Remarks:

The transformations used in the Skolemization process do not preserve the logical equivalence but preserve the inconsistency according to the following theorem:

Theorem 2: Let $U1, U2,.., Un, V$ be first-order formulas.

1. V *is* inconsistent **iff** V^P is inconsistent **iff** V^S is inconsistent **iff** V^{Sq} is inconsistent.
2. $\{U1, U2,.., Un\}$ is inconsistent **iff** $\{U1^{Sq}, U2^{Sq},.., Un^{Sq}\}$ is inconsistent.

Exemplu 7: Transform into prenex normal form and Skolem normal form the formula: A=$\neg((\forall x)(p(x) \to q(x)) \to ((\forall x)p(x) \to (\forall x)q(x)))$

Step1: replace the inner $\to$ connectives

$$\mathbf{A} \equiv \neg((\forall x)(\neg p(x) \vee q(x)) \to (\neg(\forall x)p(x) \vee (\forall x)q(x)))$$

Step1: replace $\to$ connective

$$\mathbf{A} \equiv \neg(\neg((\forall x)(\neg p(x) \vee q(x))) \vee (\neg(\forall x)p(x) \vee (\forall x)q(x))$$

Step2: rename the bound variables such that they will be distinct

$$\mathbf{A} \equiv \neg(\neg((\forall x)(\neg p(x) \vee q(x))) \vee (\neg(\forall y)p(y) \vee (\forall z)q(z))$$

Step3: apply DeMorgan laws

$$\mathbf{A} \equiv (\forall x)(\neg p(x) \vee q(x)) \wedge \neg(\neg(\forall y)p(y) \vee (\forall z)q(z))$$

Step3: apply DeMorgan laws

$$\mathbf{A} \equiv (\forall x)(\neg p(x) \vee q(x)) \wedge (\forall y)p(y) \wedge (\exists z)\neg q(z))$$

Step4: extract the quantifiers in front of the formula

$$\mathbf{A} \equiv (\exists z)(\forall x)(\forall y)((\neg p(x) \vee q(x)) \wedge p(y) \wedge \neg q(z)) = A_1 = A_1^P$$

$$\mathbf{A} \equiv (\forall x)(\exists z)(\forall y)((\neg p(x) \vee q(x)) \wedge p(y) \wedge \neg q(z)) = A_2 = A_2^P$$

$$\mathbf{A} \equiv (\forall x)(\forall y)(\exists z)((\neg p(x) \vee q(x)) \wedge p(y) \wedge \neg q(z)) = A_3 = A_3^P$$

A_1, A_2, A_3 are three prenex forms of the initial formula A.

Because the formula A contains 3 distinct bound variables, there are 3!=6 prenex normal forms logically equivalent with A.

We remark that the matrix is in CNF, and we will not apply step 5.

A_1^P, A_2^P, A_3^P **are** prenex normal forms .

After the Skolemization process, applied to the formulas A_1^P, A_2^P, A_3^P we obtain:

$A_1^S = (\forall x)(\forall y)((\neg p(x) \vee q(x)) \wedge p(y) \wedge \neg q(a))$, $[z \leftarrow a]$, $\boldsymbol{a}=$ Skolem constant

$A_2^S = (\forall x)(\forall y)((\neg p(x) \vee q(x)) \wedge p(y) \wedge \neg q(f(x)))$, $[z \leftarrow f(x)]$, $\boldsymbol{f}=$ Skolem function

$A_3^S = (\forall x)(\forall y)((\neg p(x) \vee q(x)) \wedge p(y) \wedge \neg q(g(x,y)))$, $[z \leftarrow g(x,y)]$, $\boldsymbol{g}=$ binary Skolem function

Theorem 1:

A predicate formula admits a logical equivalent conjunctive prenex normal form.

The ***prenex normal form*** is obtained by applying transformations which preserve the logical equivalence, according to the following algorithm:

Step 1: The connectives $\rightarrow$ and $\leftrightarrow$ are replaced using the connectives $\neg, \wedge, \vee$.

Step 2: The bound variables are renamed such that they will be distinct.

Step 3: Application of infinitary DeMorgan laws.

Step 4: The **extraction of quantifiers in front of the formula laws** are applied.

!!! The order of quantifiers extraction is arbitrary.

Step 5: The matrix is transformed into CNF using DeMorgan laws and the distributive laws.

Remarks:

- After step 4 we obtain the prenex normal form which is not unique. If the formula obtained after the second step contains n distinct and independent bound variables, there are $n!$ prenex normal forms, logically equivalent with the initial formula.
- A conjunctive prenex normal form (clausal normal form) is obtained after step 5.

Definition 8:

Let U be a first-order formula, and $U^p = (Q_1 x_1)\ldots(Q_n x_n)M$ be one of its conjunctive prenex normal form. To U it corresponds a formula in ***Skolem normal form***, denoted by U^S obtained as follows:

For each existential quantifier Q_r from the prefix we apply the transformation:

(1) If Q_r is the leftmost universal quantifier in the prefix, then we introduce a new constant **a**, and we replace in M all the occurrences of x_r by **a**. $(Q_r x_r)$ is deleted from the prefix.

(2) If $Q_{s1}, \ldots, Q_{sm}, 1 \leq s1 < \ldots < sm < r$, are all the universal quantifiers at the left side of Q_r, then we introduce a new m-place function symbol, **f**, and we replace in M all occurrences of x_r by $f(x_{s1}, \ldots, x_{sm})$. $(Q_r x_r)$ is deleted from the prefix.

To U it corresponds a formula in *Skolem normal form without quantifiers*(***clausal normal form***), denoted by U^{Sq}, which is obtained by deleting the universal quantifiers from U^S.

The constants and functions used to replace the variables existential quantified are called *Skolem constants* and *Skolem functions.* The prefix of the formula U^S contains only universal quantifiers, and the matrix is in conjunctive normal form.

Semidistributivity of $\exists$ over $\rightarrow$:

$$\models ((\exists x)A(x) \rightarrow (\exists x)B(x)) \rightarrow (\exists x)(A(x) \rightarrow B(x))$$

but the formula $(\exists x)(A(x) \rightarrow B(x)) \rightarrow ((\exists x)A(x) \rightarrow (\exists x)B(x))$ is not valid

Semidistributivity of $\forall$ over $\rightarrow$:

$$\models (\forall x)(A(x) \rightarrow B(x)) \rightarrow ((\forall x)A(x) \rightarrow (\forall x)B(x))$$

but the formula $((\forall x)A(x) \rightarrow (\forall x)B(x)) \rightarrow (\forall x)(A(x) \rightarrow B(x))$ is not valid

The above laws are pairs of dual equivalences. $\forall$ and $\exists$ are dual quantifiers.

Example 6.

Prove that the formula $U = (\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\exists x)(A(x) \wedge B(x))$ is not valid.

Let us consider the interpretation *I*=<*D*,m>, where:

D – the set of all straight lines belonging to a plan *P*.

Let $d \in P$, be a constant object (line) from the domain of interpretation.

$m(A)$: $D \rightarrow \{T,F\}$, $m(A)(x)$: " $x \perp d$ ";

$m(B)$: $D \rightarrow \{T,F\}$, $m(B)(x)$: " $x \parallel d$ ";

$$v^I(U) = v^I((\exists x)A(x) \wedge (\exists x)B(x)) \rightarrow v^I((\exists x)(A(x) \wedge B(x))) =$$

$$= v^I((\exists x)A(x)) \wedge v^I((\exists x)B(x)) \rightarrow v^I((\exists x)(A(x) \wedge B(x))) =$$

$$= (\exists x)_{x \in D}(x \perp d) \wedge (\exists x)_{x \in D}(x \parallel d) \rightarrow (\exists x)_{x \in D}(x \perp d \wedge x \parallel d) =$$

$$= T \wedge T \rightarrow F = T \rightarrow F = F.$$

U is evaluated as false under the interpretation *I*, *I* is an anti-model of *U*.

We conclude that *U* is not a valid formula.

Normal forms in first-order logic

The normal forms for predicate formulas are used as input dates in proof methods as: resolution and Herbrand-based procedure.

Definition 7:

1. A predicate formula U is in ***prenex normal form*** if it has the form: $(Q_1 x_1)\ldots(Q_n x_n)$M, where Q_i,i=1,...,n are quantifiers, and M is free of quantifiers. The sequence $(Q_1 x_1)\ldots(Q_n x_n)$ is called the ***prefix*** *of formula* U, and M is called the ***matrix*** *of formula* U.
2. A predicate formula is in ***conjunctive prenex normal form*** if it is in prenex normal form and the matrix is in CNF.

Logical equivalences in predicate logic

1. **Expansion laws**

- the universal quantifier is an infinitary conjunction

$(\forall x)A(x) \equiv (\forall x)A(x) \wedge A(t)$, t is an arbitrary term

- the existential quantifier is an infinitary disjunction

$(\exists x)A(x) \equiv (\exists x)A(x) \vee A(t)$, t is an arbitrary term

2. **DeMorgan infinitary laws**

$\neg(\exists x)A(x) \equiv (\forall x)\neg A(x)$ $\qquad$ $\neg(\forall x)A(x) \equiv (\exists x)\neg A(x)$

3. **Quantifiers interchanging laws**

$(\exists x)(\exists y)A(x,y) \equiv (\exists y)(\exists x)A(x,y)$ $\qquad$ $(\forall x)(\forall y)A(x,y) \equiv (\forall y)(\forall x)A(x,y)$

Remark: $(\exists x)(\forall y)B(x,y) \not\equiv (\forall y)(\exists x)B(x,y)$

4. **The extraction of quantifiers in front of the formula**

$A \vee (\exists x)B(x) \equiv (\exists x)(A \vee B(x))$ $\qquad$ $A \vee (\forall x)B(x) \equiv (\forall x)(A \vee B(x))$

$A \wedge (\exists x)B(x) \equiv (\exists x)(A \wedge B(x))$ $\qquad$ $A \wedge (\forall x)B(x) \equiv (\forall x)(A \wedge B(x))$

where A does not contain $\mathbf{x}$ as a free variable.

$(\exists x)A(x) \vee B \equiv (\exists x)(A(x) \vee B)$ $\qquad$ $(\forall x)A(x) \vee B \equiv (\forall x)(A(x) \vee B)$

$(\exists x)A(x) \wedge B \equiv (\exists x)(A(x) \wedge B)$ $\qquad$ $(\forall x)A(x) \wedge B \equiv (\forall x)(A(x) \wedge B)$

where B does not contain $\mathbf{x}$ as a free variable.

5. **Distributive laws**

$(\exists x)(A(x) \vee B(x)) \equiv (\exists x)A(x) \vee (\exists x)B(x)$ distributivity of $\exists$ over $\vee$

$(\forall x)(A(x) \wedge B(x)) \equiv (\forall x)A(x) \wedge (\forall x)B(x)$ distributivity of $\forall$ over $\wedge$

!! The distribution of $\exists$ over $\wedge$ and the distribution of $\forall$ over $\vee$, do not provide valid distributive laws, but we have semidistributive laws as follows:

Semidistributivity of $\exists$ over $\wedge$:

$\models (\exists x)(A(x) \wedge B(x)) \rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$

but the formula $(\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\exists x)(A(x) \wedge B(x))$ is not valid

Semidistributivity of $\forall$ over $\vee$:

$\models (\forall x)A(x) \vee (\forall x)B(x) \rightarrow (\forall x)(A(x) \vee B(x))$

but the formula $(\forall x)(A(x) \vee B(x)) \rightarrow (\forall x)A(x) \vee (\forall x)B(x))$ is not valid

2. Let us consider the interpretation I_2=< D_2 ,m>, where:

D_2={4,9} – the domain of interpretation;

$m(p)$: {4,9} → {T,F}, $m(p)(x)$: "$x \vdots 2$";

$m(q)$: {4,9} → {T,F}, $m(q)(x)$: "$x \vdots 3$".

To evaluate the formula U under the interpretation I_2, with the finite domain D_2={4,9}, th universally quantified subformulas are replaced by the conjunction of their instances for x= and x=9.

$$v^{I_2}(U) = v^{I_2}\,((\forall x)(p(x)\vee q(x))\,) \rightarrow v^{I_2}\,((\forall x)p(x)\vee(\forall x)q(x)\,) =$$
$$= v^{I_2}\,((\forall x)(p(x)\vee q(x))\,) \rightarrow v^{I_2}\,((\forall x)p(x)\,) \vee v^{I_2}\,((\forall x)q(x)\,) =$$
$$= (4 \vdots 2 \vee 4 \vdots 3) \wedge (9 \vdots 2 \ \vee 9 \vdots 3) \rightarrow (4 \vdots 2 \wedge 9 \vdots 2) \vee (4 \vdots 3 \wedge 9 \vdots 3) =$$
$$= (\mathrm{T} \vee \mathrm{F}) \wedge (\mathrm{F} \vee \mathrm{T}) \rightarrow (\mathrm{T} \wedge \mathrm{F}) \vee (\mathrm{F} \wedge \mathrm{T}) = \mathrm{T} \wedge \mathrm{T} \rightarrow \mathrm{F} \vee \mathrm{F} = \mathrm{T} \rightarrow \mathrm{F} = \mathrm{F}$$

I_2 evaluates the formula U as false, I_2 is an *anti-model* for U and thus U is not a vali formula, it is a contingent one (I_1 - model for U).

Example 5. Evaluate the open formula $U(z)$ under the interpretations I_1 and I_2

$$U(z) = (\exists x)(\exists y)p(f(x,y),z)$$

1. I_1=< D , m_1>, $D = \mathbf{Z}$ (the set of integer numbers),

$m_1(f)(x,y) = (x+y)^2$ and $m_1(p)(x,y)$:"$x = y$".

Because $U(z)$ is an open formula, its evaluation depends on the assignment of values (integers to z, $a \in \mathrm{As}(I_1)$, where:

$$v_a^{I_1}(U(z)) = (\exists x)_{x\in Z}(\exists y)_{y\in Z}\ \text{"}(x+y)^2 = a(z)\text{"} = \begin{cases} \mathrm{T}, \text{ if } a(z) \text{ is a square} \\ \mathrm{F}, \text{otherwise} \end{cases}$$

$U(z)$ is a consistent formula, but it's not a true formula under the interpretation I_1, therefore I_1 is not a model of the formula.

2. $I_2 =< D, m_2 >$, $D = \mathbf{Z}$ (the set of integer numbers),

$m_2(f)(x,y) = x + y$ and $m_2(p)(x,y)$:"$x = y$".

$$v_a^{I_2}(U(z)) = (\exists x)_{x\in Z}(\exists y)_{y\in Z}\text{"}x + y = a(z)\text{"} = \mathrm{T}, \ \forall a(z) \in \mathbf{Z}.$$

The formula $U(z)$ is true under the interpretation I_2, $\models_{I_2} U(z)$, therefore I_2 is a model *of U(z)*..

Definition 6:

- *A formula* A is *satisfiable (consistent)* if there is an interpretation I and an assignment function $a \in As(I)$ such that $v_a^I(A)=T$. Otherwise the formula is called *unsatisfiable (inconsistent)*.
- A formula A *is true under the interpretation* I if for any assignment function $a \in As(I)$, $v_a^I(A)=T$, notation: $\models_I A$, and I is called *model* for A.
- *A formula* A is *valid (tautology)* if A is true under all possible interpretations, notation: $\models A$.
- The *formulas* A and B are *logically equivalent* if $v_a^I(A) = v_a^I(B)$ for any interpretation I and any assignment function a, notation: $A \equiv B$.
- A set of formulas Γ *logically implies* the formula γ if all the models of the set Γ (the models of the conjunction of all formulas from Γ) are also models of the formula γ. We say that γ is a *logical consequence* of the set Γ, notation: $\Gamma \models \gamma$.
- A *set* of formulas is *consistent* if the conjunction of all its formulas has at least one model.
- A *set* of formulas is *inconsistent* if the conjunction of all its formulas does not have a model.

Remark: The evaluation of a closed formula A depends only on the interpretation in which we want to evaluate it, notation: $v^I(A)$.

Example 4:

Build a model and an anti-model for the closed predicate formula:

$$U = (\forall x)(p(x) \vee q(x)) \rightarrow (\forall x)p(x) \vee (\forall x)q(x)$$

1. Let us consider the interpretation $I_1 = \langle D_1, m \rangle$, where:

$D_1 = \mathbf{N}$ (the set of natural numbers)

$m(p): \mathbf{N} \rightarrow \{T,F\}$, $m(p)(x)$: "$x \vdots 2$";

$m(q): \mathbf{N} \rightarrow \{T,F\}$, $m(q)(x)$: "$x \vdots 3$".

$$v^{I_1}(U) = v^{I_1}((\forall x)(p(x) \vee q(x))) \rightarrow v^{I_1}((\forall x)p(x) \vee (\forall x)q(x)) =$$

$$= v^{I_1}((\forall x)(p(x) \vee q(x))) \rightarrow v^{I_1}((\forall x)p(x)) \vee v^{I_1}((\forall x)p(x)) =$$

$$= (\forall x)_{x \in \mathrm{N}}(x \vdots 2 \ \vee \ x \vdots 3) \rightarrow (\forall x)_{x \in \mathrm{N}}(x \vdots 2) \vee (\forall x)_{x \in \mathrm{N}}(x \vdots 3) =$$

$$= F \rightarrow F \vee F = F \rightarrow F = T.$$

$v^{I_1}(U) = T$, U is evaluated as true under the interpretation I_1 which is a *model* for U.

2. The semantics of first-order (predicate) logic

The semantics of predicate logic realize the connection between the constant symbols, function symbols, predicate symbols and the real constants, functions, predicates from the modeled universe. Also it is provided a meaning in terms of the modeled universe for each formula from the language.

Definition 4: An *interpretation* for a language L of predicate logic is a pair I=<*D*,*m*>, where :

- *D* is a nonempty set called the *domain of interpretation.*
- *m* is a function which assigns:
  - a fixed value $m(c) \in D$ to the constant *c*.
  - a function $m(f): D^n \rightarrow D$ to each function symbol f of arity *n*;
  - a predicate $m(P): D^n \rightarrow \{T,F\}$ to each predicate symbol *P* of arity *n*.

Notations: *I*=<*D*,*m*> be an interpretation.

1. $|I|$ is the domain of *I*, *Var* is the set of variables.
2. $I|X|$ is $m(X)$ where X is a predicate symbol or a function symbol.
3. As(*I*) is the set of assignment functions for variables over the domain of interpretation I.

   $a \in As(I)$, $a: Var \text{->} |I|$.
4. $[a]_x = \{a' | \ a' \in As(I)$ and $a'(y)=a(y)$, for every $y \neq x\}$.

Definition 5: Let *I* be an interpretation and $a \in As(I)$. The evaluation function v_a^I is defined inductively as follows:

- $v_a^I(x) = a(x)$, $x \in Var$;
- $v_a^I(c) = I|c|$, $c \in Const$;
- $v_a^I(f(t1,...,tn)) = I|f|(v_a^I(t1),..., v_a^I(tn))$, $f \in F_n$, $n>0$;
- $v_a^I(P(t1,...,tn)) = I|P|(v_a^I(t1),..., v_a^I(tn))$, $P \in P_n$, $n>0$;
- $v_a^I(\neg A) = \neg v_a^I(A)$; $\quad v_a^I(A \wedge B) = v_a^I(A) \wedge v_a^I(B)$
- $v_a^I(A \vee B) = v_a^I(A) \vee v_a^I(B)$; $\quad v_a^I(A \rightarrow B) = v_a^I(A) \rightarrow v_a^I(B)$
- $v_a^I((\exists x)A(x))$=T if and only if $v_{a'}^I(A(x))$=T for a function $a' \in [a]_x$
- $v_a^I((\forall x)A(x))$=T if and only if $v_{a}^I(A(x))$=T for any function $a' \in [a]_x$

Definition 3:

A formula $U \in F_{\Pr}$, such that $\varnothing \vdash U$ (notation: $\vdash U$) is called a **theorem.**

Remark: The theorems are the formulas deducible from the axioms, using modus ponens and the generalization rule.

Example 2: Using the definition of deduction prove that:

$$(\forall y)(\forall z)(p(y) \vee q(z)) \vdash (\forall x)(p(x) \vee q(x))$$

We build the sequence (f1,f2,f3,f4,f5 ,f6) of predicate formulas as follows:

f1: $(\forall y)(\forall z)(p(y) \vee q(z))$ -- - hypothesis

f2: $(\forall y)(\forall z)(p(y) \vee q(z)) \rightarrow (\forall z)(p(x) \vee q(z))$ --- A4 axiom, y instantiated with x

f1,f2 $\vdash_{mp}$ f3 = $(\forall z)(p(x) \vee q(z))$

f4: $(\forall z)(p(y) \vee q(z)) \rightarrow p(x) \vee q(x)$ --- A4 axiom, z instantiated with x

f3,f4 $\vdash_{mp}$ f5 = $p(x) \vee q(x)$

f5 $\vdash_{gen}$ f6= $(\forall x)\,(p(x) \vee q(x))$

(f1,f2,f3,f4,f5,f6) is the deduction (the proof) of $(\forall x)(p(x) \vee q(x))$

from $(\forall y)(\forall z)(p(y) \vee q(z))$.

Example 3: Using the definition of deduction prove that the formula $(\forall x)(p(x) \wedge q(x)) \rightarrow (\forall x)p(x)$ is a theorem.

We build the sequence (f1,f2,f3,f4,f5) of formulas as follows:

f1: $(\forall x)(p(x) \wedge q(x)) \rightarrow p(y) \wedge q(y)$ -- A4 axiom , x instantiated with y

f2: $p(y) \wedge q(y) \rightarrow p(y)$ --- theorem (from propositional logic)

f1,f2 $\vdash_{syllogism\ rule}$ f3= $(\forall x)(p(x) \wedge q(x)) \rightarrow p(y)$

f4: $((\forall x)(p(x) \wedge q(x)) \rightarrow p(y)) \rightarrow ((\forall x)(p(x) \wedge q(x)) \rightarrow (\forall x)p(x))$ --- A5 axiom

f3,f4 $\vdash_{mp}$ f5= $(\forall x)(p(x) \wedge q(x)) \rightarrow (\forall x)p(x)$

(f1,f2,f3,f4,f5) is the deduction (proof) of the theorem.

2. Axioms which define the natural numbers:

a1. Every natural number has a unique immediate successor.

existence: $(\forall x)(\exists y)equal(y, successor(x))$

unicity: $(\forall x)(\forall y)(\forall z)(equal(y, successor(x)) \wedge equal(z, successor(x)) \rightarrow equal(y, z))$

a2. The number 0 *is not* the immediate successor of a natural number.

$\neg(\exists x)egal(0, succesor(x))$

a3. Every natural number, except 0 has a unique immediate successor.

existence: $(\forall x)(\exists y)(\neg equal(0, x) \wedge equal(y, predecessor(x)))$

unicity: $(\forall x)(\forall y)(\forall z)(equal(y, predecessor(x)) \wedge equal(z, predecessor(x)) \rightarrow equal(y, z))$

Unary functions: *successor, predecessor* ; **binary predicate**: *equal*

The predicate "*equal*" is defined by the following axioms:

$(\forall x)\, equal(x, x)$ --- reflexivity

$(\forall x)(\forall y)(equal(x, y) \rightarrow equal(y, x))$ --- symmetry

$(\forall x)(\forall y)(\forall z)(equal(x, y) \wedge equal(y, z) \rightarrow equal(x, z))$ --- transitivity

The following formulas express the equality of successors and predecessors of two equal numbers:

$(\forall x)(\forall y)(equal(x, y) \rightarrow equal(successor(x), successor(y)))$

$(\forall x)(\forall y)(equal(x, y) \rightarrow equal(predecessor(x), predecessor(y)))$

Definition 2: Let U1,U2,..,Un,V be first-order formulas, U1,U2,..,Un are the hypotheses. **V is deducible from U1,...,Un,** notation: U1,..,Un |- V, if there is a sequence of formulas $(f_1, f_2, ..., f_m)$ such that f_m=V and $\forall i \in \{1, ..., m\}$ we have a) or b) or c) or d).

a) $f_i \in A_p$ (axiom of predicate logic);

b) $f_i \in$ {U1,...,Un} (hypothesis formula);

c) $f_{i1}, f_{i2} \vdash_{mp} f_i$, $i1 < i \text{ and } i2 < i$ (formula f_i is inferred, using modus ponens rule, from two formulas that are already in the sequence);

d) $f_j \vdash_{gen} f_i$, $j < i$ (formula f_i is obtained using the generalization rule from a formula that is already in the sequence).

The sequence $(f_1, f_2, ..., f_m)$ is called the ***deduction* of V from U1,U2,..,Un.**

- $A_{Pr} = \{A1, A2, A3, A4, A5\}$ is the set of axioms

  $A1$: $U \to (V \to U)$

  $A2$: $((U \to (V \to Z)) \to ((U \to V) \to (U \to Z))$

  $A3$: $(U \to V) \to (\neg V \to \neg U)$ *(modus tollens)*

  $A4$: *universal instantiation.*

  $(\forall x)U(x) \to U(a)$, where a is a term

  $A5$: $(U \to V(y)) \to (U \to (\forall x)V(x))$, where x is not free in U or V,

  y is free in V and does not appear in U.

- $R_{Pr} = \{mp, gen\}$ is the set of inference rules

  - *modus ponens* symbolized as : $U, U \to V \vdash_{mp} V$ and

  - *generalization rule (universal generalization)* symbolized as:

  $$U(x) \vdash_{gen} (\forall x)U(x)$$, x is a free variable in U.

Remark: $A_P \subset A_{Pr}$, $F_P \subset F_{Pr}$, $R_P \subset R_{Pr}$ and thus $Theorems_P \subset Theorems_{Pr}$.

The theorems in propositional logic are also theorems in predicate logic.

Definition 1:

1. In a predicate formula the variables which are within the scope of a quantifier are called *bound variables*, all the others are called *free variables.*
2. A formula is called a *closed formula* if all its variables are bound.
3. If a formula contains at least one free variable, the *formula is open.*

Example 1:

- The predicate formula $(\forall x)(\exists z)(p(x,z,a) \vee (\exists y)q(x, f(y)))$ is closed (all variables are bound), where: $x, y, z \in Var$, $a \in Const$, $f \in F_1$, $p \in P_3$, $q \in P_2$.
- The predicate formula $(\forall x)p(x,y) \wedge q(z,a)$ is open, the variables y and z are free, x is a bound variable (within the scope of $\forall$), $a \in Const$, $x, y, z \in Var$, $p, q \in P_2$.

Transform from natural language into predicate formulas the following statements:

1. If x and y are nonnegative integers and x is greater than y, then x^2 is greater than y^2.

$$(\forall x)(\forall y)(nonneg(x) \wedge nonneg(y) \wedge greater(x, y) \to greater(square(x), square(y))$$

where: - function symbol: $square \in F_1$, $square(x) = x^2$

- predicate symbols: $nonneg \in P_1$, $nneg(x)$:"$x > 0$" and

$greater \in P_2$, $greater(x, y)$:"$x > y$".

First-order logic (predicate calculus)

1. Axiomatic (formal) system of first-order logic

The axiomatic (deductive) system of predicate logic: $\mathrm{Pr} = (\Sigma_{\mathrm{Pr}}, F_{\mathrm{Pr}}, A_{\mathrm{Pr}}, R_{\mathrm{Pr}})$:

- $\Sigma_{\mathrm{Pr}} = Var \cup Const \cup (\bigcup_{j=1}^{n} F_j) \cup (\bigcup_{j=1}^{m} P_j) \cup Connectives \cup Quantifiers$

  Var - the set of *variable* symbols {*x, y, z,..*}; the variables can take different values in a specific domain *D* and they are generic terms, type definitions: *book, child, event.*

  Const - the set of *constants* {*a, b, c,..*}; the constants take fixed values in a domain *D* and usually specify objects' names, persons' names (Ex: Paul, Book_3).

  $F_i = \{f \mid f : D^i \to D\}$ - the set of function symbols of arity "*i*"

  $P_i = \{p \mid p : D^i \to \{T, F\}\}$ - the set of predicate symbols of arity "*i*", usually connection rules among variables and constants.

  $Connectives = \{\neg, \wedge, \vee, \to, \leftrightarrow\}$;

  $Quantifiers = \{\forall\ (universal\ quantifier), \exists\ (existential\ quantifier)\}$

  !! The priority of quantifiers is greater than the priorities of the connectives.

  - ***TERMS*** - the set of terms defined as follows:
    - $Var \subset TERMS$; $Const \subset TERMS$;
    - if $f \in F_k$ and $t_1,\dots,t_k \in TERMS$ then $f(t_1,\dots,t_k) \in TERMS$

    Ex: *x, a, f(x), g(x,a), g(f(x),y)*
  - ***ATOMS*** - the set of atomic formulas (atoms) defined as follows:
    - $T, F \in ATOMS$
    - if $p \in P_k$ and $t_1,\dots,t_k \in TERMS$ then $p(t_1,\dots,t_k) \in ATOMS$

    Ex: *T,F, p(x,y,a), q(f(x),a), r(a,g(f(x)),y)*
  - ***Literal*** = an atom or its negation.

    Ex: $p(f(x), a, y, g(x,b))$, $\neg q(x, a, g(x))$

- F_{Pr} = the set of well formed formulas defined as follows:

  - $ATOMS \subset F_{\mathrm{Pr}}$;

  -if $U, V \in F_{\mathrm{Pr}}$ then

  $$\neg U \in F_{\mathrm{Pr}}, U \wedge V \in F_{\mathrm{Pr}}, U \vee V \in F_{\mathrm{Pr}}, U \to V \in F_{\mathrm{Pr}},\ U \leftrightarrow V \in F_{\mathrm{Pr}}$$

  -if $U \in F_{\mathrm{Pr}}$, $x \in Var$, and x is not within the scope of a variable then

  $(\forall x)U(x) \in F_{\mathrm{Pr}}$ and $(\exists x)U(x) \in F_{\mathrm{Pr}}$.