2020 FALL MAS583 HW4

Fanchen Bu [20194185]

October 19, 2020

1 Problem 1

1.1 Question

Let X be a random variable taking integral nonnegative values, let $\mathbb{E}[X^2]$ denote the expectation of its square, and let $\mathrm{Var}[X]$ denote its variance. Prove that

$$\Pr[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X^2]}.$$

1.2 Answer

The above inequality is equivalent to the following one,

$$\Pr[X \geq 1] \geq \frac{\mathbb{E}^2[X]}{\mathbb{E}[X^2]}.$$

Let P_i denote $\Pr[X = i]$ for any nonnegative integer i, we have

$$\Pr[X \geq 1]\mathbb{E}[X^2] - \mathbb{E}^2[X]$$
$$=(\sum_{i=1}^{\infty} P_i)(\sum_{i=1}^{\infty} i^2 P_i) - (\sum_{i=1}^{\infty} i P_i)^2$$
$$=\sum_{i,j}(j^2 P_i P_j) - \sum_{i,j}(ij P_i P_j)$$
$$=\frac{1}{2}(\sum_{i,j}(i^2 P_i P_j) + \sum_{i,j}(j^2 P_i P_j) - \sum_{i,j}(2ij P_i P_j))$$
$$=\frac{1}{2}\sum_{i,j}((i-j)^2 P_i P_j)$$
$$\geq 0,$$

completing the proof.

2 Problem 2

2.1 Question

(*) Show that there is a positive constant c such that the following holds. For any n reals $a_1, a_2, ..., a_n$ satisfying $\sum_{i=1}^{n} a_i^2 = 1$, if $(\epsilon_1, .., \epsilon_n)$ is a $\{-1, 1\}$-random vector obtained by

1

choosing each ϵ_i randomly and independently with uniform distribution to be either -1 or 1, then
$$\Pr\left[\left|\sum_{i=1}^{n} \epsilon_i a_i\right| \leq 1\right] \geq c.$$

2.2 Answer

First let's consider the cases where for all i, $|a_i| < 1/2$. Let k be the minimum number such that $\sum_{i=1}^{k} a_i^2 \geq 1/2$, as k is minimum, $\sum_{i=1}^{k-1} a_i^2 < 1/2$, besides, $|a_k| < 1/2$, i.e., $a_k^2 < 1/4$, thus, $1/2 \leq \sum_{i=1}^{k} a_i^2 < 3/4$ and $1/4 < \sum_{i=k+1}^{n} a_i^2 \leq 1/2$.

Let $X = \sum_{i=1}^{k} \epsilon_i a_i$ and $Y = \sum_{i=k+1}^{n} \epsilon_i a_i$ who are independent with each other. Clearly, $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and $\text{Var}[X] = \sum_{i=1}^{k} a_i^2 \text{Var}[\epsilon_i] = \sum_{i=1}^{k} a_i^2$, similarly $\text{Var}[Y] = \sum_{i=k+1}^{n} a_i^2$. Thus, as X and Y are symmetric on the both sides of zero,

$$\Pr[0 \leq X \leq 1] \geq \frac{1}{2} \Pr[|X| \leq 1] \geq \frac{1}{2}(1 - \text{Var}[X]) > 1/8,$$

$$\Pr[-1 \leq Y \leq 0] \geq \frac{1}{2} \Pr[|Y| \leq 1] \geq \frac{1}{2}(1 - \text{Var}[Y]) \geq 1/4,$$

finally, we have

$$\Pr\left[\left|\sum_{i=1}^{n} \epsilon_i a_i\right| \leq 1\right] = \Pr[|X + Y| \leq 1] \geq \Pr[0 \leq X \leq 1]\Pr[-1 \leq Y \leq 0] > 1/32.$$

As for the cases where $|a_i| \geq 1/2$ for some i, let l be any number such that $|a_l| \geq 1/2$, and let $X = \sum_{i \neq l} \epsilon_i a_i$, $Y = \epsilon_l a_l$. Still clearly, $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, $\text{Var}[X] = \sum_{i \neq l} a_i^2 = 1 - a_l^2 \leq 3/4$, $\text{Var}[Y] = a_l^2$, $\Pr[|Y| \leq 1] = 1$. Thus, with the same symmetry,

$$\Pr[0 \leq X \leq 1] \geq \frac{1}{2} \Pr[|X| \leq 1] \geq \frac{1}{2}(1 - \text{Var}[X]) \geq 1/8,$$

$$\Pr[-1 \leq Y \leq 0] \geq \frac{1}{2} \Pr[|Y| \leq 1] = 1/2,$$

finally we have

$$\Pr\left[\left|\sum_{i=1}^{n} \epsilon_i a_i\right| \leq 1\right] = \Pr[|X + Y| \leq 1] \geq \Pr[0 \leq X \leq 1]\Pr[-1 \leq Y \leq 0] \geq 1/16,$$

completing the proof.

2.3 Reference

https://people.math.gatech.edu/~tetali/MATH7018_SPR14/HW2.pdf

3 Problem 3

3.1 Question

(*) Show that there is a positive constant c such that the following holds. For any n vectors $a_1, a_2, ..., a_n \in \mathbb{R}^2$ satisfying $\sum_{i=1}^{n} \|a_i\|^2 = 1$ and $\|a_i\| \leq 1/10$, where $\|\cdot\|$ denotes the

usual Euclidean norm, if $(\epsilon_1, .., \epsilon_n)$ is a $\{-1, 1\}$-random vector obtained by choosing each ϵ_i randomly and independently with uniform distribution to be either -1 or 1, then

$$\Pr\left[\left\|\sum_{i=1}^{n} \epsilon_i a_i\right\| \leq 1/3\right] \geq c.$$

3.2 A Failed Attempt

Theorem 3.1 (McDiarmid's Inequality) *Let $x_1, ..., x_n$ be independent random variables taking on values in a set A and let $c_1, ..., c_n$ be positive real constants. If $\phi : A^n \to \mathbb{R}$ satisfies*

$$\sup_{x_1,...,x_n,x_i' \in A} |\phi(x_1, ..., x_i, ..., x_n) - \phi(x_1, ..., x_i', ..., x_n)| \leq c_i,$$

for $i \in [n]$, then

$$\Pr[\phi(x_1, ..., x_n) - \mathbb{E}[\phi(x_1, ..., x_n)] \geq \epsilon] \leq e^{-2\epsilon^2 / \sum_{i=1}^{n} c_i^2}.$$

In our case, we have $\epsilon_1, ..., \epsilon_n$ that are independent random variables taking on values in $A = \{-1, 1\}$ and let $\phi(\epsilon_1, ..., \epsilon_n) = \|\sum_{i=1}^{n} \epsilon_i a_i\|$, then

$$\sup_{\epsilon_1,...,\epsilon_n,\epsilon_i' \in A} |\phi(\epsilon_1, ..., \epsilon_i, ..., \epsilon_n) - \phi(\epsilon_1, ..., \epsilon_i', ..., \epsilon_n)| \leq 2\|a_i\| \leq \frac{1}{5}.$$

Let $t = \mathbb{E}[\phi(\epsilon_1, ..., \epsilon_n)]$, by McDiarmid's Inequality, we have

$$\Pr\left[\left\|\sum_{i=1}^{n} \epsilon_i a_i\right\| \leq 1/3\right]$$
$$= \Pr\left[\phi(\epsilon_1, ..., \epsilon_n) - t \leq 1/3 - t\right]$$
$$\geq 1 - e^{-2(1/3-t)^2/4 \sum_{i=1}^{n} \|a_i\|^2}$$
$$= 1 - e^{-(1/3-t)^2/2}.$$

As each $\|a_i\|$ is bounded, t is also bounded (Actually, we need to prove $t < 1/3$, on which I failed), therefore, there must be some positive c such that

$$\Pr\left[\left\|\sum_{i=1}^{n} \epsilon_i a_i\right\| \leq 1/3\right] \geq c,$$

completing the proof.

3.3 Another Failed Attempt

To show
$$\left\|\sum_{i=1}^{n} \epsilon_i a_i\right\| \leq 1/3,$$

suffices to show
$$\langle \sum_{i=1}^{n} \epsilon_i a_i, v \rangle \leq 1/3, \forall v \in \mathbb{R}^2 \text{ with } \|v\| = 1.$$

Given any $v \in \mathbb{R}^2$ with $\|v\| = 1$, let $f_v : \{-1, 1\}^n \to \mathbb{R}$, where

$$f_v(\epsilon = (\epsilon_1, .., \epsilon_n)) = \langle \sum_{i=1}^n \epsilon_i a_i, v \rangle = \sum_{i=1}^n \epsilon_i \langle a_i, v \rangle$$

Theorem 3.2 (Azuma's inequality) *Let $X_0, ..., X_n$ be a Martingale with $|X_t - X_{t-1}| \le a_t$ for any $t \in [n]$, then for any $\lambda \ge 0$ one has $\Pr[|X_n - X_0| > \lambda \|a\|] \le 2e^{-\lambda^2/4}$, where $a = (a_1, ..., a_n)$.*

Let $X_0 = 0$ and $X_i = \sum_{k=1}^i \epsilon_k \langle a_k, v \rangle$ for any $i \in [n]$, easy to check $X_0, ..., X_n$ is a Martingale, where

$$|X_t - X_{t-1}| = \langle a_t, v \rangle, \forall t \in [n].$$

Note that $X_n = f_v(\epsilon)$. By Azuma's inequality and the symmetric of f_v, we have

$$\Pr[X_n > 1/3] = \frac{1}{2} \Pr[|X_n| > 1/3] \le e^{-\lambda^2/4},$$

where $\lambda = (3\sqrt{\sum_{i=1}^n \langle a_i, v \rangle^2})^{-1} \ge (3\sqrt{\sum_{i=1}^n \|a_i\|^2})^{-1} = 1/3$. Therefore, we have

$$\Pr\left[\langle \sum_{i=1}^n \epsilon_i a_i, v \rangle \le 1/3 \right] \ge 1 - e^{-1/36},$$

completing the proof (I want to claim that if we let v has the same direction with $\sum_{i=1}^n \epsilon_i a_i$ then we can complete the proof but I feel that it's not so clear).

3.4   Some Other Ideas

First, show that when $n \le c$, where c is a large constant, say 10^{100}, we can find at least 1 possible choice of $(\epsilon_1, ..., \epsilon_n)$ such that $\|\sum_{i=1}^n \epsilon_i a_i\| \le 1/3$. Then we have

$$\Pr\left[\left\| \sum_{i=1}^n \epsilon_i a_i \right\| \le 1/3 \right] \ge \frac{1}{2^{10^{100}}},$$

for all $n \le 10^{100}$. When $n > 10^{100}$, we need to show there are at least $2^{n-10^{100}}$ choices of $(\epsilon_1, ..., \epsilon_n)$ satisfy the desired condition. Intuitively, we may expect that we can find $n - 10^{100}$ a_i such that for all the $2^{n-10^{100}}$ choices of corresponding ϵ_i, we can find at least 1 possible solution for the remaining ϵ_i. However, it's obvious that when $n \to \infty$, we can hardly prove this.

3.5   Reference

Rothvoss, Thomas. "Probabilistic Combinatorics." (2019).
Kutin, Samuel. "Extensions to McDiarmid's inequality when differences are bounded with high probability." Department Computer Science, University of Chicago, Chicago, IL. Technical report TR-2002-04 (2002).
Besides, I got some ideas from Prof. Liu's lectures when he mentioned Martingale.

4 Problem 4

4.1 Question

Let X be a random variable with expectation $\mathbb{E}[X] = 0$ and variance σ^2. Prove that for all $\lambda > 0$,

$$\Pr[X \geq \lambda] \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$

4.2 Answer

For any $\lambda > 0$, let $Y_\lambda = \sigma^2 + \lambda X$, clearly $\mathbb{E}[Y_\lambda] = \sigma^2$ for all $\lambda > 0$, thus by Markov's inequality,

$$\Pr[X \geq \lambda] = \Pr[Y_\lambda \geq \sigma^2 + \lambda^2] \leq \frac{\sigma^2}{\sigma^2 + \lambda^2},$$

completing the proof.

5 Problem 5

5.1 Question

Let $v_1 = (x_1, y_1), ..., v_n = (x_n, y_n)$ be n two-dimensional vectors, where each x_i and each y_i is an integer whose absolute value does not exceed $2^{n/2}/(100\sqrt{n})$. Show that there are two disjoint sets $I, J \subset \{1, 2, .., n\}$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

5.2 Answer

First, note that "disjoint" actually makes no difference because if $\sum_{i \in I} v_i = \sum_{j \in J} v_j$ but $I \cap J \neq \emptyset$, we can let $\tilde{I} = I \setminus (I \cap J)$ and $\tilde{J} = J \setminus (I \cap J)$, which are disjoint and whose corresponding summations are still equal. When $n = 1$, clearly $v_1 = (0, 0)$ and $I = \{1\}, J = \emptyset$ are the desired sets.

Suppose for all $I, J \subset \{1, 2, .., n\}$ such that $I \neq J$, $\sum_{i \in I} v_i \neq \sum_{j \in J} v_j$. As $\{1, 2, .., n\}$ has 2^n subsets, $|\{\sum_{i \in I} v_i : I \subset \{1, 2, ..., n\}\}| = 2^n$. On the other hand, let $\{\epsilon_i\}_{i=1}^n$ be a sequence of random variables where each one is independently chosen in $\{0, 1\}$ uniformly at random, and let $X = \sum_{i=1}^n \epsilon_i x_i$ and $Y = \sum_{i=1}^n \epsilon_i y_i$, we have

$$\text{Var}[X] = \sum_{i=1}^n \frac{1}{4} x_i^2 \leq \frac{2^n}{40000},$$

similarly,

$$\text{Var}[Y] \leq \frac{2^n}{40000}.$$

Thus, by Chebyshev's inequality,

$$\Pr[|X - \mathbb{E}[X]| \geq 2\frac{2^{n/2}}{200}] \leq \frac{1}{4},$$

similarly,

$$\Pr[|Y - \mathbb{E}[Y]| \geq 2\frac{2^{n/2}}{200}] \leq \frac{1}{4},$$

by the independence between X and Y, which follows

$$\Pr[|X - \mathbb{E}[X]| \leq \frac{2^{n/2}}{100}, |Y - \mathbb{E}[Y]| \leq \frac{2^{n/2}}{100}] \geq \frac{1}{2}.$$

Note that (X, Y) is $\sum_{i \in [n], \epsilon_i = 1} v_i$, which means among all the 2^n choices of $I \subset [n]$, at least 2^{n-1} should satisfy $|\sum_{i \in I} x_i - \mathbb{E}[X]| \leq \frac{2^{n/2}}{100}$ and $|\sum_{i \in I} y_i - \mathbb{E}[Y]| \leq \frac{2^{n/2}}{100}$, where $\sum_{i \in I} x_i$ and $\sum_{i \in I} y_i$ should be distinct integers. However, the size of feasible set of this linear program is at most $((2^{n/2}/50) + 1)^2 < 2^{n-1}$ for all $n \geq 2$, which implies that there must be $I \neq J$ such that $\sum_{i \in I} v_i = \sum_{j \in J} v_j$, completing the proof.

5.3 A Failed Attempt

First, if $\exists i \neq j$ such that $v_i = v_j$, then we can just simply let $I = \{i\}$ and $J = \{j\}$ and the condition is satisfied. So we can only consider cases where each v_i is distinct. Let $N = \lfloor 2^{n/2}/(100\sqrt{n}) \rfloor$ and $\tilde{v}_i = (2N+1)(x_i+N)+(y_i+N)+1$. Clearly, each $\tilde{v}_i \in [(2N+1)^2] \subset [2^n/1000n]$ and is distinct. Furthermore, $\sum_{i \in I} v_i = \sum_{j \in J} v_j$ if and only if (WRONG!!! This only holds when $|I| = |J|$) $\sum_{i \in I} \tilde{v}_i = \sum_{j \in J} \tilde{v}_j$. Suppose such I and J don't exist, i.e., for any two disjoint sets $I, J \subset \{1, 2, .., n\}$, $\sum_{i \in I} v_i \neq \sum_{j \in J} v_j$, we have $\{\tilde{v}_i\}_{i=1}^n \subset [2^n/1000n]$ has distinct sums ("disjoint" actually makes no difference as if $\sum_{i \in I} v_i = \sum_{j \in J} v_j$ but $I \cap J \neq \emptyset$, we can let $\tilde{I} = I \setminus (I \cap J)$ and $\tilde{J} = J \setminus (I \cap J)$, which are disjoint and whose summations are still equal), thus we should have $2^n < (2^n/1000n)n$, which obviously doesn't hold. Thus, such I and J must exist.

6 Problem 6

6.1 Question

(*) Prove that for every set X of at least $4k^2$ distinct residue classes modulo a prime p, there is an integer a such that the set $\{ax \pmod{p} : x \in X\}$ intersects every interval in $\{0, 1, ..., p-1\}$ of length at least p/k.

6.2 Answer

Let $P = \{0, 1, ..., p-1\}$ and $Y = Y_a = \{ax \pmod{p} : x \in X\}$, to prove the desired result, we need to find such a so that Y intersects every interval of length $\lceil p/k \rceil$ in P.

Let $J_1, ..., J_{2k}$ be a fixed covering of P by $2k$ intervals of length $\lceil p/2k \rceil$ each. Note that if Y intersects with each J_i then it certainly satisfies the desired property because every interval of length at least $\lceil p/k \rceil$ in P must fully contain at least one J_i. Fix $i \in [2k]$, for any $x \in X$, let S_x^i be the indicating random variable of $ax \pmod{p} \in J_i$, then $S_i = \sum_{x \in X} S_x^i$ is 0 if and only if Y doesn't intersect J_i. By linearity of expectation,

$$\mathbb{E}[S^i] = \sum_{x \in X} \mathbb{E}[S_x^i] = \frac{4k^2\lceil p/2k \rceil}{p}.$$

Note that the random variables S_x^i for $x \in X$ are pairwise independent, hence

$$\mathrm{Var}[S^i] = \sum_{x \in X} \mathrm{Var}[S_x^i] = \frac{4k^2 \lceil p/2k \rceil}{p(1 - \lceil p/2k \rceil / p)} = \frac{\mathbb{E}[S^i]}{1 - \lceil p/2k \rceil / p}.$$

Therefore, by Chebyshev's Inequality,

$$\Pr(S^i = 0) \le \Pr(|S^i - \mathbb{E}[S^i]| \ge \mathbb{E}[X^i]) \le \frac{\mathrm{Var}[S^i]}{\mathbb{E}^2[S^i]} < \frac{1}{2k},$$

since there are $2k$ possible choices of i, we have that with positive probability, $S^i > 0$ for all $i \in [2k]$, completing the proof.

6.3   Reference

Alon, Noga, Igor Kriz, and J. Nesetril. "How to color shift hypergraphs." Studia
Scientiarum Mathematicarum Hungarica 30.1 (1995): 1-12.
Alon, Noga, and Yuval Peres. "Uniform dilations." Geometric & Functional Analysis
GAFA 2.1 (1992): 1-28.