Optimization Methods - Combinatorics Optimization

Bolei Zhang

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1 Combinatorics Optimization

Lemma 1: Show that LP-relaxation method has 2-approximation ratio in weighted vertex cover.

Proof: Let OPT be the optimal value of the origin problem (ILP problem), OPT_{LP} be the optimal value of the LP problem. $\bar{x}^*$ is the optimal solution of the LP problem. $\bar{S}^* = \{i \in V : \bar{x}_i^* \geq \frac{1}{2}\}$ is the set chosen by LP. There is:

$$OPT = \sum_{i \in S^*} w_i \geq \sum_{i \in \bar{S}^*} \bar{x}_i^* w_i \geq \frac{1}{2} \sum_{i \in \bar{S}^*} w_i = \frac{1}{2} OPT_{LP} \tag{1}$$

Problem 1: Get the dual problem of the following linear optimization problem:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{2}$$

Solution: The Lagrangian function is

$$L(x, \lambda_1, \lambda_2) = -c^T x + \lambda_1^T (Ax - b) + \lambda_2^T (-x), \tag{3}$$

The dual function is

$$\begin{aligned} g(\lambda_1, \lambda_2) &= \inf_x \{-c^T x + \lambda_1^T (Ax - b) + \lambda_2^T (-x)\} \\ &= \inf_x \{(-c^T + A\lambda_1 - \lambda_2)x - \lambda_1^T b\} \end{aligned} \tag{4}$$

To solve the dual function, we need that $c^T + A\lambda_1 - \lambda_2 = 0$, i.e., $\lambda_2 = -c^T + A\lambda_1$. The dual problem can be written as:

$$\begin{aligned} \max \quad & -\lambda_1^T x \\ \text{s.t.} \quad & \lambda_1 \geq 0 \\ & -c^T + A\lambda_1 \geq 0 \end{aligned} \tag{5}$$

Problem 2: Given a directed graph $G = (V, E)$, where V is the set of vertices and E is the set of edges, each edge $(i, j) \in E$ is associated with a non-negative capacity c_{ij}. The vertex set V is divided into two disjoint subsets S and T such that:

$$S \cup T = V, \quad S \cap T = \emptyset,$$

where the source vertex $s \in S$ and the sink vertex $t \in T$.

The minimum cut problem is to find a partition of V into S and T such that the total capacity of edges from S to T is minimized. The set of edges from S to T is called the **minimum cut**.

Solution: Define a binary variable x_{uv} for each edge $(u, v) \in E$:

$$x_{uv} = \begin{cases} 1 & \text{if edge } (u, v) \text{ is part of the cut,} \\ 0 & \text{otherwise.} \end{cases}$$

The LP formulation is:

$$\min \sum_{(u,v)\in E} c_{uv}x_{uv}$$

subject to:

$$x_{uv} \geq y_u - y_v, \quad \forall (u,v) \in E, u \neq s, v \neq t,$$
$$x_{xv} + y_v \geq 1, \forall (s,v) \in E, v \neq t,$$
$$x_{ut} - y_u \geq 0, \forall (u,t) \in E, u \neq s,$$
$$x_{st} \geq 1,$$
$$x_{uv} \in \{0,1\}, \forall (u,v) \in E,$$

where $y_i, i \in V/\ \{s,t\}$ indicates whether a vertex i belongs to S $(y_i = 1)$ or T $(y_i = 0)$.