CS231a PSET 2 Review

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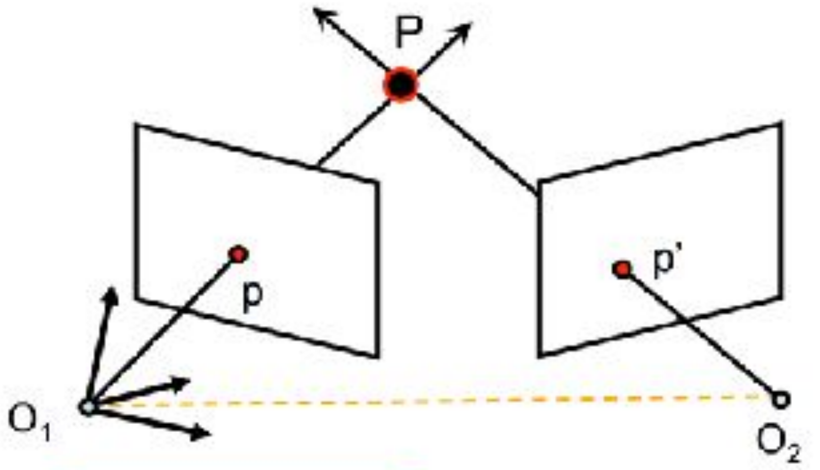
Overview

1. Problem 1 - Fundamental Matrix Estimation From Point Correspondences
2. Problem 2 - Matching Homographies for Image Rectification
3. Problem 3 - Factorization Method
4. Problem 4 - Structure from Motion

Problem 1- Fundamental Matrix Estimation

Fundamental Matrix

A matrix which maps the relationship of correspondences between stereo images



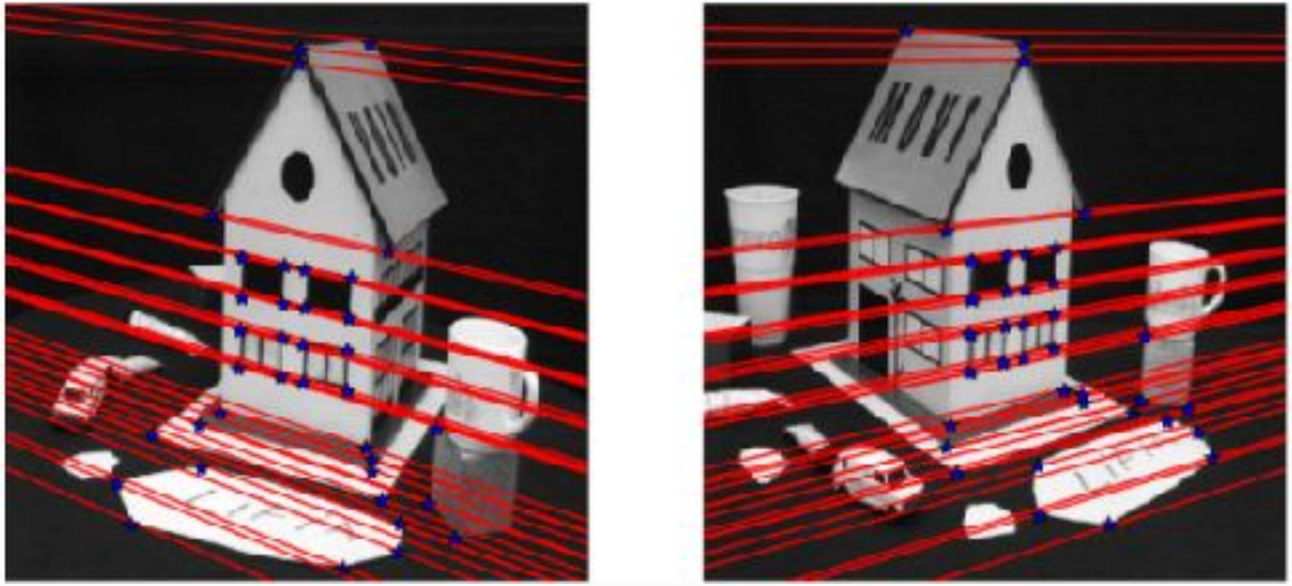
[Eq. 13] $\boxed{p^T F p' = 0}$

$$F = K^{-T} \cdot [T_\times] \cdot R K'^{-1}$$

Problem 1- Fundamental Matrix Estimation

Ex)

Image and correspondences given in the homework



Problem 1- Fundamental Matrix Estimation

How to compute F?

Eight point algorithm

Note: in our pset, please review the documentation of the method carefully. Some may define F as p^TFp' or p'^TFp.

[Eq. 13] $$\mathrm{p}^{\mathrm{T}}\,\mathrm{F}\,\mathrm{p}' = 0 \quad \Rightarrow \qquad p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

$$(u, v, 1)\begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad (uu', uv', u, vu', vv', v, u', v', 1)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

[Eq. 14]

Let's take 8 corresponding points

Problem 1- Fundamental Matrix Estimation

How to compute F?

Eight point algorithm

Problem?

W is highly unbalanced (not well conditioned)

Estimating F

$$\mathbf{W}\ \begin{pmatrix} u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix}}_{\mathbf{f}} = 0$$

[Eqs. 15]

- Homogeneous system $\mathbf{W}\mathbf{f} = 0$
- Rank 8 $\longrightarrow$ A non-zero solution exists (unique)
- If N>8 $\longrightarrow$ Lsq. solution by SVD! $\longrightarrow$ $\hat{\mathrm{F}}$

$\|\mathbf{f}\| = 1$

Problem 1- Fundamental Matrix Estimation

Final step

Reduce rank(F) to 2

$$F = U\begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}V^T$$

Where:

$$U\begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix}V^T = SVD(\hat{F})$$

[HZ] pag 281, chapter 11, "Computation of F"

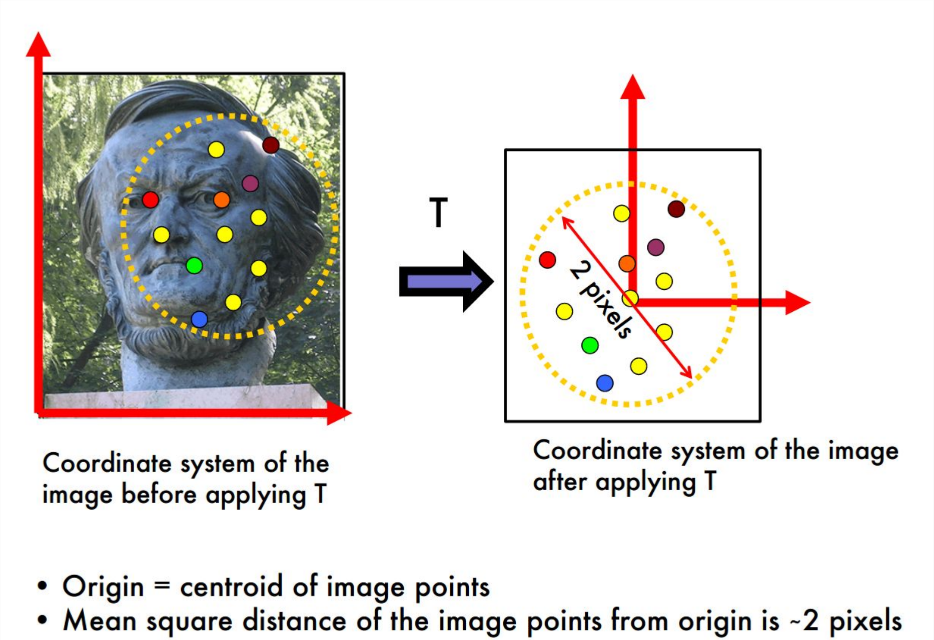
Problem 1- Fundamental Matrix Estimation

Possible improvement?

Pre-condition our linear system to get more stable result

origin = centroid of the points

mean square distance of the image points from origin is ~2px



Problem 1- Fundamental Matrix Estimation

The Normalized Eight-Point Algorithm

0. Compute T and T′ for image 1 and 2, respectively
1. Normalize coordinates in images 1 and 2:

$$q_i = T\ p_i \qquad q_i' = T'\ p_i'$$

2. Use the eight-point algorithm to compute $\hat{F}_q$ from the corresponding points q_i and q_i' .

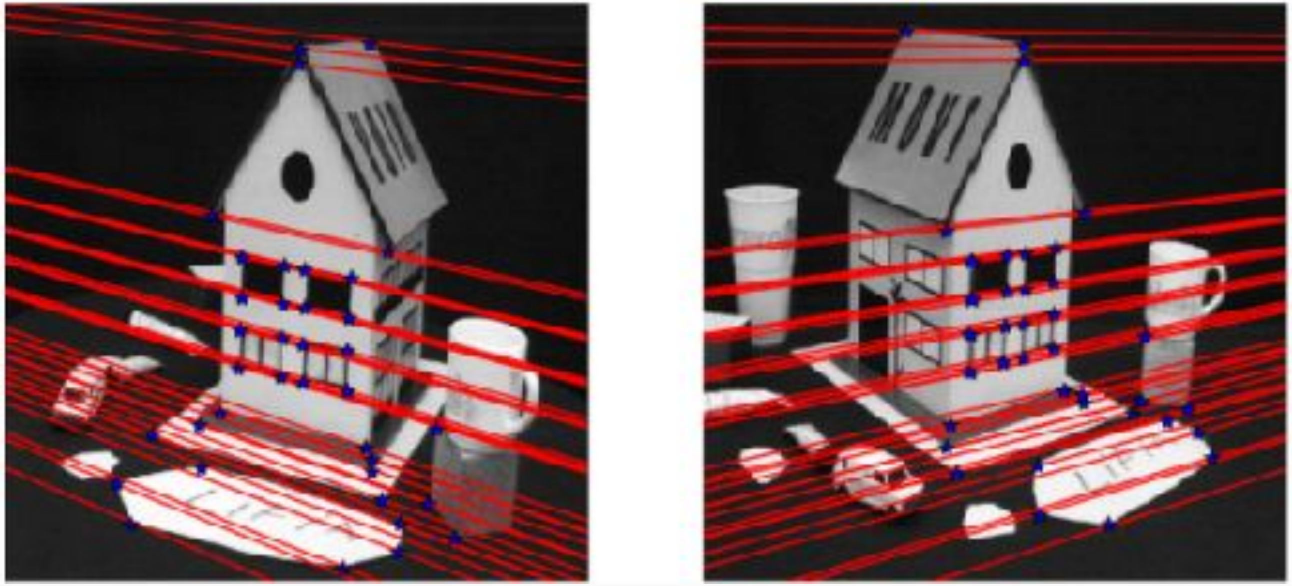
1. Enforce the rank-2 constraint: $\rightarrow \mathrm{F_q}$ such that:

$$\begin{cases} \mathrm{q^T\ F_q\ q' = 0} \\ \det(\mathrm{F_q}) = 0 \end{cases}$$

2. De-normalize F_q: $F = T'^T F_q T$

Problem 1- Fundamental Matrix Estimation

Epipolar lines



Problem 1- Fundamental Matrix Estimation

Distance to epipolar lines

Computing epipolar lines from F

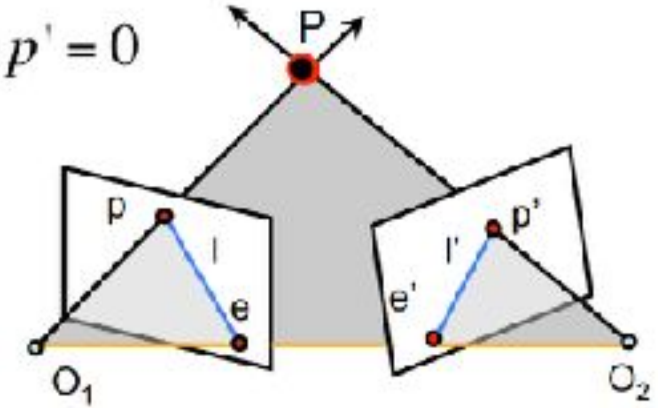
l = Fp’

l’ = F^Tp

$$\mathrm{distance}(ax + by + c = 0, (x_0, y_0)) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

Epipolar Constraint

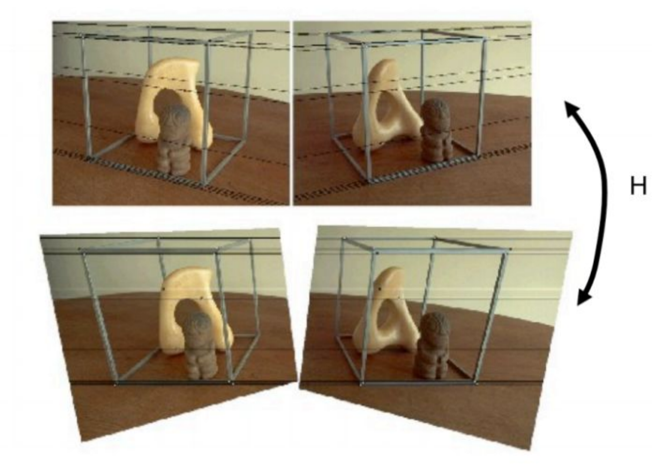
$$p^T \cdot F\, p' = 0$$



- $l = F\,p'$ is the epipolar line associated with p'
- $l' = F^T p$ is the epipolar line associated with p

Problem 2 - Image Retification

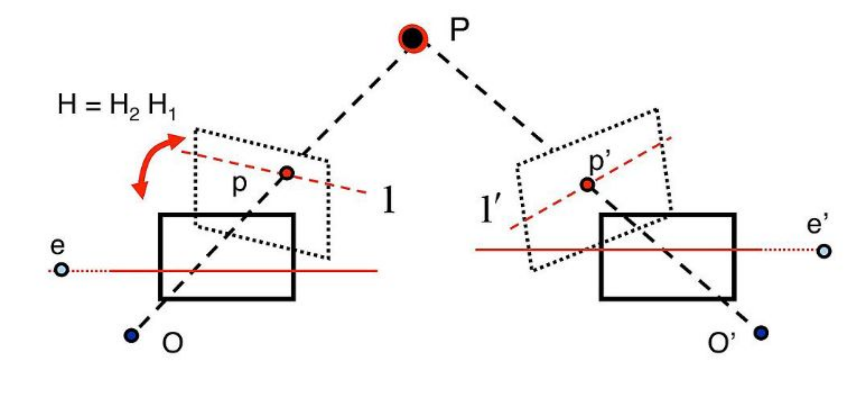
Make two images parallel to each other
⇒ epipole at infinity along the horizontal axis



Problem 2 - Image Retification

Make two images parallel to each other
⇒ epipole at infinity along the horizontal axis

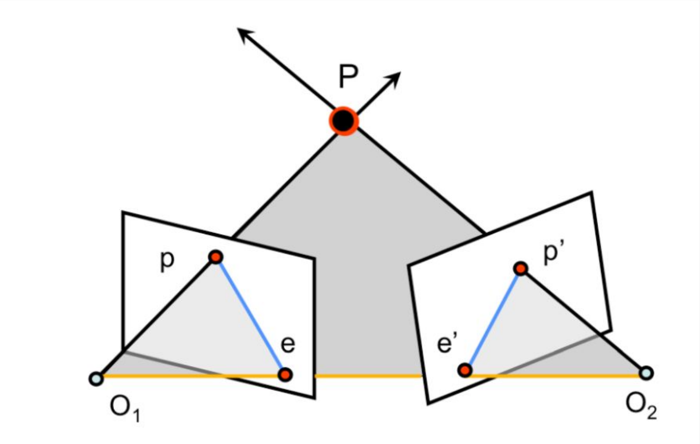
1. Find epipoles
2. Find two homographies that shift epipoles to infinity



Problem 2 - Image Retification

1. Compute epipole

- Epipolar line l = Fp'
- Epipole lies on epipolar lines l . x = 0
- Epipole is an intersection of all epipolar lines



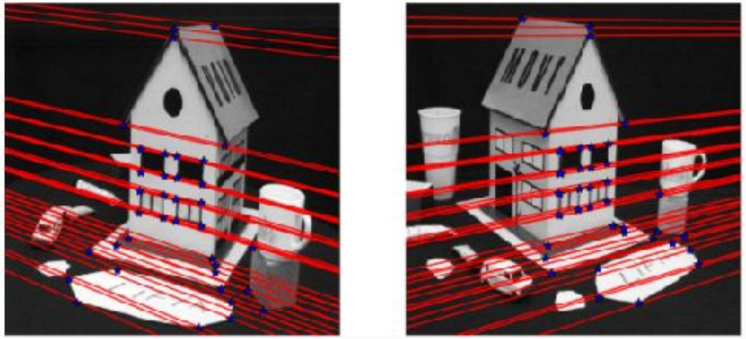
Problem 2 - Image Retification

1. Compute epipole

Due to noisy measurement, not all epipolar lines intersect in a single point

⇒ Find a point that minimizes least square error of fitting a point to all the epipolar lines

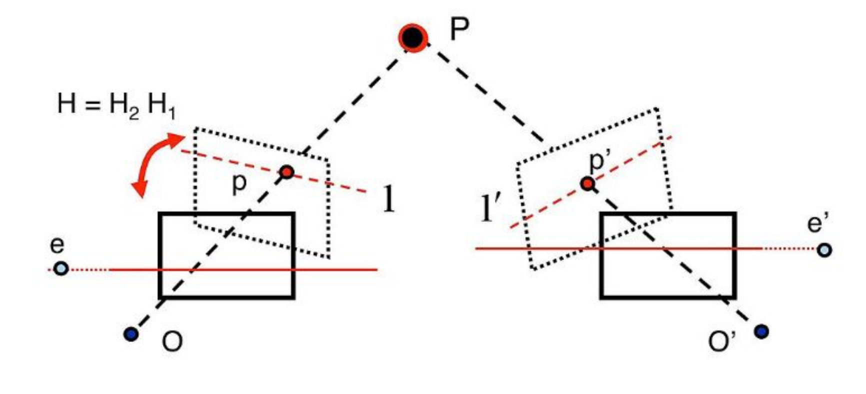
⇒ Solve least square by SVD



$$\begin{bmatrix} \ell_1^T \\ \vdots \\ \ell_n^T \end{bmatrix} e = 0$$

Problem 2 - Image Retification

2. Find two homographies that shift epipoles to infinity
   a. **Find homography H_2 that maps the second epipole e' to a horizontal axis at infinity (f, 0, 0)**
   b. Find the matching homography H_1 for the first image



Problem 2 - Image Retification

Find homography H_2 that maps the second epipole e’ to a horizontal axis at infinity (f, 0, 0)

i. Translate the second image s.t. the center is at (0, 0, 1) in homogeneous coord (***T***)

ii. Apply rotation to place the epipole on the horizontal axis (f, 0, 1) (***R***)

iii. Bring epipole at infinity on the horizontal axis (f, 0, 0) (***G***)

$$H_2 = T^{-1}GRT$$

Problem 2 - Image Retification

Find homography H_2 that maps the second epipole e' to a horizontal axis at infinity (f, 0, 0)

i. Translate the second image s.t. the center is at (0, 0, 1) in homogeneous coord (***T***)

$$T = \begin{bmatrix} 1 & 0 & -\frac{\text{width}}{2} \\ 0 & 1 & -\frac{\text{height}}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2 - Image Retification

Find homography H_2 that maps the second epipole e' to a horizontal axis at infinity (f, 0, 0)

ii. Apply rotation to place the epipole on the horizontal axis (f, 0, 1) (***R***)

The translated epipole ***T***e' = $(e'_1, e'_2, 1)$

$$R = \begin{bmatrix} \alpha \frac{e'_1}{\sqrt{e'^2_1 + e'^2_2}} & \alpha \frac{e'_2}{\sqrt{e'^2_1 + e'^2_2}} & 0 \\ -\alpha \frac{e'_2}{\sqrt{e'^2_1 + e'^2_2}} & \alpha \frac{e'_1}{\sqrt{e'^2_1 + e'^2_2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\alpha = 1$ if $e'_1 \geq 0$ and $\alpha = -1$ otherwise.

Problem 2 - Image Retification

Find homography H_2 that maps the second epipole e' to a horizontal axis at infinity (f, 0, 0)

iii. Bring epipole (f, 0, 1) at infinity on the horizontal axis (f, 0, 0) (***G***)

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{f} & 0 & 1 \end{bmatrix}$$

Problem 2 - Image Retification

Find homography H_2 that maps the second epipole e’ to a horizontal axis at infinity (f, 0, 0)

i. Translate the second image s.t. the center is at (0, 0, 1) in homogeneous coord (***T***)

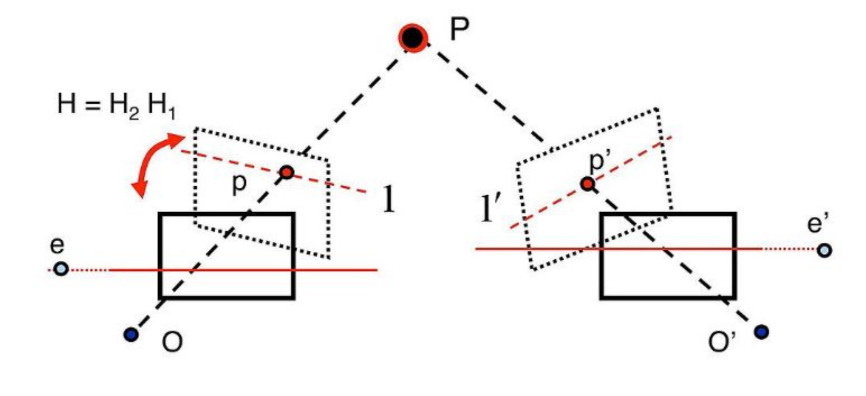
ii. Apply rotation to place the epipole on the horizontal axis (f, 0, 1) (***R***)

iii. Bring epipole at infinity on the horizontal axis (f, 0, 0) (***G***)

$$H_2 = T^{-1}GRT$$

Problem 2 - Image Retification

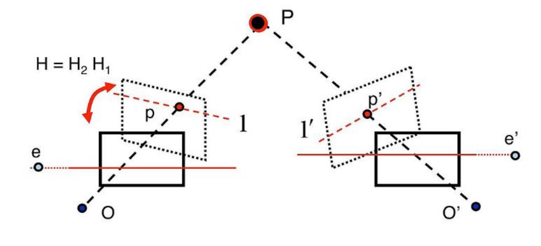
2. Find two homographies that shift epipoles to infinity
   a. Find homography H_2 that maps the second epipole e’ to a horizontal axis at infinity (f, 0, 0)
   b. **Find the matching homography H_1 for the first image**



Problem 2 - Image Retification

Find the matching homography H_1 for the first image

$$\arg\min_{H_1} \sum_i \|H_1 p_i - H_2 p_i'\|^2$$



Although the derivation is out of the scope of this class,

$$H_1 = H_A H_2 M$$

$$M = [e]_\times F + ev^T$$

$$v^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$H_A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2 - Image Retification

Find the matching homography H_1 for the first image

$$\arg\min_{H_1} \sum_i \|H_1 p_i - H_2 p'_i\|^2$$

$$H_1 = H_A H_2 M$$

$$\hat{p}_i = H_2 M p_i$$

$$\hat{p}'_i = H_2 p'_i$$

$$\arg\min_{H_A} \sum_i \|H_A \hat{p}_i - \hat{p}'_i\|^2$$

Problem 2 - Image Retification

Find the matching homography H_1 for the first image

$$\hat{p}_i = H_2 M p_i$$

$$\hat{p}'_i = H_2 p'_i$$

$$H_A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{p}_i = (\hat{x}_i, \hat{y}_i, 1) \text{ and } \hat{p}'_i = (\hat{x}'_i, \hat{y}'_i, 1)$$

$$\arg\min_{\mathbf{a}} \sum_i (a_1 \hat{x}_i + a_2 \hat{y}_i + a_3 - \hat{x}'_i)^2$$

Solving least-square $W\mathbf{a} = b$

$$W = \begin{bmatrix} \hat{x}_1 & \hat{y}_1 & 1 \\ & \vdots & \\ \hat{x}_n & \hat{y}_n & 1 \end{bmatrix} \qquad b = \begin{bmatrix} \hat{x}'_1 \\ \vdots \\ \hat{x}'_n \end{bmatrix}$$

Problem 3+4: Structure from Motion

Structure from Motion (SfM)

Estimating 3D structure from 2D images that may be coupled with local motions

Input: 2D images

Output: 3D structure
(+ camera extrinsic)

Problem 3+4: Structure from Motion

In this homework, we explore two different approaches for SfM

- Factorization Method (problem 3) - Tomasi & Kanade algorithm
- Bundle Adjustment (problem 4)

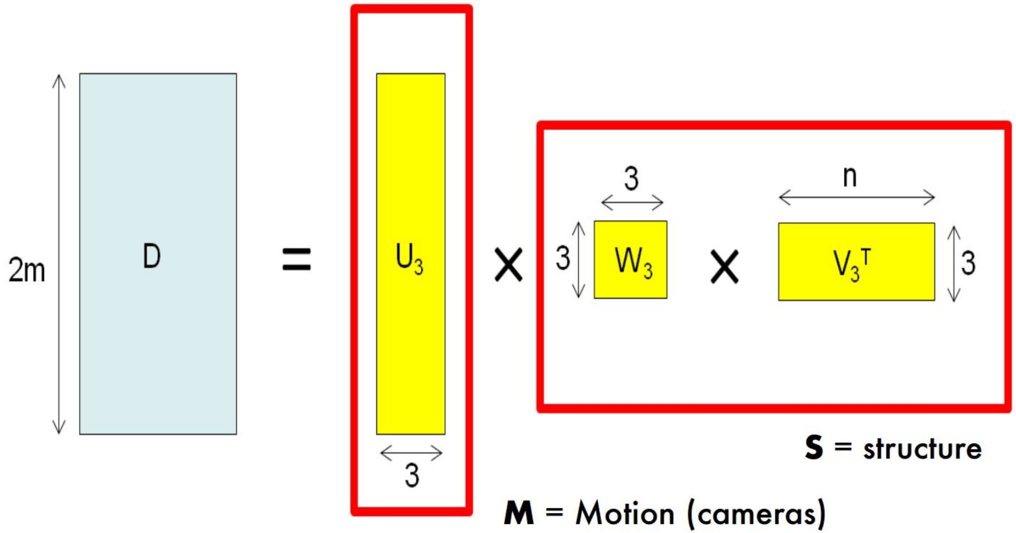
Problem 3 - Factorization Method

Centering: subtract the centroid of the image points

[Eq. 6] $$\hat{\mathbf{x}}_{ij} = \mathbf{x}_{ij} - \frac{1}{n}\sum_{k=1}^{n} \mathbf{x}_{ik}$$

$$\mathbf{D} = \begin{bmatrix} \hat{\mathbf{x}}_{11} & \hat{\mathbf{x}}_{12} & \cdots & \hat{\mathbf{x}}_{1n} \\ \hat{\mathbf{x}}_{21} & \hat{\mathbf{x}}_{22} & \cdots & \hat{\mathbf{x}}_{2n} \\ & & \ddots & \\ \hat{\mathbf{x}}_{m1} & \hat{\mathbf{x}}_{m2} & \cdots & \hat{\mathbf{x}}_{mn} \end{bmatrix}$$

Problem 3 - Factorization Method

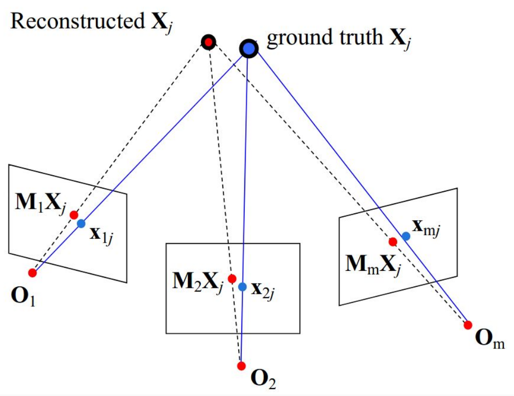


$$\mathbf{D} = \mathbf{U}_3\, \mathbf{W}_3\, \mathbf{V}_3^T = \mathbf{U}_3\, (\mathbf{W}_3\, \mathbf{V}_3^T) = \mathbf{M}\, \mathbf{S} \quad \text{[Eq. 12]}$$

Problem 4: Structure from Motion

1. Compute essential matrix E from two views
2. Use E to make initial estimate of relative rotation R and translation T
3. Estimate 3D location of the reconstruction given RT
4. Optimize (bundle adjustment)
   - Jointly optimize all relative camera motions (R's and T's)
   - Minimize total reprojection error with respect to all 3D point and camera parameters
5. Repeat 3 and 4 for pairs of frames

$$E(M,\mathbf{X}) = \sum_{i=1}^{m}\sum_{j=1}^{n} D\left(\mathbf{x}_{ij}, M_i \mathbf{X}_j\right)^2$$



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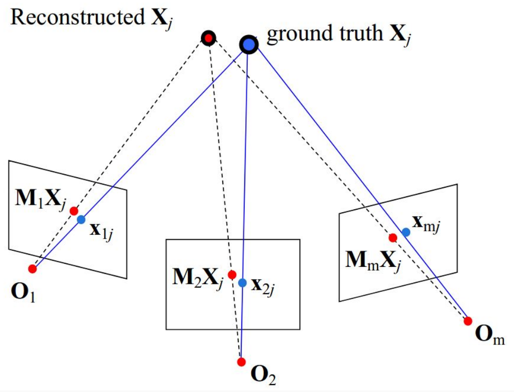

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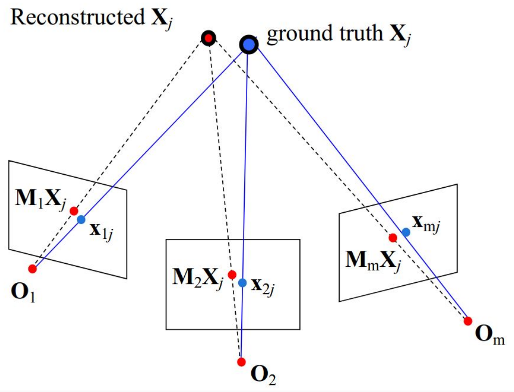


Problem 4: Structure from Motion

1. To compute R: Given the singular value decomposition $E = UDV^T$

$$Q = UWV^T \ \mathit{or} \ UW^TV^T, \text{ where}$$

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that this factorization of E only guarantees that Q is orthogonal. To find a rotation, we simply compute $R = (\det Q)Q$.

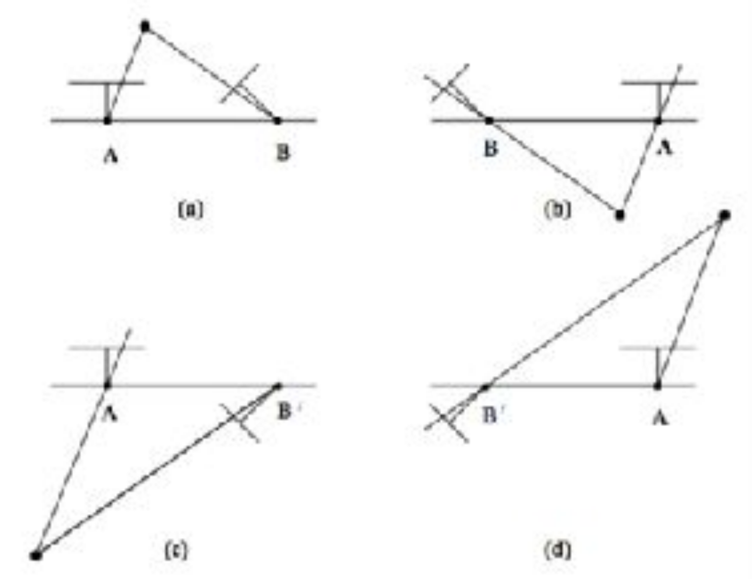
2. To compute T: Given that $E = U\Sigma V^T$, T is simply either u_3 or $-u_3$, where u_3 is the third column vector of U.

Problem 4: Structure from Motion

Use E to make initial estimate of relative rotation R and translation T

However, this gives four pairs of rotation and translation, $(R_1, R_2) \times (T, -T)$

How do we find out which R and T is the correct one?

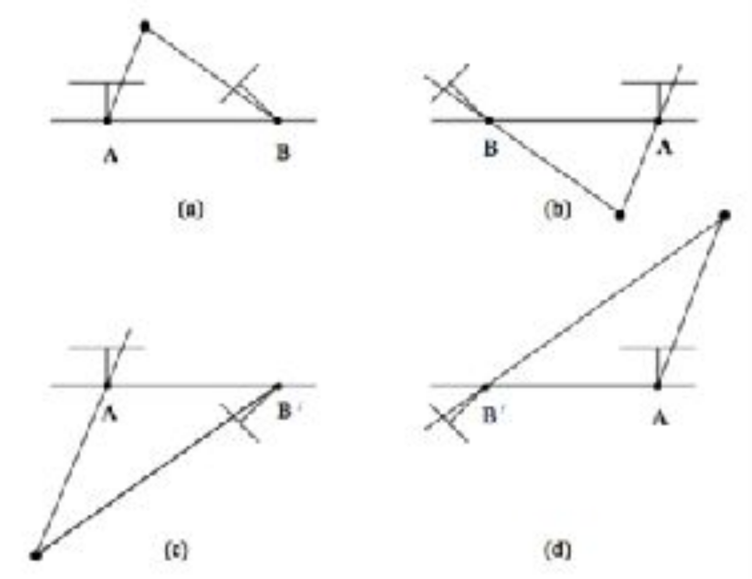


Problem 4: Structure from Motion

There exists only **one** solution that will consistently produce 3D points which are both in front of camera

Compute 3D point's location in the RT frame!

- Find 3D location of the image points given RT frame
- Chose the one which has the most 3D points with positive depth (z-coordinate) with respect to both camera frame



Problem 4: Structure from Motion

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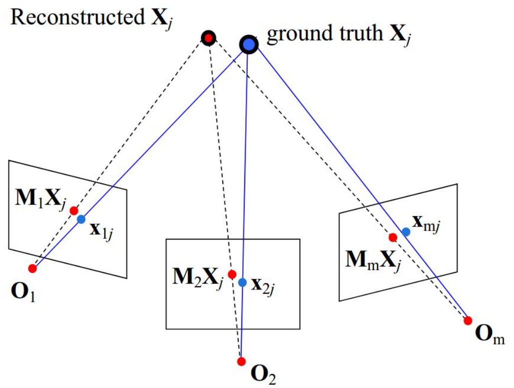
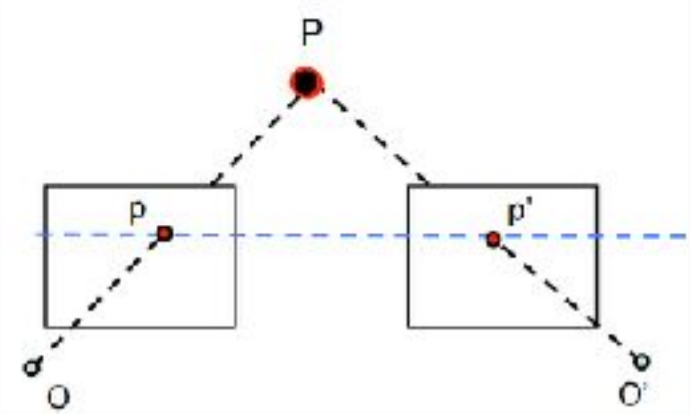


Problem 4: Structure from Motion

Estimate 3D location of the reconstruction given 1. projective camera matrix 2. their image coordinates

Two different possible approaches:

1. **Formulating a linear equation to solve**
2. Nonlinear optimization to minimize reprojection error



Problem 4: Structure from Motion

1. For each image i, we have $p_i = M_i P$, where P is the 3D point, p_i is the homogenous image coordinate of that point, and M_i is the projective camera matrix.

2. Formulate matrix

$$A = \begin{bmatrix} p_{1,1} m^{3\top} - m^{1\top} \\ p_{1,2} m^{3\top} - m^{2\top} \\ \vdots \\ p_{n,1} m^{3\top} - m^{1\top} \\ p_{n,2} m^{3\top} - m^{2\top} \end{bmatrix}$$

   where $p_{i,1}$ and $p_{i,2}$ are the xy coordinates in image i and $m^{k\top}$ is the k-th row of M.

3. The 3D point can be solved for by using the singular value decomposition.

Problem 4: Structure from Motion

Estimate 3D location of the reconstruction given 1. projective camera matrix 2. their image coordinates

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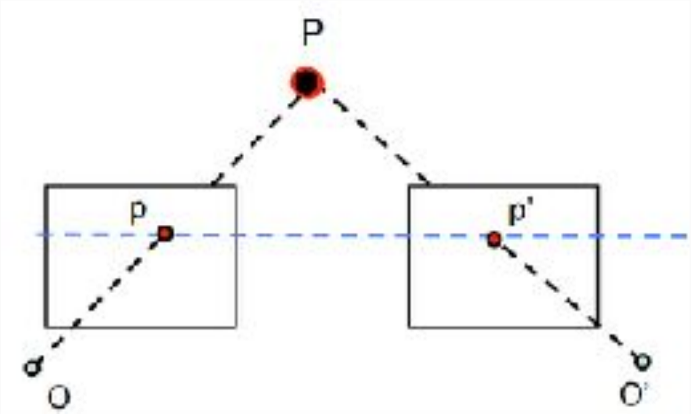
1. Formulating a linear equation to solve
2. **Nonlinear optimization to minimize reprojection error**



Problem 4: Structure from Motion

Estimate 3D location of the reconstruction
given 1. projective camera matrix 2. their image coordinates

Nonlinear optimization to minimize reprojection error

Gauss-Newton algorithm

$$\hat{P} = \hat{P} - (J^T J)^{-1} J^T e$$

Begin from linear estimation for better initialization

Problem 4: Structure from Motion

(reprojection) error: difference between the projected point ($M_i P$) and ground-truth image coordinate p_i

$$e = \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix} = \begin{bmatrix} p_1 - M_1 \hat{P} \\ \vdots \\ p_n - M_n \hat{P} \end{bmatrix}$$

Jacobian:

$$J = \begin{bmatrix} \frac{\partial e_1}{\partial \hat{P}_1} & \frac{\partial e_1}{\partial \hat{P}_2} & \frac{\partial e_1}{\partial \hat{P}_3} \\ \vdots & \vdots & \vdots \\ \frac{\partial e_N}{\partial \hat{P}_1} & \frac{\partial e_N}{\partial \hat{P}_2} & \frac{\partial e_N}{\partial \hat{P}_3} \end{bmatrix}$$

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