Acyclic Edge Colorings of Graphs

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Abstract: A proper coloring of the edges of a graph G is called *acyclic* if there is no 2-colored cycle in G. The *acyclic edge chromatic number* of G, denoted by $a'(G)$, is the least number of colors in an acyclic edge coloring of G. For certain graphs G, $a'(G) \geq \Delta(G) + 2$ where $\Delta(G)$ is the maximum degree in G. It is known that $a'(G) \leq 16\,\Delta(G)$ for any graph G. We prove that

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there exists a constant c such that $a'(G) \leq \Delta(G) + 2$ for any graph G whose girth is at least $c\Delta(G) \log \Delta(G)$, and conjecture that this upper bound for $a'(G)$ holds for all graphs G. We also show that $a'(G) \leq \Delta + 2$ for almost all Δ-regular graphs. J Graph Theory 37: 157–167, 2001

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1. INTRODUCTION

All graphs considered here are finite and simple. A coloring of the vertices of a graph is proper if no pair of adjacent vertices are colored with the same color. Similarly, an edge coloring of a graph is proper if no pair of incident edges are colored with the same color. A proper coloring of the vertices or edges of a graph G is called *acyclic* if there is no 2-colored cycle in G. In other words, if the union of any two color classes induces a subgraph of G which is a forest. The *acyclic chromatic number* of G introduced in [7] (see also [8, Problem 4.11]), denoted by $a(G)$, is the least number of colors in an acyclic vertex coloring of G. The *acyclic edge chromatic number* of G, denoted by $a'(G)$, is the least number of colors in an acyclic edge coloring of G.

1.1. Lower and Upper Bounds

For a graph G, Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in G. Any proper edge coloring of G obviously requires at least Δ colors, and according to Vizing [12] there exists a proper edge coloring with $\Delta + 1$ colors. It is easy to see that any acyclic edge coloring of a Δ-regular graph uses at least $\Delta + 1$ colors. There are cases where more than $\Delta + 1$ colors are needed for coloring the edges acyclically:

$$a'(K_{2n} \setminus F) \geq 2n + 1 = \Delta(K_{2n} \setminus F) + 2, \tag{1}$$

where K_{2n} is the complete graph on $2n$ vertices and $F \subset E(K_{2n})$ such that $|F| \leq n - 2$. This is because one color class can contain at most n edges (a perfect matching), and all other color classes can contain at most $n - 1$ edges each.

Alon et al. [2] proved that $a'(G) \leq 64\Delta$, and remarked that the constant 64 can be reduced. Molloy and Reed [10] showed that $a'(G) \leq 16\Delta$ using the same proof. The constant 16 can, in fact, be further improved. We conjecture that the lower bound in (1) is an upper bound for all graphs.

Conjecture 1. $a'(G) \leq \Delta(G) + 2$ *for all graphs G.*

Conjecture 1 is interesting for graphs G with $\Delta(G) \geq 3$. Burnstein [6] showed that $a(G) \leq 5$ if $\Delta(G) = 4$. Since any acyclic vertex coloring of the line graph $L(G)$ is an acyclic edge coloring of G and vice versa, this implies that

$a'(G) = a(L(G)) \leq 5$ if $\Delta(G) = 3$. Hence Conjecture 1 is true for $\Delta = 3$. We have found another proof for this case, which also yields a polynomial algorithm for acyclically coloring the edges of a graph of maximum degree 3 using five colors.

The only graphs G for which we know that $a'(G) > \Delta(G) + 1$ are the subgraphs of K_{2n} that have at least $2n^2 - 2n + 2$ edges (see (1)). Therefore, it might even be true that if G is a Δ-regular graph[1] then

$$a'(G) = \begin{cases} \Delta + 2 \text{ for } G = K_{2n}, \\ \Delta + 1 \text{ otherwise.} \end{cases}$$

1.2. Complete Graphs

A conjecture closely related to the problem of determining $a'(G)$ for complete graphs $G = K_n$ is the *perfect 1-factorization conjecture* (see [9,13,14]).

Conjecture 2 (perfect 1-factorization [9]). *For any $n \geq 2, K_{2n}$ can be decomposed into $2n - 1$ perfect matchings such that the union of any two matchings forms a hamiltonian cycle of K_{2n}.*

Apart from proving that the conjecture holds for certain values of n, for instance, if n is prime [9] (see [13] for a summary of the known cases), this conjecture of Kotzig [9] and others is still open. If such a decomposition of K_{2n+2} (called a *perfect* 1-*factorization*) exists, then by coloring every perfect matching using a different color and removing one vertex we obtain an acyclic edge coloring of K_{2n+1} with $2n + 1 = \Delta(K_{2n+1})+1$ colors. Such a coloring is best possible for K_{2n+1} since it is $2n$-regular.

A decomposition of K_{2n+1} into $2n + 1$ matchings each having n edges, such that the union of any two matchings forms a Hamiltonian path of K_{2n+1} is called a *perfect near*-1-*factorization.* As shown above, if K_{2n+2} has a perfect 1-factorization then K_{2n+1} has a perfact near-1-factorization, which is turn implies that $a'(K_{2n+1}) = 2n + 1$. It is easy to see that the converse is also true: if K_{2n+1} has an acyclic edge coloring with $2n + 1$ colors, then this coloring corresponds to a perfect near-1-factorization of K_{2n+1} which implies that K_{2n+2} has a perfect 1-factorization. Therefore the following holds.

Proposition 3. *The following statements are equivalent*:

1. *K_{2n+2} has a perfect 1-factorization.*
2. *K_{2n+1} has a perfect near-1-factorization.*
3. $a'(K_{2n+1}) = 2n + 1$.

[1]There always is a Δ-regular graph G' which satisfies $a'(G') = \max \{a'(G) : \Delta(G) = \Delta\}$.

By removing another vertex from the above-colored K_{2n+1}, we obtain an acyclic edge coloring of K_{2n} with $2n+1 = \Delta(K_{2n})+2$ colors, which is best possible for K_{2n}. Thus, if the perfect 1-factorization conjecture is true, then $a'(K_{2n}) = a'(K_{2n+1}) = 2n+1$ for every n. It may be possible to show the converse, i.e. if $a'(K_{2n}) = 2n+1$ then K_{2n+2} has a perfect 1-factorization. It may even be true that any acyclic edge coloring of K_{2n} with $2n+1$ colors can be completed into an acyclic edge coloring of K_{2n+1} without introducing new colors.

Alon et al. [2] observed that a' (K_p) $= a'(K_{p-1,p-1}) = p$, where $p > 2$ is prime. The fact that $a'(k_p) = p$ corresponds to the known construction proving that K_p has a perfect near-1-factorization [9]. Note that even finding the exact values of $a'(K_n)$ for every n seems hard, in view of Proposition 3 and Conjecture 2.

1.3. High Girth and Random Graphs

Using probabilistic arguments (the Lovász Local Lemma), we can show that Conjecture 1 holds for graphs having sufficiently high girth in terms of their maximum degree, and for "almost all" d-regular graphs. Recall that the girth $g(G)$ of a graph G is the minimum length of a cycle in G. Let G be a graph of maximum degree $\Delta = \Delta(G)$.

Theorem 4. *There exists $c > 0$ such that if $g(G) \geq c\Delta \log \Delta$, then $a'(G) \leq \Delta + 2$.*

Let $G_{n,d}$ denote the probability space of all d-regular simple graphs on n labeled vertices (dn is even), where all graphs have the same probability. We consider d fixed and $n \to \infty$ and say that some event is this space occurs almost surely (a.s.) if the probability of this event tends to 1 when n tends to ∞. Using known properties of random graphs we can prove the following.

Theorem 5. *Let $G \in G_{n,d}$ be the random d-regular graph on n labeled vertices. Then a.s. $a'(G) \leq d+1$ for even n and $a'(G) \leq d+2$ for odd n.*

In Section 2, we present the proof of Theorem 4, and in Section 3 present the proof of Theorem 5. Section 4 contains some concluding remarks.

2. PROOF OF THEOREM 4

Let G be a graph with maximum degree d. We do not attempt to optimize the constants here and in what follows. In this section, we show that if $g(G) \geq 2000\, d \log d$, where $g(G)$ is the girth of G (the minimum length of a cycle in G) then there exists an acyclic edge coloring of G with $d+2$ colors.

The proof is probabilistic, and consists of two steps. The edges of G are first colored properly using $d+1$ colors (by Vizing [12]). Let $c : E \mapsto \{1, \ldots, d+1\}$ denote the coloring. Next, each edge is recolored with a new color $d+2$

randomly and independently with probability $1/32d$. It remains to show that with positive probability

(A) the coloring remains proper—no pair of incident edges are colored $d+2$, and
(B) the coloring becomes acyclic—every cycle of G contains at least three different colors.

This is proved using the Lovász Local Lemma. Before continuing with the proof, we state the asymmetric form of the Lovász Local Lemma we use (cf., e.g. [3]).

The Lovász Local Lemma. *Let $A_1, \ldots, A_n$ be events in a probability space Ω, and let $G = (V, E)$ be a graph on $V = [1, n]$ such that for all i, the event A_i is mutually independent of $\{A_j : (i,j) \notin E\}$. Suppose that there exists $x_1, \ldots, x_n$, $0 < x_i < 1$, so that, for all i,* $\mathrm{Prob}[A_i] < x_i \prod_{(i,j)\in E}(1 - x_j)$. *Then* $\mathrm{Prob}[\wedge \overline{A}_i] > 0$.

The following three types of "bad" events are defined in order to satisfy (A) and (B) above.

Type I: For each pair of incident edges $B = \{e_1, e_2\}$, let E_B be the event that both e_1 and e_2 are recolored with color $d+2$.
Type II: For each cycle C which was bichromatic by the first coloring c, let E_C be the event that no edge of C was recolored with color $d+2$.

A simple cycle D having an even number of edges is called *half-monochromatic* if half its edges (every other edge) are colored the same by the first coloring c. Note that this includes cycles which are bichromatic by the first coloring.

Type III: For each half-monochromatic cycle D, let E_D denote the event that half the edges of D are recolored with color $d+2$ (all "other" edges) such that D becomes (or stays) bichromatic.

Now suppose that no event of type I, II or III holds. We claim that both (A) and (B) are satisfied. Clearly (A) is satisfied if no event of type I holds. Now suppose that (B) is not satisfied, i.e. there exists a cycle C which is bichromatic after the recoloring. If C does not contain edges of color $d+2$ then the event E_C of type II holds, otherwise C is a half-monochromatic cycle and event E_C of type III holds. Therefore, if none of these events hold, both (A) and (B) are satisfied.

It remains to show that with positive probability none of these events happen. To prove this we apply the local lemma. Let us construct a graph H whose nodes are all the events of the three types, in which two nodes E_X and E_Y (where each of X, Y is either a pair of incident edges, a bichromatic cycle or a half-monochromatic cycle) are adjacent if and only if X and Y contain a common edge. Since the occurrence of each event E_X depends only on the edges of X, H is a dependency graph for our events. In order to apply the local lemma we need estimates for the probability of each event and for the number of nodes of each type in H which are adjacent to any given node. These estimates are given in the two lemmas below, whose proofs are straightforward and thus omitted (except for a proof of Lemma 7(3)).

Lemma 6.

1. *For each event E_B of type I,* $\mathrm{Prob}[E_B] = 1/1024d^2$.
2. *For each event E_C of type II, where C is of length x,* $\mathrm{Prob}[E_C] = (1-(1/32d))^x \leq \mathrm{e}^{-x/32d}$.
3. *For each event E_D of type III, where D is of length $2x$,* $\mathrm{Prob}[E_D] \leq 2/(32d)^x$.

Lemma 7. *The following is true for any given edge e:*

1. *Less than $2d$ edges are incident to e.*
2. *Less than d bichromatic cycles contain e.*
3. *At most $2d^{k-1}$ half-monochromatic cycles of length $2k$ contain e.*

To prove part 3 of Lemma 7, note that every half-monochromatic cycle of length $2k$ that contains edge $e = (v_0, v_1)$ can be constructed as follows. First, select a vertex v_2 which is adjacent to v_1 (d possibilities). Next, decide if e or $f = (v_1, v_2)$ belong to the "monochromatic edges" (two possibilities). Suppose e was chosen. Let vertex v_3 be the vertex adjacent to v_2 such that $c((v_2, v_3)) = c(e)$, if one exists. There is at most one such vertex v_3 since the coloring c is proper. If such a vertex does not exist, the number of cycles is smaller than the bound presented in the lemma. Now continue with $i = 2, \ldots, k-1$: choose v_{2i} to be any vertex adjacent to v_{2i-1}(d possibilities), and let v_{2i+1} be the vertex adjacent to v_{2i} such that $c((v_{2i}, v_{2i+1})) = c(e)$. This completes the construction of the desired cycle. The case where f belongs to the "monochromatic edges" is treated exactly the same after swapping v_0 with v_2. Therefore, the number of half-monochromatic cycles of length $2k$ that contain edge e is at most $2d^{k-1}$. □

It follows from Lemma 7 that each event E_X where X contains x edges is adjacent (in the dependency graph H) to at most $2xd$ events of type I, at most xd events of type II and at most $2xd^{k-1}$ events E_D of type III, where D is of length $2k$, for all $k \geq 2$.

The last ingredient required for applying the Lovász Local Lemma are the real constants x_i. Let $1/512d^2, 1/128d^2$ and $1/(2d)^k$ be the constants associated with events of type I, events of type II and events E_D of type III, where D is of length $2k$, respectively. We conclude that with positive probability no event of type I, II or III occurs, provided that

$$\frac{1}{1024d^2} \leq \frac{1}{512d^2}\left(1-\frac{1}{512d^2}\right)^{4d}\left(1-\frac{1}{128d^2}\right)^{2d}\prod_k\left(1-\frac{1}{(2d)^k}\right)^{4d^{k-1}}, \tag{2}$$

$$e^{-x/32d} \leq \frac{1}{128d^2}\left(1-\frac{1}{512d^2}\right)^{2xd}\left(1-\frac{1}{128d^2}\right)^{xd}\prod_k\left(1-\frac{1}{(2d)^k}\right)^{2xd^{k-1}} \quad \text{for all } x \geq 4, \tag{3}$$

$$\frac{2}{(32d)^x} \leq \left(\frac{1}{(2d)^x}\right)\left(1-\frac{1}{512d^2}\right)^{4xd}\left(1-\frac{1}{128d^2}\right)^{2xd}\prod_k\left(1-\frac{1}{(2d)^k}\right)^{4xd^{k-1}} \quad \text{for all } x \geq 2. \tag{4}$$

Now since $(1-(1/z))^z \geq \frac{1}{4}$ for all real $z \geq 2$, the following holds for all $x, d \geq 2$:

$$\prod_k\left(1-\frac{1}{(2d)^k}\right)^{2xd^{k-1}} \geq \prod_k\left(\frac{1}{4}\right)^{x/d2^{k-1}} = \left(\frac{1}{4}\right)^{(x/d)\sum_k 1/2^{k-1}} \geq \left(\frac{1}{4}\right)^{x/256d}, \tag{5}$$

where the last inequality uses the fact that $2k \geq g(G) \geq 2000d \log d \geq 20$, and similarly

$$\left(1-\frac{1}{512d^2}\right)^{2xd} \geq \left(\frac{1}{4}\right)^{x/256d}, \tag{6}$$

$$\left(1-\frac{1}{128d^2}\right)^{xd} \geq \left(\frac{1}{4}\right)^{x/128d}. \tag{7}$$

Combining (5)–(7), we conclude that

$$\left(1-\frac{1}{512d^2}\right)^{2xd}\left(1-\frac{1}{128d^2}\right)^{xd}\prod_k\left(1-\frac{1}{(2d)^k}\right)^{2xd^{k-1}} \geq \left(\frac{1}{2}\right)^{x/32d}.$$

Thus inequality (2) holds since $2^{(1-1/16d)} \geq 1$, and inequality (4) holds since $2^{(1-5x+x+x/16d)} \leq 1$ for all $x \geq 1$. To prove inequality (3) it suffices to show that

$$e^{-x/32d} \leq \frac{1}{128d^2}\left(\frac{1}{2}\right)^{x/32d},$$

which holds for all $x \geq 2000d \log d \geq 32d\ (\log\ (128d^2)/\log(e/2))$ and $d > 2$, thereby completing the proof. $\square$

3. RANDOM REGULAR GRAPHS

In this section, we prove Theorem 5 which shows that Conjecture 1 is true for almost all d-regular graphs. We use $G_{n,d}$ to denote the probability space of all d-regular simple graphs on n labeled vertices (dn is even), where each such graph is picked uniformly at random. We consider d fixed and $n \to \infty$ and say that some event in this space occurs a.s. if the probability of this event tends to 1 when n tends to ∞.

Random d-regular graphs can be generated using the following model given in [5, pp. 48–52]. Let $W = \cup_{j=1}^{n} W_j$ be a fixed set of $2m = dn$ labeled vertices, where $|W_j| = d$ for each j. A *configuration* F is a partition of W into m pairs of vertices, called edges of F (i.e. a perfect matching). Let $\mathcal{F}_{n,d}$ be a probability space where all configurations are equiprobable. For $F \in \mathcal{F}_{n,d}$, let $\phi(F)$ be the graph on vertex set $\{1, 2, \ldots, n\}$ in which ij is an edge iff F has an edge joining W_i to W_j. Clearly $\phi(F)$ is a graph with maximum degree at most d. More importantly, the probability that $\phi(F)$ is a d-regular simple graph is bounded away from 0 as $n \to \infty$, and all such d-regular graphs are obtained in this model with the same probability. Thus, in order to study the properties of random d-regular graphs that hold a.s. we can consider the space of configurations.

By estimating the expected number of subgraphs of a given type in $\mathcal{F}_{n,d}$ it can be easily proved (as shown implicitly in [5]) that for every fixed c, random d-regular graphs a.s. contain no subgraph on c vertices with more edges than vertices. This implies the following lemma.

Lemma 8. *Let d, s and t be fixed positive integers. Then a.s. a random d-regular graph has no two cycles of length at most s connected by a path of length at most t.* $\square$

We shall also need the following result about the edge chromatic number of random d-regular graphs, obtained by Robinson and Wormald [11].

Lemma 9. *For $d \geq 3$ and even n, the edge chromatic number of $G \in G_{n,d}$ is a.s. equal to d.* $\square$

Using the above two lemmas and ideas from the proof of Theorem 4 we can deduce Theorem 5, which states that $a'(G) \leq d + 1$ a.s. for $G \in G_{n,d}$ where n is even, and $a'(G) \leq d + 2$ a.s. for $G \in G_{n,d}$ where n is odd.

Proof of Theorem 5. Let G be a random d-regular graph. We consider the case when n is even—the case of odd n can be treated similarly using Vizing's theorem [12] instead of Lemma 9. The proof is probabilistic and consists of two steps.

First, the edges of G are properly colored using d colors. By Lemma 9, this is a.s. possible. Let $c : E \mapsto \{1, \ldots, d\}$ denote the coloring. Next, an edge is selected from each bichromatic cycle and colored with a new color $d + 1$. It remains to show that with positive probability the coloring remains proper and becomes acyclic. This is proved using the symmetric form of the Lovász Local Lemma, which is stated below (cf., e.g. [3]).

The Lovász Local Lemma (symmetric case). *Let $A_1, \ldots, A_n$ be events in a probability space Ω. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most d, and that* $\mathrm{Prob}[A_i] \leq p$ *for all i. If $ep(d + 1) \leq 1$, then* $\mathrm{Prob}[\wedge \overline{A_i}] > 0$.

Call a cycle in G *short* if it has less than $800d^3$ edges, and *long* otherwise. This threshold is required later in the proof. Let $\{C_1, C_2, \ldots, C_k\}$ be the set of all short bichromatic cycles in G. From each short cycle C_j pick an arbitrary edge e_j and color it with a new color $d + 1$. By Lemma 8, the distances between these edges are a.s. at least, say, $2d^2 + 2$, since a.s. there are no two short cycles connected by a path of length at most $2d^2 + 2$. Call an edge of G *bad* if it is within distance at most 1 from some edge e_j $(1 \leq j \leq k)$, otherwise call it *good*. We claim that every long cycle X having $|X|$ edges contains at least $\frac{1}{2}|X|$ good edges. To establish this claim observe that there are at most $2d^2$ bad edges within distance at most 1 from any particular edge e_j. Therefore, if X contains more than $\frac{1}{2}|X|$ bad edges then there is a pair of bad edges in X within distance at most $2d^2$ from each other such that one is within distance at most 1 from e_i and the other within distance at most 1 from e_j, where $1 \leq i \neq j \leq k$. This implies the existence of a path of length at most $2d^2 + 2$ from e_i to e_j, a.s. a contradiction according to Lemma 8.

Let $\{D_1, D_2, \ldots, D_m\}$ be the set of all long bichromatic cycles in G. From each long cycle D_j we restrict our attention to a path p_j of at most $800d^3$ edges which contains $400d^3$ good edges. Such a path exists since at least half the edges of D_j are good, and the length of D_j is at least $800d^3$. Now we randomly pick a good edge (f_j) from each path p_j and recolor it with color $d + 1$. Let $E_{i,j}$ be the "bad" event that edges f_i, f_j are at distance at most 1 from each other $(1 \leq i, j \leq m)$. Notice that if no event $E_{i,j}$ happens then the distance between any pair of edges recolored with color $d + 1$ is more than 1, and therefore the recoloring is proper and acyclic.

The probability of each event $E_{i,j}$ can be bounded using the following observations. First notice that any two cycles D_i, D_j can intersect and share a vertex or an edge, but they cannot share a path of length greater than 1 because they are both bichromatic. At any intersection of D_i, D_j (a common vertex or edge) or edge (u, v) connecting D_i and D_j (where $u \in D_i$ and $v \in D_j$), there are at most 16 pairs of edges one from D_i and the other from D_j with distance at most 1 from each other. If two paths p_i, p_j have more than two intersections or connecting edges, then there exists a subgraph of G on at most $1600d^3 + 2$ vertices with more edges than vertices, which according to Lemma 8 a.s. does not happen. Therefore, the probability of each event $E_{i,j}$ is a.s. at most $32/(400d^3)^2 = 32/160000d^6$.

It is easy to see that there are less than $2d^3$ bichromatic (long) cycles at distance at most 1 from any given edge. Since each event $E_{i,j}$ is independent of all events $E_{p,q}$ such that $\{i,j\} \cap \{p,q\} = \emptyset$, it follows that each event is independent of all events but at most $2 \times 400d^3(2d^3 - 1) < 1600d^3$ events. Now the local lemma can be applied since $(32e/160000d^6)1600d^6 \leq 1$, implying that with positive probability a.s. no event $E_{i,j}$ holds, thereby completing the proof.

□

4. CONCLUDING REMARKS

1. The following weaker version of Theorem 4 can be proved in a similar but simpler way using the symmetric Lovász Local Lemma;

   Proposition 10. *There exists a constant $c > 0$ such that $a(G) \leq \Delta(G) + 2$ if $g(G) > c\Delta(G)^3$.*

   This can be achieved by recoloring one edge from each bichromatic cycle using one additional color, while avoiding recoloring any pair of edges which are incident or at distance 1 from each other (similar to the proof of Theorem 5).

2. By increasing the number of colors we are able to reduce the condition on the girth as follows.

   Theorem 11. *If $g(G) \geq (1 + o(1)) \log \Delta$, then $a'(G) \leq 2\Delta + 2$.*

   This can be achieved by first coloring the edges properly using $\Delta + 1$ colors $1, \ldots, \Delta + 1$, and then assigning a negative sign to the color of each edge with probability $\frac{1}{2}$.

3. For graphs G of class 1 Vizing (i.e. graphs whose edges can be properly colored using Δ (G) colors), the bound for $a(G)$ presented in Theorem 4 can be slightly improved. Indeed, the proof of this theorem shows that for graphs G of class 1 there exists a constant c such that

$$a'(G) \leq \Delta(G) + 1 \quad \text{if } g(G) \geq c\Delta \log \Delta.$$

   Note that this shows that $a'(G) = \Delta + 1$ for any Δ-regular graph G of class 1 whose girth is sufficiently large as a function of Δ.

4. Molloy and Reed [10] presented, for every fixed Δ, a polynomial-time algorithm that produces an acyclic coloring with 20Δ colors for any given input graph with maximum degree Δ. The known results about the algorithmic version of the local lemma, initiated by Beck [4] (see also [1,10]), can be combined with our method here to design, for every fixed

Δ, a polynomial algorithm that produces an acyclic $\Delta + 2$ coloring for any given input graph with maximum degree Δ whose girth is sufficiently large as a function of Δ.

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