Optimization Methods - Convex Optimzation

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November 25, 2022

1 Solutions

Theorem 1: Any locally optimal point is also (globally) optimal in the convex optimization problem.

Proof: Suppose that x is locally optimal for a convex optimization problem, i.e., x is feasible and

$$f_0(x) = \inf f_0(z)|z\text{feasible}, ||z-x||_2 \le R,$$

for some $R > 0$. Now suppose that x is not globally optimal, i.e., there is a feasible y such that $f_0(y) < f_0(x)$. Evidently $||y-x||_2 > R$, since otherwise $f_0(x) \le f_0(y)$. Consider the point z given by

$$z = (1-\theta)x + \theta y, \theta = \frac{R}{2||y-x||_2}$$

Then we have $||z-x||_2 = R/2 < R$, and by convexity of the feasible set, z is feasible. By convexity of f0 we have

$$f_0(z) \le (1-\theta)f_0(x) + \theta f_0(y) < f_0(x),$$

which is contradiction. Hence there exists no feasible y with $f_0(y) < f_0(x)$, i.e., x is globally optimal.

Problem 1: Consider the following optimization problem:

$$\begin{aligned} \min \quad & f_0(x_1, x_2) \\ s.t. \quad & 2x_1 + x_2 \ge 1 \\ & x_1 + 3x_2 \ge 1 \quad x_1 \ge 0, x_2 \ge 0, \end{aligned} \tag{1}$$

Get the feasible set of the above problem. And get the optimal solution set and optimal value w.r.t. different objective functions.
(1) $f_0(x_1, x_2) = x_1 + x_2$;
(2) $f_0(x_1, x_2) = -x_1 - x_2$;
(3) $f_0(x_1, x_2) = x_1$;
(4) $f_0(x_1, x_2) = \max\{x_1, x_2\}$;
(5) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$;

Solution: The feasible set is $\{(x_1, x_2)|2x_1 + x_2 \ge 1, x_1 + 3x_2 \ge 1, x_1 \ge 0, x_2 \ge 0\}$.

(1) This is a linear programming problem, the optimal solution is at one of the vertices of the feasible set. The optimal solution is $(2/5, 1/5)$. The optimal value is $3/5$;

(2) The optimal solution is (∞, ∞). The optimal value is $-\infty$;

(3) The optimal solution set is $\{(x_1, x_2)|x_1 = 0, x_2 \ge 1\}$. The optimal value is 0;

(4) When $x_1 \ge x_2$, the optimal value is the intersection of $x_1 = x_2$ and $2x_1 + x_2 = 1$, which is $(1/3, 1/3)$, the optimal value is $1/3$; When $x_1 \le x_2$, the optimal value and optimal solution is the same.

(5) As this is a quadratic programming with linear constraints, the optimal solution must be at the border. When $x_2 = 0$, the optimal value is 1. When $x_1 = 0$, the optimal value is 9. When $x_1 + 3x_2 = 1$, the optimal value is $1/2$. When $2x_1 + x_2 = 1$, the optimal solution does not satisfy the constraint. Therefore, the optimal solution set is $(1/2, 1/6)$. The optimal value is $1/2$.