2020 FALL MAS583 HW9 (CH15: Codes, Games and Entropy)

Fanchen Bu [20194185]

December 10, 2020

1 Problem 1

1.1 Question

Suppose that in the $(x_1, ..., x_k)$-Tenure Game of Section 15.3 the object of Paul is to maximize the number of faculty receiving tenure while the object of Carole is to minimize that number. Let v be that number with perfect play. Prove $v = \lfloor \sum_{i=1}^{k} x_i 2^{-i} \rfloor$.

1.2 Answer

Suffice it to show that for a fixed natural number v, Paul can always maximize the number to be at least v iff $\sum_{i=1}^{k} x_i 2^{-i} \geq v$.

Lemma 1.1 *If $\sum_{i=1}^{k} x_i 2^{-i} < v$ then Paul cannot always maximize the number to be at least v. In other words, regardless of Paul's strategy, Carole always has a strategy to minimize the number strictly below v. Specifically, it holds for $v = \lfloor \sum_{i=1}^{k} x_i 2^{-i} \rfloor + 1$.*

Proof. Fix a strategy for Paul. Suppose that Carole plays randomly in the sense that at each round, after Paul has selected a set S of chips Carole flips a fair coin. Carole chooses to promote all chips in S and fire the others if the coin shows head and chooses the other choice if the coin show tail. For each chip c, let I_c be the indicator random variable for c reaching level 0, and set $X = \sum_c I_c$. For a chip c starting at level j, at each round it has probability 1/2 to be promoted and probability 1/2 to be fired, which results in that $\mathbb{E}[I_c] = 2^{-j}$, implying that $\mathbb{E}[X] = \sum_{i=1}^{k} x_i 2^{-i} < v$. So we have that $X < v$ happens with positive probability, which means Carole always has some strategy to make that happen, completing the proof. ■

At some point of the game, we have y_i chips on level i, and we call $\sum_{i=1}^{k} y_i 2^{-i}$, the expected number of chips that would reach level 0 if Carole plays the rest of the game randomly in the way we mentioned, the weight of the chips.

Lemma 1.2 *For a given natural number v, if a set of chips has weight at least v, then it may split into two parts with each of weight at least $v/2$.*

Proof. There must be at least $2v$ chips at some position, otherwise the weight is less than v. If there are $2v$ chips at level 1 simply split them uniformly. If there are $2v$ chips at position $i > 1$ glue them together, and consider them as v superchips at position $i - 1$. Then by induction we can complete the proof. ■

Lemma 1.3 *If $\sum_{i=1}^{k} x_i 2^{-i} \geq v$ then Paul can always maximize the number to be at least v. Specifically, it hold for $v = \lfloor \sum_{i=1}^{k} x_i 2^{-i} \rfloor$.*

Proof. The assumption means that the initial weight is at least v. By Lemma 1.2, Paul can split the chips into two parts with each of weight at least $v/2$, and set S as any one of them. No matter what Carole chooses, there is a part of chips has its weight doubled, implying that the weight is still at least v. By repeating this, the weight never goes below v. Therefore, the game cannot end with less than v chips, completing the proof. ■
In the original statement, we proved $v \leq \lfloor \sum_{i=1}^{k} x_i 2^{-i} \rfloor$ by Lemma 1.1 and $v \geq \lfloor \sum_{i=1}^{k} x_i 2^{-i} \rfloor$ by Lemma 1.3, thus completing the whole proof.

2 Problem 2

2.1 Question

Let $A_1, ..., A_n \subseteq \{1, ..., m\}$ with $\sum_{i=1}^{n} 2^{-|A_i|} < 1$. Paul and Carole alternately select distinct vertices from $\{1, ..., m\}$, Paul having the first move, until all vertices have been selected. Carole wins if she has selected all the vertices of some A_i. Paul wins if Carole does not win. Give a winning strategy for Paul.

2.2 Answer

Let P denote the set of vertices that Paul has chosen, and let C denote the set of vertices that Carole has chosen. We define the weight of the state of the game as

$$S = S(P, C) = \sum_{i: A_i \cap P = \emptyset} 2^{-|A_i \setminus C|}.$$

Note that at first $P = C = \emptyset$, which gives $S = \sum_{i=1}^{n} 2^{-|A_i|} < 1$. Clearly, when Carole wins, say Carole has selected all the vertices of A_j, we have $S \geq 2^{-|A_j \setminus C|} = 1$. And we can observe that each move of Carole can at most double S as it can at most double each term. Suppose at some point, we have

$$\sum_{i: A_i \cap P = \emptyset} 2^{-|A_i \setminus C|} < 1$$

and now it's Paul's turn to add a vertex into P. Clearly, if there exists some A_i such that $A_i \cap P = \emptyset$ and $|A_i \setminus C| = 1$, then simply selecting the only element in $A_i \setminus C$ will make $S < 1/2$, which means Carole cannot make $S \geq 1$ in one step. In other cases, for each $v \in [m] \setminus (P \cup C)$, let

$$S_v = \sum_{i: A_i \cap P = \emptyset, v \in A_i} 2^{-|A_i \setminus C|},$$

and Paul can choose v such that S_v is maximized. By doing this, Paul actually makes S decrease by S_v and by the choice of v, Carole's next move can at most double $S_{v'} \leq S_v$, implying that after both moves, the new S is at most the previous one, thus we still have $S < 1$, completing the proof.

3 Problem 3

3.1 Question

Let $\mathcal{F}$ be a family of graphs on the labeled set of vertices $\{1, 2, ..., 2t\}$, and suppose that for any two members of $\mathcal{F}$ there is a perfect matching of t edges contained in both of them. Prove that

$$|\mathcal{F}| \leq 2^{\binom{2t}{2}-t}.$$

3.2 Answer

First, we introduce the following lemma given in the textbook as Corollary 15.7.7.

Lemma 3.1 *Let N be a finite set, and let $\mathcal{F}$ be a family of subsets of N. Let $\mathcal{G} = \{G_1, ..., G_m\}$ be a collection of subsets of N, and suppose that each element of S belongs to at least k members of $\mathcal{G}$. For each $i \in [m]$ define $\mathcal{F}_i = \{F \cap G_i : F \in \mathcal{F}\}$. Then*

$$|\mathcal{F}|^k \leq \prod_{i=1}^{m} |\mathcal{F}_i|.$$

Now, let $N = \binom{[2t]}{2}$ and consider $\mathcal{F}$ as a family of subsets of N. Let $\mathcal{G}$ be the family all subsets of N consisting of spanning stars $G_1, ..., G_{2t}$, where G_i is the spanning star at vertex i. Let $s = 2t - 1$ denote the number of edges of such a star. Clearly, each edge in N lies in precisely 2 members of $\mathcal{G}$. The crucial point is that every two graphs in $\mathcal{F}$ must have (exactly) one common edge in each $G \in \mathcal{G}$, since there intersection contains a perfect matching and there are no perfect matchings in the complement of G. Thus, we have $|\mathcal{F}_i| \leq 2^{s-1} = 2^{2t-2}$ for each i. By Lemma 3.1, we have

$$|\mathcal{F}|^2 \leq (2^{2t-2})^{2t},$$

which gives

$$|\mathcal{F}| \leq 2^{t(2t-2)} = 2^{\binom{2t}{2}-t},$$

completing the proof.

4 Problem 4

4.1 Question

(Han's Inequality.) Let $X = (X_1, ..., X_m)$ be a random variable and let $H(X)$ denote its entropy. For a subset I of $\{1, 2, ..., m\}$, let $X(I)$ denote the random variable $(X_i)_{i \in I}$. For $1 \leq q \leq m$, define

$$H_q(X) = \frac{1}{\binom{m-1}{q-1}} \sum_{Q \subset \{1,...,m\}, |Q|=q} H(X(Q)).$$

Prove that

$$H_1(X) \geq H_2(X) \geq \cdots \geq H_m(X) = H(X).$$

4.2 Answer

For $I = \{i_1, ..., i_n\} \subset [m]$, we let I_k denote $\{i_1, ..., i_{k-1}\}$ ($I_1 = \emptyset$) and let I^k denote $I \setminus \{i_k\}$ for each $k \in [n]$. Now we claim the following:

$$H(X(I)) \leq \frac{1}{n-1} \sum_{k=1}^{n} H(X(I^k)).$$

To see this, first we express

$$H(X(I)) = H(X(I^k)) + H(x_{i_k}|X(I^k)).$$

By $H(X|Y, Z) \leq H(X|Y)$, we have

$$H(x_{i_k}|X(I^k)) \leq H(x_{i_k}|X(I_k)),$$

implying that

$$H(X(I)) \leq H(X(I^k)) + H(x_{i_k}|X(I_k)).$$

By summing both sides over $k \in [n]$, we have

$$nH(X(I)) \leq \sum_{k=1}^{n} (H(X(I^k)) + H(x_{i_k}|X(I_k))).$$

And by chain rule, we have

$$\begin{aligned} H(X(I)) &= H(x_{i_n}|X(I_n)) + H(X(I_n)) \\ &= H(x_{i_n}|X(I_n)) + H(x_{i_{n-1}}|X(I_{n-1})) + H(X(I_{n-1})) \\ &= \cdots \\ &= \sum_{k=1}^{n} H(x_{i_k}|X(I_k)), \end{aligned}$$

which implies that

$$nH(X(I)) \leq \sum_{k=1}^{n} H(X(I^k)) + H(X(I)),$$

completing the proof of our claim. With that, for given $1 < q \leq m$ we have

$$\begin{aligned} \sum_{Q \subset \{1,...,m\}, |Q|=q} H(X(Q)) &\leq \sum_{Q \subset \{1,...,m\}, |Q|=q} \frac{1}{q-1} \sum_{k=1}^{q} H(X(Q^k)) \\ &= \frac{m-q+1}{q-1} \sum_{Q \subset \{1,...,m\}, |Q|=q-1} H(X(Q)) \\ &= \frac{\binom{m-1}{q-1}}{\binom{m-1}{q-2}} \sum_{Q \subset \{1,...,m\}, |Q|=q-1} H(X(Q)), \end{aligned}$$

completing the proof.

5 Problem 5

5.1 Question

Let $X_i = \pm 1$, $1 \leq i \leq n$, be uniform and independent and let $S_n = \sum_{i=1}^{n} X_i$. Let $0 \leq p \leq \frac{1}{2}$. Prove

$$\Pr[S_n \geq (1-2p)n] \leq 2^{H(p)n} 2^{-n}$$

by computing precisely the Chernoff bound $\min_{\lambda \geq 0} \mathbb{E}[e^{\lambda S_n}] e^{-\lambda(1-2p)n}$. (The case $p = 0$ will require a slight adjustment in the method though the end result is the same.)

5.2 Answer

When $p = 0$, we have $H(p) = 0$ and

$$\Pr[S_n \geq n] = 2^{-n},$$

as among all the 2^n possible cases, only one case where each $X_i = 1$ satisfies the condition. When $p > 0$, for any $\lambda > 0$, we have

$$\begin{aligned}\Pr[S_n \geq (1-2p)n] &= \Pr[e^{\lambda S_n} \geq e^{\lambda(1-2p)n}] \\ &= \Pr\left[e^{\lambda S_n} \geq \mathbb{E}[e^{\lambda S_n}] \frac{e^{\lambda(1-2p)n}}{\mathbb{E}[e^{\lambda S_n}]}\right].\end{aligned}$$

By Markov's inequality, we have

$$\Pr\left[e^{\lambda S_n} \geq \mathbb{E}[e^{\lambda S_n}] \frac{e^{\lambda(1-2p)n}}{\mathbb{E}[e^{\lambda S_n}]}\right] \leq \mathbb{E}[e^{\lambda S_n}] e^{-\lambda(1-2p)n}.$$

For $\mathbb{E}[e^{\lambda S_n}]$ we have

$$\mathbb{E}[e^{\lambda S_n}] = \prod_i \mathbb{E}[e^{\lambda X_i}] = (\frac{e^{\lambda} + e^{-\lambda}}{2})^n,$$

thus

$$\mathbb{E}[e^{\lambda S_n}] e^{-\lambda(1-2p)n} = (\frac{e^{\lambda} + e^{-\lambda}}{2})^n e^{-\lambda(1-2p)n}.$$

Therefore, we need to minimize

$$\frac{e^{\lambda} + e^{-\lambda}}{2} e^{-\lambda(1-2p)}$$

for $\lambda > 0$, which is minimized when $\lambda = \frac{1}{2} \ln \frac{1-p}{p}$, and the minimum value is

$$\frac{(\frac{1-p}{p})^p}{2(1-p)},$$

implying that

$$\Pr[S_n \geq (1-2p)n] \leq \left(\frac{(\frac{1-p}{p})^p}{2(1-p)}\right)^n = 2^{H(p)n} 2^{-n},$$

completing the proof.

6 Problem 6

6.1 Question

(Parameter optimization in the Half Liar Game.) Find, asymptotically, the $u = u(q)$ that minimizes $2^q \Pr[S_q \leq -u] + 2^{q+1}/(q-u)$ and express the minimal value in the form $2^{q+1}/q + (1+o(1))g(q)$ for some function g.

6.2 Answer

By the Chernoff bound, we have

$$2^q \Pr[S_q \leq -u] + 2^{q+1}/(q-u) \leq 2^q e^{-u^2/2q} + 2^{q+1}/(q-u).$$

Let $f_q(u) = 2^q e^{-u^2/2q} + 2^{q+1}/(q-u)$, by letting its derivative equal to zero, we have

$$\frac{2}{(q-u)^2} - \frac{e^{-u^2/2q}u}{q} = 0.$$

Using $u = o(q)$, we finally have

$$u \sim c\sqrt{q \ln q}$$

for some $c > 0$ to be optimized later. Plug that in, we have

$$\begin{aligned} 2^q e^{-u^2/2q} + 2^{q+1}/(q-u) &= 2^q/q^{c^2/2} + 2^{q+1}/(q - c\sqrt{q \ln q}) \\ &= 2^q/q^{c^2/2} + 2^{q+1}(1/q + c\sqrt{\ln q} q^{-3/2} + O(q^{-2})) \\ &= 2^{q+1}/q + (1+o(1))(\sqrt{3 \ln q} 2^q/q^{3/2}), \end{aligned}$$

where we used Taylor series and set $c = \sqrt{3}$. Actually, if we set

$$u = \sqrt{q(3 \ln q - c' \ln \ln q)}$$

in a more sophisticated way with constant c' slightly less than 1, we can have

$$2^q e^{-u^2/2q} + 2^{q+1}/(q-u) = 2^{q+1}/q + (1+o(1))(\sqrt{3 \ln q - c' \ln \ln q} 2^q/q^{3/2}),$$

which has no essential improvement though.

7 Problem 7

7.1 Question

Show that for A fixed and r sufficiently large Paul wins the $(2^r - (r+1)A, A)$, r-Chip Liar Game.

7.2 Answer

At the beginning of the game, the weight of the position is

$$x_0 B(r,0) + x_1 B(r,1) = (2^r - (r+1)A)2^{-r} + A2^{-r}(r+1) = 1,$$

so we need to show that Paul has a strategy such that the weight is kept as 1 until the end, in other words, we need to find a way such that both W^y and W^n are 1. To show that, suffice to show that we can always make $W^y = 1$. Consider the first step, suppose that Paul chooses S to be a set consisting of v_0 chips at position 0 and v_1 chips at position 1. If Carole chooses to move all chips in S one position to the left, then the new state will have $2^r - (r+1)A - v_0 + v_1$ chips at position 0 and $A - v_1$ chips at position 1, and the weight of it would be

$$(2^r - (r+1)A - v_0 + v_1)2^{-r+1} + (A - v_1)2^{-r+1}r.$$

To make it equal to 0, a viable solution would be

$$(v_0, v_1) = (2^{r-1} - A, 0),$$

which implies that the state with $r-1$ rounds left is $(2^{r-1} - rA, A)$ and we can continually repeat this with only r decreased by 1 at each round. With this choice, if Carole chooses to move all chips not in S one position to the left, then the new state would simply be $(2^{r-1}, 0)$, which is just the simplest non-liar Chip Game and clearly Paul can win the game with 2^{r-1} chips and $r-1$ rounds left. However, note that we can do this only if

$$2^r - (r+1)A \geq 2^{r-1} - A,$$

which only holds for large r. So we actually need to do that in a more sophisticated way. Again, start from the first round, this time we let

$$(v_0, v_1) = (2^{r-1} - A - (r-1)\lceil A/2 \rceil, \lceil A/2 \rceil),$$

and the choice of Carole will make the new state to be

$$(2^{r-1} - r(A - \lceil A/2 \rceil), A - \lceil A/2 \rceil)$$

or

$$(2^{r-1} - r\lceil A/2 \rceil, \lceil A/2 \rceil).$$

Now the state of the game would always be in the form

$$(2^q - (q+1)A_q, A_q)$$

with q rounds left and $A_q = A_{q+1} - \lceil A_{q+1}/2 \rceil$ or $\lceil A_{q+1}/2 \rceil$ ($A_r = A$), and we can repeat this as long as

$$2^q - (q+1)A_q \geq 2^{q-1} - A_q - (q-1)\lceil A_q/2 \rceil,$$

which holds if

$$2^{q-1} + (q-1)\lceil A_q/2 \rceil \geq qA_q.$$

To have the above inequality, let k be the unique integer such that $2^{k-1} < A \leq 2^k$, clearly, we have

$$2^{k-r+q-1} < A_q \leq 2^{k-r+q},$$

therefore, suffice it to have

$$2^{q-1} \geq (q+1)2^{k-r+q-1} \geq (q+1)A_q/2 \geq qA_q - (q-1)\lceil A_q/2 \rceil,$$

which holds for sufficiently large r as then

$$2^{r-k} \geq r+1 \geq q+1.$$

Besides, just to show that the end of the game is clear, as r is sufficiently large, there would be an earliest stage when $A_q = 1$, and we have the state

$$(2^q - q - 1, 1).$$

As designed, Paul will let

$$(v_0, v_1) = (2^{q-1} - q, 1)$$

and the two possible next states are

$$(2^{q-1} - q, 1) \text{ and } (2^{q-1}, 0).$$

If Carole keeps the state in the first form, at the end of the game, the state is $(0, 1)$ thus we know Paul wins. And once Carole makes the state become the second form, as we mentioned before, it is just the simplest non-liar Chip Game with 2^{q-1} chips and $q - 1$ round left, so Paul also can win.

To conclude, starting from $(x_0 = 2^r - (r + 1)A, x_1 = A)$, $q = r$-Chip Liar Game, at each round, Paul just lets S be a set consisting of $2^{q-1} - x_1 - (q - 1)\lceil x_1/2 \rceil$ chips currently in position 0 and $\lceil x_1/2 \rceil$ chips currently in position 1, in that way, we proved that Paul can win the game after r rounds with only one chip left on the board.