26.XI.2015

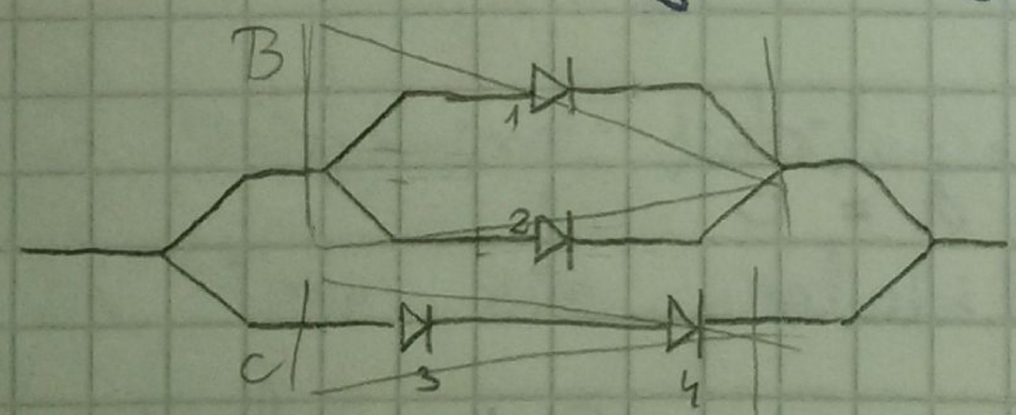
Seminar 5. Variabile aleatoare

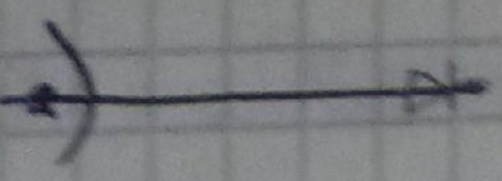
① Un circuit electric este format din 4 diode. Știind că diodele sunt mutual independente și că fiecare diodă funcționează cu probabilitate 0,9, calculați probabilitatea ca circuitul să funcționeze atunci când diodele sunt legate:

a) în serie

b) în paralel

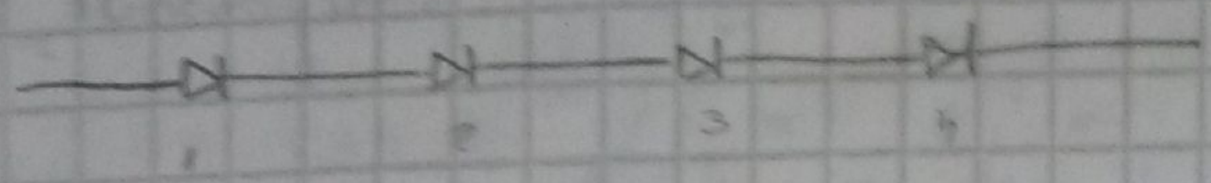
c) conform figurii:



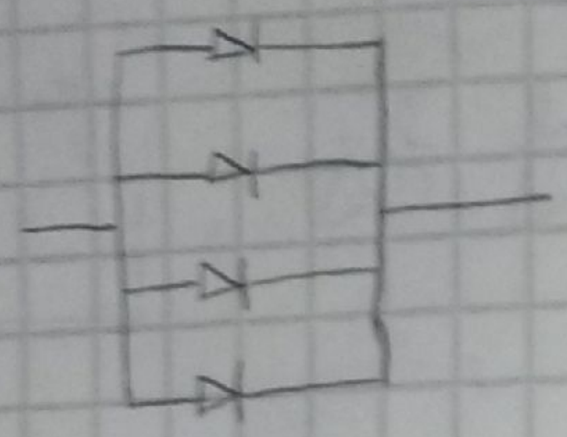


A_i : „dioda i funcționează", $i = \overline{1,4}$

A : "sistemul funcționează"

a) 

$$P(A) = P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4)$$
$$= (0,9)^4$$

b) 

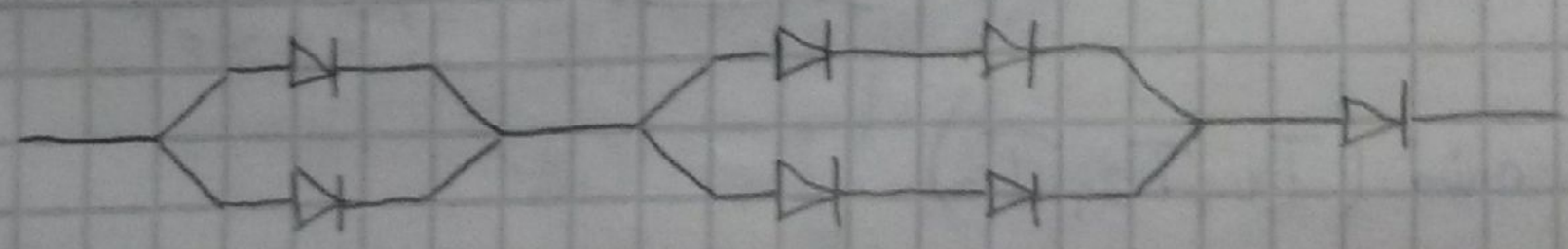
$$P(A) = 1 - P(\bar{A}) = 1 - P(\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \bar{A_4})$$
$$\left(= P(A_1 \cup A_2 \cup A_3 \cup A_4)\right)$$
$$= 1 - (0,1)^4 = 0,9999$$

c) $P(A) = P(B \cup C) = P\left((A_1 \cup A_2) \cup (A_3 \cap A_4)\right)$

$$= P(B) + P(C) - P(B \cap C)$$
$$= P(A_1 \cup A_2) + P(A_3 \cap A_4) - P\left((A_1 \cup A_2) \cap A_3 \cap A_4\right)$$
$$= P(A_1) + P(A_2) - P(A_1 \cap A_2) + P(A_3 \cap A_4) -$$
$$- P(A_1 \cup A_2) \cdot P(A_3) \cdot P(A_4)$$
$$= 2 \cdot 0,9 \cancel{- (0,9)^2} \cancel{+ (0,9)^2} - \left((2 \cdot 0,9 - (0,9)^2) \cdot (0,9)^2\right)$$
$$= 1,8 - \left((1,8 - 0,81) \cdot 0,81\right) = 1,8 - (0,99 \cdot 0,81)$$
$$= 0,9981$$

TEMĂ:

(3p) d) conform figurii



② Un sistem electric funcționează cu 3 baterii. Timpul de funcționare a fiecărei baterii este dat de o v.a. ce urmează legea exponențială cu valoarea medie de 3 ore.

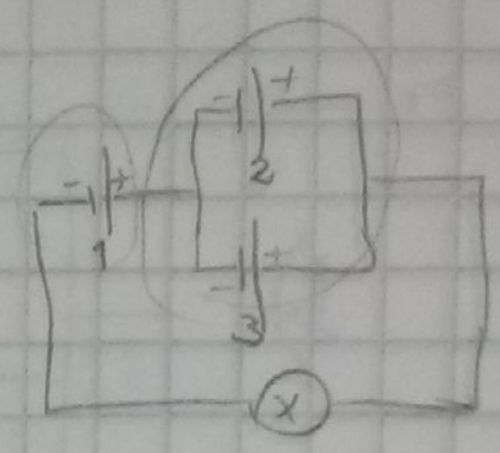
Știind că bateriile sunt mutual independente, calculați:

a) probabilitatea ca sist. să funcționeze mai mult de o oră

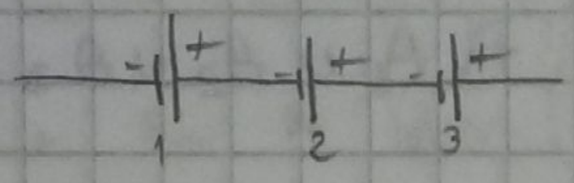
a₁) în serie

a₂) în paralel

a₃) conf. fig.



a₁)



T_i : „v.a. care indică timpul de funcționare a bat. i"

$i = \overline{1,3}$

T : „ ———————— " ———————— al sistemului"

$f_{T_i} : \mathbb{R} \to \mathbb{R}$

$$f_{T_i}(x) = \begin{cases} 0 & , x < 0 \\ \lambda e^{-\lambda x} & , x \geq 0 \end{cases} \quad (\lambda > 0)$$

$$\int_0^x \lambda e^{-\lambda t}\, dt = -\left(e^{-\lambda t}\right)\Big|_0^x = 1 - e^{-\lambda x}$$

$F_{T_i} : \mathbb{R} \to \mathbb{R}$

$$F_{T_i}(x) = \int_{-\infty}^{x} f_{T_i}(t)\, dt = \begin{cases} 0 & , x < 0 \\ 1 - e^{-\lambda x} & , x \geq 0 \end{cases}$$

$$E(T_i) = \int_{-\infty}^{\infty} x \cdot f_{T_i}(x)\, dx = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda} = 3$$

↳ val. medie

$$x \cdot \left(-e^{-\lambda x}\right)\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\, dx$$

$$= \lim_{x \to \infty} \frac{x}{-e^{\lambda x}} + \frac{1}{\lambda}\left(-e^{-\lambda x}\Big|_0^{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{1}{-\lambda e^{\lambda x}} + \left(-e^{-\infty} + 1\right)\frac{1}{\lambda}$$

$$\Rightarrow \boxed{\lambda = \frac{1}{3}}$$

$$T = \min\left(T_1, T_2, T_3\right)$$

$$P(T > 1) = P\left(\min\{T_1, T_2, T_3\} > 1\right) = P\left((T_1 > 1) \cap (T_2 > 1) \cap (T_3 > 1)\right)$$

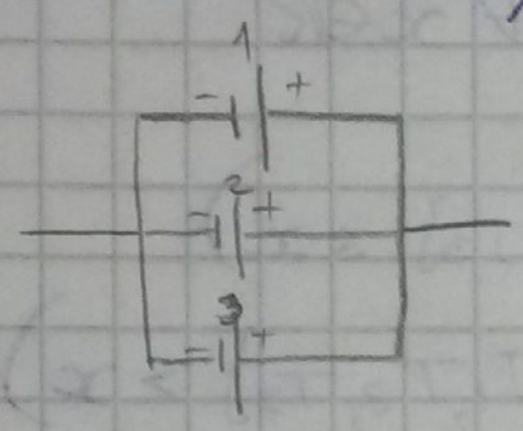
$$= P(T_1>1)\cdot P(T_2>1)\cdot P(T_3>1)$$

$$= \left(1-P(T_1\le 1)\right)\cdot\left(1-P(T_2\le 1)\right)\cdot\left(1-P(T_3\le 1)\right)$$

$$= \left(1-F_{T_1}(1)\right)\cdot\left(1-F_{T_2}(1)\right)\cdot\left(1-F_{T_3}(1)\right)$$

$$= \left(1-\left(1-e^{-\frac{1}{3}}\right)\right)^3 = e^{-1} = \boxed{\frac{1}{e}}$$

a2)



$$P(T>1) = P\left(\max\{T_1,T_2,T_3\}>1\right)$$

$$= 1-P\left(\max\{T_1,T_2,T_3\}\le 1\right)$$

$$= 1-P\left((T_1\le 1)\cap(T_2\le 1)\cap(T_3\le 1)\right)$$

$$= 1-P(T_1\le 1)\cdot P(T_2\le 1)\cdot P(T_3\le 1)$$

independente

$$= 1-F_{T_1}(1)\cdot F_{T_2}(1)\cdot F_{T_3}(1) = 1-\left(1-e^{-\frac{1}{3}}\right)^3$$

a3) $T = \min\{T_1, \max\{T_2,T_3\}\}$

$$P(T>1) = P\left(\min\{T_1, \max\{T_2,T_3\}\}>1\right)$$

$$= P(T_1>1)\cdot P\left(\max\{T_2,T_3\}>1\right)$$

$$= \left(e^{-\frac{1}{3}}\right)\cdot\left(1-P(T_2\le 1)\cdot P(T_3\le 1)\right)$$

$$= \left(e^{-\frac{1}{3}}\right)\left(1-\left(1-e^{-\frac{1}{3}}\right)^2\right)$$

b) valoarea medie a timpului de funcționare al sistemului.

b1) în serie

b2) în paralel

b3) conf. figurii a3.

$$E(T) = \int_{-\infty}^{\infty} x \, f_T(x) \, dx$$

fct. de densitate a lui T

$$F_T'(x) = f_T(x), \quad \forall\, x \in \mathbb{R}, \text{ unde } F_T \text{ este deriv.}$$

$$F_T(x) = P(T \le x) \quad , \forall x \in \mathbb{R}$$

b1) $P(T \le x) = P(\min\{T_1, T_2, T_3\} \le x)$

$$= 1 - P(\min\{T_1, T_2, T_3\} > x)$$

$$= 1 - P((T_1 > x) \cap (T_2 > x) \cap (T_3 > x))$$

$$= 1 - (1 - F_{T_1}(x)) \cdot (1 - F_{T_2}(x)) \cdot (1 - F_{T_3}(x))$$

$$= 1 - \left(e^{-\frac{1}{3}x}\right)^3 = 1 - e^{-x} \quad , x \ge 0$$

$$F_T(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-x} & , x \ge 0 \end{cases}$$

$$f_T(x) = \begin{cases} 0 & , x < 0 \\ e^{-x} & , x \ge 0 \end{cases}$$

$$-x \cdot e^{-x} \Big|_0^{\infty} + \int_0^{\infty} e^{-x} \, dx = \lim_{x \to \infty} \frac{-x}{e^x} + e^{-x} \Big|_0^{\infty}$$

$$= \lim_{x \to \infty} \frac{-1}{e^x} - (0 - 1) = \boxed{1}$$

$$E(T) = \int_0^{\infty} x \cdot e^{-x} \, dx = 1$$

b2) $F_T(x) = P(T \le x) = P(\max\{T_1, T_2, T_3\} \le x)$

$$= P(T_1 \le x) \cdot P(T_2 \le x) \cdot P(T_3 \le x)$$

$$= F_{T_1}(x) \cdot F_{T_2}(x) \cdot F_{T_3}(x)$$

$$F_T(x) = \begin{cases} 0 & , x < 0 \\ \left(1 - e^{-\frac{1}{3}x}\right)^3 & , x \ge 0 \end{cases}$$

$$f_T(x) = \begin{cases} 0 & , x < 0 \\ \cancel{3}\left(1 - e^{-\frac{1}{3}x}\right)^2 \cdot \left(\frac{1}{\cancel{3}} e^{-\frac{1}{3}x}\right) & , x \ge 0 \end{cases}$$

$$E(T) = \int_0^{\infty} x \left(e^{-\frac{1}{3}x} - 2 \cdot e^{-\frac{2}{3}x} + e^{-x} \right) dx$$

$$= 3 \cdot 3 - 3 \cdot \frac{3}{2} + 1 = 5{,}5$$

b3) $P(T \le x) = P(\min\{T_1, \max\{T_2, T_3\}\} \le x)$

$$= 1 - P(\min\{T_1, \max\{T_2, T_3\}\} > x)$$

$$= 1 - P(T_1 > x) \cdot P(\max\{T_2, T_3\} > x)$$

$$= 1 - (1 - P(T_1 \le x)) \cdot (1 - P(T_2 \le x) \cdot P(T_3 \le x$$

$$= 1 - \left(e^{-\frac{1}{3}x}\right)\left(1 - \left(1 - e^{-\frac{1}{3}x}\right)^2\right)$$

$$= 1 - e^{-\frac{1}{3}x}\left(2e^{-\frac{1}{3}x} - e^{-\frac{2}{3}x}\right)$$

$$= 1 - 2e^{-\frac{2}{3}x} + e^{-x}, \quad x \ge 0$$

$$f_T(x) = \begin{cases} 0, & x < 0 \\ \frac{4}{3} e^{-\frac{2}{3}x} - e^{-x}, & x \ge 0 \end{cases}$$

$$E(T) = \int_0^{\infty} x \left(\frac{4}{3} e^{-\frac{2}{3}x} - e^{-x} \right) dx$$

$$= \cancel{2} \cdot \frac{3}{\cancel{2}} - 1 = 2$$

c) valoarea lui $h > 0$ a.î. probab. ca sist să funcț. mai cel puțin h ore să fie 83,1% atunci când bateriile sunt legate

c_1) în serie

c_2) în paralel

c_3) conf fig a3

c1) $P(T \geq h) = \frac{89,1}{100}$

$\Leftrightarrow 1 - \underbrace{P(T < h)}_{P(T \leq h) = F_T(h)} = 0,891$

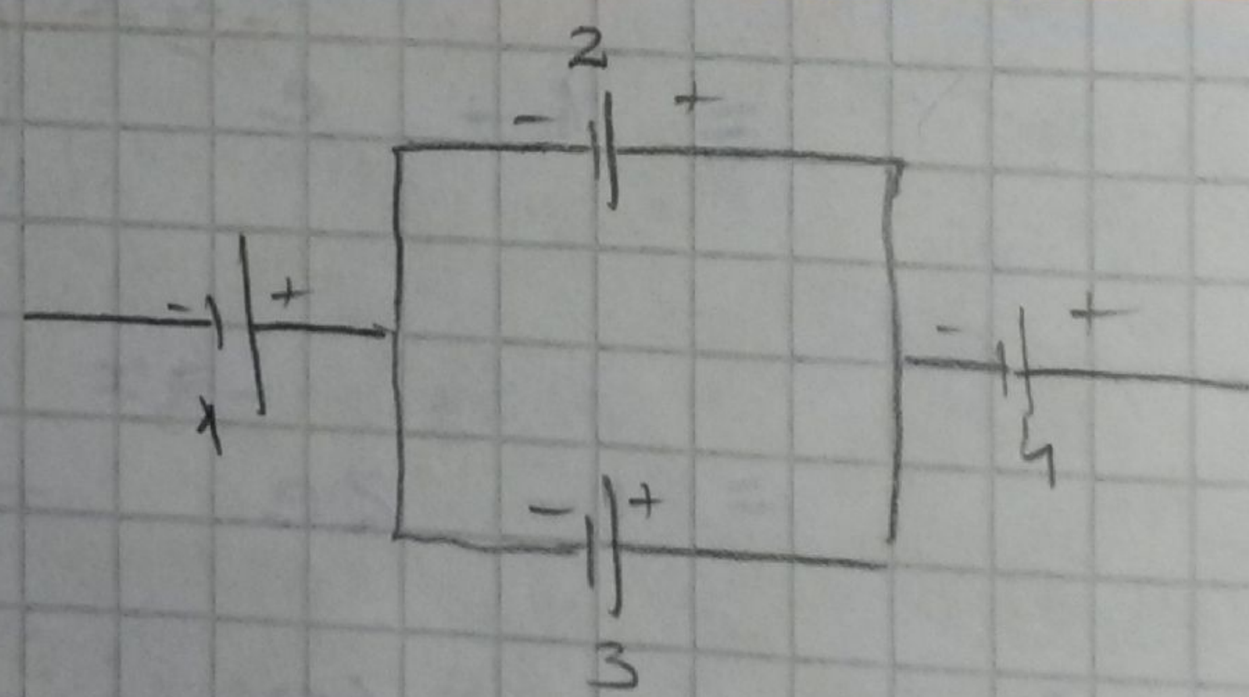
$\Leftrightarrow 1 - (1 - e^{-h}) = 0,891$

$\Rightarrow e^{-h} = 0,891 \Rightarrow h = -\ln 0,891$

$= \boxed{\ln \frac{100}{89,1}}$

TEMĀ:

C2)

C3)

a) ?i b) pt :