

# Counting integer points in polyhedra

Siddharth Bhat

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# Definitions

- ▶ Polyhedra:  $\{\vec{x} \in \mathbb{Q}^d \mid A\vec{x} \leq \vec{b}\}$
- ▶ Polytope: bounded polyhedra.
- ▶ Cone:  $\text{cone}(\vec{u}_i) = \left\{ \sum_i \lambda_i \vec{u}_i \mid \lambda_i \geq 0 \right\}$ ,  $\vec{u}_i \in \mathbb{Q}^d$
- ▶ Simple cone:  $SK = \text{cone}(\vec{u}_i)$ ,  $\vec{u}_i \in \mathbb{Z}^d$ ,  $\vec{u}_i$  are linearly independent.
- ▶ Unimodular cone:  $UK = \text{cone}(\vec{u}_i)$ ,  $\text{Volume}(\vec{u}_i) = 1$
- ▶ Line: subspace.

# Pictures of defintions!

- ▶ polytope



- ▶ cone



- ▶ polyhedra



## Example 1: valuation of line

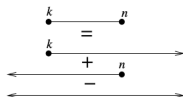
- ▶  $P$  is a polyhedra, then  $\mathcal{F}([P]) = \sum_{\vec{m} \in P \cap \mathbb{Z}^d} (x^{\vec{m}})$
- ▶  $\mathcal{F}([P])(\vec{1}) = \text{number of points.}$

$$\mathcal{F}((-\infty, \infty)) = \sum_{i \in \mathbb{Z}} x^i$$

$$\begin{aligned} \text{count}(x) &= \mathcal{F}((-\infty, \infty)) \\ &= \mathcal{F}((-\infty, 0]) + \mathcal{F}([0, \infty)) - \mathcal{F}(0) \\ &= (\dots + x^{-2} + x^{-1} + x^0) + (x^0 + x^1 + x^2 + \dots) - x^0 \\ &= \frac{1}{1 - \frac{1}{x}} + \frac{1}{1 - x} - 1 \\ &= \frac{-x}{1 - x} + \frac{1}{1 - x} = \frac{1 - x}{1 - x} - 1 = 0 \end{aligned}$$

- ▶ number of points in a line is 0!

## Example 2: valuation of interval



$$\begin{aligned}\text{count}(x) &= \mathcal{F}([0, n]) = \mathcal{F}([k, \infty)) + \mathcal{F}((-\infty, n]) - \mathcal{F}((\infty, \text{infy})) \\ &= (x^k + x^{k+1} + \dots) + \\ &\quad (\dots + x^{n-2} + x^{n-1} + x^n) + \\ &\quad (\dots + x^{-2} + x^{-1} + x^0 + x^1 + \dots) \\ &= \frac{x^k}{1-x} + \frac{x^n}{1-x^{-1}} + 0 \\ &= \frac{x^k - x^{n+1}}{1-x}\end{aligned}$$

$$\text{count}(1) = \text{L'hospital} = (n+1) - k = n - k + 1$$

# Proof outline

- ▶ Algebra of polyhedra,  $P(\mathbb{Q}^d)$
- ▶  $[ ] : \mathbb{Q}^d \rightarrow P(\mathbb{Q}^d)$
- ▶ Existence of  $\mathcal{F} : P(\mathbb{Q}^d) \rightarrow \mathbb{C}(x)$ , such that:
  - ▶  $\mathcal{F}$  is linear
  - ▶  $P$  is a polyhedra, then  $\mathcal{F}([P]) = \sum_{\vec{m} \in P \cap \mathbb{Z}^d} (x^{\vec{m}})$
  - ▶  $\mathcal{F}([\text{line}]) = 0$  (important, allows modulo line decompositions)
- ▶  $\mathcal{F}(P)(1) = \text{number of points in } P$
- ▶ reduction:  $\mathcal{F}$  for cones gives full  $\mathcal{F}$
- ▶ reduction:  $\mathcal{F}$  for simple cones gives  $\mathcal{F}$  for cones

# Caveats

- ▶ Do not understand subtleties of convergence arguments (how is evaluating at  $\vec{1}$  correct?).
- ▶ No intuition for LLL, Lattice reduction.

## Assuming $\mathcal{F}$ for cones, derive full $\mathcal{F}$ : Part 1 (Polytopes)

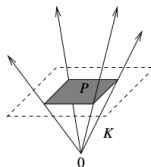


FIGURE 66. A polytope  $P \subset \mathbb{R}^d$  and a cone  $K \subset \mathbb{R}^{d+1}$  based on  $P$ .

- Write polytope as intersection of hyperplane + cone.
- $\mathcal{F}(\text{polytope}) = (\frac{d}{dx_{d+1}} \mathcal{F}(\text{cone}))(\langle \vec{1}^d, 0 \rangle)$
- $\mathcal{F}(\text{cone}) = x_{d+1}^0(\dots) + x_{d+1}(\text{POLYTOPE}) + x_{d+1}^2(\dots) + \dots$
- $\frac{d}{dx_d} \mathcal{F}(\text{cone}) = 0 + 0 \cdot (\dots) + 1 \cdot \text{POLYTOPE} + 2x_{d+1}(\dots) + \dots$
- $\frac{d}{dx_d} \mathcal{F}(\text{cone})(\langle \vec{1}^d, 0 \rangle) = \text{POLYTOPE}(\vec{1}) + 2 \cdot 0 \cdot (\dots) + \dots$
- $\frac{d}{dx_d} \mathcal{F}(\text{cone})(\langle \vec{1}^d, 0 \rangle) = \text{POLYTOPE}(\vec{1})$



## Assuming $\mathcal{F}$ for cones, derive full $\mathcal{F}$ : Part 2 (Lines)

- ▶ Line =  $\sum_{\text{dimension}} \text{cone} + \text{cone} - \text{point}$ .
- ▶ Since line can be translated:

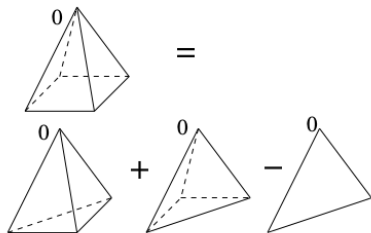
$$\begin{aligned}\forall \vec{x} \in L, L &= \vec{x} + L \\ \forall x \in L, \mathcal{F}(L) &= \mathcal{F}(L) + \mathcal{F}(\vec{x}) \\ \mathcal{F}(L) &= 0\end{aligned}$$

$$\begin{aligned}\text{count}(x) &= \mathcal{F}((-\infty, \infty)) \\ &= (\dots + x^{-2} + x^{-1} + x^0) + (x^0 + x^1 + x^2 + \dots) - x^0 \\ &= \frac{1}{1 - \frac{1}{x}} + \frac{1}{1 - x} - 1\end{aligned}$$

- ▶ In 1-d example, radius of convergence of left and right cone was 0
- ▶ Is this really well-defined? (what is this ring which admits  $f(x) = \dots + x^{-1} + x^0 + x^1 + \dots$ )

## Assuming $\mathcal{F}$ for simple cone, derive for cone

- ▶ Simple cone:  $SK = co(u_i) = \{\sum_i \lambda_i u_i | \lambda_i \geq 0\}$ ,  $u_i \in \mathbb{Z}^d$ ,  $u_i$  are linearly independent.
- ▶ Cone:  $C = co(u_i)$ ,  $u_i \in \mathbb{Q}^d$
- ▶ inclusion exclusion: decompose cone into simple cones.



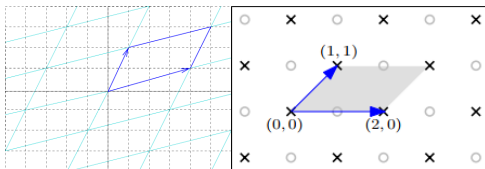
## $\mathcal{F}$ for simple cones: Part 1

- ▶ Consider the positive orthant in 3D:  $P \subset \mathbb{Q}^3 = \{(x, y, z) \mid x, y, z \geq 0\}$
- ▶  $P = \text{cone}((1, 0, 0), (0, 1, 0), (0, 0, 1))$
- ▶ this is a simple cone, and here's how we count it:

$$\begin{aligned}\mathcal{F}([P]) &= \sum_{i,j,k \in [0, \infty)} x^i y^j z^k \\ &= \sum_{i=0}^{\infty} x^i \left( \sum_{j=0}^{\infty} y^j \left( \sum_{k=0}^{\infty} z^k \right) \right) \\ &= \frac{1}{1-x} \cdot \frac{1}{1-y} \cdot \frac{1}{1-z}\end{aligned}$$

## $\mathcal{F}$ for simple cones: Part 2

- ▶ General story is similar
- ▶  $SK = co(u_i)$ ,  $u_i$  linearly independent.
- ▶ Since  $u_i$  is linearly independent, some points  $\vec{x} \in cone(u_i)$  have unique representation  $\vec{x} = \sum_i \lambda_i u_i$ ,  $\lambda_i \in \mathbb{Z}$
- ▶ fundamental parallelepiped ( $\Pi$ ) will tile the plane.
- ▶ We can count the  $\vec{x}$ , and make  $\vec{x}$  responsible for the "tile" of skipped points.



$$\mathcal{F}(SK) = \underbrace{\left( \sum_{\vec{p} \in \Pi \cap \mathbb{Z}^d} x^{\vec{p}} \right)}_{\text{per-tile points}} \underbrace{\prod_i \frac{1}{1 - x^{u_i}}}_{\text{tile starting point } \vec{x}}$$

## Performance - How?

- Write simple cone as sum of unimodular cones:

$$[K] = \sum_i \alpha_i [K_i] + \text{lower dimensional cones}$$

- We concentrate on  $\sum_i \alpha_i [K_i]$

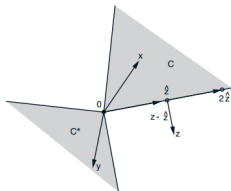
$$\alpha_i \in \{-1, 1\} \text{ and } K_i \text{ are unimodular.}$$

- use LLL to generate a new basis with a new vector  $w$
- recurse over this new cone set.

## Polar trick

- Polar:

$$P^\circ = \left\{ \vec{y} \in \mathbb{Q}^d : \forall \vec{p} \in P, \vec{p} \cdot \vec{y} \geq 0 \right\}$$



- Lower dimensional cones do not matter. First take  $[K^\circ]$ , then compute unimodular decomposition of this:

$$[K^\circ] = \sum_i \alpha_i [K_i] + \text{lower dimensional cones}$$

$$[(K^\circ)^\circ] = [K] = \sum_i \alpha_i K_i + \text{cones with lines}$$

$$\mathcal{F}([K]) = \sum_i \alpha_i \mathcal{F}(K_i) + \mathcal{F}(\text{cones with lines}) = \sum_i \alpha_i \mathcal{F}(K_i) + 0$$

# References

- ▶ Lattice Points, Polyhedra, and Complexity: Alexander Barvinok
- ▶ Integer points in polyhedra: Alexander Barvinok

# Thanks!

Questions?



Assuming  $\mathcal{F}$  for cones, derive full  $\mathcal{F}$ : Part 1.2 (Polytopes)

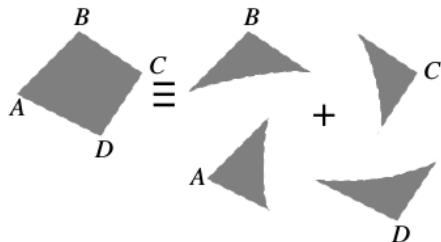


FIGURE 36. Representing the interior of a polytope as the sum of the interiors of its tangent cones at the vertices modulo polyhedra with lines.