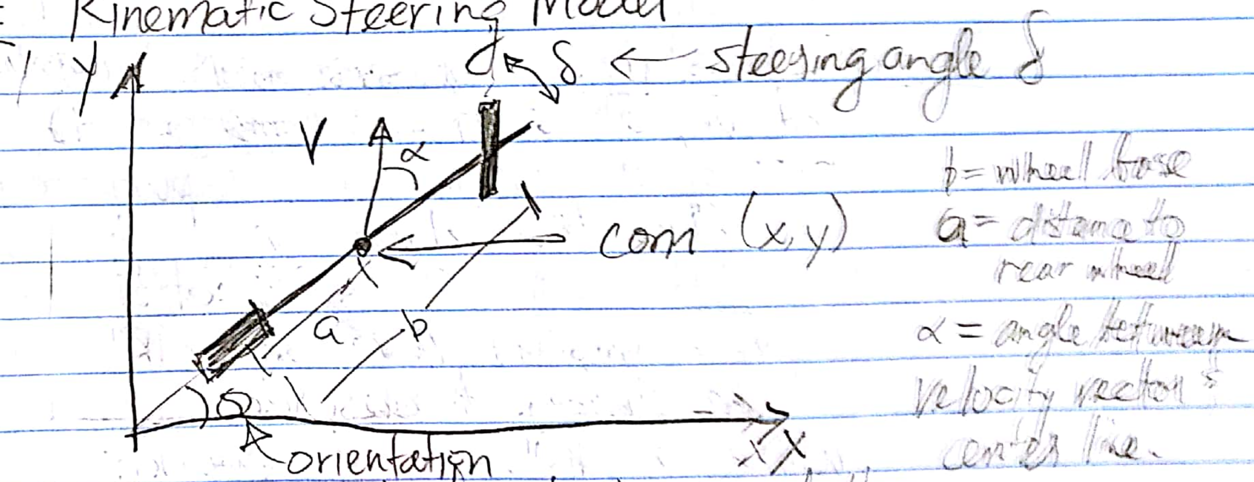
Lecture #2

Goals
1) State Space Models
2) Linearizations
3) Time Response Linear Systems
4) Stability, Connecting w/ Control
5) Control Design.

I. State Space Models

Ex: Kinematic Steering Model



Suppose we control velocity V at the center of mass and the steering angle δ. Our objective will be to ensure that we can develop control design tools to get this system in and around obstacles safely

To do this, at a minimum we will need:
1) models of motion
2) theory to design safe controllers < linear / nonlinear
3) tools to estimate state of dynamics.

To write down models of motion we can rely on physics to describe evolution of systems using ordinary differential equations.

derived in Ex 3.10 in FBS book

Ex:

$$\frac{dx}{dt} = v\cos(\alpha + \theta)$$

$$\frac{dy}{dt} = v\sin(\alpha + \theta)$$

$$\frac{d\theta}{dt} = \frac{v\cos(\alpha)\tan(\delta)}{b}$$

where $\alpha = \tan^{-1}\left(\frac{a\tan\delta}{b}\right)$

May not get to measure orientation, but only com position.

Def: a) The state of a system is a collection of variables that summarize the past of a system for the purpose of predicting the future.

b) The state vector is the collection of these states in a vector $x \in \mathbb{R}^n$.

c) A model that describes a system using a differential equation

t = time

$$\frac{dx}{dt} = f(t,x,u) \quad , \quad y = h(t,x,u)$$

where $f: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$ are smooth mappings, $u \in \mathbb{R}^m$ is a control variable, and $y \in \mathbb{R}^q$ is a set of measurements (called outputs) is called a state space model.

Ex: Do vehicle model.

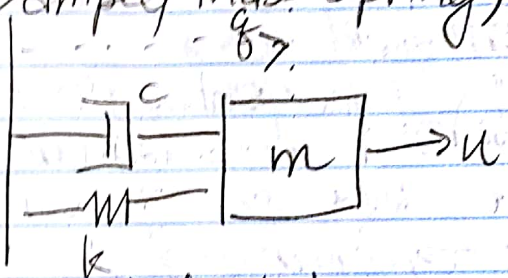
Control questions are especially easy to answer for certain classes of systems.

Def: A system is called LTI or linear time invariant if f and h do not depend on time (t) and f and h are linear in x and u.
Such LTI systems can be represented by

$$\frac{dx}{dt} = Ax + Bu \quad , \quad y = Cx + Du$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{q \times n}$, and $D \in \mathbb{R}^{q \times m}$ are constant matrices.

Ex: (Damped Mass Spring)



Force balance to write dynamics as

$$m\ddot{q} + c\dot{q} + kq = u$$

We only observe q.
Let $(q, \dot{q}) = x$, then

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u$$

$$y = x_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

II. Linearizations

As we will see LTI theory will be useful to control systems; however vehicle models are usually nonlinear.
We will use local LTI approximations of nonlinear systems around certain points to do control

We focus on constructing linearizations about specific points:

Def: Suppose we are given a time invariant state space model $\dot{x} = f(x,u)$. The <u>equillibria</u> of the dynamical system are the points x and u s.t. $\dot{x} = 0$.

Ex: Consider kinematic steering model. For sake of simplicity ignore x-coordinate and assume v is fixed then:

$S = (y, \theta)$ and $u = \delta$.

$$f(s,u) = \begin{pmatrix} v\sin(\alpha + \theta) \\ \frac{v\cos(\alpha)\tan(\delta)}{b} \end{pmatrix}$$

where $\alpha = \tan^{-1}\left(\frac{a\tan\delta}{b}\right)$

Suppose we are interested in equilibria around $\theta = 0$, then

$$0 = v\sin(\alpha) \Rightarrow \delta = 0$$

this means the equilibria of this system when $\theta = 0$ is $\delta = 0$ and y can be anything.

Def: The Jacobian Linearization of $\dot{x} = f(x,u)$ $y = h(x,u)$ at an equilibria x_e, u_e is

$$\frac{dz}{dt} = Az + Bv, \quad w = Cz + Dv$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)}, B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}, C = \left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)}, D = \left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)}$$

$\frac{1}{\cos^2(\delta}$

Ex: for vehicle steering model at $x_e = (0,0)$
$u_e = 0$, $\alpha(u) = 0$

$$\left.\frac{\partial f}{\partial x}\right|_{x_e,u_e} = \left.\begin{bmatrix} 0 & v\cos(\alpha(\delta)+y) \\ 0 & 0 \end{bmatrix}\right|_{x_e,u_e} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial u}\right|_{x_e,u_e} = \begin{bmatrix} v\cos(\alpha(\delta)+y)\left(1+\left(\frac{a\tan(\delta)}{b}\right)^2\right)^{-1}\frac{a}{b}\sec^2(\delta) \\ \frac{v\cos(\alpha)}{b}\sec^2(\delta) - \frac{v\sin(\alpha)}{b}\left(1+\left(\frac{a\tan(\delta)}{b}\right)^2\right)^{-1}\frac{a}{b}\sec^2(\delta) \end{bmatrix}$$

$$= \begin{bmatrix} av/b \\ \frac{v}{b} \end{bmatrix}$$

Remarks: a) To understand the relationship between original system and Jacobian linearization, let:

$$\delta_x(t) := x(t) - x_e, \quad \delta_u(t) = u(t) - u_e$$

then:

$$\dot{\delta}_x(t) = \dot{x}(t) = f(\delta_x(t) + x_e, \delta_u(t) + u_e)$$

Apply Taylor Expansion:

$$\dot{\delta}_x(t) \approx f(x_e, u_e) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_e \\ u=u_e}} \delta_x(t) + \left.\frac{\partial f}{\partial u}\right|_{x_e,u_e} \delta_u(t)$$

$$+ \text{h.o.t.}$$

So we call this

$$\dot{z} = Az + Bv$$

since $f(x_e, u_e) = 0$!

b) What about time varying $x_e(t), u_e(t)$?

III Time Response

A. Simulation

Hard to solve a nonlinear state space model in general starting from an initial condition (or i.c.). We can use approximations though:

Theorem: (Euler Integration) Let $\dot{x} = f(t, x, u)$ be a differentiable function, $u : [0,T] \to \mathbb{R}^m$ be square integrable, $x_0 \in \mathbb{R}^n$, and $x : [0,T] \to \mathbb{R}^n$ a solution to f under u with $x(0) = x_0$. Let $h > 0$ with $T/h \in \mathbb{N}$ and $\{\tilde{x}(k)\}_{k=0}^{T/h}$ defined as

$$\tilde{x}(k+1) = \tilde{x}(k) + h f(kh, \tilde{x}(k), u(kh))$$

with $\tilde{x}(0) = x_0$.

Suppose we linearly interpolate these states to get $\hat{x} : [0,T] \to \mathbb{R}^n$ i.e.

$$\hat{x}(t) = \begin{cases} \tilde{x}(k) + (\tilde{x}(k+1) - \tilde{x}(k))\left(\frac{t-kh}{h}\right) & t \in [kh, (k+1)h) \\ 0 & \text{o.w.} \end{cases}$$

then

$$\lim_{h \downarrow 0} \int_0^T \|x(t) - \hat{x}(t)\|_2 \, dt = 0$$

Remark: 1) Very useful for optimization too

2) ODE45 MATLAB does simulation too, just a bit more accurate.

B. Solutions to LTI Systems

Theorem: Any solution to an LTI system can be decomposed into a solution w/ zero input (homogeneous solution) plus a solution w/ zero initial condition (particular solution)

Lets start by understanding how to find homogeneous solution to LTI system:

Ex: Suppose $\dot{x} = Ax$ what is $x(t)$?

$$x(t) = e^{At}x(0)$$

What if A is a matrix?

Def: The matrix exponential to $X \in \mathbb{R}^{n\times n}$ is the infinite series:

$$e^{X} = I + X + \frac{1}{2}X^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!}X^k$$

Lemma: This series converges for any matrix $X \in \mathbb{R}^{n\times n}$

Ex: (a) $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, then $A^n = \begin{pmatrix} a_1^n & 0 \\ 0 & a_2^n \end{pmatrix}$

so $e^{A} = \sum_{k=0}^{\infty} \frac{1}{k!}\begin{pmatrix} a_1^k & 0 \\ 0 & a_2^k \end{pmatrix} = \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix}$

(b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ notice $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

then $e^{A} = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

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Note: Matlab function expm not exp.

Ex: $\frac{dx}{dt} = Ax$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!}(At)^k$$

$$\frac{d}{dt}(e^{At}) = A + A^2t + \frac{1}{2}A^3t^2 = A\sum \frac{1}{k!}(At)^k$$

$$= Ae^{At}$$

so if $x(t) = e^{At}$ then $\frac{dx}{dt} = Ax$!

Ex: Simplified steering model when linearized

$$\dot{x}(t) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} x(t)$$

Recall $A^2 = 0$ hence

$$e^{At} = \begin{pmatrix} 1 & vt \\ 0 & 1 \end{pmatrix} \text{ so}$$

$$x(t) = \begin{pmatrix} 1 & vt \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_1(0) + vtx_2(0) \\ x_2(0) \end{pmatrix}$$

What about the particular solution?

Theorem: (Convolution Integral) Let

$$\frac{dx}{dt} = Ax + Bu$$

then

$$x(t) = \underbrace{e^{At}x(0)}_{\text{homogeneous}} + \underbrace{\int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau}_{\text{particular}}$$

Def: The time response of a system when $t \to \infty$ is called the steady state response.

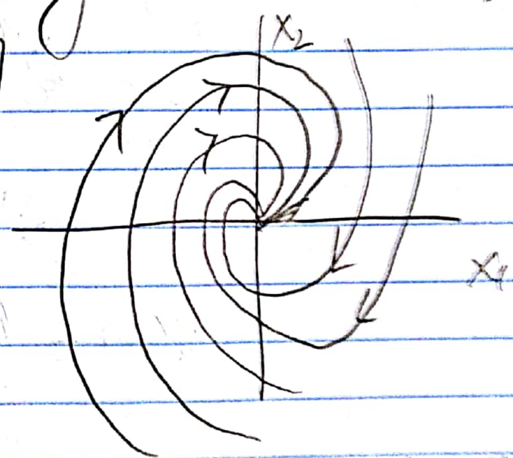
IV Stability

Lets focus our attention on systems w/o input

$$\dot{x} = f(x)$$

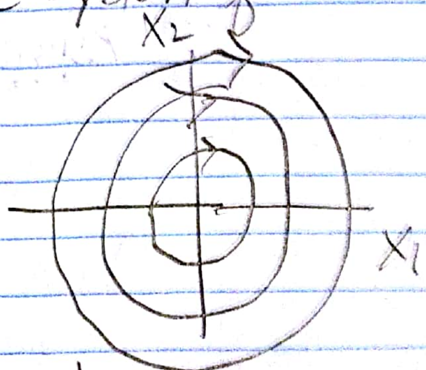
and study when we converge to equilibria. (As we will see this understanding will be critical to understanding how to control)

Ex: a) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

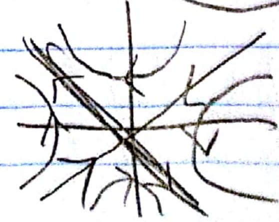


What we have drawn is a phase portrait that shows the solution of the system from that point.

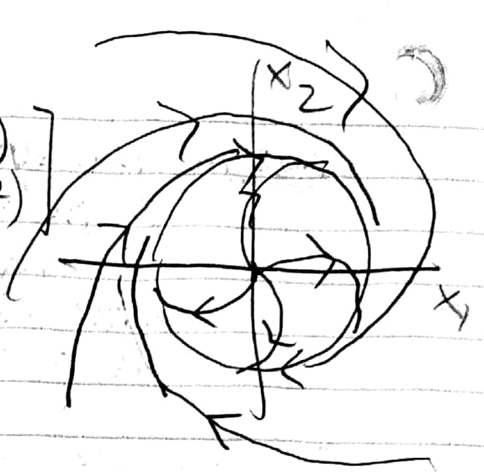
(b) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



(c) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



(d) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_1(1-x_1^2-x_2^2) \\ -x_1 + x_2(1-x_1^2-x_2^2) \end{bmatrix}$



Def: a) Let $x(t;a)$ be a solution to $\dot{x}(t)=f(x(t))$ w/ i.c. a. A solution is stable if all other solutions that start near a stay near $x(t;a)$ for all time.
(can formalize using ε-δ def)
b) If all other solutions stay close to $x(t;a)$ then the system is called globally stable otherwise it is locally stable.

Ex: Check origin of all examples
(a) and (b) are globally stable
(c) and (d) are unstable
If we negated the dynamics in (d) then the origin would be locally stable.

Def: a) A solution $x(t;a)$ is called asymptotically stable if it is stable and $x(t;b) \to x(t;a)$ as $t \to \infty$ for b sufficiently close to a.

b) Can similarly define global and local asymptotic stability.

Ex: (a) is asymptotically stable
(b) is not.

Theorem: The system $dx/dt = Ax$, at the origin is asymptotically stable iff all its eigenvalues of A all have strictly negative real part, and is unstable if any eigenvalue has a strictly positive real part.

Question: Why is this useful for control?

Ex: Suppose we have a dynamic model:

(Drug delivery model Ch3 FBS)

$$\frac{dx}{dt} = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} x + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} u$$

Model requires $k_0, k_1, k_2 > 0$

We want $y(t) = y_d \in \mathbb{R}$ as $t \to \infty$

Suppose we choose a control

$$u = -K(y - y_d) + u_d = -Kx_2 + Ky_d + u_d$$

Linear state feedback controller want to select K and u_d to ensure $y(t) \to y_d$

Substitute controller in:

$$\frac{dx}{dt} = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} x + \begin{pmatrix} b_0 \\ 0 \end{pmatrix}(u_d + Ky_d) + \begin{pmatrix} -b_0 K x_2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -k_0 - k_1 & k_1 - b_0 K \\ k_2 & -k_2 \end{pmatrix} x + \begin{pmatrix} b_0 \\ 0 \end{pmatrix}(u_d + Ky_d)$$

$$= A x + B \tilde{u}$$

$$y = Cx$$

We want to arrive at y_d and stay there.
Let's suppose there exists an x_e that generates y_d (i.e. $y_d = Cx_e$).
Then let's make x_e an equilibria that is stable!

$$\frac{dx_e}{dt} = 0 = Ax_e + Bu_e \rightarrow \text{generates equilibria}$$

$$\Rightarrow x_e = -A^{-1}Bu_e \quad (A \text{ invertible})$$

$$y_d = -CA^{-1}Bu_e$$

$$= \frac{b_0 k_2}{k_0 k_2 + b_0 k_2 k} \underbrace{(u_d + Ky_d)}_{u_e}$$

If we solve for

$$u_d = \left[\frac{k_0 + b_0 k}{b_0} - k\right] y_d$$

Now we have to ensure system is stable.
Shift equilibria to origin and then select K by ensuring that origin is asymptotically stable.

Let $z = x - x_e$

$$\frac{dz}{dt} = \frac{d}{dt}(x - x_e) = \frac{dx}{dt} = Ax + Bu$$

$u = u_e$

$$= A(z + x_e) + Bu_e$$

$$= Az + A(-A^{-1}Bu_e) + Bu_e$$

$$= Az$$

$$\frac{dz}{dt} = \begin{pmatrix} -k_0 - k, & k_1 - b_0 k \\ k_2 & -k_2 \end{pmatrix} z$$

How do we ensure eigenvalues have negative real part?

Check characteristic polynomial!

$$\det\left(sI - \begin{pmatrix} -k_0 - k_1 & k_1 - b_0 k \\ k_2 & -k_2 \end{pmatrix}\right) = 0$$

$$s^2 + (k_0 + k_1 + k_2)s + (k_0 k_2 + b_0 k_2 k) = 0$$

Want to select k to guarantee that $Re(s) > 0$!

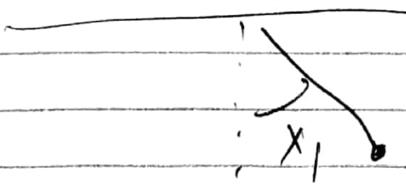
<u>Remark:</u> We don't prove it here, but we can show that since $k_0 + k_1 + k_2 > 0$ that $k_0 k_2 + b_0 k_2 k > 0$ is sufficient to ensure that $Re(s) > 0 \Rightarrow k > -k_0/b_0$ to ensure stability

<u>Theorem:</u> Let $\dot{x} = f(x)$ w/ an equilibria at x_e and f is continuously differentiable. If all the eigenvalues of

$$\left.\frac{df}{dx}\right|_{x = x_e}$$

have negative real part, then x_e is locally asymptotically stable. If any of the eigenvalues has positive real part, then x_e is unstable.

<u>Ex:</u>



$x_2 = \dot{x}_1$

Consider the pendulum w/ damping ↴

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$$\frac{dx}{dt} = \begin{pmatrix} x_2 \\ -\sin(x_1) - \gamma x_2 \end{pmatrix} \qquad \gamma > 0$$

Lets find equilibria

$$\frac{dx}{dt} = 0 \stackrel{!}{=} \begin{pmatrix} x_2 \\ -\sin(x_1) - \gamma x_2 \end{pmatrix} \Rightarrow \begin{matrix} x_{2e} = 0 \\ x_{1e} = 0, \pm\pi, \ldots \end{matrix}$$

Linearize about $x_{1e} = 0$

$$A = \begin{pmatrix} 0 & 1 \\ -\cos(x_1) & -\gamma \end{pmatrix}\Bigg|_{x_1 = 0} = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix}$$

all eigenvalues have negative real part $\Rightarrow$ locally asymptotically stable

Linearize about $x_{1e} = \pi$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -\gamma \end{pmatrix}$$

has eigenvalue w/ positive real part $\Rightarrow$ unstable

Note: 1) We don't know which points converge...
2) If eigenvalues of linearization have real part equal to zero, we know nothing!

V Control Design

Strategy: For time invariant systems, linearize about desired equilibria and ensure its stable

Problems: 1) doesn't work for TV systems, trajectories
2) don't know how good it performs (nbhd)

$\Downarrow$

3) no ability to bound inputs
4) no ability to enforce state constraints

<u>Theorem:</u> (Ackermann's Formula)

Given a single input single output LTI system

$$\frac{dx}{dt} = Ax + Bu \ , \ y = Cx + Du$$

i) Suppose $A \in \mathbb{R}^{n \times n}$ and

$$rk[B \ \ AB \ \ A^2B \cdots A^nB] = n$$

(we call systems that satisfy this requirement controllable). Suppose we want y to be equal to some reference signal r as $t \to \infty$.

ii) Suppose we design a feedback controller

$$u = -Kx + k_r r$$

which generates a closed loop system

$$\frac{dx}{dt} = (A - BK)x + Bk_r r$$

iii) Suppose we want $(A-BK)$ to have a characteristic polynomial

$$\alpha_c(s) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_0$$

where all its roots have negative real part.

Then if

$$K = [0 \cdots 0 \ 1][B \ \ AB \cdots A^{n-1}B]^{-1} \alpha_c(A)$$

and $k_r = -(C(A-BK)^{-1}B)^{-1}$, we can get the desired characteristic polynomial for $(A-BK)$ and steady state behavior.

Ex 7.6 FBS

Ex: (Vehicle Steering) We looked at a linearization of the model to look at just lateral deviation:

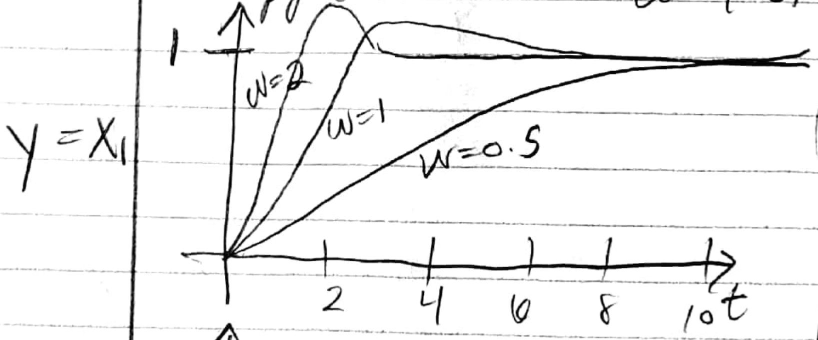
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

We can check controllability (it is full rank)
Suppose we want closed loop (A-BK) to have form $\alpha_c(s) = s^2 + 2\zeta\omega s + \omega^2$
where $\zeta = 0.7$ and we try
$\omega = 0.5, 1,$ and 2
(the real part of the eigenvalue goes from -0.35, -0.7, to -1.4)

ω	$Re(s)$	K
0.5	-0.35	[0.25 0.5750]
1	-0.7	[1 0.9]
2	-1.4	[4 0.8]

Suppose $r = 1$ and start at $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



Note: Eigenvalues → −∞ faster convergence, more overshoot, larger magnitude input