SEMINAR 10

2.V.2011

① Find the eq of the circle passing through $A(3,1)$, $B(-1,3)$ and having the center on $d: 3x-y-2=0$

$O(x_0, y_0)$

$O \in d \Rightarrow 3x_0 - y_0 - 2 = 0$

$y_0 = 3x_0 - 2$

$OA = OB$

$OA = \sqrt{(x_0-3)^2 + (3x_0-2-1)}$

$OB = \sqrt{(x_0+1)^2 + (3x_0}$

$(3-x_0)^2 + (1-y_0)^2 = (-1-x_0)^2 + (3-y_0)^2$

$9 - 6x_0 + \cancel{x_0^2} + 1 - 2y_0 + \cancel{y_0^2} = 1 + 2x_0 + \cancel{x_0^2} +$
$+ 9 - 6y_0 + \cancel{y_0^2}$

$-8x_0 + 4y_0 = 0$

$-2x_0 + y_0 = 0$

$\begin{cases} y_0 = 3x_0 - 2 \\ -2x_0 + y_0 = 0 \end{cases}$ $\begin{cases} x_0 = 2 \\ y_0 = 4 \end{cases}$

$O(2,4)$

$R = OA = \sqrt{10}$

$R^2 = 10$

$\mathcal{C}: (x-x_0)^2 + (y-y_0)^2 = R^2$

$\mathcal{C}: (x-2)^2 + (y-4)^2 = 10$

② Find eq of circle tangent to both

$d_1: 2x + y - 5 = 0$

$d_2: 2x + y + 15 = 0$

if the tangency point with d_1 is $M(3,1)$

$m_{d_1} = m_{d_2} = -2 \rightarrow d_1 \parallel d_2$

$MN: y - y_M = m_{MN}(x - x_M)$

$m_{d_1} \cdot m_{MN} = -1 \rightarrow m_{MN} = \frac{1}{2}$

$y - 1 = \frac{1}{2}(x - 3)$

$2y - 2 - x + 3 = 0$

$MN: 2y - x + 1 = 0$

$d_2 \cap MN = \{N\}$

$$\begin{cases} 2x + y + 15 = 0 \\ 2y - x + 1 = 0 \quad | \cdot 2 \end{cases}$$

$5y + 17 = 0$

$y = -\frac{17}{5} \qquad x = -\frac{29}{5}$

$x_0 = \frac{x_M + x_N}{2} = \frac{3 - \frac{29}{5}}{2} = -\frac{7}{5}$

$y_0 = -\frac{6}{5}$

$O\left(-\frac{7}{5}; -\frac{6}{5}\right)$

(3) find eq of the tangent lines to $\mathcal{C}$

$\mathcal{C}: x^2+y^2+10x-2y+6=0$ parallel to

$d: 2x+y-7=0$

$$x^2+10x+25+y^2-2y+1-25-1+6=0$$

$$(x+5)^2+(y-1)^2=20$$

$$O(-5,1)$$

$$m_d=-2 \Rightarrow m_{HP}=\frac{1}{2}$$

$$HP: y=\frac{1}{2}x+n$$

$$O\in(HP) \quad 1=-\frac{5}{2}+n \Rightarrow n=\frac{7}{2}$$

$$HP: y=\frac{1}{2}x+\frac{7}{2}$$

$$\mathcal{C}: \begin{cases} x^2+y^2+10x-2y+6=0 \\ y=\frac{1}{2}x+\frac{7}{2} \end{cases}$$

$$x^2+\left(\frac{1}{2}x+\frac{7}{2}\right)^2+10x-2\left(\frac{1}{2}x+\frac{7}{2}\right)+6=0$$

$$x^2+10x+9=0$$

$$(x+1)(x+9)=0 \Rightarrow x=\begin{cases}-1 & \Rightarrow y=3\\ -9 & y=-1\end{cases}$$

$M(-1,3)$

$N(-9,-1)$

$tg_1: y+2x-1=0$

$tg_2: y+2x+19=0$

$y-3=-2(x+1)$

Sol 4 $tg: y = -2x + m$ $O(-5, 1)$

$d(O, tg) = \sqrt{20}$

$2x + y - m = 0$

$$\left| \frac{2(-5) + 1 - m}{\sqrt{5}} \right| = \sqrt{20} \Rightarrow \begin{cases} m_1 \\ m_2 \end{cases}$$

4) Let $\mathcal{C}_\lambda : x^2 + y^2 + \lambda x + (2\lambda + 3) y = 0$

$\lambda \in \mathbb{R}$, be a family of circle. Prove that the circles from the family have 2 fixed points

Sol. Let $\lambda_1 \neq \lambda_2 \in \mathbb{R}$

$\mathcal{C}_{\lambda_1} \cap \mathcal{C}_{\lambda_2} = ?$

$$\begin{cases} x^2 + y^2 + \lambda_1 x + (2\lambda_1 + 3) y = 0 \\ x^2 + y^2 + \lambda_2 x + (2\lambda_2 + 3) y = 0 \end{cases}$$

$x(\lambda_1 - \lambda_2) + y(2\lambda_1 + 3 - 2\lambda_2 - 3) = 0$

$(\lambda_1 - \lambda_2)(x + 2y) = 0$

$x + 2y = 0 \Rightarrow x = -2y$

$4y^2 + y^2 - \lambda_1 \cdot 2y + (2\lambda_1 + 3) y = 0$

$5y^2 + y(2\lambda_1 - 2\lambda_1 + 3) = 0$

$5y^2 + 3y = 0$

$y(5y + 3) = 0 \quad \begin{cases} y_1 = 0 \Rightarrow x_1 = 0 \end{cases}$

* ⑤ find the geometric locus of the points in the plane for which the sum of the squares of the distances to the sides of an equilateral Δ is constant. *

- let O be midpoint of BC (the origin of axes) $\Rightarrow$ $B\left(-\frac{a}{2};0\right)$

$C\left(\frac{a}{2};0\right)$

$A\left(0,\frac{a\sqrt{3}}{2}\right)$

$$AB: \frac{x}{-\frac{a}{2}} = \frac{y-\frac{a\sqrt{3}}{2}}{-\frac{a\sqrt{3}}{2}}$$

$$-x\cdot\frac{a\sqrt{3}}{2} = -\frac{a}{2}\left(y-\frac{a\sqrt{3}}{2}\right)$$

$$-x\frac{a\sqrt{3}}{2} = -y\frac{a}{2}+\frac{a^2\sqrt{3}}{4} \quad \Big| : \frac{4}{a}$$

$$-2x\sqrt{3} = -2y + a\sqrt{3}$$

$$AB: -2x\sqrt{3}+2y-a\sqrt{3}=0$$

$$d(M,AB) = MC' = \frac{|-2x_M\sqrt{3}+2y_M-a\sqrt{3}|}{\sqrt{12+4}} =$$

$$= \frac{|-2x_M\sqrt{3}+2y_M-a\sqrt{3}|}{4}$$

$$AC: \frac{x}{\frac{a}{2}} = \frac{y-\frac{a\sqrt{3}}{2}}{-\frac{a\sqrt{3}}{2}}$$

$$AC: -x\frac{a\sqrt{3}}{2} = y\cdot\frac{a}{2} - \frac{a^2\sqrt{3}}{4} \quad \Big|\cdot\frac{4}{a}$$

$$AC: -2x\sqrt{3} = 2y - a\sqrt{3}$$

$$AC: -2x\sqrt{3} - 2y + a\sqrt{3} = 0$$

$$d(M, AC) = MB' = \frac{|-2x_M\sqrt{3} - 2y_M + a\sqrt{3}|}{4}$$

$$d(M, BC) = MA' = y_M$$

$$y_M^2 + \left(\frac{|-2x_M\sqrt{3} - 2y_M + a\sqrt{3}|}{4}\right)^2 + \left(\frac{|-2x_M\sqrt{3} + 2y_M - a\sqrt{3}|}{4}\right)$$

$$= k$$

$$y_M^2 + \frac{12x_M^2 + 4y_M^2 + 3a^2 + 8\sqrt{3}x_M y_M - 12ax_M - 4\sqrt{3}a y_M}{16}$$

$$+ \frac{12x_M^2 + 4y_M^2 + 3a^2 - 8\sqrt{3}x_M y_M - 4\sqrt{3}y_M a + 12ax_M}{16}$$

$$\Rightarrow 16y_M^2 + 24x_M^2 + 8y_M^2 + 6a^2 - 8\sqrt{3}ay_M = 16k$$

$$24x_M^2 + 24y_M^2 - 8\sqrt{3}ay_M + 6a^2 = 16k \quad |:24$$

$$x_M^2 + y_M^2 - \frac{\sqrt{3}}{3}ay_M + \frac{a^2}{4} = \frac{2}{3}k$$

$$x_M^2 + y_M^2 - 2\cdot\frac{\sqrt{3}}{6}ay_M + \left(\frac{a\sqrt{3}}{6}\right)^2 - \left(\frac{a\sqrt{3}}{6}\right)^2 + \frac{a^2}{4}$$

$$= \frac{2}{3}k$$

$$x_M^2 + \left(y_M - \frac{a\sqrt{3}}{\quad}\right)^2 = \frac{2}{3}k - \frac{a^2}{\quad} + \frac{a^2\cdot 3}{\quad}$$

$$x_M^2 + \left(y_M - \frac{a\sqrt{3}}{6}\right)^2 = \frac{4k+a^2}{6}$$

$\Rightarrow$ eq of circle ~~of~~ center $P\left(0, \frac{a\sqrt{3}}{6}\right)$

and $R = \sqrt{\frac{4k-a^2}{6}}$

$4k - a^2 > 0$

$k > \frac{a^2}{4}$

(6) Let P and Q be 2 fixed points and d a line, $d \perp PQ$. Two variable orthogonal lines, pass through P and Q meet d at A and B. Find geom. locus of the projection of A on BQ.

- let $\{O\} = d \cap PQ$ be the origin of the axes.

- let $P(a, 0)$, $Q(b, 0)$

$A(0, y_A) \Rightarrow d_1: \frac{x-a}{-a} = \frac{y}{y_A}$

$d_1: y_A(x-a) = -ay$

$d_1: y_A x + ay - y_A a = 0$

$m_{d_1} = -\frac{y_A}{a}$

$d_1 \perp d_2 \Rightarrow m_{d_2} = a$

$$d_2 : y = \frac{a}{y_A}(x-a) \Rightarrow$$

$$d_2 : ax - y_A y - a^2 = 0$$

$$\{B\} = d_2 \cap Oy$$

$$\Rightarrow \begin{cases} x = 0 \\ ax - y_A y - a^2 = 0 \end{cases}$$

$$\Rightarrow y = -\frac{a^2}{y_A}$$

$$B\left(0, -\frac{a^2}{y_A}\right)$$

$$BQ : \frac{x-5}{-5} = \frac{y}{-a^2}$$

$$BQ : -\frac{a^2}{y_A}x + \frac{a^2 5}{y_A} + 5y = 0$$

$$\boxed{BQ : -a^2 x + 5 y_A y + a^2 5 = 0}$$

$$m_{BQ} = \frac{a^2}{5y_A}$$

$$BQ \perp AA' \Rightarrow m_{AA'} = -\frac{5y_A}{a^2}$$

$$AA' : y - y_A = -\frac{5y_A}{a^2}x$$

$AA' : 5y_A x + a^2 y - a^2 y_A = 0$

$\{A'\} = AA' \cap BQ$

$$\begin{cases} (1) & 5y_A x + a^2 y - a^2 y_A = 0 \\ (2) & -a^2 x + 5y_A y + a^2 5 = 0 \end{cases}$$ eliminate y_A

$(2) \Rightarrow y_A = \dfrac{a^2 x - a^2 5}{5y}$ replace in (1)

$$\cancel{5} \cdot \frac{a^2 x - a^2 5}{\cancel{5} y} x + a^2 y - a^2 \cdot \frac{a^2 x - a^2 5}{5y} = 0 \quad \Big| \cdot \frac{5y}{a^2}$$

$$b(x-5)x + y - a^2 x + a^2 5 = 0$$

$$5x^2 - 5^2 x + 5y^2 - a^2 x + a^2 b = 0 \quad |:5$$

$$x^2 + y^2 - 5x - \frac{a^2}{5} x + a^2 \cancel{8} = 0$$

$$x^2 + y^2 - x\left(\frac{5^2 + a^2}{5}\right) + \cancel{4}\, a^2 = 0$$

$$y^2 + x^2 - 2x \cdot \frac{5^2 + a^2}{25} + \left(\frac{5^2 + a^2}{25}\right)^2 - \left(\frac{5^2 + a^2}{25}\right)^2 + a^2 = 0$$

$$\left(x - \frac{5^2 + a^2}{25}\right)^2 + y^2 = \frac{a^4 + 2a^2 5^2 + 5^4 - 4a^2 5^2}{45^2}$$

$$\left(x - \frac{5^2 + a^2}{25}\right)^2 + y^2 = \left(\frac{a^2 - 5^2}{25}\right)^2$$