Lecture #1

Goals
1) Notation
2) Linear Algebra.

I. Notation

Remark: a) Let $\mathbb{N}$ denote the set of natural numbers including zero.
b) Let $\mathbb{Z}$ denote the set of integers
c) Let $\mathbb{R}$ denote the set of real numbers
d) Let $\mathbb{R}^n$ denote the set of n-tuples of real numbers
e) Let $\mathbb{C}$ denote the set of complex numbers.

Def: Given two sets A and B we say that
a) $f: A \to B$ is a function if $\forall a \in A$ it assigns one and only one element $f(a) \in B$ called the value of f at a.
b) A is called the domain of f.
c) B is called the codomain of f.

Ex: Consider the function $f: \mathbb{N} \to \mathbb{R}$ where $f(k) = (1/2)^k$. It has domain $\mathbb{N}$ and codomain $\mathbb{R}$. Notice it only takes values that are in $(0,1]$.

II. Linear Algebra.

Remark: a) Given $P \in \mathbb{R}^{m \times n}$ we say that $[P]_{ij}$ is the entry in row i and column j.
b) Matrices are upper case and vectors are usually lower case, greek letters for scalars

Def: a) The *identity matrix* is a square matrix with all diagonal entries equal to 1 and all off diagonal entries equal to zero.

Given $P, Q \in \mathbb{R}^{m \times n}$, $R \in \mathbb{R}^{n \times p}$, and $S \in \mathbb{R}^{n \times n}$, then

b) $P+Q \in \mathbb{R}^{m \times n}$ where $[P+Q]_{ij} = [P]_{ij} + [Q]_{ij}$

c) The *transpose* of P or $P^T \in \mathbb{R}^{n \times m}$ where $[P^T]_{ji} = [P]_{ij}$

d) The *trace* of S or $tr(S) \in \mathbb{R}$ where $tr(S) = \sum_{i=1}^{n} [S]_{ii}$

e) $PR \in \mathbb{R}^{m \times p}$ where $[PR]_{ik} = \sum_{j=1}^{n} [P]_{ij} [R]_{jk}$

Note: $(PR)^T = R^T P^T$

A. Linear Independence

Def: A set of vectors $\{v_i \in \mathbb{R}^n\}_{i=1}^m$ is called *linearly independent* iff $\sum \alpha_i v_i = 0 \Rightarrow \alpha_i = 0 \;\forall i$. Otherwise the set of vectors is called *linearly dependent*.

Ex: (a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linearly independent

(b) $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is linearly dependent

(c) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \right\}$ is linearly dependent

since first minus second vectors equal to the third vector.

→ As we will see these will be the same.

Def: (a) Let $A \in \mathbb{R}^{m \times p}$. The rank of A denoted, $rk(A)$ is the maximum number of linearly independent column or rows

(b) A is called full rank iff $rk(A) = \min\{m,p\}$ otherwise it is called rank deficient

Ex: a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has rank 3

b) $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$ has rank 2

Note: In practice there are efficient numerical algorithms to compute the rank of matrices. Use the rank command in MATLAB

Ex: Let $A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$ be rank deficient

then $\exists \{\alpha_i\}_{i=1}^{n}$ s.t. $\sum \alpha_i a_i = 0$

with some $\alpha_i \neq 0$, which can be written as $A\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$

Theorem: A is full rank iff $Av = 0 \Rightarrow v = 0$

Ex: a) $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ Suppose we want to determine if its full rank

$Av = 0$, if $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $Av = 0$,

so not full rank.

b) Also note any scaled version of v can be used to show that A is rank deficient.

Def: The kernel of $A \in \mathbb{R}^{n \times p}$ denoted $\ker(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$

Ex: (a) $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\ker(A) = \left\{ x \in \mathbb{R}^2 \,\middle|\, \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \alpha \in \mathbb{R} \right\}$

Note we say the dimension of this kernel is 1 since it has 1 linearly independent vector describes all elements in $\ker(A)$.

(b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\ker(A) = \{0\}$

Theorem: (Rank Nullity Theorem) Let $A \in \mathbb{R}^{n \times n}$ then $rk(A) + \mathrm{Dim}(\ker(A)) = n$

Ex: (a) $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $Rk(A) = 2 - \dim(\ker(A))$
$= 2 - 1 = 1$

(b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $Rk(A) + \dim(\ker(A)) = 3$
$3 + 0 = 3$

B. Determinant

Def: The determinant of $S \in \mathbb{R}^{n\times n}$ denoted $\det(S)$ or $|S|$ is defined as

$$\det(S) = \sum_{j=1}^{n} (-1)^{j+1} [S]_{1j} \det(S_{1j})$$

where $S_{1j} \in \mathbb{R}^{(n-1)\times(n-1)}$ is obtained by deleting the first row and column j in the matrix S, where $\det(\alpha) = \alpha$ $\alpha \in \mathbb{R}$

Ex:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix}$$

$$\det(A) = 2 * \det\left(\begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}\right) = 2 * (3*9 - 4*4) = 22$$

Lemma: $\det(A) \neq 0$ iff A is full rank.

C. Eigenvalues

Def: The eigenvalues of $S \in \mathbb{R}^{n\times n}$ are scalars $\lambda \in \mathbb{C}$ such that $Sv = \lambda v$ for some nonzero $v \in \mathbb{C}^n$ which is called an eigenvector.

Note: The eigenvectors correspond to the directions of the matrix along which the matrix acts only by scaling in that direction.

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Ex: Lets try to find the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix}$. If λ is an eigenvalue the $Av = \lambda v \Rightarrow (A - I\lambda)v = 0$.
To find λ we should find when $A - \lambda I$ is rank deficient $\Leftrightarrow$ $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & 9-\lambda \end{vmatrix} = -\lambda^3 + 14\lambda^2 - 35\lambda + 22$$
$$\Rightarrow \lambda = 1, 2, 11.$$

Note: There are numerically efficient tools to compute det and eig using MATLAB.

Def: Given a matrix $S \in \mathbb{R}^{n \times n}$ the degree n polynomial $\det(S - \lambda I)$ is called the characteristic polynomial of S

Theorem: (a) If $A \in \mathbb{R}^{n \times n}$, then it has n eigenvalues

Let $A, B \in \mathbb{R}^{n \times n}$, then

(b) If $\{\lambda_i\}_{i=1}^{n}$ are the eigenvalues of A then

i) $|A| = \prod_{i=1}^{n} \lambda_i$

ii) $tr(A) = \sum \lambda_i$

(c) $|AB| = |A||B|$

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