Europ. J. Combinatorics (1985) **6**, 85–89

Even Cycles in Directed Graphs

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We discuss the complexity of finding a cycle of even length in a digraph. In particular, we observe that finding a cycle of prescribed parity through a prescribed edge is NP-complete. Also, we settle a problem of Lovász [11] and disprove a conjecture of Seymour [15] by describing, for each natural number k, a digraph of minimum outdegree k and with no even cycle. We prove that a digraph of order n and minimum outdegree $[\log_2 n]+1$ contains, for each edge set E, a cycle containing an even number of edges of E and we show that this is best possible.

A modification of the construction yields counterexamples to Hamidoune's conjecture [5] on local connectivity in digraphs of large indegrees and outdegrees.

1. INTRODUCTION

It is well-known that an undirected graph has an odd cycle unless it is bipartite and that it has an even cycle unless each block of the graph is an odd cycle or a K_2. Furthermore, it is easy to see that two vertices x and y in a connected graph are connected by an even path and also by an odd path unless each block that intersects the edge set of some (and hence each) path from x to y is bipartite. In particular, it is easy to decide whether or not a given edge is contained in an even (respectively odd) cycle. Also, a shortest even (respectively) odd cycle can be found by an efficient algorithm (as pointed out in the next section).

While it is easy to decide if a digraph has an odd cycle and to find a shortest such cycle if it is present (see the next section) it is surprisingly difficult to decide if a digraph has an even cycle. D. H. Younger (see [2]) asked if there exists a polynomially bounded algorithm for finding an even cycle. This problem is of interest in its own right but it also shows up in other contexts as described below.

Lovász [12] showed that it is NP-complete to decide whether or not a hypergraph is 2-colourable. Seymour [16] showed that a minimally 3-chromatic hypergraph has at least as many edges as vertices and he gave a surprising characterization, in terms of digraphs with no even cycles, of those minimally 3-chromatic hypergraphs that have the same number of vertices and edges. Furthermore, he conjectured in [15] that some edge of such a hypergraph must have small cardinality. Seymour's characterization shows that this is equivalent to an affirmative answer to the question of Lovász [11] if there are digraphs with no even cycles and with large minimum outdegree. However, we show by a simple recursive construction, that the answer is negative. More precisely, for each natural number $n \geq 2$, there exists a digraph of order n and minimum outdegree at least $[\frac{1}{2}\log_2 n]$ which contains no even cycle. On the other hand, if a digraph has minimum outdegree at least $[\log_2 n]+1$, then it contains, for each edge set E, a cycle containing an even number of edges of E and this becomes false if we replace $[\log_2 n]+1$ by $[\log_2 n]$.

A similar construction produces counterexamples to the conjecture of Hamidoune [5] that any digraph of minimum indegree and outdegree at least k contains an edge xy and k internally disjoint paths from y to x. The analogous statement for undirected graphs was proved by Mader [14].

Klee, Ladner and Manber [9] showed that a good algorithm for finding a cycle C containing an even number of edges of a prescribed edge set E in a digraph implies good algorithms for certain matrix problems of interest in mathematical economics. Fortune, Hopcroft and Wyllie [4] proved that the problem is NP-complete if C is supposed to

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have non-empty intersection with E (even when $|E|=2$) and using this we establish NP-completeness of the problem of finding an even (respectively odd) cycle through a given edge of a digraph (a fact which was also noted by Klee *et al.* [9]). Furthermore, we point out that another result in [4] implies a polynomially bounded algorithm for the above problem if C is not required to intersect E and if, in addition, $|E|$ is bounded by a fixed constant.

2. On the Complexity of the Even Cycle Problem

It is easy to find a shortest odd cycle in a non-bipartite (undirected) graph but less obvious how to find a shortest even cycle. The following trick, due to J. Edmonds, was communicated to the author by D. H. Younger. Let G be a graph a x and y vertices of G. Let G' be a copy of G. Then form the disjoint union $G \cup G'$ and assign the weight 1 to all the edges. Then add, for each vertex z of G, the edge between z and the corresponding vertex in G' and associate the weight 2 to that edge. Finally delete the vertices of G' corresponding to x and y. Then the resulting graph G'' has a 1-factor iff G has an odd x–y path. Furthermore, a 1-factor of G'' of maximum weight corresponds to a shortest odd path in G from x to y. Thus the problem of finding a shortest even (respectively odd) cycle through a given edge is reduced to the maximum 1-factor problem which was solved by Edmonds [3].

A digraph has an odd cycle unless each strong component is bipartite (see [6]). Furthermore, it is easy to find a shortest odd cycle in a digraph. More generally, by a simple labelling procedure one can count, in polynomial time, the number of walks of length k, $1 \leq k \leq |V(D)|$, from a vertex x to a vertex y in a digraph D. Since a shortest closed odd walk is a shortest odd cycle one can thus count, in polynomial time, the number of shortest odd cycles of a digraph that contain a specified edge. (C. Godsil has pointed out that the characteristic polynomial of the digraph also contains this information). In view of this the following may seem surprising.

Proposition 2.1. *Given an edge e in a digraph D it is NP-complete to decide if D has an odd cycle through e and it is NP-complete to decide if D has an even cycle through e.*

Proof. Clearly the two problems in Proposition 2.1 have the same complexity. Fortune, Hopcroft and Wyllie [4] showed that it is NP-complete to decide if a digraph D with prescribed vertices x_1, x_2, y_1, y_2 contains two disjoint (directed) paths P_1 and P_2 such that P_i starts at x_i and terminates at y_i for $i=1, 2$. Now form a new digraph D' by subdividing each edge of D once and then adding the edges y_2x_1 and y_1x_2. Then D' has an even cycle through y_2x_1 if and only if D contains the two paths P_1, P_2 described above and thus an NP-complete problem has been reduced to the latter problem in Proposition 2.1.

The problem mentioned in the introduction of finding a cycle containing an even number of edges of an edge set E in a digraph D is equivalent to the (seemingly more special) problem of finding an even cycle in a digraph. This can be seen simply by subdividing each edge of D–E. When E is not too large the problem can be solved by a good algorithm.

Proposition 2.2. *For any (fixed) positive constant c there exists a polynomially bounded algorithm for deciding if a digraph D of order n contains a cycle containing an even number of edges of a prescribed edge set E of cardinality $m \leq c$.*

Proof. If D – E has a cycle we have finished so assume that D – E is acyclic. Let $(e_1, e_2, \ldots, e_k)$ be an ordered k-tupel of edges where $e_i = x_iy_i$ is an edge of D for $i=1, 2, \ldots, k$.

By [4] there is a polynomially bounded algorithm for deciding if D-E contains an y_1x_2 path, a y_2x_3 path, ..., a $y_{k-1}x_k$ path, and a y_kx_1 path which are pairwise disjoint. In other words, there is a polynomially bounded algorithm for deciding, for any ordered subset E' of E, if $D-(E\setminus E')$ has a cycle containing all edges of E' in the prescribed order. Since the number of ordered subsets is less than $2^m m!$ the proof is complete.

3. Digraphs with Large Outdegrees and no Even Cycle

Koh [8] showed that there are infinitely many digraphs of minimum indegree and outdegree 2 and with no even cycle. We strengthen this result by answering Lovasz' question in [11].

Theorem 3.1. For each natural number k, there exists a digraph D_k with no even cycle such that each vertex of D_k has outdegree k.

Proof. Clearly D_1 exists. Suppose that D_k exists. Then D_{k+1} is obtained by adding, for each vertex x of D_k, a set A_x of $k+1$ vertices and a vertex x' such that each vertex of A_x dominates x and the vertices of D_k dominated by x. Furthermore we let x' dominate the vertices of A_x and we let x' be dominated by x. Clearly each vertex of D_{k+1} has outdegree $k+1$. Moreover, for each cycle of D_{k+1} there is a cycle in D_k of the same parity. Thus D_{k+1} has no even cycle.

The digraph D_k of Theorem 3.1 has vertices of indegree 1. However, if we form the disjoint union of D_k and the converse digraph and then let each vertex of the latter dominate each vertex of the former, then the resulting digraph has minimum indegree and outdegree k and it has no even cycle. We do not know if there exists a strongly connected digraph with these properties. Lovász [11] asked if a digraph of sufficiently large connectivity must contain an even cycle. We believe that the answer is affirmative. A more general conjecture is formulated in the last section.

P. Erdös (private communication) has pointed out that Theorem 3.1 leads to the following natural question: What is the smallest number $g(n)$ such that every digraph of order n and minimum outdegree $g(n)$ contains an even cycle. In connection with the aforementioned problem of Klee *et al.* it also seems worthwhile to consider the smallest natural number $h(n)$ such that, for each edge set E of a digraph D of order n and minimum outdegree at least $h(n)$, D has a cycle containing an even number of edges of E. If we think of the pair (D, E) as a *weighted digraph* where the edges of E have weight one and all other edges have weight zero, then we seek a cycle of even weight.

Clearly $g(n) \leq h(n)$ and the construction in Theorem 3.1 shows that $g(n)$ exceeds constant times $\log n/\log\log n$. We shall now improve on this.

Theorem 3.2. *For each* $n \geq 2$, $[\frac{1}{2}\log_2 n] \leq g(n) \leq h(n) = [\log_2 n]+1$.

Proof. For each natural number k we consider the weighted digraph G_k defined as follows. G_1 is a 2-cycle whose edges have weight 0 and 1, respectively. Having defined G_k, $1 \leq k$, we consider any vertex x of G_k and add a new vertex x' such that x' is dominated by x and dominates x and all vertices of G_k dominated by x. Then we assign distinct weights to the edges between x and x' and the weight of an edge $x'y$ is determined such that the paths xy and $xx'y$ have the same weight modulo 2. There are two possibilities for assigning weights to the edges between x and x' and we choose the assignment such that, for each vertex, the number of edges of weight 0 starting at the vertex exceeds the number of edges of weight 1 starting at the vertex by at most 1. Then, for each k, G_k has 2^k vertices each of which has outdegree k. Moreover, G_k has no even cycle, and if we delete the edges of weight zero we obtain a digraph of minimum outdegree at least $\frac{1}{2}(k-1)$ which has no even cycle. This shows that

$$h(n) \geq \log_2 n + 1 \qquad \text{and} \qquad g(n) \geq \tfrac{1}{2}\log_2 n$$

when n is a power of 2. The general case fqllows by considering induced subgraphs of G_{k+1} containing G_k.

To prove the equality in Theorem 3.2 we use a trick similar to a method used recently by R. Häggkvist (private communication) to obtain decomposition results on graphs with constraints on the degrees. Consider a weighted digraph of order n such that all outdegrees are at least k and assume that D has no cycle of even weight. Let $A_1 \cup A_2$ be any ordered partition of $V(D)$. We consider the subgraph of D consisting of all edges of weight 1 between A_1 and A_2 and all edges of weight 0 that do not join A_1 and A_2. Any cycle of this subdigraph has even weight so there is no such cycle. Hence the subdigraph has a vertex of outdegree zero. We call such a vertex *terminal* with respect to A_1, A_2.

The number of partitions A_1, A_2 for which a given vertex x is terminal is precisely 2^{n-m} where m is the outdegree of x so the number of partitions having at least one terminal vertex is at most $n2^{n-k}$. Since each partition has a terminal vertex,

$$2^n \leq n2^{n-k}$$

which proves that

$$h(n) \leq \log_2 n + 1.$$

4. Digraphs of Large Degrees and Small Local Connectivity

Behzad, Chartrand and Wall [1] conjectured that each r-diregular digraph with at most mr vertices contains a cycle of length at most m. Hamidoune [5] verified this for vertex-transitive digraphs and applied that result to groups. He also conjectured that every digraph of minimum indegree and outdegree at least k contains an edge yx and k internally disjoint xy paths. This would generalize a result of Mader [14] on undirected graphs and prove the above conjecture of Behzad *et al.* Inspired by the construction in Section 3 it is easy to disprove Hamidoune's conjecture.

Theorem 4.1. *For each natural number k there exists a digraph H_k of minimum outdegree and indegree k such that*
(a) *H_k does not contain a vertex x and three cycles having precisely x in common pair by pair,*
(b) *H_k does not contain a vertex x contained in two cycles of length 2, and*
(c) *H_k does not contain an edge xy and three internally disjoint paths from y to x.*

Proof. Clearly H_1 exists. Suppose H_k exists ($k \geq 1$) and let z be any vertex of H_k and let A be the set of vertices dominated by z. Now add to H_k two vertices z_1, z_2 such that z_i dominates z_{3-i} and each vertex of A for $i = 1, 2$ and let furthermore z dominate z_1. Then it is easy to see that the resulting digraph H' satisfies (a) since the vertex x in condition (a) would have to be in H_k. Clearly H' satisfies (b). If H' violates (c), then x is in A and $y = z$. Then we delete the edge zx from H' and perform again the above construction with $H' - \{zx\}$ instead of H_k and obtain thereby the digraph H'', say. Because of (b), $H' - \{zx\}$ has no cycle of length 2 containing z and hence H'' satisfies (a), (b) and (c).

If we perform successively the above construction for each vertex of H_k, we obtain a digraph of minimum outdegree $k+1$ satisfying (a), (b) and (c) and then, by the construction mentioned after Theorem 3.1 we get the desired digraph H_{k+1}.

5. A Research Problem

Theorems 3.1 and 4.1 show that a large minimum indegree and outdegree in a digraph does not imply the existence of such a rich family of configurations in the digraph as a large minimum degree implies in an undirected graphs as shown by Mader [13] and others. Therefore it is perhaps more fruitful to seek configurations in digraphs of large connectivity. We propose the following general conjecture.

Conjecture 5.1. *For each natural number k there exists a natural number $f(k)$ such that each $f(k)$-connected digraph is strongly k-linked, i.e., for any $2k$ distinct vertices x_1, $x_2, \ldots, x_k, y_1, \ldots, y_k$ the digraph contains k disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i starts in x_i and terminates in y_i for $i = 1, 2, \ldots, k$.*

The restriction of Conjecture 5.1 to undirected graphs was proved by Larman and Mani [10] and Jung [7] using a result of Mader [13], and the author [17] showed that the lengths of the paths P_l modulo a fixed odd natural number can be prescribed. If true, Conjecture 5.1 implies the existence of subdivisions of large complete symmetric digraphs in digraphs of large connectivity. In particular, it implies the existence of even cycles in such digraphs since each subdivision of the complete symmetric digraph of order 3 contains an even cycle.

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Received 2 November 1982

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