2020 FALL MAS583 HW7

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1 Problem 1

1.1 Question

Let $G = (V, E)$ be the graph whose vertices are all 7^n vectors of length n over $\mathbb{Z}_7$, in which two vertices are adjacent iff they differ in precisely one coordinate. Let $U \subset V$ be a set of 7^{n-1} vertices of G, and let W be the set of all vertices of G whose distance from U exceeds $(c+2)\sqrt{n}$, where $c > 0$ is a constant. Prove that $|W| \le 7^n \cdot e^{-c^2/2}$.

1.2 Answer

To show the inequality, suffice it to show that if we pick a vertex $x \in V$ uniformly at random, the probability that x has a distance from the given U exceeds $(c+2)\sqrt{n}$ is at most $e^{-c^2/2}$. Consider $A = \mathbb{Z}_7$ and $B = [n]$, then $\Omega = A^B$, the set of functions $g : B \to A$, is equivalent to $V = V(G)$. Let

$$p_{ab} = \Pr[g(b) = a] = 1/7,$$

for each $a \in A$ and $b \in B$. And fix a gradation

$$\emptyset = B_0 \subset B_1 \subset \cdots \subset B_n = B,$$

where $B_i = [i]$, for each $i \in [n]$. For given U, let $L : A^B \to \mathbb{R}$ be the function denoting the distance from U of the corresponding vertex. We define a martingale $X_0, X_1, ..., X_n$ by setting

$$X_i(h) = \mathbb{E}[L(g)|g(b) = h(b), \forall b \in B_i = [i]],$$

in specific $X_0 = \mathbb{E}[L(g)|g \in \Omega]$ is a constant and $X_n = L$. We can easily check that L satisfies the Lipschitz condition relative to the gradation above as changing in one coordinate can make the distance change by at most 1. Thus, our corresponding martingale satisfies

$$|X_{i+1}(h) - X_i(h)| \le 1$$

for all $0 \le i < n$ and $h \in A^B$. By setting $\mu = X_0 = \mathbb{E}[L(g)|g \in \Omega]$, we have for all $\lambda > 0$,

$$\Pr[L(g) \ge \mu + \lambda\sqrt{n}] < e^{-\lambda^2/2} \tag{1}$$

and

$$\Pr[L(g) \le \mu - \lambda\sqrt{n}] < e^{-\lambda^2/2}. \tag{2}$$

Now, with (1), we only need to show that $\mu \leq 2\sqrt{n}$ to complete the proof. Actually, by setting $\lambda = \sqrt{2\ln 7}$ in (2), we have

$$\Pr[L(g) \leq \mu - \sqrt{2\ln 7}\sqrt{n}] < 1/7,$$

which implies that $\mu \leq \sqrt{2\ln 7}\sqrt{n} < 2\sqrt{n}$ as $\Pr[L(g) \leq 0] = |U|/|V| = 1/7$, completing the proof.

2 Problem 2

2.1 Question

(*) Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = 1000$. Let $U \subset V$ be a random subset of V chosen uniformly among all $2^{|V|}$ subsets of V. Let $H = G[U]$ be the induced subgraph of G on U. Prove that

$$\Pr[\chi(H) \leq 400] < 1/100.$$

2.2 Answer

First, we show the following lemma.

Lemma 2.1 *Given $G = (V, E)$ with chromatic number $\chi(G)$, let $A \cup B = V$ be a partition of V, and let H_A and H_B be the subgraphs of G induced by A and B, respectively. Then $\chi(A) + \chi(B) \geq \chi(G)$.*

Proof. Consider a coloring where $G[A]$ is properly colored by $\chi(A)$ colors and $G[B]$ is properly colored with another $\chi(B)$ different colors, which is a proper coloring of G with $\chi(A) + \chi(B)$ colors, completing the proof. ■

As $\chi(G) = 1000$, we can find a partition of V into 1000 independent sets $\{V_1, ..., V_{1000}\}$. For any $U_0 \subset V$, we build a martingale $\mathbb{E}[\chi(G[U])] = X_0, X_1, ..., X_{1000} = \chi(G[U_0])$ by

$$X_i(H) = \mathbb{E}\left[\chi(G[H]) : H \cap \bigcup_{j\in[i]} V_j = U \cap \bigcup_{j\in[i]} V_j\right].$$

It is easy to check that the above martingale satisfies the Lipschitz condition $|X_{i+1} - X_i| \leq 1$, because at worst you can only change the coloring of the newly exposed part $H \cap V_{i+1}$, where at most one color should be used in a minimum proper coloring. By the uniformity, we have

$$\mathbb{E}[\chi(G[U])] = \mathbb{E}[\chi(G[V \setminus U])],$$

with the lemma, we further have

$$\mathbb{E}[\chi(G[U])] = \frac{1}{2}(\mathbb{E}[\chi(G[U])] + \mathbb{E}[\chi(G[V \setminus U])]) \geq \chi(G)/2 = 500.$$

Finally, by Azuma's Inequality, we have

$$\Pr[\chi(H) \leq 400] = \Pr[X_{1000} \leq 400] \leq \Pr[X_{1000} - \mathbb{E}[\chi(G[U])] \leq -100] \leq e^{-100^2/2000} < 1/100,$$

completing the proof.

2.3 References

https://math.stackexchange.com/questions/2118314

3 Problem 3

3.1 Question

Prove that there is an absolute constant c such that for every $n > 1$ there is an interval I_n of at most $c\sqrt{n}/\log n$ consecutive integers such that the probability that the chromatic number of $G(n, 0.5)$ lies in I_n is at least 0.99.

3.2 Answer

Let $\epsilon > 0$ be arbitrarily small and let $u = u(n, p, \epsilon)$ be the least integer so that

$$\Pr[\chi(G) \le u] > \epsilon.$$

Now define $Y(G)$ to be the minimal size of a set of vertices S for which $G - S$ may be u-colored. This Y satisfies the vertex Lipschitz condition since at worst one could add a vertex to S. Apply the vertex exposure martingale on $G(n, p)$ to Y. Letting $\mu = E[Y]$, for all $\lambda > 0$,

$$\Pr[Y \le \mu - \lambda\sqrt{n-1}] < e^{-\lambda^2/2}, \tag{3}$$

$$\Pr[Y \ge \mu + \lambda\sqrt{n-1}] < e^{-\lambda^2/2}. \tag{4}$$

Let λ satisfy $e^{-\lambda^2/2} = \epsilon$, with (3), we have

$$\Pr[Y \le \mu - \lambda\sqrt{n-1}] < \epsilon \le \Pr[Y \le 0],$$

which implies that $\mu \le \lambda\sqrt{n-1}$. Apply this to (4), we have

$$\Pr[Y \ge 2\lambda\sqrt{n-1}] \le \Pr[Y \ge \mu + \lambda\sqrt{n-1}] < \epsilon,$$

i.e., with probability at least $1 - \epsilon$ there is a u-coloring of all but at most $2\lambda\sqrt{n-1}$ vertices, where c_1 is a positive constant.
To complete the proof, it remains to show the following lemma.

Lemma 3.1 *Given $\epsilon > 0$, there is an absolute constant $c = c(\epsilon)$ such that with probability at least $1 - \epsilon$, every $2\lambda\sqrt{n-1}$ vertices of $G = G(n, 0.5)$ may be $(c\sqrt{n}/\log n)$-colored, where $\lambda = \lambda(\epsilon)$ satisfies $e^{-\lambda^2/2} = \epsilon$.*

Proof. As a known result, by setting $f(k) = \binom{m}{k}2^{-\binom{k}{2}}$, k_0 so that $f(k_0 - 1) > 1 > f(k_0)$, $k = k_0 - 4$ so that $k \sim 2\log_2 m$ and $f(k) > m^{3+o(1)}$, we have

$$\Pr[\omega(G(m, 0.5)) < k] = e^{-m^{2+o(1)}}.$$

By the property of $G(m, 0.5)$, we have

$$\Pr[\alpha(G(m, 0.5)) \le \log m < k] \le e^{-m^{2+o(1)}}.$$

Let $m = n^{1/3}$, it follow that the probability that there is a set of vertices S of size at least $n^{1/3}$ that does not contain an independent set of size at least $\log m = \frac{1}{3}\log n$ is at most

$$\binom{n}{n^{1/3}} e^{-n^{2/3+o(1)}} \leq n^{n^{1/3}} e^{-n^{2/3+o(1)}} \leq \exp(n^{1/3}\log n - n^{2/3+o(1)}) = o(1),$$

i.e., in $G(n, 0.5)$, almost always, every set of at least $n^{1/3}$ contains an independent set of size at least $\frac{1}{3}\log n$. In particular, with probability at least $1-\epsilon$, this holds for large n. Now, let $S_0 = S$ be a set of size at most $2\lambda\sqrt{n-1}$, when n is sufficiently large, with probability at least $1-\epsilon$, we can do the following for $i \geq 1$ until the algorithm stops:

- If $|S_i| \leq \sqrt{n}/\log n$, stop.
- Let U_i be an independent set of size at least $\frac{1}{3}\log n$ in S_i.
- Set $S_{i+1} = S_i \setminus U_i$

Clearly, the above algorithm ends in $l \leq 6\lambda\sqrt{n}/\log n$ steps. Let $W = S \setminus \bigcup_{i\in[l]} U_i$, we obtain a proper coloring of $G[S]$ with at most $(6\lambda+1)\sqrt{n}/\log n$ by coloring each U_i with color i and coloring each vertex in W with a distinct new color. It follows that, by setting $c \geq (6\lambda+1)$

$$\chi(G[S]) \leq (6\lambda+1)\sqrt{n}/\log n \leq c\sqrt{n}/\log n,$$

completing the proof. ■

Now we have

- With probability at least $1-\epsilon$, there is a u-coloring of all but at most $2\lambda\sqrt{n-1}$ vertices.
- With probability at least $1-\epsilon$, these at most $2\lambda\sqrt{n-1}$ vertices can be properly colored with $(c\sqrt{n}/\log n)$ more colors, where c is an absolute constant only depending on ϵ.
- With probability at least $1-\epsilon$, by the definition of u, $\chi(G) \geq u$.

Altogether we have

$$\Pr[u \leq \chi(G) \leq u + c\sqrt{n}/\log n] \geq 1 - 3\epsilon.$$

By setting $\epsilon \leq 1/300$, we complete the proof.

3.3 References

http://gxyau.github.io/pdf-documents/ETHZ/Autumn%202018/Probabilistic%20Methods%20in%20Combinatorics/Assignment%2012/solutions12.pdf