# Counting integer points in polyhedra

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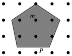
# Definitions

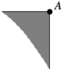
- Polyhedra: $\left\{\vec{x} \in \mathbb{Q}^d \mid Ax \leq \vec{b}\right\}$
- Polytope: bounded polyhedra.
- Cone: $cone(\vec{u_i}) = \left\{ \sum_i \lambda_i \vec{u_i} \mid \lambda_i \geq 0 \right\}$, $\vec{u_i} \in \mathbb{Q}^d$
- Simple cone: $SK = cone(\vec{u_i}), \vec{u_i} \in \mathbb{Z}^d$, $\vec{u_i}$ are linearly independent.
- Unimodular cone: $UK = cone(\vec{u_i})$, $Volume(\vec{u_i}) = 1$
- Line: subspace.

# Pictures of defintions!

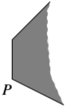
- polytope



- cone



- polyhedra



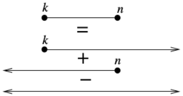
# Example 1: valuation of line

- P is a polyhedra, then $\mathcal{F}([P]) = \sum_{\vec{m} \in P \cap \mathbb{Z}^d} (x^{\vec{m}})$
- $\mathcal{F}([P])(\vec{1})$ = number of points.

$$\begin{aligned}
\mathcal{F}((-\infty, \infty)) &= \sum_{i \in \mathbb{Z}} x^i \\
\text{count}(x) &= \mathcal{F}((-\infty, \infty)) \\
&= \mathcal{F}((-\infty, 0]) + \mathcal{F}([0, \infty)) - \mathcal{F}(0) \\
&= (\ldots + x^{-2} + x^{-1} + x^0) + (x^0 + x^1 + x^2 + \ldots) - x^0 \\
&= \frac{1}{1 - \frac{1}{x}} + \frac{1}{1 - x} - 1 \\
&= \frac{-x}{1 - x} + \frac{1}{1 - x} = \frac{1 - x}{1 - x} - 1 = 0
\end{aligned}$$

- number of points in a line is 0!

# Example 2: valuation of interval



$$
\begin{aligned}
\text{count}(x) &= \mathcal{F}([0,n]) = \mathcal{F}([k,\infty)) + \mathcal{F}((-\infty,n]) - \mathcal{F}((\infty, infty)) \\
&= (x^k + x^{k+1} + \ldots) + \\
&(\ldots + x^{n-2} + x^{n-1} + x^n) + \\
&(\ldots + x^{-2} + x^{-1} + x^0 + x^1 + \ldots) \\
&= \frac{x^k}{1-x} + \frac{x^n}{1-x^{-1}} + 0 \\
&= \frac{x^k - x^{n+1}}{1-x} \\
\text{count}(1) &= \text{L'hospital} = (n+1) - k = n - k + 1
\end{aligned}
$$

# Proof outline

- Algbra of polyhedra, $P(\mathbb{Q}^d)$
- $[\,] : \mathbb{Q}^d \to P(\mathbb{Q}^d)$
- Existence of $\mathcal{F} : P(\mathbb{Q}^d) \to \mathbb{C}(x)$, such that:
  - F is linear
  - P is a polyhedra, then $\mathcal{F}([P]) = \sum_{\vec{m} \in P \cap \mathbb{Z}^d} (x^{\vec{m}})$
  - $\mathcal{F}([\text{line}]) = 0$ (important, allows modulo line decompositions)
- $\mathcal{F}(P)(1) =$ number of points in$P$
- reduction: F for cones gives full F
- reduction: F for simple cones gives F for cones
- performance: $\mathcal{F}$ for unimodular cones gives $\mathcal{F}$ for simple cones

# Caveats

- Self taught :)
- Do not understand subtleties of convergence arguments (how is evaluating at $\vec{1}$ correct?).
- No intuition for LLL, Lattice reduction.

# Assuming $\mathcal{F}$ for cones, derive full $\mathcal{F}$: Part 1 (Polytopes)

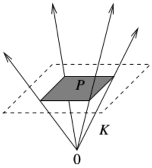


FIGURE 66. A polytope $P \subset \mathbb{R}^d$ and a cone $K \subset \mathbb{R}^{d+1}$ based on $P$.

- Write polytope as intersection of hyperplane + cone.
- $\mathcal{F}(\text{polytope}) = (\frac{d}{dx_{d+\mathbf{1}}}\mathcal{F}(\text{cone}))(\langle \vec{1}^d, 0\rangle)$
- $\mathcal{F}(\text{cone}) = {x_{d+\mathbf{1}}}^0(\ldots) + x_{d+\mathbf{1}}(\texttt{POLYTOPE}) + {x_{d+\mathbf{1}}}^2(\ldots) + \ldots$
- $\frac{d}{dx_d}\mathcal{F}(\text{cone}) = 0 + 0\cdot(\ldots) + 1\cdot\texttt{POLYTOPE} + 2x_{d+\mathbf{1}}(\ldots) + \ldots$
- $\frac{d}{dx_d}\mathcal{F}(\text{cone})(\langle \vec{1}^d, 0\rangle) = \texttt{POLYTOPE}(\vec{1}) + 2\cdot 0\cdot(\ldots) + \ldots$
- $\frac{d}{dx_d}\mathcal{F}(\text{cone})(\langle \vec{1}^d, 0\rangle) = \texttt{POLYTOPE}(\vec{1})$

## Assuming $\mathcal{F}$ for cones, derive full $\mathcal{F}$: Part 2 (Lines)

- Line = $\sum_{\text{dimension}}$ cone + cone - point.
- Since line can be translated:

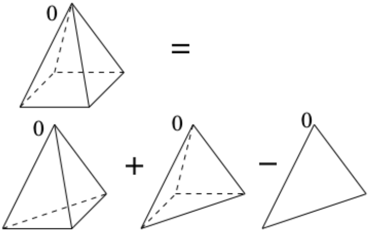
$$\begin{gathered}\forall \vec{x} \in L, L = \vec{x} + L \\ \forall x \in L, \mathcal{F}(L) = \mathcal{F}(L) + \mathcal{F}(\vec{x}) \\ \mathcal{F}(L) = 0\end{gathered}$$

$$\begin{aligned}\text{count}(x) &= \mathcal{F}((-\infty, \infty)) \\ &= (\ldots + x^{-2} + x^{-1} + x^0) + (x^0 + x^1 + x^2 + \ldots) - x^0 \\ &= \frac{1}{1 - \frac{1}{x}} + \frac{1}{1 - x} - 1\end{aligned}$$

- In 1-d example, radius of convergence of left and right cone was 0
- Is this really well-defined? (what is this ring which admits $f(x) = \ldots + x^{-1} + x^0 + x^1 + \ldots$)

# Assuming $\mathcal{F}$ for simple cone, derive for cone

- Simple cone: $SK = co(u_i) = \{\sum_i \lambda_i u_i | \lambda_i \geq 0\}$, $u_i \in \mathbb{Z}^d$, $u_i$ are linearly independent.
- Cone: $C = co(u_i)$, $u_i \in \mathbb{Q}^d$
- inclusion exclusion: decompose cone into simple cones.



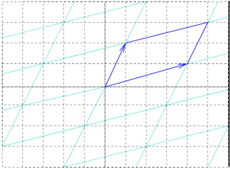
# $\mathcal{F}$ for simple cones: Part 1

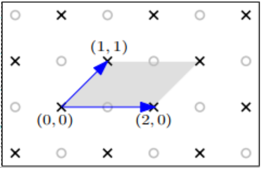
- Consider the positive orthant in 3D: $P \subset \mathbb{Q}^3 = \left\{(x, y, z) \mid x, y, z \geq 0\right\}$
- $P = cone((1,0,0),(0,1,0),(0,0,1))$
- this is a simple cone, and here's how we count it:

$$\begin{aligned}\mathcal{F}([P]) &= \sum_{i,j,k \in [0,\infty)} x^i y^j z^k \\ &= \sum_{i=0}^{\infty} x^i \left( \sum_{j=0}^{\infty} y^j \left( \sum_{k=0}^{\infty} z^k \right) \right) \\ &= \frac{1}{1-x} \cdot \frac{1}{1-y} \cdot \frac{1}{1-z}\end{aligned}$$

# $\mathcal{F}$ for simple cones: Part 2

- General story is similar
- $SK = co(u_i)$, $u_i$ linearly independent.
- Since $u_i$ is linearly independent, some points $\vec{x} \in cone(u_i)$ have unique representation $\vec{x} = \sum_i \lambda_i u_i$, $\lambda_i \in \mathbb{Z}$
- fundamental paralellopiped ($\Pi$) will tile the plane.
- We can count the $\vec{x}$, and make $\vec{x}$ responsible for the "tile" of skipped points.





$$\mathcal{F}(SK) = \underbrace{\left( \sum_{\vec{p} \in \Pi \cap \mathbb{Z}^d} x^{\vec{p}} \right)}_{\text{per-tile points}} \underbrace{\prod_i \frac{1}{1 - x^{u_i}}}_{\text{tile starting point } \vec{x}}$$

# Performance - How?

- Write simple cone as sum of unimodular cones:

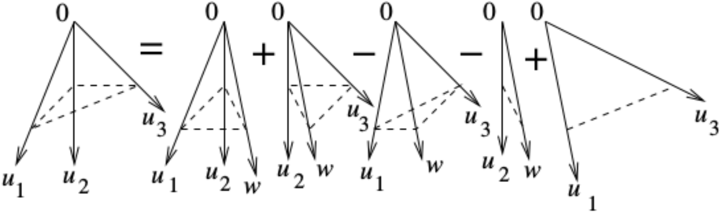
$$[K] = \sum_i \alpha_i [K_i] + \text{lower dimesnional cones}$$

- We concentrate on $\sum_i \alpha_i [K_i]$

$$\alpha_i \in \{-1, 1\} \text{ and } K_i \text{ are unimodular.}$$

- Lower dimensional cones are taken care of by a trick.

# Unimodular decomposition of a simple cone $K$: Part 1



# Unimodular decomposition of simple cone $K$: Part 2

- $\mathit{Index}(K) = \mathit{Volume}(\Pi(K))$

$\mathit{Index}(K) = 1 \leftrightarrow$ K is unimodular. $\mathit{Index}(K)$ is a measure of non-unimodularity.

- Introduce procedure which takes **polynomial steps** to **reduce Index(K)**
- Let $K = \mathit{cone}(u_1, u_2, \ldots, u_d)$, $u_i \in \mathbb{Z}^d$, $u_i$ are linearly independent.
- High level idea:
  - Pick a non-zero integer point $p$, which is shorter than the current $u_i$.
  - create $d$ new "basis sets", $\texttt{Basis}_j = \{u_\mathbf{1}, u_\mathbf{2}, \ldots, u_d\} \setminus \{u_j\} \cup \{p\}$
  - make new cones, $K_i = \mathit{cone}(\texttt{Basis}_j)$ and show that $\mathit{Index}(K_i) < \mathit{Index}(K)$
  - Intuition for shorter index: $p$ is shorter than $u_i$, parallelopiped will be smaller.
  - $K = \sum \alpha_i K_i +$ faces of $K_i$
  - show that $\mathit{Index}(K_i)$ reduces by a large enough factor that poly rounds are enough to reduce to 1
  - eliminate faces for $K_i$ with trick to kill lower dimensional objects.

# Decomposition example

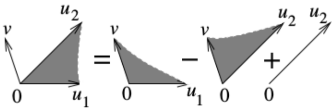
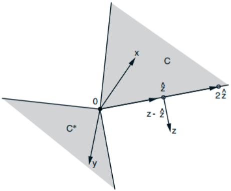


**Figure 20.** Writing the cone as a linear combination of cones with smaller indices

# Polar trick

- Polar:

$$P^\circ = \left\{ \vec{y} \in \mathbb{Q}^d : \forall \vec{p} \in P,\ \vec{p} \cdot \vec{y} \ \geq 0 \right\}$$



- Lower dimensional cones do not matter. First take $[K^\circ]$, then compute unimodular decomposition of this:

$$[K^\circ] = \sum_i \alpha_i [K_i] + \text{lower dimensional cones}$$

$$[(K^\circ)^\circ] = [K] = \sum_i \alpha_i K_i + \text{cones with lines}$$

$$\mathcal{F}([K]) = \sum_i \alpha_i \mathcal{F}(K_i) + \mathcal{F}(\text{cones with lines}) = \sum_i \alpha_i \mathcal{F}(K_i) + 0$$

# References

- Lattice Points, Polyhedra, and Complexity: Alexander Barvinok
- Integer points in polyhedra: Alexander Barvinok

# Thanks!

Questions?

# Minkowski convex body theorem

- Statement: Convex set $P \subset \mathbb{R}^d$, which is symmetric with respect to the origin ($\forall x \in P, -x \in P$), has volume greater than or equal to $2^d$ contains a non-zero integer point.
- Recap: Let $K = cone(u_1, u_2, \ldots, u_d)$, $u_i \in \mathbb{Z}^d$, $u_i$ are linearly independent.
  - Pick a non-zero integer point $p$ in $K$ **(why does this integer point exist?)**.
- Construct

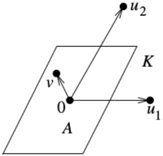
$$\Pi_0 = \left\{ \sum_i \alpha_i u_i : |\alpha_i| \leq \frac{1}{\sqrt[d]{Index(K)}} \right\}$$

  - Symmetric
  - Length per axis: $\frac{2|u_i|}{\sqrt[d]{Index(K)}}$
- Total volume:

$$\begin{aligned} Volume(\Pi_0) &= \prod_{i=1}^{d} \frac{2|u_i|}{\sqrt[d]{Index(K)}} \\ &= 2^d \frac{\prod_{i=1}^{d} |u_i|}{Index(K)} = 2^d \end{aligned}$$

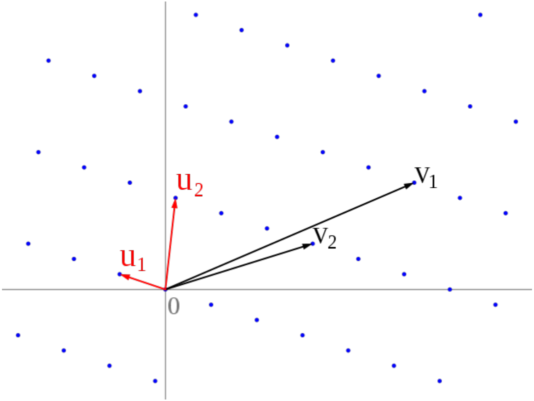
- Hence, by Minkowski convex body, we find a point $p \in \mathbb{Z}^d$ in $\Pi_0$. If this point is in the wrong direction (facing outward), pick $-p$.
- This only gives us **existence** of such a point; need to use LLL to **construct** this point.

# Minkowski convex body theorem example



# LLL (Lenstra–Lenstra–Lovász lattice basis reduction)

- Given an arbitrary basis $u_i \in \mathbb{Z}^d$ of a lattice, produces a new basis which is "nicer".
- New basis consists of shorter vectors that are "more orthogonal"
- Runs in polynomial time.
- Determining *shortest basis* is hard - used in cryptosystems IIUC.



- Black is input, red is output.

# Using LLL

- Refer to page 140 of integer points in polyhedra.
- Recap: Let $K = cone(u_1, u_2, \ldots, u_d)$, $u_i \in \mathbb{Z}^d$, $u_i$ are linearly independent.
  - Pick a non-zero integer point $p$ in $K$ **(how do we find it?)**
- Let $T : V \to V$, $T(u_i) = e_i$.
- Let $\Lambda =$ Lattice of $u_i$
- $\Lambda_0 = T(\Lambda)$
- Apply LLL on $\Lambda_0$, get shorter description
- Construct $w_0 \in \Lambda_0 \setminus \{0\}$ such that $\|w_0\|_\infty$ is minimum.
- This bounds $\|w_0\| \leq \|w_0\|_\infty \sqrt{d}$ .
- $w = T^{-1}(w_0)$
- Caveat: Why do we need this song and dance? Why not just work on the original space?

# Assuming $\mathcal{F}$ for cones, derive full $\mathcal{F}$: Part 1.2 (Polytopes)

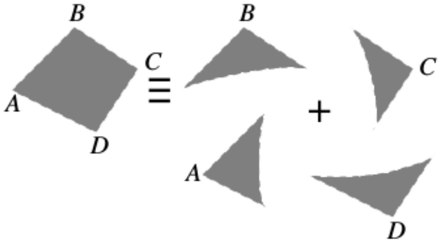


FIGURE 36. Representing the interior of a polytope as the sum of the interiors of its tangent cones at the vertices modulo polyhedra with lines.