}{32d})^x \leq e^{-\frac{x}{32d}};$$

for each event E_D, where D has length $2x$, of type 3,

$$\Pr[E_D] \leq \frac{2}{(32d)^x}.$$

The above probabilities are straightforward.

Lemma 1.7 *For any given edge e, we have*

1. *Less than $2d$ edges are incident to e*

2. *Less than d two-colored cycles contain e*

3. *At most $2d^{k-1}$ half-monochromatic cycles of length $2k$ contains e*

Proof. The first two propositions are trivial with the fact that maximum degree is d and there are $d+1$ colors in total.
To prove the third one, note that every half-monochromatic cycle of length $2k$ that contains edge $e = (v_0, v_1)$ can be constructed as follows. First, select a vertex v_2 which is adjacent to v_1 (there are at most d choices). Next, decide if e or $e' = (v_1, v_2)$ belong to the "monochromatic edges" (two choices). Suppose e was chosen, let v_3 be the vertex adjacent to v_2 such $f((v_2, v_3)) = f(e)$, at most one such v_3 exists since f is a proper edge-coloring. Now continue with $i = 2, ..., k-1$: choose v_{2i} to be any vertex adjacent to v_{2i-1} (at most d choices), and let v_{2i+1} be the vertex adjacent to v_{2i} such that $f((v_{2i}, v_{2i+1})) = f(e)$, completing the construction of the desired cycle. The case where e' belongs to the "monochromatic edges" is treated exactly the same after swapping v_0 with v_2. Therefore, the number of half-monochromatic cycles of length $2k$ that contain e is at most $2d^{k-1}$. ■
It follows from Lemma 1.7 that each event E_X where X contains x edges is adjacent to at most $2xd$ events of type 1, at most xd events of type 2, and at most $2xd^{k-1}$ events E_D with $|D| = 2k$ for all $k \geq 2$ of type 3, in H. Now we only need to define the x_i used in the Local Lemma. Let $x_1 = \frac{1}{512d^2}$ be the x_i used for all events of type 1, let $x_2 = \frac{1}{128d^2}$ be the x_i used for all events of type 2, and let $x_{3,k} = \frac{1}{(2d)^k}$ be the x_i used for events E_D with $|D| = 2k$ of

type 3. Now, let $c = c(d) = 2000d \log d$. By the Local Lemma, we only need to show the following inequalities:

$$\frac{1}{1024d^2} \leq \frac{1}{512d^2}(1 - \frac{1}{512d^2})^{4d}(1 - \frac{1}{128d^2})^{2d}\prod_k(1 - \frac{1}{(2d)^k})^{4d^{k-1}}, \tag{1}$$

$$e^{-\frac{x}{32d}} \leq \frac{1}{128d^2}(1 - \frac{1}{512d^2})^{2xd}(1 - \frac{1}{128d^2})^{xd}\prod_k(1 - \frac{1}{(2d)^k})^{2xd^{k-1}}, \forall x \geq 4, \tag{2}$$

$$\frac{2}{(32d)^x} \leq \frac{1}{(2d)^x}(1 - \frac{1}{512d^2})^{4xd}(1 - \frac{1}{128d^2})^{2xd}\prod_k(1 - \frac{1}{(2d)^k})^{4xd^{k-1}}, \forall x \geq 2. \tag{3}$$

With $(1 - (1/z))^z \geq 1/4$ for all real $z \geq 2$, we have for all $x, d \geq 2$:

$$\prod_k(1 - \frac{1}{(2d)^k})^{2xd^{k-1}} \geq \prod_k(\frac{1}{4})^{x/(d2^{k-1})} = (\frac{1}{4})^{(x/d)\sum_k 2^{1-k}} \geq (\frac{1}{4})^{x/(256d)}, \tag{4}$$

where the last inequality uses the fact that $2k \geq c \geq 2000d \log d \geq 20$, and similarly

$$(1 - \frac{1}{512d^2})^{2xd} \geq (\frac{1}{4})^{x/(256d)}, \tag{5}$$

$$(1 - \frac{1}{128d^2})^{xd} \geq (\frac{1}{4})^{x/(128d)}, \tag{6}$$

Combining (4)-(6), we conclude that

$$(1 - \frac{1}{512d^2})^{2xd}(1 - \frac{1}{128d^2})^{xd}\prod_k(1 - \frac{1}{(2d)^k})^{2xd^{k-1}} \geq (\frac{1}{2})^{x/(32d)}.$$

Thus inequality (1) holds since $2 \geq 2^{1/(16d)}$, and inequality (3) holds since $2 \leq 2^{5x-x-x/(16d)}$ for all $x \geq 1$. To prove (2), suffice it to show that

$$e^{-x/(32d)} \leq \frac{1}{128d^2}(\frac{1}{2})^{x/(32d)},$$

which holds for all $x \geq 2000d \log d \geq 32d(\log(128d^2)/\log(e/2))$ and $d > 2$, completing the proof.

1.3 A Failed Attempt

Note that if for each pair of adjacent edges e_1 and e_2 of G, e_1 and e_2 are colored with different colors (the coloring is a proper edge-coloring), then it's easy to check that any cycle with odd length must contain edges with at least 3 colors.

Let's consider cycles with even length with the assumption that the coloring is a proper edge-coloring. Clearly, if we want such a cycle to be two-colored, the coloring should be alternative, i.e., say cycle $C = v_0v_1v_2...v_{2k} = v_0$, then the color of edges (v_i, v_{i+1}) for all odd i should be the be same, and the color for all even i also should be the same. Note that we only need to care about induced cycles as each diagonal cycle can be divided into smaller induced cycles. Once we fix the $c = c(d)$ in the condition, we can claim that $2k \geq c$.

Let $G = (V = X \cup Y, E)$ be the bipartite graph, where $X \cup Y$ are the disjoint partition of $V = V(G)$. We assume that $|X|, |Y| \geq 2$, otherwise the problem is trivial. Note that if $c > 4$, then for any two vertices in the same class (X or Y), they can only have at most one common neighbour (in the other class). Thus, we can claim that for any even $l \geq c(d)$, and for any edge $e \in E$, the number of induced cycles of length l in G containing e is at most $(d-1)^{l-3}$, to see this, first, the number of simple paths of length $l-2$ starting by e and ending in another edge that is nonadjacent with e is at most $2(d-1)^{l-3}$, each such path can be completed to at most one induced cycle of length l in G, as noted. Moreover, in this manner each induced cycle of length l containing e is counted twice, which gives us the result above.

Let $f : E \to [d+1]$ be a random edge-coloring of G, where for each $e \in E$, $f(e)$ is chosen in $[d+1]$ uniformly at random. For each pair of adjacent edges (t, u), let $A_{\{t,u\}}$ be the A-type event that $f(t) = f(u)$. Similarly, for each induced even cycle C of length $2k \geq c$ in G whose edges are $e_1, e_2, ..., e_{2k}$, let B_C be the B-type event that $f(e_1) = f(e_3) = ... = f(e_{2k-1})$ and $f(e_2) = f(e_4) = ... = f(e_{2k})$. As discussed before, if none of these events holds, then f is an edge-coloring we want. Therefore, we need to show that with positive probability, none of these events occur.

Let H be the dependency graph whose vertices are all these events $A_{\{t,u\}}$ and B_C, clearly, two vertices are adjacent if and only if their corresponding edge sets intersect. With that and the discussion above, it's easy to check that each $A_{\{t,u\}}$ is adjacent to at most $4d-6$ other events of A-type in H, and at most $2(d-1)^{l-3}$ events of B-type whose corresponding cycle has length l for each $l \geq c$ in H; and each B_C, where C has length l, is adjacent to at most $l(2d-3)$ events of A-type in H, and at most $l(d-1)^{l'-3}$ events of B-type whose corresponding cycle has length l' for each $l' \geq c$ in H.

About the probability that each "A-type" event or "B-type" event happens, we have for each event A of "A-type", $\Pr[A] = \frac{1}{d+1}$, for each event B_l of "B-type" whose corresponding cycle has length l, $\Pr[B_l] = \frac{1}{(d+1)^{l-2}}$.

Now we need to define the x_i used in the Local Lemma. Suppose we let x_A be the x_i used for each A-type event and x_l be the one used for each B-type event whose corresponding cycle has length l. To use the Local Lemma, we need

$$\Pr[A] = \frac{1}{d+1} \leq x_A(1-x_A)^{4d-6} \prod_{l \geq c} (1-x_l)^{2(d-1)^{l-3}}$$

and

$$\Pr[B_l] = \frac{1}{(d+1)^{l-2}} \leq x_l(1-x_A)^{l(2d-3)} \prod_{l' \geq c} (1-x_{l'})^{l(d-1)^{l'-3}}, \forall l \geq c.$$

1.4 References

[1] Alon, Noga, Colin Mcdiarmid, and Bruce Reed. "Acyclic coloring of graphs." Random Structures & Algorithms 2.3 (1991): 277-288.

[2] Basavaraju, Manu, and L. Sunil Chandran. "A note on acyclic edge coloring of complete bipartite graphs." Discrete mathematics 309.13 (2009): 4646-4648.

[3] Alon, Noga, Benny Sudakov, and Ayal Zaks. "Acyclic edge colorings of graphs." Journal of Graph Theory 37.3 (2001): 157-167.

[4] Cai, Xing Shi, et al. "Acyclic edge colourings of graphs with large girth." Random Structures & Algorithms 50.4 (2017): 511-533.

[5] Kőnig, Dénes. "Graphok és matrixok." Mat. Fiz. Lapok 38 (1931): 116-119.

[6] Szárnyas, Gábor. "Graphs and matrices: A translation of" Graphok és matrixok" by Dénes Kőnig (1931)." arXiv preprint arXiv:2009.03780 (2020).

2 Problem 2

2.1 Question

(*) Prove that for every $\epsilon > 0$ there is a finite $l_0 = l_0(\epsilon)$ and an infinite sequence of bits $a_1, a_2, a_3, ...,$ where $a_i \in \{0, 1\}$, such that for every $l > l_0$ and every $i \geq 1$ the two binary vectors $u = (a_i, a_{i+1}, ..., a_{i+l-1})$ and $v = (a_{i+l}, a_{i+l+1}, ..., a_{i+2l-1})$ differ in at least $(\frac{1}{2} - \epsilon)l$ coordinates.

2.2 Answer

Let ϵ be fixed and let $l_0 = l_0(\epsilon)$ to be decided later. Let $A = a_1, a_2, a_3, ...$ be a random infinite sequence of bit, where each a_i is chosen in $\{0, 1\}$ independently and uniformly at random, we want to compute the probability that for every $l > l_0$ and every $i \geq 1$, the two binary vectors $u = (a_i, a_{i+1}, ..., a_{i+l-1})$ and $v = (a_{i+l}, a_{i+l+1}, ..., a_{i+2l-1})$ differ in at least $(\frac{1}{2} - \epsilon)l$ coordinates, and use the probability to decide our l_0.
For every $l > l_0$ and every $i \geq 1$, let $E_{l,i}$ be the event that the corresponding u and v differ in less than $(\frac{1}{2} - \epsilon)l$ coordinates. Clearly, any two events $E_{l,i}$ and $E_{l',i'}$ are independent to each other unless intervals $[i, i + 2l - 1]$ and $[i', i' + 2l' - 1]$ intersect. Given l and i, the two interval intersect if and only if

1. When $i' < i$, need $l' \geq (i - i' + 1)/2$ (and certainly $l' > l_0$)
2. When $i \leq i' \leq i + 2l - 1$, it holds for all $l'(> l_0)$
3. When $i' > i + 2l - 1$, impossible

And the probability that $E_{l,i}$ occurs is

$$\Pr[E_{l,i}] = \Pr[B(1/2, l) > (1/2 + \epsilon)l] = \binom{l}{\lceil 1/2 + \epsilon \rceil} 2^{-l} \leq \frac{1}{(1 + 4\epsilon^2)^{l/2}}.$$

Let H be the dependency graph whose vertices are all these events $E_{l,i}$ (for all $l > l_0$ and $i \geq 1$), where two vertices are adjacent if and only if their corresponding events depend on each other, as we analyzed. Now we need to define the x_i used in the Local Lemma. Let

$$x_{l,i} = \frac{1}{(1 + \epsilon^2)^{l/2}}$$

be the x_i for the event $E_{l,i}$. Now suffice it to show

$$\Pr[E_{l,i}] \leq \frac{1}{(1 + 4\epsilon^2)^{l/2}} \leq \frac{1}{(1 + \epsilon^2)^{l/2}} \prod_{m > l_0} (1 - \frac{1}{(1 + \epsilon^2)^{m/2}})^{i+2l-1}, \tag{7}$$

note that

$$\prod_{m > l_0} (1 - \frac{1}{(1 + \epsilon^2)^{m/2}})$$

tends to 1 as l_0 tends to infinity. To state it more clearly, we say, for any given $\epsilon_0 > 0$, we can find $M_0 = M(\epsilon_0)$ such that, when $l_0 > M_0$,

$$\prod_{m > l_0} (1 - \frac{1}{(1 + \epsilon^2)^{m/2}}) > 1 - \epsilon_0,$$

at the same time, for the same ϵ_0, we can find $M_1 = M_1(\epsilon_0)$ such that, when $l_0 > M_1$,

$$(\frac{1+\epsilon^2}{1+4\epsilon^2})^{l/2} < (\frac{1+\epsilon^2}{1+4\epsilon^2})^{l_0/2} < 1-\epsilon_0.$$

Now letting $l_0 = \max(M_0, M_1) + 1$ will solve the problem, completing the proof.

2.3 References

[1] Pegden, Wesley. "Highly nonrepetitive sequences: Winning strategies from the local lemma." Random Structures & Algorithms 38.1-2 (2011): 140-161.
[2] Beck, J. "An application of Lovász local lemma: there exists an infinite 01-sequence containing no near identical intervals." Finite and Infinite Sets. North-Holland, 1984. 103-107.

3 Problem 3

3.1 Question

Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from it class $S(v)$.

3.2 Answer

First, for each vertex $v \in V$, we randomly choose $\tilde{S}(v) \subset S(v)$ with $|\tilde{S}(v)| = 10d$, clearly, with the assumption, still for each $v \in V$ and $c \in \tilde{S}(v) \subset S(v)$ there are at most d neighbors u of v such that c lies in $\tilde{S}(u) \subset S(u)$. And suffice it to show that there is a proper coloring of G assigning to each vertex v a color from $\tilde{S}(v) \subset S(v)$.
Let $S = \bigcup_{v \in V} \tilde{S}(v)$, for each $e \in E$ and $c \in S$, let $A_{e,c}$ be the event that both endpoints of e are colored with the color c. Suppose $e = (v_1, v_2)$, note that, first, it's only possible when $c \in \tilde{S}(v_1) \cup \tilde{S}(v_2)$; second, $A_{e,c}$ is independent of $A_{e',c'}$ for any e' such that $e \cap e' = \emptyset$, i.e., for each color in $\tilde{S}(v_1)$, as at most d neighbors of v_1 share the same color, thus $A_{e,c}$ is dependent of at most $10d^2$ (including $A_{e,c}$ itself) such $A_{e',c'}$ with $v_1 \in e'$, similarly, $A_{e,c}$ is dependent of at most $10d^2$ such $A_{e'',c''}$ with $v_2 \in e''$.
Now we do the random coloring by choosing one color $c \in \tilde{S}(v)$ uniformly at random for each $v \in V$. For each $e \in E$, we only consider events $A_{e,c}$ for $c \in$ as we noted. As we mentioned, we can claim that for each $e = (v_1, v_2) \in E$ and $c \in \tilde{S}(v_1) \cap \tilde{S}(v_2)$, event $A_{e,c}$ is independent of all but at most $20d^2 - 2$ other events and $\Pr[A_{e,c}] = 1/(10d)^2 = 1/(100d^2)$. By the symmetric case of the Local Lemma, as

$$e\frac{1}{100d^2}(20d^2 - 1) \leq 1,$$

we have

$$\Pr[\bigcap_{e=(v_1,v_2) \in E, c \in \tilde{S}(v_1) \cap \tilde{S}(v_2)} \overline{A_{e,c}}] > 0,$$

which implies that with positive probability, we can find a proper coloring of G assigning to each vertex v a color from $\tilde{S}(v) \subset S(v)$, completing the proof.

4 Problem 4

4.1 Question

Let $G = (V, E)$ be a cycle of length $4n$ and let $V = V_1 \cup V_2 \cup ... \cup V_n$ be a partition of its $4n$ vertices into n pairwise disjoint subsets, each of cardinality 4. Is it true that there must be an independent set of G containing precisely one vertex from each V_i? (Prove or supply a counter example.)

4.2 Answer

Suppose $V = V(G) = [4n]$ and $E = E(G) = \{(1,2),(2,3),...,(4n-1,4n),(4n,1)\}$. Let $E = E_0 \cup E_1$ be a partition of E, where $E_0 = \{(1,2),(3,4),...,(4n-1,4n)\}$ is the set of $2n$ edges starting from odd vertices, and $E_1 = \{(2,3),(4,5),...,(4n,1)\}$ is the set of the other $2n$ edges starting from even vertices.
Given a partition $V = V_1 \cup V_2 \cup ... \cup V_n$, we arbitrarily partition each V_i again $V_i = W_{2i-1} \cup W_{2i}$ for each $i \in [n]$, where $|W_{2i-1}| = |W_{2i}| = 2$ and $W_{2i-1} \cap W_{2i} = \emptyset$. Now, we choose $w_i \in W_i$ for each $i \in [2n]$ such that if $i \neq j$, then w_i and w_j are not joined by an edge in E_0 with the following algorithm:

1. Initialize $D = \emptyset$
2. For each $i \in [2n]$, suppose $W_i = (x_i, y_i)$, if x_i and y_i are joined by an edge in E_0, then we just arbitrarily choose one of them as w_i and add i into D
3. Now, if $D = [2n]$, then we are done. Otherwise, we randomly choose $i \in [2n] \setminus D$, arbitrarily choose $w_i \in W_i$, and add i into D. Let $l = i$ and repeat the following until $D = [2n]$
4. Find the edge $e \in E_0$ containing w_l, let w' be the other endpoint of e and find j such that $w' \in W_j$ (because of what we did in step 2, we must have $i \neq j$), we choose $w_j \in W_j$ that is not w', add j into D, and let $l = j$

Now we have a set $\{w_1, ..., w_{2n}\}$ such that if $i \neq j$, then w_i and w_j are not joined by an edge in E_0. Let $\tilde{V_i} = \{w_{2i-1}, w_{2i}\} \subset V_i$ for each $i \in [n]$, we choose $v_i \in \tilde{V_i}$ for each $i \in [n]$ such that if $i \neq j$, then v_i and v_j are not joined by an edge in E_1 with a similar algorithm, where we just replace $[2n]$ with $[n]$, W_i with $\tilde{V_i}$, and E_0 with E_1. Note that for each $i, j \in [n]$ with $i \neq j$, v_i and v_j are not joined by an edge in E_0 neither, as they are also some $w_{i'}$ and $w_{j'}$ such that $i' \neq j'$. Now we can claim that for each $i, j \in [n]$ with $i \neq j$, v_i and v_j are not joined by an edge in E, as $E = E_0 \cup E_1$, completing the proof.

4.3 References

`https://math.stackexchange.com/questions/788705`

5 Problem 5

5.1 Question

(*) Prove that there is an absolute constant $c > 0$ such that for every k there is a set S_k of at least $ck \ln k$ integers, such that for every coloring of the integers by k colors there is an integer x for which the set $x + S$ does not intersect all color classes.

5.2 Answer

Definition 5.1 *For a set of integers S, let $H = H(S)$ denote the infinite hypergraph whose set of vertices is $\mathbb{Z}$ and whose set of edges is the set of all translates of S, i.e., the set $\{x + S : x \in \mathbb{Z}\}$. We call H the **shift hypergraph** of S. A k-coloring $f : \mathbb{Z} \to [k]$ is called **good** (for H), if each edge of H meets every color class, i.e., for each $i \in [k]$ and for each $x \in \mathbb{Z}$, there is an $s \in S$ so that $f(x + s) = i$.*

With the definitions above, now we need to show the following theorem.

Theorem 5.2 *There exists an absolute constant $c > 0$ such that for every k there is a set $S = S_k$ of at least $ck \ln k$ integers, such that there is no good k-coloring for the shift hypergraph $H(S)$.*

Let $g = g(k)$ denote the minimum integer so that for every set S of at least g integers there is a good k-coloring for the shift hypergraph $H(S)$. In 1975, Erdős showed that

$$g(k) \le (3 + o(1))k \ln k.$$

Proposition 5.3 *If q is a prime, and $q > l^2 2^{2l-2}$ then*

$$g(\lceil \frac{2q}{l+1} \rceil) > \frac{q+1}{2}.$$

Proof. First, note that this proposition implies that for large k:

$$g(k) \ge (\frac{1}{8} + o(1))k \log_2 k.$$

Lemma 5.4 *Let $q > l^2 2^{2l-2}$ be a prime and let $Z = \{z_1, ..., z_s\}$ be a set of $s \le l$ members of Z_q. Then there exists an element $y \in Z_q$ so that $z_i - y$ is a quadratic non-residue for all $i \in [s]$.*

Put $m = \frac{q+1}{2}$ and let S be the set of all m quadratic residues modulo q, considered here as as usual integers. Suppose $k \ge \frac{2q}{l+1}$ and let $f : \mathbb{Z} \to K = [k]$ be a k-coloring of the integers. To complete the proof we show that there exists a translate of S that misses at least one color class. To this end, consider the colors of the integers in the set $Q = \{0, 1, ..., 2q - 2\}$ and let $j \in K$ be a color assigned to at most $|Q|/k < l + 1$ members of this set. Let $y_1, ..., y_s$ with $s \le l$ be all the members of Q satisfying $f(y_i) = j$. Let z_i be the elements of Z_q defined by $z_i \equiv y_i \pmod{q}$. By lemma 5.4, there exists a $y \in Z_q$ so that $z_i - y$ is a quadratic non-residue for all $i \in [s]$. Consider, now, y as a usual integer. We claim that the translate $y + S$ of S does not intersect color class number j. To see this, suppose it is false. Since $y + S \subset Q$, this means that there is an $i \in [s]$, and there is an $x \in S$ so that $y + x = y_i$. Reducing this equation modulo q we conclude that $x \equiv (z_i - y) \pmod{q}$. But this is impossible since $z_i - y$ is a quadratic non-residue whereas x (actually, any member of S) is a quadratic residue. Thus the claim holds, completing the proof. ■

Proposition 5.3 implies the proposition we want.

5.3 Another Answer

Proposition 5.5 *For large k,*

$$g(k) \geq (1 + o(1))k \ln k.$$

Proof. To prove this, we need the following lemma.

Lemma 5.6 *For every fixed (small) $\epsilon > 0$ there exists a (small) $\delta > 0$ such that for every sufficiently large n, there exists a subset $S \subset N = [n]$ with $|S| \geq (1 - \frac{\epsilon}{10})\delta n$ so that for every set T of at most*

$$\frac{(1 - \frac{\epsilon}{10}) \ln n}{(1 + \frac{\epsilon}{10})\delta}$$

positive integers, each at most $(1 + \frac{\epsilon}{10})n$, there is an integer $0 \leq y \leq \frac{\epsilon}{10}n$ so that $y + S$ does not intersect T.

Proof. Let $\delta > 0$ satisfy

$$1 - \delta > e^{-(1+\frac{\epsilon}{10})\delta}.$$

Let $n = n(\epsilon, \delta)$ be sufficiently large and let $S \subset N = [n]$ be a random subset obtained by choosing each $i \in N$, randomly and independently, to be a member of S with probability δ. Let m denote the cardinality of S. By the standard estimates for Binomial distributions, as n is sufficiently large, with high probability, we have

$$m \geq (1 - \frac{\epsilon}{10})\delta n.$$

Fix a set T of at most

$$\frac{(1 - \frac{\epsilon}{10}) \ln n}{(1 + \frac{\epsilon}{10})\delta}$$

positive integers, each at most $(1 + \frac{\epsilon}{10})n$, and fix an integer $y \leq \frac{\epsilon}{10}n$. The probability that $y + S$ does not intersect T is the probability that $t - y$ is not in S for all $t \in T$, which is, with the assumption we made on δ, at least

$$(1 - \delta)^{|T|} > e^{-(1+\frac{\epsilon}{10})\delta \frac{(1-\frac{\epsilon}{10}) \ln n}{(1+\frac{\epsilon}{10})\delta}} = \frac{1}{n^{1-\frac{\epsilon}{10}}}.$$

For the above fixed set T consider now all the possible shift of S by an integer y satisfying $0 \leq y \leq \frac{\epsilon}{10}n$. For each such y we have the estimate above for the probability of the event E_y that $y + S$ does not intersect T. Moreover, if Y is a set of possible shifts and for every two distinct y and y' in Y, $T - y$ does not intersect $T - y'$, thus, the events E_y for all $y \in Y$ are mutually independent. It is easy to see that there is such a set Y of cardinality at least $\frac{\epsilon}{10}n/|T|^2 = \Omega(n/(\log n)^2)$, where here the constant in the Ω notation depends only on ϵ and δ. It follows that the probability that there is no shift y in the possible range so that $y + S$ does not intersect T is at most

$$(1 - \frac{1}{n^{1-\frac{\epsilon}{10}}})^{\Omega(n/(\log n)^2)} = e^{-\Omega(n^{\frac{\epsilon}{10}}/(\log n)^2)}.$$

The total number of choices for a subset T as above is only

$$\sum_{i \leq \frac{(1-\frac{\epsilon}{10}) \ln n}{(1+\frac{\epsilon}{10}\delta)}} \binom{(1 + \frac{\epsilon}{10}n)}{i} \leq e^{O((\log n)^2)}.$$

Therefore, the probability that there is a set T so that there is no shift $y+S$ of S that misses it is at most

$$e^{O((\log n)^2)}e^{-\Omega(n^{\frac{\epsilon}{10}/(\log n)^2})},$$

which tends to 0 as n tends to infinity, completing the proof. ∎

Let ϵ be a fixed small positive constant. Our objective is to show that for all sufficiently large k, $g(k) \geq (1-\epsilon)k\ln k$. Let δ, n, and S satisfy the assertion of lemma 5.6. We assume, whenever it is needed, that ϵ is sufficiently small and that n is sufficiently large. Put $m = |S|$, then

$$n \geq m \geq (1-\frac{\epsilon}{10})\delta n.$$

Let k be an integer and let $f : \mathbb{Z} \to K = [k]$ be a good k-coloring for the hypergraph $H(S)$. Clearly $k \leq m \leq n$. We claim that

$$k < \frac{(1+\frac{\epsilon}{10})^2\delta n}{(1-\frac{\epsilon}{10})\ln n} \leq \frac{(1+\epsilon)m}{\ln n} \leq \frac{(1+\epsilon)m}{\ln k},$$

and hence that,

$$m \geq (1-\epsilon)k\ln k.$$

Since $\epsilon > 0$ is arbitrarily small (and for each such ϵ any sufficiently large n can be chosen), together with the obvious monotonicity of the function $g(k)$, imply the validity of our proposition 5.5. It thus remains to prove the claim. Let Q be the set of all positive integers which do not exceed $(1+\frac{\epsilon}{10})n$. Fix a color $i \in K$ and let T be the set of all members of Q colored i. If

$$|T| \geq \frac{(1-\frac{\epsilon}{10}\ln n)}{(1+\frac{\epsilon}{10})\delta},$$

then since S satisfies the assertion of lemma 5.6, there is a translate $y+S$ of S contained in Q which misses T, contradicting the assumption that f is a good k-coloring for $H(S)$. Therefore, each of the k colors appears more than

$$\frac{(1-\frac{\epsilon}{10}\ln n)}{(1+\frac{\epsilon}{10})\delta}$$

times in Q and hence

$$(1+\frac{\epsilon}{10})n \geq |Q| \geq k\frac{(1-\frac{\epsilon}{10})\ln n}{(1+\frac{\epsilon}{10})\delta},$$

which implies the claim we made, completing the proof. ∎

5.4 References

[1] Alon, Noga, Igor Kriz, and J. Nesetril. "How to color shift hypergraphs." Studia Scientiarum Mathematicarum Hungarica 30.1 (1995): 1-12.

[2] Alon, Noga, and Yuval Peres. "Uniform dilations." Geometric & Functional Analysis GAFA 2.1 (1992): 1-28.

[3] Erdős, Paul, and László Lovász. "Problems and results on 3-chromatic hypergraphs and some related questions." COLLOQUIA MATHEMATICA SOCIETATIS JANOS BOLYAI 10. INFINITE AND FINITE SETS, KESZTHELY (HUNGARY). 1973.

[4] Graham, Ronald L., and Joel H. Spencer. "A constructive solution to a tournament problem." Canadian Mathematical Bulletin 14.1 (1971): 45-48.