# Note

## The 123 Theorem and Its Extensions

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It is shown that for every $b > a > 0$ and for every two independent identically distributed real random variables $X$ and $Y$, $\text{Prob}[|X - Y| \leqslant b] < (2\lceil b/a \rceil - 1)$ $\text{Prob}[|X - Y| \leqslant a]$. This is tight for all admissible pairs $a, b$. Higher dimensional extensions are also considered. © 1995 Academic Press, Inc.

## 1. INTRODUCTION

Our first result in this note is the following theorem, which we name after the three constants in its statement.

THEOREM 1.1 (The 123 theorem). *Let $X$ and $Y$ be two independent, identically distributed real random variables. Then*

$$\text{Prob}[|X - Y| \leqslant 2] < 3 \, \text{Prob}[|X - Y| \leqslant 1].$$

The problem of determining the smallest possible constant $C$ so that for every two independent, identically distributed (i.i.d.) real random variables the inequality

$$\text{Prob}[|X - Y| \leqslant 2] \leqslant C \, \text{Prob}[|X - Y| \leqslant 1]$$

holds was suggested by G. A. Margulis and communicated to us by Y. Peres. Several researchers, including Peres, observed that the smallest possible $C$ satisfies $3 \leqslant C \leqslant 5$, where the lower bound follows by considering two i.i.d. real random variables $X$ and $Y$ distributed uniformly on the discrete set $\{2, 4, 6, ..., 2n\}$. Here $\text{Prob}[|X - Y| \leqslant 1] = 1/n$, whereas $\text{Prob}[|X - Y| \leqslant 2] = 3/n - 2/n^2$, showing that $C \geqslant 3 - 2/n$ for every $n$.

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Theorem 1.1 thus shows that, in fact, $C = 3$. As we learned from Peres, a slightly weaker version of this theorem (without the strict inequality) has also been proved, independently of our work (and before us) by A. Kotlov in a different way. Our proof, presented in the next section, is shorter and has the additional advantage that it actually gives the following more general result.

THEOREM 1.2.   *Let* $b > a > 0$ *be two reals. Then for every i.i.d. real random variables* $X$ *and* $Y$

$$\text{Prob}[\,|X - Y| \leqslant b\,] < (2\lceil b/a \rceil - 1) \,\text{Prob}[\,|X - Y| \leqslant a\,].$$

*Moreover, the constant* $2\lceil b/a \rceil - 1$ *cannot be improved.*

The above questions can be considered for vector-valued random variables as well. Let $V = R^d$ be a finite dimensional Euclidean space, and let $b \geqslant a > 0$ be two reals. Let $C(V, a, b)$ denote the smallest possible $C$ such that for every two i.i.d. random variables $X$ and $Y$ taking values in $V$

$$\text{Prob}[\,\|X - Y\| \leqslant b\,] \leqslant C \,\text{Prob}[\,\|X - Y\| \leqslant a\,].$$

Note that $C(V, a, b) = C(V, a', b')$ whenever $b/a = b'/a'$ and hence it suffices to consider the case $a = 1$. Our method supplies rather tight estimates for the function $C(R^d, 1, b)$. However, the problem of determining this function precisely seems difficult, even for the Euclidean plane $R^2$. Yet, the technique does enable us to determine this function precisely for infinitely many values of $b$.

THEOREM 1.3.   *Let* $V = R^d$, $n \geqslant 2$, *and suppose that there is no set* $F$ *of* $n + 1$ *points in a ball of radius* $b$ *so that the center belongs to* $F$ *and the distance between any two points of* $F$ *exceeds* 1. *If there is a lattice in* $V$ *with minimum distance* 1 *so that there are* $n$ *points of it in a ball of radius smaller than* $b$ *centered at a lattice point, then* $C(V, 1, b) = n$.

This theorem, together with the main result of [1], implies that there is an absolute constant $\varepsilon > 0$ such that in the Euclidean plane $R^2$, $C(R^2, 1, b) = 7$ for all $1 < b < 1 + \varepsilon$ and $C(R^2, 1, b) = 19$ for all $2 < b < 2 + \varepsilon$. Similarly, the Leech lattice and the known bounds on kissing numbers (see [2]) imply that there exists an $\varepsilon > 0$ such that in dimension 24, $C(R^{24}, 1, b) = 196,561$ for all $1 < b < 1 + \varepsilon$. Also, for $1 < b < 1 + \varepsilon$, $C(R^3, 1, b) = 13$ and $C(R^8, 1, b) = 241$, where $\varepsilon > 0$ is an appropriate absolute constant.

The rest of this note is organized as follows. In Section 2 we consider the real case and present the proofs of Theorems 1.1 and 1.2. In Section 3 we consider higher dimensions.

## 2. REAL RANDOM VARIABLES

Our basic approach is combinatorial and (vaguely) resembles the method of Katona in [3]. See also [7], [8]. Let $T = (x_1, x_2, ..., x_m)$ be a sequence of not necessarily distinct reals. For any positive $b$, define

$$T_b = \{(x_i, x_j) : 1 \leqslant i, j \leqslant m, |x_i - x_j| \leqslant b\}.$$

We need the following simple combinatorial lemma.

LEMMA 2.1. *For any sequence $T$ as above and for every integer $r > 1$.*

$$|T_r| < (2r - 1) |T_1|.$$

*Proof.* We apply induction on $|T| = m$. The result is trivial for $m = 1$. Assuming it holds for $m - 1$, we prove it for $m$ $(>1)$. Given a sequence $T = (x_1, ..., x_m)$ let $t + 1$ be the maximum number of points of $T$ in a closed interval of length 2 centered at a member of $T$. Let $x_i$ be any rightmost point of $T$ so that there are $t + 1$ members of $T$ in the interval $[x_i - 1, x_i + 1]$ and define $T' = T \backslash \{x_i\}$. The number of members of $T'$ in the interval $[x_i - 1, x_i + 1]$ is clearly $t$ and, hence, $x_i$ appears in precisely $2t + 1$ ordered pairs of $T_1$. Thus

$$|T_1| = 2t + 1 + |T_1'|.$$

The interval $[x_i - r, x_i + r]$ is the union of the $2r - 1$ smaller intervals

$$[x_i - r, x_i - r + 1), ..., [x_i - 2, x_i - 1), [x_i - 1, x_i + 1],$$
$$(x_i + 1, x_i + 2], ..., (x_i + r - 1, x_i + r]. \tag{1}$$

By the choice of $x_i$, each of these smaller intervals can contain at most $t + 1$ members of $T$, and each of the last $r - 1$ ones, which lie to the right of $x_i$, can contain at most $t$ members of $T$. Altogether there are thus at most $(r - 1)(t + 1) + rt$ members of $T'$ in $[x_i - r, x_i + r]$ and, hence,

$$|T_r| \leqslant 2(r - 1)(t + 1) + 2rt + 1 + |T_r'| = (2r - 1)(2t + 1) + |T_r'|.$$

By the induction hypothesis $|T_r'| < (2r - 1) |T_1'|$ and hence $|T_r| < (2r - 1) |T_1|$, completing the proof. ∎

COROLLARY 2.2. *Let $X$ and $Y$ be two i.i.d. real random variables. For a positive $b$, define $p_b = \text{Prob}[|X - Y| \leqslant b]$. Then for every integer $r$, $p_r \leqslant (2r - 1) p_1$.*

*Proof.* Fix an integer $m$, and let $S = (x_1, ..., x_m)$ be a random sequence of $m$ elements, where each $x_i$ is chosen, randomly and independently, according to the distribution of $X$. By Lemma 2.1

$$|S_r| < (2r-1)\,|S_1|.$$

Therefore, the expectation of $|S_r|$ is smaller than that of $(2r-1)\,|S_1|$. However, by the linearity of expectation it follows that the expectation of $|S_b|$ is precisely $m + m(m-1)\,p_b$ for every positive $b$. Therefore,

$$m + m(m-1)\,p_r < (2r-1)(m + m(m-1)\,p_1),$$

implying that for every integer $m$,

$$p_r < (2r-1)\,p_1 + \frac{2r-2}{m-1}.$$

The desired result $p_r \leqslant (2r-1)\,p_1$ follows, by letting $m$ tend to infinity. ∎

The last corollary suffices to prove the assertions of Theorems 1.1 and 1.2, without the strict inequality. To prove the strict inequality we need an additional argument, which follows. Let $X$ and $Y$ be two i.i.d. real random variables, suppose $r > 1$ is an integer, and suppose that $p_r = (2r-1)\,p_1$, where $p_r$ and $p_1$ are defined as before. For two reals $a$ and $b$, define $\mu_b(a) = \text{Prob}[\,|X-a| \leqslant b\,]$.

CLAIM. *There exists a real $a$ so that $\mu_r(a) > (2r-1)\,\mu_1(a)$.*

*Proof.* Otherwise, $\mu_r(a) \leqslant (2r-1)\,\mu_1(a)$ for each $a$, and since $p_r = (2r-1)\,p_1$ and $p_b$ is simply the expectation of $\mu_b(a)$ when $a$ is chosen according to the distribution of $X$, it follows that $\mu_r(a) = (2r-1)\,\mu(a)$ with probability 1 (when $a$ is chosen according to the distribution of $X$). Let $A$ be the set of all real values $a$ for which $\mu_r(a) = (2r-1)\,\mu_1(a)$, and define

$$\delta = \text{Sup}\{\mu_1(a) : a \in A\}.$$

Clearly $\delta > 0$. Let $\varepsilon$ be a small positive constant such that

$$k(\delta - (2r)^k \varepsilon) > 1, \qquad \text{where} \quad k = \lceil 2/\delta \rceil. \tag{2}$$

Pick $a_0 \in A$ so that $\mu_1(a_0) > \delta - \varepsilon$. We next define a sequence of $k$ pairwise disjoint unit intervals $I_1, ..., I_k$ in the line, so that for each $i$, $\text{Prob}[X \in I_i] > \delta - (2r)^i \varepsilon$. Since in view of (2) this is impossible, the assertion of the claim will follow. The first interval $I_1$ is simply the interval $(a_0 + r - 1, a_0 + r]$. Observe that by the choice of $a_0$,

$$\text{Prob}[a_0 - r \leqslant X \leqslant a_0 + r] \geqslant (2r-1)(\delta - \varepsilon).$$

Split the interval $[a_0 - r, a_0 + r]$ into $2r - 1$ smaller intervals as in (1). Note that the definition of $\delta$ implies that the probability that $X$ lies in any one of these intervals is at most $\delta$. Therefore, for each of these smaller intervals (and in particular for the last one, $I_1 = (a_0 - r + 1, a_0 + r]$) the probability that $X$ lies in the interval is at least $(2r - 1)(\delta - \varepsilon) - (2r - 2)\delta > \delta - 2r\varepsilon$, as needed. Suppose, now, that the pairwise disjoint unit intervals $I_1, ..., I_j$ have already been defined, where $I_j$ is the rightmost interval, and $\text{Prob}[X \in I_i] > \delta - (2r)^i \varepsilon$ for all $1 \leqslant i \leqslant j \ (<k)$. We can now define $I_{j+1}$ as follows. Since $X$ attains values in $A$ with probability 1, and it lies in $I_j$ with positive probability, there is an $a_j \in I_j \cap A$. Obviously, $\mu_1(a_j) \geqslant \text{Prob}[X \in I_j] > \delta - (2r)^j \varepsilon$. Therefore, as $a_j \in A$,

$$\text{Prob}[a_j - r \leqslant X \leqslant a_j + r] \geqslant (2r - 1)\,\text{Prob}[X \in I_j] > (2r - 1)(\delta - (2r)^j \varepsilon).$$

We can thus define $I_{j+1} = (a_j + r - 1, a_j + r]$ and conclude, as before, that

$$\text{Prob}[X \in I_{j+1}] > \delta - (2r - 1)(2r)^j \varepsilon > \delta - (2r)^{j+1} \varepsilon,$$

as required. This supplies the desired contradiction and completes the proof of the claim. ∎

Returning to our two i.i.d. real random variables $X$ and $Y$ for which $p_r = (2r - 1) p_1$, observe that by the claim there is a real $a$ such that

$$\mu_r(a) = (2r - 1) \mu_1(a) + \beta, \tag{3}$$

where $\beta > 0$. Let $\alpha > 0$ be a small constant satisfying $\alpha/(1 - \alpha) < \beta/(r - 1)$. Define $X'$ as the random variable which has the distribution of $X$ with probability $(1 - \alpha)$ and with probability $\alpha$ it gets the value $a$. For any real $b$, let $p'_b = \text{Prob}[|X' - Y'| \leqslant b]$, where $X'$, $Y'$ are i.i.d. random variables with the distribution of the above $X'$. By the definition of $X'$, for every positive $b$,

$$p'_b = (1 - \alpha)^2 p_b + 2\alpha(1 - \alpha) \mu_b(a) + \alpha^2.$$

By Corollary 2.2 applied to $X'$, $p'_r \leqslant (2r - 1) p'_1$. In view of the last equality and (3) this implies that

$$(1 - \alpha)^2 p_r + 2\alpha(1 - \alpha)((2r - 1) \mu_1(a) + \beta) + \alpha^2$$
$$\leqslant (2r - 1)[(1 - \alpha)^2 p_1 + 2\alpha(1 - \alpha) \mu_1(a) + \alpha^2].$$

Therefore,

$$p_r \leqslant (2r - 1) p_1 - 2 \frac{\alpha}{1 - \alpha} \beta + (2r - 2) \frac{\alpha^2}{(1 - \alpha)^2} < (2r - 1) p_1,$$

where the last inequality follows from the choice of $\alpha$. This shows that equality is impossible in Corollary 2.2. We have thus proved the following.

PROPOSITION 2.3.   *In the notation of Corollary 2.2, for every integer $r > 1$,* $p_r < (2r - 1) p_1$.

We can now complete the proof of Theorem 1.2 (which implies, of course, Theorem 1.1).

*Proof of Theorem* 1.2.   Applying the last proposition to $X' = X/a$, $Y' = Y/a$, and $r = \lceil b/a \rceil$ we conlude that

$$\text{Prob}[\,|X - Y| \leqslant b\,] = \text{Prob}[\,|X' - Y'| \leqslant b/a\,] \leqslant \text{Prob}[\,|X' - Y'| \leqslant r\,]$$
$$< (2r - 1)\,\text{Prob}[\,|X' - Y'| \leqslant 1\,]$$
$$= (2\lceil b/a \rceil - 1)\,\text{Prob}[\,|X - Y| \leqslant a\,],$$

as required. To see that the constant $2\lceil b/a \rceil - 1$ cannot be improved, let $\gamma$ be a real satisfying $a < \gamma$ and $\gamma(\lceil b/a \rceil - 1) < b$. Let $m$ be a large integer and let the two i.i.d. random variables $X$ and $Y$ be distributed uniformly on the discrete set $\gamma, 2\gamma, ..., m\gamma$. One can easily check that the ratio between $\text{Prob}[\,|X - Y| \leqslant b\,]$ and $\text{Prob}[\,|X - Y| \leqslant a\,]$ approaches $2\lceil b/a \rceil - 1$ as $m$ tends to infinity.   ∎

## 3. VECTOR-VALUED RANDOM VARIABLES

The basic method in the previous section can be modified and extended to higher dimensions. Let $V = R^d$ be the $d$-dimensional Euclidean space. We start with the following simple observation.

LEMMA 3.1.   *Suppose there exists a lattice in $V$ with minimum distance 1 so that there are $n$ points of it in a ball of radius smaller than $b$ centered at a lattice point; then $C(V, 1, b) \geqslant n$.*

*Proof.*   Let $L$ be the above lattice, and let $\gamma > 1$ be close enough to 1 so that $\gamma L$ contains $n$ points in a ball of radius $b$ centered at a lattice point. Let $R$ be a large real, and let $X_R$ and $Y_R$ be two i.i.d. random variables, each uniformly distributed on the points of $\gamma L$ whose norm is at most $R$. It is easy to check that when $R$ tends to infinity the ratio between $\text{Prob}[\,\|X_R - Y_R\| \leqslant b\,]$ and $\text{Prob}[\,\|X_R - Y_R\| \leqslant 1\,]$ approaches the number of points of $\gamma L$ in a ball of radius $b$ centered at a point of $\gamma L$, which is at least $n$.   ∎

LEMMA 3.2.   *Suppose $n \geqslant 2$ and suppose there is no set $F$ of $n + 1$ points in a ball of radius $b$ in $V$, such that the center is in $F$ and the distance*

*between any two members of $F$ exceeds* 1. *Let* $T = (x_1, ..., x_m)$ *be a sequence of points in* $V$. *For any positive $c$ and $s$, define*

$$T_{c,s} = sm + |\{(x_i, x_j) : 1 \leqslant i \neq j \leqslant m, \|x_i - x_j\| \leqslant c\}|.$$

*If $ns > 2n + s$, then*

$$T_{b,s} < nT_{1,s}.$$

*Proof.* We apply induction on $m$. The result is trivial for $m = 1$, since in this case $T_{b,s} = T_{1,s} = s$. Assuming it holds for $m - 1$ we prove it for $m$ ($m > 1$). Given a sequence $T$ of cardinality $m$ as above, let $t + 1$ be the maximum number of members of $T$ in a ball of radius 1 centered at a point of $T$. Let $x$ be a point of $T$ with $t + 1$ members of $T$ in the radius-1 ball centered at $x$, and define $T' = T \setminus \{x\}$. Clearly,

$$T_{1,s} = T'_{1,s} + s + 2t.$$

Let $F$ be a subset of maximum cardinality of $T$ in the ball of radius $b$ centered at $x$, so that $x \in F$ and the distance between any two members of $F$ is strictly bigger than 1. By the assumption, $|F| \leqslant n$. Moreover, any point of $T$ in the ball of radius $b$ centered at $x$ lies in a radius-1 ball centered at a point of $F$. Since, by the maximality in the choice of $x$, no such ball can contain more than $t + 1$ points, it follows that there are at most $n(t + 1)$ members of $T$ in the ball of radius $b$ centered at $x$ (including $x$ itself). Therefore,

$$T_{b,s} \leqslant T'_{b,s} + s + 2(nt + n - 1).$$

By the induction hypothesis $T'_{b,s} < nT'_{1,s}$ provided $ns > 2n + s$ and hence, for such $s$

$$T_{b,s} < nT'_{1,s} + s + 2(nt + n - 1) < nT'_{1,s} + ns + 2nt = nT_{1,s},$$

completing the proof. ∎

COROLLARY 3.3. *Suppose $n \geqslant 2$ and suppose there is no set $F$ of $n + 1$ points in a ball of radius $b$ in $V$, such that the center is in $F$ and the distance between any two members of $F$ exceeds* 1. *Then $C(V, 1, b) \leqslant n$. That is, for any two i.i.d. $V$-valued random variables $X$ and $Y$*

$$\text{Prob}[\|X - Y\| \leqslant b] \leqslant n \, \text{Prob}[\|X - Y\| \leqslant 1]. \qquad (4)$$

*Proof.* For $X, Y$ as above and for a positive $c$, define $p_c = \text{Prob}[\|X - Y\| \leqslant c]$. Fix a positive $s$ satisfying $ns > 2n + s$. For a fixed, large integer $m$, let $T = (x_1, ..., x_m)$ be a random sequence of $m$ elements,

where each $x_i$ is chosen, randomly and independently, according to the distribution of $X$. By Lemma 3.2,

$$T_{b,s} < nT_{1,s}.$$

Therefore, the expectation of $T_{b,s}$ is smaller than that of $nT_{1,s}$. By linearity of expectation, the expectation of $T_{c,s}$ is precisely $sm + m(m-1)\,p_c$ for every positive $c$. Therefore,

$$sm + m(m-1)\,p_b < n(sm + m(m-1)\,p_1),$$

implying that for every integer $m$,

$$p_b < np_1 + \frac{(n-1)\,s}{m-1}.$$

The desired result (4) follows, by letting $m$ tend to infinity. ∎

Theorem 1.3 follows from Lemma 3.1 and Corollary 3.3. We next describe some consequences of this theorem.

In [1] it is shown that the minimum radius of a two-dimensional ball containing eight points, one of which is at the center, so that all mutual distances are at least 1, is $\frac{1}{2}\operatorname{cosec}(\pi/7) = 1.15....$ It is also shown that the minimum radius of a two-dimensional ball containing 20 points, one of which is its center such that all mutual distances are at least 1 is strictly bigger than 2. This, together with Theorem 1.3 (and the existence of the hexagonal lattice), implies the following.

PROPOSITION 3.4.   (i)   *If* $1 < b < \frac{1}{2}\operatorname{cosec}(\pi/7)$ *then* $C(R^2, 1, b) = 7$.

(ii)   *There exists an* $\varepsilon > 0$ *so that for all* $2 < b < 2 + \varepsilon$, $C(R^2, 1, b) = 19$.

Similarly, one can determine the asymptotic behaviour of $C(R^2, 1, b)$ as $b$ tends to infinity. It is well known (see [1]), that the maximum number of points that can be placed in a radius-$b$ two-dimensional ball so that one of the points is at the center, and all mutual distances are at least 1, is $(1 + o(1))(2\pi/\sqrt{3})\,b^2$. This is realized by the hexagonal lattice and, hence, by Theorem 1.3 the following statement holds.

PROPOSITION 3.5.   *As* $b$ *tends to infinity,*

$$C(R^2, 1, b) = (1 + o(1))\,\frac{2\pi}{\sqrt{3}}\,b^2.$$

The *kissing number* $\tau_d$ is the maximum number of points that can be placed on the boundary of a unit ball in $R^d$, so that the distance between

any two points is at least 1. By compactness it follows that the minimum radius of a ball in $R^d$ containing $\tau_d + 2$ points one of which is at the center, so that all mutual distances are at least 1, is strictly bigger than 1. The exact values of $\tau_d$ are known only for $d = 1, 2, 3, 8,$ and 24. Trivially $\tau_1 = 2$ and $\tau_2 = 6$. The value of $\tau_3$ was the subject of a discussion between Isaac Newton and David Gregory in 1694. Newton believed that $\tau_3 = 12$, and as shown by various researchers in the nineteenth century, this is indeed the case. A very short proof of this fact appears in [4]. In [6, 5] it is shown that $\tau_8 = 240$ and $\tau_{24} = 196{,}560$. In all the above cases the highest possible kissing numbers are attainable by lattices. (It is known that this is *not* the case in dimension 9.) The relevant lattices are the trivial one in dimension 1, the hexagonal lattice in dimension 2, the face-centered cubic lattice in dimension 3, the lattice $E_8$ (sometimes called the eight-dimensional diamond lattice), in dimension 8, and the well-known Leech lattice in dimension 24. See [2] for more details. Theorem 1.3 thus gives the following.

PROPOSITION 3.6. *For every $d \in \{1, 2, 3, 8, 24\}$ there exists an $\varepsilon_d > 0$ such that for all $1 < b < 1 + \varepsilon_d$, $C(R^d, 1, b) = \tau_d + 1$.*

Note that the above result for $d = 1, 2$ has already been proved (in a stronger form) in Theorem 1.2 and in Proposition 3.4(i).

The basic results in this section can be extended to other norms in finite dimensional real spaces. Some norms, like the $l_\infty$-norm, are simpler than the Euclidean one for this purpose, and it is not too difficult to determine the function $C(l_\infty^d, b, 1)$ (defined in the obvious way) precisely for many values of $b$. However, for any fixed $d \geq 2$ we are unable to determine any of the functions $C(R^d, b, 1)$ or $C(l_\infty^d, b, 1)$ for all $b$ and this problem remains open.

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