# Optimization Methods - Combinatorics Optimization

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## 1 Combinatorics Optimization

**Lemma 1:** Show that LP-relaxation method has 2-approximation ratio in weighted vertex cover.

**Proof:** Let $OPT$ be the optimal value of the origin problem (ILP problem), $OPT_{LP}$ be the optimal value of the LP problem. $\bar{x}^*$ is the optimal solution of the LP problem. $\bar{S}^* = \{i \in V : \bar{x}_i^* \geq \frac{1}{2}\}$ is the set chosen by LP. There is:

$$OPT = \sum_{i \in S^*} w_i \geq \sum_{i \in \bar{S}^*} \bar{x}_i^* w_i \geq \frac{1}{2} \sum_{i \in \bar{S}^*} w_i = \frac{1}{2} OPT_{LP} \tag{1}$$

**Problem 1:** Get the dual problem of the following linear optimization problem:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{2}$$

**Solution:** The Lagrangian function is

$$L(x, \lambda_1, \lambda_2) = -c^T x + \lambda_1^T (Ax - b) + \lambda_2^T (-x), \tag{3}$$

The dual function is

$$\begin{aligned} g(\lambda_1, \lambda_2) &= \inf_x \{-c^T x + \lambda_1^T (Ax - b) + \lambda_2^T (-x)\} \\ &= \inf_x \{(-c^T + A\lambda_1 - \lambda_2) x - \lambda_1^T b\} \end{aligned} \tag{4}$$

To solve the dual function, we need that $c^T + A\lambda_1 - \lambda_2 = 0$, i.e., $\lambda_2 = -c^T + A\lambda_1$. The dual problem can be written as:

$$\begin{aligned} \max \quad & -\lambda_1^T x \\ \text{s.t.} \quad & \lambda_1 \geq 0 \\ & -c^T + A\lambda_1 \geq 0 \end{aligned} \tag{5}$$