Kőnig's Line Coloring and Vizing's Theorems for Graphings

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Abstract

The classical theorem of Vizing states that every graph of maximum degree d admits an edge-coloring with at most $d + 1$ colors. Furthermore, as it was earlier shown by Kőnig, d colors suffice if the graph is bipartite.

We investigate the existence of measurable edge-colorings for graphings. A graphing is an analytic generalization of a bounded-degree graph that appears in various areas, such as sparse graph limits and orbit equivalence theory. We show that every graphing of maximum degree d admits a measurable edge-coloring with $d + O(\sqrt{d})$ colors; furthermore, if the graphing has no odd cycles, then $d + 1$ colors suffice. In fact, if a certain conjecture about finite graphs that strengthens Vizing's theorem is true, then our method will show that $d + 1$ colors are always enough.

1   Introduction

The old theorem of Kőnig [15] states that a bipartite graph of maximum degree d admits an edge-coloring with d colors. (Here, all edge-colorings are assumed to be *proper*, that is, no two adjacent edges have the same color.) Some 50 years later, Vizing [24] and, independently, Gupta [11] proved that, if we do not require that the graph is bipartite, then $d+1$ colors suffice. These results laid the foundation of edge-coloring, an important and active area of graph theory; see, for example, the recent book of Stiebitz, Scheide, Toft and Favrholdt [22].

In this paper, we consider measurable edge-colorings of *graphings* (which are graphs with some extra analytic structure, to be defined shortly). Although the graphs that we will consider

may have infinitely many (typically, continuum many) vertices, we will always require that the maximum degree is bounded.

If one does not impose any further structure, then Kőnig's and Vizing's theorems extend, with the same bounds, to infinite graphs by the Axiom of Choice. Indeed, every finite subgraph is edge-colorable by the original theorem so the Compactness Principle gives the required edge-coloring of the whole graph.

The first step towards graphings is to add Borel structure. Namely, a *Borel graph* (see e.g. Lovász [17, Section 18.1]) is a triple $\mathcal{G} = (V, \mathcal{B}, E)$, where $(V, \mathcal{B})$ is a standard Borel space and E is a Borel subset of $V \times V$ that defines a symmetric and anti-reflexive binary relation. As we have already mentioned, here we restrict ourselves to those graphs $\mathcal{G}$ for which the *maximum degree*

$$\Delta(\mathcal{G}) := \max\{\deg(x) : x \in V\}$$

is finite. While this definition sounds rather abstract, it has found concrete applications to finite graphs: e.g. Elek and Lippner [8] used Borel matchings to give another proof of the result of Nguyen and Onak [21] that the matching ratio in bounded-degree graphs is testable.

Define the *Borel chromatic number* $\chi_{\mathcal{B}}(\mathcal{G})$ of a Borel graph $\mathcal{G}$ to be the minimum $k \in \mathbb{N}$ such that there is a partition $V = V_1 \cup \cdots \cup V_k$ into Borel *independent* sets (that is, sets that do not span an edge of E). Also, the *Borel chromatic index* $\chi'_{\mathcal{B}}(\mathcal{G})$ is the smallest number of Borel matchings that partition E. (By a *matching* we understand a set of pairwise disjoint edges; we do not require that every vertex is covered.) A systematic study of Borel colorings was initiated by Kechris, Solecki and Todorcevic [14] who, in particular, proved the following result.

Theorem 1.1 ([14])**.** *For every Borel graph $\mathcal{G}$ of maximum degree d, we have that $\chi_{\mathcal{B}}(\mathcal{G}) \leq d+1$ and $\chi'_{\mathcal{B}}(\mathcal{G}) \leq 2d - 1$.*

Very recently, Marks [19] constructed, for every $d \geq 3$, an example of a d-regular Borel graph $\mathcal{G}$ such that $\mathcal{G}$ has no cycles, $\chi_{\mathcal{B}}(\mathcal{G}) = 2$ and $\chi'_{\mathcal{B}}(\mathcal{G}) = 2d - 1$. (Such a graph for $d = 2$ was earlier constructed by Laczkovich [16].) We see that the Borel chromatic index may behave very differently from the finite case.

Marks [19] also considered the version of the problem when, additionally, we have a measure μ on $(V, \mathcal{B})$ and ask for the *measurable chromatic index* $\chi'_\mu(\mathcal{B})$, the smallest $k \in \mathbb{N}$ for which there is a Borel partition $E = E_0 \cup E_1 \cup \cdots \cup E_k$ such that E_i is a matching for each $i \in [k] := \{1, \ldots, k\}$ while the set of vertices covered by E_0 has measure zero. In particular, Marks [19, Question 4.9] asked if

$$\chi'_\mu(\mathcal{G}) \leq \Delta(\mathcal{G}) + 1 \tag{1}$$

always holds and proved [19, Theorem 4.8] that this is the case for $\Delta(\mathcal{G}) = 3$. (It is not hard to show that (1) holds when $\Delta(\mathcal{G}) \leq 2$.)

Although we cannot answer the original question of Marks, we can improve the upper bound on the measurable chromatic index when the measure μ defines a *graphing*.

Definition 1.2. A *graphing* (or a *measure-preserving graph*) is a quadruple $\mathcal{G} = (V, \mathcal{B}, E, \mu)$, where $(V, \mathcal{B}, E)$ is a Borel graph, μ is a probability measure on $(V, \mathcal{B})$, and there are finitely many triples $(\phi_1, A_1, B_1), \ldots, (\phi_k, A_k, B_k)$ such that

$$E = \big\{ \{x, y\} : x \neq y \ \& \ \exists i \in [k] \ \phi_i(x) = y \big\} \tag{2}$$

and each ϕ_i is an invertible Borel bijection between $A_i, B_i \in \mathcal{B}$ that preserves the measure μ.

We refer the reader to Lovász [17, Section 18.2] for an introduction to graphings.

Example 1.3. Given $\alpha \in \mathbb{R}$, let $\mathcal{T}_\alpha$ be the graphing on the real unit interval $([0, 1), \mathcal{B}, \lambda)$ with the Lebesgue measure λ generated by the α-*translation* $t_\alpha : [0, 1) \to [0, 1)$ that maps x to $x + \alpha$ (mod 1).

The above simple example of a graphing exhibits various interesting properties that contradict "finite intuition" when α is irrational. Namely, E defines a 2-regular and acyclic graph while the ergodicity of t_α implies that every Borel matching misses a set of vertices of positive measure (and thus each of $\chi_\mathcal{B}(\mathcal{T}_\alpha)$, $\chi'_\mathcal{B}(\mathcal{T}_\alpha)$ and $\chi'_\lambda(\mathcal{T}_\alpha)$ is strictly larger than 2). In particular, we see that the property of being *bipartite* (that is, $\chi_\mathcal{B}(\mathcal{G}) \leq 2$) may be strictly stronger than having no odd cycles.

Graphings appear in various fields. One can view $(V, \mathcal{B}, \mu, \phi_1, \ldots, \phi_k)$ as a generalization of a dynamical system. When we pass to the graphing $\mathcal{G}$, we lose some information but many properties can still be recovered. Also, if $\phi_1, \ldots, \phi_k$ come from a measure-preserving group action (with $A_i = B_i = V$), then the connectivity components of $\mathcal{G}$ correspond to orbits. Indeed, graphings play an important role in orbit equivalence theory ([13]). For example, the well-known *Fixed Price Problem* for groups (see e.g. [10]) involves finding the infimum of the average degree $\int_V \deg(x) \, d\mu(x)$ over all graphings on $(V, \mathcal{B}, \mu)$ with the given connectivity components. We came to this topic motivated by limits of bounded-degree graphs since graphings can be used to represent a limit object for both the Benjamini-Schramm [2] (or local) convergence and the Bollobás-Riordan [4] (or global-local) convergence, as shown by Aldous and Lyons [1], Elek [7] and Hatami, Lovász and Szegedy [12].

We can make a finite graph $G = (V, E)$ into a graphing by letting $\mathcal{B} = 2^V$ consist of all subsets of V and μ be the uniform measure on V. Here, the smallest k that satisfies (2) is equal to the minimum number of graphs with degree bound 2 that decompose E. This is trivially at least $\Delta(G)/2$ and, by Vizing's theorem, is at most $\lceil (\Delta(G) + 1)/2 \rceil$. Also, if we additionally require that $A_i \cap B_i = \emptyset$ for all $i \in [k]$, then the smallest k is exactly the chromatic index $\chi'(G)$. In Section 8 we consider the smallest k in Definition 1.2 that suffices for every graphing

of degree bound d as well as its variant where a null-set of errors is allowed. It should not be surprising to the reader that Borel and measurable chromatic indices play an important role in estimating this parameter. This provides some further motivation for our main result that $\chi'_\mu(\mathcal{G}) = (1 + o(1)) \, \Delta(\mathcal{G})$:

Theorem 1.4. *If $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ is a graphing with maximum degree at most d, then its measurable chromatic index is at most $d + O(\sqrt{d})$. Moreover, if $\mathcal{G}$ has no odd cycles, then $\chi'_\mu(\mathcal{G}) \leq d + 1$.*

In fact, Theorem 1.4 is a direct consequence of Lemma 1.7 and Theorem 1.8. In order to state them, we need some further preparation.

Definition 1.5. Let $f(k)$ be the smallest $f \in \mathbb{N}$ such that for every $d \in [k]$ the following holds. Let G be an arbitrary finite graph such that every degree is at most d, except at most one vertex of degree $d + 1$. Suppose that at most $d - 1$ *leaves* (that is, edges with one of their endpoints having degree 1) are pre-colored. Then this pre-coloring can be extended to an edge-coloring of the whole graph G that uses at most $d + f$ different colors.

By definition, the function $f(k)$ is non-decreasing in k. Since we allow a vertex of degree $k + 1$ (when $d = k$), we have that $f(k) \geq 1$. We make the following conjecture which, if true, will give a strengthening of Vizing's theorem.

Conjecture 1.6. $f(k) = 1$ *for all $k \geq 1$.*

Conjecture 1.6 trivially holds for $k \leq 2$. Balázs Udvari (personal communication) proved it for $k = 3$ but his proof does not seem to extend to larger k. We note that allowing a vertex of degree $d + 1$ seems to be not an essential extension, but the pre-colored edges cause the difficulties. For general k, we can prove a weaker bound $f(k) = O(\sqrt{k})$, which follows from the following lemma.

Lemma 1.7. *Let d be sufficiently large. Then every pre-coloring of at most d leaves of a finite graph G with $\Delta(G) \leq d$ extends to an edge-coloring of G that uses at most $d + 9\sqrt{d}$ colors.*

The function f is of interest because of the following relation to the measurable chromatic index of graphings given by Theorem 1.8. Let us call a set X of vertices (in a finite or infinite graph) *r-sparse* if for every distinct $x, y \in X$ the graph distance between x and y is strictly larger than r. For example, a set is 1-sparse if and only if it is independent.

Theorem 1.8. *For every $d \geq 1$ there is $r_0 = r_0(d)$ such that if $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ is a graphing with maximum degree at most $d + 1$ such that the set J of vertices of degree $d + 1$ is r_0-sparse, then $\chi'_\mu(\mathcal{G}) \leq d + f(d)$. If, furthermore, $\mathcal{G}$ has no odd cycles, then $\chi'_\mu(\mathcal{G}) \leq d + 1$.*

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Remark 1.9. Laczkovich [16] for $d = 2$ and Conley and Kechris [5, Section 6] for every even $d \geq 4$ proved that there exists a bipartite d-regular graphing $\mathcal{G}$ such that every Borel matching misses a set of vertices of positive measure. Hence, $d + 1$ colors are necessary in Theorem 1.4 for such d, even in the bipartite case. If Conjecture 1.6 is true, then $d + 1$ colors always suffice.

This paper is organized as follows. Section 2 collects some frequently used notation. Basic properties of graphings that are needed in the proofs are discussed in Section 3. Section 4 formally describes the main inductive step (roughly, removing a matching M that covers high degree vertices) and how this yields Theorem 1.8. Section 5 shows how to construct the required matching M, provided there is a sequence of matchings $(M_i)_{i=0}^{\infty}$ that stabilizes "fast". The main bulk of the proof appears in Section 6 where we inductively construct M_{i+1} by augmenting M_i along paths of length at most $2i + 1$. The fast stabilization of M_i's is derived from a variant of the expansion property. This is relatively straightforward for the case when there are no odd cycles and is done in Section 6.2. The remainder of Section 6 deals with the general case. A few auxiliary results are moved to Section 7 in order to make the flow of argument smoother. An application of Theorem 1.8 is presented in Section 8.

When presenting the long and difficult proof of Theorem 1.8, we tried to split it into smaller steps. (For example, Theorem 1.8 follows from Theorem 4.2, which in turn follows from Theorem 6.1.) Hopefully, this makes the proof easier to follow and understand.

2 Some notation

For reader's convenience, we collect various notation here, sometimes repeating definitions that appear elsewhere.

Let $G = (V, E)$ be a graph. For $A, B \subseteq V$, the *distance* $\text{dist}(A, B)$ is the shortest length of a path connecting some vertex in A to a vertex in B. Also, $E(A, B) := E \cap (A \times B)$ denotes the set of adjacent pairs (a, b) with $a \in A$ and $b \in B$. Note that we take the ordered pairs, so that $|E(A, B)|$ counts the edges inside $A \cap B$ twice. The *complement* of $A \subseteq V$ is $A^c := V \setminus A$. The set A is r-*sparse* if no two distinct vertices of A are at distance at most r. It is r-*dense* if every vertex of V is at distance at most r from A. The k-*neighborhood* of A is

$$N_k(A) := \{x \in V : \text{dist}(\{x\}, A) \leq k\}.$$

The *degree* $\deg(x)$ of $x \in V$ is the number of edges in E containing x. The maximum degree is $\Delta(G) := \max\{\deg(x) : x \in V\}$. For a set of edges $C \subseteq E$, let $V(C) := \cup_{e \in C} e$ consist of vertices that are covered by at least one edge of C.

We may omit the set-defining brackets, for example, abbreviating $\{x, y\}$ to xy and $N_1(\{x\})$ to $N_1(x)$. Also, we write $[k] := \{1, \ldots, k\}$.

For a path p, $|p|$ will denote its *length*, i.e. the number of edges in p. The path p is called *even* (resp. *odd*) if its length $|p|$ is even (resp. odd).

3   Basic properties of graphings

This section discusses various properties of graphings that we need. Their proofs can be found in Sections 18.1–18.2 of Lovász' book [17].

Since each ϕ_i in Definition 1.2 is measure-preserving, we have that

$$\int_A \deg_B(x)\,\mathrm{d}\mu(x) = \int_B \deg_A(x)\,\mathrm{d}\mu(x), \qquad \text{for all } A, B \in \mathcal{B}, \tag{3}$$

where e.g. $\deg_A(x)$ is the number of edges that $x \in V$ sends to $A \in \mathcal{B}$. (It readily follows from Definition 1.2 that the function $\deg_A : V \to \mathbb{N}$ is Borel.) When we make a finite graph (V, E) into a graphing on $|V|$ atoms, then (3) corresponds to the trivial fact that the number of edges between sets $A, B \subseteq V$ can be counted either from A or from B.

Conversely, it is known (see [17, Theorem 18.21]) that if a measure μ on a Borel graph $(V, \mathcal{B}, E)$ satisfies (3), then $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ is a graphing. (In fact, one can take $k = 2\Delta(\mathcal{G}) - 1$ in Definition 1.2; not surprisingly, Theorem 1.1 is used in the proof.)

The following equivalence (see [17, Theorem 18.2] for a proof) is very useful.

Lemma 3.1. *Let $E \subseteq V \times V$ define a bounded degree graph on a standard Borel space $(V, \mathcal{B})$. Then $(V, \mathcal{B}, E)$ is a Borel graph if and only if $N_1(A)$ is Borel for every $A \in \mathcal{B}$.* $\qquad\square$

It follows that "locally" defined sets, such as for example the set of vertices that belong to a triangle, are Borel ([17, Exercise 18.8]). Also, Lemma 3.1 implies that if $(V, \mathcal{B}, E)$ is a Borel graph, then so is $(V, \mathcal{B}, E^k)$, where E^k consists of pairs of distinct vertices at distance at most k in E. (Indeed, the 1-neighborhood in E^k can be obtained by taking k times the 1-neighborhood in E.) Combining this with Theorem 1.1, we obtain the following useful corollary.

Corollary 3.2. *For every Borel graph $\mathcal{G}$ and $k \in \mathbb{N}$ there is a k-sparse labeling, that is, a Borel function $\ell : V \to [m]$ for some $m \in \mathbb{N}$ such that each part $\ell^{-1}(i)$ is k-sparse.* $\qquad\square$

In fact, in the above corollary it suffices to take $m = 1 + \Delta(\mathcal{G}) \sum_{i=1}^{k} (\Delta(\mathcal{G}) - 1)^{i-1}$, the maximum possible size of the k-neighborhood of a vertex.

The following proposition (see [17, Lemma 18.19]) implies that if we construct objects inside a graphing in a Borel way, then any subgraph that we encounter is still a graphing. This will be implicitly used many times here (e.g. when we remove a Borel matching from a graphing).

Proposition 3.3. *If $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ is a graphing and $E' \subseteq E$ is a Borel symmetric subset, then $\mathcal{G}' = (V, \mathcal{B}, E', \mu)$ is a graphing.*

Proof. Let measure-preserving maps $\phi_1, \ldots, \phi_k$ represent $\mathcal{G}$ as in Definition 1.2. Then their (appropriately defined) restrictions to E' represent $\mathcal{G}'$. □

Lemma 3.4. *Let $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ be a graphing of maximum degree at most $d + 1$ such that no two vertices of degree $d + 1$ are adjacent. Then we can edge-color all finite connectivity components of $\mathcal{G}$ in a Borel way, using at most $d + 1$ colors.*

Proof. For $i = 1, 2, \ldots$, we color all components with exactly $i + 1$ vertices. Given i, fix an i-sparse labeling $\ell : V \to \mathbb{N}$, which exists by Corollary 3.2. The labels in each component with $i + 1$ vertices are all different. Choose an isomorphism-invariant rule how to edge-color each labeled component. Note that at least one coloring exists by the extension of Vizing's theorem by Fournier [9] that $\chi'(G) \leq \Delta(G)$ if no two vertices of maximum degree are adjacent (see also Berge and Fournier [3] for a short proof). Apply this rule consistently everywhere. Each color class, as the countable union over $i \in \mathbb{N}$ of Borel sets, is Borel. □

One can define the measure $\mu^\#$ on $(V \times V, \mathcal{B} \times \mathcal{B})$ by stipulating that

$$\mu^\#(A \times B) = \int_A \deg_B(x)\, \mathrm{d}\mu(x), \qquad \text{for } A, B \in \mathcal{B}$$

and extending $\mu^\#$ to the product σ-algebra $\mathcal{B} \times \mathcal{B}$ by Caratheodory's theorem. It can be shown that $\mu^\#((V \times V) \setminus E) = 0$, see [17, Lemma 18.14]. Thus, in other words, $\mu^\#$ is the product of μ with the counting measure, restricted to E. Property (3) shows that $\mu^\#$ is symmetric: $\mu^\#(A \times B) = \mu^\#(B \times A)$.

If $X \subseteq V$ has measure zero, then $Y = \{y \in V : \mathrm{dist}(y, X) < \infty\}$, the union of all connectivity components intersecting X, also has measure zero. Indeed, Y is the countable union of the images of the null-set X by finite compositions of $\phi_1^{\pm 1}, \ldots, \phi_k^{\pm 1}$, where the maps ϕ_i are as in Definition 1.2. The analogous claim applies to any $\mu^\#$-null-set $X \subseteq E$. We will implicitly use this in the proof of Theorem 1.8: whenever we encounter some null-set of "errors", then we will move all edges from components with errors to the exceptional part $E_0 \subseteq E$ from the definition of $\chi'_\mu(\mathcal{G})$.

One could define yet another chromatic index $\chi^*_\mu(\mathcal{G})$, where **every** edge has to be colored but each color class is only a *measurable* subset of E (that is, belongs to the completion of $\mathcal{B} \times \mathcal{B}$ with respect to $\mu^\#$). It is easy to come up with an example when χ^*_μ is strictly larger than χ'_μ (e.g. add a null-set of high-degree vertices). However, considering χ^*_μ would give nothing new in the context of Theorem 1.8 because we can repair any null-set of errors by recoloring all components containing them via the Axiom of Choice. (Note that we need at most $d + 1 \leq d + f(d)$ colors by Fournier's theorem [9].) We restrict ourselves to $\chi'_\mathcal{B}$ for convenience, so that we can stay within the Borel universe (namely, all sets that we will encounter in the proof of Theorem 1.8 are Borel).

4 The main induction

Before we proceed with the proof of Theorem 1.8, it may be instructive to mention why known proofs of Vizing's theorem do not seem to extend to graphings. These proofs proceed by some induction, typically on $|E|$. When we extend the current edge-coloring to a new edge xy, we may need to swap colors in some maximal 2-color path p that ends in x or y. Unfortunately, we do not have any control over the length of p. This causes a problem when we do countably many such iterations in a graphing because the set of edges that flip their color infinitely often may have positive measure.

On the other hand, Kőnig's theorem for finite graphs can be proved without any backtracking: take any matching M that covers all vertices of maximum degree, color it with a new color and apply induction to the remaining graph $\mathcal{G} \setminus M$. We prove Theorem 1.8 by a similar induction on $\Delta(\mathcal{G})$. The difficulty with this approach is that even a finite (non-bipartite) graph need not have a matching covering all vertices of maximum degree. So instead we change the inductive assumption: $\mathcal{G}$ has maximum degree at most d except an $r_0(d)$-sparse set of vertices of degree $d+1$, where $r_0 : \mathbb{N} \to \mathbb{N}$ is a fast-growing function. Thus we want to find a matching M that covers all vertices of degree $d+1$ and "most" vertices of degree d, so that $\mathcal{G} \setminus M$ satisfies the sparseness assumption for $d-1$. This may still be impossible. However, if we remove all so-called stumps (to be colored later using Lemma 1.7) and, for some technical reasons, all finite components, then the desired matching M exists.

In the rest of this section, we define what a stump is, state the main inductive step (Theorem 4.2) and show how it implies Theorem 1.8.

Definition 4.1. Let $G = (V, E)$ be a graph and d be an integer such that $\Delta(G) \leq d + 1$. Call a set $A \subseteq V$ lying inside some infinite connectivity component C of G a *stump* if the number of vertices in A is finite, $|A| \geq 2$, $|E(A, A^c)| \leq d - 1$, and every vertex of A has degree d in G except at most one vertex of degree $d + 1$.

Theorem 4.2. *For every $d \geq 2$ and $r \geq 1$, there is $r_1 = r_1(d, r)$ such that the following holds. Let $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ be a graphing with degree bound $d+1$ that has no finite components. Suppose also that $\mathcal{G}$ has no odd cycles or has no stumps. If the set $J \subseteq V$ of vertices of degree $d+1$ is r_1-sparse, then there is a Borel matching M such that, up to removing a null-set, $\mathcal{G} \setminus M$ has maximum degree at most d and its set of degree-d vertices is r-sparse.*

Let us show how Theorem 4.2 implies Theorem 1.8.

Proof of Theorem 1.8. We use induction on d. For $d = 1$ it is true with $r_0(1) = 1$: each component can have at most three vertices so the required 2-edge-coloring exists by Lemma 3.4. (Note that $f(1) = 1$.)

Let $d \geq 2$. Let $r := r_0(d - 1)$ be the value returned by Theorem 1.8 for $d - 1$, using the inductive assumption. Let $r_0 := r_1(d, r)$ be the value returned by Theorem 4.2 on input (d, r). We claim that this r_0 suffices. Take any graphing $\mathcal{G}$ as in Theorem 1.8. Let J denote the r_0-sparse set of vertices of degree $d + 1$ in $\mathcal{G}$.

First, let us do the case when $\mathcal{G}$ has no odd cycles. Do *clean-up*, that is, remove all finite components from $\mathcal{G}$ and edge-color them with $d+1$ colors using Lemma 3.4 (whose assumptions are satisfied since J is an independent set). Now, Theorem 4.2 gives a Borel matching M such that, up to removing a null-set, $\mathcal{G}' := \mathcal{G} \setminus M$ has no vertices of degree larger than d while its degree-d vertices form an r-sparse set. So, by induction, we can color $\mathcal{G}'$ with d colors, and using the last color for M we get a Borel $(d + 1)$-edge-coloring of $\mathcal{G}$ a.e., as required.

In the general case, we also make sure that there are no stumps before we apply Theorem 4.2. Namely, for each integer $i \geq 1$ in the increasing order of i, we fix a $2i$-sparse labeling. For each isomorphism type of a labeled stump that has exactly $i + 1$ vertices and spans a connected subgraph, pick all such stumps A in $\mathcal{G}$ and remove all edges inside each A. (Note that we keep all edges between A and its complement A^c.) Clearly, after we have removed a stump, all its vertices have degree at most $d - 1$. In particular, none of them can belong to a stump now. Also, every two stumps that were removed simultaneously are vertex-disjoint since the labeling was sufficiently sparse. The final graphing has no stumps because for every stump A there is a stump $A' \subseteq A$ that spans a connected subgraph (and our procedure considers A' at some point).

Having removed all stumps, we do clean-up (that is, we remove and edge-color all finite components of $\mathcal{G}$). Denote the remaining graphing by $\mathcal{G}'$. It has degree bound $d+1$ and the set of vertices of degree $d+1$ is still r_0-sparse in $\mathcal{G}'$. But $\mathcal{G}'$ has no stumps nor finite components, so we can apply Theorem 4.2 as above and inductively obtain a Borel edge-coloring of $\mathcal{G}'$ a.e. with $d - 1 + f(d - 1) + 1 = d + f(d - 1)$ colors. It remains to color edges inside the stumps that we have removed. By the vertex-disjointness, we can treat each stump independently of the others. The colors on the at most $d - 1$ edges that connect the stump to its complement are already assigned. The definition of f (Definition 1.5) shows that this pre-coloring can be extended to a $(d+f(d))$-coloring of the whole stump. This again can be done in a Borel way, by applying some fixed rule consistently. Finally since $f(d) \geq f(d - 1)$, we get a Borel $(d + f(d))$-edge-coloring of $\mathcal{G}$ a.e., as desired. $\qquad\square$

5 Proof of Theorem 4.2

Here we present the proof of Theorem 4.2, by reducing it to Theorem 6.1.

It is known that a d-regular expander graphing that has no odd cycles or has no edge-cuts with fewer than d edges admits a measurable perfect matching. This has been shown by Lyons

and Nazarov [18] for the former case and Csóka and Lippner [6] for the latter case. What follows is an adaptation of these proofs to allow a sparse set of vertices of degree $d + 1$. As we will see, these "exceptional" vertices do not cause any considerable difficulties. The real problem is that our graphing $\mathcal{G}$ need not be an expander! Probably, the most crucial observation of this paper is that we can make the graphing behave like an expander at the expense of designating a small set K of vertices around each of which at most one error (an unmatched degree-d vertex) is allowed. If K is not too sparse, then $\mu(N_1(X)) = (1 + \Omega(1))\mu(X)$ for every $X \subseteq K^c$, that is, sets disjoint from K expand in measure. Theorem 6.1 then shows that such expansion is enough to obtain the matching M required in Theorem 4.2.

Proof of Theorem 4.2. Given d and r, let $r' := 3r + 7$ and let r_1 be sufficiently large. Let $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ satisfy all assumptions of Theorem 4.2. In particular, the set J of vertices of degree $d + 1$ is r_1-sparse.

Constructing the set K: First, we construct a set $K \subseteq V$ of vertices of degree at most d such that $J \cup K$ is $(r + 2)$-sparse while K is r'-*dense*, meaning that for every $x \in V$ there is $y \in K$ with $\mathrm{dist}(x, y) \le r'$. (This density requirement will later give us the desired expansion property.)

Such a set K can be constructed as follows. By Corollary 3.2, take an $(r + 2)$-sparse labeling $\ell : V \to [m]$. Next, iteratively for $i = 1, \ldots, m$, add to K all vertices $x \in V$ such that $\ell(x) = i$ and $J \cup K \cup \{x\}$ is still $(r+2)$-sparse. By the definition of ℓ, no two vertices with the same label can create a conflict to the sparseness. Thus the set $J \cup K$ remains $(r + 2)$-sparse throughout the whole procedure.

Let us verify that the final set K is r'-dense. Take any $y \notin K$. Since we did not add y, it is at distance at most $r + 2$ from $J \cup K$. Assume that $\mathrm{dist}(y, K) > r + 2$ as otherwise we are done. Then there is $z \in J$ with $\mathrm{dist}(y, z) \le r + 2$. Since we did clean-up, the component of y is infinite. Take any walk that starts at y and eventually goes away from z. Let y' be the first visited vertex that is at distance at least $r + 3$ (and thus exactly $r + 3$) from z. This vertex y' is at distance at least $r_1 - (r - 3) > r + 2$ from $J \setminus \{z\}$ by the r_1-sparseness of J. By the construction of K, we have that $\mathrm{dist}(y', K) \le r + 2$. By the triangle inequality,

$$
\begin{aligned}
\mathrm{dist}(y, K) &\le \mathrm{dist}(y, z) + \mathrm{dist}(z, y') + \mathrm{dist}(y', K) \\
&\le (r + 2) + (r + 3) + (r + 2) = r'.
\end{aligned}
$$

Thus K is indeed r'-dense.

Stars of exceptional vertices: For any $x \in K$ let us define the *star* of x to be the set

$$
D(x) := N_1(x) \cap \{y \in V : \deg(y) = d\} \tag{4}
$$

of vertices of degree d that are at distance at most 1 from x. (Note that if $\deg(x) < d$ then x itself is not included into $D(x)$.)

Given a matching $M \subseteq E$, the star of $x \in K$ can be one of three different types:

- *Complete:* if $D(x) \subseteq V(M)$, that is, all vertices of $D(x)$ are covered by the matching M.

- *Full:* if $|D(x) \setminus V(M)| = 1$, that is, exactly one vertex of $D(x)$ is not covered by M.

- *Open:* if at least 2 vertices of $D(x)$ are not covered by M.

We define the *truncated star* $D'(x) = D'_M(x)$ of $x \in K$ to consist of all but one of the uncovered vertices of the star. The excluded vertex is arbitrary: we can take, for example, the one with the largest label in a forever fixed 2-sparse labeling of $\mathcal{G}$:

$$D'_M(x) := (D(x) \setminus V(M)) \setminus \{\text{the largest remaining vertex, if any left}\}.$$

Note that the truncated star is empty if and only if the star is full or complete. We define the set of *unhappy* vertices $U = U_M$ to contain all unmatched vertices of degree at least d that are not in or adjacent to K together with all vertices in truncated stars:

$$U_M := \left(\cup_{x \in K} D'_M(x) \right) \cup \left(\{x \in V \setminus N_1(K) : \deg(x) \geq d\} \setminus V(M) \right). \tag{5}$$

Constructing the matching M: First, we will construct a sequence of Borel matchings $M_0, M_1, M_2, \ldots \subseteq E$ such that the following properties hold:

$$J \subseteq V(M_i), \qquad\qquad \text{for all } i \geq 0, \tag{6}$$

$$\mu(V(M_i \triangle M_{i+1})) \leq (2i + 2)\, \mu(U_{M_i}), \qquad\qquad \text{for all } i \geq 0, \tag{7}$$

$$\sum_{j=0}^{\infty} (2j + 2)\, \mu(U_{M_j}) < \infty. \tag{8}$$

Once we have M_i's are above, we can define $M := \cup_{j=0}^{\infty} \cap_{i=j}^{\infty} M_i$ to consist of those pairs that belong to all but finitely many matchings M_i. Clearly, $M \subseteq E$ is a Borel matching. Let us show that M satisfies Theorem 4.2.

Since $V(M_i \triangle M_{i+1})$ is the set of vertices that experience some change when we pass from M_i to M_{i+1}, the last two conditions imply by the Borel–Cantelli Lemma that the set X of vertices where the matchings M_i do not stabilize has measure zero. By (6), we conclude that $J \setminus V(M)$ is a subset of X and thus has measure zero. Also, the symmetric difference between U_M and $\cup_{j=0}^{\infty} \cap_{i=j}^{\infty} U_{M_i}$ is contained within the null-set $N_2(X)$. Since $\sum_{i=0}^{\infty} \mu(U_{M_i})$ converges by (8), each intersection $\cap_{i=j}^{\infty} U_{M_i}$ has measure zero. By the σ-additivity of μ, we conclude that U_M has measure zero too.

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Hence, if we remove all connectivity components intersecting $U_M \cup X$, then $J \subseteq V(M)$ and all unmatched degree-d vertices come from stars $D(x)$, $x \in K$, at most one vertex per star. Since the removed set has measure zero and $J \cup K$ is $(r+2)$-sparse, all conclusions of Theorem 4.2 hold.

Augmenting paths: It remains to construct the sequence $(M_i)_{i \in \mathbb{N}}$ satisfying the above three conditions. Inductively for $i = 0, 1, \dots$, we will construct M_{i+1} from M_i by flipping *alternating paths*, that is, paths that start in an unmatched vertex and whose matched and unmatched edges follow in an alternating manner. *Flipping* such a path means changing the matching along the path by swapping the matched edges with the unmatched ones. Usually one would only flip paths that end in an unmatched vertex, thus strictly increasing the set of matched vertices. In our case, however, for technical reasons, we will have to occasionally flip paths whose last edge belongs to the matching. While such a flip retains the matching property, it does not increase its size, so extra care needs to be taken.

With all these preparations we can describe what kind of alternating paths we use to improve the current matching.

Definition 5.1. An *augmenting path* is an alternating path that starts in U (that is, with an unhappy vertex) and

- either has odd length and ends in any unmatched vertex,

- or has even length and ends in a vertex of degree less then d or in a vertex of a complete star.

Claim 5.2. *The set U strictly decreases when an augmenting path is flipped.*

Proof. Clearly the vertex where the augmenting path starts ceases to be an element of U. So we only need to check that no vertex can become an element of U because of a flip. This could only be possible by uncovering a vertex of degree at least d. The only way a flip can uncover a vertex is when the path has even length: in this case the endpoint is uncovered. But the endpoint of an even length augmenting path can only have degree at least d if the augmenting path ends in a vertex of a complete star. After the flip, the star will be full but not open, so the uncovered vertex will not belong to U. $\square$

Now we are ready to describe formally the construction of the matching M_i.

We start by constructing a matching M_0 that covers all vertices in $N_{r_1/4}(J)$ of degree at least d by using Lemma 7.1. (This property will be needed later when we apply Theorem 6.1 to verify (8).) Since the neighborhoods $N_{r_1/2}(x)$ for $x \in J$ are disjoint, we can choose M_0 inside each $(r_1/2)$-neighborhood independently, for example, by taking the lexicographically smallest

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matching with respect to a fixed r_1-sparse labeling. This ensures that the obtained matching M_0 is Borel.

For $i \geq 0$, we define M_{i+1} recursively so that M_{i+1} admits no augmenting path of length at most $2i+1$. To get from M_i to M_{i+1} we keep flipping augmenting paths of length at most $2i+1$ as long as there are any of those left. This can be done in a Borel way analogously to how it was done in [8] as follows.

First, fix a $(2i+3)$-sparse labeling $\ell : V \to [m]$. Let $\mathcal{L}$ consist of all ordered sequences of labels of length at most $2i+1$. Take an infinite sequence $(\mathbf{v}_j)_{j=1}^{\infty}$ of elements of $\mathcal{L}$ such that each $\mathbf{v} \in \mathcal{L}$ appears infinitely often. Then iterate the following step over $j \in \mathbb{N}$. Let $\mathcal{P}_j$ be the set of augmenting paths whose labeling is given by $\mathbf{v}_j$. It is easy to see that $\mathcal{P}_j$ is a Borel set that consists of paths such that every two different paths are at distance at least 3 from each other. Thus we can flip all paths in $\mathcal{P}_j$ simultaneously with the updated matching being Borel. (Note that at most one path can intersect any star $D(x)$; thus every path $p \in \mathcal{P}_j$ remains augmenting even when we flip an arbitrary set of paths in $\mathcal{P}_j \setminus \{p\}$.) By Claim 5.2, all starting points of the flipped paths cease to belong to U while no new vertex can become unhappy.

Next, consider an arbitrary edge $e \in E$. Since the number of unhappy vertices within distance $2i$ from e strictly decreases every time when a path containing e is flipped, the matching eventually stabilizes on e. So let M_{i+1} consist of those edges that are always in the matching from some moment. Clearly, M_{i+1} is a Borel matching.

Suppose that M_{i+1} admits some augmenting path via vertices $p_0, \ldots, p_k$ with $k \leq 2i + 1$. There was a moment j_0 after which every edge inside the path and inside $N_3(p_0) \cup N_3(p_k)$ stabilized. The restriction of the matching to these edges determines whether $p_0, p_k \in U$. But there are infinitely many values of j when $\mathbf{v}_j = (\ell(p_0), \ldots, \ell(p_k))$ and this path should have been flipped for the first such $j \geq j_0$, a contradiction. Thus M_{i+1} has no augmenting path of length at most $2i + 1$, as desired.

Checking Conditions (6)–(8): The first condition trivially follows from our construction since $V(M_0) \supseteq J$ while no flip can unmatch a vertex of degree $d + 1$.

When constructing M_{i+1}, we flip augmenting paths that start from U_{M_i}. Each flipped path has length at most $2i + 1$ and each vertex of U_{M_i} is the initial vertex of at most one such path by Claim 5.2. It can be derived from Proposition 3.3 (or from the so-called Mass Transport Principle, see e.g. [17, Proposition 18.49]) that the total measure of vertices involved in these paths is at most $(2i + 2)\,\mu(U_{M_i})$, proving (7).

By definition, M_0 covers all vertices of degree at least d from $N_{r_1/4}(J)$. In particular, it follows that $\mathrm{dist}(U_{M_0}, J) > r_1/4$. Claim 5.2 and the fact that each edge is flipped finitely many times before we reach M_i imply that $U_{M_0} \supseteq U_{M_1} \supseteq \cdots \supseteq U_{M_i}$. Thus $\mathrm{dist}(U_{M_i}, J) > r_1/4$.

Since M_i admits no augmenting paths of length at most $2i - 1$, Theorem 6.1 shows that if r_1 is chosen large enough (namely, if $r_1 \geq r_2(d, r')$, the value returned by Theorem 6.1 on input (d, r')), then $\mu(U_{M_i})$ decreases exponentially fast with i, implying (8).

Thus we have proved Theorem 4.2 (by reducing it to Theorem 6.1). $\square$

6 Short alternating paths via expansion

In this section we state and prove Theorem 6.1, which will complete the proof of our main result. Roughly speaking, Theorem 6.1 says that if M is a matching in $\mathcal{G}$ such that there are no augmenting paths of length at most n_0 in the sense of Definition 5.1, then the set U_M of unhappy vertices is exponentially small in n_0. The proof method is adapted from [6] but is also considerably simplified since we have a dense set around which unmatched vertices are allowed. Almost all of what follows is already contained in [6]. Nevertheless we include here all the details to keep this paper self-contained. The main idea is that if the special set K is dense then subsets of K^c expand by Lemma 6.4; thus the set of vertices that we can reach by alternating paths of length at most i grows exponentially with i. This is fairly straightforward to show in the case when there are no odd cycles. This proof is presented first (in Section 6.2), to make it easier for the reader to understand the ideas that are common to both cases. The general case, however, needs some further, rather involved arguments. This is due to the fact that an alternating xy-walk cannot always be trimmed to an alternating xy-path when odd cycles are allowed.

Theorem 6.1. *For any $d \geq 2$ and r there are constants $c = c(d, r) > 0$ and $r_2 = r_2(d, r)$ such that the following holds. Let $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ be a graphing such that $\Delta(\mathcal{G}) \leq d + 1$, all components are infinite and the set J of vertices of degree $d + 1$ is r_2-sparse. Let $K \subseteq V$ be a Borel set of vertices that is r-dense in $\mathcal{G}$. Let $M \subseteq E$ be a Borel matching that admits no augmenting path of length at most n_0. Let U be the set of unhappy vertices with respect to M and K as defined in (5). If $\mathrm{dist}(U, J) > r_2/4$ and $\mathcal{G}$ has no odd cycles or has no stumps, then $\mu(U) \leq (1 + c)^{-n_0}/c$.*

6.1 Alternating breadth-first search

Given d and r, define $c_0 := d^{-r-1}$. Let $r_2 = r_2(d, r)$ be sufficiently large and then take small $c > 0$. Let $\mathcal{G}$, K and M be as in Theorem 6.1. We consider alternating paths starting from U.

Let X_n denote the set of vertices that are accessible from U via an alternating path of length at most $2n$. Our first goal is to show that subsets of X_n have large boundary if $2n \leq n_0$.

Let $\tilde{H}_n$ denote the vertices that are endpoints of odd alternating paths of length at most $2n - 1$ and $\tilde{T}_n$ those that are endpoints of even alternating paths of length at least 2 and at

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most $2n$. Then $X_n = U \cup \tilde{H}_n \cup \tilde{T}_n$. Finally, define $O_n := V \setminus X_n$.

Proposition 6.2. *If $2n \leq n_0$, then the following claims hold.*

1. *The points in $\tilde{H}_n$ are covered by M.*

2. *The edges of the matching M give a bijection between $\tilde{H}_n$ and $\tilde{T}_n$. (In particular, $\mu(\tilde{H}_n) = \mu(\tilde{T}_n)$ and U is disjoint from $\tilde{H}_n \cup \tilde{T}_n$.)*

3. *$\tilde{T}_n$ cannot contain vertices of degree less than d.*

4. *$K \cap U = \emptyset$.*

5. *There is no $x \in K$ such that $D(x) \subseteq X_n \setminus U$.*

Proof. Part 1 is clear: if a vertex in $\tilde{H}_n$ would not be matched then it would give rise to an augmenting path of length at most $2n - 1$.

Part 2 follows from Part 1 and the definition of an alternating path. (Note that $\mu(\tilde{H}_n) = \mu(\tilde{T}_n)$ because $(V, \mathcal{B}, M, \mu)$ is a graphing by Proposition 3.3.)

Part 3 is immediate from the definition of an augmenting path because a vertex of small degree in $\tilde{T}_n$ gives an augmenting path of length at most $2n$.

To see Part 4, assume that there is $x \in K \cap U$. Then all neighbors of x are matched, for otherwise we get an augmenting path of length 1. Thus $D(x)$ is a full star and $x \notin U$ by definition.

Similarly, to prove Part 5 assume on the contrary that $D(x) \subseteq X_n \setminus U$ for some $x \in K$. In particular, this means that the star of x is complete. If $x \in \tilde{T}_n$, then there is an even alternating path of length at most $2n$ ending in x. But then this path is also augmenting, a contradiction. So let $x \in \tilde{H}_n$. Let y be the last vertex before x on an alternating path of length at most $2n - 1$ from U to x. Then the path $p' = p \setminus x$ is augmenting either because $\deg(y) < d$ or because $y \in D(x)$ and $D(x)$ is complete, giving a contradiction. $\qquad\square$

Lemma 6.3. *If $2n \leq n_0$, then O_n is $(r + 1)$-dense.*

Proof. For every $x \in K$, we pick a *representative* $y_x \in D(x)$ as follows. If exactly one vertex of $D(x)$ is unmatched, let y_x be that vertex. Then, since the star of x is full, we have $y_x \notin U$ and thus $y_x \in O_n$. If all vertices of $D(x)$ are matched, then pick e.g. the largest $y_x \in O_n \cap D(x)$. (This set is non-empty by Proposition 6.2.5.) Finally, if $D(x)$ has at least two unmatched vertices, let y_x be the single element in $D(x) \setminus D'(x)$. Again, $y_x \notin U$. We conclude that

$$O_n \supseteq \{y_x : x \in K\}$$

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is $(r+1)$-dense because K is r-dense by the assumption of Theorem 6.1 and we picked one vertex from the star of each $x \in K$. □

Thus every subset of X_n has large boundary by the following result.

Lemma 6.4. *If $Q \subseteq V$ is k-dense, then the measure of edges leaving any subset $W \subseteq Q^c$ is at least $d^{-k} \mu(W)$.*

Proof. For every $w \in W$ pick a shortest path $p(w)$ from w to Q. This can be done in a Borel way by, for example, letting $p(w)$ be the lexicographically smallest such path with respect to a fixed 2-sparse labeling. Take the first edge on $p(w)$ that connects W to W^c. Clearly, any edge can arise this way for at most $1 + d + \ldots + d^{k-1} \leq d^k$ different vertices $w \in W$. The required bound now follows. □

6.2 Proof for graphings without odd cycles

We have all the ingredients to finish the proof of Theorem 6.1 in the case when $\mathcal{G}$ has no odd cycles. In the following, let $n \in \mathbb{N}$ be arbitrary with $4n + 1 \leq n_0$.

First, let us show that $\tilde{H}_n$ and $\tilde{T}_n$ are disjoint. Assume for a contradiction that $x \in \tilde{H}_n \cap \tilde{T}_n$. Then there are two vertices $u_1, u_2 \in U$ and an odd alternating path from u_1 to x and an even alternating path from u_2 to x. The concatenation of these two paths (with the second path being reversed) is an odd alternating **walk** from u_1 to u_2. Since $\mathcal{G}$ has no odd cycles, we have that $u_1 \neq u_2$. Also, we conclude that there is an odd alternating **path** from u_1 to u_2, since a shortest alternating walk in a bipartite graph is necessarily an alternating path. This path has length at most $4n - 1 \leq n_0$ and is augmenting, a contradiction.

A similar argument shows that there can be no edge within $\tilde{T}_n \cup U$ for otherwise we find an augmenting path of length at most $4n + 1 \leq n_0$.

Any vertex outside of X_n that is adjacent to $\tilde{T}_n$ will belong to $\tilde{H}_{n+1} \subseteq X_{n+1}$. We want to show that there are many such vertices, so we derive a lower bound on the measure of edges leaving $\tilde{T}_n$. By Proposition 6.2.2, we know that $\mu(\tilde{T}_n) = \mu(\tilde{H}_n)$. Also, every vertex of $\tilde{T}_n$ has degree at least d while vertices of degree $d+1$ are r_2-sparse. Since the set $\tilde{T}_n \cup U$ is independent, U sends no edges to O_n, and r_2 is large, we would expect that, say, at least 49% of the edges between X_n and O_n originate from $\tilde{T}_n$. The following inequalities make this intuition rigorous. For notational convenience, let

$$e(X, Y) := \mu^{\#}(E(X, Y)), \qquad \text{for } X, Y \subseteq V, \tag{9}$$

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denote the measure of edges between X and Y. We have

$$
\begin{aligned}
e(\tilde{T}_n \cup U, O_n) &\geq d\,\mu(\tilde{T}_n) - e(\tilde{H}_n, \tilde{T}_n \cup U) \\
&\geq d\,\mu(\tilde{H}_n) - e(\tilde{H}_n, X_n) \geq e(\tilde{H}_n, O_n) - \mu(\tilde{H}_n \cap J) \\
&= e(X_n, O_n) - e(\tilde{T}_n \cup U, O_n) - \mu(\tilde{H}_n \cap J).
\end{aligned}
$$

Hence

$$
e(\tilde{T}_n \cup U, O_n) \geq \frac{1}{2}\big(e(X_n, O_n) - \mu(\tilde{H}_n \cap J)\big).
$$

Recall that $c_0 = d^{-r-1}$. By Lemmas 6.3 and 6.4, the measure of edges leaving X_n is at least $c_0\,\mu(X_n)$.

Take, for each $x \in \tilde{H}_n \cap J$, a shortest alternating path from U to x. Its length is at least $r_2/4$ because $\operatorname{dist}(U, J) > r_2/4$ by our assumption. Moreover, since J is r_2-sparse, the final $r_2/4$ edges of this path are unique to x: for different vertices of $\tilde{H}_n \cap J$ these segments are disjoint (and, obviously, these segments belong to X_n). Since these paths can be chosen in a Borel way, we conclude that

$$
\mu(\tilde{H}_n \cap J) \leq \frac{4}{r_2}\,\mu(X_n).
$$

Assuming that $4/r_2 < c_0/2$, we have that

$$
(d+1)\,\mu(X_{n+1} \setminus X_n) \geq e(\tilde{T}_n \cup U, O_n) \geq \frac{c_0}{2}\,\mu(X_n) - \frac{c_0}{4}\,\mu(X_n) = \frac{c_0}{4}\,\mu(X_n).
$$

We get by induction on n that

$$
1 \geq \mu(X_{n+1}) \geq \left(1 + \frac{c_0}{4(d+1)}\right)\mu(X_n) \geq \left(1 + \frac{c_0}{4(d+1)}\right)^{n+1}\mu(U).
$$

In particular, by taking $n = \lfloor (n_0 - 1)/4 \rfloor$ we conclude that $\mu(U) \leq (1+c)^{-n_0}/c$, as desired.

6.3 Sketch of the proof in the general case

We continue using the notation introduced in Section 6.1 but we need a more refined analysis of different types of vertices in X_n than the one in Section 6.2. Since odd cycles are allowed, the sets $\tilde{H}_n$ and $\tilde{T}_n$ need not be disjoint. It will be convenient to introduce the following notation:

$$
\begin{aligned}
H_n &:= \tilde{H}_n \setminus \tilde{T}_n, \\
T_n &:= \tilde{T}_n \setminus \tilde{H}_n, \\
B_n &:= \tilde{H}_n \cap \tilde{T}_n.
\end{aligned}
$$

Here, H stands for "head", T stands for "tail", and B stands for "both". These sets satisfy the following simple properties in addition to those already stated in Proposition 6.2.

Proposition 6.5. *If $2n \leq n_0$, then the following properties hold.*

1. *X_n is the disjoint union of U, T_n, H_n and B_n.*

2. *$B_1 \subseteq \cdots \subseteq B_n$.*

3. *M gives a perfect matching between T_n and H_n, and also within B_n. In particular, $\mu(H_n) = \mu(T_n)$.* □

Now we are ready to sketch the proof of Theorem 6.1, pointing out the main ideas without introducing all technicalities. We encourage the reader to study the whole outline before reading the proof and to refer back to it whenever necessary. Without understanding the basic outline, some later definitions may seem unmotivated.

1. Assuming there are no short augmenting paths, we would like to show that $\mu(X_n)$ grows exponentially with n.

2. By Lemmas 6.3 and 6.4, the set X_n expands. If there are plenty of edges leaving X_n from T_n or B_n, then the other ends of these edges will be part of X_{n+1}, fueling the desired growth. If this is not the case, then there has to be many tail-tail or tail-both edges for the same reasons as in Section 6.2: $\mu(H_n) = \mu(T_n)$, every vertex of T_n has degree at least d while only a small fraction of vertices of H_n can have degree $d + 1$.

3. A tail vertex that has another tail- or both-type neighbor will normally become a both-type vertex in the next step. In this case even though X_n does not grow, the set B_n grows within X_n, still maintaining the desired expansion.

4. The problem is that certain tail-vertices will not become both-type, even though they possess a both-type neighbor. These will be called *stubborn*. The bulk of the proof is about bounding their number. The key idea here is that we can associate to each stubborn vertex x a subset of B_n called the *family* of x.

5. As we will see in Lemma 6.15, families associated to different vertices are pairwise disjoint. Thus there cannot be too many stubborn vertices with large families. On the other hand, Claim 6.17 shows that if a vertex stays stubborn for an extended amount of time, then its family has to grow. These two observations will be the basis for showing that B_n grows within X_n, thus indirectly contributing to the growth of X_n.

The proof is organized as follows. We define stubborn vertices and their families in Section 6.4 where their basic properties are stated and proved. Theorem 6.1 is proved in Section 6.5 by introducing a function $I(n)$ that exponentially grows with n for $n \leq (n_0 - 2)/2$, is bounded by a constant and satisfies $I(0) = \mu(U)$. This will give the desired upper bound on the measure of U.

6.4 Combinatorics of alternating paths

In this section we will be mainly concerned about how edges within $T_n \cup U$ and between B_n and $T_n \cup U$ contribute to the growth of B_n. We implicitly assume in all following claims that $2n + 1 \leq n_0$ (that is, that there are no augmenting paths of length at most $2n + 1$).

Lemma 6.6. If $x, y \in T_n \cup U$ and $xy \in E$, then $x \in B_{n+1}$ or $y \in B_{n+1}$.

Proof. It is sufficient to prove that x or y is in $\tilde{H}_{n+1}$. Let p and q be shortest alternating paths that witness $x \in \tilde{T}_n$ and $y \in \tilde{T}_n$ respectively. We may assume without loss of generality that $|p| \leq |q|$. (Recall that e.g. $|p|$ denotes the number of edges in the path p.) The vertex y cannot lie on p: otherwise either there would be a shorter alternating path witnessing $y \in T_n$, or we would have $y \in \tilde{H}_n$ and not in $T_n \cup U$. Hence, by adding the edge xy to p we obtain an alternating path of length at most $2n + 1$ that witnesses $y \in \tilde{H}_{n+1}$. $\square$

Edges running between $T_n \cup U$ and B_n are more complicated to handle. If $b \in B_n$ and $t \in T_n \cup U$, but all paths witnessing $b \in \tilde{T}_n$ run through t, then we cannot simply exhibit that $t \in \tilde{H}_{n+1}$ by adding the edge bt to the end of such a path since it would become self-intersecting. The following definition captures this behavior.

Definition 6.7.

- A vertex $x \in T_n \cup U$ is *stubborn* if it is adjacent to one or more vertices in B_n, but $x \notin \tilde{H}_{n+1}$.

- An edge $xy \in E$ is *stubborn* if $x \in T_n \cup U$, $y \in B_n$ and x is a stubborn vertex.

Let $S_n \subseteq T_n \cup U$ denote the set of vertices that are stubborn at time n.

We would like to bound the number of stubborn vertices. In order to do so, we will associate certain subsets of X_n to each stubborn vertex in a way that subsets belonging to different stubborn vertices do not intersect. Then we will show that these subsets become large quickly.

Remark 6.8. We think of n as the time variable, and all the sets evolve as n changes. Usually n will denote the "current" moment in this process. In the following definitions of age, descendant, and family, there will be a hidden dependence on n. When talking about the age or the family of a vertex, we always implicitly understand that it is taken at the current moment.

Definition 6.9. The *age* of a vertex $x \in S_n$ is $a(x) := n$ if $x \in U$ and $a(x) := n - \min\{k : x \in T_k\}$ otherwise.

Definition 6.10. Fix a vertex $x \in S_n$. A set $D \subseteq X_n \setminus \{x\}$ has the *descendant property* with respect to x if the following is true. For every $y \in D$ there are two alternating paths p and q starting in x and ending in y, such that

- both start with an unmatched edge, p is odd and q is even,

- $p, q \subseteq D \cup \{x\}$,

- $|p| + |q| \leq 2a(x) + 1$.

Clearly, the sets satisfying the descendant property with respect to x are closed under union.

Definition 6.11. The *family $F_n(x)$ of a vertex* $x \in S_n$ at time n is the largest subset of $X_n \setminus \{x\}$ that satisfies the descendant property. (In other words, $F_n(x)$ is the union of all sets that satisfy the descendant property.)

Claim 6.12. *If $x \in S_n$ and xy is a stubborn edge then y is in the family of x. In particular, every stubborn vertex has a non-empty family.*

Proof. Let p be a path that witnesses $y \in \tilde{T}_n$. Now if p appended by the edge yx would be a path then it would witness $x \in \tilde{H}_{n+1}$. Since this is not the case, x has to lie on p. Suppose $x = p_{2l}$ and $y = p_{2k}$. Let D denote the set of vertices that the path p visits after leaving x. For any point $z \in D$ there are two alternating paths from x to z. One is given by following p from x to z and the other by taking the edge xy and then walking backwards on p. The total length of these two paths is $2k - 2l + 1$. Since the age of x by Definition 6.9 is at least $n - l \geq k - l$ we see that $2k - 2l + 1 \leq 2a(x) + 1$. Hence the two paths satisfy all conditions of Definition 6.10 and D has the descendant property with respect to x. We conclude by Definition 6.11 that $y \in F_n(x)$. $\square$

Claim 6.13. *The family of any stubborn vertex is a subset of B_n.*

Proof. Let $x \in S_n$ be a stubborn vertex and let s be a shortest path witnessing $x \in T_n \cup U$. Let us denote $k := |s|/2$.

Let us show that the family of x is disjoint from s. Suppose that this fails. It is clear that any family consists of pairs of matched vertices. Since s is an alternating path, there is i such that $\{s_{2i-1}, s_{2i}\}$ belongs to M and lies inside $F_n(x)$. Let i be the smallest such index. Then, by Definition 6.10, there is an odd alternating path p from x to s_{2i} such that p runs within the family and its length is at most $2a(x) + 1 \leq 2n - 2k + 1$. Since i was the smallest such index, the path p is disjoint from $s_0, s_1, \ldots s_{2i-1}$. Thus by appending $s_0, s_1, \ldots, s_{2i}$ by the reverse of p we get an alternating path from U to x ending in an unmatched edge, whose length is at most $2i + 2n - 2k + 1 \leq 2n + 1$. This path witnesses $x \in \tilde{H}_{n+1}$, contradicting that x is stubborn.

Now, for any point y in the family we can take the two paths p and q from x to y as in Definition 6.10. By the age requirement in Definition 6.10, we get that $|p| + |q| \leq 2a(x) + 1 = 2n - 2k + 1$. Hence $|s| + |p| + |q| \leq 2n + 1$ and thus $|s| + |p| \leq 2n - 1$ and $|s| + |q| \leq 2n$. Since p and q run within the family (which is disjoint from s as we have just established), we

can append s with p and q respectively to get alternating paths witnessing $y \in \tilde{H}_n$ and $y \in \tilde{T}_n$ respectively. Thus $y \in B_n$, as required. □

Next we will prove that any vertex can belong to at most one family. We start with a simple lemma about concatenating alternating paths.

Lemma 6.14. *Let p be an even alternating path from x to y and q an odd alternating path from y to z. Then there is an odd alternating path from x to either y or z whose length is at most $|p| + |q|$.*

Proof. If the concatenation of p and q is a path, then we are done. Otherwise let i be the smallest index such that $p_i \in q$. Let $p_i = q_j$. Then $p_0, p_1, \ldots, p_i = q_j, q_{j+1}, \ldots, z$ is a path from x to z and $p_0, p_1, \ldots, p_i = q_j, q_{j-1}, \ldots q_0$ is a path from x to y. Both have length at most $|p| + |q|$, both of them end with non-matched edges and one of them is clearly alternating. □

Claim 6.15. *Two families cannot intersect.*

Proof. Let $x, y \in S_n$ be two stubborn vertices. Suppose that their families F and G do intersect. Let p and q be shortest alternating paths witnessing $x, y \in T_n \cup U$. Let us choose the shortest among all alternating paths from x to $F \cap G$ that runs within F. Let this path be p' and its endpoint $x' \in F \cap G$. Do the same with y to get a path q' from y to $y' \in F \cap G$ lying within G. By symmetry we may assume that $|p| + |p'| \leq |q| + |q'|$.

By the choice of p' we see that the only point on p' that is in G is its endpoint x'. From x' there are two paths, s and t, leading to y within G by Definition 6.10 one of which, say s, can be appended to p' to get an alternating path from x to y. If p' starts with a matched edge, $xz \in M$, then $z \in F$ belongs to B_n by Claim 6.13; however, an even alternating path that witnesses $z \in \tilde{T}_n$ necessarily ends with the matched edge xz, and $x \in \tilde{H}_n$ cannot be stubborn. Thus path $p' \cup s$ starts (and, by a similar argument, ends) with a non-matching edge.

Now we are in the position to apply the previous lemma. The path p leads from p_0 to x and ends with a matching edge. The path $p' \cup s$ leads from x to y and starts and ends with non-matching edges. Thus by the lemma, there is an alternating path from p_0 to either x or y which ends with a non-matching edge. The length of this alternating path is at most $|p| + |p'| + |s|$. But by the choice of p', the choice of q', and by the age requirement in Definition 6.10 we have

$$
\begin{aligned}
|p| + |p'| + |s| &\leq |q| + |q'| + |s| \\
&\leq |q| + |t| + |s| \leq |q| + 2a(y) + 1 = 2n + 1.
\end{aligned}
$$

Thus the alternating path from p_0 to x or y that we have found has length at most $2n + 1$ and it witnesses $x \in \tilde{H}_{n+1}$ or $y \in \tilde{H}_{n+1}$. But neither is possible since both x and y are stubborn, which is a contradiction. □

Claim 6.16. *There is exactly one stubborn vertex adjacent to any family.*

Proof. Let $x, y \in S_n$ and $z \in F_n(x)$. Suppose there is an edge between y and z. The vertex z is in B_n by Claim 6.13. Hence yz is a stubborn edge and z is in the family of y by Claim 6.12. But then the two families would not be disjoint, which is a contradiction to Claim 6.15. $\square$

Define $r_3 := 2d/c_0$. (Recall that $c_0 = d^{-r-1}$.) Roughly speaking, the following claim states that if a vertex remains stubborn for an extended period of time, then its family consumes its neighbors.

Claim 6.17. *Suppose that $2(n + r_3) + 1 \le n_0$, $x \in S_n$, $|F_n(x)| < r_3$, $v \in F_n(x)$ and there is an edge vw such that $w \in B_n \setminus F_n(x)$. If $x \in S_{n+r_3}$, then $w \in F_{n+r_3}(x)$.*

Proof. By definition, $x \notin \tilde{H}_{n+r_3+1}$ as x is still stubborn at the moment $n + r_3$.

First suppose that there is an even alternating path p with $|p| \le 2n$ that ends in w and does not pass through x. Let $w' \in p$ be the first even vertex on the path that is adjacent to some vertex $v' \in F_n(x)$. Then the initial segment of p up until w' has to be disjoint from $F_n(x)$. By definition, in $F_n(x)$ there has to be an alternating path from x to v' that ends in a matched edge. Extending this path through w' and then the initial segment of p, we get an alternating path from U to x. Its length is obviously at most $|p| + r_3$, hence $x \in \tilde{H}_{n+(r_3+1)/2}$ and consequently in $\tilde{H}_{n+r_3}$, which is a contradiction.

This means that every even alternating path from U to w of length at most $2n$ has to pass through x. Let p be a shortest such path. Let v' be the last vertex of p that is in $F_n(x) \cup \{x\}$. The vertex v' divides p into two segments, p_1 going from U to v' and p_2 from v' to w. Then

$$|p_2| = |p| - |p_1| \le 2n - 2\min\{k \ge 0 : x \in T_k \cup U\} = 2a(x).$$

We claim that p_2 becomes part of the family at time $n + r_3$. For any vertex $y \in p_2$ we can either go from x to y along p, or go from x to v in the even number of steps, then to w and continue backwards on p_2 to y. The total length of these two paths is at most $r_3 + |p_2| + 1 + r_3 \le 2(a(x) + r_3) + 1$. Since at moment $n + r_3$ the age of x will be exactly $a(x) + r_3$, the set $F_n(x) \cup p_2$ will satisfy the descendant property, so this whole set, including w, will be a part of $F_{n+r_3}(x)$. $\square$

Definition 6.18. We will say that at moment n the family of the vertex $x \in S_n$ is *expanding* if there is an edge vw such that $v \in F_n(x)$ and $w \in B_n \setminus F_n(x)$. For any $x \in V$, let $e_n(x)$ be the number of moments $m < n$ such that $x \in S_m$, $0 < |F_m(x)| < r_3$ and at moment m the family of x was expanding.

Claim 6.19. *For any $x \in V$ and $n \le (n_0 - 1)/2$, we have $e_n(x) \le r_3^2$.*

Proof. By Claim 6.17 we know that the number of moments in which an expanding family has a fixed size $k < r_3$ is at most r_3. This is because, within r_3 steps after the first such moment, the family either ceases to exist (as the vertex x is not stubborn anymore) or strictly grows. Thus for each possible size k there are at most r_3 moments of expansion, and thus there are at most r_3^2 such moments in all. □

6.5 Invariants of growth

Now we are ready to finish the proof of Theorem 6.1 for graphings without stumps. Let all the previous definitions and results apply (except those from Section 6.2, obviously). We restrict ourselves to those n that are at most $(n_0 - 2)/2$.

As we have seen, a fairly short computation was enough to show that $\mu(X_n)$ grows exponentially when we had no odd cycles. In the general case, we need to use a more complicated invariant than $\mu(X_n)$ as a measure of growth. Namely, we consider

$$I(n) := \mu(X_n) + \mu(B_n) + \frac{1}{2} \int_{X_n} e_n(x) \, dx.$$

Recall that $S_n \subseteq T_n \cup U$ denotes the set of stubborn vertices. Let $N_n := (T_n \cup U) \setminus S_n$ be the set of non-stubborn vertices within $T_n \cup U$. The stubborn vertices in S_n are further classified according to their families. Namely, L_n denotes those stubborn vertices whose families have size at least r_3 (are "large"). Of stubborn vertices with smaller families, E_n contains those that have expanding families and $R_n := S_n \setminus (L_n \cup E_n)$ contains the rest. Thus we have the following partitions (see Figure 1):

$$
\begin{aligned}
T_n \cup U &= N_n \cup S_n, \\
S_n &= L_n \cup E_n \cup R_n.
\end{aligned}
$$

We shall often omit the index n from our notation, except where this may lead to confusion.

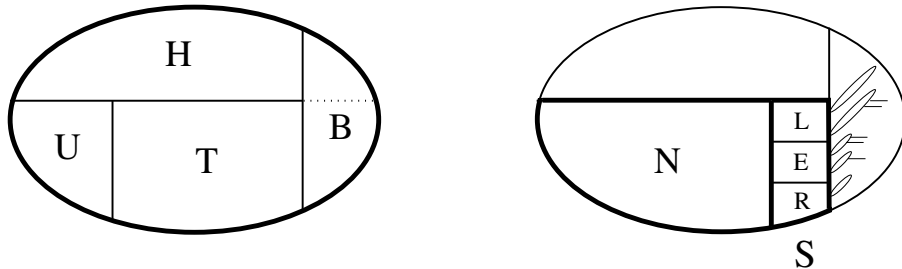
Figure 1: The structure of the set X.

Consider a stubborn vertex $x \in R$ (whose family is small and not expanding). Note that $\{x\} \cup F(x)$ consists of at least two vertices by Claim 6.12, has at most one vertex of degree $d + 1$ (since $\{x\} \cup F(n)$ spans a connected subgraph with at most $r_3 < r_2$ vertices) and each its vertex has degree at least d (as it belongs to $\tilde{T}_n \cup U$). Since $\mathcal{G}$ has no stumps, the number of edges leaving $\{x\} \cup F(x)$ is at least d. So, if k of these are adjacent to $F(x)$, then at least $d - k$ are adjacent to x and we have $|E(x, F(x))| \leq \deg(x) - (d - k) = k + \mathbb{1}_J(x)$, where $\mathbb{1}_J$ is the characteristic function of J. The set $F(x)$ cannot send any edges to $B \setminus F(x)$ because it a non-expanding family nor any edges to $S \setminus \{x\}$ by Claim 6.16. Hence the edges from $F(x)$ have to go to H, N or the outside world $O = X^c$. This gives the following edge count:

$$|E(F(x), R)| = |E(F(x), x)| \leq |E(F(x), H \cup N \cup O)| + \mathbb{1}_J(x). \tag{10}$$

By Claim 6.12 we see that any edge between R and B has to run between a vertex in R and a member of its family. Thus, by integrating (10) over $x \in R$ and using that families are pairwise disjoint subsets in B, we get that

$$e(B, R) \leq e(B, H \cup N \cup O) + \mu(J \cap R).$$

(Recall that $e(X, Y)$, as defined in (9), denotes the measure of edges between $X, Y \subseteq V$.)

We bound the number of edges between any other stubborn vertex $x \in L \cup E$ and B by the trivial bound d. (Note that if $\deg(x) = d + 1$ then $x \in J$ is covered by the current matching M, so at least one edge at x does not go to B.) Adding this to the previous equation, we get

$$e(B, S) \leq d\,\mu(L) + d\,\mu(E) + e(B, H \cup N \cup O) + \mu(J \cap R). \tag{11}$$

We have that $\mu(T) = \mu(H)$ by Proposition 6.2.2 (namely, because M gives a bijection between these two sets). Also, all vertices in T have degree at least d. Thus

$$e(H, V) - \mu(H \cap J) \leq d\,\mu(H) = d\,\mu(T) = e(T, V) - \mu(J \cap T).$$

Similarly as in Section 6.2, if we choose $r_2 > 16/c_0$ then to any vertex $x \in H \cap J$ we can associate a unique path of length $r_2/4$ that lies in X and conclude that the measure of $H \cap J$ is at most $c_0\,\mu(X)/4$. Hence

$$e(H, V) \leq e(T \cup U, V) - \mu(J \cap T) + \frac{c_0}{4}\,\mu(X).$$

The edges between $T \cup U$ and H contribute equally to the total degrees of these two sets. In the worst case there are no internal edges in H. This boils down to the following estimate:

$$\begin{aligned}
e(H, O) + e(B, H) \;\leq\; & 2\,e(T \cup U, T \cup U) + e(T \cup U, O) \\
& + e(B, S) + e(B, N) - \mu(J \cap T) + \frac{c_0}{4}\,\mu(X).
\end{aligned}$$

Combining this with (11) and subtracting $e(B, H)$ from both sides, we get

$$e(H, O) \leq 2\,e(T \cup U, T \cup U) + e(B \cup T \cup U, O)$$
$$+ 2\,e(B, N) + d\,\mu(L) + d\,\mu(E) + \mu(J \cap R) - \mu(J \cap T) + \frac{c_0}{4}\,\mu(X).$$

Clearly $\mu(J \cap R) \leq \mu(J \cap T)$. Each vertex of L has a family of size at least r_3, and these families are contained in B by Claim 6.13 and are disjoint by Claim 6.15. Thus we get that $\mu(L) \leq \mu(B)/r_3$. Using this and adding $e(B \cup T \cup U, O)$ to both sides, we obtain that

$$e(X, O) \leq 2\,e(T \cup U, T \cup U) + 2\,e(B \cup T \cup U, O)$$
$$+ 2\,e(B, N) + \frac{d}{r_3}\,\mu(B) + d\,\mu(E) + \frac{c_0}{4}\,\mu(X). \quad (12)$$

Any vertex in O_n that is adjacent to $B_n \cup T_n \cup U$ is going to be in X_{n+1}, hence

$$e(B_n \cup T_n \cup U, O_n) \leq d\,(\mu(X_{n+1}) - \mu(X_n)).$$

Since there is no augmenting path of length at most $2n + 2$, any vertex in N_n that is adjacent to an edge coming from B_n will be a part of B_{n+1}. Likewise, by Lemma 6.6, any edge in $E(T_n \cup U, T_n \cup U)$ has to be adjacent to a point in $B_{n+1} \setminus B_n$. This implies that

$$2\,e(T_n \cup U, T_n \cup U) + 2\,e(B_n, N_n) \leq 2d\,(\mu(B_{n+1}) - \mu(B_n)).$$

Plugging all this into (12) and dividing by d, we get

$$\frac{e(X_n, O_n)}{d} \leq 2\,(\mu(X_{n+1}) - \mu(X_n)) + 2\,(\mu(B_{n+1}) - \mu(B_n)) + \mu(E_n) + \frac{\mu(B_n)}{r_3} + \frac{c_0\,\mu(X_n)}{4d}. \quad (13)$$

By Definition 6.18, we have that $e_{n+1}(x) = e_n(x) + 1$ for $x \in E_n$ while $e_{n+1}(x) = e_n(x)$ otherwise. Thus

$$\int_{X_{n+1}} e_{n+1}(x)\,\mathrm{d}x = \int_{X_n} e_n(x)\,\mathrm{d}x + \mu(E_n).$$

Hence the right hand side of (13) is at most $2(I(n+1) - I(n)) + \mu(B_n)/r_3 + c_0\,\mu(X_n)/(4d)$. Furthermore, by Lemmas 6.3 and 6.4 we have

$$e(X_n, O_n) \geq c_0\,\mu(X_n),$$

which implies that

$$\frac{c_0\,\mu(X_n)}{2d} \leq I(n+1) - I(n) + \frac{\mu(B_n)}{2r_3} + \frac{c_0\,\mu(X_n)}{8d}.$$

Recall that $r_3 = 2d/c_0$. Since $\mu(B_n) \leq \mu(X_n)$, we get that

$$\frac{\mu(X_n)}{4r_3} \leq I(n+1) - I(n).$$

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On the other hand, we know from Claim 6.19 that $e_n(x) \leq r_3^2$ for every $x \in V$. Thus $\int_{X_n} e_n(x) \, dx \leq r_3^2 \, \mu(X_n)$ and

$$I(n) \leq \left(2 + \frac{r_3^2}{2}\right) \mu(X_n) \leq r_3^2 \, \mu(X_n) \leq 4r_3^3 \big(I(n+1) - I(n)\big). \tag{14}$$

This gives that $\big(1 + 1/(4r_3^3)\big) I(n) \leq I(n+1)$. We conclude by induction on n that

$$\left(1 + \frac{1}{4r_3^3}\right)^n I(0) \leq I(n) \leq r_3^2 \, \mu(X_n) \leq r_3^2,$$

as long as there are no augmenting paths of length at most $2n + 2$. Since $X_0 = U$, we have that $I(0) = \mu(U)$. In particular, taking $n = \lfloor (n_0 - 2)/2 \rfloor$, we obtain the desired exponential bound on $\mu(U)$. This finishes the proof of Theorem 6.1.

7 Some auxiliary results

7.1 Proof of Lemma 1.7

Assume that the colors on the pre-colored leaves form a subset of $[d]$. Let L consist of vertices of degree 1 whose (unique) incident edge is pre-colored. Let Y consist of those vertices of G that are incident to at least $\sqrt{d}$ pre-colored edges. Clearly, $|Y| \leq d/\sqrt{d} = \sqrt{d}$. Pick any set of $|Y|$ unused colors from $[s]$ and edge-color $G[Y]$ using these colors by Vizing's theorem, where $s := \lfloor d + 2\sqrt{d} \rfloor$.

Next, let us color, one by one, all uncolored edges that connect Y to $Z := V(G) \setminus (Y \cup L)$ by using colors from $[s]$ only. When we consider a new edge connecting $y \in Y$ to $z \in Z$ then we have at most $d - 2$ colors forbidden at y and at most $2\sqrt{d} - 1$ colors forbidden at z. (Indeed, $z \notin Y$ is incident to at most $\sqrt{d}$ pre-colored leaves and to at most $|Y| - 1$ other colored edges.) Thus the number of forbidden colors at yz is at most $s - 1$, so we can extend our coloring to yz using some color from $[s]$.

Thus it remains to color the edges in $H := G[Z]$, the subgraph induced by Z. By Vizing's theorem, we can find a proper edge-coloring $g : E(H) \to [d+1]$ of the graph H. Let H_g be a subgraph of H that consists of g-conflicting edges, i.e. those edges inside Z that are adjacent to another edge of G of the same color. (Clearly, the latter edge must have the other vertex in $L \cup Y$.)

We try to "improve" the coloring g by composing it with a permutation $\sigma : [d+1] \to [d+1]$, chosen uniformly at random. Take a vertex $z \in Z$. There are at most $|Y| + \sqrt{d} \leq 2\sqrt{d}$ edges in $E(G) \setminus E(H)$ incident to z and each of these is responsible for at most one conflicting edge at z. Next, consider the random variable $X_z = X_z(\sigma)$ which is the number of neighbors $x \in Z$ of z such that $\sigma(g(xz))$ is equal to the color at some edge between x and $L \cup Y$. In other words,

X_z counts the number of H_g-edges at z with a conflict at the other endpoint. As before, each $x \in Z$ sends at most $2\sqrt{d}$ edges to $L \cup Y$. By the linearity of expectation we have that

$$\mathbf{E}(X_z) \leq \deg(z) \frac{2\sqrt{d}}{d+1} < 2\sqrt{d}. \tag{15}$$

Note that X_z changes at most by 2 if we transpose some two elements of σ. Also, if $X_z(\sigma) \geq i$, then there are i values of σ such that $X_z(\sigma') \geq i$ for every σ' that coincides with σ on these i values. (Namely, fix the colors of some i conflicting edges at z.) Thus all assumptions of McDiarmid's concentration result [20, Theorem 1.1] are satisfied (with $c = 2$ and $r = 1$ in his notation) and we conclude that, for each $t \geq 0$, the probability of $X_z \geq m + t$ satisfies

$$\mathbf{Pr}(X_z \geq m + t) \leq 2 \exp\left(-\frac{t^2}{64(m+t)}\right), \tag{16}$$

where m is the median of X_z. Since X_z is non-negative, we have that $\mathbf{E}(X_z) \geq \frac{1}{2}m$. Thus $m < 4\sqrt{d}$ by (15). Taking, for example, $t = 0.5\sqrt{d}$ in (16) we obtain

$$\mathbf{Pr}(X_z \geq 4.5\sqrt{d}) \leq 2 \exp\left(-\frac{d}{64 \cdot 4.5\sqrt{d}}\right) = \exp(-\Omega(\sqrt{d})).$$

The Union Bound shows that there is σ such that $X_z < 4.5\sqrt{d}$ for every vertex $z \in Z$ at distance at most 2 from $L \cup Y$. (Note that there are at most $O(d^{5/2})$ such vertices z.) Since all $(\sigma \circ g)$-conflicting edges have to be at distance at most 1 from $L \cup Y$, this permutation σ satisfies that the $(\sigma \circ g)$-conflict graph $H_{\sigma \circ g} \subseteq H$ has maximum degree at most $6.5\sqrt{d}$. Recolor $E(H_{\sigma \circ g})$ with a set of new $\Delta(H_{\sigma \circ g}) + 1$ colors using Vizing's theorem. Clearly, the obtained edge-coloring of G is proper and uses at most $s + 6.5\sqrt{d} + 1$ colors, which is at most the stated bound. This finishes the proof of Lemma 1.7. ∎

7.2   Covering r-neighborhoods of high-degree vertices

Lemma 7.1. *Let r be an integer and $d \geq 2$. Let $G = (V, E)$ be an infinite connected graph of maximum degree $d + 1$. Let J be the set of vertices of degree $d + 1$ and let*

$$Z := \{z \in N_r(J) : \deg(z) \geq d\}.$$

Assume that J is $(2r + 2)$-sparse and that G is bipartite or has no stumps. Then G has a matching that covers all of Z.

Proof. Observe that the $(r + 1)$-neighborhoods around the vertices of J are pairwise disjoint. Thus it is enough, for each $x \in J$, to find a matching inside $V' := N_{r+1}(x)$ that covers $Z' := Z \cap V'$.

Let us prove the no-stumps case first. Tutte's 1-Factor Theorem [23] implies that it is enough to check that for every set $S \subseteq V'$ the number of odd components that lie entirely inside Z' is at most $|S|$. (The reduction is as follows: add, if needed, an isolated vertex to make $|V'|$ even, make all pairs in $V' \setminus Z'$ adjacent and look for a perfect matching in this new graph on V'.)

Suppose that the lemma is false; take S that violates the above condition. Each odd component C of $G - S$ that lies inside Z' sends at least d edges to S: this follows from the definition of Z if $|C| = 1$ and from the absence of stumps if $|C| \geq 2$. Thus $|E(S, S^c)| \geq (s+1)d$, where $s := |S|$. On the other hand, all vertices of S have degree at most d except at most one vertex of degree $d+1$. Hence the total degree of S is at most $sd + 1 < (s+1)d$, a contradiction since $d \geq 2$.

Let us do the bipartite case now. Split $Z' = Z_1 \cup Z_2$ into two parts according to the bipartition of G. It is enough to show that there is a matching that covers Z_1 and one that covers Z_2. Indeed, the union of these two matchings consists of even cycles and paths; moreover at least one endpoint of each path of odd order has to be outside of Z. Thus Z can be covered by cycles and paths of even order, each admitting a perfect matching.

So suppose that there is no matching that covers, say, Z_1. By the Kőnig-Hall theorem this means that there is a subset $S \subseteq Z_1$ such that the set of neighbors T of S has strictly smaller size than S. However each vertex in S has degree at least d, so the number of edges leaving S is at least $d|S|$, while the number of edges arriving in T is at most $d|T| + 1 < d|S|$ if $d \geq 2$, again a contradiction. $\qquad\square$

8 An application

As we mentioned in the Introduction, a natural question is to determine $k_\mathcal{B}(d)$ (resp. $k'_\mathcal{B}(d)$), the smallest k such that every graphing $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ of maximum degree d can be generated by k maps $\phi_1, \ldots, \phi_k$ as in Definition 1.2 (resp. where we additionally require that $A_i \cap B_i = \emptyset$). Using the results of Marks [19], we are able to determine these functions exactly.

Proposition 8.1. *We have for all $d \geq 1$ that $k_\mathcal{B}(d) = d$ and $k'_\mathcal{B}(d) = 2d - 1$.*

Proof. Since the case $d = 1$ is trivial, assume that $d \geq 2$. The lower bound in both cases can be achieved by the same construction. Namely, take the Borel graph $\mathcal{G} = (V, \mathcal{B}, E)$ constructed by Marks [19] such that $\Delta(\mathcal{G}) = d$, $\chi'_\mathcal{B}(\mathcal{G}) = 2d - 1$ and $\chi_\mathcal{B}(\mathcal{G}) = 2$, with the last property being witnessed by a partition $V = V_1 \cup V_2$.

Not every Borel graph can be made into a graphing by choosing a suitable measure. For example, neither the grandmother graph defined in [17, Example 18.36] nor any union of its vertex-disjoint copies admits such measure. However, the Borel graph constructed by Marks

can be turned into a graphing. In order to show this, we have to unfold Marks' construction, using [19, Lemma 3.11]. Namely, let $\Gamma := \Gamma_1 * \Gamma_2$ be the free product of two copies of $\mathbb{Z}/d\mathbb{Z}$. The group Γ naturally acts on $[3]^\Gamma$, the set of functions from Γ to $[3]$. Let $\mathrm{Free}([3]^\Gamma)$ be the *free part* of this action which consists of those $f \in [3]^\Gamma$ such that $\gamma \cdot f \neq f$ for all non-identity $\gamma \in \Gamma$. For $i = 1, 2$, let V_i consist of Γ_i-*equivalence classes* of $f \in \mathrm{Free}([3]^\Gamma)$ that is, sets $\{f, x \cdot f, \ldots, x^{d-1} \cdot f\}$, where x is a generator of Γ_i. Let $X_1 \in V_1$ and $X_2 \in V_2$ be adjacent in $\mathcal{G}$ if they intersect. Since we restricted ourselves to the free part, each equivalence class consists of d elements and the obtained graph $\mathcal{G}$ is d-regular. Its vertex set $V = V_1 \cup V_2$ admits the natural Borel structure coming from the product topology on $[3]^\Gamma$ as well as the natural measure μ: to sample from μ take the Γ_i-equivalence class of $f : \Gamma \to [3]$, where the index $i \in [2]$ and all values $f(\gamma) \in [3]$ for $\gamma \in \Gamma$ are uniform and independent. Let us show that we indeed have a graphing. Note that the natural projection $p_i : V_i' \to V_i$ that maps an element of $V_i' := \mathrm{Free}([3]^\Gamma)$ to its Γ_i-equivalence class is measure-preserving. Let $\mathcal{G}'$ be the bipartite graph on the disjoint union of V_1' and V_2' obtained by pulling $\mathcal{G}$ back along $p_1 \sqcup p_2$ (where each edge of $\mathcal{G}$ gives d^2 edges in $\mathcal{G}'$). A moment's thought reveals that $E(\mathcal{G}')$ can be generated as in (2) by d^2 functions $\phi_{x,y} : V_1' \to V_2'$ for $x \in \Gamma_1$ and $y \in \Gamma_2$, where $\phi_{x,y}$ acts on $f \in V_1'$ first by x and then (viewing the result as an element of V_2') by y. Clearly, each $\phi_{x,y}$ is measure-preserving and thus $\mathcal{G}'$ is a graphing. It routinely follows that $\mathcal{G}$ is a graphing too.

Now, if the bipartite graphing $\mathcal{G}$ can be defined by k Borel maps $\phi_i : A_i \to B_i$, $i = 1, \ldots, k$, as in Definition 1.2, then its edge set can be partitioned into $2k$ Borel matchings that are defined inductively on $i = 1, \ldots, k$ as follows:

$$
\begin{aligned}
M_i &:= \left\{ \{x, \phi_i(x)\} : x \in V_1 \cap A_i \right\} \setminus \cup_{j=1}^{i-1}(M_j \cup M_j'), \\
M_i' &:= \left(\left\{ \{x, \phi_i(x)\} : x \in V_2 \cap A_i \right\} \setminus M_i \right) \setminus \cup_{j=1}^{i-1}(M_j \cup M_j').
\end{aligned}
$$

Thus $2k \geq \chi_{\mathcal{B}}'(\mathcal{G}) = 2d - 1$, that is, $k \geq d$. If, furthermore, $A_i \cap B_i = \emptyset$ for all i, then we directly get a partition of E into k Borel matchings as in (2), that is, $k \geq \chi_{\mathcal{B}}'(\mathcal{G}) = 2d - 1$. This gives the desired lower bounds on $k_{\mathcal{B}}(d)$ and $k_{\mathcal{B}}'(d)$.

Conversely, let $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ be an arbitrary graphing with maximum degree d. Proposition 3.3 shows that if ϕ is an invertible Borel map between two Borel subsets $A, B \subseteq V$ such that $\{x, \phi(x)\} \in E$ for all $x \in A$ then ϕ preserves the measure μ. If particular, every Borel matching $M \subseteq E$ can be represented by one such function ϕ (by picking one element x in each $xy \in M$ in a Borel way and letting $\phi(x) = y$). Since E can be partitioned into at most $2d - 1$ Borel matchings by Theorem 1.1, we conclude that $k_{\mathcal{B}}'(d) \leq 2d - 1$.

Likewise, in order to prove that $k_{\mathcal{B}}(d) \leq d$, let us show that E can be partitioned into at most d Borel directed graphs $F_1, \ldots, F_d$, each with maximum in- and out-degree at most 1. First, take a 2-sparse labeling $\ell : V \to [m]$. Initially, let each F_i be empty. Iteratively, over pairs $uv \subseteq [m]$, take all edges of E labeled as uv and for each such edge xy pick the lexicographically smallest triple $(j, \ell(a), \ell(b))$ where $j \in [d]$, $\{a, b\} = \{x, y\}$, and when we add the ordered arc

(a, b) to F_j then both maximum in-degree and maximum out-degrees of F_j are still at most 1. Note that at least one such choice of (j, a, b) exists: if some j is forbidden, then x and y are each incident to at least one arc from F_j, which rules out at most $d - 1$ values of j. Also, the choices that we simultaneously make for some pair uv cannot conflict with each other by the 2-sparseness of ℓ. Clearly, all sets (and maps) that we obtain are Borel. This finishes the proof. □

It would be fair to say that the question addressed by Proposition 8.1 is more about Borel graphs rather than graphings. Indeed, it asks for a Borel decomposition of E into matchings (or unions of directed paths and cycles) and the role of the measure μ in the definition of $k_{\mathcal{B}}$ and $k'_{\mathcal{B}}$ is only to restrict us to those Borel graphs that can be turned into graphings. The proof of Proposition 8.1 shows that we can drop this restriction and yet the values of $k_{\mathcal{B}}$ and $k'_{\mathcal{B}}$ will not change.

On the other hand, one can ignore a set of measure zero in many applications of graphings. Note that, modulo removing a null-set of vertices, Definition 1.2 does not change if we require only that the sets A_i, B_i are in $\mathcal{B}_\mu$, the completion of $\mathcal{B}$ with respect to μ, while ϕ_i is μ-measurable. Indeed, every μ-measurable $\phi_i : A_i \to B_i$ can be made Borel by removing a null-set from A_i (and the corresponding null-set from B_i). This change of definition may bring k down. With this in mind, we define $k(d)$ (resp. $k'(d)$) as the smallest k such that for every graphing $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ with $\Delta(\mathcal{G}) = d$ there are k invertible measure-preserving maps $\phi_i : A_i \to B_i$ with $A_i, B_i \in \mathcal{B}_\mu$ for $i = 1, \ldots, k$ such that (2) holds (resp. where we additionally require that $A_i \cap B_i = \emptyset$). Note that the maps ϕ_i and ϕ_i^{-1} in the definition of $k(d)$ and $k'(d)$ are μ-measurable but not necessarily Borel.

Interestingly, this relaxation of the restrictions on ϕ_i's reduces the minimum k by factor $2 + o(1)$ as $d \to \infty$, which follows with some work from Theorem 1.8. We need an auxiliary result first.

Lemma 8.2. *Let the edge-set of a graphing $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ be partitioned into Borel sets, $E = F_0 \cup F_1 \cup \cdots \cup F_k$, so that F_j has maximum degree at most 2 for each $j \in [k]$ while F_0 is a matching. Then there is a Borel matching $M \subseteq E$ such that the measure of vertices in infinite components of each $F_j \setminus M$, $j \in [k]$, and of $F_0 \cup M$ is zero.*

Proof. Choose a fast growing sequence of integers $d_1 \ll d_2 \ll d_3 \ll \ldots$. Initially, let $M := \emptyset$. We define $F'_j := F_j \setminus M$ for $j \in [k]$ and $F'_0 := F_0 \cup M$; these are updated every time when the current matching M changes. We repeat a certain iteration step over $i \in \mathbb{N}$. Given i, pick some $j \in \{0, \ldots, k\}$ so that each j is considered for infinitely many values of i. For example, let us agree that $j = j(i)$ is always the residue of i modulo $k + 1$.

Informally speaking, given the current i we "take care" of F'_j by changing M so that the updated edge-set F'_j has only finite components, each of size at most $O(d_i)$. By doing this

carefully, we can ensure that we change M on a set of measure $O(d_i^{-1})$. Of course, some iteration step at a later moment $h > i$ may create infinite components in F_j'. But, since each F_j' is "repaired" for infinitely many moments i, a vertex y belongs to an infinite component of the final set F_j' only if some "bad" events that are related to y happen for infinitely many values of h. An application of the Borel–Cantelli Lemma will show that the measure of such y is zero.

Before we can describe the iteration step, we need some definitions. For a set $Y \subseteq E$, let $\partial Y := \{x \in V(Y) : \deg_Y(x) = 1\}$ consist of vertices that are incident to exactly one edge of Y. A subset $D \subseteq Y$ is called (Y, r)-*sparse* if, for every edge $e \in D$, the distance with respect to Y between e and $V(D \setminus \{e\}) \cup \partial Y$ is strictly larger than r.

Now, given $i \in \mathbb{N}$, the corresponding iteration step is as follows. Let $X := F_j'$ if $j \neq 0$ and $X := M$ otherwise. Take a maximal Borel (F_j', d_i)-sparse subset $D_i \subseteq X$. Such a set D_i can be constructed by the familiar argument where we take a $(d_i + 1)$-sparse labeling $V(X) \to [s]$ and, iteratively over all pairs $xy \subseteq [s]$, add to D_i all admissible edges whose label set is xy.) By definition, the obtained set D_i is a matching. If $j = 0$, then remove D_i from M. If $j \neq 0$, then add D_i to M and remove

$$D_i' := \{e \in M \setminus D_i : e \cap V(D_i) \neq \emptyset\}$$

from M. (In order words, we ensure that M is still a matching by removing the set $D_i' \subseteq M$ of the "earlier" edges that conflict with D_i.) Note that, in both cases, the updated set F_j' loses all edges from D_i. We define the final set $M_\infty \subseteq E$ to consist of those pairs that are eventually included into the current matching from some moment onwards:

$$M_\infty := \cup_{i \not\equiv 0} \, D_i \setminus \left(\left(\cup_{\substack{h \geq i \\ h \not\equiv 0}} D_i' \right) \cup \left(\cup_{\substack{h > i \\ h \equiv 0}} D_i \right) \right).$$

(Here and below all residues are modulo $k + 1$.) Clearly, M_∞ is a Borel matching. Let us show that it has all required properties.

First, we show that each set D_i has small measure. For convenience, assume that each d_i is even. By construction, D_i is (F_j', d_i)-sparse. Thus the $(d_i/2)$-neighborhoods of edges in D_i, taken with respect to F_j', are pairwise disjoint and each contains exactly $d_i + 2$ vertices as the maximum degree of F_j' is at most 2. Since all sets are Borel, we conclude by Proposition 3.3 that $\mu(V(D_i)) \leq 2/(d_i + 2)$.

Let $Y_i := \cup_{h=i+1}^\infty N_{2d_i+4}(V(D_h))$ consist of vertices that belong to the $(2d_i + 4)$-neighborhood (taken with respect the whole edge-set E) of $V(D_h)$ for at least one $h > i$. Since

$$\mu(N_{2d_i+4}(V(D_h))) \leq (\Delta(\mathcal{G}) - 1)^{2d_i+4} \, \mu(V(D_h)) \leq (2k)^{2d_i+4} \cdot \frac{2}{d_h + 2},$$

it follows that $\mu(Y_i) \leq (2k)^{2d_i+4} \sum_{h=i+1}^\infty 2/(d_h + 2)$. By letting the numbers d_h grow sufficiently fast, we can ensure that $\sum_{i=1}^\infty \mu(Y_i) < \infty$. The Borel–Cantelli Lemma implies that the set

$$Y := \cap_{i=1}^\infty \cup_{h=i+1}^\infty Y_h$$

of vertices that belong to infinitely many of the sets Y_i has measure zero.

Thus, in order to prove the lemma, it suffices to show that, for every $j \in \{0, \ldots, k\}$, each vertex y of an infinite component of the final set F'_j belongs to Y. In fact, we are going to show that $y \in Y_i$ for every $i \equiv j$. Fix any such i.

First, consider the case $j \neq 0$. Then the final set is $F'_j = F_j \setminus M_\infty$. Consider the moment when we are about to add D_i to M. Recall that D_i is a maximal (F'_j, d_i)-sparse subset of the current set F'_j. The maximality of D_i implies that if we move from y in any of at most two possible directions along F'_j (recall that $\Delta(F'_j) \leq 2$), then we encounter within $2d_i + 3$ steps an element of D_i or a vertex of F'_j-degree at most 1. Thus, at the moment right after we added D_i to M and before we removed D'_i from M, the component C of F'_j that contained y was entirely covered by $N_{2d_i+3}(y)$. At the end, the F'_j-component C of y became infinite. Consider the first time (after D_i was added) when a new edge e is attached to the current component $C \ni y$ (perhaps after some steps when C had shrunk further). Let us show that this cannot happen when D'_i is removed from M. For this, it is enough to show that $D'_i \cap F_j = \emptyset$. Now, if $D'_i \cap F_j \ni uv$ with, say, $vw \in D_i$, then v would have degree 1 in F'_j and the edge $vw \in D_i$ would be too close in F'_j to a degree-1 vertex, a contradiction. Thus there are only two ways for the edge e to be added to F'_j: for some $h > i$ either $e \in D'_h$ with $h \not\equiv 0$ (that is, e is removed from M because it intersects $V(D_h)$) or $e \in D_h$ with $h \equiv 0$. Since $\text{dist}(y, e) \leq 2d_i + 3$, we conclude that $y \in N_{2d_i+4}(V(D_h))$, that is, $y \in Y_i$.

Finally, suppose that $j = 0$ (that is, $i \equiv 0$). At the moment, when we have just removed D_i from M, the F'_j-component $C \ni y$ is a subset of $N_{2d_i+4}(y)$ by a similar argument as above. (Here, D_i is restricted to a subset of M but this can increase our distance estimates at most by 1 as M contains every second edge of each path in F'_j.) Again, consider the first moment when some new edge e gets attached to C. Here, this can happen in only one possible way, namely, e belongs to some D_h where $h > i$ is not divisible by $k + 1$ (then this set D_h is added to the matching). Here $y \in N_{2d_i+4}(V(D_h))$ and thus $y \in Y_i$. This completes the proof of the lemma. $\qquad\square$

Theorem 8.3. *Let $d \geq 1$. Then*

$$\lceil (d+1)/2 \rceil \;\; \leq \;\; k(d) \;\; \leq \;\; \lceil (d + f(d) + 1)/2 \rceil,$$
$$d + 1 \;\; \leq \;\; k'(d) \;\; \leq \;\; d + f(d).$$

In particular, by Theorem 1.4, $k(d) = d/2 + o(d)$ and $k'(d) = d + o(d)$ as $d \to \infty$.

Proof. Let $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ be a graphing with maximum degree at most d. By Theorem 1.8, there is a partition $E = M_0 \cup \cdots \cup M_m$ into Borel matchings $M_1, \ldots, M_m$ and a $\mu^\#$-null-set $M_0 \subseteq E$, where $m := d + f(d)$. We can additionally assume that M_0 is the union of some connectivity components of $\mathcal{G}$. Using the Axiom of Choice and (finite) Vizing's theorem, we

can partition M_0 into $d+1$ matchings $M_1', \ldots, M_{d+1}'$. Then the measurable matchings $M_i \cup M_i'$, $i \in [d+1]$, and M_i, $i \in \{d+2, \ldots, m\}$, can be oriented (by using some fixed 1-sparse labeling) to produce the measure-preserving maps $\phi_1, \ldots, \phi_m$ that establish the claimed upper bound $k'(d) \leq d + f(d)$.

The upper bound on $k(d)$ follows by pairing the above Borel matchings $M_1, \ldots, M_{m-1}$ into $k := \lceil (m-1)/2 \rceil$ graphs $F_1, \ldots, F_k$ of maximum degree at most 2 and taking $F_0 := M_m$. Let M be the matching returned by Lemma 8.2. We obtain a partition of E a.e. into $k+1$ subgraphs, $F_0 \cup M, F_1 \setminus M, \ldots, F_k \setminus M$ of maximum degree at most 2 with finite components. The argument of Lemma 3.4 shows that we can orient all these graphs into directed paths and cycles in a Borel way. Then we fix the null-set of errors using the upper bound of $\lceil (d+1)/2 \rceil \leq k+1$ for countable graphs. This naturally gives $k+1$ measurable maps that generate E.

The claimed lower bounds on $k(d)$ and $k'(d)$ are trivial: for example, take a finite graph G with $\Delta(G) = d$ and $\chi'(G) = d+1$ and turn it into a graphing by using the uniform measure on the vertex set. $\qquad \square$

Remark 8.4. One might think that, in the proof of Theorem 8.3, each F_i could just be oriented without removing any matching M. This is however not true, as the following example of a 2-regular graphing $\mathcal{G}$ shows. Namely, $\mathcal{G}$ cannot be oriented in a measurable way to have maximum in- and out-degree at most 1. To construct $\mathcal{G}$, take two copies of the circle, say $C_j := \{(e^{2\pi i x}, j) : 0 \leq x < 1\}$ for $j = 1, 2$. The first measure-preserving map ϕ maps $(e^{2\pi i x}, j)$ to $(e^{2\pi i x}, 3 - j)$ for $(x, j) \in [0, 1) \times [2]$, i.e. it is the natural involution between the two circles. The second map ψ has each circle as an invariant set. Namely, for $j \in [2]$, fix an axis A_j via the center of the circle C_j and let the restriction of ϕ to C_j be the reflection along A_j. Let us assume that α/π is irrational where α is the angle between A_1 and A_2. Suppose on the contrary that we can orient the edges of $\mathcal{G}$ with all in- and out-degrees being 1 a.e. Let X consist of those $x \in C_1$ such that the orientation goes from x to $\phi(x)$. The measure of X is exactly half of measure of C_1, because exactly half of edges in measure between the circles goes each way. Consider the composition $\phi \circ \psi \circ \phi \circ \psi : C_1 \to C_1$, which is a rotation by angle 2α. It follows that X is invariant a.e. with respect to this irrational rotation of C_1, contradicting its ergodicity.

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