

Counting integer points in polyhedra

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Proof outline

- ▶ Algebra of polyhedra, $P(\mathbb{Q}^d)$
- ▶ $[] : \mathbb{Q}^d \rightarrow P(\mathbb{Q}^d)$
- ▶ Existence of $\mathcal{F} : P(\mathbb{Q}^d) \rightarrow \mathbb{C}(x)$, such that:
 - ▶ \mathcal{F} is linear
 - ▶ P is a polyhedra, then $\mathcal{F}([P]) = \sum_{\vec{m} \in P \cap \mathbb{Z}^d} (x^{\vec{m}})$
 - ▶ $\mathcal{F}([\text{line}]) = 0$
- ▶ $\mathcal{F}(P)(1) = \text{number of points in } P$
- ▶ reduction: \mathcal{F} for cones gives full \mathcal{F}
- ▶ reduction: \mathcal{F} for simple cones gives \mathcal{F} for cones

Caveats

- ▶ Do not understand subtleties of convergence arguments (how is evaluating at $\vec{1}$ correct?).
- ▶ No intuition for LLL, Lattice reduction.

Assuming \mathcal{F} for cones, derive full \mathcal{F} : Part 1 (Polytopes)

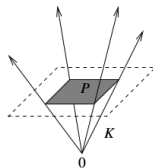


FIGURE 66. A polytope $P \subset \mathbb{R}^d$ and a cone $K \subset \mathbb{R}^{d+1}$ based on P .

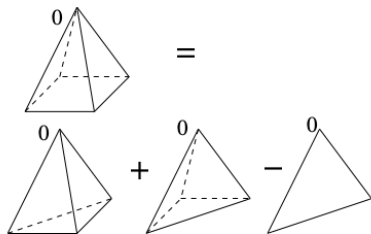
- Write polytope as intersection of hyperplane + cone.
- $\mathcal{F}(\text{polytope}) = (\frac{d}{dx_{d+1}} \mathcal{F}(\text{cone}))(\langle \vec{1}^d, 0 \rangle)$
- $\mathcal{F}(\text{cone}) = x_{d+1}^0(\dots) + x_{d+1}(\text{POLYTOPE}) + x_{d+1}^2(\dots) + \dots$
- $\frac{d}{dx_d} \mathcal{F}(\text{cone}) = 0 + 0 \cdot (\dots) + 1 \cdot \text{POLYTOPE} + 2x_{d+1}(\dots) + \dots$
- $\frac{d}{dx_d} \mathcal{F}(\text{cone})(\langle \vec{1}^d, 0 \rangle) = \text{POLYTOPE}(\vec{1}) + 2 \cdot 0 \cdot (\dots) + \dots$
- $\frac{d}{dx_d} \mathcal{F}(\text{cone})(\langle \vec{1}^d, 0 \rangle) = \text{POLYTOPE}(\vec{1})$

Assuming \mathcal{F} for cones, derive full \mathcal{F} : Part 2 (Lines)

- ▶ Line = cone + cone - point.
- ▶ Since line can be translated, $\forall \vec{x} \in L, L = \vec{x} + L$
 - ▶ $\forall x \in L, \mathcal{F}(L) = \mathcal{F}(L) + \mathcal{F}(\vec{x})$
 - ▶ $\mathcal{F}(L) = 0$

Assuming \mathcal{F} for simple cone, derive for cone

- ▶ inclusion exclusion: decompose cone into simple cones.



References

- ▶ Lattice Points, Polyhedra, and Complexity: Alexander Barvinok
- ▶ Integer points in polyhedra: Alexander Barvinok

Thanks!

Questions?

Assuming \mathcal{F} for cones, derive full \mathcal{F} : Part 1.2 (Polytopes)

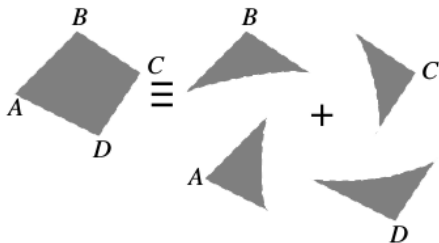


FIGURE 36. Representing the interior of a polytope as the sum of the interiors of its tangent cones at the vertices modulo polyhedra with lines.