2020 FALL MAS583 HW4

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1 Problem 1

1.1 Question

Let X be a random variable taking integral nonnegative values, let $\mathbb{E}[X^2]$ denote the expectation of its square, and let $\mathrm{Var}[X]$ denote its variance. Prove that

$$\Pr[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X^2]}.$$

1.2 Answer

The above inequality is equivalent to the following one,

$$\Pr[X \geq 1] \geq \frac{\mathbb{E}^2[X]}{\mathbb{E}[X^2]}.$$

Let P_i denote $\Pr[X = i]$ for any nonnegative integer i, we have

$$\begin{aligned}
&\Pr[X \geq 1]\mathbb{E}[X^2] - \mathbb{E}^2[X] \\
=&(\sum_{i=1}^{\infty} P_i)(\sum_{i=1}^{\infty} i^2 P_i) - (\sum_{i=1}^{\infty} iP_i)^2 \\
=&\sum_{i,j}(j^2 P_i P_j) - \sum_{i,j}(ijP_iP_j) \\
=&\frac{1}{2}(\sum_{i,j}(i^2 P_i P_j) + \sum_{i,j}(j^2 P_i P_j) - \sum_{i,j}(2ijP_iP_j)) \\
=&\frac{1}{2}\sum_{i,j}((i-j)^2 P_i P_j) \\
\geq& 0,
\end{aligned}$$

completing the proof.

2 Problem 2

2.1 Question

(*) Show that there is a positive constant c such that the following holds. For any n reals $a_1, a_2, ..., a_n$ satisfying $\sum_{i=1}^{n} a_i^2 = 1$, if $(\epsilon_1, .., \epsilon_n)$ is a $\{-1, 1\}$-random vector obtained by choosing each ϵ_i randomly and independently with uniform distribution to be either -1 or 1, then

$$\Pr\left[\left|\sum_{i=1}^{n}\epsilon_i a_i\right| \le 1\right] \ge c.$$

2.2 Answer

First let's consider the cases where for all i, $|a_i| < 1/2$. Let k be the minimum number such that $\sum_{i=1}^{k} a_i^2 \ge 1/2$, as k is minimum, $\sum_{i=1}^{k-1} a_i^2 < 1/2$, besides, $|a_k| < 1/2$, i.e., $a_k^2 < 1/4$, thus, $1/2 \le \sum_{i=1}^{k} a_i^2 < 3/4$ and $1/4 < \sum_{i=k+1}^{n} a_i^2 \le 1/2$.
Let $X = \sum_{i=1}^{k}\epsilon_i a_i$ and $Y = \sum_{i=k+1}^{n}\epsilon_i a_i$ who are independent with each other. Clearly, $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and $\mathrm{Var}[X] = \sum_{i=1}^{k} a_i^2 \mathrm{Var}[\epsilon_i] = \sum_{i=1}^{k} a_i^2$, similarly $\mathrm{Var}[Y] = \sum_{i=k+1}^{n} a_i^2$. Thus, as X and Y are symmetric on the both sides of zero,

$$\Pr[0 \le X \le 1] \ge \frac{1}{2}\Pr[|X| \le 1] \ge \frac{1}{2}(1 - \mathrm{Var}[X]) > 1/8,$$

$$\Pr[-1 \le Y \le 0] \ge \frac{1}{2}\Pr[|Y| \le 1] \ge \frac{1}{2}(1 - \mathrm{Var}[Y]) \ge 1/4,$$

finally, we have

$$\Pr\left[\left|\sum_{i=1}^{n}\epsilon_i a_i\right| \le 1\right] = \Pr[|X+Y| \le 1] \ge \Pr[0 \le X \le 1]\Pr[-1 \le Y \le 0] > 1/32.$$

As for the cases where $|a_i| \ge 1/2$ for some i, let l be any number such that $|a_l| \ge 1/2$, and let $X = \sum_{i \ne l}\epsilon_i a_i$, $Y = \epsilon_l a_l$. Still clearly, $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, $\mathrm{Var}[X] = \sum_{i \ne l} a_i^2 = 1 - a_l^2 \le 3/4$, $\mathrm{Var}[Y] = a_l^2$, $\Pr[|Y| \le 1] = 1$. Thus, with the same symmetry,

$$\Pr[0 \le X \le 1] \ge \frac{1}{2}\Pr[|X| \le 1] \ge \frac{1}{2}(1 - \mathrm{Var}[X]) \ge 1/8,$$

$$\Pr[-1 \le Y \le 0] \ge \frac{1}{2}\Pr[|Y| \le 1] = 1/2,$$

finally we have

$$\Pr\left[\left|\sum_{i=1}^{n}\epsilon_i a_i\right| \le 1\right] = \Pr[|X+Y| \le 1] \ge \Pr[0 \le X \le 1]\Pr[-1 \le Y \le 0] \ge 1/16,$$

completing the proof.

2.3 Reference

`https://people.math.gatech.edu/~tetali/MATH7018_SPR14/HW2.pdf`

3 Problem 3

3.1 Question

(*) Show that there is a positive constant c such that the following holds. For any n vectors $a_1, a_2, ..., a_n \in \mathbb{R}^2$ satisfying $\sum_{i=1}^{n} \|a_i\|^2 = 1$ and $\|a_i\| \le 1/10$, where $\|\cdot\|$ denotes the usual Euclidean norm, if $(\epsilon_1, .., \epsilon_n)$ is a $\{-1, 1\}$-random vector obtained by choosing each ϵ_i randomly and independently with uniform distribution to be either -1 or 1, then

$$\Pr\left[\left\|\sum_{i=1}^{n} \epsilon_i a_i\right\| \leq 1/3\right] \geq c.$$

3.2 Answer

First, we design an algorithm which gives the following lemma.

Lemma 3.1 *Given any t vectors $v_1, ..., v_t \in \mathbb{R}^2$ with each $\|v_i\| \leq d$, for some given constant $d > 0$. We can find $\epsilon_1, ..., \epsilon_t \in \{-1, 1\}$ such that*

$$\left\|\sum_{i=1}^{t} \epsilon_i v_i\right\| \leq 3d.$$

Equivalently, if each ϵ_i is independently chosen in $\{-1, 1\}$, uniformly at random, then we have

$$\Pr\left[\left\|\sum_{i=1}^{t} \epsilon_i v_i\right\| \leq 3d\right] \geq 1/2^t$$

Proof. *First, we assume that each $v_i \neq 0$ and divide $\mathbb{R}^2 \setminus 0$ into 6 sections $S_1, ..., S_6$, where for $i \in [6]$, S_i contains all the vectors who have argument in $[(i-1)\pi/3, i\pi/3)$. For each vector v in S_2, S_4, or S_6, we can consider $-v$ who's in S_1, S_3, or S_5, then we have t vectors where each of them is in S_1, S_3, or S_5. Now, at each step, we arbitrarily choose two vectors who are in the same section until all vectors are in different sections, when the total number of vectors is no more than 3 as we will guarantee that in the whole process, each vector is in S_1, S_3, or S_5. As the two vectors we choose, say x and y, are in the same section, their difference in argument is no more than $\pi/3$, we can easily check that among the four possible results $x + y$, $x - y$, $-x + y$, $-x - y$, there exists one that is 0 or is in S_1, S_3, or S_5 with norm no more than the maximum of $\|x\|$ and $\|y\|$. Therefore, when the algorithm ends, we have at most 3 vectors with norm no more than d, thus, any linear combination of them with coefficient -1 or 1 has norm no more than $3d$, completing the proof.* ■

Theorem 3.2 (McDiarmid's Inequality) *Let $x_1, ..., x_n$ be independent random variables taking on values in a set A and let $c_1, ..., c_n$ be positive real constants. If $\phi : A^n \to \mathbb{R}$ satisfies*

$$\sup_{x_1,...,x_n,x_i' \in A} |\phi(x_1, ..., x_i, ..., x_n) - \phi(x_1, ..., x_i', ..., x_n)| \leq c_i,$$

for $i \in [n]$, then for any $\epsilon > 0$,

$$\Pr[\phi(x_1, ..., x_n) - \mathbb{E}[\phi(x_1, ..., x_n)] \geq \epsilon] \leq e^{-2\epsilon^2/\sum_{i=1}^{n} c_i^2}.$$

Let $k_0 = 0$, and for $i = 1, 2, ...,$ let k_i be the minimum number such that

$$\sum_{j=k_{i-1}+1}^{k_i} \|a_j\|^2 \geq 1/10000,$$

until $k_i = n$ or

$$\sum_{j=k_{i-1}+1}^{n} \|a_j\|^2 < 1/10000$$

where we let $k_i = n$. For each i, as $\|a_i\| \le 1/10$, i.e., $\|a_i\|^2 \le 1/100$,

$$\sum_{j=k_{i-1}+1}^{k_i} \|a_j\|^2 < \frac{1}{100} + \frac{1}{10000}.$$

Therefore, the last i where $k_i = n$ satisfies that $99 \le i \le 10000$. Let

$$X_i = \phi_i(\{\epsilon_j\}_{j=k_{i-1}+1}^{k_i}) = \left\| \sum_{j=k_{i-1}+1}^{k_i} \epsilon_j a_j \right\|,$$

we have

$$\mathbb{E}[X_i^2] = \sum_{j=k_{i-1}+1}^{n} \|a_j\|^2,$$

thus,

$$\mathbb{E}[X_i] \le \sqrt{\mathbb{E}[X_i^2]} < \sqrt{\frac{1}{100} + \frac{1}{10000}} < 0.1005.$$

As we have

$$|\phi_i(a_{k_{i-1}+1}, ..., a_m, ..., a_{k_i}) - \phi_i(a_{k_{i-1}+1}, ..., a'_m, ..., a_{k_i})| \le 2||a_m||,$$

by McDiarmid's Inequality,

$$\begin{aligned}
\Pr[X_i \le 2/19] &\ge 1 - \Pr[X_i \ge 2/19] \\
&= 1 - \Pr[X_i - \mathbb{E}[X_i] \ge 2/19 - \mathbb{E}[X_i]] \\
&\ge 1 - \Pr[X_i - \mathbb{E}[X_i] \ge 2/19 - \sqrt{1/50}] \\
&\ge 1 - e^{-2(2/19-0.1005)^2/\sum_{j=k_{i-1}+1}^{k_i}(2||a_m||)^2} \\
&\ge 1 - e^{-(2/19-0.1005)^2/0.201} \\
&> 1/10000.
\end{aligned}$$

Therefore, with probability at least $1/20000^{10000}$, we have

$$\left\| \sum_{j=k_{i-1}+1}^{k_i} \epsilon_j a_j \right\| \le 2/19,$$

for each i, where each probability is halved as the two choices where every two counterparts are inverse number are taken into consideration. And the algorithm (Lemma 1.1) gives us a way to finally get a vector whose norm is no more than $\frac{6}{19} < 1/3$, which has a probability at least $1/2^{10000}$ given all these sub-sum of vectors with desired norm bound. Thus, we have

$$\Pr\left[\left\| \sum_{i=1}^{n} \epsilon_i a_i \right\| \le 1/3 \right] \ge 1/40000^{10000},$$

completing the proof.

3.3 A Failed Attempt

Theorem 3.3 (McDiarmid's Inequality) *Let $x_1, ..., x_n$ be independent random variables taking on values in a set A and let $c_1, ..., c_n$ be positive real constants. If $\phi : A^n \to \mathbb{R}$ satisfies*

$$\sup_{x_1,...,x_n,x_i' \in A} |\phi(x_1, ..., x_i, ..., x_n) - \phi(x_1, ..., x_i', ..., x_n)| \leq c_i,$$

for $i \in [n]$, then

$$\Pr[\phi(x_1, ..., x_n) - \mathbb{E}[\phi(x_1, ..., x_n)] \geq \epsilon] \leq e^{-2\epsilon^2/\sum_{i=1}^n c_i^2}.$$

In our case, we have $\epsilon_1, ..., \epsilon_n$ that are independent random variables taking on values in $A = \{-1, 1\}$ and let $\phi(\epsilon_1, ..., \epsilon_n) = \|\sum_{i=1}^n \epsilon_i a_i\|$, then

$$\sup_{\epsilon_1,...,\epsilon_n,\epsilon_i' \in A} |\phi(\epsilon_1, ..., \epsilon_i, ..., \epsilon_n) - \phi(\epsilon_1, ..., \epsilon_i', ..., \epsilon_n)| \leq 2||a_i|| \leq \frac{1}{5}.$$

Let $t = \mathbb{E}[\phi(\epsilon_1, ..., \epsilon_n)$, by McDiarmid's Inequality, we have

$$\begin{aligned}
&\Pr\left[\left\|\sum_{i=1}^n \epsilon_i a_i\right\| \leq 1/3\right] \\
=&\Pr\left[\phi(\epsilon_1, ..., \epsilon_n) - t \leq 1/3 - t\right] \\
\geq& 1 - e^{-2(1/3-t)^2/4\sum_{i=1}^n \|a_i\|^2} \\
=& 1 - e^{-(1/3-t)^2/2}.
\end{aligned}$$

As each $\|a_i\|$ is bounded, t is also bounded (Actually, we need to prove $t < 1/3$, on which I failed), therefore, there must be some positive c such that

$$\Pr\left[\left\|\sum_{i=1}^n \epsilon_i a_i\right\| \leq 1/3\right] \geq c,$$

completing the proof.

3.4 Another Failed Attempt

To show

$$\left\|\sum_{i=1}^n \epsilon_i a_i\right\| \leq 1/3,$$

suffices to show

$$\langle\sum_{i=1}^n \epsilon_i a_i, v\rangle \leq 1/3, \forall v \in \mathbb{R}^2 \text{ with } \|v\| = 1.$$

Given any $v \in \mathbb{R}^2$ with $\|v\| = 1$, let $f_v : \{-1, 1\}^n \to \mathbb{R}$, where

$$f_v(\epsilon = (\epsilon_1, .., \epsilon_n)) = \langle\sum_{i=1}^n \epsilon_i a_i, v\rangle = \sum_{i=1}^n \epsilon_i \langle a_i, v\rangle$$

Theorem 3.4 (Azuma's inequality) *Let $X_0, ..., X_n$ be a Martingale with $|X_t - X_{t-1}| \leq a_t$ for any $t \in [n]$, then for any $\lambda \geq 0$ one has $\Pr[|X_n - X_0| > \lambda \|a\|] \leq 2e^{-\lambda^2/4}$, where $a = (a_1, ..., a_n)$.*

Let $X_0 = 0$ and $X_i = \sum_{k=1}^{i} \epsilon_k \langle a_k, v\rangle$ for any $i \in [n]$, easy to check $X_0, ..., X_n$ is a Martingale, where

$$|X_t - X_{t-1}| = \langle a_t, v\rangle, \forall t \in [n].$$

Note that $X_n = f_v(\epsilon)$. By Azuma's inequality and the symmetric of f_v, we have

$$\Pr[X_n > 1/3] = \frac{1}{2}\Pr[|X_n| > 1/3] \leq e^{-\lambda^2/4},$$

where $\lambda = (3\sqrt{\sum_{i=1}^{n} \langle a_i, v\rangle^2})^{-1} \geq (3\sqrt{\sum_{i=1}^{n} \|a_i\|^2})^{-1} = 1/3$. Therefore, we have

$$\Pr\left[\langle\sum_{i=1}^{n} \epsilon_i a_i, v\rangle \leq 1/3\right] \geq 1 - e^{-1/36},$$

completing the proof (I want to claim that if we let v has the same direction with $\sum_{i=1}^{n} \epsilon_i a_i$ then we can complete the proof but I feel that it's not so clear).

3.5 Some Other Ideas

First, show that when $n \leq c$, where c is a large constant, say 10^{100}, we can find at least 1 possible choice of $(\epsilon_1, ..., \epsilon_n)$ such that $\|\sum_{i=1}^{n} \epsilon_i a_i\| \leq 1/3$. Then we have

$$\Pr\left[\left\|\sum_{i=1}^{n} \epsilon_i a_i\right\| \leq 1/3\right] \geq \frac{1}{2^{10^{100}}},$$

for all $n \leq 10^{100}$. When $n > 10^{100}$, we need to show there are at least $2^{n-10^{100}}$ choices of $(\epsilon_1, ..., \epsilon_n)$ satisfy the desired condition. Intuitively, we may expect that we can find $n - 10^{100}$ a_i such that for all the $2^{n-10^{100}}$ choices of corresponding ϵ_i, we can find at least 1 possible solution for the remaining ϵ_i. However, it's obvious that when $n \to \infty$, we can hardly prove this.

3.6 Reference

`Rothvoss, Thomas. "Probabilistic Combinatorics." (2019).`
`Kutin, Samuel. "Extensions to McDiarmid's inequality when differences are bounded with high probability." Department Computer Science, University of Chicago, Chicago, IL. Technical report TR-2002-04 (2002).`

Besides, I got some ideas from Prof. Liu's lectures when he mentioned Martingale.

4 Problem 4

4.1 Question

Let X be a random variable with expectation $\mathbb{E}[X] = 0$ and variance σ^2. Prove that for all $\lambda > 0$,

$$\Pr[X \geq \lambda] \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$

4.2 Answer

For any $\lambda > 0$, let $Y_\lambda = \sigma^2 + \lambda X$, clearly $\mathbb{E}[Y_\lambda] = \sigma^2$ for all $\lambda > 0$, thus by Markov's inequality,

$$\Pr[X \geq \lambda] = \Pr[Y_\lambda \geq \sigma^2 + \lambda^2] \leq \frac{\sigma^2}{\sigma^2 + \lambda^2},$$

completing the proof.

5 Problem 5

5.1 Question

Let $v_1 = (x_1, y_1), ..., v_n = (x_n, y_n)$ be n two-dimensional vectors, where each x_i and each y_i is an integer whose absolute value does not exceed $2^{n/2}/(100\sqrt{n})$. Show that there are two disjoint sets $I, J \subset \{1, 2, .., n\}$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

5.2 Answer

First, note that "disjoint" actually makes no difference because if $\sum_{i\in I} v_i = \sum_{j \in J} v_j$ but $I \cap J \neq \emptyset$, we can let $\tilde{I} = I \setminus (I \cap J)$ and $\tilde{J} = J \setminus (I \cap J)$, which are disjoint and whose corresponding summations are still equal. When $n = 1$, clearly $v_1 = (0, 0)$ and $I = \{1\}, J = \emptyset$ are the desired sets.
Suppose for all $I, J \subset \{1, 2, .., n\}$ such that $I \neq J$, $\sum_{i\in I} v_i \neq \sum_{j\in J} v_j$. As $\{1, 2, .., n\}$ has 2^n subsets, $|\{\sum_{i\in I} v_i : I \subset \{1, 2, ..., n\}\}| = 2^n$. On the other hand, let $\{\epsilon_i\}_{i=1}^n$ be a sequence of random variables where each one is independently chosen in $\{0, 1\}$ uniformly at random, and let $X = \sum_{i=1}^n \epsilon_i x_i$ and $Y = \sum_{i=1}^n \epsilon_i y_i$, we have

$$\mathrm{Var}[X] = \sum_{i=1}^n \frac{1}{4} x_i^2 \leq \frac{2^n}{40000},$$

similarly,

$$\mathrm{Var}[Y] \leq \frac{2^n}{40000}.$$

Thus, by Chebyshev's inequality,

$$\Pr[|X - \mathbb{E}[X]| \geq 2\frac{2^{n/2}}{200}] \leq \frac{1}{4},$$

similarly,

$$\Pr[|Y - \mathbb{E}[Y]| \geq 2\frac{2^{n/2}}{200}] \leq \frac{1}{4},$$

by the independence between X and Y, which follows

$$\Pr[|X - \mathbb{E}[X]| \leq \frac{2^{n/2}}{100}, |Y - \mathbb{E}[Y]| \leq \frac{2^{n/2}}{100}] \geq \frac{1}{2}.$$

Note that (X, Y) is $\sum_{i\in[n],\epsilon_i=1} v_i$, which means among all the 2^n choices of $I \subset [n]$, at least 2^{n-1} should satisfy $|\sum_{i\in I} x_i - \mathbb{E}[X]| \leq \frac{2^{n/2}}{100}$ and $|\sum_{i\in I} y_i - \mathbb{E}[Y]| \leq \frac{2^{n/2}}{100}$, where $\sum_{i\in I} x_i$ and $\sum_{i\in I} y_i$ should be distinct integers. However, the size of feasible set of this linear program is at most $((2^{n/2}/50) + 1)^2 < 2^{n-1}$ for all $n \geq 2$, which implies that there must be $I \neq J$ such that $\sum_{i\in I} v_i = \sum_{j\in J} v_j$, completing the proof.

5.3 A Failed Attempt

First, if $\exists i \neq j$ such that $v_i = v_j$, then we can just simply let $I = \{i\}$ and $J = \{j\}$ and the condition is satisfied. So we can only consider cases where each v_i is distinct. Let $N = \lfloor 2^{n/2}/(100\sqrt{n}) \rfloor$ and $\tilde{v}_i = (2N+1)(x_i+N)+(y_i+N)+1$. Clearly, each $\tilde{v}_i \in [(2N+1)^2] \subset [2^n/1000n]$ and is distinct. Furthermore, $\sum_{i\in I} v_i = \sum_{j\in J} v_j$ if and only if (WRONG!!! This only holds when $|I| = |J|$) $\sum_{i\in I} \tilde{v}_i = \sum_{j\in J} \tilde{v}_j$. Suppose such I and J don't exist, i.e., for any two disjoint sets $I, J \subset \{1, 2, .., n\}$, $\sum_{i\in I} v_i \neq \sum_{j\in J} v_j$, we have $\{\tilde{v}_i\}_{i=1}^n \subset [2^n/1000n]$ has distinct sums ("disjoint" actually makes no difference as if $\sum_{i\in I} v_i = \sum_{j\in J} v_j$ but $I \cap J \neq \emptyset$, we can let $\tilde{I} = I \setminus (I \cap J)$ and $\tilde{J} = J \setminus (I \cap J)$, which are disjoint and whose summations are still equal), thus we should have $2^n < (2^n/1000n)n$, which obviously doesn't hold. Thus, such I and J must exist.

6 Problem 6

6.1 Question

(*) Prove that for every set X of at least $4k^2$ distinct residue classes modulo a prime p, there is an integer a such that the set $\{ax \pmod p) : x \in X\}$ intersects every interval in $\{0, 1, ..., p-1\}$ of length at least p/k.

6.2 Answer

Let $P = \{0, 1, ..., p-1\}$ and $Y = Y_a = \{ax \pmod p : x \in X\}$, to prove the desired result, we need to find such a so that Y intersects every interval of length $\lceil p/k \rceil$ in P.
Let $J_1, ..., J_{2k}$ be a fixed covering of P by $2k$ intervals of length $\lceil p/2k \rceil$ each. Note that if Y intersects with each J_i then it certainly satisfies the desired property because every interval of length at least $\lceil p/k \rceil$ in P must fully contain at least one J_i. Fix $i \in [2k]$, for any $x \in X$, let S_x^i be the indicating random variable of $ax \pmod p \in J_i$, then $S_i = \sum_{x\in X} S_x^i$ is 0 if and only if Y doesn't intersect J_i. By linearity of expectation,

$$\mathbb{E}[S^i] = \sum_{x\in X} \mathbb{E}[S_x^i] = \frac{4k^2\lceil p/2k \rceil}{p}.$$

Note that the random variables S_x^i for $x \in X$ are pairwise independent, hence

$$\mathrm{Var}[S^i] = \sum_{x\in X} \mathrm{Var}[S_x^i] = \frac{4k^2\lceil p/2k \rceil}{p(1-\lceil p/2k \rceil/p)} = \frac{\mathbb{E}[S^i]}{1-\lceil p/2k \rceil/p}.$$

Therefore, by Chebyshev's Inequality,

$$\Pr(S^i = 0) \leq \Pr(|S^i - \mathbb{E}[S^i]| \geq \mathbb{E}[X^i]) \leq \frac{\mathrm{Var}[S^i]}{\mathbb{E}^2[S^i]} < \frac{1}{2k},$$

since there are $2k$ possible choices of i, we have that with positive probability, $S^i > 0$ for all $i \in [2k]$, completing the proof.

6.3 Reference

Alon, Noga, Igor Kriz, and J. Nesetril. "How to color shift hypergraphs." Studia Scientiarum Mathematicarum Hungarica 30.1 (1995): 1-12.
Alon, Noga, and Yuval Peres. "Uniform dilations." Geometric & Functional Analysis GAFA 2.1 (1992): 1-28.