Lecture #3

Goals
1) Control Design
2) Observer Design
3) LQR

I Control Design

1) Repeat Ackermann's Theorem
2) Repeat Problems
3) Do nonlinear stability

II. Observer Design
We need to know state to do Ackermann based control design, but we may only get some measurements. We may know sys dynamics, but may not know state of system perfectly at $t=0$.
Can define a system with state $\hat{x}(t)$ that estimates the state of the system by considering the error dynamics:

$$e(t) = x(t) - \hat{x}(t)$$

We want $e(t) \to 0$ as $t \to \infty$ (stable equilibria of the system)

Need to design the system $\hat{x}$ and e:
Suppose the observer is defined as:

$$\dot{\hat{x}}(t) = (A - GC)\hat{x}(t) + Gy(t) + Bu(t)$$

when $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$

Scanned by CamScanner

Then:

$$\dot{e}(t) = Ax(t) + Bu(t) - (A-GC)\hat{x}(t) - Gy(t) - Bu(t)$$
$$= (A-GC)x(t) - (A-GC)\hat{x}(t)$$
$$= (A-GC)e(t)$$

Need a theorem that allows us to design G to ensure $e \to 0$ as $t \to \infty$:

Theorem: Consider the LTI system:

$$\dot{x} = Ax + Bu \quad \text{and} \quad y = Cx$$

i) Suppose $A \in \mathbb{R}^{n \times n}$ and

$$rk \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

(we call such systems observable)

ii) Suppose we want to design an observer:

$$\dot{\hat{x}}(t) = (A-GC)\hat{x}(t) + Gy(t) + Bu(t)$$

so that the characteristic polynomial of $(A-GC)$ is equal to $\alpha_0(s)$.

Then if

$$G = \alpha_0(A) \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

we get an observer that estimates the state exactly as $t \to \infty$ w/ the desired closed loop eigenvalues.

Remarks: 1) If we place the eigenvalues of the closed loop system at $-\infty$, then get really fast convergence, but this may not always be a good idea. For example if noise is corrupting our measurements, then this strategy will typically cause us to oscillate a great deal in response to small inaccuracies

Notice G would be large in this case

2) Usually people set the eigenvalues of the observer's 3x more negative than the eigenvalues of the controller.

3) Suppose we have an LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

and $u(t) = -K\hat{x}(t)$ with

$$\dot{\hat{x}}(t) = (A-GC)\hat{x}(t) + Gy(t) + Bu(t)$$

then

$$\dot{x}(t) = Ax(t) - BK\hat{x}(t)$$

$$\dot{\hat{x}}(t) = (A-GC)\hat{x}(t) + GCx(t) - BK\hat{x}(t)$$

$$= GCx(t) + (A-GC-BK)\hat{x}(t)$$

so:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ GC & (A-GC-BK) \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

$\dot{x}(t)$

We can repeat same exercise with error dynamics:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A-BK & +BK \\ 0 & A-GC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$$

This equation has characteristic polynomial of

$$(sI-(A-BK))(sI-(A-GC))=0$$

which is the products of the closed loop control and observer system!
We can design both separately and then using $u=-K\hat{x}$ can work to ensure stability.

Theorem: (Separation Principle) If a stable controller and observer are designed for an LTI system separately then the combined system is stable.

III. Linear Quadratic Regulator (LQR)

Want to do control for broader class of systems:
1. want LTV systems since trajectory following is important if we are talking about using linearizations and stabilization theory for vehicle control
2. Hard to choose closed loop poles when we want to prevent overshoot or too much control input (its all done qualitatively i.e. make poles closer to zero... how much closer??)

$$\begin{bmatrix} x_1 & \\ x_2 & \end{bmatrix} \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}$$

First approach to overcome these problems will rely on an optimization based approach to design feedback gains, as the course moves on we will add more features

Our approach:

1. how to ensure optimization problem is well-posed
2. how to solve it.

A. Positive Semi-Definite Matrices

Def: a) A square matrix is called symmetric if it is equal to its Transpose

b) A symmetric matrix A is called positive definite if $x^T A x > 0$ for any $x \neq 0$. we denote this by $A > 0$.

c) A symmetric matrix A is called positive semi-definite if $x^T A x \geq 0$. We denote by $A \geq 0$.

Ex: a) Suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

then $x^T A x = x_1^2 + x_2^2 > 0 \quad \forall x \neq 0$

so $A > 0$.

b) Suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

then $x^T A x = (x_1 + x_2)^2 \geq 0 \quad \forall x$ so $A \geq 0$

that ~~are~~ have a bounded minimum value

Remark: We are using quadratic polynomials to show that a matrix is p.d. or p.s.d. but if we could show that a polynomial could be written as a p.s.d. matrix then we could show it's quadratic and has a bounded minima!

Theorem: a) A quadratic function $f(x) = x^T Dx + Cx + C_0$ where $D \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^n$, $C_0 \in \mathbb{R}$ and $x \in \mathbb{R}^n$ has a unique minimizer iff $D > 0$

b) A quadratic function $f(x) = x^T Dy + Cx + C_0$ where $D \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^n$, $C_0 \in \mathbb{R}$ and $x \in \mathbb{R}^n$ has one bounded minimum value iff $D \geq 0$

But how do we check psd or p.d.?

Theorem: (a) A matrix p.d. iff all its eigenvalues are real & positive

(b) A matrix is p.s.d. iff all its eigenvalues are real & non-negative

B. LQR Formulation & Solution

(LQR problem)

$$\min_{u:[0,T] \to \mathbb{R}^m} \int_0^T \left(x^T(t) Q(t) x(t) + u^T(t) R(t) u(t)\right) dt + x^T(T) Q(T) x(T)$$

$$\text{s.t.} \quad \dot{x}(t) = A(t)x(t) + B(t)u(t)$$

Scanned by CamScanner

Remarks: 1) T is time horizon
2) To ensure problem is well-formulated we will require $Q(t) = Q^T(t) \geq 0 \ \forall t \in [0,T]$ and $R(t) = R^T(t) > 0 \ \forall t \in [0,T]$.
3) R is called input cost
$Q(t)$ is called state cost $\forall t \in [0,T)$
$Q(T)$ is called final cost.
4) This is an infinite dimensional problem u is defined for all times!
5) This optimization tries to drive an LTV system to zero, can apply to tracking problems by apply a transformation that places the trajectory at origin
6) Note no i.c. is specified.

Theorem: The optimal $u^*: [0,T] \to \mathbb{R}^m$ to (LQR) is a state feedback controller:

$$u^*(t) = -K(t)x(t) \quad \forall t \in [0,T]$$

where

$$K(t) = R^{-1}(t) B^T(t) P(t) \quad \forall t \in [0,T]$$

where $P(t)$ is the solution to the Ricatti Differential Equation:

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + Q(t)$$

with $P(T) = Q(T)$

Remark: 1) To apply this we can just solve for $P(t)$ starting from $T \to 0$, then can compute $K(t)$.

2. In practice have to solve a nonlinear differential equation (use Euler)
3. Solution does not depend on i.c.! Works everywhere!!

Pf: We solve this problem using Bellman's optimality principle by defining the value function $V: \mathbb{R}^n \times [0,T] \to \mathbb{R}$ by

$$V(z,t) = \min_{u:[t,T]\to\mathbb{R}^m} \int_t^T (x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau)) d\tau + x^T(T) Q_T x(T)$$

subject to $x(t) = z$ and $\dot{x} = Ax + Bu$
(dropped time dependence for convenience, proof works with this included)

$V(z,t)$ gives the min LQR cost to go from state z at t
$V(z,T) = z^T Q_T z$

Lemma: For LQR $V(x,t) = x^T(t) P(t) x(t)$ where $P(t) \geq 0$

Pf: Andersson & Moore

Proof strategy using Bellman's Principle of optimality:

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy w/ regard to the state resulting from the first decision"

— Bellman, 1957 "Dynamic Programming"

Lets see how to apply this idea:
Suppose we start with state $x(t) = z$ and lets assume $u(t) = w \in \mathbb{R}^m$ a constant over the time interval $[t, t+h]$ for h small

The cost incurred over $[t, t+h]$ is

$$\int_t^{t+h} \left(x^T(\tau) Q x(\tau) + w^T R w\right) d\tau \approx h\left(z^T Q z + w^T R w\right)$$

and we end up at $x(t+h) = z + h(Az + Bw)$

The min cost-to-go from where we land is:

$$V(z + h(Az+Bw), t+h) = (z + h(Az+Bw))^T P(t+h)(z + h(Az+Bw))$$

$$\approx (z+h(Az+Bw))^T (P(t) + h\dot{P}(t))(z + h(Az+Bw))$$

$$\approx z^T P(t) z + h\left((Az+Bw)^T P(t) z + z^T P(t)(Az+Bw) + z^T \dot{P}(t) z\right)$$

(dropped all higher order terms w/ h^2 or higher)

According to Bellman's Principle of Optimality:

$$V(z,t) = \int_t^{t+h} \left(x^T(\tau) Q x(\tau) + w^T R w\right) d\tau + V(z + h(Az+Bw), t+h)$$

$$= z^T P_t z + h\Big(z^T Q z + w^T R w + (Az+Bw)^T P(t) z + \ldots$$
$$+ z^T P(t)(Az+Bw) + z^T \dot{P}(t) z\Big)$$

If we minimize over w (take $d/dw\, V(z,t)$ and set equal to 0 and solve):

$$2hw^T R + 2hz^T P(t) B = 0$$

so $$w^* = -R^{-1}B^T P(t) z$$
$$u^*(t) = -K(t)x(t)$$

Now to get an equation for $P(t)$, recall: $V(z,t) \triangleq z^T P(t) z$ but it is also:

$$z^T P(t) z = V(z,t) = z^T P(t) z + h\big(z^T Q z + w^{*T} R w^* + (Az + Bw^*)^T P(t) z$$
$$+ z^T P(t)(Az + Bw^*) + z^T \dot{P}(t) z\big)$$

After simplification:

$$-\dot{P}(t) = A^T P(t) + P(t)A - P(t)BR^{-1}B^T P(t) + Q$$

with $P(T) = Q_T$ $\boxtimes$

Ex: Dubins Car Examples in MATLAB.

a) y & θ + linearized (lane keeping)

b) x, y, θ + linearized (trajectory following)

Made $\alpha = 0$
$a, b = 0$
for simplicity
and fix V