Probabilitati si statistica

Euler's functions

Euler's Gamma Function

$\Gamma : (0, \infty) \to (0, \infty)$

$$\boxed{\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx}$$ convergentă are val finite

Proprietes

1. $\Gamma(1) = 1$
2. $\Gamma(a+1) = a\Gamma(a), \quad \forall a > 0$
3. $\Gamma(n+1) = n!, \quad \forall n \in \mathbb{N}$
4. $\Gamma(\frac{1}{2}) = \sqrt{2} \int_0^\infty e^{-\frac{t^2}{2}} dt = \int_{\mathbb{R}} e^{-t^2} dt = \sqrt{\pi}$

$$\int \varphi' g \, dx = f \cdot g - \int f \cdot g' \, dx$$

① $\Gamma(1) = \int_0^\infty x^0 e^{-x} dx =$

$= \int_0^\infty e^{-x} dx = -e^{-x} \Big|_0^\infty = 0 - (-1) = 1$

② $\Gamma(a+1) = \int_0^\infty x^a e^{-x} dx$

$f'(x) = e^{-x} \qquad f(x) = -e^{-x}$

$g(x) = x^a \qquad g'(x) = a x^{a-1}$

$$\Gamma(a+1) = x^a e^{-x} \Big|_0^\infty + \underbrace{\int_0^\infty e^{-x} \cdot a \cdot x^{a-1} dx}_{a \Gamma(a)}$$

$$\lim_{x \to \infty} \frac{x^a}{e^x} = \lim_{x \to \infty} \frac{a \cdot x^{a-1}}{e^x} = \dots = 0$$

$= a \cdot \Gamma(a)$

③ $\Gamma(n+1) = n \cdot \Gamma(n) = n \cdot (n-1) \Gamma(n-1) =$

$= n! \cdot \Gamma(1) = n!$

(4) $\Gamma(\frac{1}{2}) = \sqrt{2}\int_0^\infty e^{-\frac{t^2}{2}}dt = \int e^{-t^2}dt$

$\Gamma(\frac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx =$

$= \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx$

substitution
ch. of var.

$x = \frac{t^2}{2}$ /'

$\sqrt{x} = \frac{t}{\sqrt{2}}$

$dx = t\,dt$

$\Gamma(\frac{1}{2}) = \int_0^\infty \frac{e^{-\frac{t^2}{2}}}{\frac{\cancel{t}}{\sqrt{2}}} \cancel{t}\,dt = \sqrt{2}\cdot\int_0^\infty e^{-\frac{t^2}{2}}dt$

---

fast. $x = t^2$

$x = t^2$

$\sqrt{x} = t$

$dx = 2t\,dt$

$(0,\infty) \to (0,\infty)$

$\Gamma(\frac{1}{2}) = \int_0^\infty \frac{e^{-t^2}}{\cancel{t}} \cdot 2\cancel{t}\,dt$

$\to$

$= 2\int_0^\infty e^{-t^2}dt$

$\int_{-a}^{a} f(x)dx = \begin{cases} 0 & f \text{ odd} \\ & f(-x) = -f(x) \\ 2\int_0^a f(x)dx & f \text{ even} \end{cases}$

$f(-x) = f(x)$

if $a = \infty$

f bounded

e^{-t^2} – even funct.
and bounded

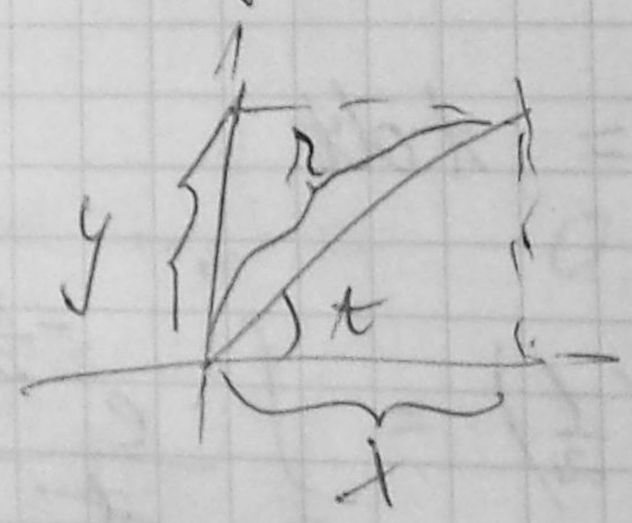
$$\Rightarrow 2\int_0^\infty e^{-t^2}dt = \int_{-\infty}^{\infty} e^{-t^2}dt = \int_{\mathbb{R}} e^{-t^2}dt$$

$$\left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 = \Gamma\left(\tfrac{1}{2}\right)\cdot\Gamma\left(\tfrac{1}{2}\right)$$

$$= \int_{\mathbb{R}} e^{-u^2}du = \int_{\mathbb{R}} e^{-v^2}dv = \iint_{\mathbb{R}^2} e^{-(u^2+v^2)}$$

Polar Coordinates

(r, t)



$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases} \qquad r^2 = x^2+y^2 \qquad \operatorname{tg} t = \frac{y}{x}$$

$$\ldots = \pi$$

Euler's Beta Function

$$\beta : (0,\infty) \times (0,\infty) \to (0,\infty)$$

$$\boxed{\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx}$$

<u>Prop.</u>

① $\beta(a,1) = \frac{1}{a}, \quad \forall\, a>0$

② $\beta(a,b) = \beta(b,a)$

③ $\beta(a,b) = \frac{a-1}{b}\beta(a-1,b+1)$

$= \frac{b-1}{a+b-1}\beta(a,b-1)$

④ $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

PS

Sem 2.

"or" $\cup$ union

"and" $\cap$ intersection

$$\overline{\bigcup A_i} = \bigcap \overline{A_i}$$

$$\overline{\bigcap A_i} = \bigcup \overline{A_i}$$

$k \geq 1$

$k \geq 1$

$(k < 1)$

$k \leq 0$

1. a.) $A = A_1 \cap A_2 \cap \dots \cap A_n =$

$$= \bigcap_{i=1}^{n} A_i$$

b.) $B = \overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_n}$

$$= \bigcap_{i=1}^{n} \overline{A_i} = \overline{\bigcup_{i=1}^{n} A_i}$$

c.) $C = \overline{B} = \bigcup_{i=1}^{n} A_i$

d.) Let D_i : only the ith ticket is winning

$$D = \bigcup_{i=1}^{n} D_i$$

$$D_1 = A_1 \setminus \left(\bigcup_{j=2}^{n} A_j \right)$$

$$= A_1 \cap \left(\overline{\bigcup_{j=2}^{n} A_j} \right)$$

$$D_i = A_i \setminus \left(\bigcup_{\substack{j=1,n \\ j \neq i}} A_j \right)$$

$$D = \bigcup_{i=1}^{m} \left(A_i \setminus \left(\bigcup_{\substack{j=1 \\ j \neq i}}^{m} A_j \right) \right)$$

e.) $$E = \bigcup_{\substack{i,j=1 \\ i \neq j}}^{m} \left[(A_i \cap A_j) \setminus \left(\bigcup_{\substack{k=1 \\ k \neq i \\ k \neq j}}^{n} A_k \right) \right]$$

f.) $\overline{F}$: 0 or 1 winning tickets

$$\overline{F} = B \cup D$$

$$F = \overline{B \cup D} = \overline{B} \cap \overline{D}$$

g.) $G = B \cup D \cup E$

2.) a.) C_{25}^{10}

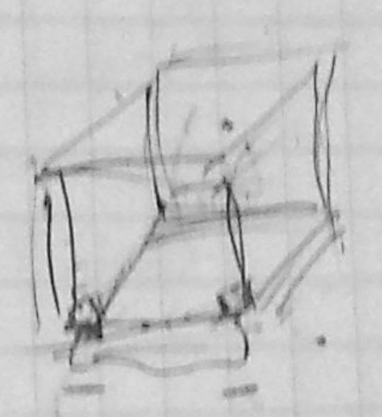
b.) - choosing three games : C_5^3

- choosing the rest : C_{20}^{7}

} indep (intuition de cazuri

$$= C_5^3 \cdot C_{20}^{7} = \ldots$$

c.) $C_5^3 \cdot C_3^2 \cdot C_{17}^5$



$40 + 8 \cdot 8 = 64$

$104 - 8$

$96 - 2$

$8 - 3$

$\boxed{64}$

$64 \cdot 6$

3.) Total nr of cases: 1000

a.) cubes on corners : 8

$P(A) = \frac{8}{1000}$

b.) cubes on edges but not on corners = 96

$P(B) = \frac{96}{1000}$

c.) cubes on faces but not on edges : 384

$P(C) = \frac{384}{1000}$

d.) $\{A, B, C, D\}$ form a partition (sistem complet de evenimente)

$P(A) + P(B) + P(C) + P(D) = 1$

$P(D) = 1 - P(A) - P(B) - P(C)$

$= 1 - \frac{8 + 96 + 384}{1000}$

$= 1 - \frac{408}{1000} = \frac{512}{1000}$

4.) total no of pos outcomes : $n! = P_n$

a.) 1, 2 .....

..... 1, 2 .....

..... 1, 2

$= P_{n-1} = (n-1)!$

$P(A) = \frac{(n-1)!}{n!} = \frac{1}{n}$

d.) $\frac{(n-2)!}{n!} = \frac{1}{n \cdot (n-1)}$

b.) by symmetry : $P(B) = 2 \cdot P(A) = \frac{2}{n}$

c.) ...1...2... ...2...1...

Exactly half the cases (by sym)

1 and 2 are placed in increasing order

$P(C) = \frac{1}{2}$

(5) (The Pigeonhole Principle)

n letters .... N mailboxes

A: exactly m letters $(0 \le m \le n)$

... → fixed mailbox

Sol

Total nr of Poss cases:

1st letter .... N poss
2nd —"— ... N poss
⋮
n^{th} —"— N poss

→ N^n

Nr of fav. cases:

- choosing those m letters
C_n^m poss.
- distributing the rest
$n-m$ letters .... $N-1$ mailboxes

indep.

$(N-1)^{n-m}$ poss.

nu contează ordinea

→ $C_n^m (N-1)^{n-m}$

$$P(A) = \frac{C_n^m (N-1)^{n-m}}{N^n}$$

Geom prob

"count" $\longrightarrow$ "measure"

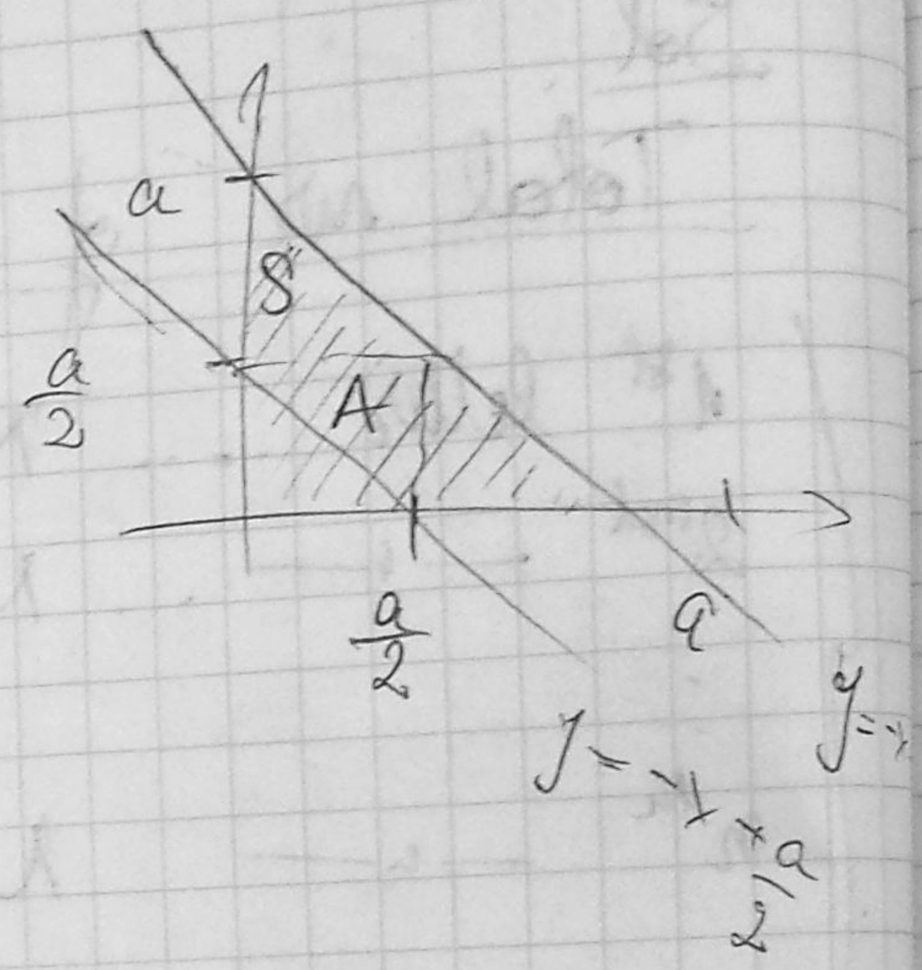
$$P(A) = \frac{\mu(A)}{\mu(S)}$$

$S \subseteq \mathbb{R}^n$

$n=1$, μ = length

$n=2$, μ = area

$n=3$, μ = volume



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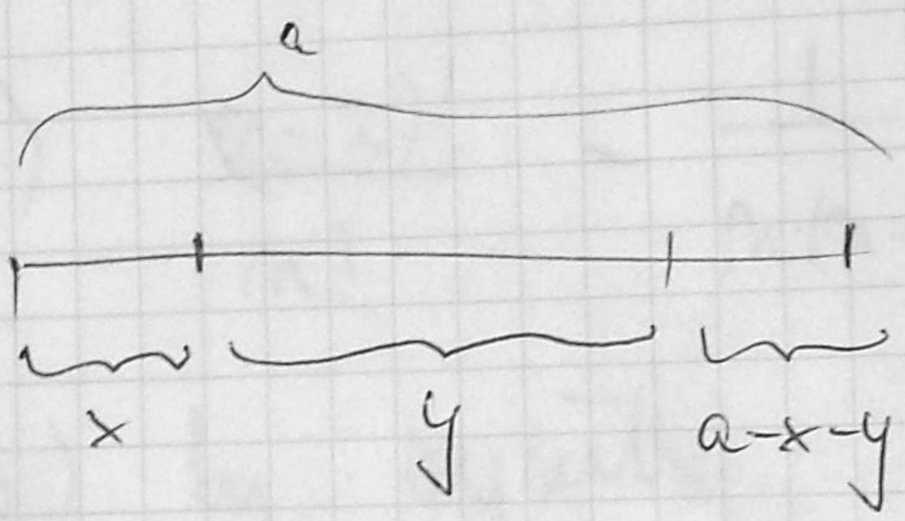


$$S = \{(x,y) \in \mathbb{R}^2 \mid x,y>0,\ x+y<a\}$$

$$A = \{(x,y) \in S \mid x,y < \tfrac{a}{2},\ x+y > \tfrac{a}{2}\}$$

$x+y > a-x-y$

$\cancel{x}+a-\cancel{x}-y > y$

$x+y > \frac{a}{2}$

$y < \frac{a}{2}$

$x < \frac{a}{2}$

$$P(A) = \frac{\text{area}(A)}{\text{area}(S)} = \frac{\frac{1}{2}\cdot\frac{a^2}{4}}{\frac{1}{2}\cdot a^2} = \frac{1}{4}$$

$$\boxed{P(A) = \frac{1}{4}}$$

*) Cond. Prob.

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

$P(B) \neq 0$

A, B indep $\Leftrightarrow$ $P(A \cap B) = P(A)P(B)$

*) C : waters show signs of chemical pollution
T : ——— " ——— thermal ———

$P(C) = 0.3$
$P(T) = 0.25$
$P(C \cap T) = 0.1$

a) $P(C)P(T) = 0.075$
$\neq 0.1 = P(C \cap T)$ NO ind.

b.) $P(A) = P(C|T) = \frac{P(C \cap T)}{P(T)}$

c.) $P(B) = P(\bar{T}|C)$
$= 1 - P(T|C)$
$= 1 - \frac{P(T \cap C)}{P(C)}$

8) <u>three Shooters Problem</u> → Model Poisson → CB

A_i : the ith shooter hits the target

$i = \overline{1,3}$

p_1, p_2, p_3

$$A_1 \setminus (A_2 \cap A_3) = (A_1 \cap \overline{A_2} \cap \overline{A_3}) \cup$$
$$(\overline{A_1} \cap A_2 \cap \overline{A_3}) \cup$$
$$(\overline{A_1} \cap \overline{A_2} \cap A_3$$

$$p_1(1-p_2)(1-p_3) + (1-p_1)p_2(1-p_3) +$$
$$(1-p_1)(1-p_2)p_3$$

75.

SEM. 3.

finite
infinite

cu
revenire
fără

= probab de success - accesti
= nr finit
= 2 posib. → success
→ insuccess

} Binomial model

$$\boxed{P(n,k) = C_n^k p^k q^{n-k}, \quad k = \overline{0,n}}$$

I. Success : silent error occurs

$n = 20$
$p = 0{,}1$ } param. modelului binomial

a.) no silent err occurs $(k=0)$

$k=0$ $\quad \boxed{q = 1-p}$

$P(A) = P(20,0)$

$= \cancel{C_{20}^0 p^0 q^{20}}$

$= C_{20}^0 \, p^0 \cdot 0{,}9^{20} = 0{,}9^{20}$

b.) $B: k \geq 1$

$\overline{B}: k < 1$

$\Rightarrow k = 0 \quad (A)$

$P(B) = 1 - P(\overline{B}) =$

$= 1 - P(A)$

$= 1 - 0{,}9^{20}$

c.) $C: k > 4$

$\overline{C}: k \leq 4$

$P(C) = 1 - P(\overline{C})$

$= 1 - P(\underbrace{k=0 \text{ or } k=1 \text{ or } k=2 \ldots \text{ or } k=4}_{\text{mutually exclusive (disjunctive)}})$

$= 1 - \sum_{k=0}^{4} P(20, k) =$

$= 1 - \sum_{k=0}^{4} C_{20}^{k} p_1^{k} \cdot 0.9^{20-k}$

$= 0{,}0432 \quad (\sim 4\%)$

yes - is unusual

5 % 0.05

1 % 0.01

0.1 % 0.001.

II Success : silent error does <u>not</u> occur

$n = 20$, $p = 0.9$ $(q = 0{,}1)$

a.) $k = 20$

$P(A) = P(20,20) = C_{20}^{20} \, 0.9^{20} \cdot 0{,}1^{0}$

$= (0.9)^{20}$

b.) 19, 18 ... 0

B: $k \leq 19$

$\bar{B}$: $k > 19$ $(20 = k)$ (A)

$P(B) = 1 - P(A)$

c.) 16, 14, ... 0

C: $k \leq 15$

$\bar{C}$: $k > 15$ $(k = 16, 17, 18, 19, 20)$

$P(C) = 1 - P(\bar{C})$

$= 1 - \sum_{k=16}^{20} P(n,k)$

$= 1 - \sum_{k=16}^{20} C_{20}^{k} (0.9)^{k} \cdot (0.1)^{20-k}$

2.) Binomial Model

$n = 40$

3.) Hypergeometric Model

Fără reinserție

$N \begin{cases} m_1 \\ N - m_1 \end{cases}$ ← total

$n \begin{cases} k \\ n-k \end{cases}$ ← este alegere

200 cards

50 people

$N = 200 \begin{cases} n_1 = 10 \\ N - n_1 = 190 \end{cases}$ pass / rest

$n = 150 \begin{cases} k = 10 \\ n - k = 140 \end{cases}$

$$P(A) = P(10, 10) = \frac{C_{10}^{10} \cdot C_{190}^{140}}{C_{200}^{150}} = \frac{C_{190}^{140}}{C_{200}^{150}}$$

4.) ns fit de rep. (3) Poisson Model

Succ.

Intern.

- m.l aulen la fnctn rep.

Guess - correct answer

$n = 3$

$p_1 = 0.7$

$p_2 = 0,8$

$p_3 = 0,6$

$k = 2$ (gets 2 points in the test)

$P(A) = P(3, 2)$

- the coeff. of λ^2 from

$(0,7\lambda + 0,3)(0,8\lambda + 0,2)(0,6\lambda + 0,4)$

5.) ns finit.

Poisson Model

6. Teoretic infinit.

success : vaccine works

$p = 0,9$

A: 4 or more tested to find 3 that work

A: 3rd success after 4 or more trials

A: 3rd success after 4 or more failures

$\bar{A}$: 3rd success, after 3 or less failures

Pascal Model

$n = 3$, $k \in \{0, 1, 2, 3\}$

$$P(A) = 1 - P(\bar{A}) = 1 - \sum_{k=0}^{3} C_{3+k-1}^{k} (0.9)^{3} \cdot (0,1)^{k}$$

0,00127 — yes, is unusal

7.) A: when he realizes that one box is empty, the other contains k $(0 \le k \le n)$ pills

a.)

A_1 : ... 1st ... empty ... 2nd k pills

A_2 : ... 2nd ... 1st k pills

$A = A_1 \cup A_2$

disjunctive event probabilité

$P(A) = 2P(A_1)$

A_1: he attempts to take the $(n+1)^{st}$ pill from the 1st / ... 2nd k

success : he attempts to take a pill from 1st box (failure : ... 2nd)

$p = 2 - \frac{1}{2}$

A_1 : the $(n+1)^{st}$ success, after $n-k$ failures

Pascal Model

"n" $= n+1$

"k" $= n-k$

$P(A_1) = C_{2n-k}^{2n-k} \left(\frac{1}{2}\right)^{n+1} \left(\frac{1}{2}\right)^{n-k}$

$$= \frac{1}{2} C_{2n-k}^{n} \left(\frac{1}{2}\right)^{2n-k}$$

$$P(A) = C_{2n-k}^{n} \left(\frac{1}{2}\right)^{2n-k}$$

b.) For $k = \overline{0,n}$

E_k : event from a.)

$\{E_k\}_{k=\overline{0,n}}$ form a partition

$$= \sum_{k=0}^{n} P(E_k)$$

$$\sum_{k=0}^{n} \left(\frac{1}{2}\right)^{2n-k} C_{2n-k}^{n}$$

$$= \frac{1}{2^{2n}} \sum_{k=0}^{n} 2^{k} C_{2n-k}^{n} = \frac{1}{2^{2n}} S_n$$

$$S_n = 2^{2n}$$

Sem 4

Discrete Random Variables and Discrete Random Vectors

- pdf

→ doar val pt care prob. nu e zero

X discr. $\quad X \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I}$

$p_i = P(X = x_i)$

$\sum_{i \in I} p_i = 1$

- cdf $\quad F : \mathbb{R} \to \mathbb{R}$

strict <

$F(x) = P(X < x)$

$\{e \mid X(e) < x\}$

eg.

$X \in \mathcal{B}(n, p) \qquad n \in \mathbb{N}$ nr incercări

$p \in (0,1)$ probab.

$X \begin{pmatrix} k \\ C_n^k p^k q^{n-k} \end{pmatrix} \quad k = \overline{0,n}$

① a) $X \in B(3, 1/2)$ } pdf

$$X \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix}$$

$$P(X=k) = C_3^k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{3-k}$$

$$= \frac{1}{8} C_3^k$$

b.) $F(-1) = P(X < -1) = 0$ — ev. imposibil probabilă 0

$$F(0) = P(X < 0) = 0$$

- $\boxed{x \leq 0}$

  $F(x) = 0$

- $\boxed{0 < x \leq 1}$

  $F(x) = P(X < x)$

  $= P(X = 0) = \frac{1}{8}$ → probabilitatea

- $\boxed{1 < x \leq 2}$

  $F(x) = P(X = 0 \cup X = 1)$

  mutualy exclusive

  $= P(X=0) + P(X=1)$

  $= \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

cumulative distr. func.

- $\boxed{2 < x \leq 3}$

$$F(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$$

- $\boxed{x > 3}$

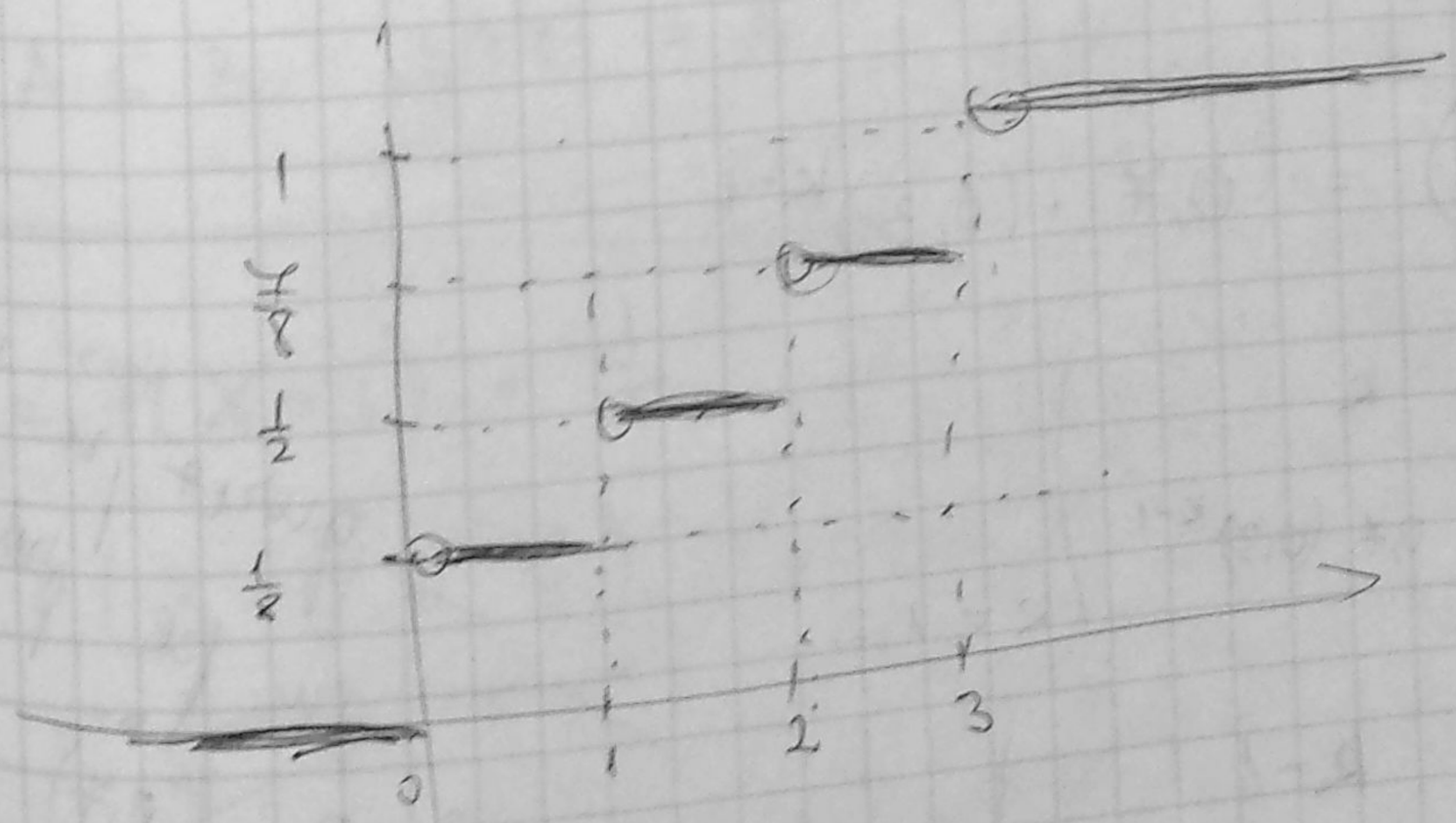
$$F(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

X discr

$$F(x) = \sum_{x_i < x} p^i \qquad \text{cdf}$$

(cumulative distr. function)

$$F(x) = \begin{cases} 0 & , \quad x \leq 0 \\ 1/8 & , \quad 0 < x \leq 1 \\ 1/2 & , \quad 1 < x \leq 2 \\ 7/8 & , \quad 2 < x \leq 3 \\ 1 & , \quad x > 3 \end{cases}$$



c.) $P(X < 2) = F(2) = \frac{1}{2}$ → cdf is strictly lin. ↗

$P(X \leq 2) = P(X < 3)$

2.5
2.9
2.1
2.0000001

$= F(3) = \frac{7}{8}$

$P(a \leq X \leq b) = F(b) - F(a)$

$P(A \cup B) = P(A)$ m.e.

2.) $p = 0.7$

a.) values : $1, 2, \dots, \infty$

$P(A \cap B) = P(A)$ ind.

~~$P(X) = 1$~~

$P(X = 1) = 0,7$

$P(X = 2) = P(1^{st}$ no, 2nd yes$) =$

$= 0,3 \cdot 0,7 = 0,7 \cdot 0,3$

$P(X = 3) = 0,7 \cdot (0,3)^2$

. . . .

$P(X = k) = 0,7 \cdot (0,3)^{k-1}$

pdf: $X \begin{pmatrix} k \\ 0,7 \cdot (0,3)^{k-1} \end{pmatrix}_{k = 1, 2 \dots}$

val →
probab →

$X - 1 \begin{pmatrix} k-1 \\ 0,7 \cdot (0,3)^{k-1} \end{pmatrix}_{k = 1, 2 \dots}$

→ aprez pe valori
nu pe probab.

$\in Geo(p = 0,7)$

b.) $P(X \leq 4) = F(5)$

c.) $P(X \geq 3) =$

$= 1 - P(X < 3) = 1 - F(3)$

cdf:
$P(X < x) = F(x)$

3.) $N \in \{1, 2, \dots 5\}$

a) $X \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$ → Discret Uniform Distribution

$X \begin{pmatrix} k \\ \frac{1}{n} \end{pmatrix}_{k=\overline{1,n}}$

$\in U(5)$

$Y \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}$

b.) $X + Y$ values: $2, \cancel{3}, 4, 5, 6, 7, 8$ $p=0$

Suriem pof - ul de final nu suriem ale lui probab 0

$P(X+Y = 2) = P(X = 1 \cap Y = 1) =$

$(X=1) \Rightarrow (Y=1) =$

$\subseteq$

$(X=1) \cap (Y=1) = (X=1)$

$A \subseteq B \quad A \cap B = A$

$= P(X=1) = \frac{1}{5}$

$P(X+Y = 3) = 0 + 0 = 0$

$\cup$ ~~1 2~~ ~~2 1~~ imposibil

$= 4$

~~1 3~~ ~~2 2~~ ✓ $= \frac{1}{5}$

XY values : 1, ~~2~~, 3, 4, 5, 6, 8, 9, 10, 12, 15

$P(X \cdot Y = 2) = 0$

1 2

2 1

(4) $N_1, N_2 \in \{1, \dots 5\}$

$X \dots N_1$

$Y \dots N_2$

$\boxed{X, Y - \text{indep.}}$

a.) same

b.)

possible values: same

$P(X+Y=2) = P(X=1, Y=1)$

$= P(X=1)\,P(Y=1) = \frac{1}{25}$

$P(X+Y=3) = \frac{3}{25} + \frac{1}{25} = \frac{4}{25}$

(5) m letters .... N mailboxes

A: m letters .... fixed mailbox

$P(A) = \frac{C_m^m (N-1)}{N^m}$

$m = 8 \qquad N = 3$

$X \begin{pmatrix} k \\ C_8^k \frac{2^{8-k}}{3^8} \end{pmatrix} \quad k = \overline{0,8}$

bolts

6.

X \ Y	0	1	2	3	
0	0.8				0.9 d.
1					0,08
2				0)	0,02
	0.91	0.0045			

indep.

$p_{ij} = P((X,Y) = (x_i, y_i))$
$= P(X = x_i, Y = y_i)$

posibil tot deauna

a.) pdf $(X,Y) \to$ pdf X, pdf Y

$X \begin{pmatrix} 0 & 1 & 2 \\ 0.9 & 0,08 & 0,02 \end{pmatrix}$ adunăm pe linii

$Y \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.91 & 0.0045 & \dots & \end{pmatrix}$ adunăm pe coloane

b.) $P(X = 2, Y = 1)$
$= 0.005$

c.) $P(X \geq 1, Y \geq 1) = 0,03$ le adunăm

d.) $P(X \leq 1) = 0,98$

(7.) $X_1, \dots X_n \in Bern(p)$, indep.

$Y = \sum_{i=1}^{n} X_i \in ?$

Cas particulier la Binomiale ou $n=$

$$X_i \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$$

Y values : $0, \dots n$

$k \in \{0, n\}$

- $Y = k$
- k of the r.v's take the value 1

  prob : p^k

- $n-k$: value 0

  prob : q^{n-k}

and there are C_n^k ways of choosing the k r.v's

$$Y \begin{pmatrix} k \\ C_n^k p^k q^{n-k} \end{pmatrix} \quad k = \overline{0,n} \in B(n,p)$$

P.S.

Sem. 5

X count r.v.

- <u>pdf</u> $f : \mathbb{R} \to \mathbb{R}$

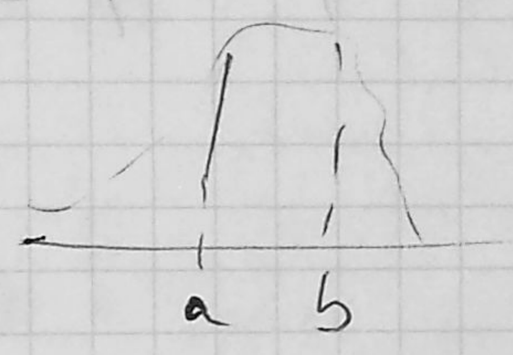
$f(x) \geq 0$

$\int_{\mathbb{R}} f(x)\,dx = 1$

- <u>cdf</u> $F : \mathbb{R} \to \mathbb{R}$

$F(x) = P(X < x)$

$\int_{-\infty}^{x} f(t)\,dt$



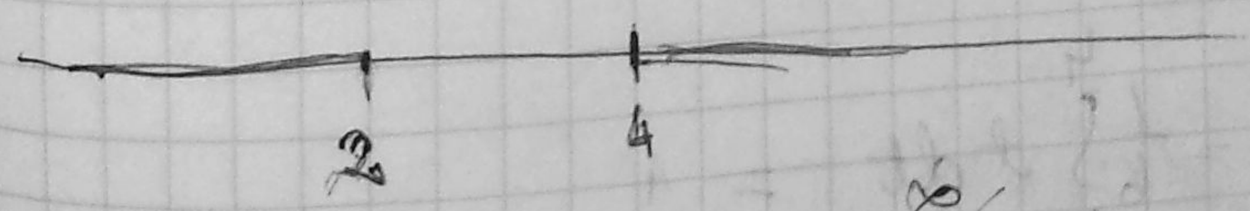
$P(X = x_0) = 0$

$P(a \underset{\leq}{<} X \underset{\leq}{<} b) = \int_a^b f(t)\,dt$

1. $f(x) = kx \quad , \quad x \in [2, 4]$

$\left(\begin{cases} 0 \quad , \quad x \notin [2,4] \end{cases} \right) \to$ se subintelege

a) $\int_{\mathbb{R}} f(x)\,dx = 1$



$\cancel{\int_{-\infty}^{2}} + \int_{2}^{4} + \cancel{\int_{4}^{\infty}}$

$$1 = \int_{\mathbb{R}} f(x)\,dx = k\int_2^4 x\,dx = k\cdot\frac{1}{2}x^2\Big|_2^4 = 6k$$

$$\cancel{1 = \int_{\mathbb{R}} k\,dx}$$ not good!

$$\Rightarrow 6k = 1 \Rightarrow \boxed{k = \frac{1}{6}}$$

$$f(x) = \begin{cases} \frac{1}{6}x, & x \in [2,4] \\ 0, & x \notin [2,4] \end{cases}$$

b.) cdf

$\underline{x \leq 2}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = 0$$

$2 < x \leq 4$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \frac{1}{6}\int_2^x t\,dt =$$

$$= \frac{1}{12}t^2\Big|_2^x = \frac{x^2-4}{12}$$

$\underline{x > 4}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \frac{1}{6}\int_2^4 t\,dt = 1$$

$$F(x) = \begin{cases} 0 & , x \leq 2 \\ \frac{x^2-4}{12} & , 2 < x \leq 4 \\ 1 & , x > 4 \end{cases}$$

c) $P(2.5 \underset{\leq}{<} X \underset{\leq}{<} 3) = F(3) - F(2.5) =$

$$= \int_{2.5}^{3} f(t)\,dt = \frac{1}{6}\int_{2.5}^{3} t\,dt = \dots$$

(2) $F(x) = a + b\,\text{arctg}\,x \quad , x \in \mathbb{R}$

$F(-\infty) = 0$

$F(\infty) = 1$

$F(-\infty) = a + b\left(-\frac{\pi}{2}\right)$

$F(\infty) = a + b \cdot \frac{\pi}{2}$

$$\begin{cases} a - b \cdot \frac{\pi}{2} = 0 \\ a + b\frac{\pi}{2} = 1 \end{cases}$$

$2a = 1 \quad , \boxed{a = \frac{1}{2}}$

$b = \frac{2a}{\pi} = \frac{1}{\pi} \rightarrow \boxed{b = \frac{1}{\pi}}$

$$\boxed{F(x) = \frac{1}{2} + \frac{1}{\pi}\operatorname{arctg} x, \quad x \in \mathbb{R}}$$ cdf

b.) $f(x) = F'(x)$

$$\underbrace{f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad x \in \mathbb{R}}$$

$X \in \text{Cauchy}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$

b.) $P(-1 \le X < \sqrt{3})$

$= F(\sqrt{3}) - F(-1) \ldots$

$= \int_{-1}^{\sqrt{3}} f(t)\,dt \ldots\ldots$

pdf (X, Y) $\quad f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) \ge 0$

$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$

cdf $\quad F: \mathbb{R}^2 \to \mathbb{R}$

$F(x,y) = P(X < x, Y < y)$

$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\,du\,dv$

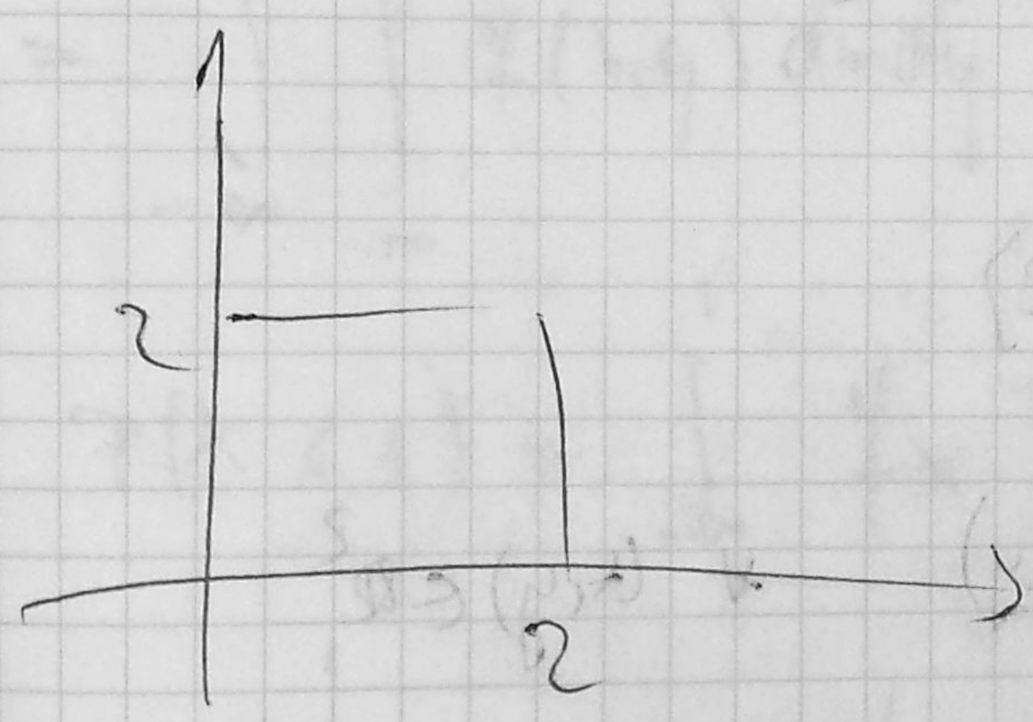
$\forall D \subseteq \mathbb{R}^2$

$$P((X,Y) \in D) = \iint_D f(u,v)\,du\,dv$$

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

(3) $f(x,y) = \frac{1}{10} x^2 y^3, \; x, y \in [0,2]$

(0, dorte)



$(x,y) \in [0,2] \times [0,2]$

a.) pdf $(X,Y) \rightarrow$ pdf X, pdf Y

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$$

For $x \notin [0,2]$

$f_X(x) = 0$

for $x \in [0,2]$

$$f_X(x) = \frac{1}{16}\int_0^2 x^3 y^3 \, dy = \frac{1}{16} x^3 \cdot \frac{1}{4} y^4 \Big|_{y=0}^{y=2}$$

$$= \frac{1}{4} x^3$$

$$f_X(x) = \begin{cases} \frac{1}{4} x^3, & x \in [0,2] \\ 0, & x \notin [0,2] \end{cases}$$

$$f_Y(y) = \frac{1}{4} y^3, \; y \in [0,2]$$

b.) $f(x,y) = f_X(x) \, f_Y(y) \quad \forall (x,y) \in \mathbb{R}^2$

If $x \notin [0,2]$ or $y \notin [0,2]$

$0 = 0 \times \ldots$

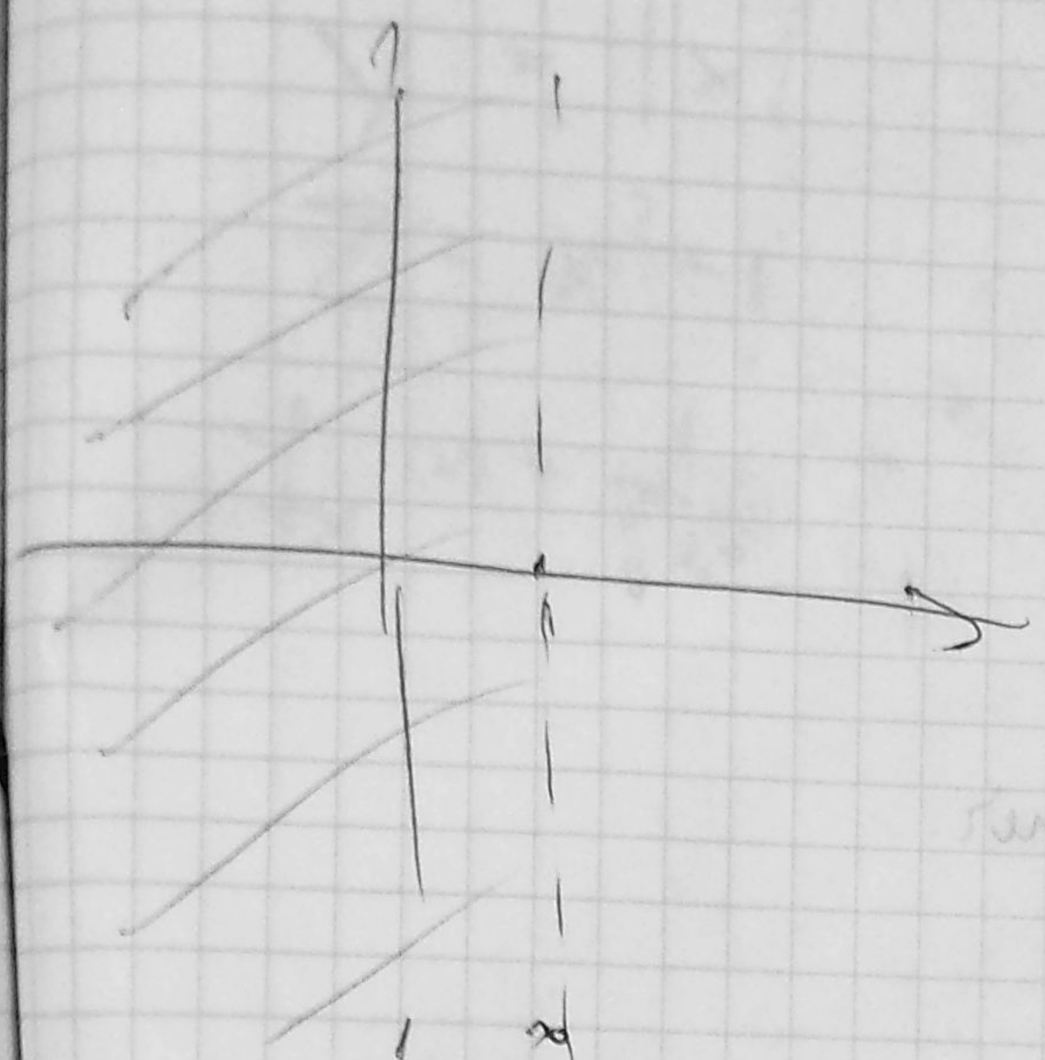
~~If $x \notin [0,2]$ or $y \notin [0,2]$~~

If $x, y \in [0,2]$

$$\frac{1}{16} x^3 y^3 = \frac{1}{4} x^3 \, \frac{1}{4} y^3$$

yes

c.) $P(X < 1)$



$$= \int_{-\infty}^{1}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \frac{1}{16}\int_{0}^{1}\int_{0}^{2} x^3 y^3\,dx\,dy$$

$$P(X \leq 1) = \int_{-\infty}^{1} f_X(x)\,dx$$

$$= \frac{1}{4}\int_{0}^{1} x^3\,dx = \frac{1}{16}$$

$Y = g(X)$

$g : \mathbb{R} \to \mathbb{R}$ differentiable, strictly mon. with

$g'(x) \neq 0, \forall x \in \mathbb{R}$

$$f_Y(y) = \begin{cases} \dfrac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}, & y \in g(\mathbb{R}) \\ 0, & \text{otherwise} \end{cases}$$

4. $f_X(x) = \frac{1}{4} x e^{-\frac{x}{2}}, \quad x \geq 0$

$y = \frac{1}{2}x + 2$

$g : \mathbb{R} \to \mathbb{R}$

$g(x) = \frac{1}{2}x + 2$

$g'(x) = \frac{1}{2} \neq 0$

$g' > 0 \Rightarrow g$ strictly incr.

$y = g(x)$

$x = g^{-1}(y)$

$y = \frac{1}{2}x + 2$

$x = 2(y-2) = g^{-1}(y)$

$f_X(g^{-1}(y)) = f_X(2(y-2))$

$= \frac{1}{4} 2(y-2) e^{-(y-2)}$

$2(y-2) \geq 0$

$y \geq 2$

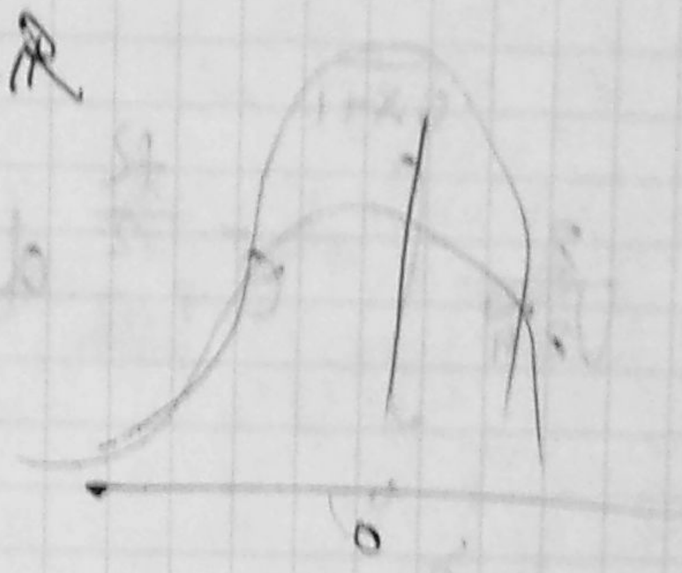
$$f_Y(y) = \begin{cases} (y-2)e^{-y} & , y \geq 2 \\ 0 & , y < 2 \end{cases}$$

(5) $X \in N(0,1)$

$Y = |X|$

$Z = X^2 - 1$

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$



Start with the cdf

$f_Z(x) = F_Z'(x)$

$F_Z(x) = P(Z < x)$

$= P(X^2 - 1 < x)$

$X^2 \geq 0$

$X^2 - 1 \geq -1$

For $x \leq -1 \quad F_Z(x) = 0$

$f_Z(x) = 0$

for $x \geq -1$

$F_Z(x) = P(X^2 < x+1)$

$x^2 \leq 4$

$= P(-\sqrt{x+1} < X < \sqrt{x+1})$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x+1}}^{\sqrt{x+1}} \underbrace{e^{-\frac{t^2}{2}}}_{\text{even}} dt$$

$$= \int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & f \text{ odd} \quad f(-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & f \text{ even} \quad f(-x) = f(x) \end{cases}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{x+1}} e^{-\frac{t^2}{2}}\,dt$$

$$F(x) = \int_0^x g(t)\,dt$$

$$F'(x) = g(x)$$

$$F(x) = \int_0^{h(x)} g(t)\,dt$$

$$F'(x) = g(h(x)) \cdot h'(x)$$

$$f_Z(x) = F_Z'(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{x+1}{2}} \cdot \frac{1}{2} \cdot (x+1)^{-1/2}$$

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{x+1}{2}}}{\sqrt{x+1}} \quad x > -1$$

(8.) $X \in \mathcal{U}(0,a)$

$Y \in \mathcal{U}(0,b)$ indep

$0 < a < b$

$Z = XY$

$$f_Z(z) = \int_{\mathbb{R}} f_X(u) \cdot f_Y\left(\frac{z}{u}\right) \cdot \frac{1}{|u|} \, du$$

$X \in \mathcal{U}(a,b)$

pdf. $f(x) = \dfrac{1}{b-a}$, $x \in (a,b)$

$f_X(x) = \dfrac{1}{a}$, $x \in (0,a)$

$f_Y(y) = \dfrac{1}{b}$, $y \in (0,b)$

$0 < u < a$

$0 < \dfrac{z}{u} < b \Rightarrow 0 < z < ub$

$\Rightarrow 0 < z < ab$

Ta $\underline{0 < z < ab}$

$\left.\begin{array}{l} 0 < u < a \\ u \geq \frac{z}{b} \end{array}\right\} \frac{z}{b} < u < a$

$$f_Z(z) = \frac{1}{ab}\int_{\frac{z}{b}}^{a} \frac{1}{u}\,du = \frac{1}{ab}\ln\frac{ab}{z}, \quad 0 < z < ab$$

P.S.

S.6

Expected Value (valoare medie)

- X discr r.v. pdf $\begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I}$

$$\boxed{E(X) = \sum_{i \in I} x_i p_i} \quad (< \infty)$$

- X cont r.v., pdf $f : \mathbb{R} \to \mathbb{R}$

$$\boxed{E(X) = \int_{\mathbb{R}} x f(x)\, dx} \quad (< \infty)$$

Variance (Dispersion) dispersie

$$V(X) = E\left[(X - E(X))^2\right] \geq 0$$
$$= E(X^2) - (E(X))^2$$

$$\sigma(X) = \sqrt{V(X)} \quad \text{std dev.}$$

---

① $X \in B(n, p)$, $n \in \mathbb{N}$, $p \in (0,1)$

pdf X $\begin{pmatrix} k \\ C_n^k p^k q^{n-k} \end{pmatrix}$ $k = \overline{0,n}$

$X_1, \dots X_n \in Bern(p)$ indep

$\left(\begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}\right) \Rightarrow X = \sum_{i=1}^{n} X_i \in B(n, p)$

$E(X) = np$

$V(X) = npq$

$$C_n^k = \frac{n}{k} C_{n-1}^{k-1} = \frac{n(n-1)}{k(k-1)} C_{n-2}^{k-2} \rightarrow \text{f. de recurentă}$$

$$\sum_{i=0}^{n} C_n^i p^i q^{n-i} = (p+q)^n = 1$$

$$E(X) = \sum_{k=0}^{n} k C_n^k p^k q^{n-k}$$

$$= \sum_{k=1}^{n} \underbrace{k C_n^k}_{n \cdot C_{n-1}^{k-1}} p^k q^{n-k}$$

$$= n \cdot \sum_{k=1}^{n} C_{n-1}^{k-1} p^k q^{n-k}$$

$$= n \cdot p \sum_{k=1}^{n} C_{n-1}^{k-1} p^{k-1} q^{n-k}$$

Let $l = k-1$

$$= np \cdot \underbrace{\sum_{l=0}^{n-1} C_{n-1}^{l} p^{l} q^{n-1-l}}_{=1}$$

$$\boxed{E(X) = np}$$

$$E(X^2) = \sum_i x_i^2 p_i$$

$$E(X^2) = \sum_{k=0}^{n} k^2 C_n^k p^k q^{n-k}$$

$$\sum_{k=1}^{n} (k^2-k) C \ldots + \sum_{k=1}^{n} k$$

$$= \sum_{k=1}^{n} (\underbrace{k^2-k}+k) C_n^k p^k q^{n-k}$$

$$= \sum_{k=2}^{n} \underbrace{k(k-1) C_n^k}_{n(n-1)C_{n-2}^{k-2}} p^k q^{n-k} + \underbrace{\sum_{k=1}^{n} k \cdot C_n^k p^k \cdot q^{n-k}}_{E(X) = np} \quad (1)$$

$$= n(n-1)p^2 \underbrace{\sum_{k=2}^{n} C_{n-2}^{k-2} p^{k-2} q^{n-k}}_{=1} + np$$

$$= n(n-1)p^2 + np$$

$$V(X) = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$\boxed{V(X) = npq}$$

varianza distr. binomiale

$|r| < 1$

Geometric Series

$$\sum_{i=0}^{\infty} a r^i = \frac{a}{1-r}, \quad |r| < 1$$

as a function of r, it can be diff any no of

$$\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{\lambda}$$

② a) pdf $f(x) = \frac{1}{\pi\sqrt{(a^2-x^2)}}$

$x \in (-a, a)$

$$E(x) = \int_{\mathbb{R}} x f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}}\, dx = 0$$

$\int_{-a}^{a} f$ — 0, f odd, $f(-x) = -f$ — $2\int_0^a f\, dx$, f $f(-x) = f$

$a = \infty$, f bounded

$$V(x) = E(x^2) = \int_{\mathbb{R}} x^2 f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-a}^{a} \frac{x^2}{\sqrt{a^2-x^2}}\, dx = \frac{2}{\pi}\int_0^a \frac{x^2}{\sqrt{a^2-x^2}}\, dx$$

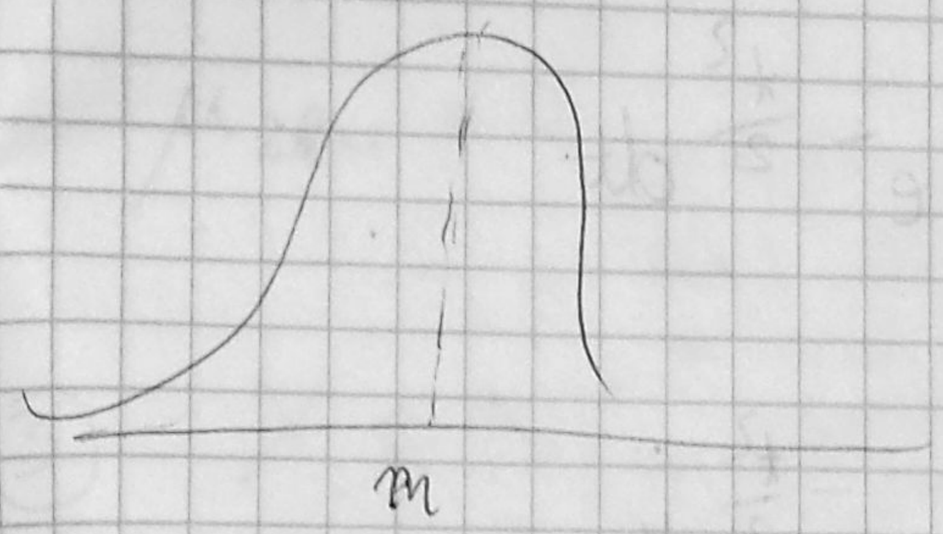
subst. $x = a \sin t$

$$V(x) = \frac{a^2}{2}$$

③ $X \in N(m, \sigma)$, $m \in \mathbb{R}$, $\sigma > 0$

$$\mu_k = E\left((X - E(X))^k \right)$$

pdf $\quad f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}, \; x \in \mathbb{R}$



$$E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} x e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

Euler's Gamma Func. (S1)

$\Gamma : (0, \infty) \to (0, \infty)$

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

$\Gamma(1) = 1$

$\Gamma(a+1) = a\Gamma(a)$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{2} \int_0^\infty e^{-\frac{x^2}{2}} dx$

$= \int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$

Ch of var's

$$t = \frac{x-m}{\sigma}$$

$$x = \sigma t + m$$

$$dx = \sigma dt$$

$$\mathbb{R} \to \mathbb{R}$$

$$E(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\sigma t + m) e^{-\frac{t^2}{2}} dt$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{\mathbb{R}} t e^{-\frac{t^2}{2}} dt + \frac{m}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{t^2}{2}} dt$$

odd and bounded

even bounded

$$= \sqrt{\frac{2}{\pi}}\, m \underbrace{\int_0^{\infty} e^{-\frac{t^2}{2}} dt}_{\frac{1}{\sqrt{2}}\Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{2}}}$$

$$\boxed{E(X) = m}$$

$$\mu_k = E\left[(X-m)^k\right] = \int_{\mathbb{R}} (x-m)^k f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} (x-m)^k \cdot e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

ch of var's $t = x - m$

$x = t + m$

$dx = dt$

$\mathbb{R} \to \mathbb{R}$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} t^k e^{-\frac{t^2}{2\sigma^2}} dt$$

$$\mu_{2\ell+1} = 0$$

③ $X \in N(\overset{E(X)}{\overset{\shortparallel}{m}}, \overset{\sigma(X)}{\overset{\shortparallel}{\sigma}}), \; m \in \mathbb{R}, \; \sigma > 0$

$$\mu_{2\ell} = \frac{1}{\sigma} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty t^{2\ell-1} e^{-\frac{t^2}{2\sigma^2}} t \, dt$$

subst. $u = \frac{t^2}{2\sigma^2}$

$t^2 = 2\sigma^2 u$

$\cancel{2} t \, dt = \cancel{2} \sigma^2 du$

$$t^{2\ell-1} = (t^2)^{\ell-\frac{1}{2}} = (2\sigma^2 u)^{\ell-\frac{1}{2}}$$

$$= 2^{\ell-\frac{1}{2}} \sigma^{2\ell-1} u^{\ell-\frac{1}{2}}$$

$$\mu_{2\ell} = \frac{1}{\sigma} \cdot \sqrt{\frac{2}{\pi}} \cdot 2^{\ell-\frac{1}{2}} \sigma^{2\ell-1} \sigma^2 \int_0^\infty u^{\ell-\frac{1}{2}} e^{-u} du$$

$$= \frac{1}{\sqrt{\pi}} 2^{\ell} \sigma^{2\ell} \underbrace{\int_0^\infty u^{(\ell+\frac{1}{2})-1} e^{-u} du}_{\Gamma(\ell+\frac{1}{2})}$$

$$= \frac{1}{\sqrt{\pi}} 2^{\ell} \sigma^{2\ell} \cdot (\ell - \tfrac{1}{2})(\ell - \tfrac{3}{2}) \cdots \tfrac{1}{2} \Gamma(\tfrac{1}{2})$$

$$= \sigma^{2l}(2l-1)(2l-3)\cdots 1$$

$$= (2l-1)!!\cdot\sigma^{2l}$$

$n! = 1\cdot 2\cdots n$

$(2n)!! = 2\cdot 4\cdots(2n)$

$(2n+1)!! = 1\cdot 3\cdot 7\cdot(2n+1)$

$$\mu_k = \begin{cases} 0 & , \quad k=2l+1 \\ \sigma^2(2l-1)!! & , \quad k=2l \end{cases}$$

In particular,

$$V(X) = \mu_2 = \sigma^2$$

st. dev. $\sigma(X) = \sqrt{V(X)} = \sigma$

④ X r.v. with mean $E(X)$, st dev. $\sigma(X) = \sqrt{V(X)}$

$$Y = \frac{X - E(X)}{\sigma(X)} = \frac{1}{\sigma(X)}(X - E(X))$$

$$\begin{cases} E(aX+b) = a\,E(X) + b \\ V(aX+b) = a^2 V(X) \end{cases}$$

$$E(Y) = \frac{1}{\sigma(X)}\left(\cancel{E(X)} - \cancel{E(X)}\right) = 0$$

$$V(Y) = \frac{1}{\cancel{(\sigma(X))^2}}\cdot\cancel{V(X)} = 1$$

Y reduced var of X

In part. $X \in N(\mu, \sigma) \Rightarrow \dfrac{X-\mu}{\sigma} \in N(0,1)$

Remark

Sample mean $\bar{X} = \frac{1}{n} \sum x_i$

$E(\bar{X}) = E(X) = \mu$

$V(\bar{X}) = \frac{V(X)}{n} = \frac{\sigma^2}{n}$

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \in N(0,1)$$

normal pop $\quad \frac{1}{n^2} \sum \underbrace{V(x_i)}_{=V(X)}$

large sample

$$\frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

(X, Y) pdf $f(x,y)$

$$E(h(X,Y)) = \iint_{\mathbb{R}^2} h(x,y) f(x,y) \, dx\, dy$$

$$\operatorname{cov}(X,Y) = E[(X - E(X))(Y - E(Y))]$$
$$= E(XY) - E(X)E(Y)$$

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}}$$

0,1)

5) $f(x,y) = x+y \,,\; x,y \in [0,1]$

a) $E(X) = \iint_{\mathbb{R}^2} x f(x,y)\,dx\,dy$

$= \int_0^1\int_0^1 x(x+y)\,dx\,dy$

$= \int_0^1 \left(\frac{1}{3}x^3 + y\cdot\frac{1}{2}x^2\right)\Big|_{x=0}^{x=1} dy$

$= \int_0^1 \left(\frac{1}{3}+\frac{1}{2}y\right)dy = \left(\frac{1}{3}y+\frac{1}{4}y^2\right)\Big|_0^1 = \frac{7}{12} = E(Y)$

$E(X^2) = \iint_{\mathbb{R}^2} x^2 f(x,y)\,dx\,dy$

$= \int_0^1\int_0^1 x^2(x+y)\,dx\,dy$

$= \dots = \frac{5}{12} = E(Y^2)$

$V(X) = V(Y) = \frac{5}{12} - \frac{49}{12^2} = \frac{11}{12^2}$

b.) $E(XY) = \iint_{\mathbb{R}^2} xy\, f(x,y)\,dx\,dy$

$= \dots = \frac{1}{3}$

$\mathrm{cov}(X,Y) = \frac{1}{3} - \frac{49}{12^2} = \frac{-1}{12^2}$

$$\rho(x,y) = \frac{-\frac{1}{122}}{\frac{11}{12^2}} = -\frac{1}{11}$$

$$-1 \leq \rho(x,y) \leq 1$$

$\rho(x,y) = 0$ x, y, uncorrelated

X, Y ind $\Rightarrow$ $\mathrm{cov}(X,Y) = \rho(X,Y) = 0$

$\not\Leftarrow$ X, Y uncorr

$$X \begin{pmatrix} -1 & 0 & 1 \\ \sin^2 a & \cos^2 a & \sin^2 a \end{pmatrix}$$

$a \in (0, \frac{\pi}{4})$

$$X^{2k+1} = X$$

$$X^2 \begin{pmatrix} 0 & 1 \\ \cos^2 a & 2\sin^2 a \end{pmatrix}$$

$$X^{2k} = X^2$$

$E(X) = E(X^{2k+1}) = 0$

$E(X^2) = E(X^{2k}) = 2\sin^2 a$

$\mathrm{cov}(X^{2k}, X^{2k+1}) = \mathrm{cov}(X^2, X)$

$= E(X^3) - E(X^2)E(X)$

$= 0$

$X, \cancel{Y} = X^2$ uncorrel.

but not ind

Chebyshev's Ineq

$$\underbrace{P(|X - E(X)| < \varepsilon) \geq 1 - \frac{V(X)}{\varepsilon^2}}_{\forall \varepsilon > 0}$$

or

$$P(|X - E(X)| \geq \varepsilon) \leq \frac{V(X)}{\varepsilon^2}$$

1) The 3σ - Rule

$E(X) = \mu$, $V(X) = \sigma^2$, $\sigma = \sqrt{V(X)}$

$$P(|X - \mu| < \varepsilon) \geq 1 - \frac{\sigma^2}{\varepsilon^2}$$

$\varepsilon = k\sigma$

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

$\underline{k=1}$

$$P(|X - \mu| < \sigma) > 1 - 1 = 0$$

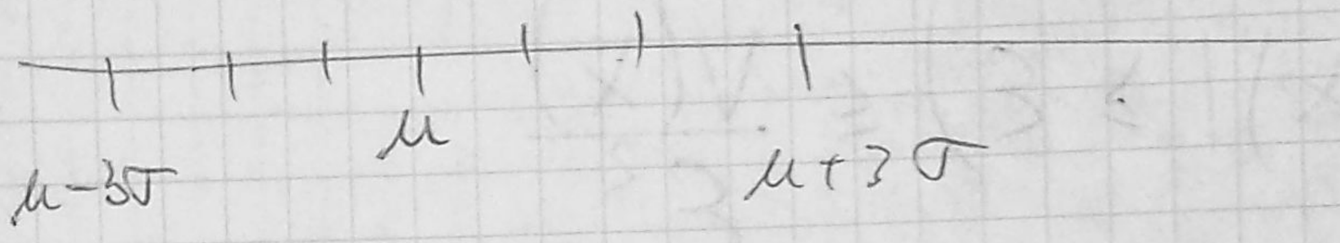
$\underline{k=2}$

$$P(|X - \mu| < 2\sigma) > \frac{3}{4} = 0.75$$

$k = 3$

$$P(|X-\mu| < 3\sigma) \geq \frac{8}{9} \approx 0.88$$

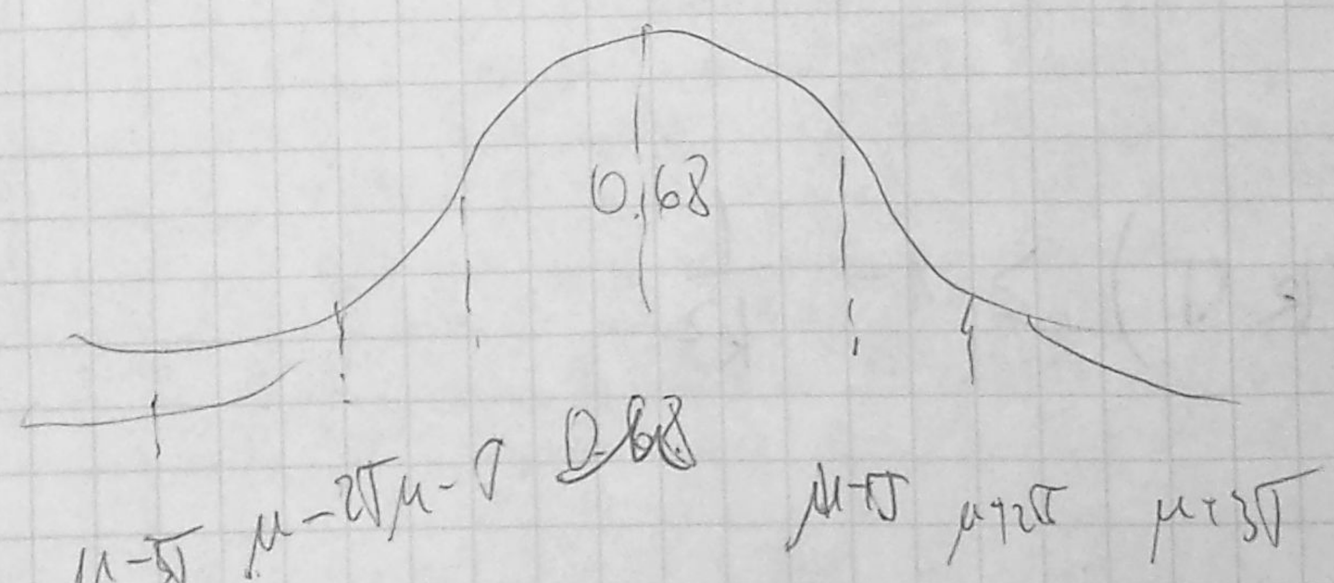
<u>The 3σ Rule</u> : Most of the values (at least 88%) that r.v. takes ~~the~~ within 3 standard deviations (3σ) away from the mean



In past, if $X \in N(\mu, \sigma)$

$$P(|X-\mu| < \sigma) \approx 0.68$$

$$\left.\begin{array}{l} P(|X-\mu| < 2\sigma) \\ P(|X-\mu| < 3\sigma) \end{array}\right\} \begin{array}{l} \approx 0.95 \\ \approx 0.99 \end{array}$$



Let X the nr of "heads" when a coin is flipped 1000 times.

$$P(450 < X < 550) \overset{?}{\geq} 0.9 \qquad E(X) = 500 \qquad V(X) = 250$$

$$X \in B(\overset{n}{1000}, \overset{p}{\tfrac{1}{2}})$$

$$\mathbb{P}$$

$X \in B(n, p$
$E(X) = np$
$V(X) = npq$

$$\mathbb{P}(|X-500| < \varepsilon) \geq 1 - \frac{250}{\varepsilon^2}$$

$-50 < X - 500 < 50$

$|X-500| < 50$

Let $\varepsilon = 50$

$$P(450 < X < 550) = P(|X-500| < 50) \geq$$

$$\geq 1 - \frac{\cancel{250}}{\cancel{2500}} = 0.9 \qquad \text{True.}$$

$S = [0,1]$

$\mathcal{K}$ - the set of all open subsets

P = length

$(S, \mathcal{K}, P)$

$X_n : S \to \mathbb{R}$ | $X_n(e) = \begin{cases} 2^n, & 0 \leq e \leq \frac{1}{n} \\ 1, & (1 > e > \frac{1}{n} \end{cases}$

$$X_n \to X = 1$$

scrierea pdf-ului

probab. 100%

$$X \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X_n \begin{pmatrix} 1 & 2^n \\ 1-\frac{1}{n} & \frac{1}{n} \end{pmatrix}$$

$$X_n \xrightarrow{a.s.} X$$

$$P\left(\lim_{n\to\infty} X_n = X\right) = 1$$

$$\neq X = 0$$

$$P\left(\lim_{n\to\infty} X_n = X\right) = P\left(\{e \in S \mid \lim_{n\to\infty} X_n(e) = X(e)\}\right)$$

$$\forall e \in S \setminus \{0\} = (0,1]$$

$$\lim_{n\to\infty} X_n(e) = 1 = X(e)$$

$$e = 0$$

$$\lim_{n\to\infty} X_n(0) = \lim_{n\to\infty} 2^n = \infty \neq 1$$

$$P\left(\lim_{n\to\infty} X_n \neq X\right) = P\left(\{e \in S \mid \lim_{n\to\infty} X_n(e) \neq X(e)\}\right)$$

$$= P(\{0\}) = 0$$

convergenta in probabilitate

$$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$$

c. in distribu

$$X_n \xrightarrow{L^r} X \quad r>0$$

$$\lim_{n\to\infty} E\left(|X_n - X|^r\right) = 0$$

$$|X_n - X|^r \begin{cases} 0 & (2^n-1)^r \\ 1-\frac{1}{n} & \frac{1}{n} \end{cases}$$

$$E\left(|X_n - X|^r\right) = \frac{(2^n-1)^r}{n} \xrightarrow{n\to\infty} \infty \neq 0$$

Generalizeta in serie de ordin r

conv. in mean of order r

$$X_n \xrightarrow{L^r} X$$

$$X_n \xrightarrow{d} X \text{ if}$$

$$\lim_{n\to\infty} F_n(x) = F(x),$$

$\forall x \in \mathbb{R}$ point of continuity for F

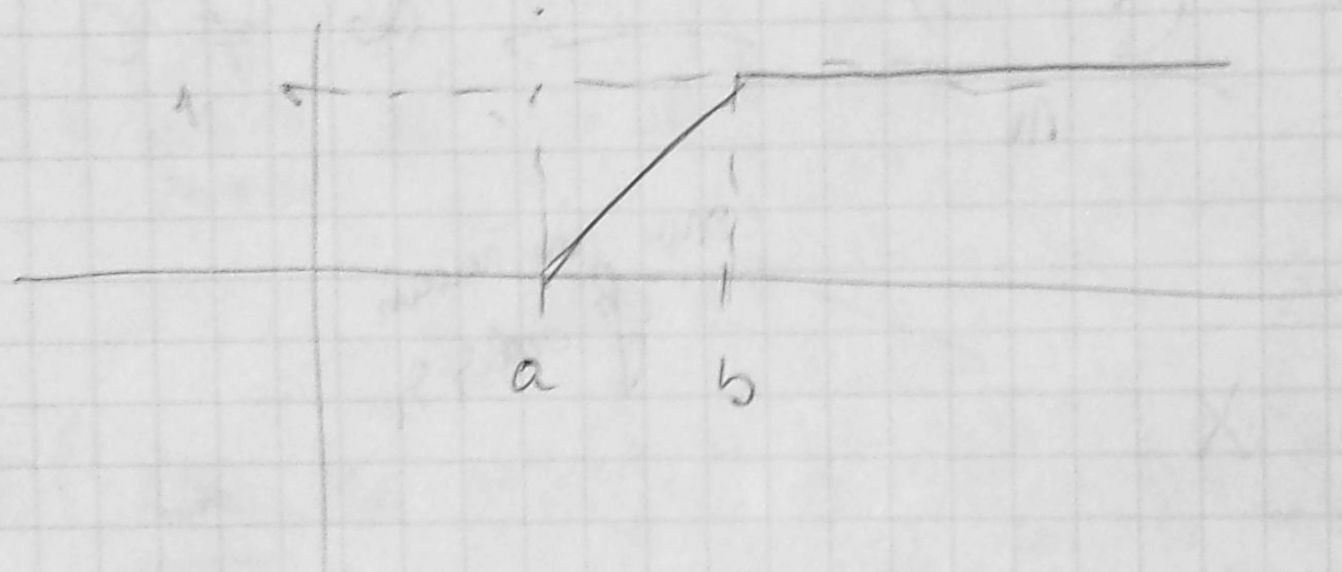
$$\left(F(x) = P(X < x)\right)$$

(4) $X_n \in \mathcal{U}(-n, n)$

$X \in \mathcal{U}(a,b)$

pdf $f(x) = \dfrac{1}{b-a}$, $x \in [a,b]$

cdf $F(x) = \displaystyle\int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & , x \le a \\ \dfrac{x-a}{b-a} & , a < x \le b \\ 1 & , x > b \end{cases}$

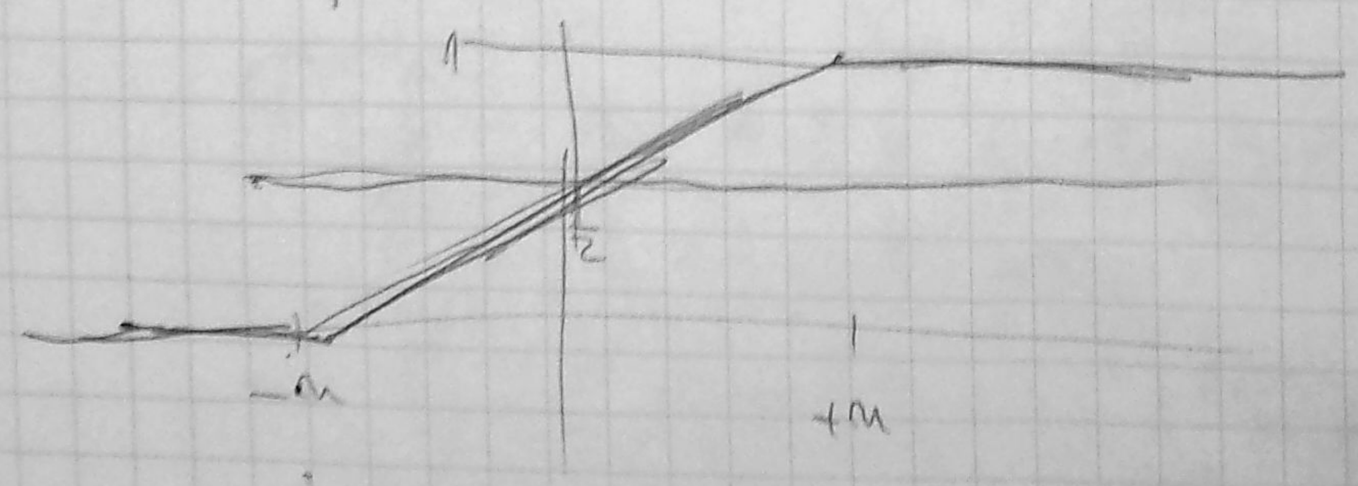


$X_n \in \mathcal{U}(-n, n)$

$F_n(x) = \begin{cases} 0 & , x \le -n \\ \dfrac{x+n}{2n} & , -n < x \le n \\ 1 & , x > n \end{cases}$

se divide :)

$\lim\limits_{n\to\infty} F_n(x) = \lim\limits_{n\to\infty} \dfrac{x+n}{2n} = \dfrac{1}{2}$



But $F(x) = \frac{1}{2}$ is not a cdf.

$\cancel{X_n \xrightarrow{d} X}$

⑤ $X_n \xrightarrow{L^r} X \Rightarrow X_n \xrightarrow{L^s} X$

$\forall\, s \le r$

Sol

$$\lim_{n\to\infty} E(|X_n - X|^r) = 0$$

Hölder : $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$

$$E(|\tilde{X}\,\tilde{Y}|) \le (E|\tilde{X}|^p)^{\frac{1}{p}} (E(|\tilde{Y}|^q))^{\frac{1}{q}}$$

Let $s < r$, $p = \frac{r}{s} > 1$, q...

$\tilde{X} = |X_n - X|^s$

$\tilde{Y} \equiv 1$

$$\therefore E(|X_n - X|^s) \le (E(|X_n - X|^r))^{\frac{s}{r}} \cdot 1$$

Take $\lim_{n\to\infty}$

$$\lim_{n\to\infty} E(|X_n - X|^s) = 0$$

⑥ $X_n \in N(\mu_n, \sigma_n)$

$X_n \xrightarrow{L^r} 0, \; r \geq 2$

$\Leftrightarrow \lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \sigma_n = 0$

<u>Sol</u>

$X_n \xrightarrow{L^r} 0, \; r \geq 2$

$\xrightarrow{⑤} X_n \xrightarrow{L^2} 0$, i.e

$E(X_n^2) \xrightarrow{n\to\infty} 0$

we use the rel for compute the variance usually

$$\begin{cases} V(X) = E(X^2) - [E(X)]^2 \\ E(X^2) = V(X) + (E(X))^2 \end{cases}$$

$E(X_n^2) = V(X_n) + (E(X_n))^2$

$= \sigma_n^2 + \mu_n^2 \xrightarrow{n\to\infty} 0$

$\Rightarrow \mu_n, \sigma_n \longrightarrow 0$

$E(X_n^2) \xrightarrow{n\to\infty} 0$

Distributions	Notation	Mean $E(X)$	Variance $V(X)$
discrete uniform	$U(m)$	$\frac{m+1}{2}$	$\frac{m^2-1}{12}$
binomial	$B(n,p)$	np	npq, $q=1-p$
hypergeometric	$H(N,m_1,n)$	$\frac{nm_1}{N}$	$\frac{nm_1(N-m_1)(N-n)}{N^2(N-1)}$
Poisson	$P(\lambda)$	λ	λ
Pascal	$NB(n,p)$	$\frac{n(1-p)}{p}$	$\frac{n(1-p)}{p^2}$
geometric	$G(p)$	$\frac{q}{p}$	$\frac{q}{p^2}$
uniform	$U(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
normal	$N(\mu,\sigma)$	μ	σ^2
gamma	$Ga(a,b)$	ab	ab^2
exponential	$Exp(\lambda)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
beta	$\beta(a,b)$	$\frac{a}{a+b}$	$\frac{ab}{(a+b+1)(a+b)^2}$
Student	$T(n)$	0, $n>1$ $\nexists$ $n=1$	$\frac{n}{n-2}$, $n>2$
χ^2 (chi squared)	$\chi^2(n)$	n	$2n$
Fisher	$F(m,n)$	$m>2$: $\frac{n}{n-2}$ $n<3$: undef.	$n>4$ $\frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$