Optimization Methods

Bolei Zhang

November 1, 2024

1 Convex Sets

Problem 1: The solution set of linear equations is affine set.

Solution: Let $C = \{x|Ax = b\}$ be the solution set of linear equations, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Suppose $x_1, x_2 \in C, i.e., Ax_1 = b, Ax_2 = b$. Then for any θ, we have

$$\begin{aligned} A(\theta x_1 + (1-\theta)x_2) &= \theta Ax_1 + (1-\theta)Ax_2 \\ &= \theta b + (1-\theta)b \\ &= b \end{aligned} \tag{1}$$

which shows that the affine combination $\theta x_1 + (1-\theta)x_2$ is also in C.

Problem 2: The set of symmetric semi-positive definite matrices is a convex cone.

Solution: Let $S^n_+ = \{X \in S^n | X \succeq 0\}$ be the set of symmetric semi-positive definite matrices. For any two points $X_1, X_2 \in S^n_+$, let $X = \theta_1 X_1 + \theta_2 X_2$, where $\theta_1 \geq 0, \theta_2 \geq 0$. Then, for any non-zero vector v, there is

$$\begin{aligned} v^T X v &= v^T(\theta_1 X_1 + \theta_2 X_2)v \\ &= \theta_1 v^t X_1 v + \theta_2 v^T X_2 v \\ &\geq 0 \end{aligned} \tag{2}$$

Therefore, S^n_+ is a convex cone.

Problem 3: The ball $B(x_c, r)$ is a convex set.

Proof: For any two points $x_1, x_2 \in B(x_c, r)$. Let $x = \theta x_1 + (1-\theta)x_2$, where $0 \leq \theta \leq 1$. We have

$$\begin{aligned} ||x - x_c||_2 &= ||\theta x_1 + (1-\theta)x_2 - x_c||_2 \\ &= ||\theta(x_1 - x_c) + (1-\theta)(x_2 - x_c)||_2 \\ &\leq \theta||x_1 - x_c||_2 + (1-\theta)||x_2 - x_c||_2 \quad \leq r \end{aligned} \tag{3}$$

Theorem 1: The intersection of any number of convex sets is a convex set.

Proof: Let $A = A_1 \cap A_2 \cap ... \cap A_k$, where $A_i, i = 1, ..., k$ is convex set. For any two points $x_1, x_2 \in A$. Let $x = \theta x_1 + (1-\theta)x_2$, where $0 \leq \theta \leq 1$. We have $x \in A_i, i = 1, ..., k$. Therefore, there is $x \in A$.

Problem 4: The affine operation of convex set is also convex set.

Solution: Recall that a function $f : \mathbb{R}_n \to \mathbb{R}_m$ is affine if it is a sum of a linear function and a constant, i.e., if it has the form $f(x) = Ax + b$, where $A \in \mathbb{R}_{m\times n}$ and $b \in \mathbb{R}_m$. Suppose $S \subseteq R_n$ is convex and $f : \mathbb{R}_n \to \mathbb{R}_m$ is an affine function. Then the image of S under f is

$$f(S) = \{f(x)|x \in S\}.$$

Consider any two points $f(x), f(y)$ in $f(S)$, with their origin points $x, y \in S$. With $0 \leq \theta \leq 1$, there is $f(\theta x + (1-\theta)y) = A\theta x + A(1-\theta)y + b = \theta(Ax + b) + (1-\theta)(Ay + b) = \theta f(x) + (1-\theta)f(y)$. Therefore, the line segment of $f(x)f(y)$ is in $f(S)$, which proves the statement.

Problem 5: Show that a set is convex if and only if its intersection with any line is convex.

Solution: The intersection of two convex sets is convex. Therefore, if S is a convex set, the intersection of S with a line is convex. Conversely, suppose the intersection of S with any line is convex. Take any two distinct points x_1 and $x_2 \in S$. The intersection of S with the line through x_1 and x_2 is convex. Therefore, convex combinations of x_1 and x_2 belong to the intersection, hence also to S.

Problem 6: Show that polyhedrons are convex sets

Solution: A polyhedron is defined as the solution set of a finite number of linear equalities and inequalities:

$$P = \{x | a_j^T x \leq b_j, j = 1, ..., m, c_j^T x = d_j, j = 1, ..., p\}.$$

For any two points $x, y \in P$, $j = 1, ..., m$, $0 \leq \theta \leq 1$, there is:

$$a_j^T(\theta x + (1-\theta)y) = \theta a_j^T x + (1-\theta) a_j^T y \leq \theta b_j + (1-\theta) b_j = b_j, \tag{4}$$

and for $j = 1, ..., p$, there is:

$$c_j^T(\theta x + (1-\theta)y) = \theta c_j^T x + (1-\theta) c_j^T y = \theta d_j + (1-\theta) d_j = d_j \tag{5}$$

Therefore, $\theta x + (1-\theta)y$ is also a point in P, showing that P is a convex set.

Problem 7: Determine whether the following sets are convex sets, polyhedra, and give a proof
(1) $\{x \in R^n | \alpha \leq a^T x \leq \beta\}$
(2) $\{x \in R^n | \alpha_1^T x \leq b_1, a_2^T \leq b_2\}$.

Solution: (1) This is an intersection of two halfspaces, hence it is a convex set (and a polyhedron).

(2) This is called a rectangle, which convex set and a polyhedron because it is a finite intersection of halfspaces.

Problem 8: Show that the affine transformation of a convex set is a convex set based on its definition.

Solution: Let S be a convex set in $\mathbb{R}^n$, and $A : \mathbb{R}^n \to \mathbb{R}^m$ be an affine transformation: $A(x) = Bx + c$, for some matrix $B \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^m$.

For any two points $u, v \in A(S)$, the points can be expressed as $u = Bx_1 + c, v = Bx_2 + c$.the line segment connecting u, v can be expressed as $w = tu + (1-t)v = t(Bx_1 + c) + (1-t)(Bx_2 + c) = B(tx_1 + (1-t)x_2) + c$. As S is convex, $tx_1 + (1-t)x_2 \in S$. Therefore, w can be expressed as $A(tx_1 + (1-t)x_2) \in S$.

Thus, the affine transformation of a convex set is also a convex set.

Problem 9: Prove that if S_1 and S_2 are convex sets in $\mathbb{R}^m \times \mathbb{R}^n$, then their Minkowski sum is also a convex set. The Minkowski sum is defined as:

$$S = \{(x, y_1 + y_2 | x \in \mathbb{R}^m, y_1 \in S_1, y_2 \in S_2)\}$$

Solution: Consider two points $(\bar{x}, \bar{y}_1 + \bar{y}_2)$,$(\tilde{x}, \tilde{y}_1 + \tilde{y}_2)$, where $(\bar{x}, \bar{y}_1) \in S_1, (\bar{x}, \bar{y}_2) \in S_2, (\tilde{x}, \tilde{y}_1) \in S_1, (\tilde{x}, \tilde{y}_2) \in S_2$

For any $0 \leq \theta \leq 1$, there is

$$\theta(\bar{x}, \bar{y}_1 + \bar{y}_2), + (1-\theta)(\tilde{x}, \tilde{y}_1 + \tilde{y}_2) = (\theta\bar{x} + (1-\theta)\tilde{x}, (\theta\bar{y}_1 + (1-\theta)\tilde{y}_1) + (\theta\bar{y}_2 + (1-\theta)\tilde{y}_2))$$

As S_1, S_2 are convex$(\theta\bar{x} + (1-\theta)\tilde{x}, (\theta\bar{y}_1 + (1-\theta)\tilde{y}_1)) \in S_1, (\theta\bar{x} + (1-\theta)\tilde{x}, (\theta\bar{y}_2 + (1-\theta)\tilde{y}_2)) \in S_2$. Therefore, their convex combination belongs to S, and S is convex.