Math778P Homework 2 Solution

Choose any 5 problems to solve.

1. Let $S_n = \sum_{i=1}^n X_i$ where $X_1, \ldots, X_n$ are n independent uniform $\{-1, 1\}$ random variables. Prove that

$$\mathrm{E}(|S_n|) = n2^{1-n}\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

Proof: Let $S_n = \sum_{i=1}^n X_i$ where $X_1, \ldots, X_n$ are n independent uniform $\{-1, 1\}$ random variables. Prove that $\mathrm{E}(|S_n|) = n2^{1-n}\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$.

$$\begin{aligned}
\mathrm{E}(|S_n|) &= \sum_{k=-n}^{n} |k| \Pr(S_n = k) \\
&= 2\sum_{k=0}^{n} k \frac{(\#\text{ of ways } S_n = k)}{2^n} \\
&= 2^{1-n} \sum_{k=0}^{n} k \left(\#\text{ of ways to have } \frac{n-k}{2} \text{ -1's and } \frac{n+k}{2} \text{ 1's.} \right) \\
&= 2^{1-n} \sum_{\substack{0 \le k \le n \\ n-k \text{ even}}} k \binom{n}{\frac{n-k}{2}} \\
&= 2^{1-n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (n-2i) \binom{n}{i} \\
&= 2^{1-n} \left[n\binom{n}{0} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} ((n-i)-i) \frac{n!}{(n-i)!i!} \right] \\
&= 2^{1-n} \left[n \cdot 1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{n!}{(n-i-1)!i!} - \frac{n!}{(n-i)!(i-1)!} \right) \right] \\
&= 2^{1-n} n \left[\binom{n-1}{0} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n-1}{i} - \binom{n-1}{i-1} \right) \right]
\end{aligned}$$

$$
\begin{aligned}
&= n2^{1-n}\left[\binom{n-1}{\lfloor\frac{n}{2}\rfloor}\right] \\
&= n2^{1-n}\binom{n-1}{(n-1)-\lfloor\frac{n}{2}\rfloor} \\
&= n2^{1-n}\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}.
\end{aligned}
$$

$\square$

2. Suppose that $p > n > 10m^2$, with p prime, and let $0 < a_1 < a_2 < \cdots < a_m < p$ be integers. Prove that there is an integer x, $0 < x < p$ for which the m numbers

$$(xa_i \mod p) \mod n, \quad 1 \leq i \leq m$$

are pairwise distinct.

Proof: Let x be chosen uniformly at random from $[p-1]$. First, note that for each a_i, xa_i is never $0 \mod p$ since x is never $0 \mod p$; thus $xa_i \mod p$ is one of $p-1$ integers. Consider a number j with $0 \leq j < n$. Let us count the maximum number of values $0 \leq i < p$ such that $i \equiv j \mod n$. Note that $i \equiv j \mod n$ if $i = j$, or $i = j + n$, or $i = j + 2n$,..., or $i = j + kn$ where $j + kn < p$ and $j + (k+1)n \geq p$. Further, one can see that $k = \lfloor \frac{p}{n} \rfloor$; hence there are at most $\lfloor \frac{p}{n} \rfloor + 1$ integers i with $0 \leq i < p$ such that $i \equiv j \mod n$. It follows that

$$\mathbb{P}((xa_i \mod p) \mod n = (xa_j \mod p) \mod n) \leq \frac{\lfloor \frac{p}{n} \rfloor + 1}{p-1}.$$

Let X_{ij} be the random variable taking on the value 1 if $(xa_i \mod p) \mod n = (xa_j \mod p) \mod n$ and 0 otherwise. We then have that the expected number of non-distinct pairs xa_i, xa_j is:

$$\begin{aligned}
\mathbb{E}(\sum_{i<j} X_{ij}) &= \sum_{i<j} \mathbb{E}(X_{ij}) \\
&= \sum_{i<j} \mathbb{P}((xa_i \mod p) \mod n = (xa_j \mod p) \mod n) \\
&\leq \sum_{i<j} \frac{\lfloor \frac{p}{n} \rfloor + 1}{p-1} \\
&= \binom{m}{2} \frac{\lfloor \frac{p}{n} \rfloor + 1}{p-1} \\
&< \left(\frac{n}{10}\right) \frac{\frac{p}{n} + 1}{p-1} \\
&= \frac{\frac{p}{10} + \frac{n}{10}}{p-1} \\
&< \frac{\frac{p}{5}}{p-1} \\
&= \frac{p}{5p-5} \\
&< 1
\end{aligned}$$

Thus, there exists some x such that the number of non-distinct pairs is smaller than 1, i.e. all the pairs are distinct. $\square$

3. Let H be a graph, and let $n > |V(H)|$ be an integer. Suppose there is a graph on n vertices and t edges containing no copy of H, and suppose that $tk > n^2 \log_e n$. Show that there is a coloring of edges of the complete graph on n vertices by k colors with no monochromatic copy of H.

   Proof: Let G be the graph on n vertices and t edges containing no copy of H. Create k copies of G, labeled $G_1, \ldots, G_k$. Color each with a different color, c_i for $1 \leq i \leq k$. Now successively and randomly lay G_i on top of K_n. i.e. color a random copy of G_1 in K_n with c_1. Then independently repeat the process for c_2. If an edge is randomly chosen to be colored by c_2 that has already been colored by c_1, then recolor it with c_2. Repeat this process k times. If at the end of the process we have managed to color each edge, then we have colored K_n with k colors. Also any monochromatic subgraph of K_n will be a subgraph of some G_i, none of which contain a copy of H. Thus there can be no monochromatic copy of H in our coloring. So it suffices to show that with positive probability we have completely colored K_n, i.e. every edge has been colored. Let e be an edge of K_n. The event $X_{e,i}$ that e has NOT been colored with c_i has probability $(1 - \frac{t}{\binom{n}{2}})$, since t out of the $\binom{n}{2}$ edges get colored. Thus the event that e does not get colored at all is the intersection $\cap_{i=1}^{k} X_{e_i}$, which has probability $(1-\frac{t}{\binom{n}{2}})^k$. Now to get the probability that at least one edge has not been colored, we take the union of this probability over all edges: Let X be the event that some edge is not colored at the end of the process. Then $X = \cup_{e\in E(K_n)}(\cap_{i=1}^{k} X_{e,i})$, which has probability $Pr(X) \leq \binom{n}{2}(1 - \frac{t}{\binom{n}{2}})^k \approx \frac{n^2}{2}(e^{-\frac{tk}{n^2}})^2 \leq \frac{n^2}{2}(\frac{1}{e^{\frac{n^2 ln(n)}{n^2}}})^2 = 1/2 < 1$. Thus with positive probability, all edges get colored. So there does exist a coloring of the edges of K_n with k colors that contains no monochromatic copy of H. $\square$

4. Prove that there is a constant $c > 0$ such that for every even $n \geq 4$ the following holds: For every undirected complete graph K on n vertices whose edges are colored red and blue, the number of alternating Hamiltonian cycles in K is at most $n^c \frac{n!}{2^n}$.

   Proof: If n is odd, then there is no alternating Hamiltonian cycle. We can assume n is even. Write $n = 2k$. Randomly partition the vertex set into two parts of size k: $V = V_1 \cup V_2$. Define a directed graph D on V as follows: a directed edge $uv \in E(D)$ if "$u \in V_1$, $v \in V_2$, and uv is red in G" or "$u \in V_2$, $v \in V_1$, and uv is blue in G".

   On one hand, a directed Hamiltonian cycle of D is always an alternating Hamiltonian cycle in G. Each alternating Hamiltonian cycle in G has probability $\frac{2}{\binom{n}{k}}$ being a directed Hamiltonian cycle of D. Let $AH(G)$ be the number of alternating Hamiltonian cycles in G. There is an partition of V so that the number of directed Hamiltonian cycle is at least $\frac{2}{\binom{n}{k}}AH(G)$.

   On the other hand, the number of directed Hamiltonian cycles can be bounded by the per(A_D), where A_D is the adjacency matrix of D. We have
$$\frac{2}{\binom{n}{k}}AH(G_D) \leq \mathrm{per}(A_D) \leq \prod_{i=1}^{n}(r_i!)^{1/r_i}.$$
   Here r_i is the out degree of i in D. The above upper bound reaches the maximum when all r_i's are about equal. Since $\sum_{i=1}^{n} r_i = k^2$, the

average of r_i is $\frac{k}{2}$.

When k is even, write $k = 2r$. Then

$$\prod_{i=1}^{n}(r_i!)^{1/r_i} \leq (r!)^{\frac{n}{r}} = (r!)^4.$$

We have

$$AH(G) \leq \frac{1}{2}\binom{n}{k}(r!)^4 = \frac{1}{2}\frac{n!}{\binom{k}{r}^2} = (1+o(1))\pi n \frac{n!}{2^n}.$$

When k is odd, write $k = 2r+1$. Then

$$\prod_{i=1}^{n}(r_i!)^{1/r_i} \leq (r!)^{\frac{n}{2r}}((r+1)!)^{\frac{n}{2(r+1)}} = (1+o(1))(r!(r+1)!)^2.$$

We have

$$AH(G) \leq (1+o(1))\frac{1}{2}\binom{n}{k}(r!(r+1)!)^2 = (1+o(1))\frac{1}{2}\frac{n!}{\binom{k}{r}^2} = (1+o(1))\pi n \frac{n!}{2^n}.$$

$\square$

5. Let X be a collection of pairwise orthogonal unit vectors in $\mathbb{R}^n$ and suppose the projection of each of these vectors on the first k coordinates is of Euclidean norm at least ϵ. Show that $|X| \leq \frac{k}{\epsilon^2}$, and this is tight for all $\epsilon^2 = k/2^r < 1$.

   Proof: Fix $\epsilon > 0$ and $k \in [n]$. Let X be a collection vectors in $\mathbb{R}^n$ as described in the problem. Let $\ell = |X|$. We can extend X to a normal basis X' for $\mathbb{R}^n$. Let A be a square matrix with the vectors in X' making the rows. Since the rows are pairwise orthogonal unit vectors, $AA^T = I$. So $A^{-1} = A^T$ and $A^TA = I$ which implies $\sum_{i=1}^{n} v_i^2(s) = 1$ for any $s \in [n]$. Hence $\sum_{i=1}^{\ell} v_i^2(s) \leq 1$. By assumption $\sum_{s=1}^{k} v_i^2(s) \geq \epsilon^2$. Putting these facts together,

$$\ell\epsilon^2 \leq \sum_{i=1}^{\ell}\sum_{s=1}^{k} v_i^2(s) = \sum_{s=1}^{k}\sum_{i=1}^{\ell} v_i^2(s) \leq k \cdot 1 = k.$$

   Therefore $|X| \leq \frac{k}{\epsilon^2}$ as desired. For the tight case, fix r and n such that $n \geq 2^r$. If we can create 2^r pairwise orthogonal unit vectors of the form

$$\left\langle \pm\frac{1}{2^{r/2}}, \ldots, \pm\frac{1}{2^{r/2}}, 0\ldots, 0 \right\rangle \in \mathbb{R}^n$$

where the first 2^r coordinates are $\pm\frac{1}{2^{r/2}}$ and the remaining $n-2^r$ are 0. Then the projection onto the first k coordinates, for $k \leq 2^r$, has Euclidean norm

$$\left(\sum_{i=1}^{k} v^2(i)\right)^{1/2} = \left(\sum_{i=1}^{k} \frac{1}{2^r}\right)^{1/2} = \left(\frac{k}{2^r}\right)^{1/2} = \epsilon$$

and for $k > 2^r$, $\left(\sum_{i=1}^{k} v^2(i)\right)^{1/2} = 1 > \epsilon.$. So it suffices to determine an assignment of ± 1's to the vectors $\left\langle \pm\frac{1}{2^{r/2}}, \ldots, \pm\frac{1}{2^{r/2}} \right\rangle \in \mathbb{R}^{2^r}$ so that we have 2^r pairwise orthogonal vectors. This assignment comes directly from the Hadamard matrices which exist for orders 2^r. $\square$

6. Let $G=(V,E)$ be a bipartite graph with n vertices and a list $S(v)$ of more than $\log_2 n$ colors associated with each vertex $v \in V$. Prove that there is a proper coloring of G assigning to each vertex v a color from its list $S(v)$.

   Proof: Let $V = L\dot{\cup}R$ be a bipartition. Define $S = \cup_{v\in V} S(v)$. Then randomly distribute each element of S into either S_L (with probability $\frac{1}{2}$) or into S_R (with probability $\frac{1}{2}$). Now for each $v \in L$, define $S_L(v) = S_L \cap S$ and likewise for each $v \in R$, define $S_R(v) = S_R \cap S$. Then, for all $v \in L$,

   $$\Pr(S_L(v) = \emptyset) = \left(\frac{1}{2}\right)^{|S(v)|} < \left(\frac{1}{2}\right)^{\log_2 n} = 2^{-\log_2 n} = \frac{1}{n}$$

   Similarly, for each $v \in S_R$, $\Pr(S_R(v) = \emptyset) = \frac{1}{n}$. And so,

   $$\begin{aligned}
   &\Pr(\{\vee_{v\in L}(S_L(v) = \emptyset)\} \vee \{\vee_{v\in R}(S_R(v) = \emptyset)\}) \\
   &= \sum_{v\in L} \Pr(S_L(v) = \emptyset) + \sum_{v\in R} \Pr(S_R(v) = \emptyset) \\
   &< \sum_{v\in L} \frac{1}{n} + \sum_{v\in R} \frac{1}{n} \\
   &= \sum_{v\in V} \frac{1}{n} \\
   &= 1
   \end{aligned}$$

   Therefore, there exists a vertex coloring where vertices from L only use colors from S_L, and vertices of R use only colors from S_r. Since $L\dot{\cup}R$ was a bipartition, such a coloring is proper. $\square$