Analogy using vector transport

Word2grass

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Analogy task looks like this - $a : b :: c : d$ where a, b, c are three words represented by points on the grassmanian manifold. Now, let me denote a, b, c with orthonormal matrices A, B, C respectively.
Steps to find the unknown subspace D representing the word d -

- Use the closed form equation of the geodesic connecting a and b to get the tangent vector at point A. The closed form equation is given by

  $$\gamma(t) = XV\cos(\Theta t) + U\sin(\theta t)$$

  where $\gamma(0) = A$, $\gamma(1) = B$ and $\Theta = \tanh\Sigma$. The quantities U, Σ, V are gotten after the SVD decomposition of projection of $B(A^TB)^{-1}$ onto the orthogonal complement of X, i.e.,

  $$U\Sigma V^T = (I - AA^T)B(A^TB)^{-1}$$

  The tangent vector that we need is basically $\dot{\gamma}(0)$. Let us give it a better name T_A. So T_A can be calculated simply using $U\Theta V^T$. Infact, Θ is the diagonal matrix having all the principal angles between subspace A and B. We also normalise T_A because it might have arbitrary length.

- We also calculate the geodesic length of the curve connecting the two subspaces. This turns out to be $\sqrt{\sum_i \theta_i^2}$ where θ_i are the principal angles between the two subspaces. Let us call it L.

- Now that we have T_A, we use the vector transport equation to parallely translate it along the geodesic connecting A and C. The closed form of the equation is given by

  $$T_A(t) = (-AV\sin(\Sigma t)U^T + U\cos(\Sigma t)U^T + (I - UU^T))T_A$$

  where $T_A(t)$ represents the tangent vector after being transported to a point $\alpha(t)$ where α represents the geodesic. U, Σ, V are gotten after SVD decomposition of T_A, i.e, $U\Sigma V^T = T_A$. We require $\tau T_A(1)$ and let us call it T_C. We rescale T_C by L.

- Next step is to use this transported vector at subspace C to move along the geodesic till we have covered a distance of length L. The closed form of this equation is given by -

  $$\beta(t) = XV\cos(\Theta t) + U\sin(\theta t)$$

  where $\beta(0) = C$ and U, Θ, V are gotten after SVD decomposition of the parallel transported tangent vector T_C, i.e, $U\Theta V^T = T_C$.

So, the point we get after traversing the curve β till length L at a velocity equal to T_C is our required subspace D. Now we search for all words that are close to D.