Optimization Methods - Convex Optimzation Problems

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1 Solutions

Theorem 1: Any locally optimal point is also (globally) optimal in the convex optimization problem.

Proof: Suppose that x is locally optimal for a convex optimization problem, i.e., x is feasible and

$$f_0(x) = \inf f_0(z)|z\text{feasible}, ||z-x||_2 \leq R,$$

for some $R > 0$. Now suppose that x is not globally optimal, i.e., there is a feasible y such that $f_0(y) < f_0(x)$. Evidently $||y-x||_2 > R$, since otherwise $f_0(x) \leq f_0(y)$. Consider the point z given by

$$z = (1-\theta)x + \theta y, \theta = \frac{R}{2||y-x||_2}.$$

Then we have $||z-x||_2 = R/2 < R$, and by convexity of the feasible set, z is feasible. By convexity of f0 we have

$$f_0(z) \leq (1-\theta)f_0(x) + \theta f_0(y) < f_0(x),$$

which is contradiction. Hence, there exists no feasible y with $f_0(y) < f_0(x)$, i.e., x is globally optimal.

Problem 1: Consider the following optimization problem:

$$\begin{aligned} \min \quad & f_0(x_1, x_2) \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \quad x_1 \geq 0, x_2 \geq 0, \end{aligned} \tag{1}$$

Get the feasible set of the above problem. And get the optimal solution set and optimal value w.r.t. different objective functions.
(1) $f_0(x_1, x_2) = x_1 + x_2$;
(2) $f_0(x_1, x_2) = -x_1 - x_2$;
(3) $f_0(x_1, x_2) = x_1$;
(4) $f_0(x_1, x_2) = \max\{x_1, x_2\}$;
(5) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$;

Solution: The feasible set is $\{(x_1, x_2)|2x_1 + x_2 \geq 1, x_1 + 3x_2 \geq 1, x_1 \geq 0, x_2 \geq 0\}$.

(1) This is a linear programming problem, the optimal solution is at one of the vertices of the feasible set. The optimal solution is $(2/5, 1/5)$. The optimal value is $3/5$;

(2) The optimal solution is (∞, ∞). The optimal value is $-\infty$;

(3) The optimal solution set is $\{(x_1, x_2)|x_1 = 0, x_2 \geq 1\}$. The optimal value is 0;

(4) When $x_1 \geq x_2$, the optimal value is the intersection of $x_1 = x_2$ and $2x_1 + x_2 = 1$, which is $(1/3, 1/3)$, the optimal value is $1/3$; When $x_1 \leq x_2$, the optimal value and optimal solution is the same.

(5) As this is a quadratic programming with linear constraints, the optimal solution must be at the border. When $x_2 = 0$, the optimal value is 1. When $x_1 = 0$, the optimal value is 9. When $x_1 + 3x_2 = 1$, the optimal value is $1/2$. When $2x_1 + x_2 = 1$, the optimal solution does not satisfy the constraint. Therefore, the optimal solution set is $(1/2, 1/6)$. The optimal value is $1/2$.

Problem 2: The Traveling Salesman Problem: A traveling salesman wants to start from home and visit the other $(n-1)$ cities at the lowest cost and finally return home. Denote the traveling cost from city i to city j as d_{ij}, and use x_{ij} to represent whether he travels from city i to city j. Please find the path with the minimum cost such that the traveling salesman visits each city exactly once.

Solution: (1) MTZ formulation

Use dummy variable u_i to represent the visiting order of each city. Count from city $1 u_i < u_j$ indicates that city i is visited before city$j$.

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_{ij} x_{ij} \\
\text{s.t.} \quad & x_{ij} \in \{0,1\} \\
& \sum_{j=1, j\neq i}^{n} x_{ij} = 1, i = 1, ..., n \\
& \sum_{i=1, i\neq j}^{n} x_{ij} = 1, j = 1, ..., n \\
& u_i - u_j + 1 \leq (n-1)(1 - x_{ij}), i \neq j, 2 \leq i \leq n, 2 \leq j \leq n. \\
& 2 \leq u_i \leq n, 2 \leq i \leq n
\end{aligned}$$

(2) DFJ formulation

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} d_{ij} x_{ij} \\
\text{s.t.} \quad & x_{ij} \in \{0,1\} \\
& \sum_{j=1, j\neq i}^{n} x_{ij} = 1, i = 1, ..., n \\
& \sum_{i=1, i\neq j}^{n} x_{ij} = 1, j = 1, ..., n \\
& \sum_{i\in Q} \sum_{j\neq i, j\in Q} x_{ij} < |Q| - 1, \forall Q \subset \{1, ..., n\}, |Q| \geq 2
\end{aligned}$$

Problem 3: The Max Flow Problem: Given a directed connected graph $G = (V, E)$, the non-negative number c_{ij} on each edge (v_i, v_j) of G is called the capacity of the edge. For any edge (v_i, v_j) in G, there is a flow f_{ij}, and the flow cannot exceed the capacity of the edge. Find the maximum flow from the source node s to the target node t. Except for the source node and the target node, the inflow of each vertex is equal to the outflow.

Solution: Denote the set of incoming edges to vertex v as $E^-(v)$ and the set of outgoing edges from vertex v as $E^+(v)$.

$$\begin{aligned}
\max \quad & \sum_{(s,v)\in E^+(s)} f_{sv}, \\
\text{s.t.} \quad & 0 \leq f_{ij} \leq c_{ij}, \forall (v_i, v_j) \in E, \\
& \sum_{(u,v)\in E^-(v)} f_{uv} = \sum_{(v,w)\in E^+(v)} f_{vw}, \forall v \in V / \ \{s, t\}
\end{aligned}$$