Saturday, February 15, 2025 5:59 PM

1-13 Prove that

$$\sum_{i=0}^{n} i\,(i+1)\,(i+2) = \frac{n\,(n+1)\,(n+2)\,(n+3)}{4}$$

$$\sum_{i=0}^{n} i\,(i+1)(i+2) = \sum_{i=0}^{N} (i^2+i)\,(i+2)$$

$$= \sum_{i=0}^{N} i^3 + 2i^2 + i^2 + 2i$$

$$= \sum_{i=0}^{N} i^3 + 3i^2 + 2i$$

$$= \sum_{i=0}^{N} i^3 + 3\sum_{i=0}^{N} i^2 + 2\sum_{i=0}^{n} i$$

$$= \frac{n^2(n+1)^2}{4} + 3\,\frac{(n)(n+1)(2n+1)}{6} + \frac{2\,(n)(n+1)}{2}$$

$$= \frac{n^2(n+1)^2 + 2\,(n)(n+1)\,(2n+1) + 4\,(n)(n+1)}{4}$$

$$= \frac{n\,(n+1)\,[n\,(n+1) + 2\,(2n+1) + 4]}{4}$$

$$= \frac{n\,(n+1)\,(n^2 + n + 4n + 2 + 4)}{4}$$

$$= \frac{n\,(n+1)\,(n^2 + 5n + 6)}{4}$$

$$(n+2)\,(n+3) = n^2 + 3n + 2n + 6$$
$$= n^2 + 5n + 6$$

$$\sum_{i=0}^{n} i\,(i+1)\,(i+2) = \frac{n\,(n+1)\,(n+2)\,(n+3)}{4}$$

1-14 Prove that

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a-1} \quad \text{on } n \geq 1,\; a \neq 1$$

$$n = 1 \Rightarrow \sum_{i=0}^{1} a^i = \frac{a-1}{a-1} + \frac{a^2-1}{a-1}$$

$$= 1 + \frac{a^2-1}{a-1}$$

$$= 1 + \frac{(a+1)\,(a-1)}{a-1}$$

$= 1 + a + 1 = a +$

if $\sum_{i=0}^{n} a^i = \dfrac{a^{n+1} - 1}{a - 1}$; We know that

1. $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$

$a^n - 1^n = (a-1)(a^{n-1} + a^{n-2} + \dots + a1^{n-2} + 1^{n-1})$

$n \geq 1$

$a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \dots$

$$\sum_{i=0}^{n+1} a^i = \frac{a^{n+2} - 1}{a - 1}$$

$$= \frac{(a-1)(a^{n+1} + a^n + \dots + a1^n + 1^{n+1})}{a-1}$$

$$= a^{n+1} + a^n + \dots + a1^n + 1^{n+1}$$

$$= a^{n+1} + (a^n + \dots + a + 1)$$

$$= a^{n+1} + \frac{a^{n+1} - 1}{a-1} \quad \text{from } 1$$

$$= a^{n+1} + \sum_{i=0}^{n} a^i$$

1-15)

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

$n = 1$ $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{2}$; Let's say $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \Rightarrow \frac{n(n+2)}{(n+1)(n+2)}$

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n+1}{n+2}$$

$$= \frac{(n+1)^2}{(n+1)(n+2)}$$

$$= \frac{n^2 + 2n + 1}{(n+1)(n+2)}$$

$$= \frac{n^2+2n}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{(n+2)n}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$1 \quad = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} \quad 2$$

$$2 \Leftrightarrow f(i) = \frac{1}{i(i+1)}$$

$$f(n+1) = \frac{1}{(n+1)(n+2)}$$

$$1 \Rightarrow \sum_{i=1}^{n} \frac{1}{i(i+1)} + f(n+1)$$

1-16 Let's prove that

$(n^3+2n) \% 3 = 0$

by induction

if $(n^3+2n) \% 3 = 0$

$\Downarrow$

$\frac{(n^3+2n)}{3} = k \quad ; k = cste$

$n^3 + 2n = 3k$

if $n = 1$;

$n^3 + 2n = 3k$

$1 + 2 = 3k$

$3 = 3k$

$k = \frac{3}{3} = cste$

lets assume that

$(n^3+2n) = 3k$

$(n+1)^3 + 2(n+1) = k_2$

$(n+1)(n+1)^2 + 2(n+1) = k_2$

$(n+1)(n^2+2n+1) + 2n+2 = k_2$

$(n+1)(n^2+2n+1) + 2n+2 = k_2$

$n^3 + 2n^2 + n + n^2 + 2n + 1 + 2n + 2$

$n^3 + 3n^2 + 5n + 3 = k_2$

$n^3 + 2n + 3n^2 + 3n + 3 = k_2$

we know that

$n^3 + 2n = 3k_1$

$3k_1 + 3n^2 + 3n + 3 = k_2$

$3[k_1 + n^2 + n + 1] = k_2$

$k_2 = 3\alpha$

$\Downarrow$

$(n+1)^3 + 2(n+1) = 3\alpha$

1-17 let's prove that a tree with n vertices has n-1 edges

$n = 1 \Rightarrow$ edges $= 0$

lets assume that

$A_2^n = \frac{n!}{(n-2)!} = \frac{n(n-1)!}{(n-2)!}$

$= \frac{n(n-1)(n-2)!}{(n-2)!}$

$A_2^n = n(n-1)$

let's say

A_2^n

$$n_{edge} = \frac{A_2^n}{n}$$

$$\text{if } \frac{A_2^n}{n} = n - 1$$

$$A_2^{n+1} = \frac{(n+1)!}{(n-1)!}$$

$$= \frac{(n+1)(n)(n-1)!}{(n-1)!}$$

$$A_2^{n+1} = (n+1)(n)$$

$$\frac{A_2^{n+1}}{n+1} = \frac{(n+1)(n)}{n+1}$$

$$\frac{A_2^{n+1}}{n+1} = n$$

$$\text{if } N = n+1$$

$$\frac{A_2^N}{N} = N - 1$$

1-18 let's prove that

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{(n^2+n)^2}{4}$$

$$= \frac{n^4+n^3+n^3+n^2}{4}$$

$$= \frac{n^4+2n^3+n^2}{4}$$

$$= n^2(n^2+2n+1)$$

$$= \frac{n^2 (n+1)^2}{4}$$

1-19 No ;

$$\sum_{i=1}^{n}$$

1-20

Be A number of word per line

Be B number of line per page

Be C number of line per page

$\bar{c}$ is the mean of line per page

$\bar{a}$ is the mean of A

$\bar{b}$ is the mean of B

number of word = $nb_{page} \times \bar{a} \times \bar{b} \times \bar{c}$

1-21

$\simeq$ 200 hours

$\simeq$ 1 day

1-22

$$e = \sum_{i=1}^{n} c_i + t_i$$

$n : n_b$ states

be $\bar{c}$ the mean of cities

and $\bar{t}$ the mean of towns

$\bar{e} = n \times (\bar{c} + \bar{t})$

1-25

a)

1s → 1000 items

for 10000 items

$\alpha = \frac{10000}{1000}$

$= 10$

time $= \alpha^2 = 100s$

b) time $= 10 \log 10$

$= 10s$

1-28

```
int div (int num, int den)
{   int count = 0
    for (int i = 0; i < num; i += den, ++count)
    {
```

```
    }
    return count
}
int div (int num , int den)
{
    int tmp = num;
    int res = 0
    while ( tmp >= den )
    {
        int count = 0;
        while ((den << (count+1)) <= tmp)
        {
            ++count;
        }
        tmp -= den << (count);
        res += 1 << count;
    }
    return res;
}
```

1-29

We need 6 races

5 to find top 5 fives

and one last to find top 3

2-1

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} j$$

$$= \sum_{i=1}^{n-1} \frac{(n-i)(n-i+1)}{2}$$

$$= \sum_{i=1}^{n-1} \frac{n^2-ni+n-ni+i^2-i}{2}$$

$$= \sum_{i=1}^{n-1} \frac{n^2-n-2ni+i^2-i}{2}$$

$$= \frac{(n-1)(n^2)-(n-1)(n)}{2} - \sum_{i=1}^{n-1}\frac{i^2-2ni-i}{2}$$

$$= \frac{(n-1)n^2-(n-1)(n)}{2} - \left[\sum_{i=1}^{n-1}\frac{i^2}{2} - n\sum_{i=1}^{n-1} i - \sum_{i=1}^{n-1}\frac{i}{2}\right]$$

$$= \frac{(n-1)n^2-(n-1)(n)}{2} - \sum_{i=1}^{n-1}\frac{i^2}{2} + n\sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1}\frac{i}{2}$$

$$= \frac{(n-1)(n^2)-(n-1)(n)}{2} - \frac{(n-1)(n)(2(n-1)+1)}{12} + \frac{n(n-1)(n)}{2} + \frac{(n-1)(n)}{4}$$

$$= \frac{6\left[(n-1)(n^2)-(n-1)(n)\right] - \left[(n-1)(n)(2(n-1)+1)\right] + 6\left[n^2(n-1)\right] + 3n(n-1)}{12}$$

$$= \frac{\left[(n^2-n)(2n+1)\right] + 6n^3-6n^2+3n^2-3n}{}$$

$$(\quad n^2 \quad n^2+n) =$$

$$= \frac{6\left(n^3 - n \ldots\right)}{12}$$

$$= \frac{6n^3 - 2n^2 + n - (2n^3 + n^2 - 2n^2 - n) + 6n^3 - 6n^2 + 3n^2 - 3n}{12}$$

$$= \frac{6n^3 - 2n^2 + n - 2n^3 + n^2 + n + 6n^3 - 6n^2 + 3n^2 - 3n}{12}$$

$$= \frac{10n^3 - 4n^2 - n}{12}$$

$$O(f(n)) = n^3$$

2-2

$$\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=j}^{i+j} 1 = \sum_{i=1}^{n} \sum_{j=1}^{i} i + j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{i} i + \sum_{i=1}^{n} \sum_{j=1}^{i} j$$

$$= \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} \frac{i(i+1)}{2}$$

$$= \frac{n(n+1)(2n+1)}{6} + \sum_{i=1}^{n} \frac{i^2}{2} + \sum_{i=1}^{n} \frac{i}{2}$$

$$= \frac{n(n+1)(2n+1)}{\ldots} + \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

$$= \frac{2n(n+1)(2n+1) + n(n+1)(2n+1) + 3n(n+1)}{12}$$

$$= \frac{(2n^2+2n)(2n+1) + (n^2+n)(2n+1) + 3n(n+1)}{12}$$

$$= \frac{4n^3+2n^2+4n^2+2n + 2n^3+n^2+2n^2+n + 3n^2+3n}{12}$$

$$= \frac{6n^3+12n^2+6n}{12}$$

$$f(n) = \frac{n^3+2n^2+n}{2}$$

$$O(f(n)) = n^3$$

2-3

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=j}^{i+j}\sum_{l=1}^{i+j-k} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=j}^{i+j} i+j-k$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=j}^{i+j} -k + \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=j}^{i+j} i + \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=j}^{i+j} j$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{i} \frac{(i+j)(i+j+1)}{2} + \sum_{i=1}^{n}\sum_{j=1}^{i} i^2+ij + \sum_{i=1}^{n}\sum_{j=1}^{i} j^2+ij$$

$$\sum_{i}^{n}\sum^{i} \frac{i^2+ij+i+ij+j^2+j}{} + \sum_{i=1}^{n}\sum_{j=1}^{i} 2ij + \sum_{i=1}^{n}\sum_{j=1}^{n} i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} j^2$$

$$= -\sum_{i=1}\sum_{j=1} \frac{\ldots}{2}$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{i} \frac{i^2+j^2+2ij+i+j}{2} + \sum_{i=1}^{n}\sum_{j=1}^{i} \frac{4ij+2i^2+2j^2}{2}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i} \frac{i^2+j^2+2ij-i-j}{2}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i} \frac{j^2}{2} + \sum_{i=1}^{n}\sum_{j=1}^{i} ij - \sum_{i=1}^{n}\sum_{j=1}^{i} \frac{j}{2} + \sum_{i=1}^{n}\sum_{j=1}^{i} \frac{i^2}{2} - \sum_{i=1}^{n}\sum_{j=1}^{i} \frac{i}{2}$$

$$= \sum_{i=1}^{n} \frac{i(i+1)(2i+1)}{12} + \sum_{i=1}^{n} \frac{i(i)(i+1)}{2} - \sum_{i=1}^{n} \frac{i(i+1)}{4} + \sum_{i=1}^{n} \frac{i^3}{2} - \sum_{i=1}^{n} \frac{i^2}{2}$$

$$= \sum_{i=1}^{n} \frac{(i^2+i)(2i+1)}{12} + \sum_{i=1}^{n} \frac{i^3+i^2}{2} - \sum_{i=1}^{n} \frac{i^2+i}{4} + \sum_{i=1}^{n} \frac{i^3}{2} - \sum_{i=1}^{n} \frac{i^2}{2}$$

$$= \sum_{i=1}^{n} \frac{2i^3+i^2+2i^2+i+6i^3+6i^2-3i^2+3i+6i^3-6i^2}{12}$$

$$= \sum_{i=1}^{n} \frac{14i^3+4i}{12}$$

$$= \frac{7}{6}\sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} \frac{i}{3}$$

$$= \frac{7}{6}\cdot\frac{(n^2)(n+1)^2}{4} + \frac{n(n+1)}{6}$$

$$n^2+n$$

$$= \frac{7(n^2)(n^2+2n+1)}{24} + \frac{n}{6}$$

$$= \frac{7(n^4+2n^3+n^2) + 4n^2+4n}{24}$$

$$= \frac{7n^4 + 14n^3 + 7n^2 + 4n^2 + 4n}{24}$$

$$= \frac{7n^4 + 14n^3 + 11n^2 + 4n}{24}$$

$$O(f(n)) = n^4$$

$$\sum_{i=1}^{n} i^2 = \frac{(n)(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{(n^2)(n+1)^2}{4}$$

2-4

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{(n-i-j+2)(n-i-j+3)}{2}$$

$$\sum^{n} \sum^{n} \frac{n^2 - ni - nj + 3n - in + i^2 + ij - 3i - nj + ij + j^2 - 3j + 2n - 2i - 2j + \ldots}{2}$$

$$= \sum_{i=1} \sum_{j=i+1} \dots$$

$$= \sum_{i=1}^{n} \sum_{j=i+1} \frac{n^2+i^2+j^2-2ni-2nj+2ij-5i-5j+5n+6}{2}$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{n^2+i^2+j^2-i(2n+5)-j(2n+5-2i)+5n+6}{2}$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{n^2+5n+6}{2} - \sum_{i=1}^{n} \frac{i^2(2n+5)}{2} - \sum_{i=1}^{n} \frac{i(i+1)(2n+5-2i)}{4}$$

$$= \frac{n^2(n^2+5n+6)}{2} - \frac{(n)(n+1)(2n+1)(2n+5)}{12} - \sum_{i=1}^{n} \frac{(i^2+i)(2n+5-2i)}{4}$$

$$= \frac{6n^4+30n^3+36n^2-(2n^3+3n^2+n)(2n+5)}{12} - \sum_{i=1}^{n} \frac{2ni^2+5i^2-2i^3+2in+\dots}{4}$$

$$= \frac{6n^4+30n^3+36n^2-(2n^4+10n^3+6n^3+15n^2+2n^2+5n)}{12} - \sum_{i=1}^{n} \frac{i^2(2n+3)-2\dots}{4}$$

$$= \frac{6n^4+30n^3+36n^2-2n^4-16n^3-17n^2-5n}{12} - \frac{(n)(n+1)(2n+1)(2n+3)}{24} - 2\dots$$

$$= \frac{12n^4+60n^3+72n^2-4n^4-32n^3-34n^2-10n-4n^4-12n^3-11n^2-3n}{24} - 2(\dots$$

its too long

and I made a

and ...

mistake somewhere

n^3 -

2-5

a)

$2n$ multiplications

n additions

b)

$2n$ multiplications

c)

Maybe improve multiplication by doing bit shifts and additions, but is it really better?

2-6.

$m_k > A[i] \mid 1 \leq i < k$

$m_2 > A[i] \mid 1 \leq i < 2$

$k < k+1$

$1 < k < k+1$

$m_{k+1} > A[k] \mid 1 \leq k < k+1$

$k = n-1$

$m_n > A[n-1] \mid 1 \leq i < n$

$i = 1$

$m_1 = max(A[1])$

$m_k = max(A[1], \ldots A[k])$

if $A[k] > x \mid x \in (A[1], \ldots A[k-1])$

$m_k = A[k]$ 1

else $m_k = max(A[1], \ldots A[k-1])$

2 $m_k = A[k] > x \mid x \in (A[1], \ldots A[k-1])$

1 $\Rightarrow A[k] > x \mid x \in (A[1], \ldots A[k-1])$

2 $\Rightarrow m_k = max(A[1] \ldots A[k-1])$ isnt the max since $A[k]$

2-7

a) Is $2^{n+1} = O(2^n)$?

$f(n) = O(g(n)) \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)$

$c_1 2^n \leq 2^{n+1}$

$c_1 \leq 2$

$c_2 2^n \geq 2^{n+1}$

$c_2 \geq 2$

$\begin{cases} c_1 = 1 \\ c_2 = 4 \end{cases}$ True

b) Is $2^{2n} = O(2^n)$

$c_1 2^n \leq 2^{2n}$

$c_1 \leq \frac{2^{2n}}{2^n}$

$c_1 \leq 2^{2n-n}$

$c_1 \leq 2^n$

$c_1 \neq$ cste

False

false

2-8

a) $f(n) = \log n^2$

$g(n) = \log n + 5$

$c_1(\log n + 5) \leq \log n^2$

$c_1(\log n + 5) \leq 2\log n$

$$c_1 \leq \frac{2\log n}{\log n + 5}$$

b) $f(n) = \sqrt{n}$; $g(n) = \log n^2$

$\sqrt{n} \geqslant \log n$

$2 \cdot \sqrt{n} \geqslant 2\log n$

$2\sqrt{n} \geqslant \log n^2$

$2 f(n) \geqslant g(n)$

$$f(n) \geqslant \frac{g(n)}{2}$$

$c_1 g(n) \leq f(n)$

$c_1 = \frac{1}{2}$ $\Rightarrow f(n) = \Omega(g(n))$

c) $f(n) = \log^2 n$, $g(n) = \log n$

let's assume there is c such that

$$\log^2 n \geq c \log n$$

for $n \geq 1$

$$\log n \geq c$$

$x = \log n$

$x^2 \geq cx$

$x^2 - cx \geq 0$

$x(x - c) \geq 0$

$x < 0$ and $x - c < 0$

$\log(n) < 0$
impossible

$x \geq 0$ and $x - c > 0$

$\log(n) \geq 0$

$\log(n) - c \geq 0$

$\log(n) \geq c$

there is a c such that

$\log(n) \geq c$

$\log^2(n) \geq c \log(n)$

$f(n) \geq c\, g(n)$

$f(n) = \Omega(g(n))$

d) $f(n) = n$; $g(n) = \log^2 n$

if $(f(n) \in \Omega(g(n)))$

$k \log^2 n \leq n$

$k \log^2 n - n \leq 0$

$n > 0$ $\quad \log(n) > 0$

$\log^2(n) > 0$

$\log^2 n < n$

$k \log^2 n < kn$; $k > 0$

$k \log^2 n < n$

$k\, g(n) < f(n)$

$f(n) \in \Omega(g(n))$

e) $f(n) = n \log n + n$

e) $f(n) = n \log n$

$g(n) = \log(n)$

$n \log n > \log(n)$

$n \log n + n > \log(n) + n$

$n > 0$; $\log(n) + n > \log(n)$

$n \log(n) + n > \log(n)$

$k = 1$

$f(n) \in \Omega(g(n))$

f) $f(n) = 10$ $g(n) = \log 10$

$\log(10) \geq 1$

$10 \log(10) \geq 10$

$10 g(n) \geq f(n)$

$f(n) \in O(g(n))$

g) $f(n) = 2^n$ $g(n) = 10n^2$

if $2^n \geq kn^2$

$10 \cdot 2^n \geq 10kn^2$

$2^n \geq \frac{k}{10}(10n^2)$

10

$f(n) \geq K(10n^2)$

$f(n) \geq K\,g(n)$

if $2^n \leq k n^2$

$10 \cdot 2^n \leq 10 k n^2$

$2^n \leq \frac{k}{10}(10n^2)$

$f(n) \leq K(g(n))$

it seems $f(n) \in \theta(g(n))$

h) $f(n) = 2^n \quad g(n) = 3^n$

$2 \leq 3 \quad ; \quad n > 0$

$2^n \leq 3^n$

$f(n) \leq g(n)$

$k = 1$

$f(n) \in O(g(n))$

2- g

a) $f(n) = (n^2 - n)/2$
$g(n) = 6n$

$n \geq 13$

$n - 1 \geq 12 \quad ; n \geq 0$

$n(n-1) \geq 12n$

$$\frac{n(n-1)}{2} \geq 6n$$

$f(n) \geq g(n)$

$f(n) \in \Omega(g(n))$

b) $f(n) = n + 2\sqrt{n}$
$g(n) = n^2$

$\sqrt{n}$

$\leq n^2 + \sqrt{n}$

if ($\sqrt{n} + 2 \geq k\, n \sqrt{n}$)

$n + 2\sqrt{n} \geq k n^2$

if ($\sqrt{n} + 2 \leq k\, n \sqrt{n}$)

$n + 2\sqrt{n} \leq k n^2$

$f(n) = \Theta(g(n))$

c) $f(n) = n \log n$

$g(n) = \frac{n\sqrt{n}}{2}$

$\log n < \sqrt{n}$

$\log(n) \leq \frac{k}{2}\sqrt{n}$; $k = 2$

$n \log(n) \leq \frac{k\, n \sqrt{n}}{2}$

$f(n) \leq k\, g(n)$

$f(n) = O(g(n))$

d) $f(n) = n + \log n$

$g(n) = \sqrt{n}$

$n \geq \sqrt{n}$

$n \geq 0$

$\log(n) \geq 0$

$n + \log(n) \geq \sqrt{n} + \log(n)$

$n + \log(n) \geq \sqrt{n}$

$f(n) \geq g(n)$

$f(n) = \Omega(g(n))$

e) $f(n) = 2(\log n)^2$

$g(n) = \log n + 1$

$2 \log n \geq 1$

$2 \log^2 n \geq \log n$

$f(n) \geq \log n$

$\log n > 1$

so $+1$ is neglectable

$f(n) = \Omega(g(n))$

f) $f(n) = 4n \log n + n$

$/2$

$g(n) = (n^2 - n)$

$\log n - 1 \leq n - 1$

$n \log n - n \leq n^2 - n$

2-10)

$\Theta(f(n)) + \Theta(g(n)) = \Theta(\max(f(n), g(n)))$

$= \Theta(n^3)$

2-18)

high

$n!$

e^n

$2^n \mid 2^{n-1}$

$n - n^3 + 7n^5$

n^3

$n^2 \mid n^2 + \lg n$

$n^{1+\varepsilon}$

$n \log n$

n

$\sqrt{n}$

$\log^2 n$

$\ln n$

$\lg n$

low $\lg(\log n)$

2.19

low

6

$\log\log n$

$\log n$

$\ln n$

$\log^2 n$

$\sqrt{n}$

$\dfrac{n}{\log n}$

$n^{1/3} + \log n$

n

$n \log n$

$n^2 \mid n^2 + \log n$

n^3

$n - n^3 + 7n^5$

$\left(\frac{1}{3}\right)^n$

$\left(\frac{3}{2}\right)^n$

2^n

high

$n!$

2.20

a) None

b) $f(n)=2n$; $g(n)=n$

c) None

d) $g(n)=n$ $f(n)=n^2$

2.21

a) True

b) false

c) true

d) False

e) true

f) True

g) $\sqrt{n} > \log n$

$$\frac{1}{\sqrt{n}} < \log n$$

false

2-22

a) $f(n) = \Omega(g(n))$

b) $f(n) = O(g(n))$

c) $f(n) = \Omega(g(n))$? not fully sure

2-23

a) True, there may be a special case for which of the inner loop is skipped

b) false, there should be at least on input representing the worst case

c) false if it $\Theta(n^2)$

it means that at least

a factor of 2

... of n is done

d) false, there should be at least a worst case of n^2 and there is no worst case in n

e) True

$$\theta(f(n)) = \theta(\max(f_{even}, f_{odd}))$$
$$= \theta(n^2)$$

2-24

a) false, $3^n > 2^n$

b) $3^n > 2^n$

$\log 3^n > \log 2^n$

False

c) True

d) True

2.25

a) $g(n) = 2\sum_{i=1}^{n} \frac{1}{i}$

b) $g(n) = n$

c) $g(n) = 2\sum_{i=1}^{n} \log i$

d) $g(n) = \ln(n!)$

2-26

$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

$f_4(n) = n+1$

$f_4(n) < f_2(n) < f_1(n) < f_3(n)$

2-27

$\sum_{i=1}^{n} \sqrt{i} = \sum_{i=1}^{n} i^{1/2}$

$\int_0^n \sqrt{x}\, dx = \frac{n^{3/2}}{\frac{3}{2}}$

$\sum_{i=1}^{n} \sqrt{i} \approx \frac{2}{3} n^{3/2}$

$$\frac{f_2}{f_3} = \frac{n^{1/2} \log n}{n \log^{1/2} n}$$

$$= n^{1/2} \log n \left(n^{-1} \log^{-1/2} n\right)$$

$$= n^{-1/2} \log^{1/2} n$$

$$= \frac{\sqrt{\log n}}{n^{1/2}} =$$

$$\lim_{n \to \infty} \frac{f_2}{f_3} = \frac{\infty}{\infty}$$

it seems like f_2 and f_3 are the same asymptotic order

$$f_2 \mid f_3 < f_1 < f_4$$

2-28

a) $f(n) = \sum_{i=1}^{n} 3i^4 + 2i^3 - 19i + 20$

$$\sum i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{10}$$

$$\simeq 6n^4 \ldots\ldots$$

$$g(n) = 6n^4$$

b) $f(n) = \sum_{i=1}^{n} 3(4^i) + 2(3^i) - i^{19} + 20$

$= 3(4^{n+1} - 1) + 2(3^{n+1} - 1) - \frac{n^{20}}{20} + 20n$

$g(n) = 20^{20}(4^n)$

c) $f(n) = \sum_{i=1}^{n} 5^i + 3^{2i}$

$f(n) = 5^{n+1} - 1 + 3^{2n-2} - 1$

$g(n) = 5^{n+1}$

2-29

c) is true

2-30

a) $g(n) = 4^n$

b) $g(n) = n\log n$

c) $g(n) = (\log n)^{10}$

d) $g(n) = n^{100}$

2-31

a) Ω

b) Ω

c) $\sqrt{n} = \Omega(g(n))$

d) Θ

e) O

f) $\Theta \Rightarrow \log(n!) = n\log n$

2-32

$$f(n) = \sum_{i=1}^{n} i^2 \cdot (-1)^{i+1}$$

$$= \sum_{\substack{i=1\\+2}}^{n} i^2 - \sum_{\substack{i=2\\+2}}^{n} i^2$$

$$= \sum_{i=1}^{n} i^2 - \sum_{\substack{i=2\\+2}}^{n} i^2 - \sum_{\substack{i=2\\+2}}^{n} i^2$$

$$= \sum_{i=1}^{n} i^2 - 2\sum_{\substack{i=2\\+2}}^{n} i^2$$

$$= \sum_{i=1}^{n} i^2 - 2\sum_{i=0}^{n/2} (2i)^2$$

$$= \sum_{i=1}^{n} i^2 - 8\sum_{i=0}^{n/2} i^2$$

$$= \frac{n(n+1)(2n+1)}{6} - 8\left[\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)(n+1)}{6}\right]$$

$$= \frac{n(n+1)(2n+1)}{6} - 8\left[\frac{\frac{n(n+2)(2n+2)}{2}}{6}\right]$$

$$= \frac{n(n+1)(2n+1)}{6} - 8\,\frac{(n)(n+2)(2n+2)}{12}$$

$$= \frac{n(n+1)(2n+1) - 4(n)(n+2)(2n+2)}{6}$$

$$= \frac{(n^2+n)(2n+1) - 4(n^2+2n)(2n+2)}{6}$$

$$= \frac{2n^3+n^2+2n^2+n-4(2n^3+2n^2+4n^2+4n)}{6}$$

$$= \frac{-6n^3-21n^2-15n}{6}$$

Failure again :(

2-33

$$m_{ij} = m_{i-1\,j-1} + m_{i-1\,j} + m_{i-1\,j+1}$$

$$m_{ij} = \sum_{k=j-1}^{j+1} m_{i-1\,k}$$

$$f(i,j) = \sum_{k=j-1}^{j+1} f(i-1,k)$$

2-34

$$sum = \sum_{i=0}^{n} \frac{i(i+1)}{2}$$

$$= \sum_{i=0}^{n} \frac{i^2+i}{2}$$

$$= \frac{1}{2}\sum_{i=0}^{n} i^2 + \frac{1}{2}\sum_{i=0}^{n} i$$

$$= \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

$$= \frac{n(n+1)\left[(2n+1)+3\right]}{12}$$

2-35

a) $\sum_{i=1}^{n} \sum_{j=i}^{2i} 1$

b) $\Rightarrow \sum_{i=1}^{n} \left[\sum_{j=0}^{2i} 1 - \sum_{j=0}^{i-1} 1 \right]$

$$\Rightarrow \sum_{i=1}^{n} \left[2i - (i-1) \right]$$

$$\Rightarrow \sum_{i=1}^{n} 2i - i + 1$$

$$\Rightarrow \sum_{i=1}^{n} i + 1$$

$$\Rightarrow \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$\Rightarrow n(n+1) + n$$

$$\frac{\quad}{2}$$

$$\Rightarrow \frac{n(n+1)+2n}{2}$$

2-36

a) $$T(n) = \sum_{i=1}^{n/2} \sum_{j=i}^{n-i} \sum_{k=1}^{j} 1$$

b) $$T(n) = \sum_{i=1}^{n/2} \sum_{j=i}^{n-i} j$$

$$= \sum_{i=1}^{n/2} \left[\sum_{j=0}^{n-i} j - \sum_{j=0}^{i-1} j \right]$$

$$= \sum_{i=1}^{n/2} \left[\frac{(n-i)(n-i+1)}{2} - \frac{(i-1)(i)}{2} \right]$$

$$= \sum_{i=1}^{n/2} \left[\frac{n^2 - ni + n - ni + i^2 - i - i^2 + i}{2} \right]$$

$$= \sum_{i=1}^{n/2} \left[\frac{n^2 - 2ni + n}{2} \right]$$

$$= \sum_{i=1}^{n/2} \frac{n^2}{2} - \sum_{i=1}^{n/2} \frac{2ni}{2} + \sum_{i=1}^{n/2} \frac{n}{2}$$

$$= \frac{n^3}{\quad} + \frac{n^2}{\quad} - n \frac{\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)}{\quad}$$

$$= \frac{n^3}{4} + \frac{n^2}{4} - \frac{n\left(\frac{n(n+2)}{2}\right)}{2}$$

$$= \frac{n^3}{4} + \frac{n^2}{4} - \frac{n^2(n+2)}{4}$$

2-37

biggest number: $b^n - 1$

in worst case we will have bigest number²

$$(b^n-1) \times (b^n-1) = \underbrace{(b^n-1) + (b^n-1) + \dots (b^n-1)}_{b^n-1}$$

$O(\text{additon}) = O(n)$: digits sum

$$O(\text{mul}(n,b)) = \underset{\text{additon}}{n} \cdot \underbrace{(b^n-1)}_{\text{times}}$$

$O(\text{mul}) = n b^n$

2-38

multiplication takes $O(1)$

$$f(a,b) = \sum_{i=0}^{n} a\, b_i \times 10^i$$

$$= a \sum_{i=0}^{n} b_i 10^i$$

nb digit$(y) = \log_b(y)$

$O(\log_b(y))$

2-39 Let's prove that

a) $\log_a (xy) = \log_a x + \log_a y$

$x' = \log_a x$

$y' = \log_a y$

$x = a^{x'}$

$y = a^{y'}$

$\log_a (xy) = \log_a (a^{x'} a^{y'})$

$= \log_a (a^{x'+y'})$

$= (x'+y') \log_a (a)$

$= x'+y'$

$= \log_a x + \log_a y$

b) $\log_a x^y = y \log_a x$

$\log_a x^y = \log_a (x \cdot x \cdot \underbrace{x \ldots x}_{y \text{ times}})$

$= \log_a (x) + \log_a (x) \ldots \log_a (x)$

$= y \log_a (x)$

c) $\log_a x = \dfrac{\log_b x}{\log_b a}$; if that's true

$= \dfrac{\log_a x}{\log_a b} \cdot \dfrac{1}{\log_b a}$

$= \dfrac{\log_a x}{\log_a b \cdot \log_b a}$ (1)

$\log_b x = \log_a x \cdot \log_b a$ (2)

(2) $\Rightarrow \dfrac{\log_a x}{\log_b b}$

$\Rightarrow \dfrac{\log_a x}{1} = \log_a x$

not absurd, so true

d) $x^{\log_b y} = y^{\log_b x}$

$$x^{\log_b y} = K$$

$$\log_y \left(x^{\log_b y}\right) = \log_y K$$

$$\log_b y \, \log_y x = \log_y K$$

$$y^{\log_b y \log_y x} = y^{\log_y K}$$

$$y^{\log_b y \log_y x} = K$$

$$\log_y x = \frac{\log_b x}{\log_b y}$$

$$\log_b x = \log_y x \cdot \log_b y$$

$$y^{\log_b x} = K$$

2.40

let's show that

$$\lceil \log(n+1) \rceil = \lfloor \lg(n) \rfloor + 1$$

$$\lg(n) + 1 = \lg(n) + \lg(10)$$
$$= \log(10\,n)$$

$$\lceil \log(n+1) \rceil = \log(10n)$$

if base $= 10$

2.41

Prove that any number

$n \geqslant 1$ has $\lg_2(n) + 1$ bits

$$n = \sum_{i=0}^{K-1} 2^i \times (1 \mid 0)$$

if a bit $\nexists$

if a bit $\exists$

let's say all bits are sets

$$n = \sum_{i=0}^{K-1} 2^i \; ; \; K = nb \text{ total bits}$$

$$n = 2^K - 1$$

$$2^K = n+1$$

$$\log_2(2^K) = \log_2(n+1)$$

$K = \log_2(n+1)$

we know that

$\lceil \log(n+1) \rceil = \lfloor \log(n) \rfloor + 1$

$K = \lfloor \log_2(n) \rfloor + 1$

2-42

- A sorting algorithm on $\sqrt{n}$ elements $\sqrt{n}$ times

$\sqrt{n}\,(\sqrt{n} \log \sqrt{n})$

- Or using a balanced tree of max height $\sqrt{n}$

2-43 · if n is known

```
int n = arr.length;
int[] selected = new int[K];
int p = 0;
for(int i = 0; i < K; i++)
{  selected[i] = arr[Math.rand() × n];
}
return selected;
```

if n is unknown

I need to count first

2-44

333 data on 3 different nodes

333 data on 3 different nodes

333 data on 3 different nodes

1 data on 1 node

if 3 random nodes fails the expected number is:

$n = \text{Prob}\left(\begin{array}{c}\text{1 data on 1 node}\\ \text{and 2 nodes}\end{array}\right) + 3 \cdot \text{Prob}\left(\begin{array}{c}\text{3 nodes with same}\\ \text{nodes}\end{array}\right)$

$$n = \frac{A_1^{1000} \times A_2^{999}}{A_3^{1000}} + 3 \cdot \frac{A_3^{333}}{A_3^{1000}}$$

$$= \frac{1000 \times 498501 + 3 \times 6099006}{166167000}$$

$$= \frac{498501000 + 18297018}{166167000}$$

$\simeq$ 3.11
$\simeq$ 3 data

2.4.5

nb orignation

$$E = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{1}{n} \cdot \ldots \; n \, \frac{1}{n}$$

$$= \frac{1}{n} \sum_{i=1}^{n} i$$

$$= \frac{1}{n} \cdot \frac{n(n+1)}{2}$$

$$E = \frac{n+1}{2}$$

2-46
With infinite supply
I can find it in $\log(n)$ time

With ~~only~~ 2 I can find it
in O(lowest floor)
O(n)
by starting from the lowest to
the greatest
I stop when I break the marble

2-47
Put all bags on balance
and start removing
one by one
at each remove if the
current weight go down
with a different delta
the removed bag is the one.

- **Take a Sample:** Take a specific number of coins from each bag, corresponding to its label:
    - From bag 1, take 1 coin.
    - From bag 2, take 2 coins.
    - From bag 3, take 3 coins.
    - ...
    - From bag 10, take 10 coins.
    - **Weigh the Sample:** Place all the collected coins on the digital balance and record the total weight.
    - **Calculate the Expected Weight:** If all the coins were real (weighing 10 grams each), the total weight would be the sum of the number of coins taken from each bag, multiplied by 10: (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) * 10 = 55 * 10 = 550 grams.
    - **Determine the False Bag:** Compare the actual weight on the balance to the expected weight (550 grams). The difference in weight (in grams) will directly correspond to the number of the bag containing the lighter coins.
        - **Example:** If the scale shows 547 grams, the difference is 3 grams. This means there are 3 lighter coins, and since you took 3 coins from bag number 3, **bag number 3 is the one with the false coins.**

2-48
- weigh 6 balls (3,3)
- if it is balanced, weigh
  the 2 remaining (1,1)
- if not balanced
  take the side weighing
  the most
  weigh 2 balls in the 3
- if they are balanced
  the ball is the remaining one
- if they are not balanced
  the ball is the heaviest

2-49

$$A_2^n + A_2^{n-1} + A_2^{n-2} + \ldots A_2^2$$

$$\sum_{i=0}^{n-2} A_2^{n-i}$$

2-50

```
fct (int n)
{
    ... Pair<int, int>>> seen = new Hashmap<>();
```

```
HashMap<int, HashSet< ...

for(int a=0; a<n; a++)
{   for(int b=0; b<n; b++)
    {     if(a= =b)
            continue;
          int tmp= Math.pow(a,3) + Math.pow(b,3);
          if(!seen.ContainKey(tmp))
          {
            seen.put(tmp, new HashSet<>());


          }
          seen.get(tmp).put( new Pair<>(a,b)) ;
    }
}
for(int key : seen.keyset())
{
        final var pairs = seen.get(key);
        for ( int i=0; i<pairs.size(); i++)
        {
              for (int j=i+1; j < pairs.size(); j++)
              {
                  if( pairs.get(i).getKey()
                      ^ pairs.get(j).getKey()
                     && pairs.get(i).getValue()
                      ^ pairs.get(j).getValue()
                     && pairs.get(i).getKey()
                      ^ pairs.get(j).getValue)

                  {
                        System.out.println(
                         pairs.get(i).getKey() ,
                         pairs.get(i).getValue(),
                        , pairs.get(j).getKey(),
                         pairs.get(j).getValue() )

                    }
                }

        }

}
}
```

2-51

3 pirates stays alive

- one take zero
- the other split

2-52

2 pirates stay alive

- the last take the money