Encyclopedia of Mathematics and its Applications 93

CONTINUOUS LATTICES AND DOMAINS

G. GIERZ, K. H. HOFMANN, K. KEIMEL, J. D. LAWSON,
M. W. MISLOVE AND D. S. SCOTT

CAMBRIDGE

This page intentionally left blank

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

FOUNDING EDITOR G.-C. ROTA

Volume 93

Continuous Lattices and Domains

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

http://publishing.cambridge.org/stm/mathematics/com

4 W. Miller, Jr. *Symmetry and separation of variables*
6 H. Minc *Permanents*
11 W. B. Jones and W. J. Thron *Continued fractions*
12 N. F. G. Martin and J. W. England *Mathematical theory of entropy*
18 H. O. Fattorini *The Cauchy problem*
19 G. G. Lorentz, K. Jetter and S. D. Riemenschneider *Birkhoff interpolation*
21 W. T. Tutte *Graph theory*
22 J. R. Bastida *Field extensions and Galois theory*
23 J. R. Cannon *The one dimensional heat equation*
25 A. Salomaa *Computation and automata*
26 N. White (ed.) *Theory of matroids*
27 N. H. Bingham, C. M. Goldie and J. L. Teugels *Regular variation*
28 P. P. Petrushev and V. A. Popov *Rational approximation of real functions*
29 N. White (ed.) *Combinatorial geometrics*
30 M. Pohst and H. Zassenhaus *Algorithmic algebraic number theory*
31 J. Aczel and J. Dhombres *Functional equations containing several variables*
32 M. Kuczma, B. Chozewski and R. Ger *Iterative functional equations*
33 R. V. Ambartzumian *Factorization calculus and geometric probability*
34 G. Gripenberg, S.-O. Londen and O. Staffans *Volterra integral and functional equations*
35 G. Gasper and M. Rahman *Basic hypergeometric series*
36 E. Torgersen *Comparison of statistical experiments*
37 A Neumaier *Intervals methods for systems of equations*
38 N. Korneichuk *Exact constants in approximation theory*
39 R. A. Brualdi and H. J. Ryser *Combinatorial matrix theory*
40 N. White (ed.) *Matroid applications*
41 S. Sakai *Operator algebras in dynamical systems*
42 W. Hodges *Model theory*
43 H. Stahl and V. Totik *General orthogonal polynomials*
44 R. Schneider *Convex bodies*
45 G. Da Prato and J. Zabczyk *Stochastic equations in infinite dimensions*
46 A Bjorner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler *Oriented matroids*
47 E. A. Edgar and L. Sucheston *Stopping times and directed processes*
48 C. Sims *Computation with finitely presented groups*
49 T. Palmer *Banach algebras and the general theory of *-algebras*
50 F. Borceux *Handbook of categorical algebra I*
51 F. Borceux *Handbook of categorical algebra II*
52 F. Borceux *Handbook of categorical algebra III*
54 A. Katok and B. Hassleblatt *Introduction to the modern theory of dynamical systems*
55 V. N. Sachkov *Combinatorial methods in discrete mathematics*
56 V. N. Sachkov *Probabilistic methods in discrete mathematics*
57 P. M. Cohn *Skew Fields*
58 Richard J. Gardner *Geometric tomography*
59 George A. Baker, Jr. and Peter Graves-Morris *Padé approximants*
60 Jan Krajicek *Bounded arithmetic, propositional logic, and complex theory*
61 H. Gromer *Geometric applications of Fourier series and spherical harmonics*
62 H. O. Fattorini *Infinite dimensional optimization and control theory*
63 A. C. Thompson *Minkowski geometry*
64 R. B. Bapat and T. E. S. Raghavan *Nonnegative matrices and applications*
65 K. Engel *Sperner theory*
66 D. Cvetkovic, P. Rowlinson and S. Simic *Eigenspaces of graphs*
67 F. Bergeron, G. Labelle and P. Leroux *Combinatorial species and tree-like structures*
68 R. Goodman and N. Wallach *Representations of the classical groups*
69 T. Beth, D. Jungnickel and H. Lenz *Design Theory volume I 2 ed.*
70 A Pietsch and J. Wenzel *Orthonormal systems and Banach space geometry*
71 George E. Andrews, Richard Askey and Ranjan Roy *Special Functions*
72 R. Ticciati *Quantum field theory for mathematicians*
76 A. A. Ivanov *Geometry of sporadic groups I*
78 T. Beth, D. Jungnickel and H. Lenz *Design Theory volume II 2 ed.*
80 O. Stormark *Lie's Structural Approach to PDE Systems*
81 C. F. Dunkl and Y. Xu *Orthogonal polynomials of several variables*
82 J. Mayberry *The foundations of mathematics in the theory of sets*
83 C. Foias, R. Temam, O. Manley and R. Martins da Silva Rosa *Navier-Stokes equations and turbulence*
84 B. Polster and G. Steinke *Geometries on Surfaces*
85 D. Kaminski and R. B. Paris *Asymptotics and Mellin–Barnes integrals*
86 Robert J. McEliece *The theory of information and coding, 2 ed.*
87 Bruce A. Magurn *An algebraic introduction to K-theory*

Continuous Lattices and Domains

G. GIERZ

K. H. HOFMANN

K. KEIMEL

J. D. LAWSON

M. MISLOVE

D. S. SCOTT

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 2RU, United Kingdom

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org

Information on this title: www.cambridge.org/9780521803380

First published in print format 2003

ISBN-13 978-0-511-06356-5 eBook (NetLibrary)
ISBN-10 0-511-06356-3 eBook (NetLibrary)

ISBN-13 978-0-521-80338-0 hardback
ISBN-10 0-521-80338-1 hardback

Contents

Preface

BACKGROUND. In 1980 we published *A Compendium of Continuous Lattices*. A continuous lattice is a partially ordered set characterized by two conditions: firstly, completeness, which says that every subset has a least upper bound; secondly, continuity, which says that every element can be approximated from below by other elements which in a suitable sense are much smaller, as for example finite subsets are small in a set theoretical universe. A certain degree of technicality cannot be avoided if one wants to make more precise what this "suitable sense" is: we shall do this soon enough. When that book appeared, research on continuous lattices had reached a plateau.

The set of axioms proved itself to be very reasonable from many viewpoints; at all of these aspects we looked carefully. The theory of continuous lattices and its consequences were extremely satisfying for order theory, algebra, topology, topological algebra, and analysis. In all of these fields, applications of continuous lattices were highly successful. Continuous lattices provided truly interdisciplinary tools.

Major areas of application were the theory of computing and computability, as well as the semantics of programming languages. Indeed, the order theoretical foundations of computer science had been, some ten years earlier, the main motivation for the creation of the unifying theory of continuous lattices. Already the *Compendium of Continuous Lattices* itself contained signals pointing future research toward more general structures than continuous lattices. While the condition of *continuity* was a robust basis on which to build, the condition of completeness was soon seen to be too stringent for many applications in computer science – and indeed also in pure mathematics; an example is the study of the set of nonempty compact subsets of a topological space partially ordered by $\supseteq$: this set is a very natural object in general topology but fails to be a complete lattice in a noncompact Hausdorff space, while a filter basis of compact sets does have a nonempty intersection. Some form of completeness therefore should be retained; the form that is satisfied in most applications is that of "directed completeness", saying that every subset in which any two element set has an upper bound has a least upper bound; the existence of either a minimal or a maximal element is not implied.

In computer science it has become customary to speak of a poset with this weak completeness property as a deeceepea-oh, written **dcpo** (for **d**irected **c**omplete **p**artially **o**rdered set). *A continuous* **dcpo** *is what we call a* **domain**. Since this word appears in the title of this book, our terminology must be stated clearly at the beginning. In that branch of order theory with which this book deals there is no terminology clouded in more disagreement and lack of precision than that of a "domain", because it has become accepted as a sort of *nontechnical* terminology.

Domains in our sense had moved into the focus of researchers' attention at the time when the *Compendium of Continuous Lattices* was written, although then they were consistently called *continuous posets*, notably in the *Compendium* itself where they appear in many exercises. When their significance was discovered, it was too late to incorporate an emerging theory in the main architecture of the book, and it was too early for presenting a theory *in statu nascendi*. So we opted at that time for giving the reader an impression of things to come by indicating most of what we knew at the time in the form of exercises. The rising trend and our perception of it were confirmed in monographs, proceedings, and texts which appeared in a steady stream trailing the *Compendium*:

1981 Bernhard Banaschewski and Rudolf-Eberhard Hoffmann, editors, *Continuous Lattices*, Springer Lecture Notes in Mathematics **871**, x+413pp.,

1982 Rudolf-Eberhard Hoffmann, editor, *Continuous Lattices and Related Topics*, Mathematik Arbeitspapiere der Universität Bremen **27**, vii+314pp.,

1982 Peter Johnstone, *Stone Spaces*, Cambridge Studies in Advanced Mathematics **3**, xxi+370pp.,

1984 H. Lamarr Bentley, Horst Herrlich, M. Rajagopalan and H. Wolff, editors, *Categorical Topology*, Heldermann Sigma Series in Pure Mathematics **5**, xv+635pp.,

1985 Rudolf-Eberhard Hoffmann and Karl Heinrich Hofmann, editors, *Continuous Lattices and Their Applications*, Marcel Dekker Lecture Notes in Pure and Applied Mathematics **101**, x+369 pp.,

1989 Steven Vickers, *Topology via Logic*, Cambridge Tracts in Theoretical Computer Science **5** (2nd edition 1990), xii+200pp.,

1990 B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, 1990, vii+248pp.,
1994 S. Abramsky and A. Jung, Domain theory, in S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science, Vol. III: Semantic Structures*, Oxford University Press,
1994 V. Stoltenberg-Hansen, I. Lindström, and E. R. Griffor, *Mathematical Theory of Domains*, Cambridge Tracts in Theoretical Computer Science **22**, xii+349pp.,
1998 R. M. Amadio and P.-L. Curien, *Domains and Lambda-Calculi*, Cambridge Tracts in Theoretical Computer Science **46**, xvi+484pp.

While some of these sources are devoted to supplementing the lattice theoretical and topological aspects of continuous lattices, the development of a more general domain theory and its computer theoretical applications predominate in this literature. From the viewpoint of pure mathematics, arguably the most prominent developments after the appearance of the *Compendium of Continuous Lattices* were

- the Lawson duality of domains (much in the spirit of Pontryagin duality of locally compact abelian groups),
- the first creation of a really satisfactory general theory of locally compact spaces in general topology via domain theory,
- other expanded connections with topology such as the the theory of sober spaces, principally the machinery surrounding the Hofmann–Mislove Theorem,
- the cross connections of domain theory and the theory of cartesian closed categories,
- the representation of topological spaces as the "ideal" or maximal points of a domain,
- and entirely new outlooks on classical analysis through domain theory.

AIMS. The *Compendium* by Gierz *et al.*, as it became known after a while, was out of print in a few years. It continued to be cited as a comprehensive reference on continuous lattices and their generalizations in spite of the cumbersome reference to a line of no less than six authors whose collaboration – notwithstanding their motley mathematical origin – was amply explained in the foreword of the *Compendium*; the five authors who had to take cover behind the hedge of "*et al.*" learned to live in hiding. The list of books which followed the *Compendium* is impressive. But somehow it seemed that the *Compendium* was not replaced or superseded, certainly not by one single book which could substitute for its expository and pedagogical drift. People felt that an attempt to overhaul the *Compendium* and to present a new edition containing the original information as well as reflecting developments of two decades of research in the larger scope of domain theory might be welcomed by readers in the area, old and young. In the fenced-in area of continuous lattices, the *Compendium* still had encyclopedic aspirations. As the vast literature of the last twenty years beyond the already respectable list of references in the *Compendium* indicates, this ambition is now beyond our grasp. It is therefore with a touch of modesty that, in the title of our book, we now drop the word *Compendium* and simply present a treatise on “Continuous Lattices and Domains”.

As was its predecessor, this book is intended to present the mathematical foundations of the theory of continuous lattices and domains from the ingredients of order theory, topology and algebra and blends of all of these. Our use of category theory remains close to the concrete categories arising in our investigations, and thus we avoid the high degree of abstraction that category theory allows. It has been our deliberate choice only to lay the groundwork for the numerous applications that the theory of domains has found in the area of abstract theories of computation, the semantics of programming languages, logic and lambda calculus, and in other branches of mathematics. In the following selective list of subject matter not treated in this book, the reader may find guidance to further sources which are concerned with these and other applications; this list is far from being comprehensive.

- Domains for semantics of lambda calculi and of programming languages (see e.g. [Scott, 1993], [Scott, 1972a], [Scott, 1976], [Plotkin, B1981], [Gunter, B1992], [Winskel, B1993], [Amadio and Curien, B1998], [Reynolds, B1998]),
- stable domain theory, Girard’s coherent spaces, hypercoherences (see e.g. [Amadio and Curien, B1998], [Girard, B1989], [Ehrhard, 1993]),
- Scott’s information systems and more generally domain theory in logical form (see e.g. [Scott, 1982c], [Abramsky, 1991b], [Jung *et al.*, 1997]),
- domains and computability, computable analysis (see e.g. [Eršov, 1972a], [Stoltenberg-Hansen *et al.*, B1994], [Escardó, 1996a], [Edalat and Sünderhauf, 1999]),
- quantitative domain theory with its many different approaches,
- categorical generalizations (see e.g. [Adámek, 1997]),
- axiomatic and synthetic domain theory (see e.g. [Hyland, 1991], [Fiore, 1997], [Fiore and Rosolini, 1997a], [Fiore and Plotkin, 1997], [Taylor, B1998]),
- applications of domain theory in classical mathematics (see [Edalat, 1997a]).

GENESIS. In the foreword to the *Compendium* we familiarized the reader with some of the background and how it was written in a time that pre-dated the actual advent of TEX as the standard for mathematical typesetting. The writing of the new book proceeded under different auspices.

As a first step, even prior to our decision to go ahead with a new printing of the compendium, Dana S. Scott secured a complete LATEX source file of the *Compendium* in its entirety at Carnegie Mellon University; this source file was kept and elaborated typographically at the Technical University of Darmstadt in the custody of Klaus Keimel. We kept a pretty good record of all typographical and mathematical errors that we and our readers found in the *Compendium*, and all of these were corrected in our master file. A first updating of the bibliography of the *Compendium* was compiled by Rudolf-Eberhard Hoffmann, Karl Heinrich Hofmann, and Dana S. Scott in 1985 and was published in the Marcel Dekker volume edited by Hoffmann and Hofmann in 1985, pp. 303–360. At a time when we seriously thought about updating our data base on the literature for this book, an electronic file of the Marcel Dekker bibliography could no longer be located. Therefore this data base had to be reconstructed, and that was done under the supervision of Klaus Keimel at Darmstadt. In 2000 he also initiated a compilation of more current literature; many people contributed to that collection; we express our gratitude to all of them. Much of this material, although not all of it, entered the bibliography of this book. An Appendix to the *Compendium* (pp. 347–349) contained a listing on 52 Memos written and circulated in the Seminar on Continuity in Semilattices (SCS) from January 1976 through June 1979, because this body of material constituted much of the history of the content of the *Compendium*. The seminar continued for a number of years through June 1986, and we include in the present book a complete list of 98 SCS Memos (see pp. 564–567).

Several visits of Dana S. Scott's to Darmstadt consolidated the plan to envisage a new edition of the *Compendium*. Yet it became obvious soon that a considerable workload of rewriting would have to be done on the existing master file in order to accommodate domain theory. For any number of reasons it was not easy to get the project on its way; one of the simplest explanations is that the mathematical biographies of all of us had diverged sufficiently that the intensive spirit of collaboration of the 1970s was almost impossible to rekindle. Yet serious planning was undertaken by Hofmann and Lawson at a meeting at Louisiana State University at Baton Rouge on March 10, 2000, by Gierz, Hofmann, Keimel, and Lawson on March 16, 2000, at a workshop organized by the University of Riverside in honor of Albert Stralka on the occasion of his sixtieth birthday, and at a meeting of Klaus Keimel and Dana S. Scott on March 22, 2000, in Pittsburgh at Carnegie Mellon University. After these initiatives the actual rewriting began in earnest at Tulane University in New Orleans, at the Technical University of Darmstadt, and at Louisiana State University. It was helpful that a conference in Cork (Ireland) in July 2000 united Keimel, Lawson, Mislove, and Scott for discussions.

With respect to the writing itself, Chapters O, I, and II were revised jointly by Hofmann, Lawson, and Keimel, Chapters III, VI, and VII by Lawson, and Chapters IV and V by Keimel. The revisions consisted primarily of reformulating and supplementing from the lattice context to the **dcpo** context, a task that frequently proved nontrivial. In addition several new sections were written to reflect some of the developmental highlights since the *Compendium* appeared: Section O-5 on T_0 spaces (Lawson and Keimel), Sections III-3 and III-5 on quasicontinuous and compact domains (Lawson), Section IV-2 on duality (Hofmann), Sections IV-5 and IV-7 on pro-continuous functors and domain equations (Keimel), Sections IV-8 and IV-9 on powerdomains (Lawson), Section V-6 on domain environments (Lawson), Section VI-6 on stably compact spaces (Lawson), and Section VII-3 on hypercontinuity (Lawson). In addition Keimel prepared the comprehensive index and other end material.

HIGHLIGHTS. It is an indication of the robust architecture of the old *Compendium* that the actual rewriting could proceed largely by retaining the chapter subdivision and revising and amplifying the old content. However, **dcpo**s replaced complete lattices wherever possible from Chapter O on and domains replaced continuous lattices where possible. This was not always possible; a good example is what we used to call "the algebraic characterization of continuous lattices" in the *Compendium*. This is attached to the monadic character of continuous lattices and simply fails for domains. Occasionally a good deal of work had to be invested to accommodate the more general viewpoint.

Chapter II on the Scott topology of domains is a case in point. We have amplified the function space aspect, described the Isbell topology on function spaces, and exposed it as a true generalization of the classical compact open topology. Furthermore we discuss the poset $Q(X)$ of compact saturated sets on a T_0 space with respect to the partial order $\supseteq$, allowing a full treatment of the Hofmann–Mislove Theorem and its various aspects and exposing some new aspects. We also elaborate on certain cartesian closed categories of domains.

Chapter III elaborates on what is known on the Lawson topology on domains and their compactness properties for this topology and thus contains much information that was not present in the *Compendium*. In the section on "Quasicontinuity and Liminf Convergence" we introduce a class of posets containing that of domains properly and call its members *quasicontinuous domains*. On a quasicontinuous domain, the Scott topology discussed in Chapter II is locally compact and sober, and the Lawson topology (the main topic of Chapter III) is regular and Hausdorff, and indeed much of the theory of domains can be recovered in this more general setting. A notion of liminf convergence is introduced, which is shown to be equivalent to topological convergence in the Lawson topology for (quasicontinuous) domains. In the old *Compendium* the concept of "quasicontinuity" was called the "GCL-property", as in "generalized continuous lattice". The section entitled "Compact Domains" is largely devoted to the question of when the Lawson topology on a domain is compact and wraps up with the theory of the Isbell topology in the context of function space topologies.

Beyond that which Chapter IV contained in the *Compendium*, it now presents a full treatment of the attractive Lawson duality of domains, which parallels Pontryagin duality – notably when it is restricted to the category of continuous semilattices (that is, domains which in addition are inf semilattices) and Scott-continuous semilattice morphisms; in that form it is a veritable character theory for domains. The Lawson duality of continuous semilattices allows us to round off the complex of the Hofmann–Mislove Theorem which was presented in Chapter II. A sort of geometric aspect of the duality between two domains is exposed in Chapter V, because it realizes a pair of dual domains as the spectrum and the cospectrum of a completely distributive complete lattice.

The section on projective limits in Chapter IV is now formulated for the category of domains (or even **dcpo**s) and morphisms appearing as pairs of a Galois adjunction; in the case of maps between complete lattices preserving arbitrary infs or sups this is automatic; in the more general setup of domains the presence of Galois adjunctions must be postulated. Wherever we had operated in the *Compendium* on a largely category theoretical level, we do not have to adjust our approach fundamentally to make it work in the more general **dcpo** framework that interests us in this book. However, the more general and updated treatment of these matters has resulted in a considerable expansion of these sections. The chapter closes with an introduction to the important topic of powerdomains, including the extended probabilistic powerdomain.

Chapter V in the *Compendium* dealt with the spectral theory of continuous lattices. Since spectral theory is largely a formalism applying to lattices, this chapter has remained largely stable, but it was augmented by a section on *domain environments* which illustrates a novel application of domain theory to that branch of analysis dealing largely with Polish spaces. It is in the nature of some of the material in the *Compendium* that it is not or only marginally affected by the general upgrading from continuous lattices to domains; sections dealing with such material remain preserved in the way they were in the old *Compendium.*

In Chapter VI the reader will find a new section, in which under the heading "Stably Compact Spaces" we discuss a concept of compact spaces which emulate in the wide class of T_0 spaces as many properties as seem reasonable of classical compact T_2 spaces. These spaces have a partner topology, called the co-compact topology, and the common refinement, called the patch topology, is a compact Hausdorff topology. The most prominent example of a stably compact space is a domain with the Scott topology such that the Lawson topology is compact; in this example the patch topology is the Lawson topology. We close Chapter VI with the spectral theory of these spaces.

Chapter VII includes a new section on "Hypercontinuity and Quasicontinuity". Hypercontinuous lattices are a special class of continuous lattices for which, among several diverse characterizations, the interval topology is Hausdorff. They stand in spectral duality to the quasicontinuous domains equipped with the Scott topology.

NOTES. The notes at the end of each section make some attempt to relate the material to the published literature, but these references are only representative, not comprehensive. Subsections entitled "Old Notes" are largely duplicated from *A Compendium of Continuous Lattices*, except for an effort to accommodate any renumbering that has taken place. Since individual contributions could at that time be identified via SCS Memos, which are listed in the bibliography, and since such a multiplicity of authors was involved, it seemed reasonable to depart from traditional practice and more or less identify some of the major contributions of various authors in the notes. Subsections entitled "New Notes" have been added to those sections that are new or significantly different from those appearing in the *Compendium.* Thus sections with little or no revision may have only "Old Notes", new sections will have only "New Notes", and old sections with significant revisions will have both.

BIBLIOGRAPHY. The literature about domain theory and continuous lattices has grown to such proportions that a comprehensive bibliography is not feasible. We have tried, however, to compile an extensive bibliography relevant to the topics treated in this book. The bibliography is subdivided into several sections:

- books, monographs, and collections, cited in the form [Gierz *et al.*, B1980], where the B refers to a book,
- conference proceedings, cited in the form [1982, Bremen] giving the year and the place of the conference,

- articles, cited simply in the form [Abramsky and Jung, 1994],
- dissertations and master's theses, cited in the form [Lawson, D1967], where the D refers to a dissertation,
- SCS Memos, cited in the form [SCS 15].

MIZAR. This is also the place to report on an activity of the MIZAR project group located primarily at the University of Białystok, Poland, the University of Alberta, Edmonton, Canada, and the Shinshu University, Nagano, Japan. It is the aim of the MIZAR project to codify mathematical knowledge in a data base. The codification means the formalization of concepts and proofs mechanically checked for logical correctness. The MIZAR language is a formal language derived from the mathematical vernacular. The principal idea was to design a language that is readable by mathematicians, and simultaneously, is sufficiently rigorous to enable processing and verifying by computer software.

Our monograph *A Compendium of Continuous Lattices* was chosen by the MIZAR group for testing their system. Since 1995, the *Compendium* has been translated piece by piece into the language MIZAR. As of August 2002, sixteen authors have worked on this specific project; they have produced fifty-seven MIZAR articles. The work is still in progress. For details one may consult the MIZAR homepage ⟨http://www.mizar.org/⟩ and the report on the work concerning the *Compendium* ⟨http://megrez.mizar.org/ccl/⟩.

The Authors
January, 2002

Acknowledgments

We thank Cambridge University Press for publishing this book and its referees for having scrutinized the project and recommended publication. David Tranah of CUP has been particularly helpful in communications and organization. Many persons interested in domain theory encouraged us to go ahead with the project of presenting the *Compendium of Continuous Lattices* in an updated version to the public.

At Carnegie-Mellon University, Staci Quackenbush carried out the task of creating a LaTeX file of the old *Compendium*. At the Technical University of Darmstadt several people worked on this master file of the old *Compendium* by proofreading, inserting pictures and diagrams, notably Michal Konečny, and Michael Marz, who were funded as Wissenschaftliche Hilfskräfte am Fachbereich Mathematik der TUD. Cathy Fischer helped with these files in May 2000 and was funded by SEFO – Frauenselbsthilfe und Fortbildungszentrum e.V. in Darmstadt. Our special thanks go to Thomas Erker for his major contributions in creating the appropriate macro apparatus for indexing, in setting up the bibliography, and in making the whole file system function smoothly.

Andrej Bauer created for us an electronic archive, at which the master copies of the main files were eventually kept. The archive made it possible for us to keep our sanity while several people were working on the same portions of the book at different locations around the globe. Diagrams were drawn with the aid of Paul Taylor's package.

Several universities and agencies supported extended periods of concentrated work on the manuscript. In February and March 2001, Klaus Keimel visited Tulane University and was funded by a grant from the Office of Naval Research to Michael Mislove. In summer 2001 he spent one month at the University of Birmingham on a visiting position of the Computer Science Department; he profited in his work on this monograph from the advice of Achim Jung and Martín Escardó. Karl H. Hofmann received travel support from the Tulane Mathematics Department to spend time at Tulane on various occasions. Jimmie Lawson visited the Technische Universität Darmstadt for two weeks during the summer of 2001 and for a month during the winter of 2002, the second visit being funded by the Alexander von Humboldt Foundation.

Foreword to *A Compendium of Continuous Lattices*

A mathematics book with six authors is perhaps a rare enough occurrence to make a reader ask how such a collaboration came about. We begin, therefore, with a few words on how we were brought to the subject over a ten-year period, during part of which time we did not all know each other. We do not intend to write here the history of continuous lattices but rather to explain our own personal involvement. History in a more proper sense is provided by the bibliography and the notes following the sections of the book, as well as by many remarks in the text. A coherent discussion of the content and motivation of the whole study is reserved for the introduction.

In October of 1969 Dana Scott was led by problems of semantics for computer languages to consider more closely partially ordered structures of function spaces. The idea of using partial orderings to correspond to spaces of partially defined functions and functionals had appeared several times earlier in recursive function theory; however, there had not been very sustained interest in structures of continuous functionals. These were the ones Scott saw that he needed. His first insight was to see that – in more modern terminology – the category of algebraic lattices and the (so-called) Scott-continuous functions is cartesian closed. Later during 1969 he incorporated lattices like the reals into the theory and made the first steps toward defining continuous lattices as "quotients" of algebraic lattices. It took about a year for the topological ideas to mature in his mind culminating in the paper published as [Scott, 1972a]. (For historical points we cannot touch on in this book the reader is referred to Scott's papers.) Of course, a large part of Scott's work was devoted to a presentation of models for the type-free lambda-calculus, but the search for such models was not the initial aim of the investigation of partially ordered structures; on the contrary, it was the avoiding of the formal and unmotivated use of lambda-calculus that prompted Scott to look more closely at the structures of the functions themselves, and it was only well after he began to see their possibilities that he realized there had to exist nontrivial T_0 spaces homeomorphic to their own function spaces.

Quite separately from this development, Karl Hofmann, Jimmie Lawson, Mike Mislove, and Al Stralka (among others) recognized the importance of compact semilattices as a central ingredient in the structure theory of compact semigroups. In his dissertation [D1967], Lawson initiated the study of a class of compact semilattices distinguished by the property that each had enough continuous semilattice morphisms into the unit interval semilattice (in its natural order) to separate points. (Such a program had already been started by Nachbin for partially ordered spaces in [Nachbin, B1965].) Lawson characterized this class of compact semilattices as those which admitted a basis of subsemilattice neighborhoods at each point (small subsemilattices): the class proved to be of considerable theoretical interest and attracted the attention of other workers in the field. In fact, it was believed for some time that all compact semilattices were members of the class, partly because the theory was so satisfactory (for example, purely "order theoretic" characterizations were discovered for the class by [Lawson, 1973]), and because no natural counterexamples seemed to exist. However, Lawson found the first example of a compact semilattice which was not in the class, one in fact which admitted only constant morphisms into the unit interval [Lawson, 1970] (see Chapter VI, Section 4).

At about the same time, Klaus Keimel had been working on lattices and lattice ordered algebras in pursuit of their spectral theory and their representation in sheaves. In his intensive collaboration with Gerhard Gierz on topological representations of nondistributive lattices, a spectral property emerged which turned out to be quite significant for compact semilattices with small subsemilattices.

The explanation for the fact that the topological algebra of Lawson's semilattices had been so satisfactory emerged clearly when Hofmann and Stralka gave a completely lattice theoretical description of the class [Hofmann and Stralka, 1976]. lt was Stralka who first recognized the relation of this class to Scott's continuous lattices, and this observation came about as follows. Two monographs on duality theories for lattices and topological structures emerged in the early seventies: One for topology and lattices by Hofmann and Keimel [B1972], and the other for compact zero dimensional semilattices and lattices by Hofmann, Mislove, and Stralka [B1974]. At the lattice theory conference in Houston in 1973, where such dualities were discussed, B. Banaschewski spoke on filters and mentioned Scott's work which was just about to appear in the *Proceedings of the Dalhousie Category Theory Conference*. Stralka checked out this hint, and while he and Hofmann were working on the algebraic theory of Lawson semilattices [Hofmann and Stralka, 1976], he realized the significance of this work as a link between the topological algebra of compact semilattices and the lattice theory of Scott's continuous lattices. This led to correspondence with Scott and much subsequent activity.

In the summer of 1976, Hofmann and Mislove spent some time collaborating with Keimel and Gierz at the Technische Hochschule in Darmstadt, and together they began a "write-in" seminar called the Seminar on Continuity in Semilattices, or SCS for short. The authors formed the core membership of the seminar, but their colleagues and students contributed greatly to the seminar by communicating their results, ideas, and problems. (A list of these seminar reports (SCS Memos) which resulted is provided at the end of this monograph.) The seminar then convened in person for several lively and well-attended workshops. The first was hosted by Tulane University in the spring of 1977, the second by the Technische Hochschule Darmstadt in the summer of 1978, and the third by the University of California at Riverside in the spring of 1979. A fourth workshop was held at the University of Bremen in the fall of 1979. We are very much indebted to all who participated in these seminars and others whose in fluence on this book is very considerable. In particular we thank H. Bauer, J. H. Carruth, Alan Day (who discovered an independent access to continuous lattices through the filter monad), Marcel Erné, R.-E. Hoffmann, John Isbell (who also gave very detailed remarks on the present manuscript), Jaime Ninio, A. R. Stralka, and O. Wyler.

lt was at the Tulane Workshop that the idea of collecting together the results of research – common and individual – was first discussed. A preliminary version of the *Compendium* worked out primarily by Hofmann, Lawson, and Gierz was circulated among the participants of the Darmstadt Workshop, and many people gave us their useful reactions. For help in typing the earlier versions of this book we would like to thank Frau Salder in Darmstadt and Mrs. Meredith Mickel at Tulane University.

The preparation of the final version of the text, which is reproduced from camera-ready copy, was carried out by and under the direction of Scott at the Xerox Palo Alto Research Center (PARC) in its Computer Science Laboratory (CSL). Scott spent the academic year 1978/79 on sabbatical as a Visiting Scientist at Xerox PARC, and the facilities of CSL, including extra secretarial aid, were very generously put at his disposal. The text was prepared on an Alto computer using the very flexible BRAVO text-editing system and a special computer-controlled printer. The typist, who in the course of the project also became a skilled computer-aided book illustrator and copy editor, was Melinda Maggiani. Without her loyal efforts and concentrated labor the book would never have been put into anywhere near the form seen here; the authors are extremely grateful to her. Special thanks are also due to many members of CSL for their interest and patience in helping Scott learn to use BRAVO, with which

he spent long, long hours; he wishes to mention with great warmth in particular Sara Dake, Leo Guibas, Jim Horning, Jeannette Jenkins, Joe Maleson, Jim Morris, HayChan Sargent, and Dan Swinehart. In the very last stages of the book make-up it was necessary to reprogram some printing routines to overcome several most irritating difficulties, and Lyle Ramshaw then stepped in and solved all the programming problems in record time. We take our hats off to him. (See especially in this regard Chapter IV, Section 3.)

The computerized editing system made it possible to produce in a very few months what were in effect two complete sets of galley proofs and two complete sets of page proofs; this is something that would never have been possible in our wildest dreams with the conventional manuscript–typescript–type style of book production. Computer-controlled editing allowed the authors to make, through the fingers of Maggiani and Scott, innumerable substantive corrections and to do extensive rewriting at every stage of the proofreading up to the last day before printing the camera-ready copy. Authors and publishers alike can only hope that such systems will soon become widely available. It was a real privilege to prepare this book at Xerox PARC, and the authors record here their heart-felt thanks to Dr. Robert J. Spinrad, Vice President and Manager of Xerox PARC, and especially to Robert Taylor, Manager of CSL. Aside from the support and cooperation, the remarkably friendly and informal atmosphere of PARC contributed much to the project.

For the support and sponsorship over the years of their research and their workshops, the authors are also happy to express their gratitude to the Alexander von Humboldt–Stiftung, the Deutsche Forschungsgemeinschaft, the National Science Foundation, the Simon Guggenheim Foundation, and the Universität Bremen, and to their own institutions, Louisiana State University, Oxford University and Merton College, Technische Hochschule Darmstadt, and Tulane University.

The Authors
January, 1980

Introduction to *A Compendium of Continuous Lattices*

Background and Plan of the Work

The purpose of this monograph is to present a fairly complete account of the development of the theory of continuous lattices as it currently exists. An attempt has been made to keep the body of the text expository and reasonably self-contained; somewhat more leeway has been allowed in the exercises. Much of what appears here constitutes basic, foundational or elementary material needed for the theory, but a considerable amount of more advanced exposition is also included.

Background and Motivation

The theory of continuous lattices is of relatively recent origin and has arisen more or less independently in a variety of mathematical contexts. We attempt a brief survey in the following paragraphs in the hope of pointing out some of the motivation behind the current interest in the study of these structures. We first indicate a definition for these lattices and then sketch some ways in which they arise.

A DEFINITION. In the body of the *Compendium* the reader will find many equivalent characterizations of continuous lattices, but it would perhaps be best to begin with one rather straightforward definition – though it is not the primary one employed in the main text. Familiarity with *algebraic lattices* will be assumed for the moment, but even if the exact details are vague, the reader is surely familiar with many examples: the lattice of ideals of a ring, the lattice of subgroups of a group.

Abstractly (and up to isomorphism) we can say that an algebraic lattice is a lattice of sets – contained, say, in the lattice of all subsets of a given set A – closed under arbitrary intersection of families of sets and under unions of directed families of sets (e.g. chains of sets). These are important closure properties of the lattice of ideals, for example. If we think of the powerset lattice as a product 2^A of A copies of the two-element lattice $2 = \{0, 1\}$, then an algebraic lattice is just a *sublattice* of 2^A with respect to the infinite operations of arbitrary pointwise inf and pointwise sup of *directed* families of lattice elements. (Note, however, that *finite* sups are different in general; so the meaning of the word "sublattice" has to be understood in a suitable sense.)

Let us now replace the discrete lattice 2 with the "continuous" lattice $[0, 1]$, the unit interval of real numbers with its natural order and familiar lattice structure. In a power $[0, 1]^A$ we can speak of sublattices with respect to arbitrary pointwise inf and pointwise sup of directed families of elements, just as before. *Up to isomorphism, these are exactly the continuous lattices*. Of course this definition gives no hint as to the internal structure of these lattices and is only a dim indicator as to their naturalness and usefulness. But it does show that they are direct generalizations of well-known kinds of lattices and that they have an important element of "continuity".

THEORY OF COMPUTATION. Often in computational schemes one employs some algorithm successively to gain increasingly refined approximations to the desired result. It is convenient to use, formally or informally, topological language – one talks about "how far" the approximation is from the desired result or how good a "fit" has been obtained. An alternate procedure is to specify at each stage a subset in which the desired result lies. The smaller the set, the better the approximation; we could say that the smaller set gives "more information". This approach leads naturally to the use of order theoretic language in discussing the partial results, and the data generated, in a way related to the containments among the sets.

Let us now abstract this approach somewhat. Let P be a partially ordered set. We think of a "computation" of an element x in P as being a sequence of increasingly larger elements – "larger" meaning "more" in the sense of information – whose supremum is x. (More generally, we could imagine a directed set whose supremum is x.) We wish to regard x as the "limit" of the sequence (or set) of approximations.

What is needed is a precise definition of how well some "stage" of the "computation" approximates the "limit" x. We take an indirect approach to this question, because there is no metric available to tell us immediately how close an approximation is to the desired limit. We define in place of a metric a notion meaning roughly: an element y is a "finite approximation" to the element x. Then, to have a well-behaved system of limits, we *assume* that every element is the sup in the partial ordering of its finite approximations. A given sequence of approximations to x is then "successful" if it eventually encompasses all the finite approximations.

Specifically, we say that an element y, which is less than or equal to x, is a *finite approximation* to x if for any directed set D with supremum x (D represents the stages of a "computation" of x) we have some member of D which is greater than or equal to y. (Hence, all the members from that stage on are greater than or equal to y.) The idea is that if we use y to measure the accuracy of computations of x, then every computation that *achieves* x must eventually be *at least as accurate as* y.

Strictly speaking the outline just given is actually not quite right. We should say that if a directed set D has supremum *greater than or equal to* x, then some member of D is greater than or equal to y. This ensures that if y is a "finite approximation" for x, then it is also one for every element larger than x – a property we would certainly want to require. In the text we use different terminology. The notion of "finite" in "finite approximation" is somewhat vague, because again there is no measure to distinguish finite elements from infinite ones in general; indeed there are lattices where *all* elements except 0 are normally thought of as infinite (as in the lattice [0, 1] for instance). This explains our feeling that another terminology was required. We have used the phrase "y is way below x" for topological and order theoretic reasons cited in the appropriate section of the book.

To recapitulate: we assume that the "finite approximations" for each element are directed and have that element for their supremum. *The complete lattices with this property are the continuous lattices*. It is the theory of these abstracted, order theoretic structures that we develop in this monograph. It should be pointed out that only the lattice case is treated in the main text; generalizations appear in the exercises.

Owing to limitations of length and time, the theory of computation based on this approach is not developed extensively here. Certain related examples are, however, mentioned in the present text or in the exercises. For instance, consider the set of all partial functions from the natural numbers into itself (or some distinguished subset such as the recursive partial functions). These can be ordered by inclusion (that is, extension). Here again the larger elements give more information. In this example f is a "finite approximation" for g if and only if g is an extension of f and the domain of f is finite. In many examples such as this the "approximating" property can be interpreted directly as a finiteness condition, since there are finite functions in the set (functions with a finite domain). This circumstance relates directly to the theory of algebraic lattices, a theme which we do cover here in great detail.

General topology. Continuous lattices have also appeared (frequently in cleverly disguised forms) in *general topology*. Often the context is that of the category of all topological spaces or of topological spaces where one assumes only a T_0 separation axiom. Such spaces have been the objects of renewed interest with the emergence of spectral theory.

In fact, a continuous lattice can be endowed in a natural way with a T_0 topology which is defined from the lattice structure; in this book we call this T_0 topology the *Scott topology*. It is shown in Section 3 of Chapter II that these spaces are exactly the "injectives" (relative injectives in the categorical sense) or "absolute retracts" in the category of all T_0 spaces and continuous functions; that is if f is a continuous function from a subspace X of a topological space Y into a continuous lattice L (equipped with the Scott topology), then there always exists a continuous extension of f from X to Y with values still in L. This property in fact gives a topological characterization of continuous lattices, since any T_0 topology of such a space is just the Scott topology of a continuous lattice naturally determined from it.

In another direction let us say that an open set U is *relatively compact* in an open set V if every open cover of V has finitely many members which cover U. If X is a topological space, then the lattice of open sets is a continuous lattice iff each open set is the union of the open sets which are relatively compact in it. In this case the "way-below" relation is viewed as just the relation of one open set being relatively compact in another. This illustrates some of the versatility of the concept of a continuous lattice.

Spaces for which the lattice of open sets is a continuous lattice prove to be quite interesting. For Hausdorff spaces it is precisely the locally compact spaces which have this property, and in more general spaces analogs of this result remain true. We investigate this situation in some detail in the context of the spectral theory of distributive continuous lattices in Section 5 of Chapter V. It is often the case that theorems concerning locally compact Hausdorff spaces extend to spaces with a continuous lattice of open sets in the category of all topological spaces (see, e.g., [Day and Kelly, 1970]). Such considerations provide another link between continuous lattices and general topology.

The dual of the lattice of open sets – the lattice of closed sets – has long been an object of interest to topologists. If X is a compact Hausdorff space, then the lattice of closed subsets under the Vietoris topology is also a compact Hausdorff space. In Chapter III we introduce a direct generalization of this topology, called here the Lawson topology, which proves to be compact for all complete lattices and Hausdorff for continuous lattices. This connection allows applications of continuous lattice theory to the topological theory of hyperspaces (cf. Example VI-3.8).

ANALYSIS AND ALGEBRA. Several applications of continuous lattices arise in analysis and functional analysis. For example, consider the family $C(X, \mathbb{R})$ of continuous real-valued functions on the locally compact space X. Using the pointwise operations and the natural order from $\mathbb{R}$, the space $C(X, \mathbb{R})$ is a lattice, but its lattice theory is rather unsatisfactory. For example, it is not even complete; however, if we consider this lattice as a sublattice of LSC(X, $\mathbb{R}^*$), the lattice of all lower semicontinuous extended real-valued functions on X, then we do have a complete lattice with which to work. In fact, although this is not at all apparent from the functional analysis viewpoint, the lattice LSC(X, $\mathbb{R}^*$) is a continuous lattice. This entails several results, not the least of which is the following: *The lattice* LSC(X, $\mathbb{R}^*$) *admits a unique compact Hausdorff function-space topology such that* $(f, g) \mapsto f \wedge g$ *is a continuous operation*. (See I-1.22, II-4.7, and II-4.20.) In light of the fact that $C(X, \mathbb{R})$ is never compact and that $C(X, \mathbb{R}^*)$ is compact only if X is finite, this result is somewhat suprising; we do not know of a “classical” proof. Indeed, lower semicontinuity motivates one of the canonical topologies on any continuous lattice, and, if we equip $\mathbb{R}^*$ with this canonical topology, then the continuous functions from X into $\mathbb{R}^*$ so topologized are exactly those extended real-valued functions on X which are lower semicontinuous relative to the usual topology on $\mathbb{R}^*$. In the same vein it emerges that the probability distribution functions of random variables with values in the unit interval form a continuous lattice; compact topologies for this example are, however, familiar from classical analysis (cf. I-2.22).

A second example is quite different and demonstrates an overlap between analysis and algebra. With a ring R one associates a topological space, called its *spectrum*, and while there are many ways of doing this, probably the most wide spread and best known is the space of prime ideals of a commutative ring endowed with the hull–kernel topology. This plays a central role in algebraic geometry (where the relevant theory deals with noetherian rings and their spectra), and this construction can also be carried out for Banach algebras, in which case the preferred spectrum is the space of closed primitive ideals (which reduces to the more familiar theory of maximal ideals if the algebra is commutative). These particular ideals are relevant since they are precisely the kernels of irreducible representations of the algebra as an algebra of operators on a Banach space or Hilbert space.

Now, the connection of these spectral theories with the theory of continuous lattices emerges more clearly if we first return to the case of a commutative ring R. In this case, the spectrum is the set of prime ideals viewed as a subset of the algebraic lattice of all ideals of the ring; in fact, the spectrum is exactly the family of prime elements of the distributive algebraic lattice of all radical ideals of R. (Recall that a radical ideal is one which is the intersection of prime ideals.) If we define the spectrum of a distributive lattice as being its family of prime elements, then we have just reduced the spectral theory of commutative rings to the spectral theory of distributive algebraic lattices.

It may not be obvious, but the situation in functional analysis is analogous. Here we consider the lattice of *closed* two-sided ideals of a Banach algebra; while this lattice is not algebraic in general, it *is* a continuous lattice (at least in the case of a C^*-algebra). Moreover, in the case where the algebra is separable, its spectrum is just the traditional primitive ideal spectrum. Again, we have reduced the spectral theory of separable C^*-algebras to the spectral theory of distributive continuous lattices. This approach to the spectral theory of C^*-algebras affords an affirmative (and perhaps more systematic) proof of the fact that the primitive spectrum of a separable C^*-algebra is a locally compact T_0 space (cf. I-1.21 and V-5.5). Indeed, the central result in the spectral theory of a continuous lattice is that its spectrum is just such a space (cf. V-5.5).

Lastly, we mention another area of functional analysis which relates to our theory. If C is a compact convex set in a locally convex topological space, it is useful to know as much as possible about the space of closed convex subsets of C. This space is a lattice, and its opposite lattice is in fact a continuous lattice whose prime elements are exactly the singleton sets containing extreme points of C. Moreover, if the upper semicontinuous affine functions on C are considered, then once again a continuous lattice is found in much the way we encountered one in the example $\mathrm{LSC}(X, \mathbb{R}^*)$ above. We have yet another instance where a function space naturally arises whose lattice and topological properties are essentially those of a continuous lattice.

CATEGORY THEORY AND LOGIC. Another area in which continuous lattices have occurred somewhat unexpectedly is the area of *category theory*. Constructions of free objects play an important role in mathematics, e.g., free groups, free semigroups, free modules. A somewhat more sophisticated construction is the construction of the free compact Hausdorff space over a set X. This turns out to be the Stone–Čech compactification of the discrete space X, which can be identified as all ultrafilters on X equipped with a suitable topology.

These constructions can all be set in a suitable categorical context: the theory of *monads* or *triples*. Here one has adjoint functors (which can be thought of as a "free" functor and a "forgetful" functor). It is then possible to define categorically the "quotients" of the free objects, which become the "algebras" of the system. lt has been found that in this abstract setting it is sometimes possible to identify free objects *before* knowing what the algebras are. A simple example is the powerset monad – but it is very easy to prove that the algebras are just the complete sup semilattices.

The question arose of identifying the monad for which the free functor is that which assigns to a set the set of all filters on that set. It is the discovery of Alan Day that the algebras for this monad are precisely the continuous lattices. (See [Day, 1975].)

In our treatment of continuous lattices we do not completely follow the categorical approach, but this is no reflection on its mathematical merit. But we do prove Day's theorem, however, and we have much to say about categorical properties of many classes of structures related to continuous lattices.

One particulary interesting categorical aspect of our work – at least in the authors' eyes – is the ease with which examples of *cartesian closed categories* can be found. (Natural examples are not so very common in mathematics or even in category theory until one comes to the theory of topoi.) The reason for the occurrence of these cartesian closed categories, as we explain in the text, has to do with the *function-space construction*.

Specifically, in the context of our considerations, we have available a rather natural notion of morphism, namely that morphisms should preserve limits of what we have called "computations". More precisely a morphism is a function between partially ordered sets with the property that the image of the supremum of a directed set is the supremum of the image of the directed set. (Such functions are treated in some detail in Chapter II, Section 2.) There are also several other interesting classes of morphisms with various properties, but this notion (called Scott continuity) works especially well in forming function spaces.

If L and M are continuous lattices, let $[L \to M]$ be the set of all morphisms from L to M. It is shown in Chapter II that, if these functions are given the pointwise ordering, then $[L \to M]$ is also a continuous lattice. Moreover, this construction is a functor adjoint to the formation of cartesian products.

By using this construction and an inverse limit procedure, examples of non-trivial continuous lattices L can be found which are actually isomorphic with their own self-function space $[L \to L]$. (This does not seem possible with any stronger separation property beyond the T_0 axiom.) Constructions of this type are treated in great generality in Sections 6 and 7 of Chapter IV.

The examples just mentioned have the striking property that their members can be interpreted either as elements or as functions, and that every self-function corresponds to some element. This is precisely the setup that one hypothesizes in the lambda-calculus approach to logic. Thus, continuous lattices provided the background for the construction of concrete models for an axiomatic logical system that had long existed without them (see [Scott, 1973]).

TOPOLOGICAL ALGEBRA. The final area that we wish to mention is that of *topological algebra*. Among the objects investigated in this field were compact topological semilattices and lattices (that is, semilattices or lattices equipped with a compact topology for which the meet or meet and join operations were continuous). In the course of study of compact semilattices it emerged that those which had a neighborhood basis of subsemilattices lent themselves more easily to mathematical investigation; in addition known examples had this property. Hence, attention was particularly focused on this class.

The amazing result of these investigations was the discovery that such semilattices were (modulo an identity or top element) continuous lattices with respect to their lattice structure. (We derive this result in Section VI-3 after first developing some of the most basic theory of topological semilattices in VI-1.) Conversely, if L is a continuous lattice, then a topology can be defined from the order which makes L into a compact topological semilattice with small semilattices. The topology in question is the Lawson topology already mentioned; it is defined and investigated in Chapter III.

This identification between compact semilattices with small subsemilattices and continuous lattices has greatly affected the development of the theory of continuous lattices. Not only do many of the results of topological semilattices transfer wholesale to continuous lattices, but also topological techniques and methods play a prominent role in their study (as opposed to most traditional lattice theory). Conversely, lattice theoretic methods frequently aid investigations of a topological nature. This interplay is illustrated in Chapters VI and VII.

Before we entirely leave our discussion of the roots of continuous lattices, a postscript concerning algebraic lattices is probably in order. Algebraic lattices provide an important link between the theory of continuous lattices and traditional lattice theory and universal algebra. Indeed they are a special class of continuous lattices; their theory has frequently suggested generalizations and directions of research for continuous lattices. They are introduced in Section I-4, but resurface on several occasions. With respect to the Lawson topology, they are precisely those continuous lattices which are totally disconnected (equivalently: zero dimensional), and so they also occupy a natural place in topological algebra.

Plan of the Work

Chapter O consists essentially of background material of an order theoretical nature. The reader may review it to the extent he feels necessary. Some familiarity with the language and notation introduced there will probably be necessary. The formalism of Galois connections explained in O-3 is vital for many things which will follow in the main body of the book.

Chapter I introduces continuous lattices from an order theoretic point of view. In Section 1 continuous lattices are defined and examples are given. The "way-below" relation is introduced, and is characterized among the auxiliary relations. Section 2 gives an equational characterization of continuous lattices and discusses their variety-like properties. Section 3 introduces prime and irreducible elements and generalizations thereof and shows the plentiful supply of such in continuous lattices. The basic properties of algebraic lattices and some of their relationships with continuous lattices appear in Section 4. Most of the material in this chapter is quite basic (except perhaps the material on auxiliary relations in Section 1).

Chapter II defines the Scott topology and develops its applications to continuous lattices. In Section 1 the Scott topology is defined and convergence in the Scott topology is characterized for continuous lattices. Section 2 gives the definition and characterizations of Scott-continuous functions. In Section 3 it is shown that continuous lattices endowed with their Scott topologies form the "injectives" in the category of T_0 topological spaces. Section 4 is concerned with function spaces (particularly the set of Scott-continuous functions between spaces and/or lattices) and questions of the categorical notion of "cartesian closedness". Of fundamental importance in this chapter are the basic properties of the Scott topology and Scott-continuous functions appearing in Sections 1 and 2 (although they are treated in greater detail than may be of interest to the general reader).

Chapter III introduces the second important topology for continuous lattices, the Lawson topology. Like the Scott topology, it is defined in an order theoretic fashion. In Section 1 it is shown that the Lawson topology is compact and T_1 for every complete lattice and compact Hausdorff for continuous lattices. Indeed in Section 2 it is shown that for meet continuous complete lattices the Lawson topology is Hausdorff if and only if the lattice is continuous. Section 3 characterizes convergence of nets in the Lawson topology. Section 4 generalizes the notion of a basis for a topology to continuous lattices and derives properties thereof. Section 1 and 2 are the basic sections of this chapter.

Chapter IV considers various important categories of continuous lattices together with certain categorical constructions. Section 1 is the important one here; it presents important duality theorems for the study of continuous lattices. Section 3 contains the important result that a continuous lattice has sufficiently many semilattice homomorphisms of the right kind into the unit interval to separate points. Since we aspired to great generality in the proof, its details may appear dry. Some of the exercises exemplify applications which were made possible on this level of generality. The next sections give general categorical constructions for obtaining continuous lattices which are "fixed-points" with respect to some self-functor of the category. This process is needed for the construction of set-theoretical models of the lambda-calculus.

In Chapter V spectral theory is taken up. An important lemma on the behavior of primes appears in Section 1. In Section 2 it is shown one always has a smallest closed generating (or order generating) set, namely the closure of the irreducible elements. Considering elements in this closure leads to generalizations of the notion of prime and irreducible in Section 3. Section 4 introduces the subject of the spectral theory of lattices in general, and Section 5 considers that of continuous lattices. There a duality is set up between the category of all distributive continuous lattices and all locally compact sober spaces (with appropriate morphisms for each category). Here probably Sections 1, 4 and 5 would be of greatest interest to the general reader.

Chapter VI begins the study of topological algebra per se. The sections of primary interest are 1 and 3. In Section 1 the most basic and useful properties of pospaces and topological semilattices are given. In Section 3 the Fundamental Theorem of Compact Semilattices is stated and proved, establishing the equivalence between the category of compact semilattices with small semilattices and the category of continuous lattices.

The rest of Chapter VI and Chapter VII center on more specialized topics in topological algebra. Section VI-2 gives an order theoretic description of convergence in an arbitrary compact semilattice. Section VI-4 presents examples of compact semilattices which are not continuous lattices (unfortunately, the construction is quite intricate). Section VI-5 covers the topic of the existence of arc chains in topological semilattices, a topic of considerable historical interest in the theory of topological semilattices and lattices. In Section VI-7 we return to spectral theory. A topology finer than the spectral topology is introduced, called the "patch topology", and the conditions under which it is compact Hausdorff are investigated.

In Section 1 of Chapter VII topological semilattices in which every open set is an upper set are considered. It is shown that under rather mild restrictions this topology must be the Scott topology. Section 2 takes up the topic of topological lattices, lattices in which both operations are continuous. In Section 4 a lattice theoretic characterization of compact topological semilattices is given, and it is shown that in such a setting separate continuity of the meet operation implies joint continuity.

O

A Primer on Ordered Sets and Lattices

This introductory chapter serves as a convenient source of reference for certain basic aspects of complete lattices needed in what follows. The experienced reader may wish to skip directly to Chapter I and the beginning of the discussion of the main topic of this book: continuous lattices and domains.

Section O-1 fixes notation, while Section O-2 defines complete lattices, complete semilattices and directed complete partially ordered sets (**dcpo**s), and lists a number of examples which we shall often encounter. The formalism of Galois connections is presented in Section 3. This not only is a very useful general tool, but also allows convenient access to the concept of a Heyting algebra. In Section O-4 we briefly discuss meet continuous lattices, of which both continuous lattices and complete Heyting algebras (frames) are (overlapping) subclasses. Of course, the more interesting topological aspects of these notions are postponed to later chapters. In Section O-5 we bring together for ease of reference many of the basic topological ideas that are scattered throughout the text and indicate how ordered structures arise out of topological ones. To aid the student, a few exercises have been included. Brief historical notes and references have been appended, but we have not tried to be exhaustive.

O-1 Generalities and Notation

Partially ordered sets occur everywhere in mathematics, but it is usually assumed that the partial order is *antisymmetric*. In the discussion of nets and directed limits, however, it is not always so convenient to assume this property. We begin, therefore, with somewhat more general definitions.

Definition O-1.1. Consider a set L equipped with a reflexive and transitive relation $\leq$. Such a relation will be called a *preorder* and L a *preordered set*. We say that a is a *lower bound* of a set $X \subseteq L$, and b is an *upper bound*, provided that

$$\begin{aligned} a \le x &\text{ for all } x \in X, \quad \text{and} \\ x \le b &\text{ for all } x \in X, \quad \text{respectively.} \end{aligned}$$

A subset D of L is *directed* provided it is nonempty and every finite subset of D has an upper bound in D. (Aside from nonemptiness, it is sufficient to assume that every *pair* of elements in L has an upper bound in L.) Dually, we call a nonempty subset F of L *filtered* if every finite subset of F has a lower bound in F.

If the set of upper bounds of X has a unique smallest element (that is, the set of upper bounds contains exactly one of its lower bounds), we call this element the *least upper bound* and write it as $\bigvee X$ or $\sup X$ (for *supremum*). Similarly the *greatest lower bound* is written as $\bigwedge X$ or $\inf X$ (for *infimum*); we will not be dogmatic in our choice of notation. The notation $x = \bigvee^{\uparrow} X$ is a convenient device to express that, firstly, the set X is directed and, secondly, x is its least upper bound. In the case of pairs of elements it is customary to write

$$\begin{aligned} x \wedge y &= \inf\{x, y\}, \\ x \vee y &= \sup\{x, y\}. \end{aligned}$$

These operations are also often called *meet* and *join*, and in the case of meet the multiplicative notation xy is common and often used in this book. □

Definition O-1.2. A *net* in a set L is a function $j \mapsto x_j : J \to L$ whose domain J is a directed set. (Nets will also be denoted by $(x_j)_{j \in J}$, by (x_j), or even by x_j, if the context is clear.)

If the set L also carries a preorder, then the net x_j is called *monotone* (resp., *antitone*), if $i \le j$ always implies $x_i \le x_j$ (resp., $x_j \le x_i$).

If $P(x)$ is a property of the elements $x \in L$, we say that $P(x_j)$ holds *eventually* in the net if there is a $j_0 \in J$ such that $P(x_k)$ is true whenever $j_0 \le k$.

The next concept is slightly delicate: if L carries a preorder, then the net x_j is a *directed net* provided that for each fixed $i \in J$ one eventually has $x_i \le x_j$. A *filtered net* is defined dually. □

Every monotone net is directed, but the converse may fail. Exercise O-1.12 illustrates pitfalls to avoid in defining directed nets. The next definition gives us some convenient notation connected with upper and lower bounds. Some important special classes of sets are also singled out.

Definition O-1.3. Let L be a set with a preorder $\le$. For $X \subseteq L$ and $x \in L$ we write:

(i) $\downarrow X = \{y \in L : y \le x \text{ for some } x \in X\}$;
(ii) $\uparrow X = \{y \in L : x \le y \text{ for some } x \in X\}$;

(iii) $\downarrow x = \downarrow\{x\}$;
(iv) $\uparrow x = \uparrow\{x\}$.

We also say:

(v) X is a *lower set* iff $X = \downarrow X$;
(vi) X is an *upper set* iff $X = \uparrow X$;
(vii) X is an *ideal* iff it is a directed lower set;
(viii) X is a *filter* iff it is a filtered upper set;
(ix) an ideal is *principal* iff it has a maximum element;
(x) a filter is *principal* iff it has a minimum element;
(xi) $\mathrm{Id}\, L$ (resp., $\mathrm{Filt}\, L$) is the set of all ideals (resp. filters) of L;
(xii) $\mathrm{Id}_0\, L = \mathrm{Id}\, L \cup \{\emptyset\}$;
(xiii) $\mathrm{Filt}_0\, L = \mathrm{Filt}\, L \cup \{\emptyset\}$. □

Note that the principal ideals are just the sets $\downarrow x$ for $x \in L$. The set of lower bounds of a subset $X \subseteq L$ is equal to the set $\bigcap\{\downarrow x \colon x \in X\}$, and this is the same as the set $\downarrow \inf X$ in case $\inf X$ exists. Note, too, that

$$X \subseteq \downarrow X = \downarrow(\downarrow X),$$

and similarly for $\uparrow X$.

Remark O-1.4. *For a subset X of a preordered set L the following are equivalent:*

(1) *X is directed;*
(2) *$\downarrow X$ is directed;*
(3) *$\downarrow X$ is an ideal.*

Proof: (2) iff (3): By Definition O-1.3.

(1) implies (2): If A is a finite subset of $\downarrow X$, then there is a finite subset B of X such that for each $a \in A$ there is a $b \in B$ with $a \leq b$ by O-1.3(i). By (1) there is in X an upper bound of B, and this same element must also be an upper bound of A.

(2) implies (1): If A is a finite subset of X, it is also contained in $\downarrow X$; therefore, by (2), there is an upper bound $y \in \downarrow X$ of A. By definition $y \leq x \in X$ for some x, and this x is an upper bound of A. □

Remark O-1.5. *The following conditions are equivalent for L and X as in* O-1.4*:*

(1) $\sup X$ *exists;*
(2) $\sup \downarrow X$ *exists.*

And if these conditions are satisfied, then $\sup X = \sup {\downarrow}X$*. Moreover, if every finite subset of* X *has a sup and if* F *denotes the set of all those finite sups, then* F *is directed, and* (1) *and* (2) *are equivalent to*

(3) $\sup F$ *exists.*

Under these circumstances, $\sup X = \sup F$*. If* X *is nonempty, we need not assume the empty sup belongs to* F*.*

Proof: Since, by transitivity and reflexivity, the sets X and ${\downarrow}X$ have the same set of upper bounds, the equivalence of (1) and (2) and the equality of the sups are clear. Now suppose that $\sup A$ exists for every finite $A \subseteq X$ and that F is the set of all these sups. Since $A \subseteq B$ implies $\sup A \leq \sup B$, we know that F is directed. But $X \subseteq F$, and any upper bound of X is an upper bound of $A \subseteq X$; thus, the sets X and F have the same set of upper bounds. The equivalence of (1) and (3) and the equality of the sups is again clear, also in the nonempty case. □

The – rather obvious – theme behind the above remark is that statements about arbitrary sups can often be reduced to statements about finite sups and sups of directed sets. Of course, both O-1.4 and O-1.5 have straightforward duals.

Definition O-1.6. A partial order is a transitive, reflexive, and antisymmetric relation $\leq$. (This last means $x \leq y$ and $y \leq x$ always imply $x = y$.) A *partially ordered set*, or *poset* for short, is a nonempty set L equipped with a partial order $\leq$. We say that L is *totally ordered*, or a *chain*, if all elements of L are comparable under $\leq$ (that is, $x \leq y$ or $y \leq x$ for all elements $x, y \in L$). An *antichain* is a partially ordered set in which any two different elements are incomparable, that is, in which $x \leq y$ iff $x = y$. □

We have remarked informally on duality several times already, and the next definition makes duality more precise.

Definition O-1.7. For $R \subseteq L \times L$ any binary relation on a set L, we define the *opposite relation* R^{op} (sometimes: the *converse relation*) by the condition that, for all $x, y \in L$, we have $x R^{\mathrm{op}} y$ iff $y R x$.

If in $(L, \leq)$, a set equipped with a transitive, reflexive relation, the relation is understood, then we write L^{op} as short for $(L, \leq^{\mathrm{op}})$. □

The reader should note that if L is a poset or a chain, then so is L^{op}. One should also be aware how the passage from L to L^{op} affects upper and lower bounds. Similar questions of duality are also relevant to the next (standard) definition.

Definition O-1.8. An *inf semilattice* is a poset S in which any two elements a, b have an inf, denoted by $a \wedge b$ or simply by ab. Equivalently, a semilattice is a poset in which every nonempty finite subset has an inf. A *sup semilattice* is a poset S in which any two elements a, b have a sup $a \vee b$ or, equivalently, in which every nonempty finite subset has a sup. A poset which is both an inf semilattice and a sup semilattice is called a *lattice*.

As we will deal with inf semilattices very frequently, we adopt the shorter expression "semilattice" instead of "inf semilattice".

If a poset L has a greatest element, it is called the *unit* or *top* element of L and is written as 1 (or, rarely, as $\top$). The top element is the inf of the empty set (which, if it exists, is the same as $\sup L$). A semilattice with a unit is called *unital*. If L has a smallest element, it is called the *zero* or *bottom* element of L and is written 0 (or $\perp$). The bottom element is the sup of the empty set (which, if it exists, is the same as $\inf L$). □

Note that in a semilattice an upper set is a filter iff it is a subsemilattice. A dual remark holds for lower sets and ideals in sup semilattices. We turn now to the discussion of maps between posets.

Definition O-1.9. A function $f\colon L \to M$ between two posets is called *order preserving* or *monotone* iff $x \leq y$ always implies $f(x) \leq f(y)$. A one-to-one function $f\colon L \to M$ where both f and f^{-1} are monotone is called an *isomorphism*. We denote by *POSET* the category of all posets with order preserving maps as morphisms.

We say that f preserves

(i) *finite sups*, or (ii) (*arbitrary*) *sups*, or (iii) *nonempty sups*, or (iv) *directed sups*

if, whenever $X \subseteq L$ is

(i) finite, or (ii) arbitrary, or (iii) nonempty, or (iv) directed,

and $\sup X$ exists in L, then $\sup f(X)$ exists in M and equals $f(\sup X)$. A parallel terminology is applied to the preservation of infs. □

In the case of (iv) above, the choice of expression may not be quite satisfactory linguistically, but the correct phrase "preserves least upper bounds of directed sets" is too long. The preservation of directed sups can be expressed in the form

$$f\left(\bigvee\nolimits^{\uparrow} X\right) = \bigvee\nolimits^{\uparrow} f(X).$$

For semilattices a map preserving nonempty finite infs might be called a *homomorphism* of semilattices. The reader should notice that a function preserving *all* finite infs preserves the inf of the empty set; that is, it maps the unit to the unit – provided that unit exists. In order to characterize maps f preserving only the nonempty finite infs (if this is the condition desired), we can employ the usual equation:

$$f(x \wedge y) = f(x) \wedge f(y),$$

for $x, y \in L$. Note that such functions are monotone, and the dual remark also holds for homomorphisms of sup semilattices.

Remark. It should be stressed that our definition of "preservation of sups" is quite strong, as we require that, whenever a set X in the domain has a sup, then its image has a sup in the range. As a consequence, if a function $f: L \to M$ preserves (directed) sups, it also preserves the order. Indeed, if $a \leq b$ in L, then $\{a, b\}$ is a (directed) set that has a sup; as f preserves (directed) sups, then $f(a) \vee f(b)$ exists and $f(b) = f(a \vee b) = f(a) \vee f(b)$, whence $f(a) \leq f(b)$.

Often in the literature a weaker definition is adopted: f "preserves sups" if whenever $\sup X$ and $\sup f(X)$ both exist, then $f(\sup X) = \sup f(X)$. In this weak sense, a one-to-one map from the two element chain to two incomparable elements preserves sups. Thus a function that preserves (directed) sups in this weak sense need not be order preserving. In order to avoid ambiguities one should keep in mind that if a map preserves (directed) sups in our sense, then it is automatically order preserving. This implies in particular that the image of a directed set is also directed.

Remark O-1.10. *Let $f: L \to M$ be a function between posets. The following are equivalent:*

(1) *f preserves directed sups;*
(2) *f preserves sups of ideals.*

Moreover, if L is a sup semilattice and f preserves finite sups, then (1) *and* (2) *are also equivalent to*

(3) *f preserves arbitrary sups.*

A dual statement also holds for filtered infs, infs of filters, semilattices and arbitrary infs.

Proof: Both conditions (1) and (2) imply the monotonicity of f. Then the equivalence of (1) and (2) is clear from O-1.4 and O-1.5. Now suppose L is a sup semilattice and f preserves finite sups. Let $X \subseteq L$ have a sup in L. By the method of O-1.5(3), we can replace X by a directed set F having the same sup. Hence, if (1) holds, then $f(\sup X) = \sup f(F)$. But since f preserves finite sups, it is clear that $f(F)$ is constructed from $f(X)$ in the same way as F was obtained from X. Thus, by another application of O-1.5(3), we conclude that $f(\sup X) = \sup f(X)$. That (3) implies (1) is obvious. $\square$

Exercises

Exercise O-1.11. Let $f\colon L \to M$ be monotone on posets L and M, and let $X \subseteq L$. Show that $\downarrow f(X) = \downarrow f(\downarrow X)$. $\square$

Exercise O-1.12. Construct a net $(x_j)_{j \in J}$ with values in a poset such that for all pairs $i, j \in J$ there is a $k \in J$ with $x_i \le x_k$ and $x_j \le x_k$ but such that $(x_j)_{j \in J}$ is *not* directed.

Hint. Consider the lattice $2 = \{0, 1\}$, let $J = \{0, 1, 2, \ldots\}$, and let the net be defined so that $x_i = 0$ iff i is even. $\square$

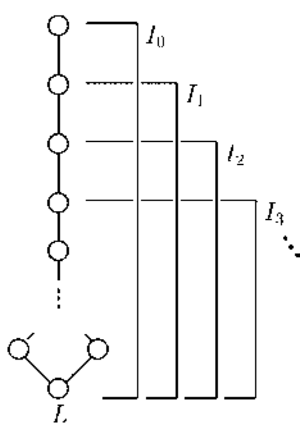
Exercise O-1.13. Modify O-1.10 so that for (3) we have only to assume that f preserves *nonempty* finite sups. $\square$

Exercise O-1.14. Is the category of preordered sets and monotone maps *equivalent* to the category of posets and monotone maps? In these categories what sort of functor is $^{\mathrm{op}}$? $\square$

Exercise O-1.15. Let L be a poset, and let the I_j for $j \in J$ be ideals of L. Prove the following.

(i) $\bigcap_j I_j$ is an ideal of L iff $\bigcap_j I_j \neq \emptyset$, for L a sup semilattice.
(ii) In general, $\bigcap_j I_j$ is not necessarily an ideal of L, even if $\bigcap_j I_j \neq \emptyset$.

Hint. Consider the semilattice and ideals in the following figure.



(iii) The intersection $I_1 \cap I_2$ of two ideals I_1, I_2 is an ideal, for L a semilattice.
(iv) If L is directed, $\bigcup_j I_j$ is contained in some ideal of L (however, even if this is the case, there need not be a smallest ideal containing $I_1 \cup I_2$) and the converse holds if this is true for any two ideals I_1, I_2.
(v) $\operatorname{Id} L$ is a sup semilattice iff L is a sup semilattice.

Hint. If L is a sup semilattice, then $I = \downarrow\{a \vee b \colon a \in I_1, b \in I_2\}$ is the sup of the ideals I_1 and I_2 of L. Conversely, if $\operatorname{Id} L$ is a sup semilattice, then we claim there is a unique element $c \in \downarrow a \vee \downarrow b$ with $a, b \leq c$. Indeed, there is at least one since $\downarrow a \vee \downarrow b$ is directed; moreover, if c and c_1 were two such elements, then $\downarrow c$ and $\downarrow c_1$ would be two ideals of L both containing a and b and both contained in $\downarrow a \vee \downarrow b$. Hence $\downarrow c = \downarrow c_1 = \downarrow a \vee \downarrow b$.

(vi) Dual statements hold for $\operatorname{Filt} L$, where one assumes L is a semilattice in part (v). □

Exercise O-1.16. Let L be a preordered set, and let $\mathcal{L}$ denote the family of all nonempty lower sets of L. Prove the following.

(i) $\operatorname{Id} L \subseteq \mathcal{L}$ and $\mathcal{L}$ is a sup semilattice.
(ii) If L is a poset, then the map $x \mapsto \downarrow x \colon L \to \mathcal{L}$ is an isomorphism of L onto the family of principal lower sets of L.
(iii) If L is a filtered poset, then $\mathcal{L}$ is a lattice with respect to intersection and union.
(iv) Let L and M be semilattices, $f \colon L \to M$ be a function, and $\mathcal{L}$ and $\mathcal{M}$ be the lattices of nonempty lower sets. Let $f_* = (A \mapsto \downarrow f(A)) \colon \mathcal{L} \to \mathcal{M}$. Then f is a semilattice morphism iff f_* is a lattice morphism. □

Old notes

The notion of a directed set goes back to the work of [Moore and Smith, 1922], where they use directed sets and nets to determine topologies. A convenient survey of this theory is provided in Chapter 2 of [Kelley, B1955]; we shall utilize this approach in our treatment of topologies on lattices, especially in Chapters II and III of this work. The material in this section is basic and elementary; a guide to additional reading – if more background is needed – is provided in the notes for Section O-2.

O-2 Completeness Conditions for Lattices and Posets

No excuse need be given for studying complete lattices, because they arise so frequently in practice. Perhaps the best infinite example (aside from the lattice of all subsets of a set) is the unit interval $\mathbb{I} = [0, 1]$. Many more examples will be found in this text – especially involving nontotally ordered lattices.

Definition O-2.1. (i) A poset is said to be *complete with respect to directed sets* (shorter: *directed complete* or also *up-complete*) if every directed subset has a sup. A **d***irected* **c***omplete* **po***set* is called a **dcpo** for short. A **dcpo** with a least element is called a *pointed* **dcpo**, or a **dcpo** *with zero* 0 or *with bottom* $\perp$.

(ii) A poset which is a semilattice and directed complete will be called a *directed complete semilattice*.

(iii) A *complete lattice* is a poset in which *every* subset has a sup and an inf. A totally ordered complete lattice is called a *complete chain*.

(iv) A poset is called a *complete semilattice* iff every nonempty (!) subset has an inf and every directed subset has a sup.

(v) A poset is called *bounded complete*, if every subset that is bounded above has a least upper bound. In particular, a bounded complete poset has a smallest element, the least upper bound of the empty set. □

We advise the reader to keep in mind that "up-complete poset" and "**dcpo**" are completely synonymous expressions; this advice is appropriate since the second terminology has become prevalent in the theoretical computer science community and since we use it in this book. We observe in the following that *a poset is a complete lattice iff it is both a* **dcpo** *and a sup semilattice with a smallest element.* In the exercises for this section we comment further on the relation of the concepts we have just introduced.

Proposition O-2.2. *Let L be a poset.*

(i) *For L to be a complete lattice it is sufficient to assume the existence of arbitrary sups (or the existence of arbitrary infs).*
(ii) *For L to be a complete lattice it is sufficient to assume the existence of sups of finite sets and of directed sets (or the existence of finite infs and filtered infs).*
(iii) *If L is a unital semilattice, then for completeness it is sufficient to assume the existence of filtered infs.*
(iv) *L is a complete semilattice iff L is a bounded complete* **dcpo**.

Proof: For (i) we observe that the existence of arbitrary sups implies the existence of arbitrary infs. Let $X \subseteq L$ and let

$$B = \bigcap\{\downarrow x \colon x \in X\}$$

be the set of lower bounds of X. (If X is empty, we take $B = L$.) We wish to show that

$$\sup B = \inf X.$$

If $x \in X$, then x is an upper bound of B; whence, $\sup B \leq x$. This proves that $\sup B \in B$; as it clearly is the maximal element of B, this also proves that X has a greatest lower bound. (There is obviously a dual argument assuming infs exist.)

For (ii) we first observe by Remark O-1.5 that the existence of finite sups and of directed sups implies the existence of arbitrary sups and then apply part (i).

For (iii), since the existence of finite infs is being assumed, the existence of all infs follows from (the dual of) (ii).

For a proof of (iv) if L is a complete semilattice and $A \subseteq L$ is bounded above, then the set of upper bounds has a greatest lower bound which will be the least upper bound of A. Conversely, for a bounded complete **dcpo** L and $\emptyset \neq A \subseteq L$ the 0 is contained in the set B of lower bounds of A. Any member of A is an upper bound of B and hence B has a least upper bound which is the greatest lower bound of A. □

Many subsets of complete lattices are again complete lattices (with respect to the restricted partial ordering). Obviously, if we assume that $M \subseteq L$ is *closed* under arbitrary sups and infs of the complete lattice L, then M is itself a complete lattice. But this is a very strong assumption on M. In view of O-2.2, if we assume only that M is closed under the sups of L, then M is a complete lattice (in itself as a poset). The well-worn example is with L equal to *all* subsets of a topological space X and with M the lattice of *open* subsets of X. This example is instructive because in general M is not closed under the infs of L (open sets are not closed under the formation of infinite intersections). Thus the infs of M (as a complete lattice) are *not* the infs of L. (**Exercise**: What is the simple topological definition of the infs of M?)

An even more general construction of subsets which form complete lattices is provided by the next theorem from [Tarski, 1955]. This theorem is of great interest in itself, as it implies that every monotone self-map on a complete lattice has a greatest fixed-point and a least fixed-point.

Theorem O-2.3. (The Tarski Fixed-Point Theorem) *Let $f\colon L \to L$ be a monotone self-map on a complete lattice L. Then the set $\mathrm{fix}(f) = \{x \in L : x = f(x)\}$ of fixed-points of f forms a complete lattice in itself. In particular, f has a least and a greatest fixed-point.* □

Proof: Let us consider first the set $M = \{x \in L : x \leq f(x)\}$ of *pre-fixed-points* of f. We first show that the sup (formed in L) of every subset $X \subseteq M$ belongs to M again. Indeed, $x \leq \sup X$ implies $x \leq f(x) \leq f(\sup X)$ by the monotonicity of f for all $x \in X$; hence $\sup X \leq f(\sup X)$ which shows that $\sup X \in M$. By O-2.2(i) we conclude that M is a complete lattice in itself. Furthermore, f maps M into itself, as $x \leq f(x)$ implies $f(x) \leq f(f(x))$ by the monotonicity of f and $M \neq \emptyset$ since $0 \in M$. Thus, restricting f yields a monotone self-map on the complete lattice M. A dual argument to the above shows that the set $F = \{x \in M : f(x) \leq x\}$ also is a complete lattice. But F is exactly the set of all fixed-points of f as the elements of F are exactly those elements of L that satisfy both inequalities $x \leq f(x)$ and $f(x) \leq x$. $\square$

If we consider again the topological example with L the powerset lattice of the space X, the mapping assigning to a subset its interior is monotone; so the completeness of the lattice of open sets also follows from O-2.3. We shall see many other examples of monotone maps. In particular, a function preserving directed sups is monotone (see Remark preceding O-1.10).

Remark O-2.4. *Let $f\colon L \to M$ be a map between complete lattices preserving sups. Then $f(L)$ is closed under sups in M and is a complete lattice in itself.*

Proof: Let $Y \subseteq f(L)$ and let $X = f^{-1}(Y)$. Then $f(X) = Y$. Also

$$\sup Y = \sup f(X) = f(\sup X),$$

because f preserves sups. Hence, $\sup Y \in f(L)$. $\square$

The above argument is not sufficient to show that if f preserves directed sups, then its image is closed under directed sups. We have to be satisfied with a special case: a self-map $p\colon L \to L$ on a poset L will be called a *projection operator* or a *projection*, for short, if it is monotone and idempotent, i.e., if $p = p \circ p$. Note that a self-map is idempotent if $p(x) = x$ for all x in the image. Projections will play a prominent role in the theory of domains.

Remark O-2.5. *For a projection p on a poset L, consider its image $p(L)$ in L with the induced ordering. Then the following properties hold.*

(i) *If X is a subset of $p(L)$ which has a sup in L, then X has a sup in $p(L)$ and*

$$\sup_{p(L)} X = p(\sup_L X).$$

The same holds for meets.

(ii) *If L is a semilattice, a lattice, a* **dcpo**, *a bounded complete* **dcpo**, *a complete lattice, respectively, the same holds for $p(L)$.*

(iii) *If, in addition, p preserves directed sups, then $p(L)$ is closed in L for directed sups, i.e., every directed subset $D \subseteq p(L)$ that has a sup in L also has a sup in $p(L)$ and*

$$\sup_{p(L)} D = \sup_L D.$$

Proof: (i) Let $X \subseteq p(L)$ have a sup in L. From $x \leq \sup_L X$ we deduce that $p(x) \leq p(\sup_L X)$ for every $x \in X$ by the monotonicity of p. By the idempotence of p, we obtain $x = p(x) \leq p(\sup_L X)$ and we we conclude that $p(\sup_L X)$ is an upper bound of X in $p(L)$. Let $a \in p(L)$ be another upper bound of X. Then $a \geq \sup_L X$, whence $a = p(a) \geq p(\sup_L X)$ again by monotonicity and idempotence of p. Thus $p(\sup_L X)$ is the least upper bound of X in $p(L)$.

Part (ii) is an immediate consequence of (i).

(iii) If $D \subseteq p(L)$ is directed and has a sup in L, then by (i), $\sup_{p(L)} D = p(\sup_L D)$. If $p\colon L \to L$ preserves directed sups, then $p(\sup_L D) = \sup_L p(D) = \sup_L D$, which finishes the proof. □

As a very simple example of the application of O-2.5, let V be a vector space (say, over the reals $\mathbb{R}$) and let L be the lattice of all *subsets* of V. For $x \in L$, define $f(x)$ to be the *convex closure* of the set x (no topology here, only convex linear combinations). The fact that an element of $f(x)$ depends on only *finitely* many elements of x is responsible for f preserving directed unions (sups) of subsets of V. Obviously we have $f(f(x)) = f(x)$. By O-2.5, the convex subsets of V form a complete lattice. Note, however, that $x \leq f(x)$ for all $x \in L$. This special property of the function f gives a special property to $f(L)$, as we shall see in Chapter I. In particular, with this property, the set of fixed-points of f is closed under infs – which is a simpler reason why $f(L)$ is a complete lattice. And, of course, this can all be verified directly for convex sets.

The next definition introduces some classical kinds of complete lattices that we shall often refer to in what follows; however, it should be noted that they only partly overlap with the class of continuous lattices.

Definition O-2.6. A *Boolean algebra* (sometimes also called *Boolean lattice*) is a lattice with 0 and 1 which is *distributive* in the sense that, for all elements x, y, z,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \tag{D}$$

and where every element x has a *complement* x' in the sense that

$$x \wedge x' = 0 \text{ and } x \vee x' = 1. \tag{C}$$

It is well known that (D) implies its dual, and that indeed every Boolean algebra is *isomorphic* to its opposite. Also well known is the fact that complements are *unique*.

A *complete Boolean algebra* (cBa for short) is a Boolean algebra that is complete as a lattice.

A *frame* (we also use the term *complete Heyting algebra* (cHa) as a synonym) is a complete lattice which satisfies the following infinite distributive law:

$$x \wedge \bigvee Y = \bigvee \{x \wedge y \colon y \in Y\}, \tag{ID}$$

for all elements x and all subsets Y. □

The proper definition of a Heyting algebra *without* completeness will emerge in the next section. From the above definition it is not immediately obvious that every cBa is a cHa, but this is the case. We return to these ideas in Exercise O-3.20.

We turn now to a list of complete lattices that, so to speak, "occur in nature". This list is far from exhaustive, and many more examples are contained in the remainder of this work. The reader may take these assertions as exercises.

Examples O-2.7. (1) We have already often referred to the *set of all subsets*, or *powerset*, of a set X. We employ the notation 2^X and of course regard this as a lattice under inclusion with union and intersection as sup and inf. It is a cBa but a rather special one. (It is atomic, for instance; and all atomic cBa's are of this form. Here, *atomic* means that every nonzero element contains a minimal nonzero element – an *atom*; for cBa's this is the same as saying that every element is the sup of atoms.)

(2) Generalizing (1), we can form the direct power L^X of any poset L; this is just the poset of *all* functions $f \colon X \to L$ under the pointwise ordering. Similarly, we can form direct products $\prod_{j \in J} L_j$ of any family of posets in the well-known way. If all the factors L_j are **dcpo**s, semilattices, lattices, complete lattices, etc., respectively, then the same holds for the direct product $\prod_{j \in J} L_j$.

(3) If X is a topological space, our notation for the *topology*, or *set of open subsets*, of X is $\mathcal{O}(X)$. It is a sublattice of 2^X closed under finite intersections and under arbitrary unions. It is clear then that $\mathcal{O}(X)$ is a frame since we know the truth of O-2.6 (ID) for the set theoretical operations. In general $\mathcal{O}(X)$ is *not* closed under arbitrary intersections, and its opposite is *not* a frame. (Consider the case of $X = \mathbb{R}$, the real line.)

The opposite of $\mathcal{O}(X)$ is a complete lattice and is obviously isomorphic to the lattice $\Gamma(X)$ of *closed* subsets of X. The isomorphism between $\mathcal{O}(X)^{\mathrm{op}}$ and $\Gamma(X)$ is by complements: $U \mapsto X \backslash U$.

Contained in $\mathcal{O}(X)$ is a very interesting complete lattice $\mathcal{O}_{\mathrm{reg}}(X)$ of *regular open sets*, that is, those sets equal to the interiors of their closures. The sup is *not* the union of the regular open sets but the *interior of the closure of the union*. The inf is the *interior of the intersection* (which is the same as the inf in $\mathcal{O}(X)$). Remarkably, $\mathcal{O}_{\mathrm{reg}}(X)$ is a cBa where the lattice complement of a $U \in \mathcal{O}_{\mathrm{reg}}(X)$ is the interior of $(X \backslash U)$. Actually this construction of a cBa can be done abstractly in any frame (cHa), and we return to it in the next section (see Exercise O-3.21).

For much more on Boolean algebras and the proof that *every* cBa is isomorphic to $\mathcal{O}_{\mathrm{reg}}(X)$ for some space X, the reader is referred to Halmos, B1963. (It is interesting to note that $\mathcal{O}_{\mathrm{reg}}(\mathbb{R})$ is an *atomless* cBa. That is to say, there are no minimal nonzero elements.)

(4) Let $\mathcal{A}$ be an abstract algebra with any number of operations. The poset (Cong $\mathcal{A}, \subseteq$) of all *congruence relations* under inclusion (of the graphs of the relations) forms a complete lattice, because congruence relations are closed under arbitrary intersections. This example includes numerous special cases:

(i) If $\mathcal{A}$ is a *group*, then Cong $\mathcal{A}$ can be identified with the lattice of all *normal* subgroups in the usual way, and if $\mathcal{A}$ is an *abelian group* (or a module or a vector space), with the lattice of *all* subgroups (submodules, vector subspaces). In general this lattice is *not* distributive.

(ii) If $\mathcal{A}$ is a *ring*, then Cong $\mathcal{A}$ is canonically isomorphic to the lattice of all two-sided ideals. If $\mathcal{A}$ is a *lattice ordered group* (lattice ordered ring), then Cong $\mathcal{A}$ can be identified with the lattice of all order convex normal subgroups (ideals) which are also sublattices. In general the ideals of a ring *do not* form a distributive lattice.

(iii) If $\mathcal{A}$ is a *lattice*, then Cong $\mathcal{A}$ cannot generally be identified with either the ideals or the filters of $\mathcal{A}$, but it *does* form a frame. (**Exercise**: Prove the distributivity.) If $\mathcal{A}$ is a Boolean algebra, then identification with the lattice of ideals is possible.

Note that in the case of algebras with finitary operations, Cong $\mathcal{A}$ is closed under directed unions. The significance of this remark will become clear in Section I-4.

(5) If $\mathcal{A}$ is an abstract algebra, then (Sub $\mathcal{A}, \subseteq$), the structure of all *subalgebras* of $\mathcal{A}$ under inclusion, also becomes a complete lattice. The reader can supply special cases easily. In the case of vector spaces, the lattice of

subspaces has complements but not unique ones owing to the failure of the distributive law.

(6) Let $\mathcal{A}$ be a compact Hausdorff topological algebra. Then the set $\mathrm{Cong}^{-}\ \mathcal{A}$ of *closed congruences* (congruences $R \subseteq \mathcal{A} \times \mathcal{A}$ closed in the product space) also forms a complete lattice. The relevance of this example is that these congruences correspond precisely to compact Hausdorff quotient algebras.

(7) Let $\mathcal{A}$ be a Hausdorff topological ring, then the set $\mathrm{Id}^{-}\ \mathcal{A}$ of *closed two-sided ideals* forms a complete lattice. Again the interest lies in the fact that the quotient rings are Hausdorff.

(8) Let $\mathcal{H}$ be a Hilbert space. Then $\mathrm{Sub}^{-}\ \mathcal{H}$, the *closed subspaces* of $\mathcal{H}$, forms a complete lattice. This generalizes to the lattice of projections in any von Neumann algebra.

(9) Every nonempty compact interval of real numbers in its natural order is a complete lattice, and all nonsingleton intervals are isomorphic to $\mathbb{I} = [0, 1]$ and to the infinite interval

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty].$$

As complete lattices are closed under direct products (see (2) above), we can form $\mathbb{I}^X$, where X is an arbitrary set. Such lattices are called *cubes*. In Exercise O-2.10, we note what can be said if X is a topological space and only *certain* functions are admitted; this connects with the ideas of semicontinuous functions and real-valued random variables, to which we return in I-1.22. An easy example of a restricted function space which is a complete lattice would be the subspace $M \subseteq \mathbb{I}^{\mathbb{I}}$ of all *monotone* functions from $\mathbb{I}$ into itself.

(10) Let $\mathcal{F}$ be the set of all *partial* functions from the set $\mathbb{N}$ of natural numbers into itself (this could be generalized to any other set besides the set $\mathbb{N}$). Thus, if the function $f \in \mathcal{F}$, then its *domain*, $\mathrm{dom}\, f$, is a subset of $\mathbb{N}$ and $f\colon \mathrm{dom}\, f \to \mathbb{N}$. The empty function $\emptyset\colon \emptyset \to \mathbb{N}$ is allowed. We define $f \leq g$ to mean that

$$\mathrm{dom}\, f \subseteq \mathrm{dom}\, g \text{ and } f = g|_{\mathrm{dom}\, f},$$

that is, whenever f is defined, then g is defined and they have the same value. This definition makes $\mathcal{F}$ into a poset with directed sups and arbitrary *nonempty* infs: $\mathcal{F}$ is a complete semilattice, it fails to be a lattice only in lacking a top.

In Exercise O-2.12 we show how to adjoin a top to such structures. Another repair would be to expand $\mathbb{N}$ to $\mathbb{N}^* = \mathbb{N} \cup \{\bot, \top\}$, which is a poset under the ordering where for $x, y \in \mathbb{N}^*$ we have

$$x \leq y \text{ iff } x = \bot \text{ or } x = y \text{ or } y = \top.$$

Then $\mathcal{F}$ can be regarded as a subset of $(\mathbb{N}^*)^{\mathbb{N}}$ under the pointwise ordering (we define $f(x) = \bot$ if $x \notin \operatorname{dom} f$). (Note that this ordering has nothing to do with natural ordering of $\mathbb{N}$.) Now $(\mathbb{N}^*)^{\mathbb{N}}$ is a complete lattice, but it is *much larger* than $\mathcal{F} \cup \{\top\}$, because for $f \in (\mathbb{N}^*)^{\mathbb{N}}$ the values taken in $\{\bot, \top\}$ and in $\mathbb{N}$ can be very mixed.

For applications to the theory of computation this proliferation of top elements is most inconvenient. If we read $f \leq g$ as an "information ordering" (roughly, f and g are consistent but g has possibly more information than f), then the only interpretation of $\top$ is to consider it as the *inconsistent* element. (The words "overdefined" for $\top$ and "underdefined" for $\bot$ have also been used.) As we generally try to keep our values "consistent" as much as possible, it seems natural to avoid $\top$. Because of the importance of the applications to computability, we should keep in mind the need to cover examples like this in our general theory. □

The following also deals with examples, but they play such a very prominent role in what follows that we separate them out.

Examples O-2.8. Let L be a poset.

(1) The family of all *lower sets* of L and the family of all *upper sets* are both complete lattices under $\subseteq$; indeed, both of these families are closed under arbitrary intersections and unions in 2^L.

(2) In any poset L, $\operatorname{Filt}_0 L$ and $\operatorname{Filt} L$ are closed under directed unions and hence **dcpo**s. If L is a semilattice, then $\operatorname{Filt}_0 L$ is a complete lattice; if L is also unital, then $\operatorname{Filt} L$ is complete. In the latter case both lattices of sets are closed under arbitrary intersections in 2^L. In a semilattice the ideals only form a semilattice, since in 2^L both $\operatorname{Id}_0 L$ and $\operatorname{Id} L$ are only closed under finite intersections. We note that the infinite intersection of ideals in a semilattice need not be an ideal (cf. O-1.15 and its figure).

(3) In a lattice, both $\operatorname{Filt}_0 L$ and $\operatorname{Id}_0 L$ are complete lattices; and if L has a top and bottom, then $\operatorname{Filt} L$ and $\operatorname{Id} L$ are complete lattices.

(4) The function $x \mapsto {\downarrow}x : L \to \operatorname{Id} L$ is an embedding preserving arbitrary infs and finite sups; it is called the *principal ideal embedding*. (There is a dual principal filter embedding.) The example $L = \mathbb{N} \cup \{\infty\}$ (with its natural ordering) shows that the principal ideal embedding need *not* preserve arbitrary (or even directed) sups.

(5) If L is a Boolean algebra, we can construe it as an algebra of "propositions" (0 is *false* and 1 is *true*, $\wedge$ and $\vee$ are *conjunction* and *disjunction*, complementation is *negation*). $\operatorname{Filt} L$ can be thought of as the lattice of *theories*. Any subset $A \subseteq L$ can be taken as a set of "axioms" generating the following "theory",

which is just a filter and corresponds to the propositions "implied" by the axioms:

$$\{x \in L : (\exists a_0, \ldots, a_{n-1} \in A) \quad a_0 \wedge \cdots \wedge a_{n-1} \leq x\}.$$

The "inconsistent" theory is L, that is, the top filter generated by $\{0\}$. If we eliminate L, then $\mathrm{Filt}\, L \backslash \{L\}$ is closed under arbitrary nonempty intersections and directed unions. This is similar to the poset of O-2.7(10). As is well known, the lattice $\mathrm{Filt} L$ is lattice isomorphic to the lattice of open subsets of the Stone space of the Boolean algebra L. □

Exercises

Exercise O-2.9. (Clopen sets) Let X be a topological space and let $\Gamma\mathcal{O}(X) = \mathcal{O}(X) \cap \Gamma(X)$ be the sublattice of 2^X of all *closed-and-open sets* (sometimes: *clopen sets*). Show that $\Gamma\mathcal{O}(X)$ is not complete in general, but it is always a Boolean algebra. For a compact totally disconnected space, show that $\Gamma\mathcal{O}(X)$ is complete iff the closure of every open set is open (such spaces are called *extremally disconnected*). (This complements Example O-2.7(3).) □

Exercise O-2.10. (Semicontinuous functions) Let X be a topological space, and let $C(X, \mathbb{R}^*)$ be the set of continuous extended real-valued functions. Verify the following assertions: under the pointwise ordering, $C(X, \mathbb{R}^*)$ is not complete, but it is a lattice with a top and bottom. For compact X, it is complete iff X is extremely disconnected.

Over an arbitrary space to have a complete lattice we must pass to a larger lattice. The *lower semicontinuous* functions $f \in \mathrm{LSC}(X, \mathbb{R}^*)$ are characterized by the condition that the set $\{x \in X\colon r < f(x)\}$ is open in X for every $r \in \mathbb{R}^*$. (For *upper semicontinuous* functions we reverse the inequality.) The lattice $\mathrm{LSC}(X, \mathbb{R}^*)$ is complete because it is closed under arbitrary pointwise sups. Notice that $\mathrm{LSC}(X, \mathbb{R}^*)$ is also closed under finite pointwise infs but not under arbitrary pointwise infs. The lattices $\mathrm{LSC}(X, \mathbb{R}^*)$ and $\mathrm{USC}(X, \mathbb{R}^*)$ are anti-isomorphic and

$$C(X, \mathbb{R}^*) = \mathrm{LSC}(X, \mathbb{R}^*) \cap \mathrm{USC}(X, \mathbb{R}^*).$$ □

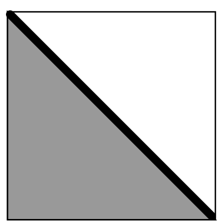
In the next exercises, and many times elsewhere in this text, we shall have occasion to discuss weaker forms of completeness as was already indicated in Definition O-2.1. In order to compare the definitions of a complete lattice and a complete semilattice we suggest that the reader recall that a complete lattice is a poset with all conceivable completeness properties which a lattice may have and which are *symmetric* (i.e., remain invariant under passage to the opposite poset); whereas a complete semilattice has, coarsely speaking, the maximal completeness properties which a semilattice may have, short of becoming a lattice. Every *finite* semilattice of course is a complete semilattice. Every complete semilattice which is, in addition, unital is clearly a complete lattice (O-2.2).

Exercise O-2.11. Let S be a poset in which every nonempty subset has an inf. Show that every $X \subseteq S$ with an upper bound has a sup. □

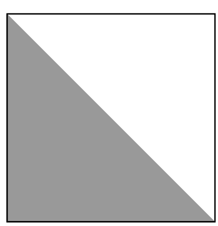
Exercise O-2.12. Let again S be a poset in which every nonempty subset has an inf. Adjoin an identity by forming $S^1 = S \cup \{1\}$ with an element $1 \notin S$ and $x \le 1$ for all $x \in S$. Show that S^1 is a complete lattice. □

As a consequence, the adjunction of an identity to a complete semilattice will produce a complete lattice.

Exercise O-2.13. Let S be the closed lower left triangle $\{(x, y) : x + y \le 1\}$ in the square $[0, 1]^2$.



Verify the following assertions: S is a complete semilattice but *not* a complete lattice. (Actually, the subsemilattice T of S consisting of the three corner points serves to illustrate this.) The interior of the triangle, $\{(x, y) : x + y < 1\}$, is a semilattice in which every nonempty subset has an inf, but it is *not* a complete semilattice.



The half open interval $]0,1]$ is a directed complete lattice, but it is *not* a complete semilattice. □

Exercise O-2.14. Prove the following.

(i) A poset is directed complete iff all ideals have sups.
(ii) A semilattice is a complete semilattice iff all filters have infs and all ideals have sups. □

Exercise O-2.15. Prove the following.

(i) Every poset may be embedded into a complete lattice with the preservation of all existing infs.
(ii) Every lattice may be embedded into a complete lattice with the preservation of all finite lattice operations and all existing infs.
(iii) Every lattice may be embedded into a complete lattice with the preservation of all existing sups and infs.

Hint. Parts (i) and (ii) are easily accomplished with the means available in Section 2. For (i) use the complete lattice of all lower sets and the embedding $x \mapsto {\downarrow}x$. For (ii) use the complete lattice $\mathrm{Id}\, L$ and the principal ideal embedding. Finally, (iii) is the so-called *MacNeille completion*, which is likewise constructed by using suitable ideals; we refer to the existing literature for details, e.g., [Balbes and Dwinger, B1974], p. 235. □

Exercise O-2.16. Prove the following.

(i) For every semilattice S, the poset $\mathrm{Id}\, S$ is a directed complete semilattice.
(ii) If S is a semilattice in which every nonempty subset has an inf, then $\mathrm{Id}\, S$ is a complete semilattice. □

Exercise O-2.17. In a Boolean algebra, is the lattice of finitely axiomatizable "theories" complete? directed complete? □

Exercise O-2.18. Let G be a group and let H be any subgroup. Let L be the lattice of all subsets of G, that is, $L = 2^G$. Let M be the collection of *double cosets* of H; that is, let

$$M = \{X \subseteq G : X = XH = HX\}.$$

Prove that M is a cBa, and discuss the closure properties of M within L with respect to sups and infs.

Hint. Consider the map $X \mapsto HXH$. □

Exercise O-2.19. Let $\mathcal{F}$ be as in O-2.7(10). Define $\mathcal{G} \subseteq \mathcal{F}$ to be the collection of all *one-to-one* partial functions. Is $\mathcal{G}$ a complete semilattice? □

Exercise O-2.20. (Least Fixed-Point Theorem for dcpos) Let L be a **dcpo** with a bottom element $\perp$. Show that every monotone self-map $f: L \to L$ has a least fixed-point.

The preceding result generalizes the fixed-point theorem O-2.3 for complete lattices. We will indicate two proofs for this fact. The first proof uses transfinite induction:

Hint. We define $a_0 = \perp$ and, by transfinite induction, $a_{\alpha+1} = f(a_\alpha)$ for every ordinal α and $a_\alpha = \sup_{\beta<\alpha} a_\beta$ for limit ordinals. As the cardinality of L is bounded, there is an ordinal γ such that $a_{\gamma+1} = a_\gamma$. This a_γ is a fixed-point of f, and it is the least one, as one can verify readily. □

A second proof avoiding transfinite or equivalent reasonings due to D. Pataraia (unpublished) is included in the following exercise.

Exercise O-2.21. Let L be a **dcpo** with a bottom element $\perp$. We denote by $\mathcal{L}$ the set of all monotone self-maps $g: L \to L$ that are *inflationary*, i.e., $x \leq g(x)$ for all $x \in L$. We equip $\mathcal{L}$ with the pointwise ordering of functions. Let f be an arbitrary monotone self-map of L. Prove the following.

(i) $\mathcal{L}$ is a **dcpo** with a greatest element T.

Hint. First, let us remark that $\mathcal{L}$ is nonempty, as it contains the identity map as least element. As $g \leq g \circ h$ and $h \leq g \circ h$ for inflationary maps g and h, we conclude that $\mathcal{L}$ is directed. It is readily verified that $\mathcal{L}$ is complete with respect to directed pointwise suprema. Hence, $\mathcal{L}$ has a greatest element that we denote by T.

(ii) For every $x \in L$, $T(x)$ is a common fixed-point of all $g \in \mathcal{L}$.

Hint. Clearly, $g \circ T \in \mathcal{L}$ for every $g \in \mathcal{L}$. Hence, $g \circ T \leq T$ as T is the top element of $\mathcal{L}$. On the other hand, $g \circ T \geq T$ for inflationary g. Consequently, $g \circ T = T$ which implies the claim.

(iii) Let $M = \{x \in L : x \leq f(x)\}$ be the set of pre-fixed-points of f. Show that (a) $\perp \in M$, (b) M is closed for directed sups, and (c) M is mapped into itself by f.

Hint. Compare the proof of O-2.3.

(iv) Every monotone self-map $f: L \to L$ has a least fixed-point.

Hint. Let S be the smallest subset of L having the following three properties: (a) $\perp \in S$, (b) S is closed in L for directed suprema, (c) S is mapped into itself by f. (For the existence of such an S, just consider the intersection of all subsets with these three properties.) As, by (iii), the set M of pre-fixed-points of f has the properties (a), (b), (c), we have $S \subseteq M$. Thus, the restriction of f is a monotone self-map of S which is inflationary. As in (i), denote by T the greatest monotone inflationary self-map of S. Then $a = T(\perp)$ is a fixed-point of f by (ii). We have to show that a is the least fixed-point f. For this, let $b \in L$ be any fixed-point of f. Then the set $\downarrow b$ satisfies the properties (a), (b), (c), whence $S \subseteq \downarrow b$. As $a \in S$, we conclude $a \leq b$. □

Exercise O-2.22. Show that every family $(g_i)_{i \in I}$ of monotone inflationary self-maps on a **dcpo** with $\perp$ has a least common fixed-point.

Hint. As in the hint for O-2.21(iv), let S be the smallest subset of L with the properties (a) $\perp \in S$, (b) S is closed in L for directed suprema, (c) S is mapped into itself by f_i for all $i \in I$. By O-2.21(ii), there exists a common fixed-point $a \in S$ for all f_i, $i \in I$. Let us show that a is the least common fixed-point of the f_i, $i \in I$. Indeed, if b is any common fixed-point of the f_i, $i \in I$, the set $\downarrow b$ satisfies the properties (a), (b), (c), whence $S \subseteq \downarrow b$. As $a \in S$, we conclude $a \leq b$. □

Old notes

It would be inappropriate to attempt a history of the material contained in this introductory chapter; it belongs to the fundamentals of almost any kind of lattice theory and is therefore presented in most sources.

However, it may serve a useful purpose to give a guide to the existing textbook and monograph literature. We disclaim any ambition to be complete in this regard.

The classic sourcebook on lattice theory is, of course, the book by Garrett Birkhoff [Birkhoff, B1967] which has inspired many generations of lattice theoreticians. The latest edition is representative of the status of the theory in 1967. The date of the first edition in 1940 points up the truly classic character of this work.

Other standard source books on lattice theory are [Grätzer, B1978], [Balbes and Dwinger, B1974] and [Crawley and Dilworth, B1973]. As far as the topic of Boolean algebras is concerned, the book by Sikorski [Sikorski, B1964] remains an effective source. The first edition dates back to 1957. The date, 1995, of appearance of the first edition of [Hermes, B1967], which experienced a second and revised edition in 1967. An introductory text to lattice theory was presented in 1953, [Dubreil-Jacotin *et al.*, B1953].

For a comprehensive treatment of general lattice theory we recommend the book [Grätzer, B1978].

New notes

Definition O-2.1 marks our first departure from *A Compendium of Continuous Lattices* with the introduction of the notion of a **dcpo**. A variety of other completeness notions arises as one introduces additional order structure, such as that of a semilattice or lattice. In particular one has the important notion of a bounded complete **dcpo**.

The elegant intuitionistic proofs of the fixed-point theorems in the exercises O-2.21 and O-2.22 are due to D. Pataraia. We have learned about these proofs through M. Escardó.

Numerous easygoing textbooks for the student are available, too. Halmos' [Halmos, B1963] has become rather well known; other textbooks from the 1960s are [Gericke, B1963], translated into English in 1966, and [Abbott, B1969]. A more recent one is that by B. A. Davey and H. A. Priestley [Davey and Priestley, B1990]. The book by Grätzer has seen an extended secoond edition in 1998.

O-3 Galois Connections

We now introduce one of the most efficient tools in dealing with complete lattices; in this sense we continue the discussion of the previous section on complete lattices. One reason for this great efficiency is that the pairs of maps of the kind we are about to single out exist in great profusion. It is therefore very helpful to know in general what properties such maps have.

Definition O-3.1. Let S and T be two posets. We shall say that a pair (g, d) of functions $g: S \to T$ and $d: T \to S$ is a *Galois connection* or *an adjunction* between S and T provided that

(i) both g and d are monotone, and
(ii) the relations $g(s) \geq t$ and $s \geq d(t)$ are equivalent for all pairs of elements $(s, t) \in S \times T$.

In an adjunction (g, d), the function g is called the *upper adjoint* and d the *lower adjoint*. □

Notice that we have to keep the order straight. Then the upper adjoint is unambiguously determined by the "greater" side in the relation $g(s) \geq t$ of (ii) above (whence the letter g), whereas the lower adjoint is given by the lower or "downward" side in the relation $s \geq d(t)$ (whence the letter d).

Terminological difficulties may arise when we recognize that Galois connections are nothing but very special cases of pairs of *adjoint functors*. For this interpretation we need to construe S and T as *categories* with their respective elements as *objects*. The question is how to link the partial orders with *morphisms*. One is tempted to read an arrow $x \to y$ for $x, y \in S$ precisely when $x \geq y$, so that the arrow and the $\geq$-sign point in the *same* direction. This was done in [Hofmann and Lawson, 1976], and as a consequence upper adjoints were called *left* adjoints and lower adjoints *right* adjoints.

However, existing practice among category theory oriented writers bears heavily upon us to choose the dual interpretation:

$$\operatorname{card}(\operatorname{Hom}(x, y)) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $x \to y$ and $y \geq x$ are now equivalent statements. The "product" in a semilattice (that is, the inf) is then a product in the categorical sense. More generally, infs are limits, sups colimits. Order preserving maps are functors, and an adjunction (g, d) is a pair of adjoint functors with g being right adjoint and d left adjoint. The entire machinery of adjoint functors is now immediately available for Galois connections. (See, for example, [Mac Lane, B1971], Chapter IV.) But for the purposes of this work we wish to give a self-contained presentation and, therefore, we offer direct, elementary arguments for the essential facts. Moreover, we try to avoid the ambiguities involved in the use of "left" and "right" by using the to-be-hoped unambiguous words "upper" and "lower" instead.

Theorem O-3.2. *Let $g\colon S \to T$ and $d\colon T \to S$ be functions between posets. Then the following conditions are equivalent:*

(1) *(g, d) is a Galois connection;*
(2) *g is monotone and $d(t) = \min g^{-1}(\uparrow t)$ for all $t \in T$;*
(3) *d is monotone and $g(s) = \max d^{-1}(\downarrow s)$ for all $s \in S$.*

Consequently, in an adjunction one map uniquely determines the other.

Proof: (1) implies (2): Since $t \leq g(s)$ iff $d(t) \leq s$ by (1), we know that $d(t)$ is a lower bound of $g^{-1}(\uparrow t)$. But O-3.1(ii) applied to $d(t) \leq d(t)$ gives us at once $t \leq g(d(t))$, that is, $d(t) \in g^{-1}(\uparrow t)$, whence (2).

(2) implies (1): Firstly, let $t \leq g(s)$. Then $s \in g^{-1}(\uparrow t)$, whence

$$s \geq \min g^{-1}(\uparrow t) = d(t).$$

Secondly, let $m = \min g^{-1}(\uparrow t)$, whence $m \in g^{-1}(\uparrow t)$, and thus $g(m) \geq t$. If now it holds that $s \geq d(t) = m$, then $g(s) \geq g(m) \geq t$, since g is monotone. The relation for d in (2) clearly makes d monotone, and thus the conditions of O-3.1 are satisfied.

The proof of (1) iff (3) is analogous. (Or alternatively, we may observe that (g, d) is an adjunction between S and T iff (d, g) is an adjunction between T^{op} and S^{op}; thus, by duality, we can use what has already been proved.) □

Theorem O-3.3. *Any upper adjoint preserves infs, any lower adjoint, sups.*

Proof: Consider an adjunction (g, d) between S and T. Let $\{s_j : j \in J\}$ be a family in S and let $s = \inf\{s_j : j \in J\}$.

Since g is order preserving, we have $g(s) \leq g(s_j)$ for all $j \in J$. Now suppose that t is an arbitrary lower bound of $\{g(s_j) : j \in J\}$. Then for all $j \in J$ we have $g(s_j) \geq t$, which means $s_j \geq d(t)$ by O-3.1(ii). Thus,

$$s = \inf\{s_j : j \in J\} \geq d(t),$$

whence $g(s) \geq t$. This shows that indeed $g(s) = \inf\{g(s_j) : j \in J\}$.

The proof that d preserves sups is dual. □

This result is very handy in establishing that certain functions preserve arbitrary infs or sups. In fact, in the presence of completeness, as we shall now see, the existence of a lower adjoint is *necessary* for the preservation of arbitrary infs.

We say that a function $g: S \to T$ into a poset is *cofinal* if for all $t \in T$ there is an $s \in S$ such that $t \leq g(s)$, i.e. if $g^{-1}(\uparrow t) \neq \emptyset$ for all $t \in T$. If g has a lower adjoint, it is immediate that g is cofinal, as $t \leq g(d(t)$ for every t.

Theorem O-3.4. *Let $g: S \to T$ be a function between posets. Assume that the following hypotheses are satisfied:*

(i) *S is a complete lattice, or S is a complete semilattice and g is cofinal, and*
(ii) *g preserves all existing infs.*

Then g has a lower adjoint $d: T \to S$ given by either of the two formulae

(1) $d(t) = \inf g^{-1}(\uparrow t)$,
(2) $d(t) = \min g^{-1}(\uparrow t)$. □

Proof: We define $d: T \to S$ by formula (1): this is possible since S is complete. Clearly, d is monotone. If $t \leq g(s)$, then $s \in g^{-1}(\uparrow t)$, and thus

$$d(t) = \inf g^{-1}(\uparrow t) \leq s.$$

Conversely, if $d(t) \leq s$, then $g(d(t)) = g(\inf g^{-1}(\uparrow t)) \leq g(s)$, since g is monotone (preserving infs); but, since g *preserves* infs, we also have

$$t \leq \inf g(g^{-1}(\uparrow t)) = g(\inf g^{-1}(\uparrow t)) \leq g(s).$$

This shows that (g, d) is an adjunction. We have also shown that $g(d(t)) \geq t$; that is, $d(t) \in g^{-1}(\uparrow t)$, which implies formula (2) in view of (1). □

Corollary O-3.5.

(i) *Let $g: S \to T$ be a function between posets of which S is a complete lattice. Then g preserves infs iff g is monotone and has a lower adjoint.*
(ii) *Let $d: T \to S$ be a function between posets of which T is a complete lattice. Then d preserves sups iff d is monotone and has an upper adjoint.*

Proof: This is clear from O-3.3 and O-3.4 and its dual. □

One can describe adjunctions in still other ways. We recall that a function $p: L \to L$ is *idempotent* iff $pp = p$.

Theorem O-3.6. *For every pair of order preserving functions between posets, $g: S \to T$ and $d: T \to S$, the following conditions are equivalent:*

(1) *(g, d) is an adjunction;*
(2) *$dg \leq 1_S$ and $1_T \leq gd$.*

Moreover, these conditions imply

(3) *$d = dgd$ and $g = gdg$,*
(4) *gd and dg are idempotent.*

Proof: (1) implies (2): For all $s \in S$ one has $g(s) \leq g(s)$, hence $d(g(s)) \leq s$ by (1); and for all $t \in T$ one has $d(t) \geq d(t)$, hence $g(d(t)) \geq t$ by (1).

(2) implies (1): Let $t \leq g(s)$; then $d(t) \leq d(g(s))$, because d is monotone. By (2), $d(g(s)) \leq s$; whence, $d(t) \leq s$. Similarly $s \geq d(t)$ implies $g(s) \geq g(d(t)) \geq t$.

(2) implies (3): $dg \leq 1_S$ implies $dgd \leq d$, since d is monotone; and $1_T \leq gd$ implies $d \leq dgd$. Thus, $d = dgd$. The rest is similar.

(3) implies (4): Trivial. □

In adjunctions, injective and surjective maps are paired off as follows.

Proposition O-3.7. *For an adjunction (g, d) between posets S and T, the following conditions are equivalent:*

(1) *g is surjective;*
(2) *$d(t) = \min g^{-1}(t)$ for all $t \in T$;*
(3) *$gd = 1_T$;*
(4) *d is injective.*

Likewise, the following statements are equivalent:

(1*) *g is injective;*
(2*) *$g(s) = \max d^{-1}(s)$ for all $s \in S$;*
(3*) *$dg = 1_S$;*
(4*) *d is surjective.*

Proof: (1) implies (2): Now $d(t) = \min g^{-1}(\uparrow t)$ by O-3.2. If g is surjective, then $g(g^{-1}(\uparrow t)) = \uparrow t$; and, since g is monotone,

$$g(d(t)) = \min g(g^{-1}(\uparrow t)) = \min \uparrow t = t.$$

Thus, $d(t) \in g^{-1}(t)$; whence $\min g^{-1}(t) = d(t)$.

(2) implies (3): From (2), we have $d(t) \in g^{-1}(t)$, i.e., $g(d(t)) = t$ for all $t \in T$.

(3) implies (4): By (3), d is a co-retraction, hence, it is injective.

(4) implies (1): By O-3.6 we have $d = dgd$, and if d is injective, we have $1_T = gd$. Thus, g is a retraction and hence surjective.

The equivalence of (1*)–(4*) is proved dually. □

We indicated in earlier examples how closure and kernel operators function in applications. Now we have a systematic framework for such maps. We begin by recalling the definition of a projection (see O-2.5):

Definition O-3.8. Let L be a poset.

(i) A *projection operator* (shortly *projection*) is an idempotent, monotone self-map $p: L \to L$.
(ii) A *closure operator* is a projection c on L with $1_L \leq c$.
(iii) A *kernel operator* is a projection k on L with $k \leq 1_L$. □

Warning: This terminology deviates from that used in [Scott, 1976], who uses 'retraction' for projection operators and 'projection' for kernel operators.

As to the nomenclature of (ii) and (iii), we remind the reader of Example O-2.7(3): If X is a topological space, then $A \mapsto \operatorname{cl} A : 2^X \to 2^X$ is a closure operator and the map $A \mapsto \operatorname{int} A : 2^X \to 2^X$ is a kernel operator of 2^X. The image of the former is $\Gamma(X)$ and that of the latter $\mathcal{O}(X)$. We note that the map $U \mapsto \operatorname{int} \operatorname{cl} U : \mathcal{O}(X) \to \mathcal{O}(X)$ is a closure operator with image $\mathcal{O}_{\text{reg}}(X)$, the regular open sets.

Notation O-3.9. For any function $f: A \to B$, we denote the *co-restriction* to the image by $f^\circ: A \to f(A)$ and then the *inclusion* of the image into B accordingly by $f_\circ: f(A) \to B$. Thus, each f has the decomposition $f = f_\circ f^\circ$. If $B = A$, then $f^\circ f_\circ$ is the restriction *and* co-restriction $f \mid f(A): f(A) \to f(A)$. □

Proposition O-3.10. *Let L be a poset and $f: L \to L$ an order preserving self-map of L. Then we have the following three groups of equivalent statements:*

(1) *f is a projection operator,*
(2) *f° is a retraction of L onto $f(L)$ with $f_\circ: f(L) \to L$ as co-retraction (that is, $f^\circ f_\circ = 1_{f(L)}$),*
(3) *there are a poset T and a monotone surjection $q: L \to T$ and a monotone injection $i: T \to L$ such that $f = iq$ and $1_T = qi$;*
(1_1) *f is a closure operator,*
(2_1) *$(f_\circ, f^\circ)$ is an adjunction between $f(L)$ and L,*
(3_1) *there is an adjunction (g, d) between some S and L where $f = gd$;*
(1_2) *f is a kernel operator,*
(2_2) *$(f^\circ, f_\circ)$ is an adjunction between L and $f(L)$,*
(3_2) *there is an adjunction (g, d) between L and some T where $f = dg$.*

Proof: We prove the equivalence of (1_2), (2_2), (3_2) only.

(1_2) implies (2_2): If f is a projection, then we have $f^\circ f_\circ = 1_{f(L)}$ and $f_\circ f^\circ = f$; if in addition, f is a kernel operator, then $f \leq 1_L$ and (2_2) follows by O-3.6.

(2_2) implies (3_2): Trivial.

(3_2) implies (1_2): By O-3.6(4), the map $f = dg$ is a projection. By O-3.6(2) we have that $f = dg \leq 1_L$, whence (1_2). □

We have in fact said that adjunctions on one hand and kernel and closure operators on the other are tightly linked: indeed, the co-restriction to the image of every closure (resp., kernel) operator *is* the lower (resp., upper) adjoint of an adjunction. Conversely, whenever (g, d) is an adjunction, then gd is a closure operator and dg is a kernel operator.

Let us now note, however, that a mere projection is the "union" of a closure and a kernel operator:

Lemma O-3.11. *Let p be a projection on a poset L. We set*

$$L_c = \{x \in L\colon x \le p(x)\} \quad \textit{and} \quad L_k = \{x \in L\colon p(x) \le x\}.$$

Then we have the following conclusions:

(i) *p maps L_c and L_k into themselves and if $p_c\colon L_c \to L_c$ and $p_k\colon L_k \to L_k$ are the two restrictions of p, then p_c is a closure operator and p_k is a kernel operator with*

$$\operatorname{im} p_c = \operatorname{im} p_k = \operatorname{im} p = L_c \cap L_k;$$

(ii) *L_c is closed under all existing sups and L_k under all existing infs;*
(iii) *if p preserves (filtered) infs, then L_c and $\operatorname{im} p$ are closed under existing (filtered) infs; analogously, if p preserves (directed) sups, then L_k and $\operatorname{im} p$ are closed under existing (directed) sups.*

Proof: (i) Straightforward.

(ii) Let $X \subseteq L_c$ be such that $\sup X$ exists in L. Since $X \subseteq L_c$ and since p is monotone, $x \le p(x) \le p(\sup X)$ for all elements $x \in X$; therefore, we find that $\sup X \le p(\sup X)$ and, consequently, $\sup X \in L_c$.

(iii) Now let X be a (filtered) subset of L_c for which $\inf X$ exists in L. If p preserves (filtered) infs, then $\inf p(X)$ exists in L and

$$p(\inf X) = \inf p(X) \ge \inf X;$$

whence, $\inf X \in L_c$. Since $\operatorname{im} p = L_c \cap L_k$ and since L_k is closed under arbitrary infs by (ii), then $\operatorname{im} p$ is also closed under (filtered) infs. □

The closure properties of $\operatorname{im} p$ may also be derived from O-2.4 and O-2.5. Notice that L_c and L_k are complete lattices as long as L is a complete lattice by (ii) and O-2.4. The second portion of (iii) will play a role when we discuss continuous lattices.

We remark next that the presence of projections makes certain preservation properties automatic, as we have seen in O-2.5: the image of a semilattice, a **dcpo**, a bounded complete **dcpo**, a complete lattice, respectively, is again of this type. For closure and kernel operators we can say more:

Proposition O-3.12.

(i) *The image of a closure operator is closed under the formation of infs, and that of a kernel operator is closed under the formation of sups (to the extent they exist).*

(ii) *The co-restriction $c^\circ\colon L \to c(L)$ of a closure operator preserves arbitrary sups; hence,* $\sup_{c(L)} X = c(\sup_L X)$ *for* $X \subseteq c(L)$.

(iii) *The co-restriction $k^\circ\colon L \to k(L)$ of a kernel operator preserves arbitrary infs; hence,* $\inf_{k(L)} X = k(\inf_L X)$ *for* $X \subseteq k(X)$.

Proof: Part (i) follows from O-3.3, O-3.10(2_1) and O-3.10(2_2).

Parts (ii) and (iii) are consequences of O-3.3 and O-3.10. □

It will be useful to think about closure operators in alternative ways. One well-known way is to associate with a closure operator a "closure system". The specifics are as follows. Let L be a poset. A subset S of L will be called a *closure system* if, for every $x \in L$, among the upper bounds of x in S there is a smallest one, i.e., $\uparrow x \cap S$ has a smallest element, which we denote by $c_S(x)$. Let $\mathcal{C}(L)$ be the set of all closure systems in L. We consider $\mathcal{C}(L)$ as a poset with respect to $\subseteq$.

Proposition O-3.13. *The function which assigns to a closure operator c on a poset L its image $c(L)$ is an order isomorphism from the set of closure operators (under the pointwise order) onto $\mathcal{C}(L)^{\mathrm{op}}$. Its inverse function $S \mapsto c_S$ associates with a closure system $S \in \mathcal{C}(L)$ the upper adjoint of the inclusion $S \to L$ followed by the inclusion $S \to L$ itself.*

Proof: We recall that the upper adjoint of the inclusion $S \to L$ is given by the formula $c_S^\circ(x) = \min(\uparrow x \cap S) = \inf(\uparrow x \cap S)$, and that $c_S^\circ(x) = c_S(x)$ for all elements $x \in L$.

The function $c \mapsto c(L)$ from the set of closure operators on L into $\mathcal{C}(L)$ is well defined. It is readily verified that $c_S(L) = S$; conversely, given a closure operator c, then by O-3.10(2_1) we know c° is the lower adjoint of the inclusion $c(L) \to L$, as is indeed the co-restriction of $c_{c(L)}$; by the uniqueness of adjoints we have $c = c_{c(L)}$. Thus, the maps $c \mapsto c(L)$ and $S \mapsto c_S$ are inverses of each other. From the formula $c_S(x) = \inf(\uparrow x \cap S)$ it is clear that the function $S \mapsto c_S$ reverses order. □

Remark. In a complete lattice L, closure systems S can be characterized very simply by the property that they are subsets closed for arbitrary infs by O-3.12(i). In a complete semilattice, closure systems S are characterized by the properties that, firstly, they are closed for nonempty infs and, secondly, every $x \in L$ has an upper bound in S.

Corollary O-3.14. *The correspondence $c \mapsto c(L)$ between closure operators and closure systems on L maps the set of closure operators preserving directed sups bijectively onto the set of those closure systems which are closed under directed sups.*

Proof: If c preserves directed sups, then $c(L)$ is closed under directed sups by O-2.5(iii). Conversely, suppose that $c(L)$ is closed under directed sups in L, and let D be a directed set in L. Then $\sup c(D) \le c(\sup D)$ because c is monotone. As c is a closure operator, $\sup D \le \sup c(D)$, whence $c(\sup D) \le c(\sup c(D))$. Finally $c(\sup c(D)) = \sup c(D)$, as c is a closure operator and as $c(D)$ is a directed set in $c(L)$ whose sup, by hypothesis, is in $c(L)$. $\square$

We conclude this section with some examples which will be of considerable importance in later chapters. In the first place, we return to Example O-2.8.

Proposition O-3.15. *Let L be a* **dcpo**. *Then*

(i) *the map $I \mapsto \sup I : \operatorname{Id} L \to L$ is lower adjoint of the principal ideal map $x \mapsto {\downarrow}x : L \to \operatorname{Id} L$; in particular, it preserves sups;*
(ii) *the map $I \mapsto {\downarrow}\sup I : \operatorname{Id} L \to \operatorname{Id} L$ is a closure operator whose image is isomorphic to L.*

Proof: If $x \in L$ and $I \in \operatorname{Id} L$, then $I \subseteq {\downarrow}x$ iff x is an upper bound of I iff $\sup I \le x$. This proves the adjointness; the rest follows from O-3.3 and O-3.10. $\square$

Secondly, Galois connections also provide an access to a class of lattices which plays an important role in logic and also in the later developments of our theory.

Lemma O-3.16. *In a semilattice S the following two conditions are equivalent:*

(1) *for all $x \in S$, the function $s \mapsto x \wedge s : S \to S$ has an upper adjoint;*
(2) $\max\{s \in S : x \wedge s \le t\}$ *exists for all $x, t \in S$.*

These conditions imply

(3) *for any family $\{x_j : j \in J\}$ with a sup and any $x \in S$ we have*

$$x \wedge \bigvee\{x_j : j \in J\} = \bigvee\{x \wedge x_j : j \in J\}.$$

If S is a lattice, then (3) *implies the distributive law* (D) *of* O-2.6. *If S is a complete lattice, then* (1)–(3) *are equivalent and equivalent to S being a frame.*

Proof: The equivalence of (1) and (2) follows from O-3.2. Condition (3) follows by O-3.3, and, trivially, (3) implies (D). If S is a complete lattice, then (3) implies (1) by O-3.5(ii) and of course is just (ID) of O-2.6. $\square$

The point of the next definition is that completeness is *not* required.

Definition O-3.17. A *Heyting algebra* is a lattice H satisfying the equivalent conditions (1), (2) of O-3.16.

The upper adjoint of the function $x \mapsto a \wedge x : H \to H$ is written

$$y \mapsto (a \Rightarrow y) : H \to H.$$

Thus, the conditions $x \geq a \wedge y$ and $(a \Rightarrow x) \geq y$ are equivalent in H. The binary operation $(a, b) \mapsto (a \Rightarrow b) : H \to H$ is called *implication*. Note that a Heyting algebra always has a unit, because $1 = a \Rightarrow a$.

If H has a zero, define $\neg a$ by $a \Rightarrow 0$ (that is, $\neg a = \max\{x \in H : a \wedge x = 0\}$). This unary operation is called *negation*. Notice that a Heyting algebra with zero satisfies $1 = \neg 0$ and $0 = \neg 1$. □

An example of a Heyting algebra without a zero is the half open interval $]0, 1]$, where $a \Rightarrow b = 1$ when $a \leq b$ but $= b$ otherwise.

We conclude this section by extending slightly the notion of lower adjoint to a kind of partial adjoint, a notion that can be useful when dealing with posets without a top element:

Remark O-3.18. *Let L^1 denote the poset obtained from an arbitrary poset L by adjoining a new "virtual" top element* 1 *and, for an order preserving map $g: L \to M$, let $g^1: L^1 \to M^1$ denote the "virtual" extension of g with* $g(1) = 1$. *We claim the following.*

For an order preserving map $g: L \to M$, the following conditions are equivalent:

(1) *the co-restriction $g^\circ: L \to \downarrow g(L)$ has a lower adjoint;*
(2) *the "virtual" extension $g^1: L^1 \to M^1$ has a lower adjoint.*

For complete semilattices L and M, these conditions are equivalent to the following one:

(3) *the map $g: L \to M$ preserves infs of nonempty sets.* □

The proof for these assertions follows from O-3.2 and O-3.5(i). An order preserving function will sometimes be called an *upper map* if it satisfies the equivalent conditions (1) and (2) above.

Exercises

We continue in the next few exercises with the discussion of Heyting algebras (see O-3.17) and their relationship to Boolean algebras.

Exercise O-3.19. Let H be a Heyting algebra with 0. Prove the following:

(i) $(\neg, \neg)$ is an adjunction between H^{op} and H; in other words, $\neg a \geq b$ iff $\neg b \geq a$ for all $a, b \in H$;

(ii) $\neg a \geq b$ iff $a \wedge b = 0$ for all $a, b \in H$;
(iii) $\neg\neg\colon H \to H$ is a closure operator, and $\neg\neg\neg = \neg$;
(iv) $\neg\neg$ preserves finite infs. □

For the following we recall from O-2.6 that a Boolean algebra is a distributive lattice with 0 and 1 in which every element has a complement.

Exercise O-3.20. Let L be a lattice with 0 and 1. Show that the following conditions are equivalent:

(1) L is a Boolean algebra;
(2) L is a Heyting algebra in which negation is an involution (i.e., L satisfies $\neg\neg x = x$ for all x).

Moreover, if these conditions are satisfied, show that $\neg x$ is the complement of x.

Hint. The implication (2) implies (1) follows from the distributivity of a Heyting algebra and the fact that $x \wedge \neg x = 0$ implies $x \vee \neg x = 1$ whenever $\neg$ is an order reversing involution: hence, $\neg x$ is a complement. For the remaining implication we first observe that (1) trivially implies

(3) for every element x there is an x^* such that for all y

$$(y \vee x^*) \wedge x \leq y \quad \text{and} \quad y \leq (y \wedge x) \vee x^*.$$

Next we observe that (3) implies (2). For, given x, y, z, if $x \leq y^* \vee z$, then $x \wedge y \leq (z \vee y^*) \wedge y \leq z$ by (3); conversely, if $x \wedge y \leq z$, then $x \leq (x \wedge y) \vee y^* \leq z \vee y^*$, again by (3). Thus, L is a Heyting algebra with $(y \Rightarrow z) = y^* \vee z$. Moreover, we find $\neg x = (x \Rightarrow 0) = x^* \vee 0 = x^*$.

Note that the proof in fact shows the equivalence of (1), (2) and (3). □

The reader should verify at this point that, in view of Lemma O-3.16, a Heyting algebra that is complete as a lattice is a cHa; also, by a related argument, that a cBa is a cHa. Besides these obvious connections, it is useful to note that with every cHa there is canonically attached a cBa; the formalism of closure operators which we discussed in this section comes in handily for this purpose.

Exercise O-3.21. Let H be a cHa (a frame) and $c\colon H \to H$ a closure operator which preserves finite infs. Prove the following: $c(H)$ is also a cHa (a frame); if $c(H) \subseteq \neg H$ and $c(0) = 0$, then $c(H)$ is a cBa; in particular, $\neg H$ is a cBa.

Hint. By O-3.12(ii), $c(H)$ is a complete lattice; and, since c° preserves finite infs by hypothesis and O-3.12(i), and arbitrary sups by O-3.12(iii), then the equation in O-3.16(3) holds in $c(H)$. Whence, $c(H)$ is a frame. Now suppose that $c(0) = 0$. If $a \in c(H)$ and $x \in H$, then $a \wedge x \leq 0$ implies

$$a \wedge c(x) = c(a) \wedge c(x) = c(a \wedge x) \leq c(0) = 0.$$

Thus, $\max\{x \in H : a \wedge x \leq 0\} \in c(H)$, and so $\neg_{c(H)} a = \neg a$. Hence, if we have $a = \neg b$ for some b, then

$$\neg_{c(H)} \neg_{c(H)} a = \neg\neg a = \neg\neg\neg b = \neg b = a,$$

by O-3.19(iii). Thus, if $c(0) = 0$ and $c(H) \subseteq \neg H$, then $c(H)$ is a Boolean algebra by O-3.20(2). By O-3.19(iii), we know that $c = \neg\neg$ is a closure operator with image $\neg H$, and $\neg\neg 0 = \neg 1 = 0$. Hence, the preceding applies to show that $\neg H$ is a cBa. □

This allows us to produce some interesting complete Boolean algebras (as we have already remarked):

Exercise O-3.22. Prove the following.

(i) If H is a frame and if L is a sublattice which is in fact closed under arbitrary sups, then L is itself a frame.

Hint. If H satisfies O-3.16(3), then so does L under the given hypotheses.

(ii) For any set X the lattice 2^X is a cBa; hence, any sublattice L of 2^X which is closed under arbitrary unions is a frame.

(iii) Let X be any topological space. Then $\mathcal{O}(X)$ is a frame (cf. O-2.7(3)). Moreover, $\neg\mathcal{O}(X) = \mathcal{O}_{\mathrm{reg}}(X)$ is a cBa. □

The following examples of frames will be of interest in our later discussions.

Exercise O-3.23. Let S be a semilattice equipped with a topology such that all translations $x \mapsto a \wedge x : S \to S$ are continuous (in this case we say that S is a *semitopological semilattice* (see also VI-1.11)). Let $L \subseteq \Gamma(S)$ be the lattice of all *closed* lower sets. Then L^{op} and L are frames. Dually, if $M \subseteq \mathcal{O}(S)$ is the lattice of all *open* upper sets, then both M and M^{op} are frames.

Hint. Since M is closed under arbitrary unions and finite intersections in 2^S, equation O-3.16(3) holds in M. The lattice M is complete, thus M and therefore $L^{\mathrm{op}} \cong M$ are frames. In order to show that L is a frame, let $\{A_j : j \in J\}$ be a family of closed lower sets; we have to show only that

$$A \cap \left(\bigcup\{A_j \colon j \in J\}\right)^- \subseteq \left(\bigcup\{(A \cap A_j) \colon j \in J\}\right)^-,$$

since the other containment is clear. Let $s \in A \cap (\bigcup\{A_j \colon j \in J\})^-$. Then

$$\begin{aligned} s \in s \wedge \left(\bigcup\{A_j \colon j \in J\}\right)^- &\subseteq \left(s \wedge \bigcup\{A_j \colon j \in J\}\right)^- \\ = \left(\bigcup\{s \wedge A_j \colon j \in J\}\right)^- &\subseteq \left(\bigcup\{(A \cap A_j) \colon j \in J\}\right)^- \end{aligned}$$

by the continuity of the translation by s and since the A_j are lower sets. □

Definition O-3.24. A map $f \colon L \to M$ between frames is called a *homomorphism of frames* iff it preserves arbitrary sups and finite infs. A subset L of a frame M is a *subalgebra* iff the inclusion $L \to M$ is a homomorphism (i.e., iff L is closed under arbitrary sups and finite infs). □

Exercise O-3.25. Show that the class of frames is closed under the formation of arbitrary direct products, subalgebras, and homomorphic images. □

We continue with some general Exercises on adjunctions.

Exercise O-3.26. Let S be a poset in which every *nonempty* subset has an inf and let T be a poset. Suppose further that $g \colon S \to T$ preserves all *existing* infs and also satisfies $T = \downarrow g(S)$. Show that g has a lower adjoint given by the formula $d(t) = \inf g^{-1}(\uparrow t)$.

Hint. Check the proof of O-3.4 in the present situation. □

Exercise O-3.27. Let L be a lattice and diag: $L \to L \times L$ the diagonal map. Show that diag is upper adjoint to the map $\vee \colon L \times L \to L$ and lower adjoint to the map $\wedge \colon L \times L \to L$. □

Exercise O-3.28. Let $\{L_i\}_{i \in I}$ be a family of complete lattices, and let $L = \prod_{i \in I} L_i$. Let $\pi_i \colon L \to L_i$ be the projection on the ith factor of L. Further define $\varepsilon_i \colon L_i \to L$ by

$$\pi_j \varepsilon_i(x) = \begin{cases} x & \text{if } i = j, \\ 1_j & \text{if } i \neq j, \end{cases}$$

and define $\delta_i \colon L_i \to L$ by

$$\pi_j \delta_i(x) = \begin{cases} x & \text{if } i = j, \\ 0_j & \text{if } i \neq j. \end{cases}$$

Show that (π_i, δ_i) is a Galois adjunction between L and L_i, while (ε_i, π_i) is a Galois adjunction between L_i and L. □

Exercise O-3.29. In the circumstances of O-3.2, show that conditions (1), (2), and (3) are also equivalent to each of the following:

(2′) g is monotone and $g^{-1}(\uparrow t) = \uparrow d(t)$ for all $t \in T$;
(3′) d is monotone and $d^{-1}(\downarrow s) = \downarrow g(s)$ for all $s \in S$.

Old notes

A part of the literature on Galois connections deals with pairs (g, d) of (antitone) maps $g\colon S \to T^{\mathrm{op}}$ and $d\colon T^{\mathrm{op}} \to S$ such that the relations $g(s) \le t$ and $d(t) \le s$ are equivalent. (This is the same as saying that $s \le dg(s)$ and $t \le gd(t)$ hold for all $s \in S$ and all $t \in T$). It is this setup which generalizes the formalism of classical Galois theory in which the order reversing correspondence is established between the lattice of fields F between two fields K and E, $K \subseteq F \subseteq E$, and the lattice of subgroups of the Galois group of $(E : K)$. In the antitone form, Galois connections were studied in [Ore, 1944]. Since that time, the general idea of Galois connections has become a pervasive theme in lattice theory literature, and it cannot be our objective to trace its precise history. One reference on such matters is the book of T.S. Blyth and M.F. Janowitz [Blyth and Janowitz, B1972]. Frequently cited contributions are by C.J. Everett [Everett, 1944], G. Pickert [Pickert, 1952], G. Aumann [Aumann, 1955], G.N. Raney [Raney, 1960], J.C. Derderian [Derderian, 1967], Shmuely [Shmuely, 1974] and H.-J. Bandelt [Bandelt, 1981].

F.W. Lawvere noticed early on that Galois connections are quite special cases of the omnipresent situation of a pair of adjoint functors. This is pointed out in [Mac Lane, B1971], pp. 93 ff. The consideration of Heyting algebras in this context is outlined by S. Eilenberg and G.M. Kelly [Eilenberg and Kelly, 1966]; see in particular pp. 555 ff. The authors credit F.W. Lawvere with this approach. There is a great variety of names under which Heyting algebras appear in the literature: Brouwerian logic, Brouwerian lattice, pseudo-Boolean lattice and relatively pseudocomplemented distributive lattice. For a complete Heyting algebra (cHa) the following names are also used: frame, local lattice, locale. We will have quite a bit to say on certain classes of frames in Chapter V; in fact, our discussion will be illustrative of the connections between frames and topological spaces.

The topic of closure and kernel operators is a lattice theoretical classic. The systematic consideration of projections as the common generalization of both of these in O-3.8 through O-3.14 is due to Scott, and we will pursue this discussion for continuous lattices in Chapter I (see I-2.2 through I-2.5, I-4.16 through I-4.18). The systematic use of Galois connections in the study of continuous lattices advocated in [Hofmann and Stralka, 1976] particularly emphasized the importance of the sup map on the ideal lattice (O-3.15), which will bear fruit in Section I-2.

O-4 Meet Continuous Lattices and Semilattices

The inf operation $(x, y) \mapsto x \wedge y : L \times L \to L$ in a lattice preserves infs (to the extent they exist); in particular, all translations $s \mapsto x \wedge s : L \times L$ preserve infs. Here (and often in what follows) we write xy in place of $x \wedge y$. Both notations abound in the literature since semilattices have been studied both as ordered sets (in which every pair of elements has an inf) and as semigroups (which are both commutative and idempotent). If L is complete, we have seen that it is precisely the frames in which this translation preserves *all* sups. Frequently this is too much to ask, since many of the examples we have listed are not even distributive. However, it occurs rather often that $s \mapsto xs$ preserves *directed* sups. The class of posets and lattices in which this is the case deserves a special designation:

Definition O-4.1. A semilattice L is called *meet continuous* if it is directed complete, i.e., a **dcpo**, and satisfies

$$x \sup D = \sup xD, \tag{MC}$$

for all $x \in L$ and all *directed* sets $D \subseteq L$. We will say that a sup semilattice L is *join continuous* iff L^{op} is meet continuous. A lattice L is *meet continuous* if it is a complete lattice satisfying (MC). □

Note that in (MC) the relation $\leq$ could replace $=$. (The same will be the case in O-4.2(7) and (8) below.) In the literature, meet continuous lattices are occasionally called “continuous lattices”; but we reserve this designation for those more special lattices which will be our principal topic. There are various equivalent ways of looking at meet continuous semilattices:

Theorem O-4.2. *In a directed complete semilattice L the following conditions are equivalent:*

(1) *the sup map for ideals $I \mapsto \sup I : \mathrm{Id}\, L \to L$ is a homomorphism of meet semilattices (preserving all sups: cf. O-3.15(i));*
(2) *for two ideals I_1, I_2 we have* $(\sup I_1)(\sup I_2) = \sup I_1 I_2$*;*
(3) *for two directed sets D_1, D_2 we have* $(\sup D_1)(\sup D_2) = \sup D_1 D_2$*;*
(4) *L is meet continuous;*
(5) *for each directed set D and each $x \leq \sup D$ we have $x \leq \sup xD$ (hence, $x = \sup xD$);*

(6) *the* inf *operation* $(x, y) \mapsto xy : L \times L \to L$ *preserves directed sups;*
(7) *for each* $x \in L$ *and each directed net* $(x_j)_{j \in J}$ *we have*

$$x \wedge \bigvee_{j \in J} x_j = \bigvee_{j \in J} (x \wedge x_j).$$

If L is in fact a lattice, then these conditions are also equivalent to the following:

(8) *for each* $x \in L$ *and any family* $(x_j)_{j \in J}$ *we have*

$$x \wedge \bigvee_{j \in J} x_j = \bigvee_{A \in \operatorname{fin} J} \left(x \wedge \bigvee_{j \in A} x_j \right),$$

where fin J *is the set of all finite subsets of* J.

Proof: (1) iff (2): Use the definition of the sup map for ideals and the fact that $I_1 I_2 = I_1 \cap I_2$ for two lower sets in a semilattice.

(2) iff (3): Notice $\downarrow(\downarrow D_1)(\downarrow D_2) = \downarrow(D_1 D_2)$ and use O-1.5 to calculate

$$\sup D_1 D_2 = \sup \downarrow(D_1 D_2) = \sup(\downarrow D_1)(\downarrow D_2) = \sup(\downarrow D_1)(\downarrow D_2).$$

Remark O-1.4 then establishes the desired equivalence.

Thus (1), (2), (3) are equivalent. Clearly (4) and (7) are equivalent and, for lattices, the equivalence of (7) and (8) is easy by Remark O-1.5.

The implications (6) implies (3) implies (4) implies (MC) implies (5) are trivial. The whole proof will be complete if we show the following.

(5) implies (6): Let $D \subseteq L \times L$ be directed and set $D_n = \pi_n D$, for $n = 1, 2$. Then $D \subseteq D_1 \times D_2$. If, on the other hand, $(d, e) \in D_1 \times D_2$, then there are elements $x, y \in L$ with $(d, y), (x, e) \in D$. Since D is directed, we find some (d^*, e^*) majorizing (d, y) and (x, e); thus, $(d, e) \leq (d^*, e^*)$. We have thus proved that $D_1 \times D_2 \subseteq \downarrow D$.

If $m \colon L \times L \to L$ is the inf map $m(x, y) = xy$, then

$$m(D) \subseteq m(D_1 \times D_2) = D_1 D_2 \subseteq m(\downarrow D) \subseteq \downarrow m(D).$$

Thus $\sup m(D) \leq \sup D_1 D_2 \leq \sup \downarrow m(D) = \sup m(D)$, by O-1.5. If $d_n = \sup D_n$, $r = 1, 2$, then $(d_1, d_2) = \sup D$. It suffices therefore to prove $d_1 d_2 = \sup D_1 D_2$.

For every $x \in D_1$ we have $x d_2 \leq \sup D_2$, hence from (5) we know

$$x d_2 = \sup x d_2 D_2 = \sup x D_2.$$

Since $d_1 d_2 \leq \sup D_1$, once more by (5) we obtain

$$d_1 d_2 = \sup d_1 d_2 D_1 = \sup D_1 d_2.$$

But then

$$\begin{aligned}\sup D_1 d_2 &= \sup\{xd_2 : x \in D_1\} = \sup_{x\in D_1} \sup x D_2 \\ &= \sup\left(\bigcup_{x\in D_1} x D_2\right) = \sup D_1 D_2.\end{aligned}$$

This proves the claim. □

Theorem O-4.2 applies, in particular, to complete lattices. We point out that condition (8) looks equational. If one imagines that directed sups are "limits" of sorts (a contention we will amply justify in Chapter II), then condition (6) is indeed a continuity assumption. This justifies the name "meet continuity". Condition (7), however, is a distributivity relation which readily compares with the distributivity relation O-3.16(3) in Heyting algebras. In fact we have

Remark O-4.3. *Let L be a lattice; then the following conditions are equivalent:*

(1) *L is a frame;*
(2) *L is meet continuous and distributive.*

Proof: That (1) implies (2) is clear from O-3.16.

(2) implies (1): By (MC), the function $s \mapsto xs : L \to L$ preserves directed sups; by O-2.6(D) it preserves finite sups. Hence, it preserves arbitrary sups (see O-1.10). Thus, O-3.16(3) holds and (1) follows. □

While frames are one source of meet continuous lattices, compact topological semilattices are another. We will develop this subject at considerably greater length in Chapter VI. But it helps now to take note at least of the examples implied by the following.

Proposition O-4.4. *Let S be a semilattice with a Hausdorff topology such that*

(i) *every directed net has a sup to which it converges,*
(ii) *the translations $s \mapsto xs : S \to S$ are continuous for all $x \in S$.*

Then S is meet continuous. If, moreover, S is compact, then condition (ii) *implies* (i).

Proof: Let $x \in S$ and suppose that $(x_j)_{j\in J}$ is directed. Then $(xx_j)_{j\in J}$ is directed, and so $\sup_J x_j = \lim_J x_j$ and $\sup_J xx_j = \lim_J xx_j$. From (ii) we know that $\lim xx_j = x \lim x_j$, and, since limits are unique for a Hausdorff topology, we deduce $\sup_J xx_j = x \sup_J x_j$. Every directed subset has a sup by (i); hence, S is a complete lattice by the dual of O-2.2(ii). Thus S is a meet continuous lattice.

Assume now that S is compact Hausdorff and satisfies (ii); we have to verify (i). Let $(x_j)_{j\in J}$ be a directed net. Since the topology is compact, this net has at least one cluster point c. Let $i \in J$. Then eventually $x_i \le x_j$, that is, $x_i = x_i x_j$. But, $\uparrow x_i = \{x \in S\colon x_i x = x_i\}$ is closed as translation by x_i is continuous, and since the net is eventually in $\uparrow x_i$, it follows that c is also. Hence $x_i c = x_i$ for each i, so c is an upper bound for the net. Moreover, if $b \in S$ is any upper bound of the net, then $bx_i = x_i$ for each i, so that $x_i \in Sb$ for each i. Again, since translation by b is continuous, Sb is closed in S, and so $c \in Sb$ also holds. Thus $c \leq b$ for each upper bound b of the net, whence $c = \sup x_i$. This also shows that c is the unique cluster point of the net $(x_j)_{j \in J}$ which implies that the net converges to c. Dually, each filtered net converges to its inf. In particular, $0 = \inf S$ exists. □

Let us now, by contrast, look at a few simple complete lattices which *fail* to be meet continuous.

Counterexamples O-4.5. (1) Let L be the subset of the square $[0, 1]^2$ consisting of its interior $]0, 1[^2$ and the points $(0, 0) = \bot$ and $(1, 1) = \top$. Then L is a complete, distributive lattice which is isomorphic to its opposite. But L is *not* meet continuous.

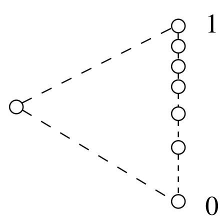
Hint. Consider $D = \{1/3\} \times]0, 1[$ with $\sup D = \top$; if $x = (2/3, 1/2)$, then we have $x \sup D = x$, but $\sup xD = (1/3, 1/2) \neq x$.

(2) Let L be the following subset of the square $[0, 1]^2$:

$$L = \left(\left\{1 - \frac{1}{n}\colon n = 1, 2, 3, \ldots\right\} \times \{0\}\right) \cup \{(0, 1), (1, 1)\}.$$

This lattice is complete but *not* meet continuous. Because of its obvious compact topology, this lattice is a very useful counterexample.

Hint. Consider $D = \{1 - \frac{1}{n}\colon n = 1, 2, \ldots\} \times \{0\}$ and $x = (0, 1)$.



□

Exercises

Exercise O-4.6. Prove the following.

(i) If S is any semilattice, then $\mathrm{Id}\, S$ is a meet continuous semilattice. (**Hint:** Utilize Exercise O-2.16.)

(ii) If L is any lattice, then $\mathrm{Id}\, L$ is a meet continuous lattice. □

Exercise O-4.7. Let S be a directed complete semilattice. Show that the following statements are equivalent (cf. G. Bruns [Bruns, 1967]).

(1) S is meet continuous;
(2) $x \leq \sup C$ always implies $x \leq \sup xC$ for every *chain* $C \subseteq S$. □

Exercise O-4.8. Show that the class of all meet continuous lattices is closed under the formation of the following operations:

(a) arbitrary products,
(b) subsets closed under arbitrary infs and directed sups,
(c) sublattices which are complete with respect to the induced order and which are closed under directed sups,
(d) surjective images by functions preserving arbitrary infs and directed sups,
(e) images of projections (O-3.8(i)) preserving directed sups.

Hint. Use O-4.2(7) and model the proof after the one that is given for I-2.5 below and in the case of O-3.11(iii). □

Exercise O-4.9. Prove the following.

(i) Let L be a lattice. Then $\mathrm{Id}_0\, L$ is a meet continuous lattice. If $f\colon L \to S$ is a function into a meet continuous lattice preserving finite infs, then the function $F\colon \mathrm{Id}\, L \to S$ given by $F(I) = \sup f(I)$ preserves finite infs and directed sups. If f is a lattice morphism, then F preserves arbitrary sups.
(ii) Let L be the opposite of the lattice of finite subsets of a set X. (This is the free semilattice generated by X). Then $\mathrm{Id}_0\, L$ contains a copy of X and is in fact the free meet continuous lattice over X in the category of meet continuous lattices and maps preserving finite infs and directed sups.
(iii) Let L be the free lattice generated by a set X. Then $\mathrm{Id}_0\, L$ contains a copy of X and is the free meet continuous lattice over X in the category of meet continuous lattices and maps preserving finite infs and arbitrary sups. (See [Isbell, 1975b].) □

Exercise O-4.10. Let L be a meet continuous lattice and H the lattice of all equivalence relations on L such that the graph $R \subseteq L \times L$ is closed under finite infs and arbitrary (!) sups. Then H is a frame. (See Isbell, *op. cit.*, p. 44.) □

Exercise O-4.11. Let L be an directed complete semilattice, and let

$$L' = \{I \subseteq L : \emptyset \neq I = I^{+} = \downarrow I\},$$

where $I^+ = \{\sup D \colon D \subseteq I \text{ is directed}\}$. Show that we have the following:

(i) L' is a lattice;
(ii) the following conditions are equivalent:
  (1) L is meet continuous;
  (2) L' is meet continuous.

Hint. (i) Straightforward. (ii) For (2) implies (1) see O-4.8(b). (1) implies (2): See O-3.22. (Cf. also II-2.1 and II-4.24 below.)

Old notes

In the literature meet continuous lattices (see [Birkhoff, B1967]) are sometimes called upper continuous lattices [Grätzer, B1978] or "nach oben stetige Verbände" ([Hermes, B1967]); lattices which are meet- and join continuous have been called continuous (see, e.g., Hermes, *op. cit.*); this notation is in conflict with what we will call continuous lattices in this book; our nomenclature seems now widely accepted.

The role played by meet continuous lattices in the literature seems to be somewhat implicit: they are rarely considered as a class by themselves. Usually it is observed that trivially all frames are meet continuous and that all algebraic lattices (which we will consider in I-4 at some length) are meet continuous. A coherent body of deep information does not appear to exist on the class of meet continuous lattices per se. Some information is provided by [Isbell, 1975b] (see Exercises O-4.9 and O-4.10), but much of Isbell's paper is concerned with continuous lattices in our sense. The choice of morphisms for a category of meet continuous lattices is not entirely clear. The definition would suggest that morphisms preserve finite infs and directed sups. In the case of the category of frames one chooses morphisms the way we did in O-3.24; Isbell considers this type of map for meet continuous lattices, and this makes his category of meet continuous lattices contain the category of frames as the full subcategory determined by the distributive objects (see O-4.3). The characterization of meet continuous lattices through the fact that the sup map $\operatorname{Id} L \to L$ is a lattice morphism is from [Hofmannand Stralka, 1976]. That meet continuity emerges in the context of compact topological semilattices (O-4.4) is well known in topological algebra.

O-5 T_0 Spaces and Order

For convenient reference we gather in one place the principal topological definitions and notions that arise in our considerations, particularly as they pertain to T_0 spaces. Indeed, in connection with order, non-Hausdorff spaces play a more significant role than Hausdorff spaces, while in classical mathematics one mainly meets Hausdorff spaces. For this reason we collect some information pertaining to non-Hausdorff spaces in this preliminary section for easy reference. In a topological space X we use the following notation:

$\mathcal{O}(X)$	for the collection of all open sets in X,
$\Gamma(X)$	for the collection of all closed subsets of X,
A^-	for the closure of a subset A,
$\operatorname{int} A$	for the interior of a subset A.

Definition O-5.1. A topological space is T_0 if given $x \neq y$, there exists an open set that contains exactly one of them. Thus a space is T_0 if no two distinct points have exactly the same family of open neighborhoods. We denote by *TOP* the category whose objects are the T_0 spaces and whose morphisms are the continuous maps between them. □

If X is a topological space, then for two elements x and y in X the following relations are easily seen to be equivalent:

(1) $\{x\}^- \subseteq \{y\}^-$;
(2) $x \in \{y\}^-$;
(3) $x \in U$ implies $y \in U$, for all open sets U.

This relation is clearly reflexive and transitive, and, if X is a T_0 space, it is antisymmetric. Hence, we have a partial order.

Definition O-5.2. The partial order $\leq$ defined on a T_0 space X by

$$x \leq y \text{ iff } x \in \{y\}^-$$

is called the *specialization order*. We denote by $\Omega X = (X, \leq)$ the poset obtained from a T_0 space X and its specialization order. □

Alternatively $x \leq y$ iff every open set that contains x must also contain y. Thus open sets are always upper sets and closed sets are always lower sets. If a space has separation properties of T_1 or higher, then the order of specialization reduces to the trivial partial order; thus it is for T_0 spaces that this order is of special interest. Several useful properties of T_0 spaces have both topological and order theoretic formulations. For example, one sees immediately that $\downarrow x = \{x\}^-$.

If $f\colon X \to Y$ is continuous, then the property $f(A^-) \subseteq f(A)^-$ when applied to singleton sets yields $f(\{a\}^-) \subseteq \{f(a)\}^-$. Hence $b \leq a$ implies $f(b) \leq f(a)$, that is, the function f is order preserving (with respect to the orders of specialization on X and Y). For the special case that f is the inclusion of a subspace X into Y, one sees that the order of specialization on X is the restriction of the specialization order of Y to X.

Definition O-5.3. A set is *saturated* if it is an intersection of open sets, or equivalently if it is an upper set in the order of specialization. (Upper sets are always intersections of open sets since their complements are unions of the closed sets $\{x\}^- = \downarrow x$, for all x in the complement.) The *saturation* sat A of a set A is the smallest saturated set containing A and consists of the intersection smallest upper set containing A. □

Definition O-5.4. Sometimes given a poset L one wishes to consider a topology (or topologies) for which the order of specialization agrees with the given order. The weakest one for which this is true arises by taking all principal ideals $\downarrow x$ as a subbasis for the closed sets (thus all closed sets arise by first taking all finite unions and then all arbitrary intersections, and adding the empty set if necessary). We call this topology the *upper topology* and we denote it by $\nu(L)$. Dually, the *lower topology* is the one that one obtains by choosing all principal filters $\uparrow x$ as a subbasis for the closed sets. We denote the lower topology by $\omega(L)$. The *interval topology* is the coarsest common refinement of the upper and the lower topology. The principal filters and principal ideals form a basis for its closed sets. □

Definition O-5.5. An arbitrary nonempty subset A of a topological space X is *irreducible* if $A \subseteq B \cup C$ for closed subsets B and C implies $A \subseteq B$ or $A \subseteq C$. □

A point closure $\{p\}^-$ is always an irreducible closed set. If an infinite set X is endowed with the cofinite topology in which a nonempty set is open iff it is the complement of finite set, then X itself is irreducible.

Definition O-5.6. A topological space X is *sober* if for every irreducible closed set C, there exists a unique $x \in X$ such that $\{x\}^- = C$. We denote by *SOB* the category of sober spaces with all continuous maps as morphisms. □

Notice that a sober space is automatically T_0 since $\{x\}^- = \{y\}^-$ always implies $x = y$. Hausdorff spaces are always sober, and sober spaces are always T_0. An infinite set with the cofinite topology is T_1 but not sober.

Definition O-5.7. A subspace K of a topological space X is *compact* if every open cover of K admits a finite subcover. □

A set K is compact if and only if its saturation sat K is compact, since a family of open sets covers K if and only if it covers its saturation. In fact, we will see that in T_0 spaces compact saturated sets play a more important role than compact sets in general. Compactness can be a rather weak property in T_0 spaces. For example a smallest element in the order of specialization is enough to ensure that a space is compact.

Definition O-5.8. A set N is a *neighborhood* of a point x in a space X if there exists an open set U such that $x \in U \subseteq N$. A collection $\mathcal{B}$ of subsets of a topological space X is a *basis* for the topology if for every $x \in U$ open, there exists a neighborhood $B \in \mathcal{B}$ of x such that $B \subseteq U$. A space is *second countable* if there is a countable basis $\mathcal{B}$ for the topology and *separable* if there exists a countable dense subset. □

Second countable spaces are always separable, and the two notions are equivalent for metric spaces.

Definition O-5.9. A space is *locally compact* if there exists a basis of compact sets. Alternatively a space is locally compact if for every x in the space and every open U containing x, there exist an open set V and a compact set K such that $x \in V \subseteq K \subseteq U$. □

In contrast to Hausdorff spaces, for local compactness in our sense of a T_0 space it is not sufficient to require that every point has at least one compact neighborhood. Indeed there are compact spaces that are not locally compact.

It should be stressed that the concept of compactness does not include the Hausdorff separation axiom as in [Bourbaki, B1966]. In this respect we also differ from the terminology adopted in *A Compendium of Continuous Lattices*, the predecessor of this work, where the term *quasicompact* was used where we now simply say *compact*. Whenever we mean *compact and Hausdorff*, this is stated explicitly.

Definition O-5.10. For a T_0 space X, the *co-compact topology* has as a basis for the closed sets all compact saturated sets. The *patch topology* is the join of the original topology and the co-compact topology, the smallest topology containing both the original topology and the co-compact topology. □

Definition O-5.11. As in O-1.2, a *net* $\{x_\alpha\colon \alpha \in D\}$ in a space X is a function $\alpha \mapsto x_\alpha$ from a directed set D into X. (We assume that the order on D is reflexive, transitive, directed, and typically, but not necessarily, antisymmetric.) A subset E of D is *residual* if it contains some $\uparrow\alpha$, $\alpha \in D$, and *cofinal* if for any $\alpha \in D$, there exists $\beta \in E$ such that $\alpha \leq \beta$. A net $\{x_\alpha\}$ *converges*, resp. *clusters*, to $x \in X$ if for any open set U containing x, the set $\{\alpha\colon x_\alpha \in U\}$ is residual, resp. cofinal, in D. Convergence points are also called *limit points*. □

A set A is closed iff it contains all convergence (or cluster) points of nets contained in the set. The set A is compact iff every net in A has a cluster point. A function between spaces is continuous at x iff it carries nets converging to x to nets converging to $f(x)$. A space is Hausdorff iff every net has a most one convergence point.

Filters are also useful generalizations of sequences for considering questions of convergence in general topological spaces.

Definition O-5.12. A *filter* $\mathcal{F}$ of subsets on a set X is a nonempty collection of nonempty subsets that is closed under finite intersections and the taking of supersets. A *filter of open sets* means a nonempty collection of nonempty open sets closed under finite intersections and supersets. A *filter base* is a collection $\mathcal{F}$ of nonempty sets satisfying that $F_1, F_2 \in \mathcal{F}$ implies that there exists $F \in \mathcal{F}$ such that $F \subseteq F_1 \cap F_2$. The set of supersets of a filter base is a filter, called the filter generated by the filter base. A filter $\mathcal{F}$ *converges* to $x \in X$ if every neighborhood of x belongs to $\mathcal{F}$ and *clusters* to x if every neighborhood of x meets every member of the filter. An *ultrafilter* is a maximal filter. $\square$

By the Hausdorff Maximality Principle every filter is contained in an ultrafilter. A filter is an ultrafilter if and only if given any subset, either the subset or its complement belongs to the filter. An ultrafilter clusters to a point iff it converges to the point.

Definition O-5.13. A metric space is *complete* if every Cauchy sequence converges. A *Polish* space is a separable topological space which is metrizable by a complete metric. A *Baire space* is a space for which any countable intersection of dense open subsets is again dense. $\square$

Polish spaces are Baire spaces (see [Bourbaki, B1966], Chapter IX, §5.3, Theorem 1). Every open subset of a Polish space is Polish, and a subset Y of a Polish space X is Polish iff Y is a G_δ-set in X, that is, iff Y is the intersection of a countable family of open subsets in X (see [Bourbaki, B1966], Chapter IX, §6.1, Theorem 1).

Exercises

Exercise O-5.14. Let X be a topological space. For any subset $Y \subseteq X$ let sat Y be its saturation, that is, the intersection of all open sets containing Y as in O-5.3. Show that

(i) every open set is saturated,
(ii) sat $Y = \{x \in X : \{x\}^- \cap Y \neq \emptyset\}$,

(iii) sat $Y = \uparrow Y$, where the latter is taken in the order of specialization,
(iv) Y is saturated iff $Y = \uparrow Y$, that is, iff Y is an upper set in the specialization order,
(v) a subset $Q \subseteq X$ is compact iff its saturation sat Q is compact. □

In the next exercises we collect some useful information about irreducible sets and sober spaces (see O-5.5 and O-5.6).

Exercise O-5.15. Let A be a subset of a T_0 space X. Show the following properties.

(i) A is irreducible in X iff it is irreducible in itself in the relative topology.
(ii) Continuous images of irreducible sets are irreducible.
(iii) A is irreducible iff its closure A^- is irreducible.
(iv) For $x \in X$, $\{x\}^- = \downarrow x$ is irreducible.
(v) X is sober iff for every irreducible set A, there exists exactly one point x such that $A^- = \{x\}^-$.
(vi) A subset directed with respect to the specialization order of X is irreducible.
(vii) If X is sober, then every subset D which is directed in the specialization order has a supremum $x = \sup D$, and D considered as a monotone net converges to x. In particular, every sober space is a **dcpo** when endowed with its specialization order.
(viii) A continuous function between sober spaces preserves the orders of specialization and directed suprema. □

Exercise O-5.16. Some topological constructions preserve sobriety: prove the following

(i) A closed subspace of a sober space is sober.
(ii) A saturated subspace of a sober space is sober.
(iii) Products of sober spaces are sober.
(iv) If Y is sober, then the set $TOP(X, Y)$ of all continuous functions $f: X \to Y$ equipped with the topology of pointwise convergence (i.e., the relative product topology) is sober.
(v) If $f, g: X \to Y$ are continuous, X is sober, and Y is T_0, then the equalizer $\{x \in X : f(x) = g(x)\}$ is sober.
(vi) A retract of a sober space is sober.

Hint. For (ii), let A be an irreducible closed subset of a saturated subspace Z of a sober space X. The closure A^- of A in X is an irreducible closed subset of X. Hence A^- is the closure in X of a point x. As Z is saturated and as $a \leq x$ for every $a \in A$ with respect to the specialization order, we conclude that $x \in Z$

and A is the relative closure of $\{x\}$ in Z. For (iii), let A be an irreducible subset of $\prod_i X_i$, where each X_i is sober. Then $A_i := \pi_i(A)$ is irreducible and hence $A_i{}^- = \{x_i\}^- = {\downarrow}x_i$ for some unique $x_i \in X_i$. If follows that $A \subseteq \prod_i A_i \subseteq {\downarrow}x$, where x is defined by $\pi_i(x) = x_i$. Now any subbasic open set $U_i \times \prod_{j \neq i} X_j$ around x must meet A (since $x_i \in A_i{}^-$), and thus all basic open sets around x must meet A since A is irreducible. It follows that $A^- = \{x\}^-$. The proof of (iv) is similar: Let A be an irreducible subset of $\mathit{TOP}(X, Y)$ with the topology induced by the product topology on Y^X. Then $A_x := \pi_x(A) = \{a(x)\colon a \in A\}$ is irreducible in Y for every $x \in X$. As Y is supposed to be sober, there is a unique element $a_x \in Y$ such that $A_x{}^- = \{a_x\}^-$. We now show that the function $f\colon X \to Y$ defined by $f(x) = a_x$ is continuous. Indeed let $x \in X$ and let V be an open neighborhood of $f(x) = a_x$ in Y. Then V intersects A_x, that is, there is an element $a \in A$ such that $a(x) \in V$. As a is continuous, there is a neighborhood U of x such that $a(z) \in V$ for every $z \in U$. As $a(z) \leq a_z = f(z)$ for the specialization order, we conclude that $f(z) \in V$ for all $z \in U$. Thus f is continuous. As in (iii) one now shows that $\{f\}^- = A^-$. □

New notes

Most of the topological concepts that we use are quite standard and may be found in textbooks like [Kelley, B1955], [Bourbaki, B1966]. The more specific features connecting order and topology related to the topics treated in this volume are treated in the texts [Johnstone, B1982], [Vickers, B1989], [Smyth, 1992c]. In the literature, no systematic account on sober spaces seems to be available. The properties collected in Exercise O-5.16 have been communicated to us by R. Heckmann (see [Heckmann, 1996]).

I

Order Theory of Domains

Here we enter into the discussion of our principal topics. Continuous lattices and domains exhibit a variety of different aspects, some are order theoretical, some are topological, some belong to topological algebra and some to category theory – and indeed there are others. We shall contemplate these aspects one at a time, and this chapter is devoted entirely to the order theory surrounding our topic.

Evidently we have first to define continuous lattices and domains. As we shall see from hindsight, there are numerous equivalent conditions characterizing them. We choose the one which is probably the simplest, but it does involve the consideration of an auxiliary transitive relation, definable in every poset, by which one can say that an element x is "way below" an element y. We will write this as $x \ll y$. We devote Section I-1 to the introduction of the way-below relation and of continuous lattices and domains. We demonstrate that the occurrence of this particular additional ordering is not accidental and explain its predominant role in the theory. We exhibit the paradigmatic examples of continuous lattices and domains; in due course we shall see many more.

In Section I-2 we show that continuous lattices have a characterization in terms of (infinitary) equations. This gives us the important information that the class of continuous lattices, as an equational class, is closed under the formation of products, subalgebras, and homomorphic images – provided we recognize from the equations *which* maps ought to be considered as homomorphisms. The essential results remain intact for bounded complete domains in place of continuous lattices. It is in the very nature of the theme of this section that very few references are made to any domains which are more general than bounded complete domains or continuous lattices.

In Section I-3 we explain why in a continuous semilattice there are always sufficiently many meet irreducible elements in the sense that every element is the infimum of the irreducibles dominating it. In Chapter V we will bring this result to fruition when we discuss the spectral theory of distributive continuous lattices.

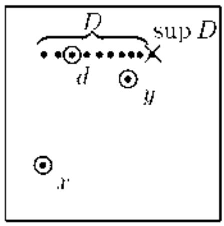
In Section I-4 we show that the familiar concept of an algebraic lattice is subsumed under the more general one of a continuous lattice, and we review some of the known aspects of algebraic lattices in this light. We also develop the parallel theory on the level of algebraic domains and their relation to continuous ones.

I-1 The "Way-below" Relation

It often happens that we encounter relations between elements of a given poset which are stronger than the simple less-than-or-equal-to relation of the partial ordering. In a linearly ordered chain, for example, we usually have need to single out the strict less-than relation. In the nonlinear case, however, this seldom proves to be a very interesting relation. Consider in this regard the lattice $\mathcal{O}(X)$ of open sets of a topological space X. To say $U \subseteq V$ but $U \neq V$ does not say very much, since the sets could differ at only one point. To say that U really is *inside* V we could say that the closure $U^- \subseteq V$. This means that U avoids the boundary of V even by limits, and in the case of compact Hausdorff spaces this is a well-known and useful relation. If, on the other hand, the space is only locally compact, the relation is not as strong as it looks. In order to say that U is *way inside* V we could require that $U^- \subseteq V$ and U^- is *compact*. This means that U avoids the boundary of V even in a compactification of the space. This relation, moreover, has a purely lattice theoretical definition, since we can define it in $\mathcal{O}(X)$ as meaning that every open covering of V has a finite subcollection that is a covering of U. (At least this works in the locally compact Hausdorff case.) What we are now going to study is the abstract generalization of this relation on **dcpo**s and on complete lattices, where the notion is nontrivial in an interesting way.

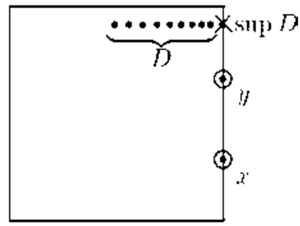
The way-below relation and continuous posets

Definition I-1.1. Let L be a poset. We say that x *is way below* y, in symbols $x \ll y$, iff for all directed subsets $D \subseteq L$ for which $\sup D$ exists, the relation $y \leq \sup D$ always implies the existence of a $d \in D$ with $x \leq d$. An element satisfying $x \ll x$ is said to be *compact* or *isolated from below*.



$x \ll y$

$L = [0,1]^2$



$x \leq y$ but not $x \ll y$ □

Remark. In a complete semilattice L, in particular in a complete lattice, the way-below relation can be defined equivalently by the following property which is modeled directly on the finite open covering property mentioned above: *$x \ll y$ iff for every subset $X \subseteq L$ the relation $y \leq \sup X$ always implies the existence of a finite subset $A \subseteq X$ such that $x \leq \sup A$.*

Indeed, if $y \leq \sup X$, consider the directed set $X^+ = \{\sup A\colon A$ is a finite subset $X\}$ for which $\sup X^+ = \sup X$. Thus, if $x \ll y$ in the sense of Definition I-1.1, then there is a finite subset $A \subseteq X$ such that $x \leq \sup A$. The converse is immediate.

Some authors prefer the term "relatively compact" to "way below", since in $\mathcal{O}(X)$ it is natural to read $U \ll V$ as "*U is relatively compact in V*". Indeed, one of the definitions of relative compactness that one finds in the literature is: U is relatively compact in V iff every open cover of V contains a finite cover of U. And this coincides exactly with $U \ll V$ according to the above alternative definition. However, important as the topological example is, it is only one out of many examples of interesting lattices; we therefore choose to emphasize the order theoretical view. The way-below relation is meaningful primarily when all directed set have sups, that is in **dcpo**s. However, its definition does not depend on that assumption.

The following properties of the relation $\ll$ follow quickly.

Proposition I-1.2. *In a poset L the following statements hold for all $u, x, y, z \in L$:*

(i) *$x \ll y$ implies $x \leq y$;*
(ii) *$u \leq x \ll y \leq z$ implies $u \ll z$;*
(iii) *$x \ll z$ and $y \ll z$ imply $x \vee y \ll z$ whenever the least upper bound $x \vee y$ exists in L;*
(iv) *$0 \ll x$ whenever L has a smallest element* 0.

Proof: Assertions (i), (ii), and (iv) are immediate. For (iii), let $z \leq \sup D$ for a directed set D. Then $x \leq d_x$ and $y \leq d_y$ for some d_x, $d_y \in D$, and then $x \vee y \leq d$ for some $d \in D$ larger than d_x and d_y. □

Clearly, $\ll$ is transitive and antisymmetric from (i) and (ii). In analogy with O-1.3 we write

$$\mathord{\Downarrow} x = \{u \in L : u \ll x\}, \qquad \mathord{\Uparrow} x = \{v \in L : x \ll v\},$$

and so on. We can then conclude from the four statements of I-1.2 and the definition of an ideal (O-1.3) that the following one holds.

For all x in a complete semilattice, the set $\mathord{\Downarrow} x$ is an ideal contained in $\downarrow x$. If $x \le y$ then $\mathord{\Downarrow} x \subseteq \mathord{\Downarrow} y$.

Examples I-1.3.

(1) As a first example let L be a *complete chain.* Then $x < y$ obviously implies $x \ll y$. Conversely, if $x \ll y$, then either $x < y$ or $x = 0$ or else $x = y$ is isolated from below – which in this case means simply that we have $\sup(\downarrow x \backslash \{x\}) < x$, so that x is the upper endpoint of a jump in the ordering. Thus, if L is the ordinary unit interval $\mathbb{I} = [0,1]$, we have $x \ll y$ iff either $x < y$ or $x = y = 0$.

(2) The way-below relation generally behaves as a type of strict less-then relation, but, as we have just seen in the case of chains, the behavior is a little more subtle than that. If L is a complete chain, and we consider the partially ordered *direct power* L^I of L in the pointwise ordering, then in the complete lattice L^I we find $x \ll y$ iff $x_i \ll y_i$ for all $i \in I$ and $x_i = 0$ for all but a finite number of indices i. When I is infinite, this circumstance obviously justifies the "way" in "way below". (Perhaps "well below" would have been less colloquial, but we wanted to make the notion more memorable.) The reader can easily explain to himself the significance of the special case when L is just the two element lattice and we can regard L^I as the powerset lattice: in the powerset of I, the relation $A \ll B$ just means that A is a finite subset of B.

(3) As a first bad example, consider the case of a *complete and atomless Boolean algebra.* Recall that "atomless" means that there are *no* minimal nonzero elements; thus, by the laws of Boolean complements, every nonzero element can be split as the join of two disjoint, nonzero parts. By employing completeness this splitting can be continued indefinitely to show that, in fact, every nonzero element is the sup of a *denumerable* family of pairwise disjoint, nonzero elements, each of which must necessarily be strictly smaller than the originally given element. By arranging these elements in a sequence and taking the joins of the initial segments, we find that the original element is the sup of a *directed* family of strictly smaller elements.

Now suppose $x \ll y$. If $x \neq 0$, then the preceding construction can be carried out in the interval $[w, y]$ where w is the relative complement of x in y (one verifies directly that all nontrivial intervals are again atomless Boolean algebras). But no element of $[w, y]$ is above x except y. Thus, the way-below relation trivializes to: $x \ll y$ iff $x = 0$.

(4) For somewhat pathological examples, think what it means for $x \ll x$ to hold for *all* $x \in L$. Clearly, every *finite poset* has this property. More generally it is necessary and suffcient that there be no strictly increasing infinite chains in the partial ordering, because the sup of such a chain cannot be isolated from below. This is just the *ascending chain condition* for L, and it is equivalent to saying that every nonempty subset contains a maximal element (and hence every directed set has a maximum). Note that if L satisfies this condition, there is no reason for L^{op} to do so: the definition of the way-below relation is, therefore, not at all symmetric with respect to the partial ordering.

(5) We should also recall at this point the examples of lattices from universal algebra, for instance those of O-2.7(4). The ring case will be suffcient for illustration: the lattice $\mathrm{Id}\,\mathcal{A}$ of *two-sided ideals* of the ring $\mathcal{A}$ is complete. If $I, J \in \mathrm{Id}\,\mathcal{A}$, then, because each ideal is the directed union of the finitely generated ideals it contains, $I \ll J$ holds iff $I \subseteq F \subseteq J$ for some *finitely generated* $F \in \mathrm{Id}\,\mathcal{A}$. We note, too, that $F \ll F$ holds iff F is finitely generated. This and related examples will be studied in full detail in Section 4 on algebraic lattices. □

Topological spaces provide other good examples – in certain cases. That is to say, in certain cases it is easy to identify the way-below relation in topological terms. In our formulations we adopt the convention of calling a space (or subset) with the Heine–Borel property *compact* (every open covering has a finite subcovering) and of saying specifically when the Hausdorff separation property is assumed in addition (see O-5.5). Accordingly, we call a space *locally compact* if for every point x and every open set U containing x there are an open set V and a compact set Q such that $x \in V \subseteq Q \subseteq U$, i.e., if every point has a neighborhood basis of compact neigborhoods (see O-5.7). Notice that a subset K of a topological space is compact if and only if for each directed set $\mathcal{D}$ of open sets whose union contains K there is a member $U \in \mathcal{D}$ containing K; the necessity is an immediate consequence of the Heine–Borel property; in order to see sufficiency for a given open cover $\{U_j : j \in J\}$ of K let $\mathcal{D}$ be the set of finite unions of the U_j; then K is contained in the union of $\mathcal{D}$ and if K satisfies our property then there is a member of $\mathcal{D}$ covering K: the Heine–Borel property holds.

Proposition I-1.4. *Let X be a topological space and let $L = \mathcal{O}(X)$.*

(i) *If $U, V \in \mathcal{O}(X)$ and if there is a compact subset $Q \subseteq X$ with $U \subseteq Q \subseteq V$, then $U \ll V$.*
(ii) *Suppose now that X is locally compact. Then $U \ll V$ in L iff there exists a compact set Q with $U \subseteq Q \subseteq V$.*

Proof: As the open subsets form a complete lattice, we may use the alternative description of the way-below relation in Definition I-1.1.

(i) Indeed, any open cover of V is an open cover of Q, and, since Q is compact, finitely many of the covering sets already cover Q, hence U. Thus, $U \ll V$.

(ii) For the converse in the case of a locally compact X, observe that each point $v \in V$ has a compact neighborhood $Q_v \subseteq V$ with interior W_v containing v. Then

$$V = \bigcup\{W_v \colon v \in V\},$$

and from $U \ll V$ we may conclude that there are finitely many elements $v_1, \ldots, v_n$ such that

$$U \subseteq W_{v_1} \cup \cdots \cup W_{v_n} \subseteq Q_{v_1} \cup \cdots \cup Q_{v_n} \subseteq V.$$

The set $Q = Q_{v_1} \cup \cdots \cup Q_{v_n}$ is the required compact set. □

One notes immediately that in Hausdorff spaces the relation $U \subseteq Q \subseteq V$ for some compact set is equivalent to saying that $U^- \subseteq V$ and U^- is compact.

The above examples and arguments suggest some alternatives to the definition of the way-below relation, notably when we are in a complete lattice. For a poset L and $x \in L$ let $J(x) = \{I \in \mathrm{Id}(L) \colon x \leq \sup I\}$.

Proposition I-1.5. (i) *In a poset L, the following conditions are equivalent:*

(1) $x \ll y$;
(2) *$x \in I$ for every ideal I of L such that $y \leq \sup I$;*
(2′) $x \in \bigcap J(y)$.

If L is a meet continuous semilattice, then conditions (1) *and* (2) *are equivalent to*

(3) *$x \in I$ for every ideal I of L such that $y = \sup I$.*

(ii) *Suppose that there exists a directed set $D \subseteq \mathop{\downarrow\!\!\!\!\downarrow} x$ with $\sup D = x$. Then $\mathop{\downarrow\!\!\!\!\downarrow} x$ is directed and $x = \sup \mathop{\downarrow\!\!\!\!\downarrow} x$. Furthermore, $y \ll x$ if $y \ll x$ in the poset $\downarrow x$ with the induced order.*

Proof: (i) That (1) $\Rightarrow$(2) is immediate from the definitions (see I-1.1 and O-1.3(vii)). (2) $\Rightarrow$ (1): Assume (2) and let D be a directed subset with $y \le \sup D$. Then $I = \downarrow D$ is an ideal, and $y \le \sup D = \sup I$. Then $x \in I$ by (2), i.e., there is a $d \in D$ such that $x \le d$. Hence $x \ll y$.

Condition (2′) is just a reformulation of (2).

For (3) we have only to remark that $y \le \sup I$ is equivalent to $y = \sup yI$ in a meet continuous lattice.

(ii) Let $y \ll x, z \ll x$. Then $y \le d_y$ and $z \le d_z$ for some $d_y, d_z \in D$. Pick $d \in D$ such that $d_y \le d$ and $d_z \le d$. Then $y \le d, z \le d$, and $d \ll x$. Thus $\twoheaddownarrow x$ is directed. If $y \ll x$ in $\downarrow x$, then $y \le d \ll x$ for some $d \in D$, so $y \ll x$ in L. □

We note that the equivalence of (1) and (2′) in I-1.5 can be expressed as follows. For all x in a **dcpo** L we have $\twoheaddownarrow x = \bigcap J(x)$.

In an arbitrary poset or even complete lattice, as we have seen, we have no guarantee that the relation $x \ll y$ is satisfied for any pairs (x, y) other than those with $x = 0$. Very roughly speaking, continuous posets are those posets for which the relation $x \ll y$ is "frequent". More precisely:

Definition I-1.6.

(i) A poset L is called *continuous* if it satisfies the *axiom of approximation*:

$$(\forall x \in L)\; x = \bigvee{}^{\uparrow} \twoheaddownarrow x, \tag{A}$$

i.e., for all $x \in L$, the set $\twoheaddownarrow x = \{u \in L \colon u \ll x\}$ is directed and $x = \sup\{u \in L \colon u \ll x\}$.

(ii) A **dcpo** which is continuous as a poset will be called a *domain*.

(iii) A domain which is a complete lattice is called a *continuous lattice*. We recall from O-2.2 that a **dcpo** which is a lattice with a smallest element is a complete lattice.

(iv) A domain which is also a semilattice is called a *continuous semilattice*.

(v) A complete semilattice (cf. O-2.1) which is a domain as a poset is called a *complete continuous semilattice* or alternatively a *bounded complete domain*. In light of O-2.2(iv) this is equivalent to saying that a bounded complete domain is a domain which is bounded complete.

(vi) A domain in which every principal ideal $\downarrow x$ is a complete lattice (in its induced order) is called an *L-domain*. □

We thus have the following chain of implications:

$$\boxed{\begin{array}{l}\text{continuous lattice } \Rightarrow \text{bounded complete domain}\\ \Rightarrow \left\{\begin{array}{l}\text{continuous semilattice}\\ L\text{-domain}\end{array}\right\} \Rightarrow \text{domain} \Rightarrow \textbf{dcpo}\end{array}}$$

The entire area of investigation of the theory and of the applications of continuous lattices and their generalizations, which we discuss here, has become rather unequivocally known as *domain theory*. However, inquiring what is really meant by a *domain* remains a highly sensitive question. We have now fixed *our* terminology by the previous definition.

Remark. Note that in a complete semilattice and in a sup semilattice, $\twoheaddownarrow x$ is automatically directed by I-1.2(iii), and thus in Definition I-1.6, we may write condition (A) simply as $x = \sup \twoheaddownarrow x$ or as

$$\text{whenever } x \not\leq y, \text{ then there is a } u \ll x \text{ with } u \not\leq y. \tag{A$_1$}$$

In other words, the axiom of approximation means that every element can be suffciently well approximated by elements way below it. Indeed, the way-below relation completely determines the partial ordering, because in a continuous poset it is the case that

$$x \leq y \Leftrightarrow \twoheaddownarrow x \subseteq \twoheaddownarrow y.$$

Remark. Often it is difficult to characterize the way-below relation on a poset completely. For proving the continuity of a poset it is sufficient to know “enough” elements that are way below. Indeed, by I-1.5(ii), a poset is continuous provided that, for every x, one can find a directed set D of elements $d \ll x$ such that $x = \sup D$.

Examples I-1.7. It is clear that all finite posets and all finite lattices are continuous. In view of the foregoing discussion in I-1.3 and I-1.4, we may assert that the following are continuous:

(1) complete chains are continuous lattices;
(2) direct powers (products) of complete chains are continuous lattices;
(3) posets satisfying the ascending chain condition are domains;
(4) the ideal lattice of a ring is a continuous lattice;
(5) the open sets of a locally compact space form a continuous lattice. □

Notice that for locally compact X the lattice $\mathcal{O}(X)$ is in fact a distributive continuous lattice by O-3.22. In the fourth section of this chapter we will see that $\mathcal{O}_{\text{reg}}(X)$ is *almost never* a continuous lattice, since Boolean algebras rarely are (cf. the atomless example of I-1.3(3)). This shows that frames (complete Heyting algebras) need not be continuous lattices. In Chapter V we shall prove that every continuous frame (= distributive continuous lattice) is of the form $\mathcal{O}(X)$ for some locally compact space X, even though not every frame is a topology. We will also exhibit there some very bad spaces X for which $\mathcal{O}(X)$ is a continuous lattice, while every compact subset of X has empty interior.

These examples cover the most immediate and obvious classes of continuous lattices. At the end of this present section we present some further types of domains occurring "in nature", where the proof of continuity is not so quick. In the next section we will find construction methods allowing us to obtain a multitude of domains by using given ones as building blocks. Before turning to these matters, let us show that that the continuity concept for posets as introduced in I-1.6 implies meet continuity as defined in O-4.1:

Proposition I-1.8. *Every continuous semilattice, hence every continuous lattice, is meet continuous.*

Proof: Let L be a continuous semilattice. We use O-4.2(5). Assume that $x \leq \sup D$, where D is directed. To show $x \leq \sup xD$, it suffices to show $\twoheaddownarrow x \subseteq {\downarrow} \sup xD$. But if $y \ll x$, then not only is $y \leq x$ but $y \leq z$ for some $z \in D$. As $y \leq xz \in xD$, it follows that $y \leq \sup xD$. □

The following theorem exhibits an important property of the way-below relation on continuous domains, the *interpolation property*.

Theorem I-1.9. (i) *If $x \ll z$ and if $z \leq \sup D$ for a directed set D in a continuous poset L, then $x \ll d$ for some element $d \in D$.*

(ii) *In a continuous poset L, the way-below relation satisfies the interpolation property*

$$x \ll z \text{ implies } (\exists y)\ x \ll y \ll z. \tag{INT}$$

Proof: (i) Let D be a directed set with $z \leq \sup D$, and let $I = \bigcup\{\twoheaddownarrow d\colon d \in D\}$. By continuity, $\sup I = \sup D$ and, being a union of a directed family of ideals, I is an ideal. Hence, if $x \ll z$ then $x \in I$ by I-1.5(2), which means that $x \ll d$ for some $d \in D$.

(ii) follows from (i) by choosing $D = \twoheaddownarrow z$ and recalling that $z = \sup \twoheaddownarrow z$ by the continuity of L. □

We continue the discussion by considering the way-below relation from a new perspective. From the context of I-1.5 we recall that for x in a poset L the set $J(x)$ is defined to be $\{I \in \operatorname{Id} L : x \leq \sup I\}$. Recall that by I-1.5(i), in a poset L we have $\twoheaddownarrow x = \bigcap J(x)$.

Theorem I-1.10. *For a* **dcpo** *L, the following conditions are equivalent:*

(1) *L is continuous;*
(2) *for each $x \in L$, the set $\twoheaddownarrow x$ is the smallest ideal I with $x \leq \sup I$;*
(3) *for each $x \in L$ there is a smallest ideal I with $x \leq \sup I$;*
(4) *the sup map $r = (I \mapsto \sup I)\colon \operatorname{Id} L \to L$ has a lower adjoint.*

These conditions imply

(5) *the sup map $r\colon \operatorname{Id} L \to L$ preserves all existing infs;*

and if L is a complete semilattice or a complete lattice, then all five conditions are equivalent.

Proof: (1) $\Rightarrow$ (2): Condition (1) holds iff for each $x \in L$, $\twoheaddownarrow x \in \operatorname{Id} L$ and $\twoheaddownarrow x \in J(x)$ by Definition I-1.6. Thus (2) follows.

Condition (2) trivially implies (3).

(3) $\Rightarrow$(1): If $J(x)$ has a smallest element M, then $M \subseteq I$ for all $I \in J(x)$ and thus $M \subseteq \bigcap J(x) \subseteq M$. Hence $M = \bigcap J(x) = \twoheaddownarrow x$.

Thus (1), (2) and (3) are equivalent.

(3) iff (4): By O-3.2, the map r has a lower adjoint iff $\min r^{-1}(\uparrow x)$ exists for all x. But $\min r^{-1}(\uparrow x)$ is precisely the smallest element of $J(x)$.

(4) implies (5): The sup map preserves infs by O-3.3.

(5) implies (4): This follows from O-3.4, as the sup map $r\colon \operatorname{Id} L \to L$ clearly is cofinal. $\square$

Auxiliary relations

To finish up our discussion of definitional matters, we take a closer look at the way-below relation and detect how it fits into a more general framework. We begin by reformulating as a definition something we already know for the way-below relation (cf. I-1.2).

Definition I-1.11. We say that a binary relation $\prec$ on a poset L is an *auxiliary relation*, or an *auxiliary order*, if it satisfies the following conditions for all u, x, y, z:

(i) $x \prec y$ implies $x \leq y$;
(ii) $u \leq x \prec y \leq z$ implies $u \prec z$;

(iii) if a smallest element 0 exists, then $0 \prec x$.

The set of all auxiliary relations on L will be denoted by $\mathrm{Aux}(L)$. □

Clearly, every auxiliary relation is transitive by (i) and (ii), and the way-below relation is an auxiliary relation by Proposition I-1.2. The set $\mathrm{Aux}(L)$ is a poset relative to the containment of graphs as subsets of $L \times L$. The largest element is the relation $\leq$ itself. If L has a smallest element 0, then $\mathrm{Aux}(L)$ has a smallest element $\bigcirc$ given by $x \bigcirc y$ iff $x = 0$. As $\mathrm{Aux}(L)$ is closed under arbitrary intersections in $2^{L \times L}$, it is therefore a complete lattice.

In order to gain better insight into the lattice $\mathrm{Aux}(L)$ we try to find an isomorphic copy.

For a poset L let Low L denote the set of all lower sets in L which are supposed to contain 0, if L has a smallest element 0.

Proposition I-1.12. *Let L be a poset and let M be the set of all monotone functions $s\colon L \to \mathrm{Low}\, L$ satisfying $s(x) \subseteq {\downarrow}x$ for all $x \in L$ – considered as a poset relative to the ordering $s \leq t$ iff $s(x) \subseteq t(x)$ for all $x \in L$. Then the assignment*

$$\prec \mapsto s_{\prec} = (x \mapsto \{y\colon y \prec x\})$$

is a well-defined isomorphism from $\mathrm{Aux}(L)$ *onto M, whose inverse associates to each function $s \in M$ the relation $\prec_s$ given by*

$$x \prec_s y \quad \textit{iff} \quad x \in s(y).$$

Proof: Let $\prec$ be an auxiliary relation. Then $s_{\prec}(x)$ is a lower set by I-1.11(ii) contained in ${\downarrow}x$ by I-1.11(i) and which contains 0 whenever L has a smallest element 0 by I-1.11(iii). If $x \leq y$, then $s_{\prec}(x) \subseteq s_{\prec}(y)$ by I-1.11(ii). Thus $s_{\prec}$ is in M, and the assignment $\prec \mapsto s_{\prec}$ is clearly order preserving.

Conversely, if $s \in M$, then $s(x) \subseteq {\downarrow}x$ implies that $\prec_s$ satisfies I-1.11(i). The relation $u \leq x \prec_s y \leq z$ implies $u \leq x$ and $x \in s(y) \subseteq s(z)$, since s is monotone. Because $s(z)$ is a lower set, $u \in s(z)$; whence, $u \prec_s z$. Thus I-1.11(ii) is satisfied. Condition I-1.11(iii) is immediate. Thus, the assignment $s \mapsto \prec_s : M \to \mathrm{Aux}(L)$ is a well-defined function, and it is obviously order preserving also.

It remains to confirm that the two assignments are inverses of each other, but this is easy. □

After this proposition we know that an auxiliary relation is essentially the same thing as assigning in a monotone fashion a lower set bounded by x to each element x of L. The largest element in M is the function $x \mapsto {\downarrow}x$. If L has a smallest element 0, then M has a smallest element, namely, the constant function $x \mapsto \{0\}$. It is easy to see directly that M is a complete lattice.

Now we can raise the question how we might locate the auxiliary relation $\ll$ within $\mathrm{Aux}(L)$. From Proposition I-1.5(i) we know that

$$\twoheaddownarrow x = \bigcap\{I \in \mathrm{Id}\, L \colon x \le \sup I\}.$$

This does not yet express the function $x \mapsto \twoheaddownarrow x$ as an inf (in M) of a recognizable collection of other functions in M. In order to approach *this* goal, let us assume that L is a **dcpo** and consider for an arbitrary ideal $I \in \mathrm{Id}\, L$ the function $m_I \colon L \to \mathrm{Low}\, L$ given by

$$m_I(x) = \begin{cases} \downarrow x \cap I = xI, & \text{if } x \le \sup I, \\ \downarrow x, & \text{otherwise.} \end{cases}$$

Then $m_I(x)$ is a lower set which is contained in $\downarrow x$, and $x \mapsto m_I(x)$ is monotone; in other words, $m_I \in M$. Now we calculate $\inf\{m_I \colon I \in \mathrm{Id}\, L\}$ in M:

$$\begin{aligned}
(\inf_{I \in \mathrm{Id}\, L} m_I)(x) &= \bigcap\nolimits_{I \in \mathrm{Id}\, L} m_I(x) \\
&= \bigcap\nolimits_{x \le \sup I} m_I(x) \cap \bigcap\nolimits_{x \not\le \sup I} m_I(x) \\
&= \bigcap\nolimits_{x \le \sup I} (\downarrow x \cap I) \cap \downarrow x \\
&= \bigcap\{I \in \mathrm{Id}\, L \colon x \le \sup I\} \\
&= \twoheaddownarrow x.
\end{aligned}$$

Definition I-1.6 of a continuous poset motivates us next to formulate the following definition.

Definition I-1.13. An auxiliary relation $\prec$ on a **dcpo** L (and the function $s_\prec \colon L \to \mathrm{Low}\, L$ associated with it) is called *approximating* iff the set $\{u \in L : u \prec x\} = s_\prec(x)$ is directed (hence an ideal) and

$$x = \sup\{u \in L \colon u \prec x\} = \sup s_\prec(x)$$

for all $x \in L$. The set of all approximating auxiliary relations is denoted by $\mathrm{App}(L)$. □

The relation $\le$ is trivially approximating, and, in a continuous poset L, the relation $\ll$ is approximating by I-1.6. Loosely speaking, the approximating auxiliary relations are those auxiliary relations which are "close" to $\le$. One does not expect a rich supply of information for auxiliary relations which are not approximating, but they do occur.

Lemma I-1.14. *In a meet continuous semilattice L, all relations belonging to the functions m_I for $I \in \mathrm{Id}\, L$ are approximating. This holds, in particular, for continuous semilattices, as these are meet continuous* (I-1.8).

Proof: Let $x \in L$. If $x \leq \sup I$, then $\sup m_I(x) = \sup xI = x \sup I = x$ by O-4.2. If $x \not\leq \sup I$, then $\sup m_I(x) = \sup {\downarrow} x = x$. □

Now we can have a lattice theoretical description of $\ll$ within $\mathrm{Aux}(L)$ – at least for meet continuous semilattices:

Proposition I-1.15. *In a* **dcpo** *L, the way-below relation $\ll$ is contained in all approximating auxiliary relations, and is equal to their intersection, if L is a meet continuous semilattice.*

Proof: Suppose that $y \ll x$ and $\prec$ is an approximating auxiliary relation. Then $\{u \in L : u \prec x\}$ is a directed set and its sup is x. This implies $y \leq u \prec x$ for some u, and hence $y \prec x$. Thus $\ll$ is contained in $\prec$.

On the other hand, for L meet continuous, we have

$$\mathord{\downarrow\!\!\downarrow} x = \bigcap\{m_I(x) \colon I \in \mathrm{Id}\, L\} \supseteq \bigcap\{s_{\prec}(x) \colon \prec \quad \text{is in App(L)}\}$$

by Lemma I-1.14. □

Notice that this does not say that $\ll$ is itself an approximating relation, because meet continuous semilattices need not be continuous. More precisely, we have

Proposition I-1.16. *Let L be a* **dcpo** *and consider the following conditions:*

(1) *L is continuous, i.e., a domain;*
(2) *the relation $\ll$ is the smallest approximating auxiliary relation on L;*
(3) *there is a smallest approximating auxiliary relation on L.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *Moreover, if L is a meet continuous semilattice, then all three conditions are equivalent.*

Proof: (1) $\Leftrightarrow$ (2): By definition, L is a domain iff $\ll$ is an approximating auxiliary relation. Thus the equivalence of (1) and (2) follows from the first part in proposition I-1.15.

That (2) $\Rightarrow$ (3) is trivial.

Let L be a meet continuous semilattice. Then $\ll$ is the intersection of all approximating auxiliary relations by I-1.15. Thus, if there is a smallest approximating auxiliary relation, this has to be $\ll$, and we see that (3) implies (1). □

We return to the discussion of the interpolation property, the single most important property of the relation $\ll$ in continuous posets from I-1.9.

Definition I-1.17. We say that an auxiliary relation $\prec$ on a poset L satisfies the *strong interpolation property*, provided that the following condition is satisfied for all $x, z \in L$:

(SI) $x \prec z$ and $x \neq z$ together imply $(\exists y)$ $(x \prec y \prec z$ and $x \neq y)$.

We say that $\prec$ satisfies simply the *interpolation property* iff the following weaker condition holds for all $x, z \in L$:

(INT) $x \prec z$ implies $(\exists y)$ $x \prec y \prec z$. □

One may look at the interpolation property as a sort of order density property. Notice that $x \prec z$ and $x = z$ (i.e., $x \prec x$) will trivially imply the existence of a y with $x \prec y \prec z$; namely, $y = x$. So clearly (SI) implies (INT); if $\prec$ is approximating, then the two conditions are equivalent. Thus, the way-below relation $\ll$ on a continuous poset also satisfies the strong interpolation property. In the theory of domains and continuous lattices much depends on the fact that $\ll$ satisfies the interpolation property. The reason for making a distinction between the two interpolation properties is that $\ll$ is generally a very irreflexive relation and that the stronger one will be needed in Chapter IV.

Lemma I-1.18. *Any approximating auxiliary relation $\prec$ in a* **dcpo** *L satisfies the following condition for all $x, z \in L$:*

$$x \prec z \textit{ and } x \neq z \textit{ together imply } (\exists y)\ (x \leq y \prec z \text{ and } x \neq y).$$

Proof: Since $z = \sup\{u\colon u \prec z\}$, there is a $u \prec z$ with $u \not\leq x$. As $s_\prec(z) = \{u\colon u \prec z\}$ is directed, there is an upper bound y of $\{x, u\}$ such that $y \prec z$; since $u \not\leq x$ we have $x \neq y$. □

The following proposition generalizes Theorem I-1.9.

Proposition I-1.19. *For any approximating auxiliary relation $\prec$ in a* **dcpo** *L the following conditions hold for all $x, z \in L$.*

(i) *If $x \ll z$, $x \neq z$ and $z \leq \sup D$ for a directed subset D of L, then $x \prec d$, $x \neq d$ for some element $d \in D$.*
(ii) *If $x \ll z$ and $x \neq z$, then there exists a y such that $x \prec y \prec z$ and $x \neq y$.*

Proof: (i) Let D be a directed set such that $z \leq \sup D$, and let $I = \bigcup\{s_\prec(d)\colon d \in D\}$. The relation $\prec$ being approximating, I is an ideal as a directed union of the ideals $s_\prec(d)$ and $\sup I = \sup\{\sup s_\prec(d)\colon d \in D\} = \sup D \geq z$. From $x \ll z$ we now conclude that $x \in I$, that is, there is an element $d \in D$ such that $x \prec d$. As $x \not\geq z$ and $z \leq \sup D$, there is an element $c \in D$ with $c \not\leq x$. Replacing d by a common upper bound of c and d in D, we have found the desired element.

(ii) Choose $D = s_{\prec}(z) = \{y \in L : y \prec z\}$. As $\prec$ is approximating, D is directed $z = \sup D$. If $x \ll z$ and $x \neq z$, by (i) we may find an element $y \in D$, that is, $y \prec z$, such that $x \prec y$ and $y \neq x$. □

In view of these lemmas, the following strengthening of the interpolation property in Theorem I-1.9(ii) is immediate.

Corollary I-1.20. *In a domain the way-below relation satisfies the strong interpolation property* (SI). □

We remark that for many purposes (INT) suffices. We also note that in the formulation of the property we could not strengthen (SI) to include $y \neq z$, because the two element chain offers a trivial counterexample.

Important examples

Let us now consider three examples of types of continuous lattices which can be fairly said "to occur in (mathematical) nature". In connection with Section 4 of this chapter, where we introduce algebraic lattices, the wide classes of lattices described here will in particular furnish examples of continuous lattices which are not generally algebraic. We then proceed to some wide classes of domains which are not lattices.

Example I-1.21. (Closed ideals of C^*-algebras). Recall the concept of a C^*-algebra, which is of central importance in functional analysis in the context of operators on Hilbert space and of operator norm closed involutive algebras of operators. Abstractly a C^*-algebra is a complex Banach algebra $\mathcal{A}$ with an involution $a \mapsto a^*$ satisfying $\|a^*a\| = \|a\|^2$. We record the following fact (cf. [Laursen and Sinclair, 1975], esp. p. 168).

Fact. *Each C^*-algebra $\mathcal{B}$ contains a unique smallest dense two-sided ideal $\mathcal{B}_0$, called the* Pedersen ideal *of $\mathcal{B}$.*

If $\mathcal{B}$ has an identity, then $\mathcal{B}_0 = \mathcal{B}$; if $\mathcal{B}$ is the algebra $\mathrm{C}_0(X)$ of all continuous complex-valued functions on a locally compact and noncompact Hausdorff space vanishing at infinity, then $\mathcal{B}_0$ is the ideal $\mathrm{K}(X)$ of all continuous functions of compact support; it was for the purposes of generalizing integration theory to the noncommutative situation that Pedersen found and investigated the ideal $\mathcal{B}_0$. If $\mathcal{B}$ is the algebra $\mathrm{LC}(\mathcal{H})$ of compact operators on the Hilbert space $\mathcal{H}$, then $\mathcal{B}_0$ is the ideal of all finite-rank operators.

For any subset X in the C^*-algebra $\mathcal{A}$ we let $\langle X\rangle$ denote the closed (!) two-sided ideal generated by X in $\mathcal{A}$. Recall that $\mathrm{Id}^-\mathcal{A}$ denotes the lattice of closed two-sided ideals (O-2.7(7)). Each $I \in Id^-\mathcal{A}$ is in itself a C^*-algebra. We now have the following proposition on the complete lattice $L = \mathrm{Id}^-\mathcal{A}$.

Proposition I-1.21.1. *For $I, J \in \mathrm{Id}^-\mathcal{A}$ the following statements are equivalent.*

(1) *$I \ll J$ (in L).*
(2) *There is an element $a \in J_0$ with $0 \leq a$ such that $I \subseteq \langle a\rangle$.*
(3) *There is a finite subset $F \subseteq J_0$ with $I \subseteq \langle F\rangle$.*

Proof: That (2) implies (3) is trivial. For the proof (3) implies (1) we formulate a lemma:

Lemma. *Let $P \in \mathrm{Id}^-\mathcal{A}$ and $Q \in \mathrm{Id}\,\mathcal{A}$ with $P \subseteq Q^-$. Then $(P \cap Q)^- = P$.*

Proof: Let $0 \leq x \in P$; then $x = \lim x_n$ with $x_n \in Q$; thus $xx_n \in P \cap Q$; hence $x^2 = \lim xx_n \in (P \cap Q)^2$ and thus $x \in (P \cap Q)^-$ by the functional calculus for C^*-algebras. □

Now we prove (3) implies (1): Let D be any directed subset of L with $J \subseteq \sup D = (\bigcup D)^-$. By the lemma, $J \cap \bigcup D$ is a dense two-sided ideal of J; hence, it contains J_0 by the Fact quoted above. If $F \subseteq J_0$ is as in (3), we may therefore conclude that there is some member $K \in D$ with $F \subseteq K$, and thus $I \subseteq \langle F\rangle \subseteq K$. This proves (1).

For a proof (1) implies (2), we take $0 \leq x, y \in J_0$, and so $0 \leq x + y \in J_0$; the observation $0 \leq x \leq x + y$ allows us to conclude $x \in \langle x + y\rangle$, since closed two-sided ideals are "hereditary". Thus the collection $\{\langle x\rangle \colon 0 \leq x \in J_0\}$ is directed in L, and its union contains J_0; whence, its sup dominates J by the Fact quoted above. Condition (1) now implies that we find some $a \in J_0$ with $0 \leq a$ and $I \subseteq \langle a\rangle$. □

We notice that the crux in defining the way-below relation in a lattice of closed ideals (or congruences, subgroups, etc.) in topological rings (algebras, groups, etc.) is the fact that the formation of the sup of a directed collection involves an eventual closure of the union, and this creates all the complication. Once again one notices that C^*-algebras constitute a class of topological algebras with particularly desirable properties. We now arrive at the main conclusion that is the point of this discussion.

Proposition I-1.21.2. *The lattice $L = \mathrm{Id}^-\mathcal{A}$ of closed two-sided ideals in the C^*-algebra $\mathcal{A}$ is a continuous lattice.*

Proof: For $I \in L$ we have $I = (\bigcup\{\langle F\rangle\colon F \text{ finite in } I_0\})^-$ by the Fact quoted above. Then apply Proposition I-1.21.1 above. □

Other proofs of this result exist, but this one relates most directly to the definitions. We remark that in fact $\mathrm{Id}^-\mathcal{A}$ is a distributive lattice, hence indeed a continuous distributive lattice by the preceding proposition. □

Example I-1.22. (Lower semicontinuous functions). Let $\mathrm{LSC}(X) = \mathrm{LSC}(X, \mathbb{R}^*)$ denote the complete lattice of all lower semicontinuous functions on a topological space X with values in the extended real line $\mathbb{R}^*$ (see O-2.10). For any function $f\colon X \to \mathbb{R}^*$ we consider its upper graph $G_f = \{(x, r)\colon r < f(x)\}$. Then f is lower semicontinuous iff G_f is open in $X \times \mathbb{R}^*$. We use the notation $x \ll y$ in $\mathbb{R}^*$, a continuous lattice itself, which of course means that $x < y$ or $x = -\infty$. □

Proposition I-1.22.1. *Suppose that X is a compact Hausdorff space. Consider the following statements for $f, g \in \mathrm{LSC}(X)$:*

(1) *$f \ll g$ in LSC(X);*
(2) *there are an open cover $\{U_j\colon j \in J\}$ of X and a family $\{y_j\colon j \in J\}$ of elements in $\mathbb{R}^*$ where $f(x) \leq y_j \ll g(x)$ for all $j \in J$ and $x \in U_j$;*
(3) *for each element $x \in X$ there are an open neighborhood U in X and an element $y \in \mathbb{R}^*$ where $f(x_1) \leq y \ll g(x_1)$ for all $x_1 \in U$;*
(4) *$G_f^- \subseteq G_g$ in $X \times \mathbb{R}^*$;*
(5) *there is a continuous function $h \in \mathrm{C}(X, \mathbb{R}^*)$ where for all $x \in X$ we have $f(x) \leq h(x) \ll g(x)$.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

Remark. The implications (1) implies (2) and (2) iff (3) maintain their validity for regular X.

Proof of proposition: (2) iff (3): Clear.
(1) implies (3): Let $\mathcal{F}(g)$ be the set of all functions $s\chi_U$ such that

(i) χ_U is the characteristic function of an open set U, and
(ii) $s \ll g(x)$ for all $x \in U^-$.

Then $\mathcal{F}(g)$ is a subset of $\mathrm{LSC}(X)$ with $g = \sup \mathcal{F}(g)$, since X is regular and g is lower semicontinuous. Because $f \ll g$ we find functions

$$s_1\chi_{U_1}, \ldots, s_n\chi_{U_n} \in \mathcal{F}(g) \text{ with } f \leq \sup s_j\chi_{U_j}.$$

Let $x \in X$ be arbitrary, and set $I(x) = \{j\colon x \notin U_j^-\}$. Then $i \in I(x)$ implies $s_i \ll g(x)$. Set $s(x) = \max\{s_j\colon j \in I(x)\}$, then $s(x) \ll g(x)$. The set

$$V(x) = X \setminus \bigcup \{U_j^-\colon j \notin I(x)\}$$

is an open neighborhood of x. Since g is lower semicontinuous, there is an open neighborhood $U(x)$ of x in $V(x)$ such that $u \in U(x)$ implies $s(x) \ll g(u)$. But $u \in U(x)$ entails that $u \notin U_j$ for $j \notin I(x)$; whence,

$$f(u) \le \sup\bigl\{s_j \chi_{U_j}(u)\colon j = 1, \ldots, n\bigr\} = \sup\bigl\{s_j \chi_{U_j}\colon j \in I(x)\bigr\} \le s(x).$$

(3) implies (1): Let h_j be a directed net in LSC(X) with $g \le \sup h_j$. For each $x \in X$ we find an index $j = j(x)$ and an element $s(x) \in \mathbb{R}$ where $s(x) \ll h_j(x)$, and $f(y) \le s(x)$ for all y in an open neighborhood $U(x)$ of x. Since h_j is lower semicontinuous, there is an open neighborhood $V(x) \subseteq U(x)$ with $s(x) \ll h_j(v)$ for all $v \in V(x)$. By the compactness of X, we find finitely many $x_1, \ldots, x_n$ with $X = V(x_1) \cup \cdots \cup V(x_n)$. Let k be an index with $j(x_1), \ldots, j(x_n) \le k$. Then for each $x \in X$ there is an i with $f(x) \le s(x_i) \le h_k(x)$; whence, $f \le h_k$.

(4) iff (3): Direct verification.

(2) implies (5): Let $\{V_i\colon i \in I\}$ be a finite cover such that for each $i \in I$ we have a $j(i) \in J$ with $V_i^- \le U_{j(i)}$. We write $z_i = y_{j(i)}$. Let $\{p_i\colon i \in I\}$ be a partition of unity subordinate to $\{V_i\colon i \in I\}$. Define $h = \sum z_i p_i$. For $x \in X$ note

$$h(x) = \sum \{z_i p_i(x)\colon x \in V_i\} \ll \sum \{g(x) p_i(x)\colon x \in V_i\} = g(x).$$

Similarly $f(x) \le h(x)$. □

If X is $\mathbb{N}$ with the discrete topology and f is the constant function with value $1/2$ and g that with the value 1, then (3) is evidently satisfied. But (1) fails: Consider the directed family of characteristic functions of finite subsets of $\mathbb{N}$; its sup is g, but no member dominates f. Thus the compactness of X is indispensable to conclude (3) implies (1).

Proposition I-1.22.2. *If X is a compact space, then* LSC(X) *is a continuous lattice.*

Proof: Consider that $f = \sup\{r\chi_U\colon r < f(u)$ for all $u \in U.\}$. □

The description we have given for $f \ll g$ in LSC(X) and the proofs we used were those of an analyst. Proposition I-1.22.2 will be vastly generalized in Chapter II (see II-4.6); the methods will be more lattice theoretical and topological and will not be based on a frontal attack via the way-below relation.

(See also Exercises I-2.21 and I-2.22 below.) Indeed, the examples of I-1.21 and I-1.22 illustrate that in many circumstances the way-below relation is difficult to describe explicitly, and this points to the need of these other tools in order to deal effectively with continuous lattices. □

Example I-1.23. (Convex compact subsets of a compact convex set). Let K be a compact convex subset of a locally convex topological vector space. Denote by $\mathrm{Con}(K)$ the lattice of all closed convex subsets of K (including the empty set). Recall that $\mathrm{Con}(K)^{\mathrm{op}}$ is the lattice with the reverse inclusion.

Proposition I-1.23.1. *The lattice* $\mathrm{Con}(K)^{\mathrm{op}}$ *is a continuous lattice, in which we have* $A \ll B$ *iff* $B \subseteq \mathrm{int}(A)$ *with the interior being taken in the relative topology of* K.

Proof: Since $\mathrm{Con}(K)$ is closed under arbitrary intersections, it – and hence $\mathrm{Con}(K)^{\mathrm{op}}$ – is a complete lattice. If $B \subseteq \mathrm{int}\,(A)$, then if the intersection of a descending family of closed convex sets is contained in B, it follows by compactness that one of them is a subset of A. Thus $A \ll B$ in $\mathrm{Con}(K)^{\mathrm{op}}$. The local convexity of the vector space implies that B is the intersection of its compact convex neighborhoods in K. Hence $\mathrm{Con}(K)^{\mathrm{op}}$ is continuous.

Finally assume that $A \ll B$ in $\mathrm{Con}(K)^{\mathrm{op}}$. Since we have just seen that B is the intersection of its compact convex neighborhoods in K, and since this family of neighborhoods is descending, we conclude from the definition of $\ll$ that one of them must be a subset of A. Therefore, A is a compact convex neighborhood of B, and this completes the proof. □

We remark that this example illustrates how we sometimes encounter lattices L where L^{op} instead of L is the continuous lattice even though L is more naturally given. However, there is no point in taking the dual definition, since there are just as natural examples that conform to the convention we have adopted. □

We have seen several natural examples of continuous lattices, notably in the field of topology. We now proceed to exhibit a couple of natural examples of domains which are not lattices in general. First, let us note that the nonempty closed convex subsets of a compact convex in a locally convex vector space K form a bounded complete domain under reverse inclusion. This follows immediately from the above. The following example will be of crucial interest to the theory.

Example I-1.24. (Compact saturated sets) As in O-5.3, a subset K of a topological space is called *saturated* if it is the intersection of its open neighborhoods. If A is any subset of X, the intersection sat A of all of its open neighborhoods is a saturated set called its *saturation*. If A is compact, its saturation sat A is compact, too (see after O-5.7). In so far as the order theoretical properties of the set of compact sets and their relationships with open sets are concerned, we may just as well restrict our attention to the saturated compact sets. We shall denote by $Q(X)$ the poset of compact saturated subsets of X with the order reverse to containment, i.e., $K_1 \leq K_2$ iff $K_2 \subseteq K_1$. The empty set is included in $Q(X)$, it is the top element of $Q(X)$. The sub-poset of nonempty compact saturated subsets of X is denoted by $Q^*(X)$.

We notice that $Q^*(X)$ is a **dcpo** *if and only if in the topological space X the intersection of any filter basis of nonempty compact saturated sets is nonempty and compact. In this case, $Q(X)$ will be a* **dcpo** *in which the top element (the empty set) is isolated.*

There are lots of spaces for which $Q(X)$ is not a **dcpo**. For example the set $E = \{\pm\frac{1}{n}\colon n = 1, 2, \ldots\}$ with the collection of all upper sets as a topology is a T_0 space in which the filter basis $\{\uparrow\frac{-1}{n}\colon n = 1, 2, \ldots\}$ of saturated compact sets has the noncompact intersection $\{\frac{1}{n}\colon n = 1, 2, \ldots\}$.

For every Hausdorff space X the poset $Q^*(X)$ is a **dcpo**. We shall encounter in I-1.24.2 below a much wider class of spaces X for which $Q(X)$ is a **dcpo**.

In the following proposition we use the following.

Definition I-1.24.1. We shall say that a space X is *well-filtered* if for each filter basis $\mathcal{C}$ of compact saturated sets and each open set U with $\bigcap \mathcal{C} \subseteq U$ there is a $K \in \mathcal{C}$ with $K \subseteq U$. □

The example E above is not well-filtered. However, standard topological arguments yield that a Hausdorff space is well-filtered, and we will see in Theorem II-1.21 of Chapter II that all sober spaces (see O-5.6) are well-filtered and among locally compact spaces these are the only well-filtered spaces.

Proposition I-1.24.2. *Let X be a topological space.*

(i) *$Q(X)$ and $Q^*(X)$ are semilattices (the semilattice operation being $\cup$).*

(ii) *Let $K_1, K_2 \in Q(X)$ and consider the following assertions:*

(a) *there is an open set U such that $K_1 \supseteq U \supseteq K_2$, i.e.,* $\mathrm{int}(K_1) \supseteq K_2$*;*
(b) *$K_1 \ll K_2$ in $Q(X)$.*

If X is well-filtered, then (a) $\Rightarrow$ (b)*, and if X is locally compact, then* (b) $\Rightarrow$ (a).

(iii) *If X is well-filtered, then $K = \bigcap \mathcal{C}$ is a nonempty compact saturated set for each filter base $\mathcal{C}$ of nonempty compact saturated sets C. Hence, $Q^*(X)$ is a* **dcpo** *and the top element in $Q(X)$ is isolated.*

(iv) *If X is locally compact and well-filtered, then $Q(X)$ and $Q^*(X)$ are continuous semilattices, in particular, domains.*

Proof: (i) The union of two compact sets is compact. Now let K_1 and K_2 be two saturated compact sets, and let $x \notin K_1 \cup K_2$. Since the K_n are saturated, there are open neighborhoods U_n of K_n, respectively, such that $x \notin U_n$. Hence $x \notin U_1 \cup U_2$, and this is an open neighborhood of $K_1 \cup K_2$. Thus this compact set is saturated.

(ii) We assume that X is well filtered and prove that (a) $\Rightarrow$ (b): Let $\mathcal{C}$ be a filter basis of compact saturated sets with $\bigcap \mathcal{C} \subseteq K_2$, i.e., $K_2 \leq \sup \mathcal{C}$. Then, as X is well-filtered, there is a $C \in \mathcal{C}$ such that $C \subseteq U$ which implies $K_1 \leq C$. This proves $K_1 \ll K_2$. Next we assume that X is locally compact and prove (b) $\Rightarrow$ (a): Let U be an open set containing K_2. Every point of $x \in K_2$ has a compact saturated neighborhood $C_x \subseteq U$. As K_2 is compact, there is a finite sequence of points $x_1, \ldots, x_n \in K_2$ such that K_2 is contained in the interior of $C_{x_1} \cup \cdots \cup C_{x_n} \subseteq U$. Hence K_2 is the intersection of the filter basis $\mathcal{C}$ of its compact saturated neighborhoods. Thus $K_2 = \sup \mathcal{C}$. From $K_1 \ll K_2$ we conclude the existence of a $C \in C$ such that $K_1 \leq C$. If U denotes the interior of C we have $K_1 \supseteq U \supseteq K_2$.

(iii) Let $\mathcal{C}$ be a filter basis of nonempty compact saturated sets and let $K = \bigcap \mathcal{C}$. If $K = \emptyset$, then by the property of being well-filtered, $C \subseteq \emptyset$ for come some $C \in \mathcal{C}$, a contradiction. Thus $K \neq \emptyset$. That K is an intersection of open sets, hence is saturated, is immediate. Let $\mathcal{U}$ be a collection of open sets covering K. Again by the property of being well-filtered, $C \subseteq \bigcup \mathcal{U}$ for some $C \in \mathcal{C}$. Thus finitely many members of $\mathcal{U}$ cover C and hence K. Thus K is compact, and hence $Q^*(X)$ is a **dcpo** and the empty set, the top element of $Q(X)$, is isolated.

(iv) We have seen in (ii) that in a locally compact space, every saturated compact set K is the intersection of the filter basis of its compact neighborhoods C, which we may assume to be saturated; by (ii) we have $C \ll K$. This shows that $K = \sup \twoheaddownarrow K$. Since by (i) $Q(X)$ and $Q^*(X)$ are semilattices and by (iii) **dcpo**s, we conclude that $Q(X)$ and $Q^*(X)$ are continuous semilattices. □

Proposition I-1.24.2 is the second example, after Proposition I-1.4, that illustrates the connections existing between the concepts of continuous lattices and domains on the one hand, and of general topology, in particular the theory of locally compact spaces, on the other. In this particular example for X locally compact and well-filtered, domains (in the form of continuous semilattices) have a particular place that cannot be taken by continuous lattices, since $Q(X)$ in general is not even a bounded complete domain. The subject of continuous semilattices will be pursued further in Chapter IV in the context of duality. □

We shall see another class of domains and continuous semilattices related to topology at the end of the first section of Chapter II.

An example of a domain dating back to the origins of domain theory is the union $\mathbb{N}_\perp = \mathbb{N} \cup \{\perp\}$ where the natural numbers form an antichain and $\perp$ is an extra element below each $n \in \mathbb{N}$. In this example, the set of natural numbers can be replaced by any set M. The resulting domains $M_\perp$ are called the *flat domains*. These flat domains are of a particular kind: they are bounded complete domains. To be continuous lattices they just need a top element.

Proposition I-1.25. *Let D be a semilattice and let $D^1 = D \cup \{1\}$ denote the semilattice obtained from D by adjoining an identity* 1 *(whether D has one or not). (Cf.* O-2.12.*) We have the following conclusions.*

(i) *The following statements are equivalent.*

(1) *D is a bounded complete domain, i.e., complete continuous semilattice* (O-2.2(iv)).
(2) *D^1 is a continuous lattice satisfying* $1 \ll 1$.
(3) *D is a domain and every finite set with an upper bound has a least upper bound.*
(4) *D is a domain and every subset with an upper bound has a least upper bound.*

(ii) *If L is a continuous lattice with* $1 \ll 1$*, that is,* 1 *is isolated from below, then $D = L \setminus \{1\}$ is a bounded complete domain.*

(iii) *If L is a continuous lattice and $X = \uparrow X$ is an upper set such that $L \setminus X$ is closed under directed sups, then $L \setminus X$ is a bounded complete domain.*

Proof: (1) $\Rightarrow$ (2): Assume (1). By O-2.12, D^1 is a complete lattice. Since D is closed under directed sups by (1) we have $1 \ll 1$. In particular, this shows $\twoheaddownarrow 1 = \downarrow 1 = D^1$. Hence for all $x \in D^1$ the set $\twoheaddownarrow x$ is directed and $\sup \twoheaddownarrow x = x$.

(2) $\Rightarrow$ (3): This immediate.

(3) $\Rightarrow$ (4): Bottom is the least upper bound of the empty set, which is a finite set for which any element of D is an upper bound. Now let B be a nonempty set with an upper bound. For each finite subset $F \subseteq B$, by (3), $x_F = \sup F$ exists. Then $D = \{x_F \colon F \subseteq B \text{ finite}\}$ is a directed set which has a sup since D is a domain, hence a **dcpo**. Since $\sup D = \sup B$, (4) follows.

(4) $\Rightarrow$ (1): Assume (4). We conclude that D has a bottom $0 = \sup \emptyset$. Let $\emptyset \neq X \subseteq D$. Let B be the set of all lower bounds of X. This set is not empty as it contains $0 = \perp$. Since $X \neq \emptyset$ there is an $x \in X$. For $b \in B$ we have $b \leq x$. Thus B has an upper bound. Then $\sup B$ exists by (4). Since $\sup B \leq x$ for all $x \in X$ we have $\sup B \in B$ and thus $\sup B = \max B = \inf X$.

(ii) If L is a continuous lattice with $1 \ll 1$, set $D = L\backslash\{1\}$. Then $L = D^1$ and the assertion follows from (i).

(iii) Under the circumstances of (iii) set $D = L\backslash X$. Since the subset D is a lower set and is closed under directed sups, it is a complete semilatte. It is a continuous semilattice since the way-below relation of D is the way-below relation of L induced on D. □

Proposition I-1.25 shows that bounded complete domains are very close to continuous lattices. Indeed, if a bounded complete domain fails to be a continuous lattice, then it only fails by lacking an isolated top. Therefore, bounded complete domains, as a rule, can be fitted into the theory of continuous lattices by adjoining an identity. Or at least this is true when only *one* semilattice is being considered at a time, and then it does not matter whether the "ideal" elements like 1 are "really there" or considered fictions (as with $\pm\infty$ in the reals). When *several* semilattices are being combined (in, say, a direct product), then the inclusion of ideal elements makes a big difference to the outcome by their entering into combination with other elements (recall, for example, LSC(X), where the values $\pm\infty$ can be utilized in a function quite often). The exact way extra elements can occur sometimes subtly affects the properties of a construction; one example is contained in Exercise I-1.31 below. There are in addition important applications in which the adjunction of an identity is simply *unnatural*, such as Example O-2.7(10). We review this matter in the light of the definitions of this section by discussing a more general situation in I-1.32 below.

In the past, in some circles, bounded complete domains used to be called "domains". According to our terminology introduced in Definition I-1.6, a domain is a continuous **dcpo**; our class of domains therefore is much larger.

Example I-1.26. (Subcontinua of a continuum) A space X is called *locally connected* if each point has a basis of open connected neighborhoods. A *continuum* is a compact connected Hausdorff space. For a space X denote by *Cont*(X) the set of nonempty subcontinua of X ordered by reverse inclusion.

Example I-1.26.1. (The interval domain) Consider the set of subintervals of the unit interval [0, 1] ordered by reverse inclusion. This is a bounded complete domain with bottom element [0, 1] and the singleton intervals the maximal points. The relation $\ll$ is characterized by $[a, b] \ll [c, d]$ if and only if $[c, d]$ is contained in the open interval $]a, b[$. If we consider computational algorithms for computing some real number in the interval, then we may envision the intervals $[a, b]$, $a < b$, as states of partial knowledge of the numbers that arise along the way, with smaller intervals representing higher stages of knowledge. Thus the ordering of reverse inclusion can be viewed as an information ordering.

Example I-1.26.2. (Subarcs of the unit circle) Consider all subcontinua of the unit circle $\mathbb{S}^1$ consisting of all complex numbers of modulus 1. These consist of singleton sets, subarcs, and S^1 itself, which we order by reverse inclusion. If one picks two disjoint subarcs, then there exist two minimal subcontinua containing both of them, not one. Thus this example is not a semilattice, hence not a bounded complete domain. It is readily verified to be an L-domain. We establish this in much greater generality in the next proposition.

Proposition I-1.26.3. *For a locally connected continuum X, the poset $Cont(X)$ of subcontinua is an L-domain. We have $A \ll B$ in $Cont(X)$ iff $B \subseteq \mathrm{int}(A)$.*

Proof: Given any nonempty collection $\mathcal{C}$ of subcontinua containing a continuum A, the infimum of $\mathcal{C}$ is $(\bigcup \mathcal{C})^-$ and the supremum of $\mathcal{C}$ is the component of $\bigcap \mathcal{C}$ containing A. Thus each principal ideal is a complete lattice. Given any open set U containing $A \in Cont(X)$, for each $x \in A$ pick V_x open and connected such that $(V_x)^-$ is compact and contained in U. Finitely many of the V_x cover A, and the union of their closures is a subcontinuum contained in U and containing A in its interior. From this point the argument that $Cont(X)$ is a domain parallels other arguments involving compact subsets of locally compact spaces (cf. e.g. I-1.23). □

Exercises

We begin by mentioning a few general results on the interpolation property.

Exercise I-1.27. Let L be a poset and $\prec$ an auxiliary relation. Let $s_\prec\colon L \to \mathrm{Low}\, L$ be the function in M corresponding to $\prec$ according to I-1.12. Prove the following.

(i) The following statements are equivalent:

(1) $\prec$ satisfies the strong interpolation property (SI);
(2) no lower set of the form $s_\prec(x)$ has a maximal element with respect to $\prec$ unless $x \prec x$.

(ii) Also the following statements are equivalent:

(3) $\prec$ satisfies the interpolation property (INT);
(4) each lower set of the form $I = s_\prec(x)$ has the property that for every $x \in I$ there is a $y \in I$ with $x \prec y$. □

(iii) If $\prec$ is approximating, then (INT) implies (SI). □

Exercise I-1.28. Let L be a poset and $\prec$ an auxiliary relation on L. Define a binary relation $\prec\!\bullet$ as follows:

$x \prec\!\bullet y$ iff there is a $\prec$-chain C such that $x, y \in C$, $x \prec y$, and $\prec$ restricted to C satisfies (SI).

Show that we have:

(i) $\prec\!\bullet$ is an auxiliary relation which satisfies (SI);
(ii) the given relation $\prec$ satisfies (SI) iff $\prec\, = \,\prec\!\bullet$;
(iii) moreover, $\prec\!\bullet$ is the largest auxiliary relation contained in $\prec$ satisfying (SI).
(iv) Show that analogs of the preceding results hold if L is only a set and the words "auxiliary relation" are replaced by "transitive relation" throughout. □

Exercise I-1.29. Let L be a poset and $\prec$ an auxiliary relation on L. Define a binary relation $\prec^{\sup}$ as follows:

$x \prec^{\sup} y$ iff there exist systems of elements $\{x_i : i \in I\}$ and $\{y_i : i \in I\}$ such that $x_i \prec y_i$ for all $i \in I$ and $x = \sup x_i$ and $y = \sup y_i$.

We can call $\prec^{\sup}$ the *sup closure* of $\prec$. Show that we have:

(i) if L is meet continuous, then $\prec^{\sup}$ is an auxiliary relation;
(ii) $\prec$ is approximating iff $\prec^{\sup} = \leq$. □

Exercise I-1.30. Prove the following.

(i) Every closed interval of a bounded complete domain is a continuous lattice.
(ii) The following example is a domain with a closed interval which is not continuous. Take monotone increasing sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converging to their sups x, y, z, respectively. The poset P has the additional order defining relations that $z_n < y_n < x_n$ for each n, $z < y < x$, and $z < x_1$. Then P is a domain, but $[z, x]$ is not because y is not a supremum of elements way below it in $[z, x]$.
(iii) If L is a domain and $M \subseteq L$ satisfies

$$M = \uparrow M = \twoheaduparrow M = \bigcup\{\twoheaduparrow u : u \in M\}$$

then M is also a domain. □

Exercise I-1.31. Let L and M be two posets. Define the following five kinds of "disjoint" sums:

(1) **(Disjoint sum)** $L \sqcup M$, the disjoint union of L and M (with the obvious partial ordering: elements $x \in L$ and $y \in M$ are incomparable);
(2) **(Coalesced sum)** $L \oplus M$, the disjoint sum $L \sqcup M$ with the bottom elements identified, if they have them;
(3) **(Separated sum)** $L + M = (L \sqcup M)_\perp$, that is, the disjoint sum with a *new* bottom element adjoined;
(4) $L +_1 M = (L \oplus M)^1$;
(5) $L +_2 M = L \oplus M$ with the 1 elements identified.

If L and M are **dcpo**s, show that all of these "sums" are **dcpo**s, too. If L and M are domains, show that $L \sqcup M$, $L \oplus M$, $L + M$ and $L +_1 M$ are also domains, but $L +_2 M$ need not be continuous. (The case $L = M = [0, 1]$ gives a counterexample. Some pictures will help!) Which of the properties of being a continuous lattice, a bounded complete domain, an L-domain are preserved by these constructions? □

Exercise I-1.32. Let X be a topological space and Y a set. Let L be the set of partial functions with values in Y defined on an *open* subset $\operatorname{dom} f \subseteq X$. Consider on L the partial order given by

$$f \le g \Leftrightarrow \operatorname{dom} f \subseteq \operatorname{dom} g \text{ and } f = g \mid \operatorname{dom} f.$$

Prove the following.

(i) The nowhere-defined function $\emptyset \colon \emptyset \to Y$ is the bottom of L; every nonempty subset has an inf and every directed subset has a sup.
(ii) If $\mathcal{O}(X)$ is a continuous lattice, then L is a bounded complete domain. (Recall that $\mathcal{O}(X)$ is a continuous lattice if X is locally compact by I-1.7(5).)
(iii) The semilattice L has no top if Y has more than one element. □

Exercise I-1.33. Let $X = [0, 1]$ with the usual Hausdorff topology and define $f, g \in \mathrm{LSC}(X, X)$ by $f(x) = x/2$, $g(x) = x$. Show that $f(x) \ll g(x)$ for all x, but $f \ll g$ fails. So in I-1.22.1, condition (5) does not imply (1)–(4). □

Exercise I-1.34. If the space X is regular and $\mathcal{O}(X)$ is a continuous lattice, show that X is locally compact and, hence, locally compact Hausdorff if T_0.

Hint. Consider $x \in X$ and any open neighborhood V of x. Since $\mathcal{O}(X)$ is a continuous lattice, there is a $U \in \mathcal{O}(X)$ such that $x \in U \ll V$ by definition. Since X is regular, there is an open neighborhood W of x with $W^- \subseteq U$. Now let $\mathcal{W} = \{W_j\colon j \in J\}$ be an open cover of W^-. Then $\mathcal{W} \cup \{X \backslash W^-\}$ is an open cover of V. Since $U \ll V$, a finite subcover thereof covers U. But then a finite subcover of $\mathcal{W}$ covers W^-. Hence, W^- is compact. □

We have seen in Proposition I-1.8 that every continuous lattice is meet continuous; the converse is incorrect. Indeed not even a complete Boolean algebra need be a continuous lattice. It is therefore useful to have suffcient conditions which will ensure that meet continuity implies continuity.

Exercise I-1.35. Show that a meet continuous lattice L is continuous if it satisfies at least one of the following two conditions.

(a) L does not contain a free semilattice with infinitely many generators.
(b) L does not contain an infinite antichain.

Hint. (a) This is established by proving the following theorem.

Theorem. *Let L be a meet continuous lattice and let x and y be elements such that $\twoheaddownarrow x \subseteq \downarrow y$ but $x \notin \downarrow y$. Then $(\downarrow x \backslash \downarrow y) \cup \{1\}$ contains a free semilattice with infinitely many generators.*

Proof: Suppose that F is a free semilattice such that $F\backslash\{1\}$ is contained in $\downarrow x \backslash \downarrow y$ and that X is the generating set in $F\backslash\{1\}$. Suppose that X is finite. Then $z = \inf F$ is not in $\twoheaddownarrow x$. Hence, since L is meet continuous, there is a directed set D with $\sup D = x$ but $D \cap \uparrow z = \emptyset$. For each $f \in F$ we know that $\sup fD = f$. Since F is finite, we find an element $b \in D$ such that $b \leq d \in D$ implies $fd \neq gd$ for all $f \neq g$ in F. Since $\sup zD = z$ and $z \not\leq y$, there is some $p \in D$ with $b \leq p$ such that $pz \not\leq y$. Then $F \cup \{p\}$ is a free set, and the semilattice F' generated by this set is contained in $\downarrow x \backslash \downarrow y$. By induction we obtain a countably generated infinite free semilattice contained in $\downarrow x \backslash \downarrow y$. □

(b) The generating set of a free semilattice is an antichain. □

The notion of an L-domain is a bit tricky in a way not completely obvious from the definition. We are going to illustrate this in the following exercises.

Exercise I-1.36. Let $L = \{\perp, a_1, a_2, b_1, b_2\}$ be the five element poset with a least element $\perp$ and the order relations $a_i < b_j$ for all $i, j = 1, 2$. But a_1 and a_2 are incomparable and likewise b_1, b_2. Show that both of the principal ideals $\downarrow b_1$ and $\downarrow b_2$ are four element lattices: b_1 is the sup of a_1 and a_2 in $\downarrow b_1$ and b_2 is the sup of a_1 and a_2 in $\downarrow b_2$, although a_1 and a_2 do not have a sup in L, and thus L is an L-domain but not a lattice. □

Let us introduce the following notation. Let a_1, a_2 be elements of a poset L with an upper bound b. We denote by $a_1 \vee_b a_2$ the least upper bound of a_1 and a_2

in the poset $\downarrow b$, whenever it exists. The preceding example shows that $a_1 \vee_b a_2$ may depend on the specific upper bound b.

Exercise I-1.37. Let L be a **dcpo** with the property that $a_1 \vee_b a_2$ exists, whenever a_1 and a_2 are bounded above by b. (This applies to every L-domain.) Prove the following.

(i) If b_1, b_2 have an upper bound c in L, then $a_1 \vee_{b_1} a_2 = a_1 \vee_{b_2} a_2$.
(ii) If $a_1 \ll b$ and $a_2 \ll b$, then $a_1 \vee_b a_2 \ll b$. (Compare I-1.2(iii).)

Hint. (i) As $b_1 \leq c$, one sees that $a_1 \vee_{b_1} a_2 = a_1 \vee_c a_2$ and similarly for b_2.

(ii) Let D be a directed set with $b \leq \sup D$. From the hypotheses one finds an element $d \in D$ such that $a_1 \leq d$ and $a_2 \leq d$. As $\sup D$ is an upper bound of b and d, using (i) one obtains $a_1 \vee_b a_2 = a_1 \vee_d a_2 \leq d$. □

Exercise I-1.38. Show that the following conditions are equivalent in a **dcpo** M:

(i) $\downarrow x$ is a continuous lattice for each $x \in M$;
(ii) M is a domain in which every nonempty set that is bounded above has a greatest lower bound;
(iii) M is an L-domain. □

Hint. Proposition I-1.5(ii) is useful in proving that M is a domain in the implication (i) $\Rightarrow$ (ii). □

Old notes

Continuous lattices were introduced by Dana Scott, who discovered the idea as a generalization of algebraic lattices in the fall of 1969. He presented the first coherent picture at the Dalhousie Category Theory Conference in 1971; this presentation appears in [Scott, 1972a] and is the first source on continuous lattices in the accessible literature. In an expository paper Scott [Scott, 1973] details his motivation for the invention of continuous lattices; there he repeats his original definition and says: “Such lattices I call continuous lattices, and their mathematical theory is highly satisfactory.” What he meant was that everything seemed to fall neatly in place, and considering the extensive mathematical development of the theory of continuous lattices since 1974, this claim is a modest understatement.

Whether the choice of nomenclature was an entirely wise one will remain contested in some quarters. The name is reminiscent of that of von Neumann’s continuous geometries (a certain type of lattice), which – strictly speaking – have nothing to do with continuous lattices in our sense. Nevertheless, the passage from the more discrete (more precisely: zero dimensional) algebraic lattices to the continuous lattices has a certain analogy to the passage from a discrete range of dimensions to a continuous dimension function. Furthermore, Scott had in mind the circumstance that continuous functions (in what we call in Chapter II the Scott topology) on a continuous lattice are well behaved and exist in profusion; in particular, the lattice operations are continuous. There is considerable sense to calling a lattice "continuous" just when its lattice operations are continuous, but actually the known classes of such lattices are very wide (e.g. meet continuous lattices). Continuous lattices in the sense of this monograph have the advantages of being restricted enough to have a good theory, general enough to capture important examples, and natural enough that we can argue that the class ought to be singled out for many different reasons.

It is noteworthy that the concept of a continuous lattice (and that of a semilattice) was rediscovered independently by other authors working in other areas. Rather extensive work was carried out by Yu. L. Ershov (see the bibliography), part of which was independent of and essentially contemporary with Scott's work and part of which answered many questions Scott left open. The notion of what he called f-spaces arose as a useful tool in his study in the early 1970s of computable functionals of finite type. Their completions, which turn out to be bounded complete algebraic domains equipped with the Scott topology in the case that the f-space has a least element, were important for extending in a natural fashion the domain of definition of these computable functionals to noncomputable arguments. The notion of an A-space appeared as his attempt to find a general context for both f-spaces and the continuous lattices introduced by Scott. See the "New Notes" at the end of Section III-4 for further comments, particularly concerning A-spaces.

In 1973–74, Karl H. Hofmann and Albert Stralka studied the algebraic (i.e., lattice theoretical) foundations of a class of compact topological semilattices known to workers in the field of compact semigroups as *Lawson semilattices*; they found that continuous lattices and *compact* Lawson semilattices were one and the same thing, although at the time they were not aware of Scott's article and, thus, did not phrase their results in this language. Their paper appeared as [Hofmann and Stralka, 1976], and the not too succinct title was soon contracted for everyday use to ATLAS (using the initial letters of "Algebraic Theory of Lawson Semilattices").

In 1975 Alan Day identified the monadic algebras associated with the so-called *filter monad* (in the same sense as compact spaces emerge as monadic algebras of the ultrafilter monad) and found them to be continuous lattices (cf. [Day, 1975]). Independently, Oswald Wyler also studied these algebras in 1975 and described and discussed them thoroughly: only towards the end of

1976, due to a hint by John Isbell, was it discovered that indeed Wyler's algebras were precisely the continuous lattices (see [Wyler, 1981b]).

The definition given here, strictly speaking, is not the principal one given by Scott, nor does it correspond explicitly to either the characterization given in ATLAS or that given by Day and Wyler. The first published version of this definition is to be found in the note [Lea, 1976a] (though see the remark in the next paragraph). The point is that the way-below relation of I-1.1 is not Scott's auxiliary relation $\prec$ derived from a topology (which we will consider extensively in Chapter II). It is immediately seen from Scott's definition that $x \prec y$ implies $x \ll y$ in any complete lattice. Scott *defines* a continuous lattice to be a complete lattice in which his relation $\prec$ is approximating. By I-1.16 this means that *on continuous lattices*, Scott's relation and the way-below relation agree. On complete lattices they are different in general (see II-1.33).

The way-below relation had been implicitly introduced by Hofmann and Stralka by saying that "*x is relatively compact under y*" iff $x \ll y$ (*op. cit.*, p. 27) and they also introduced the notation $\ll$ (*op. cit.*, p. 42). Isbell used the terminology "*x is compact in y*" in his paper on meet continuous lattices, [Isbell, 1975b]; his identification of Scott's relation with the way-below relation on meet continuous lattices is not convincing, and in fact to our knowledge it is not known in general on what class of complete lattices the two agree. That is one good reason for our employing the more understandable relation here in the definition. Scott, however, had originally defined continuous lattices in this more lattice theoretic way and refers to the characterization in his paper (*op. cit.*, p. 110). In writing up his paper he chose the other definition in order to emphasize the topological simplicity of the notion, and he did not feel the need to consider whether the two definitions of the way-below relation agreed in a wider context.

Propositions I-1.15 and I-1.16 first appeared in the *Compendium.* The strong interpolation property has been recognized as useful for some time, although I-1.20 and a complete proof had not been published before the *Compendium.* But J. Isbell had recognized the interpolation property in his paper (*op. cit.*).

The axioms of auxiliary orders with the interpolation property were formulated by M. B. Smyth (cf. Scott [scs 4] and [Smyth, 1978]) for sup semilattices; later (see Definition III-4.16 and subsequent exercises) we will describe his motivation. The equivalence between auxiliary relations and functions $L \to \operatorname{Id} L$ discussed in I-1.12 ff. was anticipated by Smyth (*op. cit.*) and by Gierz, Hofmann, Keimel and Mislove [scs 12]; in this report the details of Exercise I-1.27 are introduced and elaborated.

The literature contains several forerunners of the way-below relation in the context of the representation theory of lattices; see in particular [Raney, 1953], notably p. 520; [Papert, 1959], notably pp. 174 ff.; also [Bruns, 1962a], [Bruns, 1962b], notably Part II, pp. 4 ff.

The fact that locally compact spaces X give rise to a continuous lattice $\mathcal{O}(X)$ was known to [Day and Kelly, 1970], Proposition 5.

The observation that the lattice L of closed two-sided ideals in a C^*-algebra is a continuous lattice and its proof via Pedersen's ideals are from Hofmann [SCS 31]. There are at least two alternative proofs, one requiring the entire spectral theory of C^*-algebras and the results of Chapter V, another requiring an observation due to J. M. G. Fell [Fell, 1962] to the effect that L is a compact subsemilattice of a product of unit-interval semilattices together with the equivalence of continuous lattices with compact Lawson semilattices (see Chapter VI).

Example I-1.22 of classical lower semicontinuous functions and its discussion are from Hofmann [SCS 17].

The concept of a continuous semilattice (I-1.6) is from Lawson [scs 30], which theory of duality we will explore in Chapter IV. Continuous posets are also discussed in [Markowsky, 1976], R.-E. Hoffmann [Hoffmann, 1979b], and [Wilson, D1978].

The example in I-1.30(ii) is due to Marcel Erné.

The results in Exercise I-1.35 are due to Lawson [SCS 30]; however, the suffcient condition (b) was independently rediscovered by Heiko Bauer with an independent (and more complicated) proof in [SCS 45].

The fact that all continuous lattices are meet continuous (I-1.8) was in principle known in Scott [1972a, p. 106, Prop. 2.7], as Isbell points out [Isbell, 1975b], p. 46.

New notes

Continuous lattices were the first class of domains introduced by Dana Scott with a view toward applications in theoretical computer science. However, it soon became apparent to researchers in that area that more general classes of domains were needed. Both to broaden the class of domains and to make the theory more accessible, Scott [Scott, 1982a] introduced "information systems", structures for which the ideal completions gave the class of bounded complete algebraic domains (= complete algebraic semilattices). The bounded complete algebraic domains have often been called "Scott domains" in the literature. In some sense bounded complete domains constitute a minimal generalization, since they all arise from continuous lattices with the element 1 compact by removing the 1, i.e., they are continuous lattices without a top (see I-1.25). Other investigations led to other important classes of domains. For example the category of L-domains turned out to be a cartesian closed category of domains, and was shown by Jung [Jung, B1989] to be one of two maximal cartesian closed categories of pointed domains (domains with a 0). A prominent topological example of a continuous semilattice which is not in general a continuous lattice is the sup semilattice $Q(X)$ of compact saturated subsets of a locally compact sober space; this topic emerges in this section for the first time in I-1.24 but will be a recurrent theme later.

I-2 Products, Substructures and Quotients

In this section we investigate the construction of new domains, new continuous semilattices, new continuous lattices etc. from known ones by means of

forming direct products with the product (= pointwise) order,
taking subsets closed under appropriate operations,
taking images under maps preserving appropriate operations.

The reader should be warned that some obvious conjectures turn out to be wrong:

an infinite product of domains need not be a domain;
a subset of a domain closed under directed sups need not be a domain;
the image of a domain under a map preserving directed sups need not be a domain.

Counterexamples will be given in the exercises. Thus, we have to restrict our attention to more special situations.

Products, projection, kernel and closure operators on domains

For products, the situation is quite simple:

Proposition I-2.1. (i) *The direct product $L_1 \times \cdots \times L_n$ of finitely many domains $L_1, \ldots, L_n$ is a domain. For elements $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $L_1 \times \cdots \times L_n$ the way-below relation is given by*

$$x \ll y \quad \text{iff} \quad x_i \ll y_i \text{ for all } i = 1, \ldots, n.$$

(ii) *If L_i, $i \in I$, is a family of domains with least element 0, then the direct product $\prod_{i\in I} L_i$ is also a domain. For elements $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ in $\prod_{i\in I} L_i$ the way-below relation is given by*

$$x \ll y \text{ iff } x_i \ll y_i \text{ for all } i \in I \text{ and } x_i = 0 \text{ for all but finitely many } i \in I.$$

For continuous semilattices, continuous lattices, bounded complete domains and L-domains, the same statements hold. (Note that continuous lattices and bounded complete domains always have a least element.)

Proof: Note that the properties of being a **dcpo**, a semilattice, a complete semilattice, a complete lattice, etc., are preserved under direct products, as sups and infs are formed pointwise. As (i) is a special case, we only have to prove (ii). Let us first show that the characterization of the way-below relation holds in any product of **dcpo**s L_i with 0.

Suppose first that $x \ll y$. For every finite set $F \subseteq I$ define y^F to be the element of $\prod_{i \in I} L_i$ with $y^F_i = y_i$ for $i \in F$ and $y^F_i = 0$ for $i \notin F$. The family of the y^F is directed and its supremum is y. As $x \ll y$, there is some finite subset $F \subseteq I$ such that $x \leq y^F$, whence $x_i = 0$ for all $i \notin F$. In order to show that $x_i \ll y_i$ for all $i \in I$, fix i and consider any directed set D in L_i such that $y_i \leq \sup D$. To every $d \in D$ we associate the element $\bar{d} \in \prod_{i \in I} L_i$ defined by $\bar{d}_i = d$ and $\bar{d}_j = y_j$ for all $j \neq i$. The family $(\bar{d})_{d \in D}$ is directed and $y \leq \sup_{d \in D} \bar{d}$. As $x \ll y$, there is some $d \in D$ such that $x \leq \bar{d}$, whence $x_i \leq d$.

For the converse, suppose that $x_i \ll y_i$ for all $i \in I$ and that there is a finite set $F \subseteq I$ such that $x_i = 0$ for all $i \notin F$. Let D be any directed set in $\prod_{i \in I} L_i$ such that $y \leq \sup D$. Then $y_i \leq \sup_{d \in D} d_i$ for every $i \in I$. As $x_i \ll y_i$, for every $i \in I$, there is a $d^i \in D$ such that $x_i \leq d^i_i$ (= ith component of d^i). As D is directed, there is a $d \in D$ such that $d^i \leq d$ for all $i \in F$. Thus $x_i \leq d_i$ for all $i \in F$. As $x_i = 0$ for all $i \notin F$, we conclude that $x \leq d$. This proves that $x \ll y$.

If all the L_i are continuous, the set of all $x \ll y$ is easily seen to be directed and to have y as its supremum by the above characterization of the way-below relation. Hence, $\prod_{i \in I} L_i$ is continuous, too. □

If L is a domain, neither subsets closed under sups of directed sets nor images of L under a map preserving sups of directed sets have to be domains. It is remarkable that the image of a continuous poset under a projection operator (O-2.5, O-3.8) preserving directed sups is again continuous:

Theorem I-2.2. *Let L be a continuous poset and p: $L \to L$ a projection preserving sups of directed sets. Then the image $p(L)$ with the order induced from L is a continuous poset, too. For $x, y \in p(L)$, we have*

$$x \ll_{p(L)} y \textit{ iff there is an element } u \in L \textit{ such that } x \leq p(u) \textit{ and } u \ll_L y.$$

Proof: From O-2.5 we know that $p(L)$ is closed in L under passing to directed sups. Let $y \in p(L)$ be given. As L is continuous, the set $\twoheaddownarrow_L y = \{u \in L : u \ll_L y\}$ is directed and $\sup \twoheaddownarrow_L y = y$. As p preserves directed sups, the set

$p(\,\Downarrow_L y)$ is directed and $\sup\ p(\,\Downarrow_L y) = p(y)$. As $y \in p(L)$, we have $p(y) = y$. Thus, for the continuity of $p(L)$, it suffices to prove that $p(u) \ll_{p(L)} y$ whenever $u \ll_L y$. For this, let u be an element of L such that $u \ll_L y$. Consider any directed subset $D \subseteq p(L)$ such that $y \le \sup D$. As $u \ll_L y$, we find a $d \in D$ such that $u \le d$. Then $p(u) \le p(d) = d$ by the monotonicity and idempotency of p. This shows that $p(u) \ll_{p(L)} y$. For the second part of the claim, let $x, y \in p(L)$ be such that $x \ll_{p(L)} y$. As $y = \sup p(\Downarrow_L y)$ by the above, there is a $u \in L$ with $u \ll_L y$ such that $x \le p(u)$. The converse has already been shown in the first part of the proof. □

As the various completeness properties are preserved under projections (O-2.5), we immediately have

Corollary I-2.3. *Let M be the image of a projection $p\colon L \to L$ preserving sups of directed sets. If L is a domain, an L-domain, a bounded complete domain, a continuous semilattice, a continuous lattice, respectively, the same is true for the image M.*

As kernel and closure operators are particular kinds of projections (see O-3.8), the previous preservation results hold for images under kernel and closure operators provided that they preserve sups of directed sets. The characterization of the way-below relation on the image can be simplified:

Remark. *For a kernel operator k and a closure operator c on a continuous poset L both preserving sups of directed sets the following hold.*

(i) *For all $x, y \in k(L)$, one has $x \ll_{k(L)} y$ iff $x \ll_L y$.*
(ii) *For all $x, y \in L$, one has $x \ll_L y \Rightarrow c(x) \ll_{c(L)} c(y)$.* □

For a closure operator $c\colon L \to L$, the image $c(L)$ is closed in L for infs and the co-restriction $c^\circ\colon L \to c(L)$ preserves arbitrary sups (to the extent that they exist) by O-3.12. This does not imply that c as a function from L into L preserves arbitrary sups or sups of directed sets. But we can say the following.

Lemma I-2.4. *A closure operator $c\colon L \to L$ on a* **dcpo** *L preserves sups of directed sets if and only if its image $c(L)$ is closed in L with respect to sups of directed sets.*

Proof: Assume first that $c(L)$ is closed in L for sups of directed sets. Let D be a directed subset of L. As c is order preserving, $\sup_L c(D) \le c(\sup_L D)$. As the co-restriction $c^\circ\colon L \to c(L)$ preserves sups, $c(\sup_L D) = \sup_{c(L)} c(D)$. If we assume that $c(L)$ is closed in L for sups of directed sets, $\sup_{c(L)} c(D) = \sup_L c(D)$ and we conclude that $\sup_L c(D) = c(\sup_L D)$, that is, c preserves sups of directed sets. The converse follows from O-2.5. □

We have seen that images of closure operators are characterized by the property of being closure systems (O-3.13), that is, subsets M of L such that, for every element $x \in L$, the set of upper bounds of x in M has a least element. By the lemma above, the bijective correspondance betweeen closure operators and closure systems induces a bijection between closure operators preserving directed sups and closure systems closed for directed sups. From I-2.3 and I-2.4 we may conclude that the following holds.

Corollary I-2.5. *If M is a closure system closed under sups of directed sets in a domain L, then M is a domain for the induced order.* □

Let us summarize the consequences of the previous results for continuous lattices and bounded complete domains: subalgebras and homomorphic images of bounded complete domains and of continuous lattices are bounded complete domains and continuous lattices, respectively. Here subalgebras should be understood as images under closure operators and homomorphic images as images under kernel operators, both preserving directed sups.

Theorem I-2.6. (i) *A subset M of a bounded complete domain that is closed under infs of nonempty subsets and under directed sups is a bounded complete domain. If M has a top element, it is a continuous lattice.*

(ii) *If M is the image of a bounded complete domain, respectively a continuous lattice L under a map preserving infs of nonempty subsets and sups of directed sets, then M is a bounded complete domain, respectively a continuous lattice.*

Proof: For (i) we take a bounded complete domain L and a subset M closed under infs of nonempty sets and directed sups. Adjoining a new top element yields a continuous lattice L^1 (see I-1.25), and $M \cup \{1\}$ is closed in L^1 for arbitrary infs and directed sups, i.e., $M \cup \{1\}$ is a closure system on L^1 closed under directed sups. Hence, $M \cup \{1\}$ is a continuous lattice by I-2.5. As the top element 1 is isolated, M is a bounded complete domain (see I-1.25).

For (ii), let L be a bounded complete domain and $g\colon L \to M$ a surjection preserving infs of nonempty subsets and sups of directed sets. Then g has a lower adoint $d\colon M \to L$ (O-3.4) which is injective (O-3.7) and preserves sups (to the extent that they exist, O-3.5). Thus, $k = dg$ is a kernel operator preserving sups of directed sets. Hence M, which is order isomorphic to the image of k, is a bounded complete domain by I-2.3. If L is a continuous lattice, then it has a top element, hence M has a top element, too, and consequently is a continuous lattice. □

Equational theory of continuous lattices

The property (MC) defining meet continuity in semilattices and also the characterizations of meet continuity in O-4.2, in particular properties (7) and (8), are of an equational character. We have seen that continuous lattices are meet continuous (I-1.8). Now we are going to characterize continuous lattices in a similar, although more technical, vein. The equational description of continuous lattices enables us to discern clearly what kind of homomorphisms should be considered between continuous lattices, namely those functions preserving arbitrary infs and directed sups. As in universal algebra, one may deduce directly from the equational description that products, subalgebras and homomorphic images of continuous lattices are again continuous lattices, results that we have already obtained above in greater generality.

The type of equation we will use is a form of the infinite distributive law which we shall call the *directed distributive law*. This law is stronger than meet continuity but weaker than the law of *complete distributivity* that we will discuss in detail later (see I-2.8). It is well known that the lattice of all subsets of a set is completely distributive (with respect to arbitrary unions and intersections). Note that in I-1.10 above we have shown that every continuous lattice L is the image of a lattice of sets – namely, Id L – by a map preserving infs and sups. But Id L is closed under set theoretical intersection and directed union; hence, any equations these operations satisfy on set theoretical grounds will transfer to L. What we now exhibit is a basis for these equations.

Theorem I-2.7. *For a complete semilattice L, the following conditions are equivalent.*

(1) *L is a bounded complete domain.*
(2) *Let $\{x_{j,k}\colon j \in J, k \in K(j)\}$ be a nonempty family of elements in L such that $\{x_{j,k}\colon k \in K(j)\}$ is directed for all $j \in J$. Then the following identity holds:*

$$\text{(DD)} \qquad \bigwedge_{j\in J} \bigvee_{k\in K(j)}^{\uparrow} x_{j,k} = \bigvee_{f\in M}^{\uparrow} \bigwedge_{j\in J} x_{j,f(j)},$$

where M is the set of all choice functions $f\colon J \to \bigcup_{i\in J} K(j)$ with $f(j) \in K(j)$ for all $j \in J$.

If L is a complete lattice, then these conditions are also equivalent to

(3) *Let $\{x_{j,k}\colon (j,k) \in J \times K\}$ be any family in L. Then the following identity*

holds:

(DD*) $$\bigwedge_{j\in J}\bigvee_{k\in K} x_{j,k} = \bigvee_{f\in N}^{\uparrow}\bigwedge_{j\in J}\bigvee_{k\in f(j)} x_{j,k},$$

where N is the set of all functions $f\colon J\to$ fin K into the finite subsets of K.

Remark. Note that all the sups in (DD) are directed sups. Strictly speaking, (DD) is *not* an equation because its validity requires the hypothesis that certain sets are directed. The point of formulating (DD*) is that it, on the other hand, *is* a pure lattice equation in (infinite) infs and sups. Note, too, that we could write $\leq$ in place of $=$ in (DD) and (DD*) since the reverse inequality holds in any complete lattice.

Proof of theorem: (1) implies (2): For convenience let *lhs* denote the left hand side of (DD) and *rhs* the right hand side. It is obvious that in any complete semilattice $lhs \geq rhs$. Assuming that L is continuous, all we have to do to prove the reverse inequality is to show that whenever $t \ll lhs$, then $t \leq rhs$.

Suppose then that $t \ll lhs$; we conclude that $t \ll \bigvee^{\uparrow}_{k\in K(j)} x_{j,k}$ for all $j \in J$. By the definition of $\ll$, we can therefore choose a $g(j) \in K(j)$ with $t \leq x_{j,g(j)}$ for all $j \in J$. But then we see that $t \leq \bigwedge_{j\in J} x_{j,g(j)}$, and so $t \leq rhs$ must follow.

(2) implies (1): We are going to establish the approximation axiom (A) of the original Definition I-1.6: namely, $x = \sup\{u \in L : u \ll x\}$. Indeed let $x \in L$ be a given element, and let J be the set of all directed subsets j of L with $\sup j \geq x$. For each $j \in J$ let $K(j) = j$, in other words j is indexing itself. Further, consider the family of elements $x_{j,k} = k$ for $j \in J$ and $k \in K(j)$. The hypothesis of (2) is thus satisfied.

Suppose $f \in M$ and let $t = \bigwedge_{j\in J} x_{j,f(j)} = \bigwedge_{j\in J} f(j)$. Then we claim that $t \ll x$. Indeed if D is a directed set with $x \leq \sup D$, then $t \leq f(D) \in D$ – because $D \in J$ and t is defined to make this so.

Looking now at (DD), we see that $x = lhs$, because $\{x\} \in J$. But the equation $x = rhs$ implies (A_1) in view of what we checked in the last paragraph. Hence, L is continuous.

We have (2) iff (3) provided that L is a complete lattice: We note first that condition (2) is equivalent to the following variant:

(2′) for any family $\{x_{j,k}\colon (j,k) \in J \times K\}$ in L such that $\{x_{j,k}\colon k \in K\}$ is directed for all $j \in J$, the following identity holds:

(DD_0) $$\bigwedge_{j\in J}\bigvee^{\uparrow}_{k\in K} x_{j,k} = \bigvee^{\uparrow}_{f\in K^J}\bigwedge_{j\in J} x_{j,f(j)}.$$

Indeed (DD) obviously implies (DD$_0$); conversely suppose that (DD$_0$) is satisfied and that $\{x_{j,k}\colon j \in J, k \in K(j)\}$ is given as in (2). Then define the set $K = \bigcup_{j\in J} K(j)$ and for $(j,k) \in J \times K$ define

$$y_{j,k} = \begin{cases} x_{j,k}, & \text{if } k \in K(j), \\ 0, & \text{otherwise.} \end{cases}$$

Then $\bigvee^{\uparrow}_{k\in K(j)} x_{j,k} \bigvee^{\uparrow}_{k\in K} y_{j,k}$ for all $j \in J$, and

$$\bigvee\nolimits^{\uparrow}_{f\in M} \bigwedge\nolimits_{j\in J} x_{j,f(j)} = \bigvee\nolimits^{\uparrow}_{g\in K^J} \bigwedge\nolimits_{j\in J} y_{j,g(j)},$$

and thus the desired equation (DD) for the $x_{j,k}$ follows from (DD$_0$) for the $y_{j,k}$.

The equivalence of (DD$_0$) and (DD*) is easily seen via O-1.5, and we leave the details to the reader. □

We remark that an alternative proof that (2) implies (1) in I-2.7 can be given utilizing the characterization of continuity of I-1.10(3). For this purpose let J be the set of all ideals j of L with $x \le \sup j$. Then (DD) shows that the ideal I that is the intersection of all the ideals in J also satisfies $x \le \sup I$. This is just what we need to apply I-1.10(3).

Before we move on to applications, we observe how the equations in Theorem I-2.7 relate to traditional lattice theoretical concepts. In O-4.3 we noted that meet continuity generalized the property of being a frame (that is, of satisfying a general distributive law O-3.16(3)). In a similar vein we observe that I-2.7(2) generalizes the most restrictive type of general distributivity, which we put on record here:

Definition I-2.8. A lattice is called *completely distributive* iff it is complete and for any family $\{x_{j,k} : j \in J, k \in K(j)\}$ in L the identity

$$\bigwedge_{j\in J} \bigvee_{k\in K(j)} x_{j,k} = \bigvee_{f\in M} \bigwedge_{j\in J} x_{j,f(j)} \tag{CD}$$

holds, where M is the set of choice functions defined on J with values $f(j) \in K(j)$. As usual, we could write $\le$ in place of $=$. □

As the complete distributivity law (CD) immediately implies the directed distributivity law (DD) in I-2.7, we conclude that the following holds.

Corollary I-2.9. *Every completely distributive lattice is continuous.* □

Completely distributive lattices are fairly special; various characterizations are known which do not properly belong to our topic. However, later we will give a characterization theorem for complete distributivity in continuous lattices (see Section VII-2).

Now we exploit Theorem I-2.7. Our first observation is that the algebraic operations occurring in the equational characterization of bounded complete domains and of continuous lattices through equation (DD) are *arbitrary* infs and *directed* sups. If we want to define homomorphisms between continuous lattices at this point, it is then clear that these are the algebraic infinitary operations which should be preserved by such homomorphisms. Therefore we make the following definition.

Definition I-2.10. If S and T are continuous lattices, then a function $g\colon S \to T$ is called a *morphism of continuous lattices*, or, briefly, a *homomorphism*, if it preserves arbitrary infs and directed sups.

A continuous lattice T is called a *homomorphic image* of a continuous lattice S iff there is a surjective homomorphism $g\colon S \to T$.

A subset S of a continuous lattice T is called a *subalgebra* iff the inclusion map $S \to T$ is a homomorphism (that is, iff S is closed in T under the formation of arbitrary infs and directed sups.)

For bounded complete domains, a homomorphism is only required to preserve infs of nonempty subsets (and directed sups), and a subalgebra is only required to be closed for nonempty infs (and directed sups). □

In due time, notably in Chapter IV, we will use more systematically the language of category theory; the present terminology suffices for the purposes at hand, and we are able to formulate some direct consequences of I-2.7. (We have obtained these results in a more general context already in I-2.1 and I-2.6):

Theorem I-2.11. *The class of continuous lattices is closed under the formation of arbitrary products, subalgebras and homomorphic images. Specifically we have the following conclusions:*

(i) *if* $\{L_j\colon j \in J\}$ *is a family of continuous lattices, then the cartesian product* $\prod_{j\in J} L_j$ *is a continuous lattice (relative to the componentwise partial order);*
(ii) *if* L *is a continuous lattice and* S *a subalgebra of* L*, then* S *is a continuous lattice in the induced order;*
(iii) *if* L *is a continuous lattice and* S *a poset and if* $g\colon L \to S$ *is a surjective homomorphism, then* S *is a continuous lattice.*

These statements remain intact if we replace continuous lattices *by* bounded complete domains.

Proof: (i) The product of complete lattices is a complete lattice. Since all operations in the cartesian product are componentwise, then any equation which holds in each factor holds in the product. Thus, I-2.7 proves the claim.

(ii) Firstly, S is a complete lattice, since it is closed in L under infs (O-2.4). If $x_{j,k} \in S$ is a family satisfying the conditions of I-2.7(2), then both sides of the equation (DD) are contained in S (since S is closed under infs and directed sups). Since the equation holds in L, it then holds in S, and thus S is a continuous lattice by I-2.7.

(iii) Let $X \subseteq S$ and set $Y = g^{-1}(X)$. Since g is surjective, $X = g(Y)$. As L is a complete lattice, $y = \inf Y$ exists in L. Since g preserves infs, then $\inf g(Y)$ exists in S and

$$g(y) = g(\inf Y) = \inf g(Y) = \inf X.$$

Hence, S is a complete lattice by O-2.2. Now let $x_{j,k}$ be a family in S satisfying the conditions of I-2.7(2). Let d be the lower adjoint of g, which exists by O-3.5. If we set $y_{j,k} = d(x_{j,k})$, then the family $y_{j,k}$ satisfies the hypotheses of I-2.7(2) in L, since d is order preserving. Thus the $y_{j,k}$ satisfy equation (DD) of I-2.7. Now we apply g to both sides of this equation and obtain equation (DD) for the family $x_{j,k}$ (since g commutes with all $\bigwedge$ and all $\bigvee^{\uparrow}$ and satisfies $gd = 1$ by O-3.7). It follows that S is a continuous lattice by O-2.3.

To bounded complete domains one adds an isolated top element and one uses the above results. $\square$

We are now going to make the correspondence between closure and kernel operators on the one side and subalgebras and quotients on the other side more explicit.

We turn to closure operators first. We recall from O-3.13 and the definition preceding it that a closure system in a complete lattice L is a subset which is closed under arbitrary infs. In O-3.14 together with I-2.4 it is established that there is a bijective correspondence between the closure operators of L which preserve directed sups and the closure systems which are closed under directed sups. If L is a continuous lattice, then these closure systems are precisely the subalgebras by I-2.10. This gives immediately the following classification of closure operators preserving directed sups by subalgebras.

Proposition I-2.12. *Let L be a continuous lattice. Then the assignment $c \mapsto c(L)$, which associates with a closure operator $c\colon L \to L$ of L its image, induces a bijection from the set of all closure operators of L which preserve directed sups onto the set of all subalgebras of L.* $\square$

The next step is to consider kernel operators. They are related to kernels of homomorphisms.

Proposition I-2.13. *Let $g\colon L \to T$ be a homomorphism between bounded complete domains, or between continuous lattices. Then the kernel $R = \{(x, y) \in L \times L : g(x) = g(y)\} = (g \times g)^{-1}(\Delta)$, where Δ is the diagonal in $T \times T$, is a subalgebra of $L \times L$.*

Proof: As $g\colon L \to T$ is a homomorphism, also $g \times g\colon L \times L \to T \times T$ is a homomorphism. As the diagonal Δ is a subalgebra in $T \times T$, its inverse image under $g \times g$ is a subalgebra, too. □

The kernel of a homomorphism should be called a congruence. We will see below that every congruence also gives rise to a quotient structure.

Definition I-2.14. If L is a bounded complete domain, or a continuous lattice, then a subset R of $L \times L$ is called a *congruence relation* on L, if it is an equivalence relation on L and a subalgebra of $L \times L$. □

We now can give a more complete picture of congruences on bounded complete domains and continuous lattices and how they relate to kernel operators (while, as we recall from I-2.12, closure operators relate to subalgebras).

Theorem I-2.15. *Let L be a bounded complete domain, respectively a continuous lattice, and $R \subseteq L \times L$ an equivalence relation. Let L/R be the quotient set. Then the following conditions are equivalent.*

(1) *R is a congruence relation on L.*
(2) *There is a kernel operator k on L preserving sups of directed sets such that $R = (k \times k)^{-1}(\Delta)$ where Δ is the diagonal of $L \times L$.*
(3) *L/R is a bounded complete domain respectively a continuous lattice, in such a way that the quotient map $g\colon L \to L/R$ is a homomorphism.*
Moreover, L/R is isomorphic to $k(L)$.

Proof: (1) implies (2): Let $R(x) = \{y \in L\colon xRy\}$, the equivalence class of x. Define the map $k\colon L \to L$ by $k(x) = \inf R(x)$. As R is closed for infs of nonempty sets, $k(x) \in R(x)$ and $k(x) \leq x$. In fact, $k(x)$ is the smallest element of the congruence class $R(x)$. But this implies that $k(k(x)) = k(x)$ for all $x \in L$, i.e., k is idempotent. Suppose next that $x \leq y$, then $xy = x$. Since R is a semilattice congruence, $R(x)R(y) \subseteq R(xy)$; whence

$$k(x) = k(xy) \leq k(x)k(y) \leq k(y).$$

This shows that k is monotone. Thus k is a kernel operator.

Let D be directed in L and set $d = \sup D$. Let $d^* = \sup k(D) \leq k(d)$. We claim that $k(d) \leq d^*$. Indeed for all $x \in D$ we have $(x, k(x)) \in R$ and the set $\{(x, k(x)) : x \in D\}$ is directed. Since R is closed with respect to sups of directed sets we conclude $(d, d^*) = \sup_{x \in D}(x, k(x)) \in R$. Thus $d^* \in R(d)$, whence $k(d) = \min R(d) \leq d^*$ as was claimed. Thus k preserves sups of directed sets.

Finally, if $(x, y) \in R$, then $k(x) = k(y)$, and vice versa, whence we see that $(k \times k)^{-1}(\Delta) = R$.

(2) implies (3): The co-restriction $k^\circ : L \to k(L)$ factors canonically through the quotient map $g : L \to L/R$ with a bijection $f : L/R \to k(L)$. This means that $k^\circ = fg$ with $f(R(x)) = k(x)$. By I-2.3, $k(L)$ is a bounded complete domain, and by O-3.12(iii), k° preserves existing infs. Since k preserves directed sups and $k(L)$ is sup closed in L by O-3.12(i), then k° preserves sups of directed sets. If we transport the order structure of $k(L)$ to L/R via f^{-1}, then L/R is a bounded complete domain such that g preserves all nonempty infs and sups of directed sets.

(3) implies (1): Immediate from I-2.13. □

Evidently, the homomorphic images of a bounded complete domain and its quotients are practically the same thing on account of the canonical factorization theorem for homomorphisms.

The following is now a parallel to I-2.12.

Corollary I-2.16. *Let L be a bounded complete domain. Then the rule $k \mapsto (k \times k)^{-1}(\Delta)$, where Δ is the diagonal of $L \times L$, associates with a kernel operator its kernel congruence and induces a bijection from the set of all kernel operators of L which preserve directed sups onto the set of all congruences of L.* □

The last result of this section is a surprising one: we obtain the continuity of the image of a continuous lattice under a projection without supposing that the projection preserves sups of directed sets. This result is a counterpart to I-2.2.

Proposition I-2.17. *If L is a bounded complete domain and $p : L \to L$ is a projection operator preserving nonempty infs, then $p(L)$ is a bounded complete domain. Moreover $p(L)$ is closed in L under nonempty infs.*

Proof: By O-3.11(ii) and (iii), the set $L_c = \{x \in L : x \leq p(x)\}$ is closed under existing sups and infs. Thus L_c is a bounded complete domain by I-2.6(i). The map $p_c : L_c \to L_c$ induced by p is a closure operator which preserves existing infs, since p preserves infs and L_c is inf closed.

The co-restriction $p_c^\circ : L_c \to \operatorname{im} p_c$ then preserves nonempty infs since $\operatorname{im} p_c$ is inf closed in L_c by O-3.12(i), and it preserves sups by O-3.12(iii). Thus I-2.6(ii) applies to show that $\operatorname{im} p_c$ is a bounded complete domain. But im $p_c = p(L)$ by O-3.11(i). By O-3.12(i), $p(L) = \text{im}\, p_c$ is closed in L_c under infs, and L_c is closed in L under infs, thus $p(L)$ is closed in L under infs. □

Exercises

In the first exercises we present a necessary and suffcient condition for a product of domains to be a domain and we propose counterexamples to preservation properties of domains that one might conjecture.

Exercise I-2.18. Prove the following.

(i) If L_i, $i \in I$, is a family of domains, where all but finitely many of the L_i have a least element, then $\prod_{i \in I} L_i$ is a domain.
(ii) The half open unit interval]0, 1] is a domain, but a direct product of infinitely many copies of]0, 1] is not continuous.
(iii) Let A be the three element poset obtained from a two element antichain by adjoining a top element. The direct product of infinitely many copies of A is not a domain.
(iv) The converse of (i) does not hold: every antichain is a domain without least element, and an infinite product of antichains is an antichain, hence a domain.
(v) The direct product $\prod_{i \in I} L_i$ of a family of domains is again a domain if and only if all but a finite number of the domains L_i are discrete unions of domains with least elements. (By a discrete union of posets M_j, $j \in J$, we mean a disjoint union of the M_j such that no elements in different components M_j are comparable.) □

Exercise I-2.19. Find an example of a subset M of a domain L which is closed for sups of directed sets and finite infs but not a domain.

Hint. Consider the rationals $\mathbb{Q}$ with the usual topology induced from $\mathbb{R}$. Let $L = 2^{\mathbb{Q}}$ be the set of all subsets and M the set of all open subsets of $\mathbb{Q}$. □

Exercise I-2.20. Find examples of a domain L and of a **dcpo** M which is not a domain but where there exists a surjective map $c: L \to M$ preserving sups of directed sets.

Hint. Let $L = 2^{\mathbb{Q}}$, let M be the set of all closed subsets of $\mathbb{Q}$ and $c(A)$ the closure of an arbitrary subset of $\mathbb{Q}$. □

In the following exercises we construct some further examples of continuous lattices by utilizing kernel operators. We will, however, re-prove these results in Chapter II with different methods.

Exercise I-2.21. Prove the following.

(i) Let S be any poset and T a continuous lattice. Let $(S \to T)$ denote the poset of all order preserving maps $f\colon S \to T$ with the pointwise order. Then $(S \to T)$ is a continuous lattice.

(ii) Let S be a continuous poset. For $f \in (S \to T)$ define $k(f)\colon S \to T$ by

$$k(f)(s) = \sup f(\twoheaddownarrow s).$$

Then $k(f) \in (S \to T)$ and $k\colon (S \to T) \to (S \to T)$ is a kernel operator. Further, k preserves directed sups. In particular, im k is a continuous lattice (I-2.3).

(iii) Let S be a continuous poset. $[S \to T]$ denotes the poset of all those maps $f \in (S \to T)$ with $f = k(f)$. Each such f preserves directed sups. In fact, for $g \in (S \to T)$, the function $k(g)$ is the greatest function below g which preserves directed sups.

Hint. (i) Note that $(S \to T)$ is a subalgebra of T^S, which is a continuous lattice by I-2.11(ii).

(ii) Use the interpolation property of $\ll$ for the proof of $k^2 = k$.

(iii) Let D be a directed set with $\sup D = s$ in S. Clearly we have then $\sup f(D) \le f(s)$. The converse requires a frequently used trick: Take an arbitrary element

$$x \ll f(s) = k(f)(s) = \sup f(\twoheaddownarrow s)$$

and show $x \le \sup f(D)$; since x is arbitrary, $f(s) \le \sup f(D)$ will follow. To accomplish the claim, use I-1.1 to find $s^* \ll s$ with $x \le f(s^*)$. Since $s^* \ll \sup D$, there is a $d \in D$ with $s^* \ll d$ (I-1.9). Thus

$$x \le f(s^*) \le f(d) = k(f)(d) \le \sup f(D). \qquad \square$$

We summarize the main outcome of this discussion in the following exercise and indicate some applications.

Exercise I-2.22. Prove the following.

(i) If S is a continuous poset and T a continuous lattice, then the poset $[S \to T]$ of *all* functions $S \to T$ preserving directed sups is a continuous lattice. (Note that it is closed in T^S under arbitrary sups but not under infs in general!)

(ii) Since $\mathbb{R}$ is a continuous poset and $\mathbb{I} = [0, 1]$ a continuous lattice, as are their opposites, then $[\mathbb{R}^{\mathrm{op}} \to \mathbb{I}^{\mathrm{op}}]$ is a continuous lattice. This is the space of all nondecreasing, upper semicontinuous functions $F\colon \mathbb{R} \to \mathbb{I}$. Those functions $F \in [\mathbb{R} \to \mathbb{I}]$ which are such that $F(x) \to 0$ as $x \to -\infty$ and $F(x) \to 1$ as $x \to +\infty$ are precisely the distribution functions of real random variables. They form a sublattice $P(\mathbb{R})$ which is closed under pointwise infs (which are the sups in the function space) of sets with lower bounds (upper bounds in the function space). But there are no elements $F, G \in P(\mathbb{R})$ with $F \ll G$. Thus $P(\mathbb{R})$ fails totally be a continuous poset.

(iii) By contrast, however, the set $P(\mathbb{I}) \subseteq [\mathbb{I}^{\mathrm{op}} \to \mathbb{I}^{\mathrm{op}}]$ consisting of all $F\colon \mathbb{I} \to \mathbb{I}$ with $F(1) = 1$ is the set of distribution functions of probability measures on the unit interval, and it is closed under arbitrary pointwise infs (sups in the function space); the inf in the function space of a family is the upper semicontinuous envelope of the pointwise sup, which is also the inf in the continuous lattice $[\mathbb{I}^{\mathrm{op}} \to \mathbb{I}^{\mathrm{op}}]$. Hence, $P(\mathbb{I})$ is a continuous lattice. □

One could call $P(\mathbb{I})$ *the random unit interval*. Notice, however, that the partial order on the corresponding probability measures is the order induced from $[\mathbb{I}^{\mathrm{op}} \to \mathbb{I}^{\mathrm{op}}]$; indeed if X and Y are random variables on $\mathbb{I}$ with distribution functions F_X and F_Y , then $F_X \le F_Y$ means that X is likely to take larger values than Y: we should write $Y \le X$.

Definition I-2.23. If S and T are L-domains, then a function $g\colon S \to T$ is called a *homomorphism of L-domains* if it preserves infs of nonempty sets bounded above and directed sups. The stipulations of Definition I-2.10 are extended accordingly from bounded complete domains to L-domains. □

Exercise I-2.24. Prove that Theorems I-2.11, I-2.15 and Corollary I-2.16 persist for L-domains. □

Despite these successes, there remain some questions:

Problem. Is there an equational characterization of continuous semilattices? □

Problem. Do any of the conclusions of Theorems I-2.11 and I-2.15 and of Proposition I-2.13 and Corollary I-2.16 persist for continuous semilattices? □

The following exercise discusses some material which is related to the context of large distributive laws and auxiliary relations.

Exercise I-2.25. Let L be a complete lattice. Let $\mathcal{M}$ be a set of subsets of L

satisfying the following conditions:

(a_1) if $\mathcal{A} \subseteq \mathcal{M}$ and $\{\sup X \colon X \in \mathcal{A}\} \in \mathcal{M}$, then $\bigcup \mathcal{A} \in \mathcal{M}$;
(a_2) if $\mathcal{A} \subseteq \mathcal{M}$, then $\{\inf f(\mathcal{A}) \colon f \in \mathrm{Sel}(\mathcal{A})\} \in \mathcal{M}$,

where $\mathrm{Sel}(\mathcal{A}) = \{f \colon f$ is a selection function $\mathcal{A} \to \bigcup \mathcal{A}\}$.

G. Bruns [Bruns, 1962a] calls such a set *distributively closed*. One calls a lattice $\mathcal{M}$-*distributive* if for any set $\mathcal{A} \subseteq \mathcal{M}$

$$\inf\{\sup X \colon X \in \mathcal{A}\} = \sup\{\inf f(\mathcal{A}) \colon f \in \mathrm{Sel}(\mathcal{A})\}.$$

If $\mathcal{M} = 2^L$, then we retrieve complete distributivity. If $\mathcal{M}$ is the smallest distributively closed subset containing all finite sets, then one obtains the (F)-distributivity of S. Papert [Papert, 1959]. The set $\mathcal{D}$ of all directed subsets of L is *not* distributively closed; however, the distributive law (DD) in I-2.7 would be called $\mathcal{D}$-distributivity in our present context.

Define two relations on L as follows (the first follows S. Papert, *op. cit.*, p. 174, the second G. Bruns, *op. cit.*, p. 4):

$x \swarrow y$ iff for all $X \in \mathcal{M}$ with $y = \sup X$ one has $x \leq a$ for some $a \in X$;
$x \dashv y$ iff for all $X \in \mathcal{M}$ with $y \leq \sup X$ one has $x \leq a$ for some $a \in X$.

(i) Then $\swarrow$ and $\dashv$ are auxiliary relations satisfying the interpolation property (INT).
(ii) Both relations are approximating in any $\mathcal{M}$-distributive lattice.
(iii) For $x \in L$ there is a set $X(x) \in \mathcal{M}$ such that $\downarrow X(x) = \{y \colon y \prec x\}$ with $\prec$ equal to $\swarrow$ or $\dashv$, respectively. □

Old notes

The characterization of continuous lattices through equations as expressed in Theorem I-2.7 is due to Alan Day [Day, 1975]. His proof is different from ours. He obtained the equational characterization in the course of identifying the class of continuous lattices and the class of homomorphisms of continuous lattices (I-2.10) as a category which is equivalent to the category of algebras of the filter monad over sets (and the open filter monad over T_0 spaces). Of course one may interpret this setup as identifying free continuous lattices over a set, as was pointed out by D. Scott [SCS 15]. A. Day explored the issue further in [SCS 18]. Independently of these developments, O. Wyler also identified continuous lattices as algebras of filter monads some time in 1975–76.

The distributive law (DD) of I-2.7 is of a type considered in a systematic fashion by G. Bruns [Bruns, 1961] and [Bruns, 1962a]. However, the case of continuous lattices is not subsumed in Bruns' work, and before him S. Papert used certain (almost) auxiliary relations which are approximating and satisfy the interpolation property (see I-2.25). In the case of completely distributive lattices, the relation $\dashv$ of I-2.25 was introduced by G. N. Raney [Raney, 1953]. He showed that it was approximating iff the lattice was completely distributive, and he observed that it satisfied the interpolation property in this case. With these tools he showed that a complete lattice is completely distributive iff it can be embedded into a product of complete chains under preservation of arbitrary sups and infs. (See IV-3.31 and IV-3.32, and for more on complete distributivity see I-3.16 ff.)

That the closure properties of the class of continuous lattices which are expressed in Theorem I-2.11 would be important from the viewpoint of universal algebra was remarked by Scott [SCS 15]. The fact that quotients of continuous lattices are continuous is probably the hardest of the closure properties; the stability under formation of products and subalgebras could be derived relatively easily directly from our Definition I-1.6; this is not the case with the quotients. This had also been the harder part of the theory of compact semilattices with small subsemilattices [Lawson, 1969], which we know today is equivalent to the theory of continuous lattices.

Scott had emphasized all along the significance of projections (retracts) on continuous lattices, notably those which preserve directed sups. Theorem I-2.2 is from [SCS 15]. The useful statements I-2.12 and I-2.16 concerning closure and kernel operators were published in the *Compendium* for the first time.

The result in Exercise I-2.21 is a mild generalization of a principal result of [Scott, 1972a], p. 112, Theorem 3.3. A systematic treatment will follow in Chapter II. The random unit interval was discussed by Hofmann and Liukkonen in [SCS 16].

New notes

In the *Compendium* this section was entirely devoted to continuous lattices and their equational characterization and the associated universal algebra. By utilizing the machinery of retractions an analogous theory can be developed for the wider class of domains, and this has been the program of the earlier part of the section. The latter part resumes the old theme.

There are several variations of the notion of "continuity" of a poset obtained by replacing directed sets by other types of subsets. Taking arbitrary subsets instead of directed ones leads to the way-way-below relation and to completely distributive lattices (see I-2.25). Both of these cases are covered by the notion of Z continuous posets, where Z is some class of subsets of posets. This generalization has been studied by several authors and its theory surveyed in [Erné, 1999].

I-3 Irreducible Elements

In a semilattice an element is irreducible if it is not the meet of two larger elements. These elements play an important role in lattice theory, notably for distributive lattices, where they are exactly the prime elements; they are at the basis of all of the spectral theory and of the representation theorems of distributive lattices. Irreducible elements exist abundantly in all finite lattices, and it is one of the important features of continuous lattices that this property persists.

Open filters and irreducible elements

We first introduce some necessary machinery to prepare the way for the development of the theory of irreducible elements. This early material will be better motivated in Chapter II when topologies are introduced.

Definition I-3.1. Let L be a **dcpo**. An upper set $U = \uparrow U \subseteq L$ will be called *open* iff for each directed set D in L the relation $\sup D \in U$ implies $D \cap U \neq \emptyset$. Filters are particular upper sets (O-1.3), and we will make use of *open filters*, that is, filters that are open in the sense just defined. The set of all open filters of L is denoted by

$$\mathrm{OFilt}(L) = \{F\colon F \text{ is an open filter of } L\}. \qquad \square$$

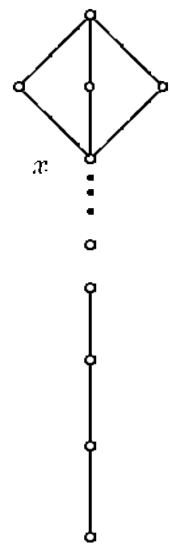
Example I-3.2. *Assume that we have a descending sequence*

$$\cdots \ll y_n \ll \cdots \ll y_2 \ll y_1$$

in a **dcpo** *L. Then* $U = \bigcup_{n=1}^{\infty} \uparrow y_n$ *is an open filter.*

Proof: Since the subset U is an ascending union of principal filters it is a filter. Now let D be a directed subset of L such that $\sup D \in U$. Then there is a natural number n such that $y_n \leq \sup D$. Since $y_{n+1} \ll y_n$, we find a $d \in D$ with $y_{n+1} \leq d$, and thus $d \in \uparrow y_{n+1} \subseteq U$. Hence U is open. $\square$

Using the interpolation property (I-1.9) we see immediately that in a domain, all sets $\twoheaduparrow x$ are open. One must be careful to note that even in a continuous lattice the sets $\Uparrow x$ are not generally filters: the picture shows a continuous lattice in which $\Uparrow x$ is not a filter. Nevertheless, there are still enough open filters:



Proposition I-3.3. *In a domain the following hold.*

(i) *If $x \ll y$, then there is an open filter U with $y \in U \subseteq \Uparrow x$.*
(ii) *If $y \not\leq z$, then there is an open filter U containing y but not z.* □

Proof: (i) By the interpolation property (I-1.9) we construct inductively a decreasing sequence of elements y_n with

$$x \ll \cdots \ll y_n \ll y_{n-1} \ll \cdots \ll y_1 = y.$$

Set $U = \bigcup\{\uparrow y_n \colon n = 1, 2, \ldots\}$. Clearly, $y \in U$ and $U \subseteq \Uparrow x$. By Example I-3.2, U is an open filter.

(ii) If $y \not\leq z$, then there is an x such that $x \ll y$, but $x \not\leq z$. If we choose an open filter as in (i), it will have the desired property. □

The significance of the openness of an upper set U is that its complement must have maximal elements:

Lemma I-3.4. *Let U be an open upper set in a* **dcpo**. *Then for any $x \in L \backslash U$ there is an $m \in L \backslash U$ with $x \leq m$ and m maximal in $L \backslash U$.*

Proof: By the Hausdorff Maximality Principle, there exists a maximal chain $C \subseteq L \backslash U$ containing x. Let $m = \sup C$. If $m \in U$, then $U \cap C \neq \emptyset$ by I-3.1 which contradicts the hypothesis on C. Thus $m \in L \backslash U$. Since C is a maximal chain, not only is $m \in C$ but m is maximal in $L \backslash U$. □

Definition I-3.5. An element p in a poset is called *irreducible* iff p is maximal or $\uparrow p \backslash \{p\}$ is a filter. The set of all irreducible elements is written $\mathrm{IRR}\, L$. □

We note at once that in a semilattice, an element p is irreducible iff the relation $p = xy$ always implies $x = p$ or $y = p$. Here they are also rightfully called *meet irreducible*. In a sup semilattice *join irreducible* elements are defined dually.

Proposition I-3.6. *Let L be a poset and $p \in L$. Assume that F is a filter in L such that $\uparrow p \cap F$ is also a filter. (In a semilattice, this is automatic.) If p is maximal in $L \backslash F$, then p is irreducible.*

Proof: Since p is maximal in $L \backslash F$, we see that $\uparrow p \backslash \{p\} = \uparrow p \cap F$. Since this is a filter, we conclude that p is irreducible. □

The next theorem is important because it guarantees an abundance of irreducibles in any continuous lattice or semilattice.

Theorem I-3.7. *Suppose that x and y are elements of a continuous semilattice with $y \not\leq x$. Then there is an irreducible element p with $x \leq p$ and $y \not\leq p$.*

Proof: By Proposition I-3.3(ii), there is an open filter U with $y \in U$ and $x \notin U$. By I-3.4 and I-3.6, there is an irreducible element p with $x \leq p \notin U$. Since $y \in U$, we then have $y \not\leq p$. □

This result may be rephrased in another convenient fashion (see I-3.10).

Definition I-3.8. A subset X of a poset L is said to be *order generating* iff $x = \inf(\uparrow x \cap X)$ for all $x \in L$. □

Note that it is also true that $\inf(\uparrow x \cap X) = \inf(\uparrow x \cap (X \backslash \{1\}))$.

Proposition I-3.9. *For a subset X of any poset L, the following statements are equivalent:*

(1) *X is order generating;*
(2) *every element of L can be written as an inf of a subset of X;*

(3) *L is the smallest subset containing X closed under arbitrary infs;*
(4) *whenever $y \not\leq x$, then there is a $p \in X$ with $x \leq p$ but $y \not\leq p$.*

Proof: (1) implies (2): Immediate from the definition.

(2) iff (3): A standard lattice theoretical argument.

(2) implies (4): Let $y \not\leq x$. By assumption $x = \inf P$ for some $P \subseteq X$. But then $y \not\leq p$ for some $p \in P$, and the conclusion follows.

(4) implies (1): Clearly, x is a lower bound of the set $\uparrow x \cap X$. Let y be any lower bound of $\uparrow x \cap X$. We claim that $y \leq x$. Suppose not, that is, $y \not\leq x$. Then by (4) there is a $p \in \uparrow x \cap X$ with $y \not\leq p$ which contradicts the assumption that y is a lower bound of $\uparrow x \cap X$. This proves that $x = \inf(\uparrow x \cap X)$. □

Corollary I-3.10. *In a continuous semilattice L, the set* $\mathrm{IRR}\, L \backslash \{1\}$ *of non-identity irreducibles is order generating.* □

It is noteworthy that Corollary I-3.10 holds in the environment of a continuous semilattice, where the existence of arbitrary infs is not guaranteed in general.

Distributivity and prime elements

At this point we specialize our discussion to *distributive semilattices*. This requires, firstly, that we recall the appropriate definition of distributivity for semilattices which agrees with the concept of distributivity (see O-2.6) for lattices, and, secondly, that we introduce a new type of element, called prime; every prime element is irreducible, but the converse fails without distributivity.

Definition I-3.11. (i) A semilattice S is said to be *distributive* if $ab \leq x$ implies the existence of elements c, d with $a \leq c$, $b \leq d$ and $x = cd$.

(ii) An element p in a poset L is called *prime* iff $p = 1$ or $L \backslash {\downarrow} p$ is a filter. An element is *co-prime* iff it is a prime of L^{op}. The sets of prime and co-prime elements are denoted by $\mathrm{PRIME}\, L$ and $\mathrm{COPRIME}\, L$, respectively. □

Note that a lattice is distributive as a semilattice in the sense of the above definition iff it satisfies the usual distributivity law (see Exercise I-3.30):

$$(\forall x, y, z) \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{D}$$

Notice that we allow 1 to be prime – not all authors agree. For the record we note some general facts about primality. We shall write $2 \stackrel{def}{=} \{0, 1\}$ for the two element lattice.

Proposition I-3.12. *Let $p \neq 1$ in a semilattice L. Then the following statements are equivalent:*

(1) *p is prime;*
(2) $(\forall x, y \in L)\ xy \leq p \Rightarrow (x \leq p$ *or* $y \leq p)$*;*
(3) *the function $f\colon L \to 2$ defined by*

$$f(x) = \begin{cases} 0 & \text{if } x \leq p, \\ 1 & \text{otherwise} \end{cases}$$

is a semilattice homomorphism.

If L is a distributive semilattice, then the above are equivalent to

(4) *p is maximal in the complement of an open filter,*
(5) *p is irreducible.*

If L is a Boolean lattice, then these conditions are equivalent to

(6) *p is a co-atom, that is, a maximal nonunit element.*

Proof: The equivalence of (1), (2) and (3) is straightforward. If p is prime, then $L \backslash {\downarrow} p$ is a filter. It is clearly open according to definition I-3.1. Thus (1) implies (4). If p is maximal in the complement of an open filter U, then p is irreducible by I-3.6 and we see that (4) implies (5).

Now assume that L is a distributive semilattice and let p be irreducible. Assume $xy \leq p$. By I-3.11(ii) there are elements $x' \geq x$ and $y' \geq y$ such that $p = x'y'$. Since p is irreducible, $x' = p$ or $y' = p$. In the first case $x \leq p$, in the second, $y \leq p$. Thus, (5) implies (2).

We leave the equivalence of (1) and (6) in a Boolean lattice as an exercise. $\square$

From I-3.10 and the above we deduce

Corollary I-3.13. *In a distributive continuous semilattice L, the set* $\mathrm{PRIME}\, L \backslash \{1\}$ *of nonidentity primes is order generating.* $\square$

A well-studied class of complete lattices is that of topologies $\mathcal{O}(X)$ of topological spaces X. We discussed the semilattice $Q(X)$ of compact saturated subsets of a topological space ordered by reverse inclusion $\supseteq$ (see I-1.24). What are the prime elements in these cases? For a subset A of a topological space we called its *saturation*

$$\operatorname{sat} A \stackrel{def}{=} \bigcap \{U \in \mathcal{O}(X)\colon\ A \subseteq U\}$$

and we said that A is *saturated* iff $\operatorname{sat} A = A$ (see O-5.3).

Example I-3.14. Let X be a topological T_0 space.

(i) Consider the following statements for a proper open subset U of X and $A = X \backslash U$.

(0) $(\exists a \in A)\ A = \{a\}^-$.
(1) $U \in \mathrm{PRIME}(\mathcal{O}(X))$.
(2) $U \in \mathrm{IRR}(\mathcal{O}(X))$.
(3) The set $\{V \in \mathcal{O}(X)\colon\ U \subset V, U \neq V\}$ is a filter in L.
(4) A is not the union of any two of its proper closed subsets.

Then (0) $\Rightarrow$ (1) and conditions (1) through (4) are equivalent.

(ii) Consider the following statements for a nonempty compact saturated subset C of X.

(0) $(\exists c \in C)\ C = \mathrm{sat}\{c\}$.
(1) $C \in \mathrm{PRIME}(Q(X))$.
(2) $C \in \mathrm{IRR}(Q(X))$.
(3) The set $\{K \in S\colon\ K \subset C, K \neq C\}$ is a filter in $Q(X)$, i.e., is a $\cup$-semilattice.

Then (0)$\Rightarrow$(1)$\Rightarrow$(2)$\Leftrightarrow$(3), and if X is locally compact, then (0)$\Leftrightarrow$(1). If, in addition, the intersection of two compact saturated sets is compact in X, then all four conditions are equivalent.

Proof: (i) An open set U is irreducible in $\mathcal{O}(X)$ iff it is not the intersection of two properly larger open sets. By de Morgan's Rules, this is equivalent to (4). Since $\mathcal{O}(X)$ is distributive, an element U is prime iff it is irreducible by I-3.12 ((1)$\Leftrightarrow$(5)); by Definition I-3.5, U is irreducible in $\mathcal{O}(X)$ iff (3) holds. Hence (1) through (4) are equivalent. The implication (0)$\Rightarrow$(4) is straightforward: If $A_1 \cup A_2 = A = \{a\}^-$, then $a \in A_1$ or $a \in A_2$. Then $A = \{a\}^- \subseteq A_1 \subseteq A$, i.e., $A = A_1$ or, similarly, $A = A_2$.

(ii) By Definition I-3.5, (2) and (3) are equivalent, and (1)$\Rightarrow$(2) is clear. (0)$\Rightarrow$(1): If sat{c} = $C \subseteq C_1 \cup C_2$ with $C_1, C_2 \in S$, then $c \in C_1$ or $c \in C_2$, whence $C = \mathrm{sat}\{c\} \subseteq \mathrm{sat}\, C_1 = C_1$, or, similarly, $C \subseteq C_2$. By I-3.12 (1) iff (3); this establishes (1).

Now assume that X is locally compact. We will show (1)$\Rightarrow$(0). So let $C \in$ PRIME Q(X). Then the set $F = \{K \in Q(X)\colon\ C \not\subseteq K\}$ is a $\cup$-semilattice. Consequently, the sets of the form $C \backslash K$ with $C \not\subseteq K$ form a filter F of subsets on C. As C is compact, this filter has a cluster point $c \in C$. Now, let U be any open set containing c. By the local compactness, there is a compact saturated neighborhood Q of c contained in U. We have $C \subseteq Q$, whence $C \subseteq U$; suppose indeed, $C \not\subseteq Q$; this implies that $C \backslash Q$ belongs to the filter F, whence $Q \cap (C \backslash Q) \neq \emptyset$, as Q is a neighborhood of the cluster point c of F, a contradiction. Thus C is contained in every open neighborhood of c, hence C is contained in the saturation $\operatorname{sat}\{c\}$. But as $c \in C$ and as C is supposed to be saturated, we conclude that $C = \operatorname{sat}\{c\}$.

Finally, assume that finite intersections of compact sets are compact. Then $Q(X)$ is a distributive lattice. Hence (1) and (2) are equivalent by I-3.12. $\square$

According to O-5.5 a nonempty closed subset $A \subseteq X$ of a topological space is called *irreducible* iff it satisfies the condition in I-3.14(i)(4). Any set of the form $\{a\}^-$ is irreducible. A space is called *sober* (see O-5.6) iff every irreducible closed set has a unique dense point. (We will return to these concepts in a more systematic way in Chapter II and in Chapter V.) From what has been said it is clear that a space X is sober iff the function

$$x \mapsto X \backslash \{x\}^- : X \to (\mathrm{PRIME}\, \mathcal{O}(X)) \backslash \{X\}$$

is bijective. By I-3.14(ii), for a locally compact space X, we have a bijection

$$x \mapsto \operatorname{sat}\{x\} : X \to \mathrm{IRR}\, Q(X) \backslash \{\emptyset\}.$$

Theorem I-3.15. *Let L be a complete lattice. Consider the following conditions:*

(0) *L is a frame;*
(1) *L is a Heyting algebra;*
(2) *L is distributive;*
(3) PRIME *L is order generating.*

Then (0)⇒(1)⇒(2) *and* (3)⇒(0). *If L is a continuous lattice, all four conditions are equivalent.*

Proof: That (0)⇒(1)⇒(2) is trivial from the definitions. That (2)⇒(3) follows from I-3.13. (3)⇒(0): By I-3.12(3) and I-3.9 the maps $f : L \to 2$ determined by primes separate the points of L. If H is the set of all of these maps, then the function $x \mapsto (f(x))_{f \in H} : L \to 2^H$ is injective. As the maps $f \in H$ preserve finite meets and arbitrary joins, the same holds for the function $x \mapsto (f(x))_{f \in H} : L \to 2^H$. But 2^H is a frame, hence so is L. $\square$

The sublattice $[0, 1[^2 \cup \{(1, 1)\}$ of the square $[0, 1]^2$ in the product order is a complete distributive lattice in which $(1, 1)$ is the only prime (cf. O-4.5(1)); thus, I-3.15 may very well fail for a complete distributive lattice. In fact, it may fail for a frame and even for a complete Boolean algebra. Because $L^{\mathrm{op}} \cong L$ for any Boolean algebra, we find nontrivial primes in a Boolean algebra iff we find atoms. The Boolean algebra $\mathcal{O}_{\mathrm{reg}}([0, 1])$ of regular open subsets of the unit interval is an atomless complete Boolean algebra, hence without primes. Hence, I-3.15(3) fails even in very special complete distributive lattices. It is fundamental for the theory of distributive continuous lattices that this converse holds. (Boolean algebras which are continuous lattices are characterized in the next section.) All of this shows that continuity is quite essential in I-3.15.

If L is a chain, then $L = \mathrm{IRR}\, L$, and this property is evidently characteristic for chains. If $L = [0, 1]$, let X be the set of rationals, Y the set of irrationals. Then each of the disjoint sets X and Y is order generating and neither is minimal relative to this property. In general, there is no minimal set which order generates L. We will see later that *algebraic* lattices have a minimal order generating set (I-4.26). We will also show that the closure of IRR L relative to a suitable topology is the smallest *closed* order generating set for this topology.

We saw in I-2.9 that every completely distributive lattice is continuous. Now we can give a sharper description of completely distributive lattices in terms of continuous lattices and primes and co-primes (primes of the dual).

Theorem I-3.16. *Let L be a complete lattice. Then the following statements are equivalent:*

(1) *L is completely distributive;*
(2) *L is distributive and both L and* L^{op} *are continuous lattices;*
(3) *L is continuous and every element is the sup of co-primes.*

Proof: (1) implies (2): One knows that L is completely distributive iff L^{op} is completely distributive. Then I-2.9 yields the desired implication.

(2) implies (3): By I-3.15 applied to L^{op}.

(3) implies (1): We remark first that, in a lattice satisfying (3), every element is the sup of co-primes way below it; because *all* the elements way below it are sups of co-primes. Thus, to verify the equation (CD) of I-2.8, it is suficient to show that every *co-prime* way below the left hand side (*lhs*) is less than or equal to the right side (*rhs*).

Let $x_{j,k}$ be given as in I-2.8 and suppose that p is a co-prime element with $p \ll$ *lhs*. Then $p \ll \bigvee_{k\in K(j)} x_{j,k}$ for all $j \in J$. By I-1.1 we find, for each $j \in J$, a finite set $F \subseteq K(j)$ with $p \leq \bigvee_{k\in F} x_{j,k}$. But, since p is co-prime, there is in fact some element $k \in F$ with $p \leq x_{j,k}$; we denote this k by $f(j)$. By these choices we have found a function $f \in M$ such that $p \leq \bigwedge_{j\in J} x_{j,f(j)}$. This proves that $p \leq$ *rhs*, and the proof is complete. □

It is interesting to remark that from G. N. Raney's classical theory of completely distributive lattices one knows that the equivalent conditions of I-3.16

are also equivalent to embedding L into a direct product of complete chains under preservation of arbitrary sups and infs. (See e.g. [Balbes and Dwinger, B1974], p. 248. We will present this result as Exercises IV-3.31 and IV-3.32.)

We also know that for any lattice L the complete lattice $\mathrm{Id}\,L$ of ideals is distributive iff L is distributive. In view of the importance for continuous lattices of the sup map on ideals, $I \mapsto \sup I : \mathrm{Id}\,L \to L$ (see I-1.10 and compare I-4.17), we need to view $\mathrm{Id}\,L$ in our present context. First a simple remark:

Remark I-3.17. *For an ideal I of a poset L the following statements are equivalent:*

(1) $I \in \mathrm{PRIME}(\mathrm{Id}\,L)$;
(2) $L\backslash I$ *is a filter or is empty.*

If L is a semilattice, then these conditions are equivalent to the following one:

(3) *If $xy \in I$, then $x \in I$ or $y \in I$ for all $x, y \in L$.*

Proof: (2) iff (3): Is obvious for semilattices.

(1) implies (2): Suppose that $s, s^* \in L\backslash I$. Then the principal ideals $\downarrow s$ and $\downarrow s^*$ are not contained in I. By (1), there is an ideal $J \subseteq \downarrow s \cap \downarrow s^*$ such that $J \not\subseteq I$. Hence there is an element $s^{**} \in J\backslash I$ and for this element we have $s^{**} \leq s, s^*$.

(2) implies (1): Let $I_1, I_2 \in \mathrm{Id}\,L$. If neither I_1 nor I_2 is contained in I, then we find elements $x_n \in I_n\backslash I, n = 1, 2$, and by (2) we find a $y \leq x_n, n = 1, 2$, with $y \in L\backslash I$. But then also $y \in I_n, n = 1, 2$, since I_n is an ideal, and so $\downarrow y \in (I_1 \cap I_2)\backslash I$. □

Definition I-3.18. A *prime ideal* in a poset is an ideal satisfying the equivalent conditions (1) and (2) of I-3.17. For a semilattice, a prime ideal is characterized by property (3) in I-3.17 which is more familiar than the previous characterizations. *Prime filters* in posets and sup semilattices are defined dually. □

It will be useful at this point to recall a basic mathematical concept, the well-known notion of a *filter of sets*. We recall from Definition O-1.1 that a filter of a poset is not empty. But we do *not* require that a filter on a poset L must necessarily be a *proper* subset. Sometimes the entire poset L itself is a filter, sometimes it is not: If $L = \{(0, 1), (1, 1), (1, 0)\}$ with the componentwise order, then L itself is not a filter. If L is singleton, it is itself a filter. All of this applies to $\mathrm{OFilt}(L)$ for a **dcpo** L. It is important to keep these things in mind because our terminology collides with the traditional usage of the concept of a filter of sets. Specifically, if $L = 2^X$ is the lattice of all subsets of a nonempty set X, common terminology usually rules out that a filter can contain the empty set. This eliminates L itself from being a filter in *that* terminology; however, the powerset $L = 2^X$ itself is a filter in the terminology of this book.

Remark I-3.19. *Let $\mathcal{F}$ be a filter on a set X. The following statements are equivalent:*

(1) *$\mathcal{F}$ is a proper prime filter in 2^X;*
(2) *for any subset of X, either it or its complement belongs to $\mathcal{F}$;*
(3) *$\mathcal{F}$ is a maximal proper filter in 2^X.* □

Recall, too, that filters satisfying the equivalent conditions of I-3.19 are usually called *ultrafilters*. We record a standard lemma:

Lemma I-3.20. *Let L be a distributive lattice, I an ideal and F a filter in L with $I \cap F = \emptyset$. Then there is a prime ideal P in L with $I \subseteq P$ and $P \cap F = \emptyset$.*

Proof: By Zorn's Lemma we find a maximal ideal P containing I and missing F. We claim that P is a prime ideal. To prove this we let $x, y \notin P$. The ideal generated by P and x as well as the ideal generated by P and y both meet F by the maximality of P. Thus there are some elements $u, v \in P$ with $u \vee x, v \vee y \in F$. Let $w = u \vee v$. Then $w \in P$, since P is an ideal; and we also have $w \vee x, w \vee y \in F$, since F is an upper set. From the fact that L is distributive and F is a filter we conclude that $w \vee (x \wedge y) = (w \vee x) \wedge (w \vee y) \in F$. Since $w \in P$, we cannot have $x \wedge y \in P$, because otherwise we would have $w \vee (x \wedge y) \in P \cap F = \emptyset$; but $x \wedge y \notin P$ is what we had to show. □

Note that if $\mathcal{F}$ is a filter on X and $\mathcal{I}$ is an ideal of 2^X disjoint from $\mathcal{F}$, then by applying I-3.20 to $(2^X)^{\mathrm{op}}$, we conclude that there is an ultrafilter containing the given filter $\mathcal{F}$ but missing the ideal $\mathcal{I}$. In the special case where $\mathcal{I} = \{\emptyset\}$, we obtain the well-known fact that *every* proper filter may be extended to an ultrafilter.

Ultrafilters are frequently useful tools in the theory of domains. We illustrate this with the following two propositions, which also show how compactness theorems in topology often have more general formulations in terms of the way-below relation.

Proposition I-3.21. *Let U and V be open subsets in a topological space X, with $U \subseteq V$. The following statements are equivalent:*

(1) *$U \ll V$ in the lattice $\mathcal{O}(X)$;*
(2) *every proper filter containing U has a cluster point in V;*

(3) *every ultrafilter containing U has a convergence point in V.*

Proof: (1) implies (2): Let $\mathcal{F}$ be a proper filter with $U \in \mathcal{F}$. If no member of V is a cluster point, then for every element $x \in V$, we can find an open set W_x containing x and a set $F_x \in \mathcal{F}$ such that $W_x \cap F_x = \emptyset$. By hypothesis, because the W_x cover V, there are finitely many of them covering U. The finite intersection of the corresponding F_x has an empty intersection with U, which is a contradiction since this finite intersection is in $\mathcal{F}$.

(2) implies (3): Immediate – since by I-3.19(2) cluster points are points of convergence.

(3) implies (1): Let $\mathcal{U}$ be an open cover of V, and assume that U has no finite subcover. Then the family of sets $U \setminus W$ with $W \in \mathcal{U}$ generates a proper filter. Extend this to an ultrafilter $\mathcal{F}$; it is the case that $U \in \mathcal{F}$. By assumption, let $p \in V$ be a point of convergence. Now for some $W \in \mathcal{U}$ we have $p \in W$; but $\mathcal{F}$ converges to p, so $W \in \mathcal{F}$. It then follows that both $U \cap W$ and $U \setminus W$ belong to $\mathcal{F}$, a contradiction. □

The following result is a mild generalization of the classical result known as Alexander's Lemma. The reader should recall the difference between a base and a subbase for a topology.

Proposition I-3.22. (Alexander's Lemma) *Let $\mathcal{B}$ be a collection of open subsets forming a subbase for the topology of a space X, and let U and V be open sets with $U \subseteq V$. Then a necessary and sufficient condition for $U \ll V$ is that every cover of V by members of $\mathcal{B}$ has a finite subcover of U.*

Proof: The necessity is clear. For the sufficiency we use I-3.21(3). Let $\mathcal{F}$ be an ultrafilter with $U \in \mathcal{F}$. Suppose no element of V is a convergence point of $\mathcal{F}$. Then, if $x \in V$, there is a basic open set W_x containing x but not in $\mathcal{F}$. Since W_x is a finite intersection of elements of $\mathcal{B}$, and since $\mathcal{F}$ is a filter, we can assume W_x to be subbasic; that is, $W_x \in \mathcal{B}$. Because $\mathcal{F}$ is an ultrafilter, it follows that $U \setminus W_x \in \mathcal{F}$. Because the W_x cover V, it follows by assumption that finitely many cover U. This means that a finite intersection of the $U \setminus W_x$ is empty, which is impossible because $\mathcal{F}$ is a proper filter. □

The next proposition is essentially an abstract version of I-3.22.

Proposition I-3.23. *Let x and y be elements in a complete distributive lattice. Then $x \ll y$ if and only if for every prime ideal P with $y \ll \sup P$ we have $x \in P$.*

Remark. Compare this statement with I-1.5(i) in order to note that prime ideals suffice here to test the relation $x \ll y$.

Proof of proposition: "Only if" is clear from I-1.5(i). In order to see that the new condition is sufficient, let I be an arbitrary ideal with $y \leq \sup I$; we must show that $x \in I$. Suppose not. Then we set $F = \uparrow x$ and apply I-3.20 to find a prime ideal P with $I \subseteq P$ and $x \notin P$. But $I \subseteq P$ impliees $y \leq \sup I \leq \sup P$. Hence, by our hypothesis, $x \in P$, and this is a contradiction, which proves the claim. □

Pseudoprime elements

Each prime element p of L gives rise to a prime ideal $\downarrow p$ (as is immediate from I-3.17(3)); but, conversely, if P is a prime ideal, then $\sup P$ *need not* be a prime element. If we look at the example following I-3.2, then $x = \sup \twoheaddownarrow x$ is not prime, but $\twoheaddownarrow x$ is a prime ideal. This motivates the formulation of the following definition.

Definition I-3.24. An element p of a semilattice is called *pseudoprime* if $p = \sup P$ for some prime ideal P. The set of pseudoprimes is called $\Psi\mathrm{PRIME}\, L$. □

By the preceding remarks $\mathrm{PRIME}\, L \subseteq \Psi\mathrm{PRIME}\, L$, and the containment is proper in general. In view of I-1.10(2) an element p in a continuous semilattice is pseudoprime iff there is a prime ideal P with $\twoheaddownarrow p \subseteq P \subseteq \downarrow p$. It is clear that for any directed complete semilattice L the sup map *on ideals* maps $\mathrm{PRIME}(\mathrm{Id}\, L)$ onto $\Psi\, \mathrm{PRIME}\, L$. In continuous semilattices we have the following characterization of pseudoprimes.

Proposition I-3.25. *Let L be a semilattice and $1 \neq p \in L$. Consider the following conditions:*

(1) *p is pseudoprime;*
(2) *in any finite collection $x_1, \ldots, x_n \in L$ with $x_1 \cdots x_n \ll p$ there is one of the elements with $x_j \leq p$;*
(3) *the filter generated by $L \backslash \downarrow p$ does not meet $\twoheaddownarrow p$.*

Then (2)⇔(3)*; if L is continuous, then* (1)⇒(2)*, and if L is in addition a distributive lattice, all three conditions are equivalent.*

Proof: Condition (2) says that no finite meet of elements from $L \backslash \downarrow p$ is ever way below p. Therefore (2) and (3) are always equivalent.

(1) implies (2): Let p be pseudoprime and suppose that $x_1 \cdots x_n \ll p$. Let P be a prime ideal with $\sup P = p$. By I-1.10(2) we have $\Downarrow p \subseteq P$, hence $x_1 \cdots x_n \in I$. Since P is prime, there is one x_j with $x_j \in P \subseteq \downarrow p$.

(3) implies (1): Suppose now that L is a distributive lattice. Let F be the filter generated by $L \backslash \downarrow p$; by (3) we have $\Downarrow p \cap F = \emptyset$. By Lemma I-3.20, there is a prime ideal P with $\Downarrow p \subseteq P$ and $P \cap F = \emptyset$. Since $L \backslash \downarrow p \subseteq F$, we have $P \subseteq L \backslash F \subseteq \downarrow p$, whence $p = \sup \Downarrow p \leq \sup P \leq \sup \downarrow p = p$. Thus $p = \sup P$ (where we used continuity of L via I-1.6), and so p is pseudoprime. □

We draw the reader's attention to the fact that condition (2) is a "weak" analog of the definition of a prime, which may be formulated by saying that p is prime if in any collection $x_1, \ldots, x_n \in L$ with $x_1 \cdots x_n \leq p$ there is one of these elements with $x_j \leq p$. The hard implication, involving the Axiom of Choice, is (3) implies (1).

At a later point we will give yet another characterization of pseudoprimes in a continuous lattice, but topology will be needed for that result; it will in effect say that pseudoprimes are exactly those elements which can be approximated by primes in a suitable sense. (See V-2.)

It is a natural question to ask for circumstances under which every pseudoprime is in fact prime. In order to establish a sufficient condition we record the following lemma.

Lemma I-3.26. *In a semilattice L the following conditions are all equivalent for any auxiliary relation $\prec$ (see* I-1.11)*:*

(1) *for all a, x, $y \in L$, the relations $a \prec x$ and $a \prec y$ imply $a \prec xy$;*
(2) *for all $x \in L$, the set $\{y \in L : x \prec y\}$ is a filter;*
(3) *for all a, b, x, $y \in L$ the relations $a \prec x$ and $b \prec y$ imply $ab \prec xy$;*
(4) *the graph of $\prec$ is a subsemilattice of $L \times L$;*
(5) *the function $x \mapsto s_\prec(x) : L \to \mathrm{Low}\, L$ is a semilattice morphism, where $s_\prec(x) = \{y \in L : y \mapsto x\}$.*

Proof: The connections (1) iff (2), (3) implies (1), and (3) iff (4) are trivial. If $a \prec x$ and $b \prec y$, then $ab \prec x$ and $ab \prec y$. If (1) holds then this implies $ab \prec xy$. Thus (1) implies (3). For $x, y \in L$ one has $s_\prec(xy) \subseteq s_\prec(x) \cap s_\prec(y)$. Thus (5) means that for all $a, x, y \in L$ with $a \in s_\prec(x) \cap s_\prec(y)$ one has $a \in s_\prec(xy)$; but this is the same as (1). □

Definition I-3.27. We will say that an auxiliary relation $\prec$ on L is *multiplicative* iff it satisfies the equivalent conditions of I-3.26 This terminology applies in particular to the way-below relation. □

Proposition I-3.28. *Let L be a continuous semilattice. If $\ll$ is multiplicative, then the following conditions are equivalent for an element $p \in L$:*

(1) *p is pseudoprime;*
(2) *if $ab \ll p$, then $a \leq p$ or $b \leq p$ for all $a, b \in L$;*
(3) *p is prime.*

Conversely, if L is, in addition, a distributive lattice, then $\mathrm{PRIME}\, L = \Psi\mathrm{PRIME}\, L$ *implies that $\ll$ is multiplicative.*

Proof: Clearly (3) implies (1). By I-3.25 we have (1) implies (2).

(2) implies (3): By way of contradiction suppose that p is not prime. Then there are elements $x, y \not\leq p$ with $xy \leq p$. By 1.6 we find elements $a, b \not\leq p$ with $a \ll x$ and $b \ll y$. Since $\ll$ is multiplicative, we conclude $ab \ll xy \leq p$; that is, $ab \ll p$ by I-1.2(ii). But this contradicts (2).

Now assume that L is a distributive lattice and $\ll$ is not multiplicative. We wish to show $\mathrm{PRIME}\, L \neq \Psi\mathrm{PRIME}\, L$. There are elements a, x, y with $a \ll x$ and $a \ll y$ but not $a \ll xy$. The ideal $\mathord{\downarrow\!\!\!\downarrow} xy$ and the filter $\uparrow a$ are disjoint; hence, by I-3.20, there is a prime ideal P containing $\mathord{\downarrow\!\!\!\downarrow} xy$ missing a. Then it follows that $p = \sup P \in \Psi\mathrm{PRIME}\, L$. But $xy = \sup \mathord{\downarrow\!\!\!\downarrow} xy \leq \sup P = p$. Consider the representative case $x \leq p = \sup P$. Then $a \ll x$ implies $a \in P$ by I-1.5, which is impossible. Thus, $x \not\leq p$ and $y \not\leq p$; whence, $p \notin \mathrm{PRIME}\, L$. □

Exercises

The first exercise is a variant of Proposition I-3.6.

Exercise I-3.29. Let L be a modular lattice and $p \in L$. (A lattice L is called *modular* if for all $x, y, z \in L$ the relation $x \geq z$ implies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.) Show that the following conditions are equivalent:

(1) p is irreducible;
(2) p is maximal in the complement of every filter maximal with respect to missing p;
(3) p is maximal in the complement of some filter maximal with respect to missing p. □

Problem. Is every irreducible element in a continuous lattice maximal in the complement of some open filter? □

The following exercises deal with distributivity in semilattices. For easy reference, we repeat Definition I-3.11(i): a semilattice S is *distributive* if $ab \leq x$ always implies the existence of c, d with $a \leq c$, $b \leq d$, and $x = cd$.

Exercise I-3.30. Show that if a semilattice S is actually a lattice, then S is distributive as a semilattice iff it is distributive as a lattice. $\square$

Exercise I-3.31. Show that if S is a sup semilattice, then S is distributive iff $\operatorname{Id} S$, the lattice of ideals, is distributive. $\square$

Exercise I-3.32. Show that the condition of distributivity is equivalent to $\uparrow(uv) = (\uparrow u)(\uparrow v)$ for all $u, v \in S$. $\square$

We have seen in Corollary I-3.13, that in a distributive continuous semilattice, any element is the inf of primes. The following example shows that the converse, which is valid if L is a lattice by I-3.15, fails for semilattices.

Exercise I-3.33. Consider the following subsets of $\mathbb{N}$: all singleton sets $\{n\}$, $n \in \mathbb{N}$, the set a of all even, the set b of all odd, numbers, and the set x of all natural numbers $n \geq 100$. Let L be the collection of all finite unions of these sets ordered by reverse inclusion $u \leq v$ iff $v \subseteq u$. Show that L is a continuous semilattice in which every element is an inf of primes, but which is not distributive. $\square$

Hint. It is clear that L is a $\cup$-semilattice and a domain, as there are no infinite ascending chains. The singletons $\{n\}$ are prime in L and every element in L is a union of singletons, hence an inf of primes. But $x \subseteq a \cup b$ and there are no $a' \subseteq a$, $b' \subseteq b$ in L such that $x = a' \cup b'$. Hence, L is not distributive. $\square$

Problem. Is OFilt(S) distributive for a distributive continuous semilattice? $\square$

In I-3.14 we have identified the prime elements in topologies. Let us identify the primes in a few specific examples.

Exercise I-3.34. Let $L = \operatorname{Id}^{-}\mathcal{A}$ be the lattice of closed two-sided ideals of a C^*-algebra. Show that $I \in \operatorname{PRIME} L$ iff I is a closed prime ideal in the ring theoretical sense. $\square$

Note that in a C^*-algebra, for two closed two-sided ideals one has $IJ = I \cap J$. Every primitive ideal (i.e., the kernel of an irreducible representation) is prime and thus is an element of $\operatorname{PRIME} L$. If $\operatorname{Prim}\mathcal{A}$ denotes the set of all primitive ideals of $\mathcal{A}$, we have $\operatorname{Prim}\mathcal{A} \subseteq \operatorname{PRIME} L$. For separable C^*-algebras equality is known to hold (and we will suggest an independent proof: see Exercise V-5.30). This is false for nonseparable algebras (see [Weaver, 2001]). One knows from the theory of C^*-algebras that $\operatorname{Prim}\mathcal{A}$ is an order generating set for L (in the sense of I-3.8).

Exercise I-3.35. Let $\{L_j : j \in J\}$ be a family of lattices. Show that an element $(x_j)_{j\in J} \neq 1$ is irreducible (resp., prime) in the product $\prod_{j\in J} L_j$ iff there is an index $k \in J$ such that $x_j = 1$ for $j \neq k$ and x_k is irreducible (resp., prime) in the lattice L_k. □

Exercise I-3.36. Let K be a compact convex subset of a locally convex topological vector space over the reals. Let $\mathrm{Con}(K)$ be the lattice of all compact convex subsets of K ordered by inclusion; by I-1.23 we know that $\mathrm{Con}(K)^{\mathrm{op}}$ is a continuous lattice. Prove the following.

(i) $A \in \mathrm{Con}(K)$ is co-irreducible ($A \in \mathrm{IRR}\,\mathrm{Con}(K)^{\mathrm{op}}$) iff A has at most one point;
(ii) A is co-prime iff A either is empty or consists of a single element which is an extreme point of K.

Hint. (i) Obviously a singleton is co-irreducible. Conversely, suppose $A \in \mathrm{Con}(K)$ has more than one point. Let f be a continuous linear functional into the reals which is nonconstant on A. Find a real number r such that the inverse images under f of the intervals $]-\infty, r]$ and $[r, +\infty[$ have nonempty intersections with A. This will give a decomposition of A that will show it is not co-irreducible in $\mathrm{Con}(K)$.

(ii) If A is the singleton of an extreme point, then it is easy to check that, when it is contained in the convex hull of the union of two sets in $\mathrm{Con}(K)$, it must be contained in one of them. Conversely, if A is co-prime, it is co-irreducible and so must consist of at most one point. If it does have a point and is not extreme, then it lies between two other points. This will show that A is not co-prime. □

Exercise I-3.37. Let L be a semilattice and $m\colon L \times L \to L$ be the meet function given by $m(x, y) = xy$. Let $G = \{(x, y) \in L \times L : x \leq y\}$, the graph of the relation $\leq$, and let G^{-1} be the graph of $\geq$. Prove the following:

(i) $m((L \times L)\backslash(G \cup G^{-1})) = L\backslash \mathrm{IRR}\,L$;
(ii) $m^{-1}(\mathrm{IRR}\,L) \subseteq G \cup G^{-1}$.

Hint. Clearly (ii) follows from (i). If $(x, y) \notin G \cup G^{-1}$, then xy is different from x and from y, and thus $xy \notin \mathrm{IRR}\,L$. If $z \notin \mathrm{IRR}\,L$, then there are elements $z < x$ and $z < y$ with $z = xy$. Then $(x, y) \notin G$ and $(x, y) \notin G^{-1}$. □

The following exercise concerns pseudoprimes in a continuous lattice. We know that a lattice must be distributive if it is order generated by primes. How nearly is a continuous lattice distributive if it is order generated by pseudoprimes? I-3.38 gives some sort of answer.

We say that $p \in L$ is a *weak prime* iff condition I-3.25(2) is satisfied. (Thus if L is distributive, then p is a weak prime iff it is a pseudoprime by

I-3.25.) The set of weak primes is denoted by WPRIME L (cf. V-3.1 and V-3.4).

Exercise I-3.38. Let L be a continuous lattice. Show that the following statements are equivalent:

(1) IRR $L \subseteq$ WPRIME L;
(2) WPRIME L is order generating;
(3) for all finite sequences of elements $a_1, \ldots, a_n, x \in L$ the relation $a_1 \ldots a_n \ll x$ implies $(a_1 \vee x) \ldots (a_n \vee x) = x$;
(4) for all finite sequences of elements $a_1, \ldots, a_n, x \in L$ the relation $a_1 \ldots a_n \ll x$ implies the existence of elements $b_k \geq a_k$, $k = 1, \ldots, n$, such that $b_1 \ldots b_n = x$;

Show that moreover these conditions imply

$$\text{for all } X \subseteq L \text{ and } x \in L \text{ we have } \Uparrow(x \sup X) = \Uparrow(\sup xX). \qquad \text{(WH)}$$

Hint. See [Hofmann and Lawson, 1976], p. 337. □

Notice that, according to our definitions,

$$\text{PRIME}\, L \subseteq \Psi\text{PRIME}\, L \subseteq \text{WPRIME}\, L.$$

In Chapter V we will return to these concepts.

In the following exercise we begin to discuss the connection between completely distributive lattices (see I-2.8, I-2.9, I-3.16) and continuous posets; only in Chapter V will we be able to complete this discussion (V-1.10 ff.).

Exercise I-3.39. Let L be a complete lattice. Let $P = \text{COPRIME } L \backslash \{0\}$ be the set of nonzero co-primes with the induced partial order. Prove the following.

(i) P is a **dcpo**.
(ii) If L is a completely distributive lattice, then for each $x \in P$

$$x = \sup_L(\Downarrow_L x \cap P) = \sup_P \Downarrow_P x.$$

Hint. (i) An element $p \in L$ is co-prime iff $L \backslash \uparrow p$ is an ideal. If $(p_j)_{j \in J}$ is a monotone net of co-primes (see O-1.2), then for $p = \sup\, p_j$ the set

$$L \backslash \uparrow p = \bigcup_{j \in J} (L \backslash \uparrow p_j)$$

is an ideal since the union is ascending.

(ii) Since $\Downarrow_L x \cap P \subseteq \Downarrow_P x$, we need only show the first equality. Let $x \not\leq y$. Then there is a $u \ll x$ with $u \not\leq y$. Now $u = \sup(\downarrow u \cap P)$ by I-3.16; but $\downarrow u \cup P \subseteq \Downarrow_L x \cap P$, and so $\sup(\Downarrow_L x \cap P) \not\leq y$. □

If we compare this with Definition I-1.6, then we see that only the directedness of $\Downarrow_P x$ is lacking for P to be recognized as a continuous poset. We will show this condition in Section 1 of Chapter V (V-1.6).

Exercise I-3.40. (Generalized Baire Category Theorem) This exercise presents in a sequence of results an application of the theory of irreducible elements to derive a very general Baire Category Theorem. However, we need some of the early results of this section in a sharpened form, and we state them in the first four propositions.

Proposition I-3.3 was one of the keys to the principal results. It says in effect that, in a domain, for any open upper set V of the form $\Uparrow x$ and for any element $v \in V$, there is an open filter $U \subseteq V$ with $v \in U$. A considerably stronger result is true – but it is noteworthy that countability enters here.

Proposition I-3.40.1. *Let L be a continuous semilattice and V an open upper set. If F is a countably generated filter in L with $VF \subseteq V$, then for each $v \in V$ there is an open filter $U \subseteq V$ containing both v and F.*

Remark. With $F = \{1\}$ we obtain a version of I-3.3.

Hint. Suppose $a_1, a_2, \ldots$ is a sequence of generators of F, which we may suppose decreasing without loss of generality. Let $v \in V$ be given – we must find an open filter U so that $va_n \in U$ for all n. Inductively select a sequence $b_n \in V$ as follows. Let $b_0 = v$. Since $a_1 \in F$, we have $va_1 \in V$; and since V is open, we find a $b_1 \ll va_1$. Suppose that $b_0, \ldots, b_n$ have been selected. Since $a_{n+1} \in F$ we have $b_n a_{n+1} \in V$; and thus, since V is open, we find a $b_{n+1} \ll b_n a_{n+1}$. Since

$$\cdots \ll b_n \ll b_{n-1} \ll \cdots \ll b_1 \ll v,$$

then U, the filter generated by the b_n, is an open filter containing v (I-3.2). Because of $b_n \leq a_n$ all a_n are contained in U. □

Corollary I-3.40.2. *Let L be a continuous semilattice and V an open upper set. If N is any countable subset of L with $VN \subseteq V$, then for each $v \in V$ there is an open filter U with $vN \subseteq U$.*

Hint. Let F be the filter generated by N. Then I-3.40.1 applies to F and yields the assertion. □

The basic result I-3.10 may be formulated as follows: if V is the open upper set $L \backslash \downarrow s$, then for every $y \in V$ and every $x \notin V$ there is a $p \in \mathrm{IRR}\, L$ such that $x \leq p$ and $y \not\leq p$. This we now sharpen:

Proposition I-3.40.3. *Let L be a continuous semilattice and V an open upper set. If N is a countable set with $VN \subseteq V$, then for every $y \in V$ and every*

$x \notin V$ there is an irreducible element p such that $x \leq p$ and $yn \not\leq p$ for all $n \in N$.

Hint. Let U be the open filter constructed in I-3.40.2. Then by I-3.4 there is a maximal element p in $\uparrow x \backslash U$, and by I-3.6 p is irreducible. □

Corollary I-3.40.4. *Let L be a continuous semilattice, $x \in L$, and $N \subseteq L$ a countable set such that $y \not\leq x$ and $n \in N$ always imply $yn \not\leq x$. Then for any y with $y \not\leq x$ there is an irreducible p with $x \leq p$ and $yn \not\leq p$ for all $n \in N$.*

Hint. Apply I-3.40.3 with $V = L \backslash \downarrow x$. □

Now consider the lattice of open subsets $\mathcal{O}(X)$ of a topological space X. An open set $U \in \mathcal{O}(X)$ is *dense* in X iff for all $V \in \mathcal{O}(X)$ with $V \neq \emptyset$ we have $U \cap V \neq \emptyset$. Accordingly we are motivated to make the following definition.

Definition I-3.40.5. An element u in a semilattice L with a smallest element 0 is called *dense* if $v \neq 0$ implies $uv \neq 0$ for all $v \in L$. □

Let us momentarily assume that X is a Hausdorff space. Then by I-3.14 there is a bijection between the points $x \in X$ and the nontrivial primes of $\mathcal{O}(X)$ given by $x \mapsto X \backslash \{x\}$. For an element $U \in \mathcal{O}(X)$ we have $x \in U$ iff $U \not\subseteq X \backslash \{x\}$. This motivates the following definition.

Definition I-3.40.6. In a complete lattice L we define a binary relation ε between IRR $L \backslash \{1\}$ and L by setting $p \,\varepsilon\, u$ iff $u \not\leq p$. □

At this point we return to Corollary I-3.40.4 which we specialize to the case $x = 0$. The hypothesis on N then says that all members of N are dense. This leads to

Theorem I-3.40.7. ***(Baire Category Theorem for Continuous Lattices)*** *Let L be any continuous semilattice with a smallest element 0 and D a countable collection of dense elements. Then for any nonzero element u there is a point $p \in \mathrm{IRR}\, L \backslash \{1\}$ such that $p \,\varepsilon\, (u \wedge v)$ for all $v \in D$.*

As a consequence we have the following result.

Theorem I-3.40.8. ***(Baire Category Theorem for Locally Compact Spaces)*** *Let X be any locally compact space and $\mathcal{D}$ a countable collection of dense open sets. Then for any nonempty open set U there is an irreducible closed set A such that $A \cap U \cap V \neq \emptyset$ for all $V \in \mathcal{D}$.*

Hint. Recall that $\mathcal{O}(X)$ is a continuous lattice by I-1.7(5). □

Notice that if $A = \{x\}^-$, then the conclusion of I-3.40.8 implies that $x \in A \cap U \cap W$ and that $U \cap W \neq \emptyset$, where W is the intersection of the sets in $\mathcal{D}$. We recall next a definition from general topology (see O-5.13):

A topological space X is called a *Baire space* iff the intersection of any countable collection of dense open subsets is dense (or, equivalently, iff the union of a countable collection of nowhere dense closed sets is nowhere dense).

As an immediate consequence of the above we have a generalization of Baire's well-known classical theorem on locally compact Hausdorff spaces:

Corollary I-3.40.9. *Every locally compact sober space is a Baire space.*

The following counterexamples are instructive; both are first countable.

Let $X = \mathbb{N}$ with upper sets in the usual ordering open. Then X is locally compact T_0, but it is not a Baire space.

Let X be the set of all ordinals less than the first uncountable ordinal with upper sets open. Then X is a locally compact T_0 Baire space which is not sober.

Old notes

The study of irreducibles and primes in continuous lattices was begun in [Hofmann and Lawson, 1976] and [Hofmann and Lawson, 1978]. Substantial contributions were made by [Gierz and Keimel, 1977]; this paper as well as the second one mentioned above will contribute to Chapter V.

The proof of Proposition I-3.3 is due to Lawson (folklore tradition in SCS). The results in Exercise I-3.40 are due to [Hofmann, 1980] (see also [SCS 43]). The result that a locally compact space is a Baire space can also be excavated from [Isbell, 1975a], p. 334, Section 4.2. Isbell asks for a "pointless" generalization of Baire category; our Baire category theorem for continuous lattices and for locally compact spaces may be considered such a "pointless" theory. The characterization of completely distributive lattices in terms of continuous lattices in Theorem I-3.16 is a result of [Kamara, 1978]. The theory of pseudoprimes was first developed in the second of the two papers by Hofmann and Lawson cited above. Proposition I-3.28, however, in slightly different language is due to Keimel and Mislove [SCS 19].

The current chapter deals exclusively with the purely lattice theoretical aspects of the theory of continuous lattices. Many of the finer results on the spectra of continuous lattices require a better understanding of various topologies on a continuous lattice and will be treated in Chapter V.

I-4 Algebraic Domains and Lattices

In universal algebra, algebraic lattices have become familiar objects as lattices of congruences and lattices of subalgebras of an algebra. As a consequence, they have been extensively studied and it cannot be our purpose here to survey this classical field. However, algebraic lattices are continuous, and they fit perfectly into the general theory. It is this fit which is the object of our present discussion.

Compact elements, algebraic and arithmetic domains

We have noticed that the auxiliary relations need not be reflexive, and that in fact the way-below relation rarely is. Nevertheless there are elements x such as $x = 0$ which satisfy $x \ll x$. It is those elements which now come into focus. In Definition I-1.1 they were called *isolated* (from below) or *compact*. Let us recall the definition:

Definition I-4.1. In any poset L, an element k is called *compact* (or *isolated*) iff $k \ll k$, i.e., whenever D is a directed subset of L such that $\sup D$ exists and $k \leq \sup D$, then $k \leq d$ for some $d \in D$. *The subset of all compact elements* is denoted by $K(L)$. □

In a complete semilattice and in particular in a complete lattice an element k is compact iff whenever C is a (bounded) subset such that $k \leq \sup C$, then there is a finite subset $F \subseteq D$ such that $k \leq \sup F$. Thus, this notion of compactness is an order theoretical version of the Heine–Borel covering property characterizing compactness in topological spaces: every open covering contains a finite open covering. In particular, an open set U is compact in the lattice $\mathcal{O}(X)$ of open subsets of a topological space in the sense of Definition I-4.1 iff U is compact as a subset of the topological space X.

If L is the unit interval, then $K(L) = \{0\}$. If L is the standard Cantor chain in the unit interval, then $K(L)$ consists exactly of those elements which are isolated from below in the topological sense. In this example compact elements are so abundant that every element is approximated from below by them. The general idea of this kind of abundance is formalized in the following definition.

Definition I-4.2.

(i) A poset L is called *algebraic* iff it satisfies the *Axiom of Compact Approximation*

$$(\forall x \in L) \quad x = \bigvee{}^{\uparrow}(\downarrow x \cap K(L)), \tag{K}$$

i.e., for all $x \in L$ the set $\downarrow x \cap K(L)$ is directed and $x = \sup(\downarrow x \cap K(L))$.

(ii) A directed complete algebraic poset L is called an *algebraic domain*.
(iii) An algebraic domain which is a lattice is called an *algebraic lattice*.
(iv) An algebraic domain which is a semilattice is called *an algebraic semilattice*.
(v) A complete semilattice (cf. 0-2.1(iv)) which is an algebraic domain as a poset is called a *bounded complete algebraic domain*.
(vi) An algebraic domain in which every principal ideal $\downarrow x$ is a complete lattice (in its induced order) is called an *algebraic L-domain*. □

For a poset we thus have the following chain of implications:

$$\boxed{\begin{array}{l}\text{algebraic lattice} \Rightarrow \text{bounded complete algebraic domain}\\ \Rightarrow \left\{\begin{array}{l}\text{algebraic semilattice}\\ \text{algebraic } L\text{-domain}\end{array}\right\} \Rightarrow \text{algebraic domain.}\end{array}}$$

Proposition I-4.3. *In a poset L, the following statements are equivalent:*

(1) *L is algebraic;*
(2) *L is continuous, and $x \ll y$ iff there is a $k \in K(L)$ with $x \leq k \leq y$.*

In particular every algebraic poset is a continuous poset and every algebraic (semi)lattice is a continuous (semi)lattice.

Proof: (1)⇒(2): Assume (1) and $x, y \in L$. If $x \ll y$, then, since $y = \sup D$ with the directed set $D = \downarrow y \cap K(L)$ by (1), there is a $k \in D$ with $x \leq k$. Hence $x \leq k \leq y$ with $k \in K(L)$. Conversely, if there is a compact element k with $x \leq k \leq y$, then $x \leq k \ll k \leq y$, whence $x \ll y$ by I-1.2(ii).

The continuity of L now follows directly from the second Remark after Definition I-1.6 and from the compact approximation property (K).

(2)⇒(1): Assume (2) and let $y \in L$. Then $y = \sup \twoheaddownarrow y$ and $\twoheaddownarrow y$ is directed. As, also by (2), for every $x \ll y$, there is a compact element k such that $x \leq k \leq y$, we conclude that $y = \sup(\downarrow y \cap K(L))$. Further, $(\downarrow y \cap K(L))$ is directed. For if k_1, k_2 are two compact elements below y, then there is an $x \in \twoheaddownarrow y$ dominating both of these elements as $y = \sup \twoheaddownarrow y$. □

We now turn to the properties of the set of compact elements with the induced ordering.

Remark I-4.4. *Let L be a* **dcpo**. *If L has a least element* 0, *then* $0 \in K(L)$. *If two elements $x, y \in K(L)$ have a sup in L, then $x \vee y \in K(L)$.*

Proof: If $x \ll x$ and $y \ll y$, then $x \vee y \ll x \vee y$ by I-1.2(iii), and $0 \in K(L)$ by I-1.2(iv). □

The preceding remark gives rise to the following definition.

Definition I-4.5. A poset L will be called a *conditional sup semilattice*, if any two elements x, y with a common upper bound have a least upper bound $x \vee y$ in L. □

From I-4.4 we conclude that the following holds.

Remark I-4.6.

(i) *In every complete semilattice, in particular in any bounded complete algebraic domain, the set $K(L)$ of compact elements is a conditional sup semilattice with a smallest element.*
(ii) *In every complete lattice, in particular in every algebraic lattice, the set $K(L)$ of compact elements is a sup semilattice with a smallest element.*

□

A frequently encountered subclass of algebraic semilattices is introduced in the next definition.

Definition I-4.7. A semilattice L is called an *arithmetic semilattice* iff it is algebraic and if $K(L)$ is a subsemilattice of L, i.e., if $x \wedge y \in K(L)$ for all $x, y \in K(L)$. An *arithmetic lattice* is an algebraic lattice in which the set of compact element is a subsemilattice. □

Proposition I-4.8. *Let L be an algebraic semilattice. Then the following statements are equivalent:*

(1) *L is arithmetic;*
(2) *$K(L)$ is a semilattice;*
(3) *the way-below relation $\ll$ is multiplicative* (I-3.27).

Proof: That (1) implies (2) is trivial.

(2) implies (1): Let $a, b \in K(L)$, $c = a \wedge_{K(L)} b$. Then $c \leq ab$ $(= a \wedge_L b)$. But if $X = \downarrow(ab) \cap K(L)$, then $c = \sup_{K(L)} X$, and $ab = \sup_L X$, since L is algebraic. Thus $ab \leq c$, whence $a \wedge_{K(L)} b = ab$.

(1) implies (3): Let $a \ll x$ and $a \ll y$. Then there are $c, k \in K(L)$ with $a \leq c \leq x$ and $a \leq k \leq y$ by I-4.3. Thus $a \leq ck \leq xy$, and since $ck \in K(L)$ by (1), we have $a \ll xy$ by I-4.3.

(3) implies (1): If $a, b \in K(L)$, then $a \ll a$, $b \ll b$, hence $ab \ll ab$ by (3). Thus $ab \in K(L)$. □

Corollary I-4.9. *Every pseudoprime in an arithmetic semilattice is prime. Conversely, if in a distributive algebraic semilattice L we have* $\Psi\mathrm{PRIME}\,L = \mathrm{PRIME}\,L$*, then L is arithmetic.*

Proof: I-4.8 and I-3.28. □

Proposition I-4.10. *Let S be a poset and $L = \mathrm{Id}\,S$ the set of all ideals of S ordered by inclusion.*

(i) *L is an algebraic domain whose compact elements are the principal ideals.*

(ii) *The principal ideal map is an isomorphism $x \mapsto {\downarrow}x\colon S \to K(L)$;*

(iii) *If S is a conditional sup semilattice, a sup semilattice, a semilattice, a lattice, respectively, with a least element 0, then L is a bounded complete algebraic domain, an algebraic lattice, an arithmetic semilattice, an arithmetic lattice, respectively.*

Conversely, let L be an algebraic domain and $S = K(L)$ the poset of compact elements.

(iv) *The map $x \mapsto {\downarrow}x \cap S : L \to \mathrm{Id}\,S$ is an isomorphism.*

(v) *The ideal ${\downarrow}x \cap S$ is principal iff $x \in S$.*

(vi) *If L is a bounded complete algebraic domain, an algebraic lattice, an arithmetic semilattice, an arithmetic lattice, respectively, then S is a conditional sup semilattice, a sup semilattices, a semilatice, a lattice, respectively, with a least element 0.*

Proof: Most of the facts recorded here have already been established. We add a few hints. (i) By invoking O-2.8(2) we note that L is closed in 2^S under directed unions and hence a **dcpo**. Clearly, principal ideals are compact. Conversely, let I be compact in $\mathrm{Id}\,S$; then, as I is the union of the directed set of principal ideals ${\downarrow}x$, $x \in S$, it follows that $I = {\downarrow}x$ for some x. The proof of (ii) is clear. For (iii) we notice that, in a conditional sup semilattice S with 0, the intersection of any nonempty family of ideals is an ideal, whence L is a complete semilattice. If S is a lattice, then the intersection of two principal ideals is a principal ideal; hence, L is an arithmetic lattice.

(iv) To prove that $x \mapsto {\downarrow}x \cap S : L \to \mathrm{Id}\,S$ is bijective, we claim that sup: $\mathrm{Id}\,S \to L$ is the inverse of this map. Since the latter is clearly surjective, it suffices to show that ${\downarrow}(\sup I) \cap S = I$ for each $I \in \mathrm{Id}\,S$, and since $\supseteq$ is clear, we must show $\subseteq$. Let $k \in {\downarrow}(\sup I) \cap S$; that is, $k \ll k \leq \sup I$. Thus $k \ll \sup I$ by I-1.2(ii). Hence we have an $x \in I$ with $k \leq x$. But since I is an ideal in S, we have $k \in I$. Part (v) is clear from the fact that $\sup({\downarrow}x \cap S) = x$. Part (vi) follows from I-4.6 and I-4.7. □

In particular, we have the following corollary.

Corollary I-4.11. *Every algebraic domain L admits an injection $g\colon L \to 2^{K(L)}$ preserving directed sups such that $(pr_k g)^{-1}(1) = \uparrow k$ for $k \in K(L)$. For an algebraic lattice, respectively a bounded complete algebraic domain, the injection also preserves arbitrary, respectively nonempty, infs.*

Proof: By I-4.10, $L \cong \mathrm{Id}(K(L))$, and $\mathrm{Id}(K(L)) \subseteq 2^{K(L)}$ is closed under unions of directed sets. If L is an algebraic lattice, respectively bounded complete, then $K(L)$ is a sup semilattice, respectively conditional sup semilattice, and $\mathrm{Id}(K(L)) \subseteq 2^{K(L)}$ is also closed under arbitrary, respectively nonempty, intersections. If the injection g is interpreted in terms of characteristic functions, then $g(x)(k) = 1$ iff $k \in \downarrow x \cap K(L)$ iff $k \leq x$, and this is the assertion. □

Products, kernel and closure operators

It is now natural to investigate the closure properties of the class of algebraic lattices within the class of continuous ones, following the lines of Section 2. This will allow us to exhibit some characteristic examples of algebraic domains. As finite posets are algebraic domains and as finite lattices are algebraic, even arithmetic, lattices, products will allow us to construct algebraic domains and arithmetic lattices. As flat domains, i.e., antichains M with a bottom element adjoined, are algebraic and bounded complete, products allow us to construct bounded complete algebraic domains:

Proposition I-4.12. *If $\{L_j\colon j \in J\}$ is a family of algebraic domains which have a least element* 0 *(except, perhaps, for finitely many $j \in J$), then the cartesian product $\prod_{j\in J} L_j$ is an algebraic domain; the same holds for cartesian products of algebraic semilattices, bounded complete algebraic domains, algebraic lattices, arithmetic lattices, and for algebraic L-domains.*

Proof: An element $(x_i)_{i\in I}$ of the product is compact iff $x_i \in K(L_i)$ for all $i \in I$ and $x_i = 0$ for all but a finite number of indices. Since every factor is algebraic, every element of L is the sup of such elements. □

Now recall that if a closure operator (O-3.8) c on a domain L preserves directed sups, then $c(L)$ is a domain (see I-2.2). We have a parallel for the algebraic case:

Proposition I-4.13. *Let L be an algebraic domain.*

(i) *If $c\colon L \to L$ is a closure operator preserving sups of directed sets, then*

(1) *the image $c(L)$ is an algebraic domain (relative to the induced order) closed for sups of directed sets in L,*

(2) *$c(K(L)) = K(c(L))$.*

(ii) *If M is a closure system which is closed in L with respect to directed sups, then M is an algebraic domain.*

The statements remain true if we replace algebraic domain *by any of the concepts* algebraic semilattice*,* algebraic lattice, algebraic bounded complete domain, algebraic L-domain.

Proof: (i) By I-2.2, $c(L)$ is continuous. We also note that, for every directed subset $D \subseteq c(L)$, one has $\sup_L D = \sup_{c(L)} D$ so that we can omit the subscripts when we take sups of directed sets. The Remark following I-2.3 allows us to say that $k \ll_L k$ implies $c(k) \ll_{c(L)} c(k)$, whence $c(K(L)) \subseteq K(c(L))$. Then since c preserves directed sups, we have

$$c(x) = c(\sup(\downarrow x \cap K(L))) = \sup c(\downarrow x \cap K(L)),$$

and, as $c(\downarrow x \cap K(L))$ is a directed set in $K(c(L)$, this shows that $c(L)$ is algebraic. Thus we have shown (1).

As we have seen that $c(K(L)) \subseteq K(c(L))$, for (2), it remains to show that the converse containment holds. For this purpose let $a \in K(c(L))$, that is, $a \ll_{c(L)} a$ and $c(a) = a$. By the characterization of the way-below relation in $c(L)$ contained in I-2.2, there is a $u \in K(L)$ with $u \leq a$ such that $a \leq c(u) \ll_{c(L)} a$, and this implies $c(u) = a$.

(ii) Recall that a closure system is the image of a closure operator. By Lemma I-2.4, a closure operator preserves directed sups iff its image is closed under directed sups. Thus, (ii) is a consequence of (i). □

Corollary I-4.14. *Let L be an algebraic lattice. Then the assignment $c \mapsto c(L)$, which associates with a closure operator $c \colon L \to L$ its image, induces a bijection from the set of all closure operators of L preserving directed sups onto the set of subalgebras of L.*

Moreover, all of the subalgebras, i.e., subsets M closed under (nonempty) infs and directed sups, of an algebraic lattice or an algebraic bounded complete domain L are algebraic. □

Proof: This is immediate from I-4.13 and I-2.12 and from the fact that, in algebraic lattices, closure systems are precisely subalgebras. □

Examples I-4.15.

(1) For any set X the lattice 2^X of all subsets of X is algebraic; the compact elements of 2^X are the finite subsets $F \subseteq X$.
(2) If L is a subalgebra of 2^X, that is, L is a subset which is closed under arbitrary (or nonempty) intersections and directed unions, then L is an algebraic lattice (respectively an algebraic bounded complete

domain) and

$$E \in K(L) \text{ iff } E = \bigcap\{Y \in L : F \subseteq Y\} \text{ for some finite } F \in 2^X.$$

The compact elements just described are also called the *finitely generated* elements of L.

(3) As a special case of this last example, consider 2^{2^X}. There are many well-known subalgebras: Filt 2^X and Id 2^X, to name two. In the case of filters,

$$P \in K(\text{Filt } (2^X)) \text{ iff } P \text{ is a principal filter.}$$

Likewise, the compact elements in Id 2^X are the principal ideals. This example generalizes considerably, as we have already seen in I-4.10. The importance of the filter lattice Filt 2^X for the class of continuous lattices will appear later in this section. □

The algebraic lattices occurring as subalgebras of powerset lattices are archetypical:

Theorem I-4.16. *Let L be a poset. Then the following statements are equivalent:*

(1) *L is an algebraic lattice;*
(2) *for some set X, the lattice L is isomorphic to a subset of* 2^X *which is closed under arbitrary intersections and directed unions;*
(3) *L is isomorphic to the image of some closure operator* $c\colon 2^X \to 2^X$ *which preserves directed unions.*

Conditions (1) *and* (2) *remain equivalent if one replaces* algebraic lattice *by* bounded complete algebraic domain *and if one restricts to intersections of nonempty instead of arbitrary families of sets.*

Proof: (1) implies (2): I-4.11.
(2) implies (3): I-4.14.
(3) implies (1): I-4.13. □

In I-2.2 we have seen that the continuity of a poset or a lattice is preserved if we pass to images under projections which preserve directed sups. In particular, images of algebraic posets or lattices under projections preserving directed sups will be continuous. It is noteworthy that algebraicity is not inherited by images under projections preserving directed sups, in general, except if we restrict our attention to closure operators (see I-4.13).

It belongs to the same circle of ideas that the class of algebraic lattices is not closed under the formation of homomorphic images (see I-2.10). The ordinary Cantor set C in the unit interval $\mathbb{I} = [0, 1]$ is an algebraic lattice. The Cantor function $g: C \to \mathbb{I}$ which maps C continuously, surjectively and in a monotone fashion onto the unit interval illustrates this phenomenon. Of course, all homomorphic images of an algebraic lattice are continuous by I-2.11(iii), and in I-4.17 below we will see that all continuous lattices are so obtained. From this viewpoint it is correct to say that the class of continuous lattices is the smallest class closed under the formation of products, subalgebras, and homomorphic images and which contains all algebraic lattices (or even just the two element lattice as a generator).

If $d: \mathbb{I} \to C$ is the lower adjoint of the Cantor function g just mentioned, then $k = dg: C \to C$ is a kernel operator preserving sups whose image is not algebraic. Thus, a sharp analog of Corollary I-2.3 for algebraic lattices is not available. Corollary I-4.14 provides a substitute. We utilize this observation further in showing how continuous domains can be derived from algebraic ones via kernel operators.

Theorem I-4.17. *Let L be a* **dcpo**. *Then the following statements are equivalent.*

(1) *L is continuous, i.e., a domain.*
(2) *There are an algebraic domain A and a map $r: A \to L$ which is surjective, preserves directed sups, and has a lower adjoint.*
(3) *There are an algebraic domain A and a kernel operator $k: A \to A$ preserving directed sups such that $L \cong \operatorname{im} p$.* □

Remark. We could rephrase (3) in words as: L is (isomorphic to) a retract of some algebraic domain A under some kernel operator preserving directed sups.

Proof of theorem: (1) implies (2): Take $A = \operatorname{Id} L$ and let $r(I) = \sup I$. Then A is an algebraic domain by I-4.10, r is surjective and has a lower adjoint by I-1.10, (1) implies (4).

(2) implies (3): Let $d: L \to A$ be the lower adjoint of r, then $k = dr$ is a kernel operator preserving directed sups. As r is surjective, the lower adjoint d is injective and in fact an isomorphism from L onto $\operatorname{im} k$.

(3) implies (1): I-2.2. □

Analogous statements as in the theorem above hold for L-domains, bounded complete domains, continuous semilattices and continuous lattices. In the case of continuous semilattices L, one should notice that A can be chosen to be arithmetic, as the intersection of two principal ideals of L is a principal ideal.

For bounded complete domains and for continuous lattices, finally, the fact that r has a lower adjoint can be expressed by saying that it preserves (nonempty) infs. We finally arrive at the following.

Corollary I-4.18. *Let L be a lattice. Then the following statements are equivalent.*

(1) *L is continuous.*
(2) *There are an arithmetic lattice A and a surjective map $r\colon A \to L$ preserving arbitrary infs and directed sups.*
(3) *There are an algebraic lattice A and a surjective map $r\colon A \to L$ preserving infs and directed sups.*
(4) *There are a set X and a projection operator $p\colon 2^X \to 2^X$ preserving directed sups such that $L \cong \operatorname{im} p$.*

Proof of: In view of the theorem above and the subsequent remarks, we just have to show that (3) implies (4): By I-4.13, there is a closure operator c preserving directed sups on some powerset 2^X such that the algebraic lattice A is isomorphic to the image of c. As above, let $d\colon L \to A$ be the lower adjoint of r and $k = dr$ the kernel operator on A preserving directed sups. Define $p\colon 2^X \to 2^X$ by $p = c_\circ k c^\circ$ (see O-3.9). By O-3.12(iii), c° preserves arbitrary sups, and $c_\circ$ preserves directed sups, as $\operatorname{im} c$ is closed for directed sups. Hence p preserves directed sups, and $\operatorname{im} p \cong rc_\circ(A) = r(A) = L$. □

In proving I-4.18 we obtained a given continuous lattice L as a quotient of its ideal lattice $A = \operatorname{Id} L$ which is arithmetic. In this construction A depends rather heavily on L, but there is in fact a choice of an arithmetic lattice that depends only on the *cardinality* of L. As we have already proved that the class of continuous lattices is equationally characterizable in Section 2, we could guess at which lattice this is: the free continuous lattice of $\operatorname{card}(L)$ generators. Indeed, it follows from quite general theorems that such a lattice exists. Instead of invoking the general theory, however, we can construct free lattices for this class directly and see at once why they are arithmetic.

Theorem I-4.19. *For any set X, the lattice $\operatorname{Filt} 2^X$ of all filters is the free continuous lattice generated by X. More precisely: For every $x \in X$, let $\mathcal{F}(x) = \{Y \subseteq X : x \in Y\}$ be the fixed ultrafilter generated by x. Then, for every continuous lattice L and every map $f\colon X \to L$, there is one and only one homomorphism of continuous lattices $f^*\colon \operatorname{Filt} 2^X \to L$ such that $f^*(\mathcal{F}(x)) = f(x)$ for all $x \in X$.*

Proof of: Let f be any map from X into an arbitrary continuous lattice L. We have to show that there is one and only one map $f^*\colon \operatorname{Filt} 2^X \to L$ preserving arbitrary infs and directed sups such that $f^*(\mathcal{F}(x)) = f(x)$ for all $x \in X$.

If F is any filter, then

$$F = \sup\{\inf\{\mathcal{F}(x)\colon x \in Y\}\colon Y \in F\},$$

because the set in the inside is just $\uparrow Y$ in 2^X. As the sup (which in 2^X is just a union) is directed, this shows that F belongs to the subalgebra generated by the $\mathcal{F}(x)$. Therefore, any map on Filt 2^X which preserves infs and directed sups is uniquely determined by its action on the $\mathcal{F}(x)$.

The definition of f^* can be given on a filter F as follows:

$$f^*(F) = \sup\{\inf\{f(x)\colon x \in Y\}\colon Y \in F\} = \sup\{\inf f(Y)\colon Y \in F\}.$$

On the right hand side the sups (directed!) and infs are to be calculated in L. The map f^* is well defined and it satisfies $f^*(\mathcal{F}(x)) = \sup\{\inf f(Y)\colon x \in Y\} = f(x)$. Obviously f^* preserves directed sups, because in the filter lattice directed sups *are* unions. We must prove that f^* preserves infs; it will be better to calculate backwards.

$$\begin{aligned}\inf\{f^*(F_i)\colon i \in I\} &= \bigwedge\nolimits_{i\in I} \bigvee\nolimits_{Y\in F_i} \bigwedge\nolimits_{x\in Y} f(x)\\ &= \bigvee\nolimits_{Z\in P} \bigwedge\nolimits_{i\in I} \bigwedge\nolimits_{x\in Z_i} f(x)\\ &= \bigvee\nolimits_{Z\in P} \bigwedge \left\{f(x)\colon x \in \bigcup\nolimits_{i\in I} Z_i\right\}.\end{aligned}$$

Here P is the cartesian product of the F_i for $i \in I$, and we have applied the distributive law (DD) from I-2.7 to the lattice L. Now note that for $Z \in P$ we have

$$\left(\bigcup_{i\in I} Z_i\right) \in \left(\bigcap_{i\in I} F_i\right),$$

and thus every element of the intersection of the filters comes up in this way. Thus, the right hand side of the above equation reduces to $f^*(\bigcap_{i\in I} F_i)$ as desired. $\square$

It follows at once from what we have done that if $A = \text{Filt}\ 2^L$, then L is the quotient of the arithmetic lattice A by the map which sends $\mathcal{F}(x)$ to x for $x \in L$.

This is a good time to characterize continuous Boolean algebras:

Theorem I-4.20. *Let L be a Boolean algebra. Then the following statements are equivalent:*

(1) *$L \cong 2^X$ for some set X;*
(2) *L is arithmetic;*
(3) *L is algebraic;*
(4) *L is continuous;*
(5) *L and L^{op} are continuous;*

(6) *L is completely distributive;*
(7) *every element in L is the sup of atoms and L is complete.* □

Proof of: That (1) implies (2) implies (3) implies (4) is trivial, and since in a Boolean algebra $x \mapsto \neg x : L \to L^{\mathrm{op}}$ is an isomorphism, (4) implies (5) is clear. Since a Boolean algebra is distributive, and since co-primes are precisely atoms, the equivalences of (5), (6), and (7) follow from I-3.16.

(7) implies (1): Let X be the set of atoms. Define two functions:

$$f = (A \mapsto \sup A)\colon 2^X \to L;$$
$$g = (x \mapsto {\downarrow} x \cap X)\colon L \to 2^X.$$

Then $fg = 1_L$ by (7). In order to show that f is an isomorphism it is sufficient to understand that f is injective. For this it suffices to observe that for $A \subseteq X$, $a \in X \backslash A$, one has $\sup(\{a\} \cup A) = a \vee \sup A > \sup A$; and this may be deduced from the fact that in a Boolean algebra the function

$$x \mapsto (x \wedge a, x \wedge (\neg a)) : L \to [0, a] \times [0, \neg a]$$

is an isomorphism. □

This shows that continuous Boolean algebras are quite simple. Continuous Heyting algebras are much less trivial; they can nevertheless be completely characterized, as is shown in Chapter V.

Completely irreducible elements

The issue of irreducibility and order generation which we discussed in Section 3 for domains and continuous lattices can be rendered even more precise for algebraic lattices and bounded complete algebraic domains. The key is the fact that bounded complete algebraic domains contain an ample supply of special irreducibles which we introduce in the next definition.

Definition I-4.21. Let L be a poset. An element $p \in L$ is called *completely irreducible* iff either p is maximal in L but different from the top element or the set ${\uparrow} p \backslash \{p\}$ has a least element which we shall denote by p^+. The set of all completely irreducible elements of L will be written $\mathrm{Irr}\, L$. □

Clearly by definition $1 \notin \mathrm{Irr}\, L \subseteq \mathrm{IRR}\, L$.

Remark I-4.22. *If X is an order generating subset of a poset L, then* $\mathrm{Irr}\, L \subseteq X$.

Proof of: Since X is order generating, for every $p \in L$ we have $p = \inf({\uparrow} p \cap X)$. Now assume that $p \in \mathrm{Irr}\, L$. If p is maximal in L, then clearly $p \in X$. If p is not maximal, then $p \notin X$ would imply $\inf(\uparrow p \cap X) \geq p^+$ which contradicts our hypothesis that X is order generating. □

We now establish a sufficient condition for complete irreducibility under suitable conditions:

Remark I-4.23. *Let L be a complete semilattice and $p \in L$. If there is a $k \in L$ such that p is maximal in $L \backslash \uparrow k$, then p is completely irreducible.* □

Proof of: If p is already maximal in L, then it is completely irreducible, as it cannot be the top element in L because $k \not\leq p$. If p is not maximal in L, then $\emptyset \neq \uparrow p \backslash \{p\} \subseteq \uparrow k$. Hence $p^+ = \inf(\uparrow p \backslash \{p\})$ exists, and $p^+ > p$, as $p^+ \geq k$. □

If L is a complete chain, then $\operatorname{Irr} L = K(L^{\mathrm{op}}) \backslash \{1\}$. Thus, for the unit interval, $\operatorname{Irr} L$ is empty. The important fact for bounded complete algebraic domains is that there are enough complete irreducibles. In the proof we shall use the following characterization of compact elements.

Remark I-4.24. *For an element k in a* **dcpo** *L the following statements are equivalent:*

(1) *$\uparrow k$ is an open filter (in the sense of* I-3.1*);*
(2) *k is compact.*

Proof of: (2) implies (1): If $u \in \uparrow k$ and $k \ll k$, then $k \ll u$ by I-1.2(ii).
(1) implies (2): If D is a directed set such that $k \leq \sup D$, then $\sup D \in \uparrow k$. By (1), there is a $d \in D$ such that $d \in \uparrow k$, i.e., $k \leq d$. Hence, k is compact. □

Theorem I-4.25. *Suppose that x and y are elements of a bounded complete algebraic domain with $y \not\leq x$. Then there is a completely irreducible element p with $x \leq p$ and $y \not\leq p$.*

Proof of: The proof is analogous to that of I-3.9. By I-4.2, there is an element $k \in K(L)$ with $k \leq y$ and with $k \not\leq x$. By I-4.24 and I-3.4 there is a maximal element p in $\uparrow x \backslash \uparrow k$. By I-4.23, p is completely irreducible. □

Theorem I-4.26. *In any bounded complete algebraic domain,* $\operatorname{Irr} L$ *is the unique smallest order generating set. In particular, $s = \inf(\uparrow s \cap \operatorname{Irr} L)$ for all $s \in L$.*

Proof of: By I-4.25 and I-3.9, $\operatorname{Irr} L$ is order generating. By I-4.22, $\operatorname{Irr} L$ is the unique smallest order generating set. □

Recall that in a continuous lattice in general there is no smallest order generating set, as the example of the unit interval demonstrates. As the example of the Cantor lattice C shows, we have in general $\operatorname{Irr} L \neq \operatorname{IRR} L$ (since

$\operatorname{Irr} C = K(C^{\mathrm{op}}) \neq C = \operatorname{IRR} L$). In Section V-2 we will learn more about generating sets in continuous lattices.

We finish this section by proving the converse of I-4.23 under two different hypotheses:

Proposition I-4.27. (i) *In a bounded complete algebraic domain, an element p is completely irreducible iff p is maximal in $L\backslash\uparrow k$ for some compact element k.*

(ii) *In a join continuous distributive complete lattice, an element p is completely irreducible iff p is maximal in $L\backslash\uparrow k$ for some compact element k.*

Proof of: (i) Let $p \in \operatorname{Irr} L$. By the proof of I-4.24,

$$p = \inf\{x \in \uparrow p\colon \text{there is a } k \in K(L) \text{ with } x \text{ maximal in } L\backslash\uparrow k\}.$$

Since p is completely irreducible, p is maximal in $L\backslash\uparrow k$ for some $k \in K(L)$.

(ii) A completely irreducible element p is irreducible and hence prime in a distributive lattice by I-3.12. Therefore $U = L\backslash\downarrow p$ is an open filter. Set $k = \inf U$. Since L is join continuous (O-4.1), then

$$p \vee k = p \vee \inf U = \inf(p \vee U) \geq \min(\uparrow p\backslash\{p\}),$$

since $p \vee U \subseteq \uparrow p\backslash\{p\}$. Thus $k \in U$, and so $\uparrow k = U$. This shows $k \in K(L)$ by I-4.24. $\square$

Exercises

We know from I-1.7(5) that for a locally compact topological space X the lattice $\mathcal{O}(X)$ is continuous, and that in the case of Hausdorff spaces by I-1.9 that X is locally compact iff $\mathcal{O}(X)$ is a continuous lattice. Let us look at these facts in the light of the algebraic lattices considered in the present section.

Exercise I-4.28. Let X be a topological space. Prove the following.

(i) An open set is a compact element of the lattice $\mathcal{O}(X)$ iff it is compact.
(ii) The lattice $\mathcal{O}(X)$ is algebraic iff the space X has a basis of compact open sets.
(iii) The lattice $\mathcal{O}(X)$ is arithmetic iff the space X has a basis of compact open sets which is closed under finite intersections.
(iv) If X is Hausdorff, then $\mathcal{O}(X)$ is algebraic iff $\mathcal{O}(X)$ is arithmetic iff X is totally disconnected and locally compact. Moreover, $K(\mathcal{O}(X))$ is a complete lattice iff X is compact extremally disconnected.

Hint. We note that there is a small point in the proof of the characterization of arithmetic topologies which needs to be observed. One may wish first to note the following lemma. Let L be a distributive algebraic lattice (i.e., an algebraic frame). If $B \subseteq K(L)$ is such that $BB \subseteq B$ and $x = \sup(\downarrow x \cap B)$ for all $x \in L$, then L is arithmetic. $\square$

We remark that the terminology of calling the elements x in a lattice with $x \ll x$ *compact* is motivated by the example of $\mathcal{O}(X)$. Example I-4.15 would suggest we call these elements *finite*, and indeed this terminology has also been utilized.

Exercise I-4.29. Let G be a locally compact group with identity component G_0 and suppose that G/G_0 is compact. Let L be the lattice of compact normal subgroups. Show that $(L \cup \{G\})$ is algebraic.

Hint. This requires considerable insight into the structure of locally compact groups. First, L is a complete lattice; in particular, it has a maximal element. Second, one shows that N is compact in L^{op} iff G/N is a Lie group. Third, one applies the fact that every locally compact group H with H/H_0 compact (e.g., $H = G/N$) is a projective limit of Lie groups. This is tantamount to saying that every compact normal subgroup N is the intersection of compact normal subgroups M for which G/M is a Lie group. $\square$

Exercise I-4.30. A *gap* in a totally ordered set is a pair of elements $u < v$ with nothing strictly in between. Show that a totally ordered and complete set L is an algebraic lattice iff whenever $x < y$ there is a gap where $x \leq u < v \leq y$. (In this case there is no distinction between algebraic and arithmetic.) $\square$

Exercise I-4.31. Let L be an algebraic domain, respectively an algebraic semilattice. Show that the poset of open filters OFilt(L) is an algebraic domain, respectively an algebraic semilattice. $\square$

Notice that even if we start with an algebraic lattice L, then OFilt(L) in general is just an algebraic semilattice. We will see later that every algebraic lattice can be represented as OFilt(S) for some algebraic semilattice S.

Exercise I-4.32. From I-1.10(5) we know that the sup map $r\colon \mathrm{Id}\, L \to L$ preserves all existing sups and infs, if L is a domain. In this sense, show that each continuous semilattice is a quotient of an arithmetic domain.

Exercise I-4.33. Let L be a poset. If P is a **dcpo** and $f\colon L \to P$ is an order preserving function, show that there exists a unique $F\colon \mathrm{Id}\, L \to P$ such that F preserves directed sups and $F(\downarrow x) = f(x)$ for each $x \in L$. (See [Markowsky and Rosen, 1976].) $\square$

Exercise I-4.34. Let L be a **dcpo** and $S = k(L)$ the image of a kernel operator $k: L \to L$ preserving directed sups. Show that an element $x \in S$ is compact in S iff x is compact in L, that is $K(S) = S \cap K(L)$.

Hint. Use the Remark after I-2.3. □

Old notes

This section links the framework of continuous lattices which we discussed in Sections I-1, I-2 and I-3 with the classical theory of algebraic lattices. These appear now as a special case of continuous lattices. Algebraic lattices were invented in the 1940s by G. Birkhoff and O. Frink [Birkhoff and Frink, 1948] and L. Nachbin [Nachbin, 1949], who independently and in their own ways conceived of the idea of compact elements in a lattice. In the thirty years of their history, algebraic lattices have become a part of the textbook literature of lattice theory and universal algebra, notably because of their applications to the theory of congruence lattices and lattices of subalgebras in universal algebras. The close relationship between algebraic lattices and the topological algebra of compact semilattices and their character theory was emphasized in [Hofmann *et al.*, B1974]. These matters will be touched upon in Chapters III and IV; in the meantime, I-4.17 gives a flavor of this theory.

The examples of algebraic lattices given in I-4.10 and I-4.15 are more or less standard. The fact that the class of algebraic lattices is not closed under the formation of quotients (I-4.17 ff.) is the source of complications which were recognized in topological algebra by A. D. Wallace and R. J. Koch. The classification of those algebraic lattices all of whose quotients are likewise algebraic was accomplished by [Hofmann *et al.*, B1974] and by [Hofmann and Mislove, 1977]. The facts about closure operators on algebraic lattices (I-4.13, I-4.14, I-4.16) are classical. Closure operators preserving directed sups have been called *inductive* or *algebraic* in the literature (see also [Scott, 1976], notably pp. 549–553). The representation theorem I-4.18 of continuous lattices is a combination of results of [Scott, 1972a] and [Hofmann and Stralka, 1976]. The results in I-4.25 and I-4.27 are classics due to R. P. Dilworth and P. Crawley [Dilworth and Crawley, 1960].

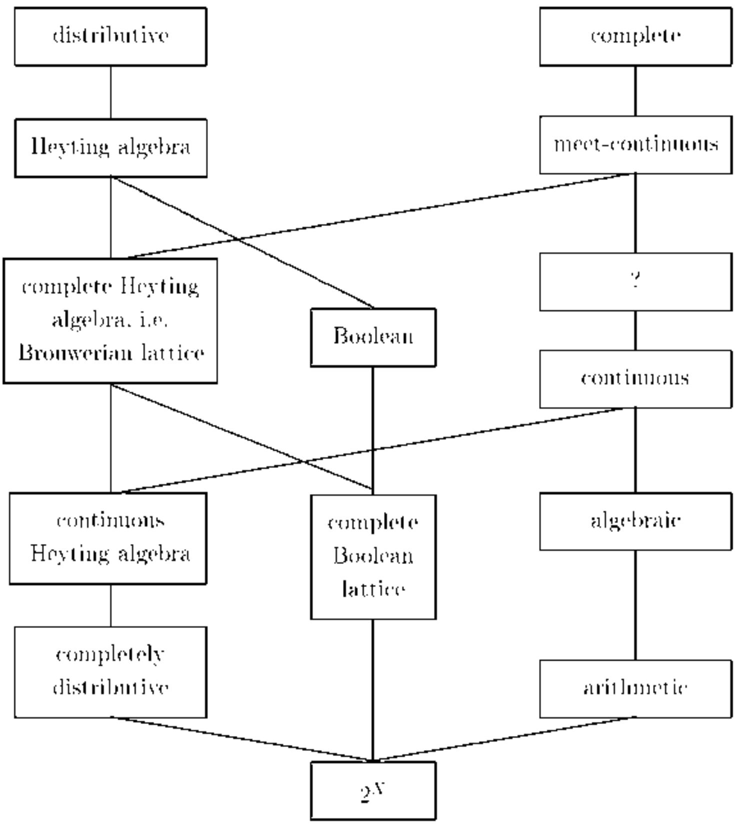
The concept of an algebraic poset was formulated by R.-E. Hoffmann [Hoffmann, 1979a].

New notes

In a recent paper K. V. Adaricheva, V. A. Gorbunov and M. V. Semonova [Adaricheva *et al.*, 2001] consider noncomplete algebraic lattices and exhibit interesting connections to the algebraic theory of lattices: Finitely presented lattices, finitely generated free lattices and lattices freely generated by a finite partially ordered set are algebraic, dually algebraic and linked bicontinuous (see VII-2.5).

We conclude the chapter by depicting the hierarchy of some of the classes of complete lattices which we discussed.

The Hierarchy of Lattices According to Their Distributivity and Continuity Properties



In place of the question mark the hierarchy will be refined in Chapters V and VI. The most noteworthy block in this subhierarchy will be that of compact semilattices.

II

The Scott Topology

In Chapter I we encountered the rich order theoretic structure of complete lattices and of continuous lattices. Wherever it was feasible to express statements on the level of generality of **dcpo**s and domains we did so. Perhaps even more typical for these partially ordered sets is their wealth of topological structure. The aim of the present chapter is to introduce topology into the study – a program to be continued in Chapter III.

Section II-1 begins with a discussion of the Scott topology and its connection with the convergence given in order theoretic terms by lower limits, or liminfs. This leads to a characterization theorem for domains in terms of properties of their lattices of Scott open sets (II-1.14) – a type of theorem that will become a recurrent theme (see Chapter VII). One motivation for such considerations arises from the appearances of domain theory in theoretical computer science: one typically needs the generality of domains to model the structures and constructions under consideration, while continuous lattices enter the scene as their lattices of open sets.

In Section II-2 we determine that the functions continuous for the Scott topology are those preserving directed sups. We can thus express one and the same property of a function between **dcpo**s either in topological or in order theoretical terms. The space $[S \to T]$ of all Scott-continuous functions between *continuous* lattices is itself a continuous lattice, and the category of continuous lattices proves to be *cartesian closed*. We also identify other more general cartesian closed categories of domains.

At this point we know that every continuous lattice is a topological space in the Scott topology; it is T_0, compact, locally compact, and sober. But exactly which T_0 spaces arise in this fashion? Section II-3 presents the answer: they are precisely the *injective* ones.

In Section II-4 we consider spaces of continuous functions $[X, \Sigma L]$ from a space X into a nonsingleton complete lattice L equipped with the Scott topology.

The function space $[X, \Sigma L]$ carries a generalization of the compact–open topology, the Isbell topology, and an associated order, the order of specialization. Relative to this partial order $[X, \Sigma L]$ is a continuous lattice if and only if both $\mathcal{O}(X)$, the lattice of open sets of X, and L are continuous lattices. Spaces X for which $\mathcal{O}(X)$ is a continuous lattice have been given various names in the literature: quasilocally compact (as opposed to "locally compact"), semilocally bounded, core compact, CL-spaces. In this section we take another tack: we refuse to name them at all – even though they appear in several significant places (II-4.2, II-4.5, II-4.7, II-4.10, II-4.13).

II-1 The Scott Topology

The definition of the Scott topology on a **dcpo** will characterize rather than exhibit open sets; in general topology this type of definition is common in associating open sets with a class of nets given as convergent. Since we wish to make a strong case for this parallel and illustrate at the same time the relation of the Scott topology with the classical idea of semicontinuity, we take some time at first to dwell on the concept of lower semicontinuous functions.

Scott convergence

Consider an extended real-valued function $f\colon X \to \mathbb{R}^*$ on, say, a metric space X. It is *lower semicontinuous* (cf. also O-2.10 and I-1.22) if and only if it satisfies any of the following equivalent conditions:

(1) for each real number t, the set $f^{-1}(]t, \infty])$ is open in X;
(2) for any sequence x_n converging to x in X, the cluster points c of the sequence $f(x_n)$ satisfy $f(x) \leq c$;
(3) for any sequence x_n converging to x in X, $f(x) \leq \underline{\lim}_n f(x_n)$, where $\underline{\lim}_n f(x_n) = \sup_n \inf_{m \geq n} f(x_m)$.

In the above, sequences are adequate because X is metric; in more abstract settings nets would be required. Note that the range $\mathbb{R}^*$ is a complete (and, of course, continuous) lattice. In order to treat the concepts emerging in the conditions (1), (2) and (3) in a systematic fashion, we describe on an arbitrary **dcpo** that structure of convergence (with its associated topology) which pertains precisely to the idea of lower semicontinuity. Evidently, the lower limit (often referred to as liminf or $\underline{\lim}$) is a vital ingredient. We make it the subject of our first definition.

Definition II-1.1. Let L be a complete semilattice. For any net $(x_j)_{j\in J}$ we write

$$\underline{\lim}_j\, x_j = \sup_j \inf_{i\geq j} x_i,$$

and call $\underline{\lim}_j\, x_j$ the *lower limit* or the *liminf* of the net. Let $\mathcal{S}$ denote the class of those pairs $((x_j)_{j\in J}, x)$ such that $x \leq \underline{\lim}_j\, x_j$. For each such pair we say that that x is an $\mathcal{S}$*-limit* of $(x_j)_{j\in J}$ and write briefly $x \equiv_S \lim x_j$.

More generally, let L be a **dcpo**. A point $y \in L$ is an *eventual lower bound* of a net $(x_j)_{j\in J}$ in L if there exists $k \in J$ such that $y \leq x_j$ for all $j \geq k$. Let $\mathcal{S}$ denote the class of those pairs $((x_j)_{j\in J}, x)$ such that $x \leq \sup D$ for some directed set D of eventual lower bounds of the net $(x_j)_{j\in J}$. For each such pair we again say that that x is an $\mathcal{S}$*-limit* of $(x_j)_{j\in J}$ and write $x \equiv_{\mathcal{S}} \lim x_j$.

If the set of all eventual lower bounds of $(x_j)_{j\in J}$ has a supremum which is also a directed supremum of some subset of the set of eventual lower bounds (i.e., is an $\mathcal{S}$-limit of $(x_j)_{j\in J}$), then this supremum is called the *lower limit* or the *liminf* of the net, written $\underline{\lim}_j\, x_j$. □

The second definition of $\mathcal{S}$ and of the liminf for **dcpo**s, when applied to complete semilattices, agrees with the first definition of $\mathcal{S}$ and of the liminf for complete semilattices. Indeed if $\inf_{i\geq j} x_i$ exists for all $j \in J$, write $y_j = \inf_{i\geq j} x_i$. Then the collection Y of all such y_j is directed and the set of all eventual lower bounds is equal to $\downarrow Y$. Thus $\sup Y = \underline{\lim}_j x_j$. Furthermore, $((x_j)_{j\in J}, x) \in \mathcal{S}$ iff $x \leq \underline{\lim}_j\, x_j$. See Exercise II-1.27 for more details.

We remark that for any (eventually) constant net x_j with value x we have $x = \underline{\lim}\, x_j$, and that more generally for any net with $x = \underline{\lim}\, x_j$, if eventually $x_j \leq y$, then $x \leq y$ (the same holds with $\leq$ replaced by $\geq$). In the case of monotone nets (cf. O-1.2), the liminf is just the supremum. Keep in mind that $\mathcal{S}$-limits, by this definition, are far from being unique; the liminf, if it exists, is the *largest* and the set of $\mathcal{S}$-limits in this case is the lower set of the liminf.

We recall next the general relation between *convergence* and *topology*. If on any set L one is given an arbitrary class $\mathcal{L}$ of pairs $((x_j)_{j\in J}, x)$ consisting of a net and an element of L, then associated with $\mathcal{L}$ is a family of sets

$$\begin{aligned}\mathcal{O}(\mathcal{L}) = \{U \subseteq L :\ &\text{whenever } ((x_j)_{j\in J}, x) \in \mathcal{L} \text{ and } x \in U,\\ &\text{then eventually } x_j \in U\}.\end{aligned}$$

Clearly both Ø and L belong to $\mathcal{O}(\mathcal{L})$, which is closed under the formation of arbitrary unions and finite intersections; that is to say, $\mathcal{O}(\mathcal{L})$ is a topology.

By the very definition we know that, for any $((x_j)_{j\in J}, x) \in \mathcal{L}$, the element x is a limit of the net x_j relative to the topology $\mathcal{O}(\mathcal{L})$. Since, however, Ø and L may very well be the only elements of $\mathcal{O}(\mathcal{L})$, we are obviously not saying very much; specific information on $\mathcal{L}$ must become available before one can hope to get a close link between $\mathcal{L}$ and $\mathcal{O}(\mathcal{L})$. (A canonical reference for the relation between convergence and topology in this framework is [Kelley, B1955], Chapter II.) Fortunately, in our present situation, we do have specific information about our class $\mathcal{S}$. We begin exploiting it by characterizing the sets $U \in \mathcal{O}(\mathcal{S})$.

Lemma II-1.2. *Let L be a* **dcpo** *and $U \subseteq L$. Then $U \in \mathcal{O}(\mathcal{S})$ iff the following two conditions are satisfied:*

(i) $U = \uparrow U$;
(ii) $\sup D \in U$ *implies* $D \cap U \neq \emptyset$ *for all directed sets* $D \subseteq L$.

In (ii) *directed sets may be replaced by ideals.*

Proof: First, suppose $U \in \mathcal{O}(\mathcal{S})$. To prove (i), assume $u \in U$ and $u \leq x$. Then $u \leq x = \underline{\lim}\, x$ with the constant net (x) with value x, so by definition $((x), u) \in \mathcal{S}$. Since we have that $u \in U \in \mathcal{O}(\mathcal{S})$, we conclude from the definition of $\mathcal{O}(\mathcal{S})$ that the net (x) must be eventually in U. This means $x \in U$.

In order to prove (ii), let D be a directed set in L with $\sup D \in U$. Consider the net $(x_d)_{d \in D}$ with $x_d = d$. Now $\inf_{c \geq d} x_c = d$, and thus $\underline{\lim}\, x_d = \sup D \in U \in \mathcal{O}(\mathcal{S})$. Since $((x_d)_{d \in D}, \sup D) \in \mathcal{S}$, we conclude that $d = x_d$ is eventually in U; whence $D \cap U \neq \emptyset$.

Second, suppose that U satisfies (i) and (ii). We take $((x_j)_{j \in J}, x) \in \mathcal{S}$ with $x \in U$, and we must show that x_j is eventually in U. By the definition of $\mathcal{S}$, we have $x \leq \sup D$ for some directed set D of eventual lower bounds of $(x_j)_{j \in J}$. Then $x \in U$ implies $\sup D \in U$ by (i), and then $d \in U$ for some $d \in D$ by (ii). By definition $d \leq x_i$ for all $i \geq j$ for some $j \in J$. Again by (i), $x_i \in U$ for all $i \geq j$. Thus $U \in \mathcal{O}(\mathcal{S})$.

The equivalence of (ii) with ideals in place of directed sets is immediate in the presence of condition (i). □

From our previous remarks we know that the sets U satisfying the conditions in II-1.2 form a topology, and it is simple enough to verify this directly. The point of our discussion was to show that this is a naturally arising topology, because liminf convergence is natural in any complete lattice. This topology will thus be officially named.

Definition II-1.3. A subset U of a **dcpo** L is called *Scott open* iff it satisfies the conditions of II-1.2. The complement of a Scott open set is called *Scott closed*. The collection of all Scott open subsets of L will be called the *Scott topology* of L and will be denoted by $\sigma(L)$.

We say that a subset X of a **dcpo** L *has the property* (S) provided that the following condition is satisfied:

(S) If $\sup D \in X$ for any directed set D, then there is a $y \in D$ such that $x \in X$ for all $x \in D$ with $x \geq y$. $\square$

Remark. Back in I-3.1 we had introduced a notion of openness for upper sets in order to be able to talk about open filters. This now becomes fully justified, as the open upper sets introduced there are precisely the Scott open sets. In the following, when we talk about an open filter, we shall mean a Scott open filter which is the same as an open filter in the sense of I-3.1.

We have thus far motivated the Scott topology from the classical notion of semicontinuity, but there are also strong motivations coming from theoretical computer science. Suppose that members of a **dcpo** stand for states of information or knowledge and the partial order is the information order: $x \leq y$ if and only if y represents at least as high a state of knowledge as x. One may view a directed set as stages of a computation and its supremum as the total information uncovered by all stages of the computation. It is then natural to view the states of the directed set as converging toward the supremum. But not only that, the directed set converges also to all lesser states, since they also uncover (in the limit) all information in those lower states also. By these considerations one is again led to a family of convergent nets (much sparser than our earlier collection $\mathcal{S}$), but one that again yields the Scott topology (see Exercise II-1.28).

Remark II-1.4. *In any* **dcpo** L *we have the following conclusions:*

(i) *a set is Scott closed iff it is a lower set closed under directed sups;*
(ii) $\downarrow x = \{x\}^-$ *(closure with respect to* $\sigma(L)$*) for all* $x \in L$*;*
(iii) $\sigma(L)$ *is a* T_0*-topology;*
(iv) *every upper set is the intersection of its Scott open neighborhoods;*
(v) *a set is Scott open iff it is an upper set satisfying* (S)*;*
(vi) *every lower set has property* (S)*;*
(vii) *the collection of all subsets having property* (S) *is a topology.*

Proof: (i) $A \subseteq L$ is a lower set iff $L \backslash A$ is an upper set, and $L \backslash A$ satisfies II-1.2(ii) iff A is closed under directed sups.

(ii) We have that $\downarrow x$ is the smallest lower set containing x, and it happens to be closed under directed sups.

(iii) If $\{x\}^- = \{y\}^-$, then $\downarrow x = \downarrow y$ by (ii); thus $x = y$.

(iv) Every upper set B is the intersection of the sets $L \backslash \downarrow x$ where $x \in L \backslash B$. These sets are open in view of (ii).

(v) and (vi): Immediate.

(vii) The intersection of two sets satisfying (S) will again satisfy (S), and any union of sets satisfying (S) will satisfy (S). Since Ø and L clearly satisfy (S), the assertion follows. □

The definition of Scott open sets provides a characterization but not a procedure for building Scott open sets in general – except for the rather meager information of II-1.4(ii) above. It is therefore important that we familiarize ourselves with some examples.

Examples II-1.5. Let L be a **dcpo**.

(1) If L is finite, then the Scott open sets are just the upper sets.

(2) If L is a chain, then the sets $]x, 1] = \uparrow x \backslash \{x\} = L \backslash \downarrow x$ are Scott open for any $x \in L$; and together with L these are all the Scott open sets.

(3) For the chain $2 = \{0, 1\}$, we have $\sigma(2) = \{\emptyset, \{1\}, \{0, 1\}\}$. The space 2 with this topology is well known under the name of *Sierpiński space*.

(4) If $L = 2^X$, the powerset space, we have $\sigma(L)$ equal to the well-known collection of families of sets called *families of finite character*. These are families $\mathcal{F}$ such that $S \in \mathcal{F}$ iff $F \in \mathcal{F}$ for some finite subset F of S.

(5) If $L = [0, 1]^2$, the square with the componentwise order, a subset U is Scott open iff it is an upper set and is open in the ordinary topology induced by the plane. Here is a typical picture:

We leave this characterization as an exercise, since later on we will have enough theory to make this an easy consequence. We can see at this point, however, that every Scott open set of $[0, 1]^2$ is the union of open upper rectangles. Note that these rectangles are the intersection of two sets of the form $L \backslash \downarrow x$. □

Proposition II-1.6. *If L is a domain, then all sets $\twoheaduparrow x$ for $x \in L$ are Scott open. Conversely, if L is a* **dcpo** *and $y \in \mathrm{int}(\uparrow x)$, then $x \ll y$.*

Proof: Let D be a directed set with $\sup D \in \twoheaduparrow x$. Then Theorem I-1.9(i) implies the existence of a $d \in D$ such that $x \ll d$. Hence $\twoheaduparrow x$ is open by Definition II-1.3.

Suppose L is a **dcpo** and $y \in \mathrm{int}(\uparrow x)$. If D is a directed set with $y \leq \sup D$, then $\sup D \in \mathrm{int}(\uparrow x)$ by Lemma II-1.2(i), and hence $d \in \mathrm{int}(\uparrow x)$ for some $d \in D$ by II-1.2(ii). Thus $x \leq d$, and it follows that $x \ll y$. □

We will show below in II-1.10 that, in a domain, the sets $\Uparrow x$, $x \in L$, form a basis for $\sigma(L)$, and that $\Uparrow x$ is in fact the interior of $\uparrow x$ with respect to this topology. Thus, in a domain, the way-above sets of single points provide a good supply of relatively small Scott open sets (small, given the restriction that they all have to be upper sets). Note that it suffices for the openness of all $\Uparrow x$ that $\ll$ satisfy the interpolation property (INT) from I-1.9(ii). Observe also, that, in general, the Scott topology on a **dcpo** is neither the coarsest nor the finest of all of the T_0-topologies for which $\downarrow x = \{x\}^-$.

In order to complete the story we must return to the discussion of the concept of convergence and investigate whether the Scott topology (which we derived from a convergence concept) is in fact adequate to describe in topological terms the $\mathcal{S}$ (or liminf) convergence. If $\mathcal{S}$ is precisely the class of convergent nets for the Scott topology, then we say that $\mathcal{S}$ is topological.

Proposition II-1.7. *Let L be a domain. Then*

$$x \equiv_{\mathcal{S}} \lim x_j \textit{ iff the net } (x_j)_{j\in J} \to x \textit{ with respect to the Scott topology } \sigma(L).$$

In particular, $\mathcal{S}$-convergence is topological.

Proof: By definition of the Scott topology, if $x \equiv_{\mathcal{S}} \lim x_j$, then $(x_j)_{j\in J} \to x$ with respect to $\sigma(X)$. Conversely, suppose that we have a convergent net $(x_j)_{j\in J} \to x$ in the Scott topology. For each $y \in \Downarrow x$, we have that $\Uparrow y$ is a Scott open set containing x by Proposition II-1.6. Thus the net $(x_j)_{j\in J}$ is eventually in $\Uparrow y$, and hence y is an eventual lower bound for the net. Since $\Downarrow x$ is directed and has supremum x, we have $((x_j)_{j\in J}, x) \in \mathcal{S}$. $\square$

The converse is also true.

Lemma II-1.8. *Let L be a* **dcpo**. *If the $\mathcal{S}$-convergence is topological, then L is a domain.*

Proof: By Lemma II-1.2 the topology arising from $\mathcal{S}$-convergence is the Scott topology. Thus if $\mathcal{S}$-convergence is topological, we must have

$$x \equiv_{\mathcal{S}} \lim x_j \text{ iff the net } (x_j)_{j\in J} \to x \text{ with respect to } \sigma(L).$$

Let $x \in L$. Define

$$I = \{(U, n, a) \in \mathcal{N}(x) \times \mathbb{N} \times L : a \in U\},$$

where $\mathcal{N}(x)$ consists of all Scott open sets containing x, and define an order on I to be the lexicographic order on the first two coordinates, that is, $(U, m, a) < (V, n, b)$ iff V is a proper subset of U or $U = V$ and $m < n$. Let $x_i = a$ for $i = (U, n, a) \in I$ define the net. Then it is easy to see that $(x_i)_{i\in I}$ converges to x in the Scott topology. Thus $x \equiv_{\mathcal{S}} \lim x_i$, and we conclude that there exists a directed set D of eventual lower bounds of $(x_i)_{i\in I}$ such that $x \leq \sup D$. Let $d \in D$. Then there exists $i = (U, m, a) \in I$ such that $(V, n, b) = j \geq i$ implies $d \leq b$. In particular, we have $(U, m+1, b) > (U, m, a)$ for all $b \in U$, and thus d is a lower bound for U, i.e., $x \in \operatorname{int} \uparrow d$. By Proposition II-1.6, $d \ll x$. Since D is directed with supremum greater than or equal to x, we conclude that x is the directed supremum of $D \subseteq \,\twoheaddownarrow x$. Since x was arbitrary, we conclude that L is a domain. □

What we now have proved is the following characterization of domains.

Theorem II-1.9. *For a* **dcpo** L *the following statements are equivalent:*

(1) *$\mathcal{S}$-convergence is topological convergence for the Scott topology; that is, for all $x \in L$ and all nets (x_j) on L*

$$x \equiv_{\mathcal{S}} \lim x_j \text{ iff } (x_j)_{j\in J} \text{ converges to } x \text{ with respect to } \sigma(L);$$

(2) *L is a domain.* □

The Scott topology of domains

Having recognized convergence as the essential ingredient in the study of lower semicontinuity, we can say after Theorem II-1.9 that among **dcpo**s it is precisely the domains that allow the study of lower semicontinuity completely in topological terms. Nevertheless, the Scott topology in itself remains a highly useful tool in treating arbitrary **dcpo**s.

Proposition II-1.10. *Let L be a domain.*

(i) *An upper set U is Scott open iff for every $x \in U$ there is a $u \in U$ such that $u \ll x$.*
(ii) *The sets of the form $\twoheaduparrow u$, $u \in L$, form a basis for the Scott topology. In particular, each point $x \in L$ has a $\sigma(L)$ neighborhood basis consisting of the sets $\twoheaduparrow u$ with $u \ll x$.*
(iii) *With respect to $\sigma(L)$, we have* $\operatorname{int} \uparrow x = \,\twoheaduparrow x$.
(iv) *With respect to $\sigma(L)$, we have for any subset $X \subseteq L$*

$$\operatorname{int} X = \bigcup \{\twoheaduparrow u \colon \twoheaduparrow u \subseteq X\}.$$

Proof: (i) Let U be Scott open and $x \in U$. As in a domain the set $\twoheaddownarrow x$ is directed and has x as its sup, we conclude that there is a $u \ll x$ with $u \in U$ by II-1.2(ii). If conversely for every $x \in U$ there is a $u \in U$ such that $u \ll x$ then U is the union of the sets $\twoheaduparrow u$, $u \in U$, which are Scott open by II-1.6; hence, U is Scott open.

Part (ii) is an immediate consequence of (i).

(iii) If $y \in \operatorname{int}\uparrow x$, then by (i) there is a $u \in \uparrow x$ with $u \ll y$. But then $y \in \twoheaduparrow x$. Obviously $\twoheaduparrow x \subseteq \operatorname{int}\uparrow x$.

(iv) This follows directly from (ii). $\square$

We recall at this point that every topology is a lattice, and indeed a frame (remember O-2.7(3)!). It is therefore meaningful to search for prime and co-prime elements in $\sigma(L)$ (see I-3.11–I-3.16).

To formulate one of our conditions it is useful to speak of the continuity of an operation (the main topic of the next section). We say that the sup operation is *jointly continuous with respect to the Scott topology* provided that the mapping

$$(x, y) \mapsto x \vee y : (L, \sigma(L)) \times (L, \sigma(L)) \to (L, \sigma(L))$$

is continuous in the product topology.

We wish to warn the reader about a subtlety concerning the joint continuity of the sup operation above. We cannot be satisfied by saying that the sup operation is a continuous function $(L \times L, \sigma(L \times L)) \to (L, \sigma(L))$; this continuity is weaker, since in general we have a proper containment of topologies: $\sigma(L \times L) \supset \sigma(L) \times \sigma(L)$. We will return to this question at greater length in Section II-4 below (see II-4.13 ff.).

Remark. In the process of classifying co-primes we shall need the concept of open filters which we encountered for the first time in I-3.1 through I-3.3. We have already remarked before that open as defined there is the same as Scott open. In the following, an open filter is always understood to be Scott open. We remind the reader that we always assume that filters are nonempty. Recall that, for a **dcpo** L,

$$\mathrm{OFilt}(L) = \{F \subseteq L : F \text{ is an open filter}\}$$

denotes the set of open filters of L. We will always consider $\mathrm{OFilt}(L)$ to be ordered by inclusion. Since the union of a directed family of open filters is again an open filter, $\mathrm{OFilt}(L)$ is a **dcpo**. We note that the intersection of two (open) filters is not in general a filter even if nonempty. However, on a semilattice the intersection of two filters is a filter, if nonempty. Thus if L is a semilattice with a top element, then the poset $\mathrm{OFilt}(L)$ is a semilattice, too. Similarly, if L is a sup semilattice, then the intersection of two filters is a filter and, consequently, the poset $\mathrm{OFilt}(L)$ is also a semilattice.

Now we are ready for the characterization of the primes and the co-primes of $\sigma(L)$:

Proposition II-1.11. *Let L be a* **dcpo** *and U a Scott open subset of L.*

(i) *U is a co-prime in $\sigma(L)$ iff $U \in \mathrm{OFilt}(L)$.*
(ii) *If $U = L \backslash \downarrow a$ for some $a \in L$, then U is a prime in $\sigma(L)$, and all primes $U \neq L$ in $\sigma(L)$ are of this form provided that L is* (1) *a domain, or* (2) *a sup semilattice with a jointly Scott-continuous sup operation. Hence, for a domain,*

$$\text{CO-PRIME}(\sigma(L)) = \mathrm{OFilt}(L), \quad \textit{and} \quad \text{PRIME}(\sigma(L)) = \{L \backslash \downarrow u \colon u \in L\}.$$

Proof: (i) Firstly suppose that $U \in \sigma(L)$ is a filter and that U is not a co-prime in $\sigma(L)$. Then there are $V, W \in \sigma(L)$ such that $U \subseteq V \cup W$ and elements $v \in U \backslash V$ and $w \in U \backslash W$. Let $z \in U$ satisfy $z \leq v$ and $z \leq w$. Since V and W are upper sets we have $z \notin V \cup W$, a contradiction to $z \in U$.

Secondly, suppose that U is a co-prime in $\sigma(L)$. To show that U is a filter, note first that it is an upper set. Suppose $v, w \in U$. Then $U \not\subseteq L \backslash \downarrow v$ and $U \not\subseteq L \backslash \downarrow w$. By II-1.4(ii), the sets $L \backslash \downarrow v$ and $L \backslash \downarrow w$ are Scott open. Thus, since U is co-prime,

$$U \not\subseteq ((L \backslash \downarrow v) \cup (L \backslash \downarrow w)) = L \backslash (\downarrow v \cap \downarrow w).$$

Thus there is a $u \in U$ such that $u \leq v$ and $u \leq w$.

(ii) Now let $U \in \sigma(L), U \neq L$. Assume first that $U = L \backslash \downarrow a$ for some $a \in L$. Recall that $\downarrow a = \{a\}^-$ by II-1.4(ii) and that complements of singleton closures are prime in any topology (see I-3.14).

For the converse, assume that U is prime in $\sigma(L)$, and let $A = L \backslash U$ be its complement which is Scott closed. We have to show that A has a largest element a; since A is a lower set, this will show $A = \downarrow a$ as desired.

Let us begin with case (1), where L is a domain. Let

$$A^* = \{b \in A : \text{ there is an } a \in A \text{ with } b \ll a\} = \bigcup \{ \twoheaddownarrow a \colon a \in A\} = \twoheaddownarrow A.$$

We claim that A^* is directed: Let $b, c \in A^*$; we first show $\twoheaduparrow b \cap \twoheaduparrow c \cap A \neq \emptyset$. If not, then $\twoheaduparrow b \cap \twoheaduparrow c \subseteq U$; but $\twoheaduparrow b, \twoheaduparrow c \in \sigma(L)$ by II-1.6. Since U is prime, we conclude $\twoheaduparrow b \subseteq U$ or $\twoheaduparrow c \subseteq U$; but $\twoheaduparrow b$ contains an $a \in A = L \backslash U$ which is impossible; similarly $\twoheaduparrow c \subseteq U$ is impossible.

Pick $a \in \twoheaduparrow b \cap \twoheaduparrow c \cap A$. Since $\twoheaduparrow b \cap \twoheaduparrow c$ is Scott open, there exists $d \ll a$ such that $d \in \twoheaddownarrow b \cap \twoheaddownarrow c$ by II-1.10(i). Thus $d \in A^*$ and d is a common upper bound for b and d. Thus A^* is directed and $a = \sup A^*$ exists. Since A is Scott closed,

$a \in A$. Now let $x \in A$. Then $x = \sup \Downarrow a$ since L is a continuous poset; but $\Downarrow a \subseteq A^*$ implies $x = \sup \Downarrow x \leq \sup A^* = a$. Thus $a = \max A$ as was desired.

Now let us turn to case (2), where L is a sup semilattice and the sup operation is jointly continuous relative to the Scott topology. We will verify that A is directed. Then $a = \sup A$ exists, and it belongs to A, as A is Scott closed.

By way of contradiction assume that A is not directed. Then there are elements $b, c \in A$ with $b \vee c \in U$. By the continuity of the sup operation we would find Scott open neighborhoods V and W of b and c, respectively, such that $V \vee W \subseteq U$. But since V and W are upper sets, we have $V \vee W = V \cap W$. Since U is prime, the relation $V \cap W \subseteq U$ implies $V \subseteq U$ or $W \subseteq U$. This would entail $b \in U$ or $c \in U$, which would contradict $b, c \in A = L \backslash U$. □

Remark. *In a continuous lattice L, the sup operation is jointly continuous with respect to the Scott topology. (Thus the preceding proposition contains two alternative proofs for the characterization of primes in $\sigma(L)$ for continuous lattices.)*

Proof: In order to show the continuity of the sup operation at (a, b) we pick some $u \ll a \vee b$. By I-1.6 we have

$$a \vee b = (\sup \Downarrow a) \vee (\sup \Downarrow b) = \sup(\Downarrow a \vee \Downarrow b).$$

Since $\Downarrow a \vee \Downarrow b$ is directed with supremum $a \vee b$, we find some $x \ll a$ and $y \ll b$ with $u \ll x \vee y$. But then $\Uparrow x$ and $\Uparrow y$ are Scott open neighborhoods of a and b, respectively, such that

$$\Uparrow x \vee \Uparrow y \subseteq \uparrow(x \vee y) \subseteq \Uparrow u.$$

Since the $\Uparrow u$ with $u \ll a \vee b$ form a basis of $\sigma(L)$ neighborhoods of $a \vee b$ by Proposition II-1.10(ii), the desired continuity is established. □

We can immediately rephrase II-1.11 in topological terminology, if we recall the concept of a *sober space* (see O-5.6). Remember from O-5.5 that a nonempty closed subset A of a topological space X is called *irreducible* iff it is not the union of two proper nonempty closed subsets (that is, the complementary set $X \backslash A \in \mathrm{PRIME}\ \mathcal{O}(X)$). A space X is called *sober* iff every irreducible closed set A has a unique dense point (that is, $A = \{a\}^-$ for a unique $a \in A$). We now have the following corollaries of II-1.11 with a slight sharpening.

Corollary II-1.12. *If L is* (1) *a domain, or* (2) *a* **dcpo** *and a sup semilattice such that the sup operation is jointly Scott-continuous, then* $(L, \sigma(L))$ *is a sober space.*

Proof: Immediate from II-1.11 and the definitions. □

Corollary II-1.13. *If L is a domain, then $(L, \sigma(L))$ is a locally compact sober space. In particular, $(L, \sigma(L))$ is a Baire space. If L has a smallest element, then $(L, \sigma(L))$ is compact.*

Proof: If $x \in U \in \sigma(U)$, then by II-1.10(ii) there exists $y \in U$ such that $x \in \twoheaduparrow y \subseteq \uparrow y \subseteq U$. Since $\uparrow y$ (and hence, in particular $L = \uparrow 0$) is trivially compact with respect to any topology whose open sets are upper sets, the assertion is proved. That $(L, \sigma(L))$ is a Baire space follows from I-3.40.9. □

We know enough about the Scott topology now to use it for yet another characterization theorem for domains.

Theorem II-1.14. *For any* **dcpo** *L, the following conditions are equivalent:*

(1) *L is a domain;*
(2) *each $\twoheaduparrow x$ is open, and if $U \in \sigma(L)$, then $U = \bigcup\{\twoheaduparrow x \colon x \in U\}$;*
(3) OFilt(L) *is a basis of $\sigma(L)$ and $\sigma(L)$ is a continuous lattice;*
(4) *$\sigma(L)$ has enough co-primes and is a continuous lattice;*
(5) *$\sigma(L)$ is completely distributive;*
(6) *both $\sigma(L)$ and $\sigma(L)^{\mathrm{op}}$ are continuous.*

If L is a complete semilattice then these conditions are equivalent to

(7) *for each point $x \in L$ we have $x = \sup\{\inf U \colon x \in U \in \sigma(L)\}$.*

Proof: (1) implies (2): Use II-1.6 and II-1.10.

(2) implies (1): Let $x \in L$. If $u \ll x$, $v \ll x$, then there exists $w \in \twoheaduparrow u \cap \twoheaduparrow v$ with $x \in \twoheaduparrow w$ by hypothesis. Thus $\twoheaddownarrow x$ is directed. Set $y = \sup \twoheaddownarrow x \le x$. If $y < x$, then $L \setminus \downarrow y$ is a Scott open neighborhood of x; hence by (2) it contains an open neighborhood $\twoheaduparrow z$ of x with $z \in L \setminus \downarrow y$. But then $z \ll x$, and thus $z \le \sup \twoheaddownarrow x = y$, a contradiction.

(2) implies (3): By (2), x has arbitrarily small neighborhoods of the form $\twoheaduparrow y$ with $y \ll x$. By II-1.10 and I-3.3, we know then that x has arbitrarily small Scott open neighborhoods which are filters. In order to prove the continuity of $\sigma(L)$, we let U be Scott open. For any $x \in U$ we find a $y \in U$ with $y \ll x$ by (2). Then $x \in \twoheaduparrow y \in \sigma(L)$, and we claim that $\twoheaduparrow y \ll U$: Indeed, if D is a directed family of Scott open sets covering U, then one of its members must contain y, hence $\uparrow y$, since Scott open sets are upper sets, and thus it contains $\twoheaduparrow y$. We have shown $U = \bigcup\{V \colon V \ll U\}$.

(3) implies (1): Let $x \in U \in \sigma(U)$. There exists $V \in \sigma(L)$ such that $x \in V \ll U$, since $\sigma(L)$ is continuous. Pick an open filter F such that $x \in F \subseteq V$. Suppose that for each $y \in U$, it is not the case that $F \subseteq \uparrow y$. Then $y \in L \setminus \downarrow z$ for some $z \in F$, and hence there exists an open filter F_y such that $y \in F_y \subseteq L \setminus \downarrow z$,

since the open filters form a basis. Finitely many of the F_y, say $F_{y_1}, \ldots, F_{y_n}$, must cover V. Pick $z_i \in F \backslash F_{y_i}$ for each i and pick $z \in F$ such that $z \leq z_i$ for all i (remember F is a filter). Then $z \in F_{y_i}$ would imply $z_i \in F_{y_i}$, so none of the F_{y_i} contain z, a contradiction. Thus there exists $y \in U$ such that $x \in F \subseteq \uparrow y$.

For $x \in L$, consider $D = \{y \in L : x \in \operatorname{int}(\uparrow y)\}$. Then $y \ll x$ for each $y \in D$ by Proposition II-1.6. Furthermore since by the previous paragraph every Scott open set containing x contains a member of D, it follows easily that D is directed. Finally $\sup D = x$ by an argument similar to that given in (2) implies (1). Thus L is a domain.

(3) iff (4): Clear from II-1.11.

(4) iff (5) iff (6): Consequence of I-3.16.

(3) implies (7): Now assume that L is a complete semilattice. Then for each $x \in L$ and each $\sigma(L)$ neighborhood U of x the element $\inf U$ exists. For $x \in L$ set

$$y = \sup\{\inf U \colon x \in U \in \sigma(L)\} \leq x.$$

If $y < x$, then $L \backslash \downarrow y$ is a Scott open neighborhood of x. Let V be a Scott open neighborhood of x with $V \ll L \backslash \downarrow y$, which exists since $\sigma(L)$ is continuous by (3). Now use (3) to find a Scott open filter neighborhood U of x within V. By the definition of y we have $\inf U \leq y$. Then

$$L \backslash \downarrow y \subseteq L \backslash \downarrow \inf U = L \backslash \bigcap\{\downarrow u \colon u \in U\} = \bigcup\{L \backslash \downarrow u \colon u \in U\}.$$

Since U is a filter, the $L \backslash \downarrow u$ for $u \in U$ form a directed family of Scott open sets. Since $V \ll L \backslash \downarrow y$, there must be a $u \in U$ such that $V \subseteq L \backslash \downarrow u$ and so $u \notin V$. This is a contradiction to $U \subseteq V$. Thus $x = y$.

(7) implies (1): Clear since for every Scott open neighborhood U of x one has $\inf U \ll x$. $\square$

A parallel result to II-1.14 for algebraic lattices reads as follows.

Corollary II-1.15. *For any* **dcpo** *L, the following conditions are equivalent.*

(1) *L is an algebraic domain.*
(2) *The Scott topology has a basis of sets $\uparrow k$ where $k \in K(L)$.*
(3) *$\sigma(L)$ is algebraic and has enough co-primes.*
(4) *$\sigma(L)$ is algebraic and completely distributive.*

Proof: (1) implies (2): Let U be a Scott open neighborhood of x. We recall that $x = \sup(\downarrow x \cap K(L))$ and $\downarrow x \cap K(L)$ is directed. Hence by II-1.2(ii), we find a $k \in \downarrow x \cap K(L) \cap U$. Then $\uparrow k = \twoheaduparrow k$ is a Scott open neighborhood of k, hence of x, with $\uparrow k \subseteq U$.

(2) implies (3): If $k \in K(L)$, then $\uparrow k \in \sigma(L)$ since $\Uparrow k = \uparrow k$. Now $\uparrow k$ is a compact set (if $\uparrow k$ is covered by Scott open sets, then one of them must contain k, hence $\uparrow k$ by II-1.2(i)). Thus $\sigma(L)$ is algebraic by I-4.28. Since all $\uparrow k$ are filters, hence co-primes by II-1.11, we are done.

As (3) and (4) are equivalent by I-3.16, it remains to show that (3) implies (1): Since $\sigma(L)$ is algebraic, the Scott topology has a basis of compact sets U. Since there are enough co-primes, U is a union of open filters by II-1.11, and thus, by compactness, $U = U_1 \cup \cdots \cup U_n$ with open filters U_k. It is no loss of generality to assume that none of the U_k is contained in the union of the others. Then $V = U_1 \backslash (U_2 \cup \cdots \cup U_n)$ is compact and filtered. We claim that V has a smallest element u_1.

For if not, then $V \subseteq \bigcup\{L \backslash \downarrow v \colon v \in V\}$; and by compactness and the fact that the $L \backslash \downarrow v$ form a directed family, there would be a $v \in V$ with $V \subseteq L \backslash \downarrow v$, notably $v \notin V$, which is impossible. Since $U_2 \cup \cdots \cup U_n$ is an upper set, it cannot contain $\inf U_1$. Hence $u_1 = \min U_1$. Since U_1 is an upper set, $U_1 = \uparrow u_1$. Since U_1 is Scott open, $u_1 \in K(L)$. Similarly $U_i = \uparrow u_i$ for each i.

We have shown that $\sigma(L)$ has a basis of sets $\uparrow k$ where $k \in K(L)$. It follows that the compact elements below any fixed element form a directed set. Now let $x \in L$, set $y = \sup(\downarrow x \cap K(L)) \leq x$. If $y < x$, then the Scott open neighborhood $L \backslash \downarrow y$ of x would contain a basic neighborhood $\uparrow k$ of x. Then there would be a $k \in K(L)$ with $k \leq x$ and $k \not\leq y$ which contradicts the definition of y. □

The Hofmann–Mislove Theorem

We close this section by showing that for a domain L the open filter **dcpo** OFilt(L) is again a domain and by pointing out that open filter **dcpo**s yield an important alternative approach to the study of the poset $Q(X)$ of compact saturated subsets (including the empty set) of a topological space ordered by reverse inclusion. This approach leads to basic topological theorems concerning sober spaces and demonstrates again close connections between domain theory and topology.

Lemma II-1.16. *Assume that L is a* **dcpo** *such that the following condition is satisfied:*

(F) *if $u \in U \in$* OFilt(L) *then there are an $u_* \in U$ and a $V \in$* OFilt(L) *such that*

$$u \in V \subset \uparrow u_*.$$

Then the poset OFilt(L) *is a domain, and*

$$V \ll U \text{ iff } (\exists x \in U) V \subseteq \uparrow x. \qquad (*)$$

Proof: We begin by proving (∗). Firstly, if $U, V \in \mathrm{OFilt}(L)$ are such that there is an $x \in U$ with $V \subseteq \uparrow x$ then $V \ll U$; indeed if $\{U_j : j \in J\}$ is a family with $U \subseteq \bigcup_{j \in J} U_j$, then there is $j \in J$ such that $x \in U_j$ and then $V \subseteq \uparrow x \subseteq U$. This proves the claim. Conversely, assume that $V \ll U$. Then by (F) for each $u \in U$ there are a $u_* \in U$ and a $V_u \in \mathrm{OFilt}(L)$ such that $u \in V_u \subseteq \uparrow u_*$. If $u_1, u_2 \in U$, then since U is a filter, there is a $u \in U$ such that $u \leq (u_1)_*, (u_2)_*$. Then $(u_n)_* \in V_u$ and thus $V_{u_n} \subseteq \uparrow (u_n)_* \subseteq V_u$ for $n = 1, 2$. Hence $\{V_u : u \in U\}$ is directed and $U \subseteq \bigcup_{u \in U} V_u$. Thus $V \ll U$ implies the existence of a $u \in U$ with $V \subseteq V_u$. Thus $V \subseteq V_u \subseteq \uparrow u_* \subseteq U$. Therefore (*) is proved.

Now let $U \in \mathrm{OFilt}(L)$. From (∗) and (F) it follows at once that $U = \sup \twoheaddownarrow U$. We must still show that $\twoheaddownarrow U$ is directed. If $V_1, V_2 \ll U$, then since $V_1, V_2 \ll U$, there are elements $u_1, u_2 \in U$ such that $V_n \subseteq \uparrow u_n$. Since U is a filter we have a $u \in U$ with $u \leq u_1, u_2$. Then by (F) there are a $u_* \in U$ and a $V \in \mathrm{OFilt}(L)$ such that $u \in V \subseteq \uparrow u_*$. Since $V_n \subseteq \uparrow u_n \subseteq \uparrow u \subseteq V$ we see that $\twoheaddownarrow U$ is indeed directed. This completes the proof. □

The set of open filters on a poset is a very natural object to study. Some caution is in order, however: the interior of a filter may not have a unique maximal open subfilter, as we have seen in the example following I-3.2.

Theorem II-1.17. *For a domain L, the poset* $\mathrm{OFilt}(L)$ *is a domain and the following statements hold.*

(i) *For* $x \ll y$ *in L there is a* $U \in \mathrm{OFilt}(L)$ *such that* $y \in U \subseteq \twoheaduparrow x$.
(ii) *For* $U, V \in \mathrm{OFilt}(L)$, $V \ll U$ *iff* $(\exists u \in U) V \subseteq \uparrow u$.

If L is a continuous semilattice, then $\mathrm{OFilt}(L)$ *is a continuous semilattice.*

Proof: Statement (i) is Proposition I-3.3(i).

Given $u \in U \in \mathrm{OFilt}(L)$, by Proposition II-1.10(i), there is a $v \in U$ such that $v \ll u$; then by (i) we find an open filter V of L such that $u \in V \subseteq \twoheaduparrow v$. Hence condition (F) of the preceding lemma is satisfied. Thus by that lemma, $\mathrm{OFilt}(L)$ is a domain and (ii) is satisfied. □

We saw in Example I-1.7(5) that $\mathcal{O}(X)$ is a continuous lattice if X is a locally compact space. Hence $\mathrm{OFilt}(\mathcal{O}(X))$ is a continuous semilattice (ordered by inclusion) by the preceding theorem. There is a noteworthy connection between this continuous semilattice and the continuous semilattice $Q(X)$ of compact saturated subsets of X (ordered by reverse inclusion) according to I-1.24.2(iv) which we shall exhibit in the following.

Lemma II-1.18. *Let X be a T_0-space. If $K \subseteq X$ is compact, then*

$$\Phi(K) = \{U \in \mathcal{O}(X) \colon K \subseteq U\}$$

is an open filter in $\mathcal{O}(X)$.

Proof: Clearly, $\Phi(K)$ is a filter. If a directed union of open sets contains a compact set, then one of the open sets must contain the compact set. Hence, this filter is open. □

This motivates the question whether all open filters of $\mathcal{O}(X)$ arise in this fashion. In the following we prove the Hofmann–Mislove Theorem which, among other things, asserts that this is true for sober spaces. The following lemma contains an essential ingredient. Recall that in our unconventional sense a filter of sets may contain the empty set.

Lemma II-1.19. *Let X be a sober topological space and $\mathcal{F}$ an open filter in the lattice $\mathcal{O}(X)$ of open subsets of X. Then*

(i) *every open set U containing $\Psi(\mathcal{F}) = \bigcap \mathcal{F}$ already belongs to $\mathcal{F}$,*
(ii) *the intersection $K = \bigcap \mathcal{F}$ is compact, saturated, and nonempty, if all $U \in \mathcal{F}$ are nonempty.*

Proof: (i) Let $\mathcal{F}$ be an open filter of nonempty open subsets of X and let K be its intersection. Let U be an open set containing K. Suppose that U is not in $\mathcal{F}$. Then there exists an open set V containing U maximal with respect to not being in $\mathcal{F}$ (by openness of $\mathcal{F}$, cf. I-3.12). One verifies from maximality that V is prime, and hence that the complement of V is an irreducible closed set, thus the closure of some point p, as X is supposed to be sober (see O-5.6). Then every $F \in \mathcal{F}$ must contain p, for otherwise F misses the closure of $\{p\}$, and hence $F \subseteq V$, which would imply $V \in \mathcal{F}$, a contradiction. Thus $\mathcal{F}$ consists of all open sets containing K.

(ii) As an intersection of open sets, K is saturated. To see that K is compact, let $\mathcal{U}$ be an open cover of K. Then $U := \bigcup \mathcal{U}$ is an open set containing K, hence $U \in \mathcal{F}$ by (i). The finite unions of members of $\mathcal{U}$ form a directed family with union U. Since $\mathcal{F}$ is Scott open, some finite union belongs to $\mathcal{F}$, and hence covers K. If $K = \bigcap \mathcal{F} = \emptyset$, then $\emptyset \in \mathcal{F}$ by (i). □

We summarize the preceding lemmas.

Theorem II-1.20. (The Hofmann–Mislove Theorem I) *Let X be a sober space. Then the mapping*

$$\Phi \colon Q(X) \to \mathrm{OFilt}(\mathcal{O}(X)), \quad \Phi(K) = \{U \in \mathcal{O}(X) : K \subseteq U\}$$

which assigns to a compact saturated subset K of X the open filter of all open sets containing K is an order isomorphism between $Q(X)$ (ordered by reverse inclusion) and OFilt$(O(X))$. The inverse sends an open filter of open sets to its intersection:

$$\Psi\colon \mathrm{OFilt}(\mathcal{O}(X)) \to Q(X), \quad \Psi(\mathcal{F}) = \bigcap \mathcal{F}.$$

Proof: The filter of open sets containing a given compact saturated set K is open by Lemma II-1.18 and has intersection K (from saturation). Conversely suppose that $\mathcal{F}$ is a Scott open filter of open sets with intersection K. Then K is compact and saturated and $\mathcal{F}$ consists precisely of those open sets containing K by Lemma II-1.19. Thus the two mappings of the theorem are inverses of each other and hence bijections. One verifies readily that they are order preserving. □

The following theorem shows that the Hofmann–Mislove Theorem holds for sober spaces only, and it relates sober spaces to well-filtered ones (see Definition I-1.24.1).

Theorem II-1.21. *Let X be a T_0 space. Consider the following statements.*

(1) *X is sober.*
(2) *Any open filter $\mathcal{F}$ of open sets consists of all open sets U containing the intersection of the filter (which is a compact saturated set).*
(3) *X is well-filtered, that is, for each filter basis $\mathcal{C}$ of compact saturated sets and each open set U with $\bigcap \mathcal{C} \subseteq U$, there is a $K \in \mathcal{C}$ with $K \subseteq U$.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3)*, and all three are equivalent if X is locally compact.*

Proof: That (1) implies (2) follows directly from the Hofmann–Mislove Theorem.

Conversely assume (2), and suppose that A is an irreducible closed set. Then $\mathcal{F} = \{U \in \mathcal{O}(X) : U \cap A \neq \emptyset\}$ is a filter (by irreducibility) and is open (since a directed union meets A if and only if some member of the union does). If A is not the closure of a singleton, then for every $x \in A$, $X \backslash {\downarrow} x = X \backslash \{x\}^-$ is an open set meeting A and is hence in $\mathcal{F}$. Thus we have

$$K := \bigcap \mathcal{F} \subseteq \bigcap_{x \in A} (X \backslash {\downarrow} x) \subseteq V := X \backslash A.$$

By hypothesis V is in $\mathcal{F}$ and thus meets A, which contradicts $V = X \backslash A$. We have shown (2) $\Rightarrow$ (1).

Assume (2) and let $\mathcal{C}$ be a filter basis of compact saturated sets and let U be an open set with $\bigcap \mathcal{C} \subseteq U$. Let $\mathcal{F}$ be the collection of all open sets containing some

$C \in \mathcal{C}$. As $\mathcal{F}$ is the directed union of the open filters $\Phi(C) = \{U \in \mathcal{O}(X) : C \subseteq U\}$, $C \in \mathcal{C}$ (see Lemma II-1.18), it is an open filter, too. Note that since each $C \in \mathcal{C}$ is saturated, it must be the case that $\bigcap \mathcal{F} = \bigcap \mathcal{C}$. By (2) we have $U \in \mathcal{F}$, and thus $K \subseteq U$ for some $K \in \mathcal{C}$. We have proved that X is well-filtered.

Finally assume that X is locally compact and well-filtered. Let $\mathcal{F}$ be an open filter of open sets with intersection K. Let $\mathcal{C}$ be the collection of all compact saturated sets C such that $U \subseteq C$ for some $U \in \mathcal{F}$. We show that $\mathcal{C}$ is a filter basis and that each $U \in \mathcal{F}$ contains some member of $\mathcal{C}$. Let $C_1, C_2 \in \mathcal{C}$ and let $U_1, U_2 \in \mathcal{F}$ be such that $U_i \subseteq C_i$ for $i = 1, 2$. Then $U_1 \cap U_2 \in \mathcal{F}$. For each $x \in U_1 \cap U_2$, pick (by local compactness) a compact neighborhood K_x of x such that $K_x \subseteq U_1 \cap U_2$. The family of all finite unions of $\operatorname{int}(K_x)$, $x \in U_1 \cap U_2$, is a directed family of open sets whose union is $U_1 \cap U_2$. Hence there exist $x_1, \ldots, x_m \in U_1 \cap U_2$ such that $\operatorname{int}(K_{x_1}) \cup \cdots \cup \operatorname{int}(K_{x_m})$ is in $\mathcal{F}$ (since $\mathcal{F}$ is open). Then $K_{x_1} \cup \cdots \cup K_{x_n}$ is a compact set in $\mathcal{C}$ that is contained in $U_1 \cap U_2$, which in turn is contained in $C_1 \cap C_2$. Hence $\mathcal{C}$ is a filter basis of compact saturated sets. It follows from the preceding argument that $\bigcap \mathcal{C} = \bigcap \mathcal{F}$, since a member of the collection on either side contains a member of the other side.

Let W be an open set containing $\bigcap \mathcal{F} = \bigcap \mathcal{C}$. By hypothesis, there exists $K \in \mathcal{C}$ such that $K \subseteq W$. Since $K \in \mathcal{C}$, there exists some $U \in \mathcal{F}$ such that $U \subseteq K$. Thus $U \subseteq W$ and hence $W \in \mathcal{F}$, since the latter is a filter. Thus (2) is satisfied. □

The Hofmann–Mislove Theorem II-1.20 concerns sober spaces and does not require local compactness. Theorem II-1.21 shows that for sober and locally compact sober spaces all of Proposition I-1.24.2 applies and thus proves the following for the collections $Q^*(X)$ and $Q(X)$ of all nonempty, respectively all, compact saturated subsets ordered by $\supseteq$.

Corollary II-1.22. *Let X be a sober space. Then the intersection $K = \bigcap \mathcal{C}$ is nonempty, compact and saturated for every filter base $\mathcal{C}$ of nonempty compact saturated sets $C \subseteq X$. The posets $Q^*(X)$ of nonempty compact saturated subsets and $Q(X) = Q^*(X) \cup \{\emptyset\}$ are directed complete semilattices. Both $Q^*(X)$ and $Q(X)$ are continuous semilattices for each locally compact sober space.* □

Theorem II-1.20 deals satisfactorily with the function

$$K \mapsto \{U \in \mathcal{O}(X) : K \subseteq U\} : Q(X) \to \operatorname{OFilt}(\mathcal{O}(X)) \tag{†}$$

and shows that it is bijective. However, there is a function

$$U \mapsto \{K \in Q(X) : K \subseteq U\} : \mathcal{O}(X) \to \operatorname{OFilt}(Q(X)) \tag{‡}$$

which we should consider as well.

Lemma II-1.23. *For a sober topological space X and an open subset U the set*

$$\Phi'(U) = \{K \in Q(X) : K \subseteq U\}$$

is an open filter in the directed complete semilattice $(Q(X), \supseteq)$

Proof: Firstly, the set $\Phi'(U)$ is closed under finite unions, and if $C \subseteq K$ in $Q(X)$ and $K \subseteq U$ then $C \subseteq U$. As $\emptyset \in \Phi'(U)$, the set $\Phi'(U)$ is nonempty. Hence $\Phi(U)$ is a filter in $(Q(X), \supseteq)$. Secondly, let $\mathcal{C}$ be a directed set in $(Q(X), \supseteq)$ whose sup is a member of $\Phi'(U)$. This means that $\mathcal{C}$ is a filter basis of compact saturated sets whose intersection is contained in U. Then by Theorem II-1.21(3) above there is a $K \in \mathcal{C}$ such that $K \subseteq U$, i.e., $K \in \Phi'(U)$. Hence $\Phi'(U)$ is an open filter. □

By this lemma the function

$$\Phi' : \mathcal{O}(X) \to \mathrm{OFilt}((Q(X), \supseteq)), \quad \Phi'(U) = \{K \in Q(X) : K \subseteq U\}$$

is well-defined, it is clearly monotone and indeed preserves directed unions and finite intersections.

Theorem II-1.24. **(The Hofmann–Mislove Theorem II)** *Let X be a sober space. Then*

$$\Phi' : \mathcal{O}(X) \to \mathrm{OFilt}((Q(X), \supseteq)), \quad \Phi'(U) = \{K \in Q(X) : K \subseteq U\}$$

is an injective semilattice homomorphism preserving directed unions between directed complete semilattices. If X is locally compact, then it is an isomorphism of continuous semilattices whose inverse associates with an ideal of saturated compact sets with respect to $\subseteq$ its union.

Proof: Assume that $U_1 \neq U_2$, say there is a $u \in U_2 \backslash U_1$. Let $K = \mathrm{sat}(u) = \bigcap\{V \in \mathcal{O}(X) : u \in V\}$ be the saturation of $\{u\}$. Then $K \in Q(X)$ and $K \subseteq U_2$ but $K \not\subseteq U_1$. Hence $\Phi'(U_2) \not\subseteq \Phi'(U_2)$. Hence Φ' is injective.

Next we assume that X is locally compact and prove the surjectivity of Φ'. For this purpose we let $\mathcal{U}$ be an open filter of $(Q(X), \supseteq)$. Then $\mathcal{U}$ is a directed lower set of compact saturated subsets of X and we set $U = \bigcup \mathcal{U}$. Now let $u \in U$. Then there is a $K \in \mathcal{U}$ such that $u \in K$. Since $\mathcal{U}$ is a lower set, $\mathrm{sat}(u) \subseteq K$ is a member of $\mathcal{U}$. Let $\mathcal{D}$ be the set of all compact neighborhoods of u; since X is locally compact, $\mathcal{D}$ is a nonempty directed subset of $(Q(X), \supseteq)$ with supremum $\sup \mathcal{D} = \bigcap \mathcal{D} = \mathrm{sat}(u) \in \mathcal{U}$. As $\mathcal{U}$ is Scott open, there is a member V of $\mathcal{D}$ such that $V \in \mathcal{U}$. Thus $V \subseteq \bigcup \mathcal{U} = U$. Hence u is an interior point of U and therefore U is open. Then $\mathcal{U} \subseteq \Phi'(U)$. But now let $K \in \Phi'(U)$, i.e. $K \subseteq U$.

We have just seen that every $x \in K$ has a neighborhood $V_x \in \mathcal{U}$. Since K is compact we have elements $x_1, \ldots, x_n$ such that $K \subseteq V_{x_1} \cup \cdots \cup V_{x_n} \in \mathcal{U}$ since $\mathcal{U}$ is a directed lower set. Thus $\Phi'(U) \subseteq \mathcal{U}$ and we conclude $\mathcal{U} = \Phi'(U)$. Hence Φ' is surjective and thus bijective. $\square$

Let us observe that for any set X the set $\mathrm{Fin}(X)$ of all finite subsets is an algebraic domain with respect to $\supseteq$. Indeed, every directed set has a largest element. Thus every element is compact, and every filter is open, and there is a bijection between the set of all subsets U of X and the set OFilt(Fin(X)) of filters of $\mathrm{Fin}(X)$ which associates with U the set of finite subsets of U.

Example II-1.25. (a) *Let X be a nondiscrete Hausdorff space in which $Q(X)$ is the set of finite subsets. Then*

(i) *$Q(X)$ is an algebraic domain,*
(ii) *the function $\Phi' \colon \mathcal{O}(X) \to \mathrm{OFilt}(Q(X))$ is not surjective,*
(iii) *X is sober (since Hausdorff) but not locally compact.*

(b) *Let p be a point in $\beta(\mathbb{N})\backslash\mathbb{N}$ where $\beta(\mathbb{N})$ is the Stone–Čech compactification of the discrete space of natural numbers, and consider on $X = \mathbb{N} \cup \{p\}$ the induced topology. Then the space X is completely regular Hausdorff, and every compact subset is finite.*

Proof: (a) We have just observed (i) and (ii) in view of the fact that X has subsets which are not open. Part (iii) follows from Theorem II-1.24 and (ii).

(b) Let K be a compact subset of $X = \mathbb{N} \cup \{p\}$. If $K \subseteq \mathbb{N}$ then K is finite as a discrete and compact space. Assume $p \in K$ and let $\mathcal{U}$ be the neighborhood filter of p. If K were infinite then we would have infinitely many nonfixed ultrafilters on K, each of these is contained in an ultrafilter on $\mathbb{N}$ but only one at most could converge in K, namely $\{U \cap K \colon U \in \mathcal{U}\}$ if all of the $U \cap K$ are infinite. This is a contradiction to the fact that on a compact Hausdorff space every ultrafilter converges. $\square$

The combination of Theorem II-1.17, II-1.20 through II-1.24 and Example II-1.25 sheds additional light on our first discussion of the poset $(Q(X), \supseteq)$ of compact saturated subsets of a space which we started in Proposition I-1.24.2. In Section 2 of Chapter IV, when we have a good duality theorem for domains, we shall resume and conclude the discussion of $Q(X)$ by showing that for a sober space X the **dcpo** $Q(X)$ is a domain only if X is locally compact. This will be the completion of the Hofmann–Mislove Theorem IV-2.18.

Exercises

Exercise II-1.26. Let L be a **dcpo**. Prove the following.

(i) If $U \neq \emptyset$ is a Scott open subset of L, then U is a sub**dcpo**, and a subset of U is Scott open in U iff it is Scott open in L. In particular, the Scott topology on U agrees with the relative Scott topology from L.
(ii) If $A \neq \emptyset$ is a Scott closed subset of L, then A is a sub**dcpo**, and a subset of A is Scott closed in A iff it is Scott closed in L. In particular, the Scott topology on A agrees with the relative Scott topology from L.
(iii) If $X \neq \emptyset$ is the intersection of a Scott open subset U and a Scott closed subset A of L, then X is a sub**dcpo**, and the Scott topology on X agrees with the relative Scott topology from L. □

Exercise II-1.27. Let L be a **dcpo**. Let $(x_j)_{j\in J}$ be a net in L for which $y_j := \inf_{i\geq j} x_i$ exists for $j \in J'$, a cofinal subset of J. Prove the following.

(i) The set $Y := \{y_j\colon j \in J'\}$ is directed.
(ii) A point $z \in L$ is an eventual lower bound of $(x_j)_{j\in J}$ iff $z \leq y_j$ for some $j \in J'$.
(iii) The set of eventual lower bounds is equal to $\downarrow Y$, and hence is directed. Thus the directed supremum $\sup Y$ is also the directed supremum $\underline{\lim}_j x_j$ of the set of eventual lower bounds.
(iv) For $x \in L$, $x \leq \sup D$ for some directed set D of eventual lower bounds for $(x_j)_{j\in J}$ iff $x \leq \sup Y = \underline{\lim}_j x_j$. Thus the set of $\mathcal{S}$-limits of (x_j) in Definition II-1.1 are the same for both parts of the definition.
(v) The above considerations apply in an arbitrary **dcpo** to any constant net or to any monotone net, and in a complete semilattice to any net. Thus in a complete semilattice the alternative definitions for $\mathcal{S}$ and for the liminf in the cases of a complete semilattice and a **dcpo** agree. □

Exercise II-1.28. Let L be a **dcpo**. For each directed subset D of L consider the net $(x_d)_{d\in D}$ defined by $x_d = d$. Let $\mathcal{D}$ denote the class of all pairs $((x_d)_{d\in D}, x)$, where D ranges over all directed subsets and $x \leq \sup D$. Define

$$\mathcal{O}(\mathcal{D}) = \{U \subseteq L : ((x_d)_{d\in D}, x) \in \mathcal{D}, x \in U \Rightarrow x_d \in U \text{ eventually}\}.$$

Show that $\mathcal{O}(\mathcal{D})$ is the Scott topology. □

Exercise II-1.29. The following is a standard characterization for a class $\mathcal{L}$ of convergent nets to be topological, i.e., to be precisely the class of all convergent nets in the resulting topology $\mathcal{O}(\mathcal{L})$ (see, for example, [Kelley, B1955]).

Fact. *Given a class $\mathcal{L}$ of convergent nets on a set X, we have* $((x_j)_{j\in J}, x) \in \mathcal{L}$ *iff the net* $(x_j)_{j\in J}$ *converges to* x *with respect to* $\mathcal{O}(\mathcal{L})$ *precisely when the following axioms are satisfied.*

(constants) *For every constant net one has* $((x)_{j\in J}, x) \in \mathcal{L}$.
(subnets) *If* $(y_i)_{i\in I}$ *is a subnet of* $(x_j)_{j\in J}$ *and* $((x_j)_{j\in J}, x) \in \mathcal{L}$, *then* $((y_i)_{i\in I}, x) \in \mathcal{L}$.
(divergence) *If* $((x_j)_{j\in J}, x)$ *is* not *in* $\mathcal{L}$, *then* $(x_j)_{j\in J}$ *has a subnet* $(y_i)_{i\in I}$ *no subnet* $(z_k)_{k\in K}$ *of which ever has* $((z_k)_{k\in K}, x) \in \mathcal{L}$.
(iterated limits) *If* $((x_i)_{i\in I}, x) \in \mathcal{L}$, *and if* $((x_{i,j})_{j\in J(i)}, x_i) \in \mathcal{L}$ *for all* $i \in I$, *then* $((x_{i,f(i)})_{(i,f)\in I\times M}, x) \in \mathcal{L}$, *where* $M = \prod_{i\in I} J(i)$ *is a product of directed sets.* □

Now prove the following.

(i) The class $\mathcal{S}$ of Definition II-1.1 satisfies the axioms **(constants)** and **(subnets)** for any **dcpo**.
(ii) If a complete semilattice S satisfies the axiom **(iterated limits)**, then S is a bounded complete domain.

Hint. For (ii), use the axiom **(iterated limits)** to show the validity of equation (DD) of I-2.7. □

Part (ii) gives an alternative proof that if the convergence given by $\mathcal{S}$ is topological on a complete semilattice L, then L is a bounded complete domain, a special case of Lemma II-1.8.

For the following exercise we need a definition:

Definition II-1.30. A topology on a **dcpo** L is said to be *order consistent* if

(i) $\{x\}^- = \downarrow x$ for all $x \in L$,
(ii) every monotone net $(x_j)_{j\in J}$ converges to $x = \sup_j x_j$.

In (ii) we could say equivalently: if $x = \sup I$ for an ideal I, then $x = \lim I$. □

Recall that the *upper topology* $\nu(L)$ is the topology generated by the collection of sets $L\backslash\downarrow x$ (see O-5.4).

Exercise II-1.31. Prove the following.

(i) Both the Scott and upper topologies are order consistent on any **dcpo** L, and for any order consistent topology τ on L, we have

$$\nu(L) \subseteq \tau \subseteq \sigma(L).$$

In other words, the upper topology is the coarsest and the Scott topology is the finest of all order consistent topologies.

(ii) If $(x_j)_{j\in J}$ is a monotone net and τ is order consistent, then the set of all limit points of this net is precisely $\downarrow\sup x_j$.

(iii) If τ is order consistent and if $(x_j)_{j\in J}$ is a net and $(z_j)_{j\in J}$ is a directed net such that $z_j \le x_j$ for all j, then $\downarrow\sup z_j$ is contained in the set of τ-limit points of the net $(x_j)_{j\in J}$.

(iv) If L is a complete semilattice and τ an order consistent topology on L, then $\downarrow(\underline{\lim}_j x_j)$ is contained in the set of all τ-limit points of the net $(x_j)_{j\in J}$.

(v) If, in addition, the translations $x \mapsto a \wedge x : L \to L$ are τ continuous for all $a \in L$ (in this case we say that L is a *semitopological semilattice* (see also VI-1.11)), then L is meet continuous. □

In Sections 2 and 4 of Chapter VII we will describe those complete lattices for which $\nu(L) = \sigma(L)$.

Problem. Can one characterize those complete lattices on which the Scott topology has the property that each point has a neighborhood basis of open filters, or, alternatively, those complete lattices L for which $\{\uparrow x \colon x \in L\}$ is a basis of $\sigma(L)$? □

Exercise II-1.32. In a **dcpo** L, let $\mathrm{int}_\sigma X$ denote the $\sigma(L)$-interior of a set. Define $x \prec y$ iff $y \in \mathrm{int}_\sigma \uparrow x$. Prove the following:

(i) $\prec$ is an auxiliary relation;

(ii) $x \prec y$ implies $x \ll y$;

(iii) we have the equivalence of $x \prec y$ and $x \ll y$ for all $x, y \in L$ iff $\uparrow x$ is Scott open for all $x \in L$ (This is the case if $\ll$ satisfies the interpolation property (cf. I-1.17).);

(iv) the relation $\prec$ is approximating (see I-1.13) iff $\ll$ is approximating, that is, iff L is a domain.

Hint. Use II-1.6 for the proof of (ii). For the proof of (iv), use (ii) for one implication and (iii) for the converse. □

Problem. For which complete lattices do we have $x \prec y$ iff $x \ll y$? □

Exercise II-1.33. The following example shows that $x \ll y$ need not imply $x \prec y$ in a complete lattice: Let $L = \{\bot\} \cup (\mathbb{N} \times [0, 1]) \cup \{\top\}$ with $\bot$ as smallest and $\top$ as greatest element and $\uparrow(n, r) = (\{n\} \times [r, 1]) \cup \{(p, 1) \colon p \ge n\} \cup \{\top\}$. Show that $(1, 0) \ll \top$ but not $(1, 0) \prec \top$, in fact $\mathrm{int}_\sigma \uparrow(1, 0) = \emptyset$. □

In Exercise II-1.32 we have associated an auxiliary relation with the Scott topology on any **dcpo**. We are generalizing this to more general topologies on **dcpo**s. We obtain characterizations of the Scott topology on meet continuous semilattices and their continuity that are in perfect analogy to the characterizations of the way-below relation on meet continuous semilattices and their continuity in Propositions I-1.15 and I-1.16.

Exercise II-1.34. Let τ be a topology on a **dcpo** L such that the specialization order (see O-5.2) agrees with the original order on L. Such a topology may be called *order compatible*. Define

$$x \prec_\tau y \text{ iff } \uparrow x \text{ is a } \tau \text{ neighborhood of } y.$$

(i) Show that $\prec_\tau$ is an auxiliary relation on L.
(ii) If the auxiliary relation $\prec_\tau$ is approximating, show that τ is finer than the Scott topology on L.
(iii) Show that for a meet continuous semilattice L, the Scott topology is the intersection of the order compatible topologies τ with $\prec_\tau$ approximating.
(iv) Show that a meet continuous semilattice is continuous iff it admits a coarsest order compatible topology τ with $\prec_\tau$ approximating.

Hint. Part (i) is straightforward. (ii) Let U be Scott open and $y \in U$. As $\prec_\tau$ is approximating, there is a directed set D of elements $x \prec_\tau y$ such that $y = \sup D$. As U is Scott open, there is some $x \in D$ such that $x \in U$. As $x \prec_\tau y$, we have that $\uparrow x$ is a τ neighborhood of y which is contained in U. Thus the topology τ is finer than the Scott topology.

(iii) For a fixed ideal I of L, we define a subset U of L to be τ_I open, if U is an upper set and if $\sup I \in U$ implies $x \in U$ for some $x \in I$. It is easily seen that the τ_I open sets form a topology which is order compatible. The Scott topology clearly is the intersection of the topologies τ_I, when I ranges over all ideals of L. It remains to show that the associated auxiliary relations $\prec_{\tau_I}$ are approximating. But this follows from Lemma I-1.14 and the fact that $\prec_{\tau_I}$ is the auxiliary relation corresponding to the function m_I considered there.

(iv) By (iii), there is a coarsest order compatible topology τ with $\prec_\tau$ approximating iff the auxiliary relation $\prec$ associated with the Scott topology is approximating, and this is the case iff L is continuous by Exercise II-1.32(iv). $\square$

Exercise II-1.35. Let L be a **dcpo**. Show that L is a domain iff for for every Scott open set U and $x \in U$, there exist $y \in U$ and a Scott open set V such that $x \in V \subseteq \uparrow y$.

Hint. If L is a domain, use II-1.14(2). Conversely, define $x \prec y$ iff $y \in \mathrm{int}_\sigma \uparrow x$ as in Exercise II-1.32. We saw in that exercise that $x \prec y$ implies $x \ll y$. That the set $\{y\colon y \prec x\}$ is directed follows immediately from our hypothesis. Given $x \not\leq w$, then $L \backslash \downarrow w$ is a Scott open set containing x, so again by hypothesis there exists $y \prec x$ such that $y \not\leq w$. Hence $\prec$ is approximating, and thus L is a domain. □

Remark. The preceding exercise provides a sometimes useful approach for showing that a **dcpo** is a domain, becauses it finesses the need to show the directedness of the way-below set.

In II-1.12 we have seen that every domain is a sober space with respect to its Scott topology. In the following exercise we present an example of a **dcpo** which is not sober for its Scott topology.

Exercise II-1.36. Let $L = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ with the partial order defined by

$$(m, n) \leq (m', n') \text{ iff either } m = m' \text{ and } n \leq n' \leq \infty \text{ or } n' = \infty \text{ and } n \leq m'.$$

(i) Show that the relation $\leq$ defined above is indeed a partial order and that the elements of the form (m, ∞) are the maximal elements of L.

(ii) Show that a directed subset of L either has a greatest element or is contained in $\{m\} \times (\mathbb{N} \cup \{\infty\})$ for some m. Conclude that $(L, \leq)$ is a **dcpo**.

(iii) Show that any two nonempty Scott open subsets of L have a nonempty intersection.

(iv) Conclude that L itself is an irreducible Scott closed set which is not the closure of any point, whence $(L, \sigma(L))$ is not sober.

(v) Show that there is no sober topology on L which has the given order as its specialization order.

Problem. Find an order theoretical characterization for those **dcpo**s which are sober with respect to their Scott topology.

Old notes

The topology introduced on a **dcpo** L (in particular, on any complete lattice L) in this section was first concisely formulated for the lattice $L = \mathcal{O}(X)$ of open sets of a topological space by B. J. Day and G. M. Kelly (see [Day and Kelly, 1970], p. 51). But it is clearly the merit of D. Scott [Scott, 1972a] to have defined this topology in all generality and to have demonstrated its usefulness in his article on “Continuous lattices”. The name *Scott topology* was first used by Isbell [Isbell, 1975b], p. 41, and [Isbell, 1975a], p. 317, and the name was used in the Seminar on Continuity in Semilattices (SCS) for several years. Our detailed discussion of the relation between the Scott topology, liminf convergence and lower semicontinuity is an elaboration of a motive proffered by Scott in his 1972 article (see p. 104), by J. D. Lawson [Lawson, 1973] and by K. H. Hofmann and A. Stralka (see [Hofmann and Stralka, 1976], p. 16).

The results on the sobriety of the Scott topology discussed in II-1.12 ff. were published in the *Compendium* as was most of the Characterization Theorem II-1.14, but only for the case of complete semilattices. For further results on sobriety and order, see [Isbell, 1975a] and [Hoffmann, 1979a], [Hoffmann, 1981b]. For complete lattices, the equivalence of (1) and (7) in II-1.14, however, was (practically) used by Scott to define a continuous lattice in [Scott, 1972a]. Thus it is really Theorem II-1.14 which establishes the equivalence of our definition of continuous lattices in I-1.6 with Scott's original definition. Scott used the auxiliary relation $\prec$ of II-1.32 for his definition, and the precise statement of the equivalence of Scott's definition of a continuous lattice with Definition I-1.6 is given in II-1.32. The characterization in Exercise II-1.35 of domains is also much in the spirit of Scott's earliest work and Eršov's topological approach [Eršov, 1973]. The example in II-1.33 is due to C. E. Clark [scs 21].

New notes

In the first subsection, the analysis of the Scott topology in terms of liminf convergence has been extended from complete lattices to **dcpo**s. The second subsection leads up to the Characterization Theorem II-1.14, no longer restricted to continuous lattices, but proved for domains in general. The third subsection presents the complete machinery of the Hofmann–Mislove Theorem, which orginated from [Hofmann and Mislove, 1981] and which constitutes a significant contribution of domain theory to the theory of sober spaces. The original proof rested strongly on spectral theory; the proof given here is more topological in nature along the lines given by Keimel and Paseka [Keimel and Paseka, 1994]. Our treatment of well-filtered spaces, notably in Theorem II-1.21, is original. The equivalence of condition (3) with sobriety for locally compact spaces was noted by [Kou, 1999], who also gave a counterexample in the absence of local compactness. We complete the subject matter of the Hofmann–Mislove Theorem in Section IV-2 when the full apparatus of Lawson duality is available.

The results collected in Exercise II-1.34 have been communicated to us by M. Escardó and R. Heckmann.

The example of a **dcpo** which is not sober in its Scott topology II-1.36 is due to Johnstone [Johnstone, 1981]. Isbell [Isbell, 1982a] has used Johnstone's example to construct a complete lattice which is not sober for its Scott topology.

II-2 Scott-Continuous Functions

The next task is to characterize those functions between **dcpo**s and between complete lattices which are continuous with respect to the Scott topology. Our original motivations must now bear fruit: after all the Scott topology was introduced to describe the classical concept of lower semicontinuous functions.

Scott-continuous functions

Proposition II-2.1. *For a function f from a* **dcpo** *S into a* **dcpo** *T, the following conditions are equivalent:*

(1) *f is continuous with respect to the Scott topologies, that is, $f^{-1}(U) \in \sigma(S)$ for all $U \in \sigma(T)$;*
(2) *f preserves suprema of directed sets, that is, f is order preserving and $f(\sup D) = \sup f(D)$, for all directed subsets D of S;*
(3) *f is order preserving and $f(\underline{\lim}_{j\in J} x_j) \leq \underline{\lim}_{j\in J} f(x_j)$, for any net $(x_j)_{j\in J}$ on S such that $\underline{\lim}_{j\in J} x_j$ and $\underline{\lim}_{j\in J} f(x_j)$ both exist (which is always the case if S and T are complete semilattices).*

If S and T are domains, then (1), (2), *and* (3) *are equivalent to each of the following two conditions:*

(4) *$y \ll f(x)$ iff for some $w \ll x$ one has $y \ll f(w)$, for all $x \in S$ and $y \in T$;*
(5) *$f(x) = \sup\{f(w) : w \ll x\}$, for all $x \in S$.*

If S and T are algebraic domains, then the following conditions are also equivalent to the preceding ones:

(6) *$k \leq f(x)$ iff for some $j \leq x$ with $j \in K(S)$ one has $k \leq f(j)$, for all $x \in S$ and $k \in K(T)$;*
(7) *$f(x) = \sup\{f(j) : j \leq x$ and $j \in K(S)\}$, for all $x \in S$.*

Proof: (1) implies (2): First we show that (1) implies that f is order preserving: Suppose that $f(x) \not\leq f(y)$; then the Scott open set $V = T\backslash{\downarrow} f(y)$ contains $f(x)$. Thus $U = f^{-1}(V)$ is a Scott open neighborhood of x by (1) not containing y. But then $x \not\leq y$ as U is an upper set. Thus $x \leq y$ implies $f(x) \leq f(y)$.

Now let D be a directed subset of S. Then $f(D)$ is directed and $\sup f(D) \leq f(\sup D)$, since f is order preserving. Set $x = \sup D$ and $t = \sup f(D)$. We claim $f(x) \leq t$. Suppose $f(x) \not\leq t$. The Scott open set $T\backslash{\downarrow}t$ contains $f(x)$; thus, the inverse image $U = f^{-1}(T\backslash{\downarrow}t)$ is a Scott open neighborhood of x in

S by (1). It follows that there is a $d \in D$ such that $d \in U$. Then $f(d) \in T \setminus \downarrow t$, that is, $f(d) \not\leq t = \sup f(D)$. This contradiction proves our claim.

(2) implies (1): Let A be a Scott closed subset of T. In order to show that $f^{-1}(A)$ is Scott closed in S we take a directed subset D of $f^{-1}(A)$. Then by (2) $f(\sup D) = \sup f(D)$. But $\sup f(D) \in A$ by II-1.4(i), since A is Scott closed and $f(D)$ is directed owing to the monotonicity of f. Then $f(\sup D) \in A$, and hence $\sup D \in f^{-1}(A)$. Thus $f^{-1}(A)$ is Scott closed by II-1.4(i).

(3) implies (2): Let D be a directed set in S. If we set $x_d = d$, $d \in D$, then, by II-1.27, $\underline{\lim} x_d = \sup D$ and, as $f(D)$ is directed by the monotonicity of f, similarly $\underline{\lim} f(x_d) = \sup f(D)$. Thus (3) implies $f(\sup D) \leq \sup f(D)$. As the converse inequality is true by the monotonicity of f, we deduce $f(\sup D) = \sup f(D)$.

(2) implies (3): Suppose that $x = \underline{\lim}_j x_j$ for a net $(x_j)_{j \in J}$ in S and $y = \underline{\lim}_j f(x_j)$. Then by II-1.1 there is a directed set D of eventual lower bounds of the net $(x_j)_{j \in J}$ such that $x = \underline{\lim}_j x_j = \sup D$. By the monotonicity of f, every $f(d)$, $d \in D$, is an eventual lower bound of the net $(f(x_j))_{j \in J}$ and the set $f(D)$ is directed, whence $\sup f(D) \leq \lim_j f(x_j) = y$. From (2) we conclude $(\underline{\lim}_j x_j) = f(\sup D) = \sup f(D) \leq \underline{\lim}_j f(x_j)$.

From now on we assume that S and T are domains.

(2) implies (5): Clear, since $\twoheaddownarrow x$ is directed and $x = \sup \twoheaddownarrow x$ by I-1.6.

(5) implies (4): From (5) we can conclude that f is monotone. Indeed, if $x \leq y$, then $\twoheaddownarrow x \subseteq \twoheaddownarrow y$ and consequently $f(x) = \sup f(\twoheaddownarrow x) \leq \sup f(\twoheaddownarrow y) = f(y)$.

Now let $y \ll f(x) = \sup f(\twoheaddownarrow x)$; since f is monotone, $f(\twoheaddownarrow x)$ is directed. Thus, by I-1.9, there is a $w \ll x$ with $y \ll f(w)$. Conversely, if $y \ll f(w)$ for some $w \ll x$, then $y \ll f(x)$ by monotonicity of f and I-1.2(ii) for $\ll$.

(4) implies (1): Let $U \in \sigma(T)$ and $x \in f^{-1}(U)$. By II-1.10 there is a $y \in U$ with $y \ll f(x)$. By (4) we find a $w \ll x$ such that $y \ll f(w)$; we will have finished the proof if we show that $f(\twoheaduparrow w) \subseteq U$.

Now, take a z with $w \ll z$. For every $y' \ll f(w)$ we have $y' \ll f(z)$ by (4); and consequently $f(w) = \sup \twoheaddownarrow f(w) \leq f(z)$. But $y \leq f(w)$ by I-1.2(i) and $y \in U$; hence, $f(z) \in U$ by II-1.2(i).

Now let S and T be algebraic domains. Note that in an algebraic domain we have $x \ll y$ iff there is a compact element k with $x \leq k \leq y$ (cf. I-4.3). Thus the equivalences (4) iff (6) and (5) iff (7) follow easily. □

Definition II-2.2. A function $f\colon S \to T$ between **dcpo**s is *Scott-continuous* iff it satisfies the equivalent conditions II-2.1(1),(2),(3). The category whose objects are **dcpo**s and whose morphisms are Scott-continuous maps will be denoted by *DCPO*, and the full subcategory of complete lattices by *UPS* (preservation of *UP*-directed *S*ups).

The full subcategories of domains and continuous lattices are called *DOM* and *CONT* respectively. The full subcategories of algebraic domains and algebraic lattices are called *ALGDOM* and *ALG* respectively. □

In Chapter IV we will talk about the category *SUP* of all complete lattices and maps preserving all sups. This will be a proper subcategory of *UPS*. There the reader will also find other useful categories of continuous and algebraic domains and lattices that are not full subcategories of *DCPO*.

Let us pause to record next those examples of continuous functions which, implicitly, we have encountered before.

Examples II-2.3. (1) Every map preserving arbitrary sups is Scott-continuous. In particular, lower adjoints are Scott-continuous, as they preserve sups by O-3.3. Notably, the co-restriction of any closure operator to its image is Scott-continuous (O-3.12(ii)).

(2) We had specific occurrences of maps preserving directed sups in O-3.11, O-3.14, O-4.2(6), I-2.2 through I-2.6, I-2.12, I-2.15, I-2.17, I-4.11, I-4.13, I-4.14, I-4.16 through I-4.18. (It might be a useful exercise for the reader to rephrase these propositions and theorems in terms of Scott continuity.)

(3) A function $f: X \to \mathbb{R}^*$ from a topological space into the extended set of reals is lower semicontinuous iff it is continuous with respect to the Scott topology on $\mathbb{R}^*$. With this remark we have closed the circle which we began with the motivating observations preceding II-1.1. It is in this light that we prefer to view examples like I-1.22.

(4) As we remarked in II-2.1, every Scott-continuous function is monotone. The converse is obviously false; counterexamples $f: \mathbb{R}^* \to \mathbb{R}^*$ are trivial to construct. (This is so whenever the domain of the function contains proper limits: on a finite poset monotonicity and continuity come to the same thing.) However, there is an interesting circumstance where "in effect" monotonicity implies continuity. Consider the question of defining continuous functions $f: S \to T$, where S is *algebraic* (and T is just assumed to be complete). In view of II-2.1(7), the function f is completely determined by its restriction to the poset $K(S)$; on $K(S)$, moreover, all we can say about f is that it is monotone. To see this suppose we are given any monotone $f_0: K(S) \to T$. We then employ the formula of II-2.1(7) as a *definition* of an extension to all of S:

$$f(x) = \sup\{f_0(j)\colon\ j \le x \quad \text{and} \quad j \in K(S)\}, \quad \text{for all } x \in S.$$

The reader can easily prove that the f so defined is continuous and agrees exactly with f_0 on $K(S)$. □

The following fixed-point theorem for Scott-continuous self-maps on **dcpo**s is extremely useful although it is more elementary and less general than the Tarski Fixed-Point Theorem O-2.3 for monotone self-maps on complete lattices and the Least Fixed-Point Theorem O-2.20 for monotone self-maps on **dcpo**s. The proof is simple and constructive. One may notice that we do not need the full strength of directed completeness and Scott continuity here, but only ω-completeness and ω continuity by which we mean that $\sup x_n$ exists and $f(\sup_n x_n) = \sup_n f(x_n)$ for all inceasing sequences $x_0 \le x_1 \le x_2 \le \cdots$.

Proposition II-2.4. (Least Fixed-Point Theorem for Scott-Continuous Functions) *Let L be a* **dcpo** *with a least element $\perp$.*

(i) **Existence:** *Every Scott-continuous self-map $f\colon L \to L$ has a least fixed-point* $\mathrm{LFP}(f)$.
(ii) **Construction:** *The least fixed-point can be approximated by the recursively defined* Kleene chain

$$x_0 \;=\; \perp, x_{n+1} = f(x_n) = f^{n+1}(\perp)$$

in the sense that

$$\mathrm{LFP}(f) \;=\; \sup_n x_n \;=\; \sup_n f^n(\perp).$$

(iii) **Preservation:** *Let M be a second* **dcpo** *with bottom and let*

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, "f"'] & M \arrow[d, "g"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

be a commuting diagram of Scott-continuous maps. Then

$$h(\mathrm{LFP}(f)) = \mathrm{LFP}(g \mid {\uparrow} h(\perp)).$$

If h is strict, i.e., if $h(\perp) = \perp$, then

$$h(\mathrm{LFP}(f)) = \mathrm{LFP}(g).$$

Proof: As $x_0 = \perp \le f(\perp) = x_1$ and as f is order preserving, we conclude that $x_1 = f(x_0) \le f(x_1) = x_2$, and by induction that $f(x_n) \le f(x_{n+1})$ for all n. As the sequence (x_n) is increasing, it has a least upper bound $x = \sup_n x_n$ in L. By the continuity of f, we have $f(x) = f(\sup_n x_n) = \sup_n f(x_n) = \sup_n x_{n+1} = x$. Thus x is a fixed-point of f. It is the smallest fixed-point of f. Indeed, let $y = f(y)$ be another fixed-point. As $x_0 = \perp \leq y$, we get $x_1 = f(\perp) \leq f(y) = y$ and, by induction, $x_n \leq y$ for all n, whence $x = \sup_n x_n \leq y$. This proves (i) and (ii). For (iii) we first remark that $\uparrow h(\perp)$ is a **dcpo** with a smallest element $h(\perp)$ and that g maps $\uparrow h(\perp)$ into itself, as $x \geq h(\perp)$ implies $g(x) = gh(\perp) = hf(\perp) \geq h(\perp)$. Hence the restriction of g to $\uparrow h(\perp)$ has a least fixed-point and

$$\begin{aligned} h(\mathrm{LFP}(f)) &= h(\sup_n f^n(\perp)) && \text{by (ii)} \\ &= \sup_n hf^n(\perp) && \text{as } h \text{ is Scott-continuous} \\ &= \sup_n g^n h(\perp) && \text{as the above diagram commutes} \\ &= \mathrm{LFP}(g \mid \uparrow h(\perp)) && \text{by (ii).} \end{aligned}$$

□

Function spaces and cartesian closed categories of dcpos

Now we are in a position to build a general theory which subsumes the investigation of objects like $\mathrm{LSC}(X, \mathbb{R}^*)$. This will be accomplished in the remainder of this chapter.

One of the most noteworthy features of the category *DCPO* is that it is *cartesian closed* (we will say presently what this means explicitly). Not only is this fundamental for the applications of continuous lattices and domains to logic and computing, but it also provides evidence of the mathematical naturalness of the notion.

As a first step toward showing why the category is cartesian closed, we discuss hom-sets. Let S and T be **dcpo**s. As is common, we denote by $DCPO(S, T)$ the *set* of Scott-continuous functions from S into T, that is, the set of maps in the category *DCPO* with the indicated domain and codomain. This set is also a poset under the pointwise partial ordering

$$f \leq g \text{ iff } f(x) \leq g(x) \text{ for all } x \in S.$$

For the moment we will not distinguish notationally between $DCPO(S, T)$ as an object in *SET*, the category of sets, and as an object in some other (concrete) category.

Lemma II-2.5. *Let S and T be* **dcpo***s. Let L be the poset $DCPO(S, T)$ of Scott-continuous functions from S into T. Then L is closed in T^S under the formation of all existing sups, in particular, of all directed sups; consequently L is always a* **dcpo***, and indeed a complete lattice if T is a complete lattice.*

Proof: Let $F \subseteq L$ and assume that $f(s) = \sup_{g\in F} g(s)$ exists for all $s \in S$. Now take an arbitrary directed set D in S. Then

$$\begin{aligned}\sup f(D) &= \sup_{d\in D} \sup_{g\in F} g(d) = \sup_{g\in F} \sup g(D)\\ &= \sup_{g\in F} g(\sup D) = f(\sup D).\end{aligned}$$ □

The following will fix our notation for the function space we have just constructed by putting it in the proper category.

Definition II-2.6. We denote by $[S \to T]$ the set $DCPO(S, T)$ of all Scott-continuous functions $f\colon S \to T$ considered as a **dcpo**. Furthermore if $f\colon S_1 \to S_2$ and $g\colon T_1 \to T_2$ are $DCPO$-maps, then we denote by $[f \to g]\colon [S_2 \to T_1] \to [S_1 \to T_2]$ the map $h \mapsto ghf$. □

Proposition II-2.7. *The construction of* II-2.6 *defines a functor*

$$[\cdot \to \cdot]\colon DCPO^{\mathrm{op}} \times DCPO \to DCPO$$

restricting to a functor

$$UPS^{\mathrm{op}} \times UPS \to UPS.$$

Proof: The map $[f \to g]$ is certainly well defined in II-2.6 and carries functions in the first set to functions in the second set. It also obviously behaves properly under composition. We need only observe that $[f \to g]$ preserves directed sups to make sure the map resides in the right category. But if $h = \sup H$, a directed sup, then

$$\begin{aligned}[f \to g](h)(s) &= g(h(f(s))) = g(\sup_{k\in H} k(f(s)))\\ &= \sup_{k\in H} g(k(f(s))) \quad \text{(since g preserves directed sups)}\\ &= \sup_{k\in H}[f \to g](k)(s)\end{aligned}$$

and since sups are calculated pointwise, the assertion follows. □

Having provided a function space construction for $DCPO$ and UPS, we now turn to a discussion of some properties of products appropriate to the proof that all these categories are cartesian closed.

Lemma II-2.8. *Let R, S, T be* **dcpo***s. A function $f\colon R \times S \to T$ is Scott-continuous on the product $R \times S$ iff f is Scott-continuous in each variable separately; that is,*

(a) *for all $s \in S$, the function $r \mapsto f(r, s) : R \to T$ is Scott-continuous,*
(b) *for all $r \in R$, the function $s \mapsto f(r, s) : S \to T$ is Scott-continuous.*

Proof: If f is Scott-continuous on the product $R \times S$, it clearly is separately continuous. Now assume f to be separately continuous, and let D be a directed subset of $R \times S$. For $d \in D$ we write $d = (d_1, d_2)$. It is easy to check that $\sup D = (a_1, a_2)$ with $a_k = \sup_{d \in D} d_k$, $k = 1, 2$. Then

$$\begin{aligned} f(\sup D) &= f(a_1, a_2) = f(\sup_{d \in D}(d_1, a_2)) \\ &= \sup_{d \in D} f(d_1, a_2) && \text{(by (a))} \\ &= \sup_{d \in D} f(d_1, \sup_{c \in D} c_2) \\ &= \sup_{d \in D} \sup_{c \in D} f(d_1, c_2) && \text{(by (b)).} \end{aligned}$$

Since D is directed, we find for $d, c \in D$ an $e \in D$ with $d, c \leq e$. Since f is monotone by the separate continuity, we conclude

$$f(\sup D) \leq \sup_{e \in D} f(e_1, e_2) = \sup f(D).$$

The inequality $\sup f(D) \leq f(\sup D)$ holds for any monotone map f. □

Lemma II-2.9. *For arbitrary* **dcpo***s we have:*

(i) *The evaluation map* $(f, x) \mapsto f(x)$*, which we can call* eval: $[R \to S] \times R \to S$*, is Scott-continuous.*
(ii) *The composition map* $(f, g) \mapsto f \circ g : [S \to T] \times [R \to S] \to [R \to T]$ *is Scott-continuous.*
(iii) *There is a natural isomorphism* $[R \to (S \times T)] \cong [R \to S] \times [R \to T]$.

Proof: (i) By the preceding lemma, for the continuity of the evaluation map it suffices to show that $f(\sup D) = \sup f(D)$ for every directed subset $D \subseteq R$ and $(\sup F)(x) = \sup_{f \in F} f(x)$ for every directed set $F \subseteq [R \to S]$. The first statement is true as f is Scott-continuous, and the second is true as sups are calculated pointwise in $[R \to S]$.

(ii) Again by the preceding lemma it suffices to show that $(\sup F) \circ g = \sup_{f \in F}(f \circ g)$ and $f \circ (\sup G) = \sup_{g \in G}(f \circ g)$ for directed subsets $F \subseteq [R \to S]$ and $G \subseteq [S \to T]$: For every $x \in R$ one has $((\sup F) \circ g)(x) = (\sup F)\,(g(x)) = \sup_{f \in F} f(g(x)) = \sup_{f \in F}(f \circ g)(x)$ and $(f \circ (\sup G))(x) = f((\sup G)(x)) = f(\sup_{g \in G} g(x)) = \sup_{g \in G} f(g(x)) = \sup_{g \in G}(f \circ g)(x)$ firstly, because sups of functions are defined pointwise, and secondly, because f preserves directed suprema.

Part (iii) holds for Scott-continuous functions into arbitrary products of **dcpo**s, not only into finite products. The proof is simple, as projections are Scott-continuous, and is left to the reader. □

A category A with a terminal object and finite products is called *cartesian closed* iff there are an *internal hom-functor* $(Y, Z) \mapsto Z^Y : A^{\mathrm{op}} \times A \to A$

and a natural isomorphism $A(X \times Y, Z) \simeq A(X, Z^Y)$. (More information on cartesian closed categories may be found in [Mac Lane, B1971], pp. 95 ff.)

We shall first see that the categories *DCPO* and *UPS* are cartesian closed, and we remark that $R \times S$ provides the cartesian product functor and $[R \to S]$ the internal hom-functor. We shall then see that certain full subcategories such as *CONT* and *ALG* and others are cartesian closed, too. The crucial point to check will be whether these categories are stable for the function space construction $[R \to S]$.

Theorem II-2.10. *Let R, S, T be* **dcpo***s, and let*

$$T^{(R\times S)} \underset{F}{\overset{E}{\rightleftarrows}} (T^S)^R$$

be the canonical pair of mutually inverse bijections given by

$$E(f)(r)(s) = f(r,s) \quad \textit{and} \quad F(g)(r,s) = g(r)(s).$$

Then E and F induce mutually inverse bijections that are in fact isomorphisms of **dcpo***s:*

$$[(R \times S) \to T] \rightleftarrows [R \to [S \to T]].$$

In particular, DCPO and UPS are cartesian closed categories.

Proof: Let $f \in [(R \times S) \to T]$; then f preserves directed sups in each argument separately, whence $E(f)(r) \in [S \to T]$ for each $r \in R$. Since sups of functions are calculated pointwise, and $r \mapsto E(f)(r)(s) = f(r,s)$ preserves directed sups for each $s \in S$, then $E(f) \in [R \to [S \to T]]$.

On the other hand, let $g \in [R \to [S \to T]]$. Then $F(g)$ is equal to the composition of maps $(r,s) \mapsto (g(r), s) \mapsto g(r)(s) : R \times S \to [S \to T] \times S \to T$. The first map is Scott-continuous as g is. The second map is evaluation which is Scott-continuous by II-2.9(i). Hence $F(g) \in [(R \times S) \to T]$.

This proves that the restrictions of E and F relate the desired **dcpo**s. But E and F are clearly monotone, and so they are isomorphisms, since they are inverse to one another. The reader may check that the isomorphism of functors obtained in this way is natural. □

We wish to show now the important result that the categories *CONT* and *ALG* are cartesian closed, too. Before we give the argument, however, it is useful to identify the lattice $[S \to 2]$, where of course 2 is the two element chain. The easy proof is left to the reader. (Recall that in the Scott topology of 2 the singleton set $\{1\}$ is *open*.)

Lemma II-2.11. *For a* **dcpo** *S the function $f \mapsto f^{-1}(1) : [S \to 2] \to \sigma(S)$ is an isomorphism of lattices.* □

Theorem II-2.12. *If S is a domain and T a continuous lattice, respectively if S is an algebraic domain and T an algebraic lattice, then $[S \to T]$ is a continuous lattice, respectively an algebraic lattice. In particular, the functor $[\cdot \to \cdot]$ maps $CONT^{\mathrm{op}} \times CONT$ into CONT, and $ALG^{\mathrm{op}} \times ALG$ into ALG, and CONT and ALG are cartesian closed categories.*

Proof: Suppose that T is a continuous lattice, respectively algebraic lattice. Then T is the image of some 2^X under a Scott-continuous projection (resp., closure) operator by I-4.18 (resp., I-4.16). Every functor preserves idempotent morphisms; hence so does $[S \to \cdot]$. By II-2.7 $[S \to \cdot]$ therefore preserves Scott-continuous projection operators. If $c^* \leq c$ in $[T^* \to T]$, then $[1_S \to c^*] \leq [1_S \to c]$ in $[[S \to T^*] \to [S \to T]]$ by II-2.7; hence, $[S \to \cdot]$ also preserves Scott-continuous closure operators (recall O-3.8(ii)). Hence $[S \to T]$ is the image of $[S \to 2^X]$ under a Scott-continuous projection (resp., closure) operator. But $[S \to \cdot]$ preserves products, and so $[S \to 2^X] \simeq [S \to 2]^X \simeq \sigma(S)^X$ by Lemma II-2.11. Since $\sigma(S)$ is continuous if S is continuous by II-1.14 (resp., algebraic if S is algebraic by II-1.15), and since products of continuous (resp., algebraic) lattices are continuous (resp., algebraic) by I-2.1 (resp., I-4.12), then $[S \to 2^X]$ is continuous (resp., algebraic). Thus $[S \to T]$ is continuous by I-2.3 (resp., algebraic by I-4.13). □

The argument for II-2.12 is typical of proofs that give the *answer* without directly giving the *reason*. Another proof of the last statement of II-2.12 was indicated in I-2.21. We return to the question again in the exercises below; see II-2.31. The attentive reader will have noticed that the only property of S that we have used is that its lattice of Scott open sets is continuous. The same proof then shows that for a topological space X the set $[X \to T]$ of all continuous functions $f : X \to (T, \sigma(T))$ with its pointwise order is a continuous lattice iff the topology $\mathcal{O}(X)$ and T both are continuous lattices.

FS-domains and bifinite domains

We next consider a rather large cartesian closed category of domains. For their study we introduce a useful concept.

Definition II-2.13. An *approximate identity* for a **dcpo** S is a directed set $\mathcal{D} \subseteq [S \to S]$ satisfying $\sup \mathcal{D} = 1_S$, the identity on S. □

Lemma II-2.14. *Approximate identities are preserved under the following constructions.*

(i) *If* $\mathcal{D} \subseteq [S \to S]$ *is an approximate identity for* S*, then* $\mathcal{D}' = \{\delta^2 = \delta \circ \delta \colon \delta \in \mathcal{D}\}$ *is also an approximate identity.*

(ii) *If for all* $i \in I$, $\mathcal{D}_i$ *is an approximate identity for* S_i*, then* $\prod_{i \in I} \mathcal{D}_i := \{\prod_{i \in I} \delta_i \colon \delta_i \in \mathcal{D}_i\}$ *is an approximate identity for* $\prod_{i \in I} S_i$*. If all but finitely many of the* S_i *have a least element* 0*, then the family consisting of products of members of the* $\mathcal{D}_i$ *in finitely many coordinates and the constant* 0 *mapping in the remaining coordinates is also an approximate identity on* $\prod_{i \in I} S_i$*.*

(iii) *If* $\mathcal{D} \subseteq [S \to S]$ *is an approximate identity for* S *and* $\mathcal{E} \subseteq [T \to T]$ *is an approximate identity for* T*, then* $[\mathcal{D} \to \mathcal{E}]$ *is an approximate identity for* $[S \to T]$*, where members of* $[D \to \mathcal{E}]$ *are denoted by* $[\delta \to \varepsilon]$ *for* $\delta \in \mathcal{D}$ *and* $\varepsilon \in \mathcal{E}$ *and defined by* $[\delta \to \varepsilon](g) = \varepsilon g \delta$ *for* $g \in [S \to T]$*.*

(iv) *The restriction of an approximate identity on* S *to a nonempty Scott closed subset* A *of* S *is an approximate identity for* A*.*

(v) *If a* **dcpo** S *has an approximate identity* $\mathcal{D}$ *such that* $\delta(x) \ll x$ *for all* $\delta \in \mathcal{D}$ *and for all* $x \in S$*, then* S *is a domain.*

Proof: (i) The map $\delta \mapsto (\delta, \delta) \mapsto \delta \circ \delta : [S \to S] \to ([S \to S] \times [S \to S]) \to [S \to S]$ is Scott-continuous, as the first map is trivially Scott-continuous and the second map is composition, which is Scott-continuous by II-2.9(ii). Thus $\sup_{\delta \in \mathcal{D}} \delta = 1_S$ implies $\sup_{\delta \in \mathcal{D}} \delta \circ \delta = 1_S$, too.

(ii) Straightforward.

(iii) Using the continuity of composition, one may deduce this item along the lines of part (i).

(iv) Immediate.

(v) By hypothesis for each $x \in S$, x is the directed supremum of the set $\{\delta(x) \colon \delta \in \mathcal{D}\}$. By hypothesis $\delta(x) \ll x$ for each $\delta \in \mathcal{D}$. Thus by I-1.5(ii) S is a domain. □

Definition II-2.15. A Scott-continuous function $\delta \colon S \to S$ on a **dcpo** S is *finitely separating* if there exists a finite set F_δ such that for each $x \in S$, there exists $y \in F_\delta$ such that $\delta(x) \le y \le x$. A **dcpo** S is *finitely separated* if there is an approximate identity for S consisting of finitely separating functions. A finitely separated **dcpo** that is also a domain will be called an *FS-domain*. □

Note that all finite posets are FS-domains, since the identity mapping is an approximate identity of finitely separating maps. We shall see in Section 4 that all bounded complete domains and hence all continuous lattices are FS-domains.

Lemma II-2.16. *Let S be a* **dcpo**. *If $\delta \in [S \to S]$ is finitely separating, then $\delta(x) \ll x$ for all $x \in S$. Thus a finitely separated* **dcpo** *is a domain, hence an FS-domain.*

Proof: Let D be a directed set such that $w = \sup D \geq x$. There exists a finite set F such that for each $x \in S$, there exists $y \in F$ such that $\delta(x) \leq y \leq x$. Since F is finite, there exist a cofinal subset $D_0 \subseteq D$ and a $y \in F$ such that $\delta(d) \leq y \leq d$ for all $d \in D_0$. By continuity of δ we have $\delta(w) \leq y \leq w$. Since δ is monotone, $\delta(x) \leq \delta(w) \leq y$. For $d_0 \in D_0$ and $d \geq d_0$, we have $\delta(x) \leq y \leq d_0 \leq d$. Thus $\delta(x) \ll x$.

It now follows from Lemma II-2.14(v) that a finitely separated **dcpo** is a domain, and hence an *FS*-domain. □

The property of being an *FS*-domain is stable under a variety of constructs.

Proposition II-2.17.

(i) *A finite product of FS-domains is again an FS-domain.*
(ii) *If S is an FS-domain and $p\colon S \to S$ is a Scott-continuous projection, then $p(S)$ is an FS-domain.*
(iii) *If A is a nonempty Scott closed subset of an FS-domain S, then A is an FS-domain.*

Proof: (i) The general finite case follows from the case for two by induction. Let S and T be *FS*-domains, and let $\mathcal{D}$ and $\mathcal{E}$ be directed families of Scott-continuous mappings giving the *FS*-structure. Then one checks directly that the family $\mathcal{D} \times \mathcal{E}$ shows that $S \times T$ satisfies the conditions to be an *FS*-domain (the finite sets in $S \times T$ that separate are the products of the corresponding ones in S and T respectively).

(ii) Let $\mathcal{D}$ be the family in $[S \to S]$ arising in the definition of an *FS*-domain. Then it is straighforward to verify that the family $\{p\delta|_{p(S)}\colon \delta \in \mathcal{D}\}$ is a family on $p(S)$ that shows that it is finitely separated.

(iii) Take for the family $\mathcal{D}$ on A the restriction to A of the corresponding family on S. □

Proposition II-2.18. *Let S and T be FS-domains. Then $[S \to T]$ is an FS-domain.*

Proof: Let $\mathcal{D}$ and $\mathcal{E}$ be the directed family of mappings giving the *FS*-structure to S and T respectively. We define a family $\mathcal{D} \otimes \mathcal{E}$ on $[S \to T]$ by $g \mapsto \varepsilon^2 g \delta^2$ for $\delta \in \mathcal{D}$ and $\varepsilon \in \mathcal{E}$. From parts (i) and (iii) of Lemma II-2.14 it follows that the collection of all such maps on $[S \to T]$ is an approximate identity. We show that each such function is finitely separating.

Let F_δ and F_ε be the finite sets guaranteed for δ and ε respectively. Define a relation $\sim$ on $[S \to T]$ by $f \sim g$ if $\varepsilon f(x) \leq y \leq f(x) \Leftrightarrow \varepsilon g(x) \leq y \leq g(x)$ for all $x \in F_\delta$ and $y \in F_\varepsilon$. Since F_δ and F_ε are finite, we conclude that there are only finitely many equivalence classes for $\sim$. Pick one representative from each class, say $\{f_1, \ldots, f_n\}$. We claim that the finite family $\{\varepsilon f_1 \delta, \ldots, \varepsilon f_n \delta\}$ is the one needed to establish finite separation.

Let $g \in [S \to T]$. Pick $f_i \sim g$. Given $s \in S$, there exists $x \in F_\delta$ such that $\delta(s) \leq x \leq s$. Then $g\delta(s) \leq g(x)$. There exists $y \in F_\varepsilon$ such that $\varepsilon g(x) \leq y \leq g(x)$. Then $\varepsilon f_i(x) \leq y \leq f_i(x)$. Thus

$$\varepsilon f_i \delta(s) \leq \varepsilon f_i(x) \leq y \leq g(x) \leq g(s),$$

i.e., $\varepsilon f_i \delta \leq g$. A symmetric argument yields that $\varepsilon g \delta \leq f_i$, and hence $\varepsilon^2 g \delta^2 \leq \varepsilon f_i \delta$. This establishes the claim.

Since $[S \to T]$ is a **dcpo**, it follows from Lemma II-2.16 that $[S \to T]$ is an *FS*-domain. □

Corollary II-2.19. *The category FS of FS-domains is a full cartesian closed subcategory of DCPO.*

Proof: This follows immediately from the preceding results on *FS*-domains and Theorem II-2.10. □

An interesting class of **dcpo**s emerges when we consider the algebraic *FS*-domains.

Proposition II-2.20. *For a* **dcpo** *L, the following properties are equivalent:*

(1) *L is an algebraic FS-domain;*
(2) *L is an algebraic domain and has an approximate identity consisting of maps with finite range;*
(3) *L has an approximate identity consisting of kernel operators with finite range.*

Proof: The implication (2) implies (1) is immediate. For the implication (3) implies (2) it suffices to show that (3) implies that L is algebraic. For this, let $\mathcal{D}$ be a directed set of Scott-continuous kernel operators with finite range such that $\sup \mathcal{D} = 1_L$. Then $\{\delta(x)\colon \delta \in \mathcal{D}\}$ is directed and $x = \sup\{\delta(x)\colon \delta \in \mathcal{D}\}$ for every $x \in L$. As the range of $\delta \in \mathcal{D}$ is finite, all of its elements are compact in the finite **dcpo** $\operatorname{im} \delta$. From Exercise I-4.35, it follows that all the elements of $\operatorname{im} \delta$ are compact in L. Thus, every $x \in L$ is the sup of a directed set of compact elements, and we have shown that L is algebraic.

Now we establish that (1) implies (3). Let $\mathcal{D}$ be an approximate identity on L such that each $\delta \in \mathcal{D}$ is finitely separating. For each $\delta \in \mathcal{D}$, set $G_\delta = \{k \in L : \delta(k) = k\}$. Note that it must be the case that $G_\delta \subseteq F_\delta$, the finite separating set, and hence G_δ is finite. Also, all elements of G_δ are compact by II-2.16.

We claim that for each $x \in L$, there exists a largest member of G_δ in $\downarrow x$. Indeed pick a minimal member z of F_δ in $\downarrow x$ (since there is a member of F_δ between x and $\delta(x)$, the finite set F_δ meets $\downarrow x$, and hence has a minimal element in the intersection). Then there exists a member of F_δ between $\delta(z)$ and $\delta(\delta(z))$, and this must be z by minimality of z. It follows that $z = \delta(z)$. Thus $\downarrow x \cap G_\delta \neq \emptyset$.

Pick $k_1, k_2 \in G_\delta \cap \downarrow x$. Then $k_i = \delta(k_i) \leq \delta(x)$ for $i = 1, 2$. There exists $y \in F_\delta$ such that $\delta(x) \leq y \leq x$, and thus $k_i \leq y \leq x$ for $i = 1, 2$. Pick a minimal element $k \in F_\delta \cap \downarrow x$ such that $k_i \leq k$ for $i = 1, 2$. Then $k_i = \delta(\delta(k_i)) \leq \delta(\delta(k)) \leq \delta(k)$ for $i = 1, 2$. We argue again that there must be an element of F_δ between $\delta(\delta(k))$ and $\delta(k)$, and this element must be equal to k by minimality of k. We conclude that $\delta(k) = k$. Thus the finite set $G_\delta \cap \downarrow x$ is directed, and hence must have a largest element.

For $\delta \in \mathcal{D}$, define a function $\kappa = \kappa_\delta$ by $\kappa(x)$ is the largest compact element $k \leq x$ such that $\delta(k) = k$. The preceding paragraphs guarantee the existence of such a function. One verifies easily that κ is a Scott-continuous kernel operator. Also the family $\{\kappa_\delta \colon \delta \in \mathcal{D}\}$ is directed, since as δ becomes larger, the set of fixed-points grows. Since $\mathcal{D}$ is an approximate identity, for each $k \in K(L)$, there exists $\eta \in \mathcal{D}$ such that $\delta(k) = k$ for $\delta \geq \eta$. Thus the supremum of the κ_δ restricted to the compact elements is the identity. By Scott continuity the supremum is the identity on all of L. It follows that the family $\{\kappa_\delta \colon \delta \in \mathcal{D}\}$ is an approximate identity. □

Definition II-2.21. A domain satisfying any of the equivalent conditions of Proposition II-2.20 is called a *bifinite* domain. We denote by BF the category of all bifinite domains and Scott-continuous maps between them. □

Theorem II-2.22. *If L and M are bifinite domains, then $L \times M$ and $[L \to M]$ are also bifinite domains.*

Proof: Let $\mathcal{D}$ and $\mathcal{E}$ be approximate identities for L and M, respectively, consisting of kernel operators with finite range (see II-2.20(3)). Then $\mathcal{D} \times \mathcal{E} = \{\delta \times \varepsilon \colon \delta \in \mathcal{D} \text{ and } \varepsilon \in \mathcal{E}\}$ clearly is an approximate identity for $L \times M$ consisting of projections with finite range (whence $L \times M$ is bifinite). Let us show that on $[L \to M]$ the same holds for the collection $[\mathcal{D} \to \mathcal{E}]$ of self-maps $[\delta \to \varepsilon]$ defined by $f \mapsto \varepsilon f \delta$ for $\delta \in \mathcal{D}$, $\varepsilon \in \mathcal{E}$.

As composition of maps is Scott-continuous, $[\delta \to \varepsilon]$ is Scott-continuous. Again by the Scott continuity of composition, $\sup_{\delta\in\mathcal{D}} \sup_{\varepsilon\in\mathcal{E}} \varepsilon f \delta = (\sup_{\varepsilon\in\mathcal{E}} \varepsilon)$ $f(\sup_{\delta\in\mathcal{D}} \delta) = 1_M f 1_L = f$ for all $f \in [L \to M]$. Thus, $[\mathcal{D} \to \mathcal{E}]$ is an approximate identity on $[L \to M]$. As δ and ε are idempotent, the same follows for $[\delta \to \varepsilon]$ and consequently this is a kernel operator. Its range is finite, as it can be viewed to be the set of all monotone functions from the finite poset im δ into the finite poset im ε. Thus, $[\mathcal{D} \to \mathcal{E}]$ is an approximate identity on $[L \to M]$ consisting of kernel operators, which implies that $[L \to M]$ is a bifinite domain. □

Corollary II-2.23. *The category BF of bifinite domains is a full cartesian closed subcategory of DCPO.*

Proof: This follows immediately from the preceding theorem and Theorem II-2.10. □

It is convenient to record in conclusion a few obvious functors between the various categories which we have already encountered implicitly or explicitly and which we will often use in the following developments.

We know that a function $f: S \to T$ between dcpos belongs to $[S \to T]$ iff it is Scott-continuous (see II-2.1). The assignment, which associates with a **dcpo** L the topological space $(L, \sigma(L))$ defines in an obvious way the functor Σ from *DCPO* into the category *TOP* of T_0 topological spaces. The restrictions of this functor to *CONT* and *ALG* are very interesting, and in the next section we shall describe the subcategories of *TOP* thereby obtained.

If X is a T_0 space, then its topology $\mathcal{O}(X)$ is a frame (O-3.22). In view of the infinite distributivity law in O-2.6, which singles out frames, it is reasonable to consider the *category FRM* of frames and functions preserving *arbitrary* sups and *finite* infs (cf. O-3.24). If we are given a continuous map $f: X \to Y$, then the map $U \mapsto f^{-1}(U)$, which we shall call $\mathcal{O}(f): \mathcal{O}(Y) \to \mathcal{O}(X)$, preserves arbitrary unions and finite intersections. Thus $\mathcal{O}: TOP \to FRM$ is a well-defined contravariant functor.

If L is a **dcpo**, then we have

$$\mathcal{O}(\Sigma L) = \mathcal{O}(L, \sigma(L)) = \sigma(L).$$

Thus we also have a contravariant functor $\sigma: DCPO \to FRM$ (which on functions operates just like $\mathcal{O}$). By Theorem II-1.14 the functor σ maps *CONT* contravariantly into the category *CONT* $\cap$ *FRM* of continuous frames (distributive continuous lattices), and by II-1.15, it maps *ALG* contravariantly into the category *ALG* $\cap$ *FRM* of algebraic frames (distributive algebraic lattices).

Notation II-2.24. We record the following functions:

$$\begin{array}{lll} \sigma\colon \mathit{DCPO}^{\mathrm{op}} \to \mathit{FRM}, & \sigma(L) = \text{Scott topology}, & \sigma(f)(U) = f^{-1}(U); \\ \Sigma\colon \mathit{DCPO} \to \mathit{TOP}, & \Sigma L = (L, \sigma(L)), & \Sigma(f) = f; \\ \mathcal{O}\colon \mathit{TOP}^{\mathrm{op}} \to \mathit{FRM}, & \mathcal{O}(X) = \text{topology of } X, & \mathcal{O}(f)(U) = f^{-1}(U). \end{array}$$

We note $\sigma = \mathcal{O}\Sigma$. □

Exercises

Exercise II-2.25. Let L be a complete lattice and let I be a given directed set. Then each of the following three infinitary operations can be viewed as a mapping defined on the direct power:

$$\underline{\lim}, \sup, \inf : L^I \to L.$$

All these functions are monotone, but which are Scott-continuous? □

Exercise II-2.26. Carry out the suggestion mentioned in II-2.3(2). □

Exercise II-2.27. Consider functions of several variables which for simplicity are defined on and take values in a fixed **dcpo** S. Let f be an n-place function, and let $g_1, \ldots, g_n$ all be m-place functions. Define the m-place function h by composition:

$$h(x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)).$$

(i) Use a generalization of II-2.8 to prove that h is Scott-continuous if f and the g_i are Scott-continuous in each variable separately.
(ii) Give a direct proof of the same result.

Hint. For (ii) it is sucient to consider the two special cases $f(x, x)$ and $(f(g(x))$ by first making all variables distinct and then identifying them one occurrence at a time. □

A function is continuous if the preimage of every open set is open. It is clear that it is sufficient for continuity that the preimage of every open set belonging to a subbasis for the topology is open. In the case of Scott continuity this condition can be weakend considerably:

Exercise II-2.28. For a map $f\colon L \to M$ of **dcpo**s to be Scott-continuous show that the following is sufficient: $f^{-1}(U)$ is Scott open for every U belonging to a collection $\mathcal{U}$ of Scott open sets in M that separates the points of M, that is, whenever $x \not\leq y$ in M, there is a $U \in \mathcal{U}$ such that $x \in U$ but $y \notin U$.

Hint. f is order preserving: Indeed, let $u, v \in L$. If $f(u) \not\leq f(v)$, by hypothesis there is a $U \in \mathcal{U}$ containing $f(u)$ but not $f(v)$. Then u belongs to $f^{-1}(U)$ but not v. As $f^{-1}(U)$ is Scott open and hence an upper set, we conclude that $u \not\leq v$.

f preserves directed sups: Let D be directed in L. Then $\sup f(D) \leq f(\sup D)$ by the monotonicity of f. Assume that $\sup f(D) < f(\sup D)$; by hypothesis there is a $U \in \mathcal{U}$ containing $f(\sup D)$ but not $\sup f(D)$. As U is an upper set, the latter implies that $f(D) \cap U = \emptyset$. Then $\sup D$ is contained in $f^{-1}(U)$ but $f^{-1}(U) \cap D = \emptyset$. As $f^{-1}(U)$ is Scott open, the latter implies $\sup D \notin f^{-1}(U)$, a contradiction. □

For a **dcpo** with a least element we have seen in II-2.4 that every Scott-continuous self-map f has a least fixed-point LFP(f). The next exercise shows that this least fixed-point depends continuously on f and that least fixed-points are preserved under appropriate hypotheses.

Exercise II-2.29. Let L be **dcpo** with a least element $\perp$. Show that the *least fixed-point operator*

$$\mathrm{LFP} = (f \mapsto \mathrm{LFP}(f)) : [L \to L] \to L$$

is Scott-continuous.

Hint. The least fixed-point operator is monotone, for if $f \leq g$ are self-maps of L, then $f(\perp) \leq g(\perp)$ and, by induction, $f^n(\perp) \leq g^n(\perp)$, whence $\mathrm{LFP}(f) = \sup_n f^n(\perp) \leq \sup_n g^n(\perp) = \mathrm{LFP}(g)$. Now let (f_i) be a directed family of Scott-continuous self-maps of L. Then $\mathrm{LFP}(\sup_i f_i) = \sup_n(\sup_i f_i)^n(\perp) = \sup_i \sup_n f_i^n(\perp) = \sup_i \mathrm{LFP}(f_i)$. Thus, the least fixed-point operator is Scott-continuous. Note that we have used that composition of maps is Scott-continuous (see II-2.9) for the third equality. □

The next exercise contains some further information on fixed-points of Scott-continuous self-maps (compare II-2.4).

Exercise II-2.30. Let L be an arbitrary **dcpo** and $f : L \to L$ a Scott-continuous self-map. Denote by L_f the set of all *pre-fixed-points* $x \in L$ with $x \leq f(x)$ and by L_f° the subset of all fixed-points of f.

(i) Show that for every pre-fixed-point x the element $\tilde{g}(x) = \sup_n g^n(x)$ is the smallest fixed-point of g above x.
(ii) Show that L_f is closed in L with respect to directed sups, hence a **dcpo**, and that $\tilde{g} : L_f \to L_f$ is a Scott-continuous closure operator whose image is the set of fixed-points. □

The following exercise provides an alternative proof for Theorem II-2.12.

Exercise II-2.31. Let L and M be domains and suppose that M has a least element. For $x \in L$ and $y \in M$ define a function $(\!| x \Rightarrow y |\!)$ on L by the formula

$$(\!| x \Rightarrow y |\!)(z) = y, \quad \text{if} \quad x \ll z, \quad \text{and} \quad = 0_M \text{ otherwise.}$$

Prove the following:

(i) this function is always Scott-continuous;
(ii) if $f \in [L \to M]$ and $y \ll f(x)$, then $(\!| x \Rightarrow y |\!) \ll f$ in $[L \to M]$;
(iii) if M is a bounded complete domain, respectively a continuous lattice, then $f = \sup\{(\!| x \Rightarrow y |\!)\colon y \ll f(x)\}$ for all $f \in [L \to M]$; hence, $[L \to M]$ is a bounded complete domain, respectively a continuous lattice. □

Now, let L and M be algebraic domains and suppose that M has a least element. For compact elements k in L and j in M define a function $(\!| k \Rightarrow j |\!)$ on L by the formula

$$(\!| k \Rightarrow j |\!)(z) = j, \quad \text{if} \quad k \le z, \quad \text{and} \quad = 0_M \text{ otherwise.}$$

Prove the following:

(iv) this function is always Scott-continuous;
(v) the function $(\!| k \Rightarrow j |\!)$ is compact in the **dcpo** $[L \to M]$;
(vi) if M is an algebraic bounded complete domain, respectively an algebraic lattice, then $f = \sup\{(\!| k \Rightarrow j |\!)\colon j \le f(k)\}$ for all $f \in [L \to M]$; hence, $[L \to M]$ is an algebraic bounded complete domain, respectively an algebraic lattice. □

Conclude that the categories of bounded complete domains and algebraic bounded complete domains with Scott-continuous maps are cartesian closed.

Exercise II-2.32. If L and M are (algebraic) L-domains, show that $L \times M$ and $[L \to M]$ are (algebraic) L-domains, too, and conclude that the categories *LDOM* of L-domains and *ALGLDOM* of algebraic L-domains and Scott-continuous maps are cartesian closed.

Hint. For the function space proceed as in the previous exercise. Two changes are necessary. Firstly, one has to be careful when taking sups which are not directed sups: one shows that $f(x) = \sup\{(\!| x \Rightarrow y |\!)\colon y \ll f(x)\}$ where for all $x \in L$ the sup has to be taken in the complete lattice $\downarrow f(x)$. Secondly, an L-domain does not necessarily have a least element. Thus, we have to modify the definition of the "step functions":

$$(\!| x \Rightarrow y |\!)(z) = \begin{cases} y \text{ if } x \ll z, \\ 0_z \text{ otherwise,} \end{cases}$$

where 0_z is the least element of the complete lattice $\downarrow z$. □

Exercise II-2.33. Let A be a full cartesian closed subcategory of *DCPO*, in the sense that A contains the terminal object (the one element **dcpo**) and, whenever L, M are objects in A, then $L \times M$ and $[L \to M]$ also belong to A.

Now let *RA* be the full subcategory of *DCPO* whose objects are the retracts of objects in A, that is, the images of objects in A under Scott-continuous projections (up to isomorphism). Show that *RA* also is a cartesian closed subcategory of *DCPO*. □

An example for the situation indicated in the preceding exercise is the following. In II-2.10 we have shown that the full subcategories *CONT* and *ALG* of continuous and algebraic lattices, respectively, are cartesian closed. As a lattice is continuous iff it is isomorphic to the image of an algebraic lattice under a Scott-continuous projection (see I-4.18), the preceding exercise shows that the cartesian closedness of *CONT* can be derived from the cartesian closedness of *ALG*.

In II-2.24 we have seen that the full subcategory of bifinite domains is cartesian closed. By the preceding exercise it follows that the full subcategory of *DCPO* whose objects are the images of bifinite domains under Scott-continuous projections is also cartesian closed. As the bifinite domains are precisely the algebraic *FS*-domains by II-2.20 one might conjecture that every *FS*-domain can be obtained from a bifinite domain as the image under a Scott-continuous projection. This conjecture is neither proved nor disproved to date:

Problem. Is there an *FS*-domain that is not isomorphic to the image of any bifinite domain under a Scott-continuous projection? □

The following example due to Lawson may serve for a test.

Exercise II-2.34. Let L be the collection of all closed discs in the Euclidean $\mathbb{R}^2$ including the singleton sets. Show that L ordered by reverse inclusion $\supseteq$ is a domain, even an *FS*-domain, in which $C \ll D$ iff $\operatorname{int} C \supseteq D$. □

Problem. Is the *FS*-domain L of closed discs in the plane in the example above isomorphic to the image of some bifinite domain under a Scott-continuous projection? □

Exercise II-2.35. Let L and M be algebraic lattices. Define maps

$$\text{fun}\colon 2^{K(L)\times K(M)} \to [L \to M] \quad \text{and} \quad \text{graph}\colon [L \to M] \to 2^{K(L)\times K(M)}$$

by the formulae

$$\begin{aligned} \text{fun}(F)(x) &= \sup\{j\colon (k, j) \in F \text{ and } k \leq x\}, \\ \text{graph}(f) &= f(k, j)\colon j \leq f(k)\}. \end{aligned}$$

Prove the following:

(i) these maps are Scott-continuous (assuming only that we know that $[L \to M]$ is a complete lattice);
(ii) (fun, graph) is an adjunction with the first map surjective and the second injective (cf. O-3.7);
(iii) $[L \to M]$ is isomorphic to the range of a Scott-continuous closure operator on $2^{K(L)\times K(M)}$; hence, it is algebraic;
(iv) part (iii) can be used to describe $K([L \to M])$ explicitly. □

Exercise II-2.36. Prove the following.

(i) Let *SEMI* be the category of sup semilattices with 0 and monotone maps. Then we can construe the construction of I-4.10 as a functor

$$\text{Id}\colon \textit{SEMI} \to \textit{ALG}$$

provided we define Id f to be the map $I \mapsto \downarrow f(I)$.
(ii) This construction is *not* an equivalence of categories, because not every continuous map is obtained.
(iii) Expand *SEMI* to the category *GRAPH* using the idea of II-2.35 by adding more maps. Specifically, for $R, S \in$ *SEMI* define $F\colon R \to S$ to mean that F is a sup subsemilattice of $S \times R$ which is monotone in the sense that yFx and $y_1 \leq y$ always imply $y_1 F x$. (Every $f\colon R \to S$ in *SEMI* is represented in *GRAPH* by the relation $\{(y, x)\colon y \leq f(x)\}$.) Composition *GF* for $G\colon S \to T$ is just the ordinary composition of relations. In this way *GRAPH* is a category and *SEMI* is a subcategory with the same collection of objects. Now define Id: *GRAPH* $\to$ *ALG* where Id F is the map

$$I \mapsto \{y\colon yFx \text{ for some } x \in I\}.$$

This construction provides an equivalence of categories. □

Old notes

The results in this section are largely based on Scott's ideas. (See [Scott, 1972a] and also [Scott, 1976] and the bibliography contained therein.)

New notes

Cartesian closed categories of domains have attracted considerable attention as they are appropriate for models of various typed and untyped lambda-calculi and functional programming languages. The subsection on *FS*-domains and bifinite domains provides us with maximal classes of cartesian closed categories of domains, and of algebraic domains. The bifinite domains were introduced and studied by G. Plotkin [Plotkin, 1976] in the context of powerdomains (see also Section IV-8). *FS*-domains were introduced by A. Jung [Jung, 1990b].

As the category *DOM* of all domains is not cartesian closed, there has been interest in classifying cartesian closed categories of domains. M. B. Smyth has shown that among the full subcategories of pointed countably based algebraic domains there is a unique largest one: the category of pointed countably based bifinite domains [Smyth, 1983b]. (An algebraic domain is countably based iff it has countably many compact elements only; see also Section III-4.) For the general case the problem has been solved by A. Jung [Jung, B1989; Jung, 1990a; Jung, 1990b]: there are exactly two maximal full subcategories of pointed domains: the category of all pointed *FS*-domains and the category of all pointed L-domains. In the algebraic case, *FS*-domains have to be replaced by bifinite domains and L-domains by algebraic ones. In the general case (of domains without a least element), there are exactly four maximal full cartesian closed subcategories of (algebraic) domains: one considers either disjoint sums of pointed *FS*-domains or L-domains, or else certain finite amalgams of *FS*-domains or L-domains.

II-3 Injective Spaces

In the previous sections we associated with a complete lattice a canonical topology, and we further pursue the relation between topological spaces and lattices in this section. Our goal here is to characterize *continuous* lattices via the Scott topology in purely topological terms. The question is: which topological spaces are of the form ΣL for L a continuous lattice? We find a complete (and brief!) answer. Furthermore we show that ΣL as a space completely determines L as a lattice.

We work entirely in the category *TOP* of T_0-spaces and continuous maps and will never consider topological spaces which do not at least satisfy the T_0-separation axiom.

Injective and densely injective spaces

We begin by recalling the idea of *relative injectives* in a category A. One is given a class J of monomorphisms which is closed under the pre- and post-multiplication with isomorphisms. Then an object Z is called a *J-injective* iff for any map $j\colon X \to Y$ in J and every morphism $f\colon X \to Z$ in A there is a morphism $f^*\colon Y \to Z$ with $f = f^* \circ j$, that is, the following diagram commutes.

$$\begin{tikzcd} Z & \\ X \arrow[u, "f"] \arrow[r, "j"'] & Y \arrow[ul, dotted, "f^*"'] \end{tikzcd}$$

In the category *TOP* we wish to consider J-injectives for the class J of all *subspace embeddings* (that is, continuous maps whose co-restriction to their image is a homeomorphism). For future reference we restate the definition in this special case.

Definition II-3.1. *A* T_0-space Z is called *injective* iff every continuous map $f\colon X \to Z$ extends continuously to any space Y containing X as a subspace. □

It is useful to record that there are some purely arrow-theoretical facts about relative injectives (whose proof we leave as an exercise on the manipulation of injectives in any category).

Lemma II-3.2.

(i) *Products of J-injectives are J-injectives.*
(ii) *Retracts of J-injectives are J-injectives.*
(iii) *If Z is a J-injective and $j\colon Z \to Y$ is a J-monomorphism, then Z is a retract of Y.* □

The immediate question is now whether in *TOP* we have any injectives. We give at first a rather modest answer, but it will be the key for all that follows.

Lemma II-3.3. *The Sierpinski space $\Sigma 2$ is injective.*

Proof: Suppose that X is a subspace of Y and that $f: X \to \Sigma 2$ is a continuous map. Then $U = f^{-1}(1)$ is open in X since $\{1\} \in \sigma 2$. By the definition of the induced topology on X, there is an open set V on Y with $U = V \cap X$. Define g: $Y \to \Sigma 2$ to be the characteristic function of V (that is, $g^{-1}(1) = V$); it is continuous, and clearly $g \mid X = f$. □

In order to see how far this will lead us we make the following remarks.

Lemma II-3.4.

(i) *For every set M we have $\Sigma(2^M) = (\Sigma 2)^M$; that is, the Scott topology on 2^M and the product topology agree. Moreover, $\Sigma(2^M)$ is injective.*
(ii) *Every T_0-space X is embedded in some $(\Sigma 2)^M$.*
(iii) *Every injective T_0-space X is a retract of some $(\Sigma 2)^M$; that is, there is a continuous $f: (\Sigma 2)^M \to (\Sigma 2)^M$ with $f^2 = f$ and* $\operatorname{im} f$ *homeomorphic to X.*

Proof: (i) As 2^M is an algebraic lattice, we recall by II-1.15 that $\Sigma(2^M)$ has as a basis for its topology the sets of the form $\uparrow k$ where k is compact in 2^M. But by I-4.15(1) – expressed in terms of characteristic functions rather than sets – k is just a function that takes on the value 1 only finitely often. The set $\uparrow k$, then, is exactly the class of functions that take the value 1 at least at the places that k does.

Turning now to the product space $(\Sigma 2)^M$, we remark that, because $\{1\}$ is the only nontrivial open set of $\Sigma 2$, a basis for the open sets is given by putting $\{1\}$ on finitely many coordinates and $\{0, 1\}$ on the remainder. But as we just noted the sets formed this way are the sets of the form $\uparrow k$. Thus, the two topologies have the same basis and must be the same.

The last assertion of (i) then follows from II-3.3 and II-3.2(i).

(ii) For a given space X we take $M = \mathcal{O}(X)$ and define j: $X \to 2^M$ by having $j(x)(U) = 1$ iff $x \in U$. Since X is T_0, it follows that j is injective.

Let W be a basic open set of 2^M. By our description in the proof of (i), W is determined by a finite number of coordinates $U_1, \ldots, U_n \in M$. We have

$$j(x) \in W \text{ iff } x \in U_1 \cap \cdots \cap U_n;$$

whence, j is continuous.

Let V be any open subset of X. Then it is easy to see that

$$j(V) = \{f \in \operatorname{im} j : f(V) = 1\}.$$

As this is the intersection of a basic open subset of 2^M with im j, this shows that j is an embedding.

(iii) This is now a consequence of (ii), (i) and II-3.2(iii). $\square$

It is useful at this point to recall the various formal aspects of the concept of a retract. If $j\colon X \to Y$ and $e\colon Y \to X$ are morphisms in a category with $ej = 1_X$, then X is called a *retract* of Y. The map e is a *retraction*, the map j a *co-retraction*.

If, in a given category, every morphism $f\colon A \to B$ may be decomposed into a composition $f = f_\circ f^\circ$ with an epimorphism f° and a monomorphism $f_\circ$, then any projection $f = f^2$ on an object Y gives rise to a retract where $X =$ domain $f_\circ =$ codomain f°. Indeed $f_\circ f^\circ = f = f^2 = f_\circ f^\circ f_\circ f^\circ$ implies that $f^\circ f_\circ = 1_X$, since $f_\circ$ is monic and f° is epic. In such categories the retracts X of an object Y are, up to canonical isomorphism, in bijective correspondence with the projections on Y. We have made use of this situation in Section I-2 (in I-2.4, I-2.17) and in II-3.2(iii) and II-3.4(iii) above. We will use it further now. All the categories we consider have the required epic–monic factorization property. A direct proof of the next proposition is given in Proposition II-3.9.

Proposition II-3.5. *If L is a continuous lattice, then ΣL is an injective space.*

Proof: By I-4.18(4), L is a retract in *UPS* of some 2^M. Since functors preserve retracts, ΣL is a retract of $\Sigma(2^M)$. By II-3.4(i), $\Sigma(2^M)$ is injective. Hence, by II-3.2(ii), ΣL is injective. $\square$

So far we operated exclusively in terms of topology, using, where lattices arose, the canonical Scott topology. We now associate with each T_0-space a canonical (and well-known) poset structure.

Recall that in a T_0-space X, for two elements x and y in X the following relations are equivalent (see before O-5.2):

(1) $\{x\}^- \subseteq \{y\}^-$;
(2) $x \in \{y\}^-$;
(3) $x \in U$ implies $y \in U$, for all open sets U.

The relation

$$x \leq y \text{ iff } x \in \{y\}^-$$

is a partial order that we have called the specialization order in O-5.2. Furthermore if $f\colon X \to Y$ is a continuous map in *TOP*, then it is obvious from (3) that the relation is preserved; that is, f is a monotone map. We thus have a functor from *TOP* into the category *POSET* of posets and monotone maps.

Definition II-3.6. We denote by Ω: *TOP* $\to$ *POSET* the functor which associates with a space X the poset $\Omega X = (X, \le)$, where $\le$ is the specialization order, and with $\Omega f = f$. $\square$

Note that, with respect to the specialization order, $\{x\}^- = \downarrow x$, closed sets are lower sets and open sets are upper sets. If L is a **dcpo**, then $\Omega\Sigma L = L$ by I-1.4(ii); that is to say, the Scott topology determines the partial ordering by means of a purely topological definition. We are now ready for a counterpart of II-3.5.

Proposition II-3.7. *If X is an injective T_0-space, then ΩX is a continuous lattice.*

Proof: By II-3.4(iii), there is a continuous function $f = f^2 \colon (\Sigma 2)^M \to (\Sigma 2)^M$ such that we may identify the space X with im f. We apply the functor Ω and note $\Omega(\Sigma 2)^M = \Omega\Sigma(2^M) = 2^M$ by II-3.4(i). We thus obtain a projection operator $f \colon 2^M \to 2^M$ which preserves directed sups by I-2.1. But then im f is a continuous lattice in the *induced* partial order by I-4.18. However, the *specialization* order of a space induces on a subspace the specialization order of this subspace (indeed if A is a subspace of B and $P \subseteq A$, then the closure of P in A is $P^- \cap A$, where P^- is the closure of P in B). Thus, ΩX is a continuous lattice. $\square$

If we apply to the diagram

$$\begin{tikzcd} 2^M \arrow[rr, "f"] \arrow[dr, "\overline{f}"'] & & 2^M \\ & \Omega X \arrow[ur, "\underline{f}"'] & \end{tikzcd} \qquad \overline{f}\,\underline{f} = 1_X$$

of *UPS*-maps the functor Σ, we obtain, in view of II-3.4(i), the commutative diagram

$$\begin{tikzcd} (\Sigma 2)^M \arrow[rr, "f"] \arrow[dr, "f^\circ"'] & & (\Sigma 2)^M \\ & \Sigma\Omega X \arrow[ur, "f_\circ"'] & \end{tikzcd} \qquad f^\circ f_\circ = 1_X$$

of continuous maps. Since $f_\circ \colon X \to (\Sigma 2)^M$ is an embedding, then the identity map $1_X \colon \Sigma\Omega X \to X$ is continuous; since the retraction $f^\circ \colon (\Sigma 2)^M \to X$ is a quotient map (as all retractions are), the identity map in the other direction $1_X\colon X \to \Sigma\Omega X$ is continuous. Hence $\Sigma\Omega X = X$.

Taking this remark into account, we may summarize the principal results of this subsection in the following theorem.

Theorem II-3.8.

(i) *If L is a continuous lattice, then $\Sigma L = (L, \sigma(L))$ is an injective space and $\Omega\Sigma L = L$.*
(ii) *If X is an injective T_0-space, then $\Omega X = (X, \leq)$ is a continuous lattice (with respect to the specialization order) and $\Sigma\Omega X = X$.* □

There is, therefore, a canonical bijection between continuous lattices and injective topological T_0-spaces. In fact we have shown that *INJ*, the full subcategory of *TOP* consisting of injective spaces and all continuous maps, is essentially the same category as *CONT*. This allows not only a purely topological description of continuous lattices (injective spaces under the specialization order), but also a complete answer to the question which spaces are of the form ΣL with L continuous. The results of Section 4 will serve as a first illustration of how useful this knowledge can be. (Note that by II-3.8 and II-1.13 all injective T_0-spaces are locally compact and sober. The converse is clearly incorrect as the two point *discrete* space shows.)

We turn now to a topological characterization of bounded complete domains. First we give an explicit construction of continuous extensions.

Proposition II-3.9. *Let L be a continuous lattice, resp. a bounded complete domain, equipped with the Scott topology, let X be a subspace, resp. a dense subspace, of a topological space Y, and let $f\colon X \to L$ be continuous. Then*

$$f^*(y) := \sup\{\inf f(U \cap X)\colon U \text{ is open, } y \in U\}$$

is a continuous extension of f to Y, and is the supremum of all such extensions.

Proof: The infimum $\inf f(U \cap X)$ exists for L a continuous, hence complete, lattice. If X is dense, then $U \cap X$ is nonempty, so the infimum $\inf f(U \cap X)$ exists for L a bounded complete domain. For $y \in U \subseteq V$, $\inf f(V \cap X) \leq \inf f(U \cap X)$, and hence the supremum is a directed supremum and thus exists since L is a **dcpo**.

Suppose that $q \ll f^*(y)$. Pick p such that $q \ll p \ll f^*(y)$, a directed supremum; then $p \leq \inf f(U \cap X)$ for some open set U containing y. It follows that $f^*(U) \subseteq \uparrow p \subseteq \twoheaduparrow q$. Since sets of the form $\twoheaduparrow q$ form a basis for the Scott topology, we conclude that f^* is continuous.

Let $x \in X$. Clearly $f^*(x) \leq f(x)$. If $q \ll f(x)$, then by continuity there exists U open containing x such that $f(U \cap X) \subseteq \uparrow q$. Then $q \leq f^*(x)$. Since $f(x) = \sup \Downarrow f(x)$, we conclude $f(x) \leq f^*(x)$.

Finally suppose that $g\colon Y \to L$ is continuous and extends f. Let $y \in Y$, and let $q \ll g(y)$. Then there exists U open containing y such that $g(U) \subseteq \uparrow q$. It follows that

$$q \leq \inf g(U) \leq \inf g(U \cap X) = \inf f(U \cap X) \leq f^*(y).$$

Thus $g(y) = \sup \Downarrow g(y) \leq f^*(y)$. □

Definition II-3.10. A T_0 space Z is called *densely injective* if every continuous map $f\colon X \to Z$ extends continuously to any space Y containing X as a dense subspace. □

Note that the densely injective T_0-spaces are precisely the J-injectives for the class J of all dense subspace embeddings. Thus the general remarks we made on J-embeddings at the beginning of the section apply to densely injective spaces.

Proposition II-3.11. *A space is a densely injective T_0-space iff it is a bounded complete domain equipped with the Scott topology.*

Proof: It follows from Proposition II-3.9 that a bounded complete domain equipped with its Scott topology is densely injective. Conversely let Z be a densely injective T_0 space. Then Z can be topologically embedded in some $(\Sigma 2)^M$, the continuous lattice 2^M equipped with the Scott topology by Lemma II-3.4(i),(ii); let $j\colon Z \to X = j(Z)$ be the embedding. The closure Y of the embedded image X is a Scott closed subset of 2^M, hence closed under arbitrary nonempty infs and directed sups, and thus a bounded complete domain (see Theorem I-2.11). By the dense embedding property there exists a continuous mapping $f\colon Y \to Z$ extending $j^{-1}\colon X \to Z$. Then jf is a continuous projection operator from Y onto X, and thus X is a bounded complete domain by Corollary I-2.3. Since j is an order isomorphism with respect to the order topologies on Z and X, Z is also a bounded complete domain. Since Z is a retract of Y, it follows from Proposition II-3.15(iii) below that the topology of Z is the Scott topology. □

Monotone convergence spaces

For an arbitrary space X, very little can be said in general of the poset structure of ΩX. Directed nets need not have sups. Even if directed nets always have sups, they need not converge to their sups (as the unit interval in which all upper sets are open shows). The following definition is therefore natural.

Definition II-3.12. A T_0 space X is called a *monotone convergence space* iff every subset D directed relative to the specialization order (II-3.6) has a sup, and the relation $\sup D \in U$ for any open set U of X implies $D \cap U \neq \emptyset$. □

Clearly, a space is a monotone convergence space iff every directed net in ΩX has a sup and converges to this sup; whence the name. In particular, every monotone convergence space is a **dcpo** with repect to its specialization order and every open subset of X is Scott open in ΩX. By II-1.2(ii), every space ΣL for a **dcpo** L is a monotone convergence space. All injective spaces are monotone convergence spaces. By O-5.15, every sober space is a monotone convergence space.

Lemma II-3.13. *A continuous function $f\colon X \to Y$ from a monotone convergence space X to any space Y preserves directed sups in the specialization orders (that is, Ωf preserves directed sups).*

Proof: Let D be a directed subset of ΩX. Since f is monotone, it follows that $f(\sup D)$ is an upper bound of $f(D)$. Let a be also an upper bound of $f(D)$ and a $\not\geq f(\sup D)$. Then $U = Y \setminus {\downarrow} a$ is an open neighborhood of $f(\sup D)$; that is, $f^{-1}(U)$ is an open neighborhood of $\sup D$, by the continuity of f. Since X is a monotone convergence space, there is a $d \in D$ with $d \in f^{-1}(U)$; that is, $f(d) \in U$ and $f(d) \not\leq a$, a contradiction to $f(D) \leq a$. □

Lemma II-3.14. *Let X be a space and Y a monotone convergence space. Let $(f_j)_{j \in J}$ be a net of continuous functions $f_j\colon X \to Y$ such that $(f_j(x))_{j \in J}$ is a directed net of ΩY for each x. Let $f\colon X \to Y$ be the pointwise sup of the net f_j. Then f is continuous.*

Proof: Let $x \in X$ and let U be an open set in Y containing $f(x)$. Since $f(x) = \sup_j f_j(x)$ and Y is a monotone convergence space, there is a $j \in J$ with $f_j(x) \in U$; that is, $x \in V = f_j^{-1}(U)$. As f_j is continuous, $V = f_j^{-1}(U)$ is open. For all $z \in V$ we have $f_j(z) \leq f(z)$. As $f_j(z) \in U$ and as open sets are upper sets with respect to the specialization order, we conclude $f(z) \in U$ and hence $f(V) \subseteq U$. □

The set $TOP(X, Y)$ of all continuous functions $f\colon X \to Y$ may be considered as a subset of $(\Omega Y)^X$ with the induced ordering, that is, $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$ with respect to the specialization order on Y. With this convention we have

Proposition II-3.15. *Let X, Y, Z be T_0 spaces.*

(i) *If Y is a monotone convergence space, then the subset $TOP(X, Y)$ of $(\Omega Y)^X$ is closed under directed sups, this is, $TOP(X, Y)$ is a* **dcpo**.

(ii) *If Y is a monotone convergence space and $f: Y \to Z$ a continuous map, then the function $TOP(X, f) = (g \mapsto fg): TOP(X, Y) \to TOP(X, Z)$ preserves directed sups.*

(iii) *If X is a continuous retract of a monotone convergence space Y, then X must be T_0 and is a monotone convergence space. In the special case that Y is a* **dcpo** *equipped with the Scott topology, then X must also have the Scott topology.*

Proof: Part (i) follows immediately from II-3.14. In order to prove part (ii) it suffices to point out that f preserves directed sups by Lemma II-3.13 and that sups of functions are calculated pointwise.

(iii) Let $f: Y \to X$ and $j: X \to Y$ be continuous functions such that fj is the identity on X. For $y = j(x) \in j(X)$, $j(f(y)) = j(fj(x)) = j(x) = y$, and thus j is a homeomorphic embedding of X into Y (with inverse f restricted to $j(X)$). Since Y is T_0, it follows that X is T_0.

Now f and j are order preserving with respect to the orders of specialization, and f preserves directed sups by Lemma II-3.13. Let D be a directed set in X. Then $j(D)$ is a directed set in Y and hence has a supremum d. Then $f(d)$ is the supremum for $(fj)(D) = D$, and thus X is a **dcpo** with respect to the order of specialization (Lemma II-3.13). Furthermore by continuity of f, the directed set D converges to its supremum $f(d)$, and thus X is a monotone convergence space.

Now let X be a **dcpo** and U a Scott open set in X. Then $f^{-1}(U)$ is Scott open in Y, since f preserves directed sups and hence is Scott-continuous. Thus $j^{-1}(f^{-1}(U)) = (1_X)^{-1}(U) = U$ is open in X if Y has the Scott topology. Conversely every open set in X is Scott open, as X has been proved to be a monotone convergence space. □

We come now to a topological analog of Theorem II-1.14.

Theorem II-3.16. *For a monotone convergence space X and its order of specialization, the following conditions are equivalent.*

(1) *ΩX is a domain and the topology of X is the Scott topology.*
(2) *For each $U \in \mathcal{O}(X)$, $U = \bigcup\{\mathrm{int}(\uparrow x) : x \in U\}$.*
(3) *Each point has a neighborhood basis of open filters, and $\mathcal{O}(X)$ is a continuous lattice.*
(4) *$\mathcal{O}(X)$ has enough co-primes and is a continuous lattice.*
(5) *$\mathcal{O}(X)$ is completely distributive.*
(6) *Both $\mathcal{O}(X)$ and $\mathcal{O}(X)^{\mathrm{op}}$ are continuous.*

If X is a complete semilattice then these conditions are equivalent to

(7) *for each point $x \in X$ we have $x = \sup\{\inf U : x \in U \in \mathcal{O}(X)\}$.*

Proof: That (1) implies (2)–(7) follows from Theorem II-1.14.

(2) implies (1): Let $x \in X$ and consider $D = \{d \in X : x \in \operatorname{int}(\uparrow d)\}$. It follows directly from the assumption (2) that D is directed. Suppose that its supremum t satisfies $t < x$. Then $X\backslash{\downarrow}t = X\backslash\{t\}^-$ is an open set containing x and missing t. By assumption there exists $d \in D$ such that $d \in X\backslash{\downarrow}t$, contradicting the fact that t is their supremum. The argument of Proposition II-1.6 yields that $d \ll x$ for each $d \in D$. Thus ΩX is a domain. Furthermore, for any Scott open set V containing x, it must be the case that $d \in V$ for some $d \in D$. Hence $x \in \operatorname{int}(\uparrow d) \subseteq \uparrow d \subseteq V$. We conclude that the identity is continuous from X to ΩX endowed with the Scott topology. But since X is a monotone convergence space, its topology is contained in the Scott topology, and so the two are equal.

(3) implies (2): Let $x \in U \in \mathcal{O}(X)$. There exists $V \in \mathcal{O}(X)$ such that $x \in V \ll U$, since $\mathcal{O}(X)$ is continuous. Pick an open filter F such that $x \in F \subseteq V$. Suppose that for each $y \in U$, it is not the case that $F \subseteq \uparrow y$. Then $y \in X\backslash{\downarrow}z$ for some $z \in F$, and hence there exists an open filter F_y such that $y \in F_y \subseteq X\backslash{\downarrow}z$, since the open filters form a basis. Finitely many of the F_y, say $F_{y1}, \ldots, F_{yn}$, must cover V. Pick $z_i \in F\backslash F_{yi}$ for each i and pick $z \in F$ such that $z \le z_i$ for all i (remember F is a filter). Then $z \in F_{yi}$ would imply $z_i \in F_{yi}$, so none of the F_{yi} contain z, a contradiction. Thus there exists $y \in U$ such that $x \in F \subseteq \operatorname{int}(\uparrow y)$.

(3) iff (4): The argument in the proof of Proposition II-1.11(i) applies equally to $\mathcal{O}(X)$ as to σL to show that an open set is a filter iff it is a co-prime in $\mathcal{O}(X)$. The equivalence of (3) and (4) is then immediate.

(4) iff (5) iff (6): A consequence of I-3.16.

(7) implies (2): The proof follows directly from the fact that X is a monotone convergence space. □

Exercises

Exercise II-3.17. For a T_0 space X, show that the following conditions are equivalent:

(1) X is injective;
(2) X is a retract of every space of which it is a subspace;
(3) $X = \Sigma\Omega X$, ΩX is a complete lattice, and $\mathcal{O}(X)$ is completely distributive.

Hint. Use II-3.4(ii) and II-3.2(i) for (1) iff (2). For (1) iff (3) recall II-1.14. □

We remark that II-3.17(3) gives a characterization of injective spaces that is completely intrinsic, in the sense that relationships to other spaces are not involved in the statement of the property.

Exercise II-3.18. Which spaces are of the form ΣL for some *algebraic* lattice L?

Answer: They are the injective spaces for which the monogeneric open sets form a basis for the topology. (A subset is called *monogeneric* if it has a smallest element in the order of specialization, so that any neighborhood of it covers the whole subset. For an open set, this is the same as being compact and a co-prime in the lattice of open sets.)

Hint. Recall II-1.15. □

Exercise II-3.19. Let X be a subspace of Y, L a continuous lattice, and $f: X \to \Sigma L$ an arbitrary function. Define $f^*: X \to Y$ by

$$f^*(y) := \sup\{\inf f(U \cap X): U \text{ is open}; y \in U\}.$$

Show that f^* is continuous, that it is the largest continuous function such that $f^*(x) \leq f(x)$ for all $x \in X$, and that $f(x) = f^*(x)$ at every point of continuity of f. State and derive an analogs result for bounded complete domains.

Hint. Adapt the proof of II-3.9. □

Exercise II-3.20. Let X be a compact space and L a **dcpo**. If $V \neq \emptyset$ is a Scott open subset of L, show that $\{f \in TOP(X, \Sigma L): f(X) \subseteq V\}$ is a Scott open subset of $TOP(X, \Sigma L)$, and its Scott topology agrees with the relative Scott topology from $TOP(X, \Sigma L)$.

Hint. Suppose that $f(X) \subseteq V$, and let f be the directed supremum of a family D in $TOP(X, \Sigma L)$. For each $x \in X$, pick $g_x \in D$ such that $g_x(x) \in V$, and then pick U_x, a Scott open set containing x, such that $g_x(U_x) \subseteq V$. Finitely many of the U_x cover X, and if g is chosen in D larger than the corresponding g_x, then $h(X) \subseteq V$ for $h \in D$, $g \leq h$. Since $\{f \in TOP(X, \Sigma L): f(X) \subseteq V\}$ is also an upper set in the pointwise order, it follows that it is Scott open. Hence by Exercise II-1.26(i) its Scott topology agrees with the relative Scott topology. □

Exercise II-3.21. Show that a topological space X is a monotone convergence space iff it is a **dcpo** with respect to the order of specialization and its topology is order consistent with respect to that order (see Exercise II-1.31).

Old notes

The idea of characterizing continuous lattices exclusively in terms of T_0 spaces as injectives in the category of T_0 spaces and continuous functions was one of the basic results in [Scott, 1972a]; thus, Theorem II-3.8 was a core result of that treatise.

The specialization order of II-3.6 has been traditionally used in the spectral theory of rings. The idea of monotone convergence spaces (II-3.12) was used by O. Wyler in a seminar report [SCS 35]; he called them d-spaces.

New notes

Proposition II-3.11 was proved in [Eršov, 1973]. Densely injective spaces have been studied in more detail in [Escardó, 1998b]. The investigation of monotone convergence spaces has been taken up in [Eršov, 1999c] recently.

II-4 Function Spaces

Throughout this section we assume that all topological spaces under consideration are T_0 spaces. Recall that *TOP* denotes the category of all T_0 spaces and all continuous functions.

In Section 2 we introduced the poset $[S \to T]$ for two **dcpo**s as the set of Scott-continuous functions from S to T equipped with the pointwise order induced from T. By II-2.1 we have $[S \to T] = \mathit{TOP}(\Sigma S, \Sigma T)$, so this suggests that there is a topological description of the poset.

The Isbell topology

A common function space topology is the *compact–open topology*. We need a modification of this topology for treating general spaces X and Y. We normally define subbasic open subsets for the compact–open topology on the set $\mathit{TOP}(X, Y)$ of all continuous functions from X into Y to be sets of the form

$$N(K \to V) := \{f \in \mathit{TOP}(X, Y) : f(K) \subseteq V\},$$

where K is compact in X and V is open in Y. Note that $f(K) \subseteq V$ iff $K \subseteq f^{-1}(V)$, and the latter is true iff the saturation of K, the intersection of all open sets containing K, is contained in $f^{-1}(V)$. Hence one obtains exactly the same collection if one restricts to compact saturated sets. For any compact saturated set K, the collection of open sets containing K is a Scott open filter F_K by Lemma II-1.18, and we observe that $f \in N(K \to V)$ iff $f^{-1}(V) \in F_K$. Thus if we define

$$N(F_K \leftarrow V) := \{f \in TOP(X, Y) : f^{-1}(V) \in F_K\},$$

then $N(F_K \leftarrow V) = N(K \rightarrow V)$. If we assume additionally that X is sober, then by the Hofmann–Mislove Theorem (II-1.20), for each open subset V of Y, the assignments $N(K \rightarrow V) \mapsto N(F_K \leftarrow V)$ and $N(F \leftarrow V) \rightarrow N(\bigcap F \rightarrow V)$ implement mutually inverse bijections between the sets $\{N(K \rightarrow V)$: K a compact saturated subset of $X\}$ and $\{N(F \leftarrow V)$: F a Scott open filter in $\mathcal{O}(X)\}$. This provides motivation for the following definition.

Definition II-4.1. For two spaces X and Y, let H be a Scott open subset of the complete lattice $\mathcal{O}(X)$, let V be an open subset of Y, and set

$$N(H \leftarrow V) = \{f \in TOP(X, Y) : f^{-1}(V) \in H\}.$$

As H ranges over $\sigma(\mathcal{O}(X))$ and V ranges over $\mathcal{O}(Y)$, the sets $N(H \leftarrow V)$ form a subbasis for a topology on $TOP(X, Y)$, called the *Isbell topology*. Let $[X, Y]$ denote the set $TOP(X, Y)$ endowed with the Isbell topology. Let, in addition, $[f, h](g) = hgf$, thus defining a functor:

$$[\cdot, \cdot]\colon TOP^{\mathrm{op}} \times TOP \rightarrow TOP. \qquad \square$$

We verify in Lemma II-4.2 that $[f, h]$ is indeed a continuous function.

We recall that the order of specialization on any space can be characterized by $x \leq y$ iff for all open sets U, $x \in U$ implies $y \in U$ (see remarks preceding O-5.3). It is straightforward that it is sufficient to check this property for all open sets U belonging to a subbasis for the topology.

Lemma II-4.2. *Let X and Y be spaces.*

(i) *The Isbell topology on $[X, Y]$ is finer than the compact–open topology which in turn is finer than the topology of pointwise convergence (which is the point open topology, or equivalently the relative product topology from Y^X restricted to $TOP(X, Y)$). If X is sober and $\mathcal{O}(X)$ is a continuous lattice, then the Isbell topology and the compact–open topology agree.*

(ii) *Let $\Omega[X, Y]$ denote $[X, Y]$ with its order of specialization. Then $f \leq g$ in $\Omega[X, Y]$ iff $f(x) \leq g(x)$ in ΩY for all $x \in X$ iff $f^{-1}(V) \subseteq g^{-1}(V)$ for all $V \in \mathcal{O}(Y)$. Thus $[S \rightarrow T] = \Omega[\Sigma S, \Sigma T]$ for* **dcpo***s.*

(iii) *If $f\colon X_1 \rightarrow X$ and $h\colon Y \rightarrow Y_1$ are continuous, then $[f, h]\colon [X, Y] \rightarrow [X_1, Y_1]$ is continuous.*

Proof: (i) The comments preceding Definition II-4.1 imply that every subbasic open set in the compact–open topology is a subbasic open set in the Isbell topology. Since points are compact, the topology of pointwise convergence with subbasic open sets $N(\{p\} \to V)$ is contained in the compact–open topology. Suppose that X is sober and $\mathcal{O}(X)$ is a continuous lattice. If $f \in N(H \leftarrow V)$, then $f^{-1}(V) \in H$, and since $\mathcal{O}(X)$ is continuous, by Theorem II-1.14(3) there exists an open filter F such that $f^{-1}(V) \in F \subseteq H$. Again by the remarks preceding Definition II-4.1 there exists a compact saturated set K such that $N(K \to V) = N(F \leftarrow V)$ and thus $V \in N(K \to V) = N(F \leftarrow V) \subseteq N(H \leftarrow V)$. Thus the Isbell topology is also contained in the compact–open topology in this case, and hence the two are equal.

(ii) Let $f \leq g$ in $\Omega[X, Y]$. Then for $x \in X$ and V open in Y, $f(x) \in V$ iff $f \in N(\mathcal{N}(x) \leftarrow V)$ implies $g \in N(\mathcal{N}(x) \leftarrow V)$ iff $g(x) \in V$, where $\mathcal{N}(x)$ is the Scott open filter of open neighborhoods of x. Hence $f(x) \leq g(x)$.

If $f(x) \leq g(x)$ in ΩY for all x, then for V open in Y, $x \in f^{-1}(V)$ iff $f(x) \in V$ implies $g(x) \in V$ iff $x \in g^{-1}(V)$. Thus $f^{-1}(V) \leq g^{-1}(V)$.

Suppose that $f^{-1}(V) \subseteq g^{-1}(V)$ for all $V \in \mathcal{O}(Y)$. If $f \in N(H \leftarrow V)$, an Isbell open set, then $f^{-1}(V) \in H$, and thus $g^{-1}(V) \in H$, since H is closed under supersets. Thus $g \in N(H \leftarrow V)$. Since the $N(H \leftarrow V)$ form a subbasis, $f \leq g$.

(iii) For the continuity of $[f, h]$ it is sufficient to take any Scott open subset H_1 of $\mathcal{O}(X_1)$ and any open subset V_1 of Y_1 and to show that $[f, h]^{-1}(N(H_1 \leftarrow V_1)) = N(H \leftarrow V)$ for some Scott open subset H of $\mathcal{O}(X)$ and some open subset V of Y: We take $V = h^{-1}(V_1)$ which is open as g is continuous, and $H = (\mathcal{O}f)^{-1}(H_1) = \{U \in \mathcal{O}(X) : f^{-1}(U) \in H_1\}$. As $\mathcal{O}f$ preserves arbitrary unions, it is Scott-continuous. Thus the preimage H of the Scott open set H_1 is Scott open too. Finally $g \in [f, h]^{-1}(N(H_1 \leftarrow V_1))$ iff $(hgf)^{-1}(V_1) \in H_1$ iff $f^{-1}(g^{-1}(h^{-1}(V_1))) = f^{-1}(g^{-1}(V)) \in H_1$ iff $g^{-1}(V) \in H$ iff $g \in N(H \leftarrow V)$. $\square$

Remark. In later parts of this section (see II-4.10) and notably in Chapter V we undertake a detailed study of spaces X for which $\mathcal{O}(X)$ is continuous. By I-1.7(5) we know that this class contains all locally compact spaces, and, among regular T_0 spaces, by I-1.34 only the locally compact Hausdorff spaces are in this class. If L is a domain, then ΣL is in this class because $\mathcal{O}(\Sigma L) = \sigma(L)$ is continuous by II-1.13. We shall see in Chapter V that a sober space is locally compact iff $\mathcal{O}(X)$ is a continuous lattice.

Note that the topological space $[X, Y]$ and the poset $\Omega[X, Y]$ have the same underlying set $TOP(X, Y)$.

We frequently regard $[X, \cdot]$ as a functor, on TOP and thus given $h\colon Y \to Y_1$, write $[X, h]$ for $[1_X, h]\colon [X, Y] \to [X, Y_1]$. In a similar fashion we have a functor $[\cdot, Y]$ on TOP^{op}.

Lemma II-4.3. (i) *If Y is a monotone convergence space, then $[X, Y]$ is closed in $(\Omega Y)^X$ under the formation of directed sups and is a monotone convergence space.*

(ii) *If $f: Y \to Z$ is a continuous function, resp. an embedding, then $[X, f]$: $[X, Y] \to [X, Z]$ is continuous, resp. an embedding. If Y is a monotone convergence space, then $[X, f]$ is also Scott-continuous.*

(iii) *The mapping $f \mapsto f^{-1}(1) : [X, \Sigma 2] \to \Sigma\mathcal{O}(X)$, where the $\Sigma 2$ is Sierpinski space, is a homeomorphism. In particular, the Isbell topology on $[X, \Sigma 2]$ agrees with the Scott topology.*

(iv) *If X is a space and Y is a retract of a monotone convergence space Z, then $[X, Y]$ is a Scott-continuous retract of $[X, Z]$. In particular, if $\Omega[X, Z]$ is a domain, or a continuous lattice, then so is $\Omega[X, Y]$.*

Proof: The first assertion of (i) is a reiteration of II-3.15(i). Suppose that f is the pointwise supremum of a directed set $D \subseteq [X, Y]$, and that $f \in N(H \leftarrow V)$, a subbasic open set in the Isbell topology. Then from the fact that Y is a monotone convergence space and II-4.2(ii), one deduces readily that $f^{-1}(V) = \bigcup_{g\in D} g^{-1}(V)$. Since this union is directed and $f^{-1}(V) \in H$, a Scott open set, it follows that $g^{-1}(V) \in H$ for some $g \in D$, i.e., $g \in N(H \leftarrow V)$. It follows that D converges to f in $[X, Y]$.

As to (ii), the continuity of $[X, f]$ is a special case of Lemma II-4.2(iii). The embedding assertion follows from the observation that for any U open in Z, $N(H \leftarrow f^{-1}(U))$ in $TOP(X, Y)$ is equal to the inverse image under $[X, f]$ of $N(H \leftarrow U)$ in $TOP(X, Z)$. The final assertion follows from part (i) and Lemma II-3.13.

For (iii), $f \mapsto f^{-1}(1)$ is a bijection as in Lemma II-2.11, and then one verifies directly that $N(H \leftarrow \{1\})$ in $[X, \Sigma 2]$ corresponds to the Scott open set H in $\mathcal{O}(X)$. Thus the correspondence is continuous both ways, hence a homeomorphism, and thus an order isomorphism, since the orders are the orders of specialization. Since $\mathcal{O}(X)$ has the Scott topology, so does $[X, \Sigma 2]$.

For (iv), note that $[X, Z]$ is a continuous retract of the monotone convergence space $[X, Y]$ (since functors preserve retracts), and hence a Scott-continuous retract by Lemma II-3.13. The last assertion follows from I-2.3. □

Spaces with a continuous topology

We now investigate the situation that $\Omega[X, Y]$ is a domain.

Proposition II-4.4. *Let X be a space and Y a monotone convergence space, and suppose that $\Omega[X, Y]$ is a domain. Then*

(i) *ΩY is a domain,*

(ii) *if the order of specialization on Y is nontrivial, that is, if there are elements $y < y^*$, then $\mathcal{O}(X)$ is a continuous lattice.*

Proof: (i) We fix a point $b \in X$ and consider the retraction $f: X \to X$ given by $f(x) = b$ for all x. Then $[f, Y]: [X, Y] \to [X, Y]$ is a continuous retraction onto the set of constant maps, and hence Scott-continuous by Lemma II-4.3(ii). Thus the image of $[f, Y]$ is a domain by I-2.3, which is order isomorphic to ΩY.

(ii) Let U be any open neighborhood of y^* which does not contain y. Its characteristic function is a retraction $Y \to \Sigma 2$ with right inverse $i: \Sigma 2 \to Y$, where $i(1) = y^*$ and $i(0) = y$. (Note for continuity of i that for any open set $V \subseteq Y$ with $0 \in i^{-1}(V)$ we have $y \in V$; hence $y^* \in V$, and thus $1 \in i^{-1}(V)$.) Thus $[X, \Sigma 2]$ is a Scott-continuous retract of $[X, Y]$, hence a domain by Lemma II-4.3(iv), and thus a continuous lattice since it is a complete lattice. But $\mathcal{O}(X)$ is isomorphic to $[X, \Sigma 2]$. □

Let us observe that the continuity of ΩY *does not* allow us to conclude that Y has the Scott topology; the given topology of Y may be coarser than the Scott topology of ΩY. Since Y is a *monotone convergence space*, then its topology is coarser than or equal to the Scott topology.

Given sets X, Y, Z, we consider the exponential or currying function E: $Z^{X \times Y} \to (Z^Y)^X$ by $Ef(x)(y) = f(x, y)$ for $f: X \times Y \to Z$ as in II-2.10. Then E is a bijection with inverse $(E^{-1}g)(x, y) = g(x)(y)$. For topological spaces X, Y, Z, we consider the restriction of E to *TOP*$(X \times Y, Z)$.

Proposition II-4.5. *Let X, Y, Z be spaces.*

(i) *If $f: X \times Y \to Z$ is continuous, then $Ef: X \to [Y, Z]$ is continuous.*
(ii) *If the evaluation map* eval: $[Y, Z] \times Y \to Z$ *given by* eval$(f, y) = f(y)$ *is continuous, then $E^{-1}g: X \times Y \to Z$ is continuous for every continuous function $g: X \to [Y, Z]$ and E: TOP$(X \times Y, Z) \to$ TOP$(X, [Y, Z])$ is a bijection.*
(iii) *If $\mathcal{O}(Y)$ is a continuous lattice, then the evaluation map* eval: $[Y, Z] \times Y \to Z$ *is continuous.*
(iv) *If $\mathcal{O}(Y)$ is a continuous lattice, then $E^{-1}g: X \times Y \to Z$ is continuous for every continuous function $g: X \to [Y, Z]$ and E: TOP$(X \times Y, Z) \to$ TOP$(X, [Y, Z])$ is a bijection.*

Proof: (i) Let $f: X \times Y \to Z$ be continuous. For $x \in X$, the mapping $Ef(x)$: $Y \to Z$ is continuous since $Ef(x)$ is the composition $y \mapsto (x, y) \mapsto f(x, y)$. Let $N(H \leftarrow V)$ be a subbasic open set in $[Y, Z]$ containing $g := Ef(x)$. Then $g^{-1}(V) \in H$. For each $y \in g^{-1}(V)$, we have $f(x, y) = Ef(x)(y) = g(y) \in V$. Hence there exist open sets U_y containing x and W_y containing y such that $f(U_y \times W_y) \subseteq V$. Since H is Scott open and $\bigcup_y W_y \supseteq g^{-1}(V) \in H$, there exist finitely many of the W_y whose union W is a member of H. Let U be the intersection of the corresponding U_y. Then $f(U \times W) \subseteq V$. Thus for each $u \in U$, $W \subseteq (Ef(u))^{-1}(V)$, and hence $(Ef(u))^{-1}(V) \in H$. Therefore $Ef(U) \subseteq N(H \leftarrow V)$ and thus Ef is continuous at x. Since x was arbitrary, Ef is continuous.

(ii) Let $\hat{f}: X \to [Y, Z]$ be continuous. Set $f := E^{-1}\hat{f}: X \times Y \to Z$. Then f can be written as the composition

$$(x, y) \mapsto (\hat{f}(x), y) \mapsto f(x, y) = \mathrm{eval}(\hat{f}(x), y): X \times Y \to [Y, Z] \times Y \to Z,$$

where both maps are continuous by hypothesis. Hence f is continuous.

We know that at the set level E and E^{-1} are inverse functions. It then follows from (i) and the previous paragraph that when restricted to $TOP(X \times Y, Z)$ and $TOP(X, [Y, Z])$, respectively, they are mutually inverse functions.

(iii) Let $\mathcal{O}(Y)$ be a continuous lattice, let $f \in [Y, Z]$, $y \in Y$, and let V be an open set in Z containing $f(y)$. Then there exists an open set U in Y such that $y \in U \ll f^{-1}(V)$. Set $H := \Uparrow U = \{B \in \mathcal{O}(Y) : U \ll B\}$. By Proposition II-1.6, H is Scott open, and hence $N(H \leftarrow V)$ is an open neighborhood of f. If $g \in N(H \leftarrow V)$, then $U \ll g^{-1}(V)$; thus for $u \in U$, $\mathrm{eval}(g, u) = g(u) \in V$. We conclude that the evaluation mapping is continuous.

Part (iv) follows from (ii) and (iii). □

In topological parlance part (i) of Proposition II-4.5 establishes that the Isbell function space topology is always splitting, while parts (ii) and (iii) establish that it is admissible (or conjoining) in the case that $\mathcal{O}(Y)$ is a continuous lattice.

Proposition II-4.6. *If Y is a space such that $\mathcal{O}(Y)$ is a continuous lattice and Z is an injective space, resp. a densely injective space, then $[Y, Z]$ is injective, resp. densely injective. In particular, $\Omega[Y, Z]$ is a continuous lattice, resp. a bounded complete domain, and $[Y, Z] = \Sigma\Omega[Y, Z]$, i.e., the Isbell topology is the Scott topology.*

Proof: Suppose that Z is a densely injective space. Let X be a dense subset of X_1 and suppose that $\hat{h}: X \to [Y, Z]$ is continuous. By Proposition II-4.5(iv), $h = E^{-1}\hat{h}: X \times Y \to Z$ is continuous. Since $X \times Y$ is dense in $X_1 \times Y$, there exists a continuous extension H of h, $H: X_1 \times Y \to Z$. Then $EH = \hat{H}$: $X_1 \to [Y, Z]$ is continuous by II-4.5(i) and is easily verified to be an extension of $\hat{h}$. Thus we have established that $[Y, Z]$ is also a densely injective space. By Proposition II-3.11 it must be a bounded complete domain equipped with the Scott topology.

A similar argument establishes the case that Z is an injective space, or one can use the previous paragraph and the fact that a bounded complete domain that is a complete lattice is a continuous lattice. □

If S is a **dcpo** such that $\sigma(S)$ is a continuous lattice (e.g., if S is a domain as in II-1.14), and if T is a continuous lattice, we deduce that $[S \to T] = \Omega[\Sigma S, \Sigma T]$ is a continuous lattice. This re-proves II-2.12.

Remark. In light of Theorems II-1.14(3), II-2.10, and II-3.8 and Propositions II-3.11 and II-4.6, we have that the topological category of injective spaces, or that of densely injective spaces, and continuous maps is cartesian closed, where the function space object is $[X, Y] = \Sigma[\Omega X \to \Omega Y]$. Indeed this assertion is a topological reformulation of Theorem II-2.10, as it pertains to bounded complete domains and continuous lattices.

From our main propositions we can now extract the following theorem.

Theorem II-4.7. *Let X be a space and L a complete* nonsingleton *lattice. Then the following statements are equivalent:*

(1) $\Omega[X, \Sigma L]$ *is a continuous lattice;*
(2) *both $\mathcal{O}(X)$ and L are continuous lattices.*

Proof: We set $Y = \Sigma L$; then Y is a monotone convergence space. If condition (1) holds, then the hypotheses of II-4.4 are satisfied. Hence, $\mathcal{O}(X)$ and $\Omega Y = \Omega\Sigma L = L$ are continuous lattices; thus (2) follows. If (2) holds, then, by II-3.8 and II-4.6, condition (1) follows. □

The function spaces of the form $[X, \Sigma L]$ deserve more attention.

Lemma II-4.8. *Let X and Y be topological spaces and $f\colon X \to \Sigma\mathcal{O}(Y)$ be a function. If the set $G_f = \{(x, y) \in X \times Y : y \in f(x)\}$ is open in $X \times Y$, then f is continuous.*

Proof: Let $U \in \sigma(\mathcal{O}(Y))$ and suppose that $f(x) \in U$. Since G_f is open, for each $y \in f(x)$ there are an open neighborhood $A(y)$ of x in X and an open neighborhood $B(y)$ of y in Y such that $A(y) \times B(y) \subseteq G_f$. If J is the directed set of finite subsets of $f(x)$ and $F \in J$, we set $B(F) = \bigcup_{y\in F} B(y)$. Then $(B(F))_{F\in J}$ is a directed net in $\mathcal{O}(Y)$ with $\sup_F B(F) = f(x) \in U$. By II-1.2(ii) we then find a finite set $F \subseteq f(x)$ such that $B(F) \in U$. Set $A(F) = \bigcap_{y\in F} A(y)$. Then $A(F)$ is an open neighborhood of x in X, and for each $a \in A(F)$ we have

$$\{a\} \times B(F) \subseteq A(F) \times B(F) \subseteq G_f;$$

hence, $B(F) \subseteq f(a)$. But also $B(F) \in U$, hence $f(a) \in U$. □

Proposition II-4.9. *For topological spaces X and Y, there is a natural monotone function $\theta\colon \mathcal{O}(X \times Y) \to \Omega[X, \Sigma\mathcal{O}(Y)]$ given by*

$$\theta(W)(x) = \{y \in Y : (x, y) \in W\}.$$

Proof: We have to show that $\theta(W)\colon X \to \Sigma\mathcal{O}(Y)$ is continuous. But

$$G_{\theta(W)} = \{(x, y) \in X \times Y : y \in \theta(W)(x)\} = W \in \mathcal{O}(X \times Y).$$

Hence, $G_{\theta(W)}$ is open, and Lemma II-4.8 shows that $\theta(W)$ is continuous. It is evident that θ is monotone. □

In general, there is no reason to believe that the map θ is an isomorphism. We will now give necessary and sucient conditions for this to be the case.

Theorem II-4.10. *Let Y be a T_0 space. Then the following statements are equivalent:*

(1) *for all spaces X and all continuous lattices L, the pair E and E^{-1} of mutually inverse bijections induce by restriction bijections*

$$\mathit{TOP}(X, \Sigma\Omega[Y, \Sigma L]) \rightleftarrows \mathit{TOP}(X \times Y, \Sigma L);$$

(1′) *for all spaces X and all continuous lattices L, the pair E, E^{-1} of mutually inverse bijections $Z^{X\times Y} \rightleftarrows (Z^Y)^X$ induce by restriction order isomorphisms*

$$\Omega[X, \Sigma\Omega[Y, \Sigma L]] \rightleftarrows [X \times Y, \Sigma L];$$

(2) *for all spaces X, the function $\theta\colon \mathcal{O}(X \times Y) \to \Omega[X, \Sigma\mathcal{O}(Y)]$ of* II-4.9 *is an isomorphism;*
(3) *for all continuous $f\colon X \to \Sigma\mathcal{O}(Y)$ the set G_f of* II-4.8 *is open in $X \times Y$;*
(4) *the set $\{(U, y) \in \mathcal{O}(Y) \times Y : y \in U\}$ is open in $\Sigma\mathcal{O}(Y) \times Y$;*
(5) *for each $y \in U \in O(Y)$ there is a Scott open neighborhood $H \in \sigma(\mathcal{O}(Y))$ containing U such that $\bigcap H$ is a neighborhood of y in Y;*
(6) *$\mathcal{O}(Y)$ is a continuous lattice;*
(7) *for all spaces Z the evaluation mapping $(f, y) \mapsto f(y) : [Y, Z] \times Y \to Z$ is continuous;*
(8) *for all spaces X, Z the mapping $E\colon \mathit{TOP}(X \times Y, Z) \to \mathit{TOP}(X, [Y, Z])$ is a bijection.*

Proof: (1) iff (1′): Clearly (1′) implies (1). Conversely if (1) holds, then one can use the fact that the order on the function spaces arises from the pointwise order relative to ΩZ (Lemma II-4.2(ii)) to check that E is an order isomorphism.

(1) implies (2): For a space Z we denote by α_Z: $[Z, \Sigma 2] \to \mathcal{O}(Z)$ the isomorphism given by $\alpha_Z(f) = f^{-1}(1)$. A straightforward calculation shows the following diagram to be commutative:

$$\begin{tikzcd}
{[X \times Y, \Sigma 2]} \arrow[r, "E"] \arrow[d, "\alpha_{X\times Y}"'] & \Omega[X, [Y, \Sigma 2]] \arrow[d, "{[X, \alpha_Y]}"] \\
\mathcal{O}(X \times Y) \arrow[r, "\theta"'] & \Omega[X, \Sigma\mathcal{O}(Y)]
\end{tikzcd}$$

Condition (2) is therefore equivalent to the following:

(2′) for all spaces X

$$TOP(X \times Y, \Sigma 2) \xrightarrow{E} TOP(X, \Sigma[Y, \Sigma 2])$$

is a bijection with inverse E^{-1}.

Thus (1) clearly implies (2′).

(2) implies (3): Condition (2) says that all continuous $f\colon X \to \Sigma\mathcal{O}(Y)$ are of the form $\theta(W)$ for some open set $W \subseteq X \times Y$; that is, they are given by the equation $f(x) = \{y \in Y : (x, y) \in W\}$ for some open set W of $X \times Y$. This says precisely that $G_f = W$ for some open set W of $X \times Y$.

(3) implies (4): Take $X = \Sigma\mathcal{O}(Y)$ and $f = 1_{\sigma\mathcal{O}(Y)}$, then we find that $G_f = \{(U, y)\colon y \in U\}$.

(4) implies (5): Let $y \in U \in \mathcal{O}(Y)$; then by (4) there are an open neighborhood H of U in $\Sigma\mathcal{O}(Y)$ and an open neighborhood V of y in Y such that $(W, v) \in H \times V$ implies $v \in W$. Thus $V \subseteq \bigcap H$.

(5) implies (6): We have to show that for each $U \in \mathcal{O}(Y)$ and each $y \in Y$ there is a $V \in \mathcal{O}(Y)$ with $y \in V \ll U$ in order to satisfy I-1.6 for $L = \mathcal{O}(Y)$. By (5) there are an open neighborhood H of U in $\Sigma\mathcal{O}(Y)$ and an open neighborhood V of y such that $V \subseteq \bigcap H$. Then $V \ll U$ by Proposition II-1.6.

(6) implies (7): This is Proposition II-4.5(iii).

(7) implies (8): This is Proposition II-4.5(ii).

(6) implies (1): We have already that (6) implies (8), and it follows from Proposition II-4.6 (giving equality of the Scott and Isbell topologies) and condition (8) that (1) is satisfied.

(8) implies (2): In light of Lemma II-4.3(iii) and Proposition II-4.6, condition (2′) (given in the proof of (1) implies (2)) is a special case of condition (8). But it was already shown that (2′) is equivalent to (2).

The next to last implication establishes the equivalence of (1)–(6), so (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (2) completes the proof. $\square$

Definition II-4.11. An object Y in a category A with finite products is called *exponentiable*, if the functor $- \times Y: A \to A$ has a right adjoint $-^Y$, that is, if for all objects X and Z there is a natural bijection $E: A(X \times Y, Z) \to A(X, Z^Y)$. $\square$

The equivalence of (6) and (8) in the preceding theorem shows that in the category *TOP* of T_0-spaces all spaces Y are exponentiable for which the lattice $\mathcal{O}(Y)$ of open subsets is continuous; for an arbitrary space Z, the exponential Z^Y is the space $[Y, Z]$ of all continuous functions from Y to Z with the Isbell topology. We shall show that these are all exponentiable objects:

Theorem II-4.12. *For a space Y the following properties are equivalent.*

(1) *The lattice $\mathcal{O}(Y)$ of open subsets of Y is continuous.*
(2) *The space Y is exponentiable in the category TOP of T_0-spaces.*
(3) *The functor $- \times Y$ preserves quotient maps.*

Proof: We have seen from II-4.10 that (1) implies (2). In the category *TOP* quotient maps are precisely coequalizers (see [Herrlich and Strecker, B1973], 16.3). As coequalizers are particular colimits (*op. cit.*, 20.3) and as left adjoints preserve colimits, (2) implies (3).

(3) implies (1): We show that (3) implies that the set $G = \{(U, y) \in \mathcal{O}(Y) \times Y : y \in U\}$ is open in $\Sigma\mathcal{O}(Y) \times Y$, which is condition (4) in II-4.10.

For this we choose an arbitrary ideal I of the lattice $\mathcal{O}(Y)$ and define a topology τ_I on $\mathcal{O}(Y)$ as in the proof of II-1.34 by defining $H \subseteq \mathcal{O}(Y)$ to be τ_I open, if H is an upper set (that is, $U \in H, U \subseteq V$ together imply $V \in H$) and if $\bigcup\{W: W \in I\} \in H$ implies that $V \in H$ for some $V \in I$. We denote by $\mathcal{O}(Y)_I$ the set $\mathcal{O}(Y)$ endowed with the topology τ_I. For these topologies we have the following properties.

(a) The Scott topology on the lattice $\mathcal{O}(Y)$ clearly is the intersection of the topologies τ_I, where I ranges over all ideals of $\mathcal{O}(Y)$.

(b) G is open in $\mathcal{O}(Y)_I \times Y$. Indeed, let $(U, y) \in G$, that is $y \in U \in \mathcal{O}(Y)$. First case: $U \not\subseteq \bigcup\{W: W \in I\}$. Then the set H of all $V \in \mathcal{O}(Y)$ with $U \subseteq V$ is τ_I open and $z \in V$ for all $(V, z) \in H \times U$, that is, $H \times U$ is an open neighborhood of (U, y) in $\mathcal{O}(Y)_I \times Y$ which is contained in G. Second case: $U \subseteq \bigcup\{W: W \in I\}$. Then there is $W \in I$ with $y \in W$. After replacing W by $U \cap W$ we may suppose that $W \subseteq U$. Then the set H of all $V \in \mathcal{O}(Y)$ with $W \subseteq V$ is τ_I open and $z \in V$ for all $(V, z) \in H \times W$, that is, $H \times W$ is an open neighborhood of (U, y) in $\mathcal{O}(Y)_I \times Y$ which is contained in G.

In order to prove our claim, we consider the disjoint union $\sum_I \mathcal{O}(Y)_I$ of the spaces $\mathcal{O}(Y)_I$ with the obvious topology, where I ranges over all ideals of the lattice $\mathcal{O}(Y)$. The identity maps $\mathcal{O}(Y)_I \to \Sigma\mathcal{O}(Y)$ yield a function q: $\sum_I \mathcal{O}(Y)_I \to \Sigma\mathcal{O}(Y)$ which is a quotient map by property (a) above. By our hypothesis (3), then

$$q \times 1\colon \sum_I \mathcal{O}(Y)_I \times Y \to \Sigma\mathcal{O}(Y) \times Y$$

is a quotient map, too. Note that $\sum_I \mathcal{O}(Y)_I \times Y \simeq \sum_I (\mathcal{O}(Y)_I \times Y)$. As G is open in each $\mathcal{O}(Y)_I \times Y$ by property (b) above, we conclude that $(q \times 1)(\sum_I G) = G$ is open in $\Sigma\mathcal{O}(Y) \times Y$. $\square$

On dcpos with a continuous Scott topology

We apply this information in the proof of the following result which touches a subtle point: If S and T are **dcpo**s, the Scott topology on $S \times T$ need not be the product of the Scott topologies $\sigma(S)$ and $\sigma(T)$, in general. One always has $\sigma(S) \times \sigma(T) \subseteq \sigma(S \times T)$, but the containment may be proper. For an example see Exercise II-4.26. As a consequence, a Scott-continuous function defined on $S \times T$ need not be continuous for the product topology $\sigma(S) \times \sigma(T)$. This fact is easily overlooked and has been a source of confusion in the past.

Theorem II-4.13. *Let L be a* **dcpo**. *Then the following statements are equivalent.*

(1) $\sigma(L)$ *is a continuous lattice.*
(2) *For every* **dcpo** *or complete lattice S one has $\sigma(S \times L) = \mathcal{O}(\Sigma S \times \Sigma L)$.*
(3) *For every* **dcpo** *or complete lattice S one has $\Sigma(S \times L) = \Sigma S \times \Sigma L$.*

Proof: It is obvious that (2) and (3) are equivalent.

(1) implies (2): We have $\sigma(S \times L) \cong [S \times L \to 2]$ (sending a Scott open set of $S \times L$ to its characteristic function) $\cong [S \to [L \to 2]]$ (by II-2.10) $\cong$ $[S \to \sigma(L)] = \Omega[\Sigma S, \Sigma\mathcal{O}(\Sigma L)]$ (since $\sigma(L) = \mathcal{O}(\Sigma L)$, and in view of II-2.1 and II-4.2(ii)); and the order isomorphism $\theta\colon \sigma(S \times L) \to \Omega[\Sigma S, \Sigma\mathcal{O}(\Sigma L)]$ is given by $\theta(W)(s) = \{y \in L : (s, y) \in W\}$. But under the hypothesis (1), by Theorem II-4.10 we have $\mathcal{O}(\Sigma S \times \Sigma L) \cong \Omega[\Sigma S, \Sigma\mathcal{O}(\Sigma L)]$ under the same map. Hence $\mathcal{O}(\Sigma S \times \Sigma L) = \sigma(S \times L)$.

(2) implies (1): We apply condition (2) with $S = \sigma(L)$. Then the topology of $\Sigma\sigma(L) \times \Sigma L$ is the Scott topology of $\sigma(L) \times L$. Now we verify condition II-4.10(4) with $Y = \Sigma L$ which will prove the continuity of $\mathcal{O}(Y) = \sigma(L)$. We

must show that the set W of all $(U, y) \in \sigma(L) \times L$ with $y \in U$ is open in $\Sigma\sigma(L) \times \Sigma L$; but by the preceding this is tantamount to showing that this set is Scott open in $\sigma(L) \times L$. This is not hard to see: If the net (U_j, y_j) is directed with $U = \bigcup_j U_j$ and $y = \sup y_j$, and if $(U, y) \in W$, that is, $y \in U$, then $y_j \in U$ for some j, since U is Scott open, and thus $y_j \in U_k$, for some k. But then, if $m > j, k$ we obtain $y_m \in U_m$, and thus $(U_m, y_m) \in W$. $\square$

We reformulate this in a slightly weaker fashion:

Corollary II-4.14. *The functor* Σ: *DOM* $\to$ *TOP preserves finite products.*

Proof: This follows from II-4.13, since $\sigma(L)$ is continuous whenever L is a domain by II-1.14. $\square$

There are other useful corollaries:

Corollary II-4.15. *If L is a complete lattice such that $\sigma(L)$ is continuous, then the sup operation* $\vee: \Sigma L \times \Sigma L \to \Sigma L$ *is continuous.*

Proof: By II-4.13, $\Sigma L \times \Sigma L = \Sigma(L \times L)$. But we know $\vee: \Sigma(L \times L) \to \Sigma L$ is continuous, since $\vee: L \times L \to L$ preserves arbitrary sups. $\square$

Corollary II-4.16. *If L is a complete lattice such that $\sigma(L)$ is continuous, then ΣL is a sober space.*

Proof: II-1.12 and II-4.15. $\square$

In II-4.15 we found a sufficient condition for a complete lattice to be a topological sup semilattice with respect to the Scott topology. What do we know about the inf operation? By O-4.2(6) and II-2.1 we know that a complete lattice L is meet continuous (O-4.1) iff $\wedge: L \times L \to L$ is Scott-continuous; that is, $\wedge: \Sigma(L \times L) \to \Sigma L$ is continuous. Thus, if $\sigma(L)$ is a continuous lattice, then L is meet continuous iff ΣL is a topological inf semilattice iff (in view of II-4.15) ΣL is a topological lattice. The question remains whether the meet continuity of L can be recognized from properties of $\sigma(L)$. The answer is yes:

Proposition II-4.17. *Let L be a complete lattice. Then the following conditions are equivalent:*

(1) *L is meet continuous* (O-4.1)*;*
(2) *$\sigma(L)$ is join continuous* (O-4.1)*;*
(3) *$\sigma(L)^{\mathrm{op}}$ is a frame.*

Remark. $\sigma(L)$ is always a frame by O-3.22.

Proof of proposition: (1) implies (3): By O-4.1 and II-2.1, all translations $x \to a \wedge x : \Sigma L \to \Sigma L$ are continuous. We apply O-3.23 with $S = \Sigma L$ and $M = \sigma(L) = \mathcal{O}(\Sigma L)$ and conclude that (3) holds.

(2) iff (3): By O-4.3.

(2) implies (1): By II-1.4(i), $\sigma(L)^{\mathrm{op}}$ is isomorphic to the lattice T of all lower sets which are closed under directed sups. Let $S \subseteq T$ be the subset of all principal ideals $\downarrow x$ for $x \in L$. Then S is closed in T under directed sups; indeed, if t_j is a directed net in L with $\sup t_j = t$, then $\downarrow t$ is an upper bound in T of the net $\downarrow t_j$. And if $A \in T$ is an upper bound on all $\downarrow t_j$, then $t_j \in A$ and thus $t \in A$, since A is closed under directed sups; hence, $\downarrow t \subseteq A$.

Since $\bigcap_{x \in X} \downarrow x = \downarrow \inf X$, then S is closed in T under arbitrary infs. Thus, if T satisfies the relation (MC) of O-4.1, then so does S. But L is isomorphic to S under the map $x \mapsto \downarrow x$. □

Corollary II-4.18. *Let L be a complete lattice such that $\sigma(L)$ is a continuous lattice. Then the following statements are equivalent:*

(1) *L is meet continuous;*
(2) *ΣL is a topological lattice;*
(3) *$\sigma(L)$ is join continuous;*
(4) *$\sigma(L)$ is a continuous frame (= continuous distributive lattice) and $\sigma(L)^{\mathrm{op}}$ is a frame.* □

In Chapter VII we will identify those L satisfying the conditions of II-4.18 as precisely the underlying lattices of Haudorff compact topological semilattices with identity. In Section VI-4 we will give examples to show that these need not be continuous lattices.

Let us note in passing that II-4.17 allows us to express the following fact: if $\tau(L)$ denotes the lattice of all lower sets which are closed under directed sups, then for a complete lattice L, the following statements are equivalent.

(1) L is meet continuous.
(2) For some $n = 1, 2, 3, \ldots$, the lattice $\tau^n(L)$ is meet continuous.
(3) For all $n = 1, 2, 3, \ldots$, the lattice $\tau^n(L)$ is meet continuous.

Here, of course, we set $\tau^{n+1}(L) = \tau(\tau^n(L))$, inductively.

The results in II-4.17 and II-4.18 should be compared with II-1.14 above. These theorems express properties, such as the continuity, or meet continuity, of L, exclusively in terms of properties of the Scott topology $\sigma(L)$. We will pursue this further in Chapter VII. But dwelling just a bit longer on complete lattices L with continuous Scott topology $\sigma(L)$ and on function spaces $[X, \Sigma L]$, we note a companion to II-4.7.

Proposition II-4.19. *Let X be a topological space and L a complete lattice such that $\sigma(L)$ is a continuous lattice. Then*

(i) $\Omega[X, \Sigma L]$ *is a meet continuous lattice iff L is meet continuous,*
(ii) $\Omega[X, \Sigma L]$ *is a frame if and only if L is a frame.*

Proof: If L is meet continuous, then ΣL is a topological lattice by II-4.18 and thus $[X, \Sigma L]$ is closed in L^X under finite infs and sups. Since ΣL is a monotone convergence space, $[X, \Sigma L]$ is closed under directed sups in L^X by II-3.15. Hence $[X, \Sigma L]$ is closed in L^X under *arbitrary* sups (O-1.5), and thus it is a complete lattice. Since L^X as a product of meet continuous lattices is meet continuous and since $\Omega[X, \Sigma L]$ is closed in L^X under finite infs and arbitrary (hence, in particular, directed) sups, then $[X, \Sigma L]$ is meet continuous. If, in addition, L is distributive (see O-4.3), then so are L^X and the sublattice $[X, \Sigma L]$. Conversely, if $[X, \Sigma L]$ is meet continuous (and distributive), then so is the sublattice of all constant functions. This last is isomorphic to L. Since the order in $\Omega[X, \Sigma L]$ agrees with the pointwise order (Lemma II-4.2), the proposition follows. □

The entire function-space theory of this section has been developed without recourse to the relation $\ll$ and thus without referring directly to the original definition of a continuous lattice. In the following we elucidate the way-below relation on function spaces. In a special case we undertook an investigation of this kind in Example I-1.22.

Let X be a space such that $\mathcal{O}(X)$ is a continuous lattice and let L be a domain with smallest element 0. For $g \in [X, \Sigma L]$ and $U \subseteq X$, we define $g\chi_U$ by $g\chi_U(x) = g(x)$ if $x \in U$ and 0 otherwise. Note that $g\chi_U$ is again continuous. If $s \in L$, then we identify s with the constant function with value s in order to define $s\chi_U$ to be the characteristic function with value s on U and value 0 else. For $f \in [X, \Sigma L]$, we define a function $s\chi_U$ to be an *approximating characteristic function* if U is open and there exist V open and $t \in L$ with $U \ll V$ and $s \ll t \leq f(v)$ for all $v \in V$.

Proposition II-4.20. *Let X be a space such that $\mathcal{O}(X)$ is a continuous lattice, and let L be a domain with smallest element* 0.

(i) *If $f \ll g$ in $\Omega[X, \Sigma L]$, then f vanishes outside a set $U \in \mathcal{O}(X)$ with $U \ll X$.*
(ii) *If $s \ll t$ in L and $U \ll V$ in $\mathcal{O}(X)$, then $s\chi_U \ll t\chi_V$ in $\Omega[X, \Sigma L]$.*
(iii) *We have $s\chi_U \ll f$ for each approximating characteristic function of $f \in \Omega[X, \Sigma L]$ and f is the sup of the set of its approximating characteristic functions $s\chi_U$.*

(iv) *Let L be an L-domain, resp. bounded complete domain, resp. continuous lattice. Then, $\Omega[X, \Sigma L]$ is an L-domain, resp. bounded complete domain, resp. continuous lattice, where for $f, g \leq h$, the supremum $f \vee g(x) = f(x) \vee g(x)$ is calculated in $\downarrow h(x)$ for each x. Furthermore, $f \ll g$ in $\Omega[X, \Sigma L]$ if and only if there exist finitely many approximating characteristic functions $\{c_i \chi_{U_i} : 1 \leq i \leq n\}$ for g with $f \leq \sup\{c_i \chi_{U_i} : 1 \leq i \leq n\}$, where the finite suprema are calculated in $\downarrow g$.*

Proof: (i) Consider the directed family of functions $(g\chi_U)_{U \ll X}$, U open, where $g\chi_U(x) = g(x)$ if $x \in U$ and 0 otherwise, and note $g = \sup_{U \ll X} g\chi_U$. Thus $f \leq g\chi_U$ for some U, and hence vanishes outside U.

(ii) If $(f_j)_{j \in J}$ is a directed family with $t\chi_V \leq \sup f_j$ in $\Omega[X, \Sigma L]$, then $U_j = f_j^{-1}(\Uparrow s)$ is a directed family with $V \subseteq \bigcup U_j$ in $\mathcal{O}(X)$. Hence there exists $j \in J$ with $U \subseteq U_j$, and then $s\chi_U \leq f_j$.

(iii) Given any approximating characteristic function $s\chi_U$ for $f \in [X, \Sigma L]$, pick t and V such that $U \ll V$ and $s \ll t \leq f(v)$ for all $v \in V$. From (ii) we have $s\chi_U \ll t\chi_V \leq f$.

If $f \in [X, \Sigma L]$ and $y \in X$, let $s \ll f(y)$ be arbitrary. By the interpolation property I-1.9, find t with $s \ll t \ll f(y)$. The set $V = f^{-1}(\Uparrow t)$ is open, hence we find $U \in \mathcal{O}(X)$ with $y \in U \ll V$, using the continuity of $\mathcal{O}(X)$. Then $s\chi_U$ is an approximating characteristic function for f such that $s = s\chi_U(y)$. From the continuity of L we conclude that f is the supremum of its approximating characteristic functions.

(iv) Let L be an L-domain. For $f, g, h \in [X, \Sigma L]$ with $f, g \in \downarrow h$, define $f \vee_h g$ by $(f \vee_h g)(x) = f(x) \vee_{h(x)} g(x)$, where the second supremum is calculated in $\downarrow h(x)$. That these pairwise suprema are continuous follows from Corollary II-4.15 for the case that L is a continuous lattice, and from Exercise II-4.25 for the general case. The verification that $f \vee_h g$ is the supremum in $\downarrow h$ is straightforward; hence L is an L-domain if it is a domain. It follows from Exercise I-1.37(ii) that $f \vee_h g \ll h$ whenever $f, g \ll h$. In particular, finite suprema of approximating characteristic functions of h are in $\twoheaddownarrow h$ and form a directed family. It follows from part (iii) that L is a domain. The last assertion now follows easily, since h is a directed supremum of the finite suprema of approximating characteristic functions.

If L is a bounded complete domain or continuous lattice, then it is an L-domain, so the preceding derivations apply. In these cases $[X, \Sigma L]$ is also a complete semilattice, resp. a complete lattice that is also a domain, and hence a bounded complete domain, resp. a continuous lattice. $\square$

Remark. Roughly speaking, (i) means that f has compact support; in view of I-1.4(ii) this is indeed the case if X is locally compact. Later we see that under our present conditions X has to be locally compact as long as it is a sober space (Chapter V). Notice that in part (iv) above we have a characterization of the way-below relation in $\Omega[X, \Sigma L]$ entirely in terms of the way-below relation in $\mathcal{O}(X)$ and L. We refine this in the exercises. The preceding yields another proof of the continuity of $\Omega[X, \Sigma L]$ (see II-4.7) which uses directly the definition of the continuity of a lattice. See also II-2.12, II-2.31, II-2.32.

For a bounded complete domain L, consider the function space $[L \to L]$. Each $s\chi_U$ that is an approximating characteristic function for 1_L has a two element range, and hence a finite supremum of approximate characteristic functions (which exists since they are all bounded by 1_L) has finite range. It follows that there is an approximate identity for L consisting of functions with finite range, a strengthened version of an *FS*-domain.

Corollary II-4.21. *A bounded complete domain, in particular a continuous lattice, has an approximate identity consisting of continuous functions with finite range, and hence is an FS-domain.* □

We remark that the proof of Proposition II-4.20 readily generalizes to arbitrary L-domains, not just those with 0. For every element $x \in L$, there exists a least element 0_x in $\downarrow x$. Then for $U \subseteq X$, $g \in [X, \Sigma L]$, we define $g\chi_U$ by $g\chi_U(x) = g(x)$ if $x \in U$ and 0_x otherwise. Since $g\chi_U$ is constant on directed sets outside of U, one verifies directly that $g\chi(U)$ is Scott-continuous if g is. One can the proceed as in the preceding proposition, replacing the approximating characteristic functions by the newly defined $s\chi_U$. One then directly obtains the following generalization, which provides another large class of domains that is cartesian closed in addition to the category of *FS*-domains (see also Exercise II-2.32).

Corollary II-4.22. *Let X be a space such that $\mathcal{O}(X)$ is a continuous lattice and let L be an L-domain. Then $\Omega[X, \Sigma L]$ is an L-domain. Hence the category LDOM of all L-domains and Scott-continuous functions is cartesian closed, with function spaces $\Omega[\Sigma L, \Sigma M] = [L \to M]$.* □

We conclude the section by identifying the irreducible elements $\mathrm{IRR}[X, \Sigma L]$ of $\Omega[X, \Sigma L]$. The reader should recall Definition I-3.5.

Lemma II-4.23. *Let L be a continuous lattice and X a space. For a function $f \in [X, \Sigma L]$ the following assertions are equivalent:*

(1) $f \in \mathrm{IRR}[X, \Sigma L]$;
(2) *there are a prime element U of $\mathcal{O}(X)$ and an irreducible element p of L such that $f = \chi_U \vee \mathrm{const}_p$, where χ_U is the characteristic function of the set U and const_p is the constant function on X with value p in L.*

Proof: (2) implies (1): Suppose $f = a \wedge b$. Let $A = a^{-1}(\downarrow p)$ and $B = b^{-1}(\downarrow p)$. Then A and B are closed, and if $p < 1$ is irreducible, $A \cup B = X \backslash U$. But as U is prime, that is, $X \backslash U$ is an irreducible closed set, $A = A \backslash U$ or $B = X \backslash U$ follows, and so $f = a$ or $f = b$. The case $p = 1$ yields $f = \text{const}_1$.

(1) implies (2): We may assume that $f \neq \text{const}_1$. Hence we find a $t < 1$ in $f(X)$. We take an arbitrary $s \ll t$ in L and set $U = f^{-1}(\twoheaduparrow s)$. Since $\twoheaduparrow s$ is open in L by II-1.6 and f is continuous, U is open and the two functions $a = f \vee \chi_U$ and $b = f \vee \text{const}_s$ are in $[X, \Sigma L]$.

If $x \in U$ then $a(x) = f(x) \vee 1 = 1$ and $b(x) = f(x) \vee s = f(x)$, since $s \ll f(x)$; hence $(a \wedge b)(x) = f(x)$. If, however, $x \notin U$, then $a(x) = f(x) \vee 0 = f(x)$, and $b(x) = f(x) \vee s$; that is,

$$(a \wedge b)(x) = f(x) \wedge (f(x) \vee s) = f(x).$$

Hence $a \wedge b = f$, and since f is irreducible by (1), we have $a = f$ or $b = f$.

In the first case, $f = a = f \vee \chi_U$, that is, $\chi_U \leq f$. Let x be such that we have $t = f(x)$. Then $s \ll f(x)$, that is, $x \in U$, and thus $1 \leq f(x) = t < 1$, a contradiction. Hence we must have $f = b = f \vee \text{const}_s$. Thus $s \leq f(x)$ for all $x \in X$. Since we chose $s \ll t$ arbitrarily, and since L is continuous, we conclude $t \leq f(x)$ for all $x \in X$ (see I-1.6). This means that f takes at most one value $t < 1$. Thus we have shown that $f \neq 1$ implies $f = \chi_U \vee \text{const}_p$ for a $p < 1$ and for the open set $U = f^{-1}(1)$.

We claim that $p < 1$ is irreducible: If $p = v \wedge w$, set $a = \chi_U \vee \text{const}_v$ and set $b = \chi_U \vee \text{const}_w$, and observe $a \wedge b = f$. By (1) we have $f = a$ or $f = b$; that is, $p = v$ or $p = w$. Now we claim that U is prime: Indeed if $U = V \cap W$, then set $a = \chi_V \vee \text{const}_p$ and $b = \chi_W \vee \text{const}_p$. If $x \in U$, then $(a \wedge b)(x) = 1 \wedge 1 = 1$; if $x \notin U$, then $x \notin V$, say, and then $(a \wedge b)(x) = p \wedge 1 = p$. Hence, we have shown that $a \wedge b = f$. The irreducibility of f implies either $f = a$ (that is, $U = V$) or $f = b$ (that is, $U = W$). The proof is complete. □

We recall that in a sober space (see remarks preceding II-1.12) the prime elements of $\mathcal{O}(X)$ are precisely the sets X and $X \backslash \{x\}^-$ where $x \in X$. We will note that each T_0-space X can be naturally embedded into a sober space Y such that every continuous function $f \colon X \to S$ into a sober space extends uniquely to Y (see Chapter V). For a continuous lattice L, the space ΣL is sober by II-1.12. Hence $[X, \Sigma L] = [Y, \Sigma L]$, and it is therefore no loss of generality if we now talk about sober spaces only in our present context.

Proposition II-4.24. *Let X be a sober space and L a continuous lattice. Then there is a bijection*

$$(x, p) \mapsto \chi_{X \backslash \{x\}^-} \vee \text{const}_p : X \times ((\text{IRR } L) \backslash \{1\}) \to \text{IRR}[X, \Sigma L] \backslash \{1\}. \quad \square$$

This allows us to identify the irreducible elements < 1 in $\Omega[X, \Sigma L]$ in a canonical way with the product $X \times (\mathrm{IRR}\ L \backslash \{1\})$. From Theorem II-4.7 we know that $\Omega[X, \Sigma L]$ is a continuous lattice iff $\mathcal{O}(X)$ is continuous. (If X is arbitrary, we can still assert meet continuity by I-1.8 and II-4.19.) Thus Corollary I-3.10 applies to $\Omega[X, \Sigma L]$ under these circumstances. However, the explicit characterization of $\mathrm{IRR}[X, \Sigma L]$ in II-4.24 allows us to draw the conclusion of Corollary I-3.10 – the irreducibles are order generating in $\Omega[X, \Sigma L]$ – even without hypothesis on X, other than its sobriety. In fact, if for each $x \in X$ we have a set $P(x) \subseteq \mathrm{IRR} L \backslash \{1\}$ with $s = \inf(\uparrow_s \cap P(x))$ for all $s \in L$, then the image P under the canonical function in II-4.24 of the set $\bigcup\{\{x\} \times P(x): x \in X\}$ satisfies $f = \inf(\uparrow f \cap P)$ for all $f \in [X, \Sigma L]$. This allows the construction of rather bizarre order generating sets, which we will use in Chapter V for the construction of pathological spaces X with $\mathcal{O}(X)$ continuous.

Exercises

Exercise II-4.25. Let L be a domain for which $\downarrow x$ is a sup semilattice in the relative order for every $x \in L$. Given $y, z \in \downarrow x$, we denote their supremum in $\downarrow x$ by $y \vee_x z$, and call it the relative sup operation with respect to x. Prove the following.

(i) Set $G := \{(x, y, z) \in L^3 : x, y \in \downarrow z\}$, and endow G with the subspace topology from $(L, \sigma(L))^3$. Then the relative sup operation $(x, y, z) \mapsto x \vee_z y : G \to L$ is Scott-continuous.

(ii) If $f, g, h \in \mathrm{TOP}(X, \Sigma L)$ satisfy $f(x), g(x) \in \downarrow h(x)$ for all $x \in X$. Then $f \vee_h g$ defined by $(f \vee_h g)(x) = f(x) \vee_{h(x)} g(x)$ is continuous.

Hint. (i) Let $x \vee_z y \in V$ be open. Since $x \vee_z y$ is the directed supremum of $\{a \vee_z b: a \ll x, b \ll y\}$, there exist $a \ll x$ and $b \ll y$ such that $a \vee_z b \in V$. By I-1.37(ii), $c := a \vee_z b \ll x \vee_z y \le z$. Pick $(u, v, w) \in (\twoheaduparrow a \times \twoheaduparrow b \times \twoheaduparrow c) \cap G$, a neighborhood of (x, y, z) in G. Then

$$u \vee_w v \ge a \vee_w b = a \vee_c b = a \vee_z b = c \in V$$

(using twice the observation that for $p, q \in \downarrow r \subseteq \downarrow s$, $p \vee_r q = p \vee_s q$). Thus $u \vee_w v \in \uparrow V = V$.

(ii) Compose $f \times g \times h$ with the relative sup operation. $\square$

Note that this exercise gives an alternative proof of the continuity of the sup operation in a continuous lattice (see the remark before II-1.12 and Corollary II-4.15), since in that case the mapping $(x, y) \mapsto (x, y, 1)$ followed by the relative sup operation will be continuous.

Exercise II-4.26. Let L be the **dcpo** defined in Exercise II-1.36 and $M = \sigma(L)$ the lattice of Scott open subsets of L. Show that the Scott topology on $M \times L$ is properly finer than the product topology $\Sigma M \times \Sigma L$.

Hint. One first verifies that $\sigma(L)$ is not a continuous lattice. Indeed, the empty set is the only Scott open set U with $U \ll L$. From Theorem II-4.10(4) it follows that $G = \{((U, x) \in \sigma(L) \times L : x \in U\}$ is not open in the product topology of $\Sigma(\sigma(L)) \times \Sigma L$, but one sees directly that it is open in the Scott topology of the product. □

This preceding exercise illustrates in a nice way the claims on the relation between the Scott topology on a product and the product of Scott topologies in Theorem II-4.13.

Notice that in this present section we have presented a general theory of function spaces of lower semicontinuous functions. If X is a (locally) compact Hausdorff space, then by II-2.3(3) we have $\mathrm{LSC}(X, \mathbb{R}^*) \simeq [X, \Sigma\mathbb{I}]$, where $\mathbb{I} = [0, 1]$, the unit interval with its natural order. In this sense, the present theory supersedes earlier discussions such as Examples I-1.22, I-2.21, I-2.22.

We now want to complement the information about the way-below relation in function spaces given in Proposition II-4.20. Surprisingly enough, no satisfactory characterization of the way-below relation in the function space $[X, \Sigma L]$ is known except for special cases as in $[X, \Sigma\mathbb{I}]$ for compact spaces in I-1.22. In the next exercise we present the best result we know. For proofs one may consult [Erker *et al.*, 1998], where one finds more information on this topic. The notations are those from II-4.20.

Exercise II-4.27. Let X be a locally compact space in which the intersection of any two compact saturated subsets is compact. (Such spaces will be called stably locally compact in Section III-5.) Let L be a bounded complete domain. For $f\colon X \to L$, we denote by supp f the set of all $x \in X$ such that $f(x) \neq 0$, and $Q(X)$ will denote the set of all compact saturated subsets of X. Then the following statements for two functions $f, g \in [X, \Sigma L]$ are equivalent:

(1) $f \ll g$;
(2) supp $f \ll X$ and there are finitely many $V_i \in \mathcal{O}(X)$, $Q_i \in Q(X)$, $t_i \in L$ such that (a) $t_i \ll g(v)$ for all $v \in V_i$, (b) $f(w) \leq t_i$ for all $w \notin Q_i$ and (c) $X = \bigcup_i (V_i \setminus Q_i)$;
(3) there are finitely many $V_i \in \mathcal{O}(X)$, $Q_i \in Q(X)$, $t_i \in L$ such that (a) $V_i \ll g^{-1}(\uparrow t_i)$, (b) $f(x) \leq t_i$ for all $x \notin Q_i$ and (c) supp $f \subseteq \bigcup_i (V_i \setminus Q_i)$. □

Problem. Investigate systematically under what circumstances $[X, \Sigma L]$ is a domain. $\square$

Exercise II-4.28. Show that Proposition II-4.17 holds for **dcpo**s that are semilattices. $\square$

Exercise II-4.29. Prove the following.

(i) If $\mathcal{O}(Y)$ is a continuous lattice, then for all spaces X and Z, composition $(f, g) \mapsto g \circ f : [X, Y] \times [Y, Z] \to [X, Z]$ is continuous.
(ii) If L is a continuous lattice, then $\Sigma[L \to L]$ is a topological monoid relative to composition $(f, g) \mapsto f \circ g$.

Hint. For (i), let $g \circ f \in N(H \leftarrow V)$. Use the continuity of $\mathcal{O}(Y)$ to pick $U \ll g^{-1}(V)$ such that $f^{-1}(U) \in H$. Then $N(\mathord{\uparrow}U \leftarrow V) \circ N(H \leftarrow U) \subseteq N(H \leftarrow V)$.

For part (ii), use II-4.13 and II-2.9(ii). $\square$

Exercise II-4.30. Let X, Y be T_0 spaces such that $\mathcal{O}(X)$ is a continuous lattice. Show that the Isbell topology on $[X, Y]$ has a basis of open sets of the form $N(U, V) := \{f \in [X, Y] : U \ll f^{-1}(V)\}$.

Hint. See the proof of Proposition II-4.5(iii). $\square$

Old notes

The train of thought leading to the main result II-4.7 and this result itself are due to John Isbell [Isbell, 1975a; Isbell, 1975b], as is the formulation of what is commonly called the Isbell topology. (The former of the two sources is the one to consult according to Isbell's own recommendation.)

In II-4.10 we have a first characterization theorem for spaces Y to have a continuous lattice as topology $\mathcal{O}(Y)$; later we will see others. This result appeared for the first time in the *Compendium*, although certain equivalences had been known previously: (4) iff (5) was in the paper of B. J. Day and G. M. Kelly [Day and Kelly, 1970]; in fact this paper as well as Isbell's second paper above contains additional information concerning the context of II-4.10. In particular, the equivalence of (1) and (3) in II-4.12 is due to [Day and Kelly, 1970]. An elementary proof has been given by [Richter, 1997]. Various names were used in the literature for spaces Y for which $\mathcal{O}(Y)$ is a continuous lattice: "semilocally bounded" in [Isbell, 1975b], "quasilocally compact" in [Ward, 1969], "*CL*-spaces" in [Hofmann, 1978] and "core compact" in [Hofmann and Lawson, 1978].

For complete lattices, the results in II-4.13 through II-4.16 were new in the *Compendium.* For continuous lattices one finds essential portions of them in [Scott, 1972a], see p. 107, 2.9. Results such as II-4.13 are vital if one wishes to determine the (joint) continuity of finitary operations (as in the example of II-4.15). Also II-4.17 and II-4.18 were new in [Gierz *et al.*, B1980]; these results will be applied in Chapter VII. They belong to the type of statement in which lattice theoretical properties of a complete lattice L are characterized in terms of lattice theoretical properties of the Scott topology $\sigma(L)$. An earlier example of this is II-1.14. The identification of irreducible (or prime) elements of a function space in II-4.23 and II-4.24 was new in the *Compendium* after it had appeared in the SCS Memo 41 of 1977 by Hofmann and Scott [SCS 41]. We will refer back to this result in Chapter V.

New notes

In contrast with the *Compendium*, from the beginning of this section through II-4.6 we discuss the Isbell topology on the space $TOP(X, Y)$ of all continuous functions from a T_0 space X to a T_0 space Y; it is a recurring theme in this section. We also include a discussion of exponentiable objects in the category of T_0 spaces in II-4.12 following the work of Day and Kelly [Day and Kelly, 1970].

The way-below relation on function spaces cannot be characterized easily (see II-4.20, II-4.27). A careful analysis of the way-below relation on function spaces can be found in [Erker *et al.*, 1998].

III

The Lawson Topology

The first topologies defined on a lattice directly from the lattice ordering (that is, Birkhoff's order topology and Frink's interval topology) involved "symmetrical" definitions – the topologies assigned to L and to L^{op} were identical. A guiding example was always the unit interval of real numbers in its natural order, which is of course a highly symmetrical lattice. The initial interest was in such questions as which lattices became compact and/or Hausdorff in these topologies. The Scott topology stands in strong contrast to such an approach. Indeed it is a "unidirectional" topology, since, for example, all the open sets are always upper sets; thus, for nontrivial lattices, the T_0 separation axiom is the strongest it satisfies. Nevertheless, we saw in Chapter II that the Scott topology provides many links between domains and general topology in such classical areas as the theory of semicontinuous functions and in the study of lattices of closed (compact, convex) sets (ideals) in many familiar structures.

In this chapter we introduce a new topology, called the Lawson topology, which is crucial in linking continuous lattices and domains to topological algebra. Its definition is more in the spirit of the interval and order topologies, and indeed it may be viewed as a mixture of the two. However, it remains asymmetrical – the Lawson topologies on L and L^{op} need not agree. But, even if one is seeking an appropriate Hausdorff topology for continuous lattices, this asymmetry is not at all surprising in view of the examples we have developed. We also show that the new topology is closely related to the earlier topology, because in any **dcpo** a set is Scott open iff it is a Lawson open upper set (Proposition III-1.6). Though the Scott topology determines the underlying partial order, the Lawson topology does not do so, however.

In Section III-1 it is shown that the Lawson topology on complete lattices is always compact and T_1, and that it is Hausdorff for domains. In Section III-2 we see that for meet continuous complete lattices the Lawson topology is Hausdorff if and only if the lattice is continuous. (Hence the asymmetry, because L may be continuous when L^{op} is not.) In fact continuous lattices equipped with the Lawson topology give compact Hausdorff topological semilattices which have a basis of subsemilattices – a most important class of semilattices in topological algebra. This interplay culminates in the Fundamental Theorem VI-3.4, which equates the two classes.

In Section III-3 we introduce a class of **dcpo**s called quasicontinuous domains. Though not actually domains, their theory nonetheless exhibits many parallels to the theory of domains including the Hausdorffness of the Lawson topology. Indeed it is shown that the quasicontinuous complete lattices are precisely the complete lattices for which the Lawson topology is Hausdorff. We also show that the Lawson topology can be defined in terms of convergence, where the notion of convergence is, once again, given in order theoretical terms that involve the liminf. This resumes and concludes a theme that we began to investigate in Section II-1.

An important objective of Section III-4 is to clarify when the Scott and Lawson topologies have *countable bases*. We first introduce the important notion of a basis for a domain and prove that the existences of a countable domain basis, a countable basis for the Scott topology, and a countable basis for the Lawson topology are all equivalent; in fact, we treat arbitrary infinite cardinalities.

Although the Lawson topology is always compact for complete, in particular for continuous, lattices, it is typically not compact for general domains. In Section 5 we consider the class of domains that are compact in the Lawson topology and develop criteria for determining whether a domain is compact.

III-1 The Lawson Topology

As we have seen in the previous chapter, the Scott topology is well suited for many aspects of domain theory, including the encoding of the partial order as the order of specialization. However, its coarseness limits the usefulness of many of the classical notions from topology: for example, compactness in the Scott topology of a complete lattice is trivial, since any open cover must cover the bottom element 0, and the open set containing 0 must be the whole space. To utilize the tools such as closures and compactness more efficiently, a refinement of the Scott topology must be found.

One effective way to refine the Scott topology is via the consideration of *dual topologies*, topologies for which the order of specialization is $\geq$, the opposite of the given order. The coarsest of these is the one that takes as a subbasis for the closed sets all principal filters $\uparrow x$. We begin by recalling the definition of this auxiliary topology (cf. O-5.4), which we will use more extensively in Chapter V. At the moment it serves to refine the Scott topology by taking the join or *patch* of the two topologies.

Definition III-1.1. Let L be a poset. We call the topology generated by the complements $L\backslash\uparrow x$ of principal filters (as subbasic open sets) *the lower topology* and denote it by $\omega(L)$. $\square$

Let us note that the lower topology $\omega(L)$ on a **dcpo** L is generally coarser than $\sigma(L^{\mathrm{op}})$, since the principal filters $\uparrow x$ are always closed for the dual Scott topology. The sets of the form $\uparrow F$ are $\omega(L)$ closed for finite F, but one should note that not every $\omega(L)$ closed set is necessarily of this form – even in a complete lattice where the collection of all principal filters $\uparrow x$ is closed under arbitrary intersections. The closure of a singleton $\{x\}$ is $\uparrow x$, and in particular the lower topology is T_0.

Lemma III-1.2. *Let S and T be posets and $f\colon S \to T$ any function.*

(i) *If f is an upper map, i.e., if the co-restriction $f\colon S \to \downarrow f(S)$ has a lower adjoint (see* O-3.18*), then f is continuous relative to the lower topologies. In particular, if S and T are complete semilattices and f preserves arbitrary nonempty infs, then f is continuous relative to the lower topologies.*

(ii) *If f is continuous with respect to the lower topologies, then f preserves filtered infs, that is, f is monotone and whenever F is a filtered set such that* $\inf F$ *exists, then* $\inf f(F)$ *exists and* $\inf f(F) = f(\inf F)$.

(iii) *If S and T are complete semilattices and if f is a semilattice homomorphism which is continuous with respect to the lower topologies, then f preserves arbitrary nonempty infs.*

Proof: (i) Suppose that $g\colon \downarrow f(S) \to S$ is a lower adjoint for the co-restriction of f from S to $\downarrow f(S)$. We must show that the inverse image of a subbasic closed set of the form $\uparrow t$ in T is closed in S. If $t \notin \downarrow f(S)$, then $f^{-1}(\uparrow t) = \emptyset$, which is closed. Otherwise let $s = g(t)$. Then

$$g(t) = s \le x \Leftrightarrow t \le f(x) \Leftrightarrow x \in f^{-1}(\uparrow t),$$

and thus $x \in \uparrow g(t)$ iff $x \in f^{-1}(\uparrow t)$, i.e., $\uparrow g(t) = f^{-1}(\uparrow t)$.

In the case that S and T are complete semilattices and f preserves arbitrary nonempty infs, then f is an upper map by O-3.18.

(ii) If f is continuous relative to the lower topologies, then it is order preserving for their orders of specialization, which are the dual orders $\ge$, and hence is monotone.

Now, let F be filtered in S and suppose that $s = \inf F$ exists. As f is monotone, $f(s) \le f(u)$ for all $u \in F$. We claim that $f(s) = \inf f(F)$. For this let v be a lower bound of $f(F)$. Whenever $x \notin {\downarrow}s$, then F cannot be contained in ${\uparrow}x$; hence, it is eventually in $L\backslash{\uparrow}x$. Thus, F converges to s relative to the lower topology. Then $f(F)$ converges to $f(s)$ by the continuity of f. Hence, $f(s)$ belongs to the closed set ${\uparrow}v$ which contains $f(F)$. Hence, $v \le f(s)$, that is, $f(s)$ is the greatest lower bound of $f(F)$.

(iii) If, in addition, f preserves finite nonempty infs, then f preserves arbitrary nonempty infs (by (ii) and O-1.10). $\square$

In words we can say that a *semilattice* homomorphism between two complete semilattices is continuous relative to the lower topologies iff it preserves infs of filtered sets. This is the same as saying that it preserves arbitrary nonempty infs, so that the extra assumption of continuity allows the passage from the finite infs to the infinite ones.

Lemma III-1.3. *If S and T are posets, then $\omega(S \times T)$ is the product topology of the topologies $\omega(S)$ and $\omega(T)$.*

Proof: This is immediate from the following two relations:

$$(S\backslash{\uparrow}s) \times (T\backslash{\uparrow}t) = (S \times T)\backslash((S \times {\uparrow}t) \cup ({\uparrow}s \times T)), \quad \text{and}$$
$${\uparrow}(s,t) = (S \times {\uparrow}t) \cap ({\uparrow}s \times T).$$

$\square$

Note that the situation here is considerably simpler than the corresponding one for the Scott topologies (see II-4.13).

Lemma III-1.4. *If L is a semilattice, then $(L, \omega(L))$ is a topological semilattice, that is, the inf operation*

$$(x, y) \mapsto x \wedge y : (L, \omega(L)) \times (L, \omega(L)) \to (L, \omega(L))$$

is continuous.

Proof: The inf operation has lower adjoint the diagonal map $z \mapsto (z, z)$. The assertion follows from III-1.2(i) and III-1.3. $\square$

We now proceed to the essential definition:

Definition III-1.5. Let L be a **dcpo**. Then the common refinement $\sigma(L) \vee \omega(L)$ of the Scott topology and the lower topology is called the *Lawson topology* and is denoted by $\lambda(L)$. The space $(L, \lambda(L))$ is written ΛL. $\square$

In other words, the Lawson topology has as a *subbasis* the sets U, with $U \in \sigma(L)$, together with the sets $L\backslash{\uparrow}x$, for $x \in L$. The sets $U\backslash{\uparrow}F$, where $U \in \sigma(L)$

and F is finite in L, form a *basis* for $\lambda(L)$. Note that both U and $L\backslash\uparrow F$ satisfy property (S) (see II-1.3 and II-1.4(v), (vi)); hence, all $U\backslash\uparrow F$ and all Lawson open sets satisfy property (S) by II-1.4(vii). All one appears to be able to say beyond this about the structure of Lawson open sets in general is the following.

Proposition III-1.6. *Let L be a* **dcpo**.

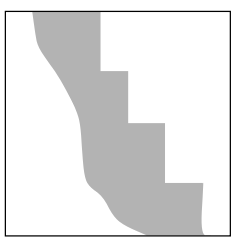
(i) *An upper set U is Lawson open iff it is Scott open.*
(ii) *A lower set is Lawson closed iff it is closed under sups of directed sets.*
(iii) *If A is Scott closed in L, then the relative lower topology, resp. relative Lawson topology, on A is just the lower, resp. Lawson, topology of A.*

Proof: (i) Since $\sigma(L) \subseteq \lambda(L)$ we have to show that every Lawson open upper set is Scott open. But by the preceding remarks such a set satisfies property (S), and then as an upper set it is Scott open by II-1.4(v).

(ii) The second assertion follows from the first in view of II-1.4(i).

(iii) One sees easily that A has the same subbasic closed sets in its lower topology as in the relative lower topology from L. A subset of A that is Scott closed in A is easily seen to be Scott closed in L and hence the relative Scott topology on L agrees with the Scott topology of L. The assertion for the Lawson topology follows from these two. □

A picture may help in the visualization of the open sets in the Lawson topology:



It will be seen that the Lawson topology on the unit square $[0, 1]^2$ is just the ordinary Euclidean topology.

The examples in O-4.5 show that for a Lawson open set U the upper set $\uparrow U$ need not be open and that the Lawson interior of an upper set need not be an upper set. When we talk about meet continuous lattices, then things improve as we will see in III-2.5.

Lemma III-1.7. *Let L be a* **dcpo** *and F a filtered subset. Then* $\lim F$ *exists with respect to $\lambda(L)$ iff* $\inf F$ *exists, and in this case* $\inf F = \lim F$ *and is the* *unique cluster point. Similarly if D is a directed subset, then* $\lim D$ *exists and* $\lim D = \sup D$*, and is the unique cluster point.*

Proof: Suppose that $\inf F$ exists. The sets $L\setminus\uparrow x$ with $x \not\leq \inf F$ are subbasic $\omega(L)$ neighborhoods of $\inf F$, and F is eventually contained in any of them. Hence F converges to $\inf F$ with respect to $\omega(L)$. It also converges to $\inf F$ with respect to $\sigma(L)$ trivially, since every Scott neighborhood of $\inf F$ is an upper set and hence contains F. As a consequence, F converges to $\inf F$ with respect to $\lambda(L)$.

Now let y be any cluster point of F with respect to $\lambda(L)$. If $u \in F$, then $\downarrow u$ is $\lambda(L)$ closed and F is eventually in $\downarrow u$. Hence $y \leq u$ for all $u \in F$. If z is a lower bound for F, then $\uparrow z$ is $\lambda(L)$ closed and contains F. Thus $y \in \uparrow z$, and this proves $y = \inf F$.

Let D be a directed set. Then $x = \sup D$ exists since L is a **dcpo**. The directed set converges to x in the Scott topology, trivally so in the lower topology, hence in the Lawson topology. Let y be any cluster point. Since the Lawson closed set $\uparrow d$ eventually contains D, it must contain y. Hence y is an upper bound for D, i.e., $x \leq y$. Since the Lawson closed set $\downarrow x$ contains D, it must also contain y and hence $y = x$. □

Theorem III-1.8. *Let S and T be complete semilattices and $f\colon S \to T$ a semilattice homomorphism. Then the following statements are equivalent:*

(1) *f is Lawson continuous (that is, $\lambda(S)$-$\lambda(T)$ continuous);*
(2) *f preserves arbitrary nonempty infs and directed sups;*
(3) *f preserves liminfs;*
(4) *f is Scott-continuous and an upper map.* □

Remark. Condition (3) means of course that $f(\underline{\lim}\, x_j) = \underline{\lim}\, f(x_j)$ for all nets $(x_j)_{j\in J}$ on S; compare this with II-2.1(3).

Proof of theorem: (2) implies (1): Assume that f preserves arbitrary nonempty infs and directed sups. Then f is $\omega(S)$-$\omega(T)$ continuous by III-1.2 and $\sigma(S)$-$\sigma(T)$ continuous by II-2.1. Hence f is Lawson continuous.

(1) implies (2): We suppose that f is $\lambda(S)$-$\lambda(T)$ continuous. Let F be a filtered set in S. Then $\inf F = \lim F$ (with respect to $\lambda(S)$) by III-1.7. Since f is λ continuous we have $f(\inf F) = f(\lim F) = \lim f(F)$. The latter is $\inf f(F)$, since $f(F)$ is filtered because f is a semilattice morphism and III-1.7 applies once more. Thus, f preserves infs of filtered sets and, hence, infs of arbitrary nonempty sets (compare O-1.10). If $U \in \sigma(T) \subseteq \lambda(T)$, then $f^{-1}(U) \in \lambda(S)$. Because $f^{-1}(U)$ is an upper set owing to the monotonicity of f, we have therefore the conclusion $f^{-1}(U) \in \sigma(L)$ by III-1.6. Hence f is Scott-continuous, and so it preserves directed sups by II-2.1.

(2) implies (3): Immediate from the definition of $\underline{\lim}$ in II-1.1.

(3) implies (2): If D is directed, then $f(\sup D) = f(\underline{\lim}\, D) = \underline{\lim}\, f(D) = \sup f(D)$, since f is order preserving as a semilattice morphism. If F is filtered, then $\inf(\downarrow x \cap F) = \inf F$ for all $x \in F$; hence,

$$\underline{\lim}\, F = \bigvee_{x \in F} \inf(\downarrow x \cap F) = \inf F.$$

Thus $f(\inf F) = f(\underline{\lim}\, F) = \underline{\lim}\, f(F) = \inf f(F)$. This shows that f preserves filtered infs. Since f is a semilattice morphism, f also preserves arbitrary nonempty infs (see O-1.10).

We have (2) $\Leftrightarrow$ (4) by O-3.18 and II-2.1. □

Remark. The morphisms in Theorem III-1.8 are very natural ones to consider for complete semilattices. Condition (4) suggests a generalization to general **dcpo**s, namely those functions that are Scott-continuous upper maps. By Lemma III-1.2(i) such maps are continuous for the lower topology, and hence are also Lawson continuous.

Theorem III-1.9. *For a complete (semi)lattice L, the Lawson topology $\lambda(L)$ is a compact T_1 topology.* □

Proof: Firstly, for $x \in L$ we have $\{x\} = \downarrow x \cap \uparrow x$. Now, $\downarrow x$ is Scott closed, while $\uparrow x$ is closed in the lower topology. Hence, $\{x\}$ is Lawson closed; that is, $\lambda(L)$ is a T_1 topology.

To prove that $\lambda(L)$ is compact, we firstly suppose that L is a complete lattice and we use the Alexander Subbasis Lemma: a space is compact if every open cover consisting of *subbasic* open sets contains a finite subcover.

Thus assume $\{U_j \in \sigma(L) : j \in J\}$ and $\{L \backslash \uparrow x_k \colon k \in K\}$ together form a cover of L. Let $x = \sup\{x_k \colon k \in K\}$. Then

$$\bigcup\{L \backslash \uparrow x_k \colon k \in K\} = L \Big\backslash \bigcap\{\uparrow x_k \colon k \in K\} = L \backslash \uparrow x.$$

But $x \notin L \backslash \uparrow x$; therefore, there is a j such that $x \in U_j$. Since U_j is Scott open, there are indices $k_1, \ldots, k_n$ such that $x_{k_1} \vee \cdots \vee x_{k_n} \in U_j$. Then

$$U_j \cup \big(L \backslash \uparrow x_{k_1}\big) \cup \cdots \cup \big(L \backslash \uparrow x_{k_n}\big) = L,$$

and we are finished.

If L is a complete semilattice, then L^1 is a complete lattice, hence compact in the Lawson topology. Since $\{1\}$ is Scott, hence Lawson, open, L is compact in the relative topology. But the relative Lawson topology of the Scott closed lower set L is the Lawson topology of L by III-1.6(iii). □

This compactness theorem shows in a certain sense that we have not refined the Scott topology too much, even though we have improved the T_0 separation property of the Scott topology to the T_1 separation of the Lawson topology. It will be of paramount importance to understand when in fact the Lawson topology is Hausdorff; most of the following observations serve to study this question. In particular the next theorem shows the suitability of the Lawson topology for domains and continuous lattices.

Theorem III-1.10. *For a domain L, the Lawson topology $\lambda(L)$ is a Hausdorff topology.*

Proof: Suppose that $x \neq y$ in L, and assume that $x \not\leq y$. Then by I-1.6 there is a $u \ll x$ with $u \not\leq y$. Then $\uparrow u$ is a Scott (hence, Lawson) open neighborhood of x (see II-1.6), and $L \backslash \uparrow u$ is an $\omega(L)$ (hence, Lawson) open neighborhood of y. Clearly these two neighborhoods are disjoint. □

The two previous theorems imply immediately the following.

Corollary III-1.11. *For any complete continuous semilattice, in particular for any continuous lattice, the Lawson topology is compact and Hausdorff.* □

A finer analysis of the Lawson topology is possible for meet continuous lattices. We make it the subject of the *next* section. We close *this* section with a discussion of the relationship between subalgebras and the Lawson topology (in the same spirit as Theorem III-1.8 for morphisms).

Theorem III-1.12. *Let L be a complete continuous semilattice, and let S be a subsemilattice. The following conditions are equivalent:*

(1) *S is closed in the Lawson topology;*
(2) *S is closed with respect to the formation of arbitrary nonempty infs and directed sups in L;*
(3) *for all nets $(x_j)_{j \in J}$ in S, we have $\underline{\lim}\, x_j \in S$, where $\underline{\lim}\, x_j$ is taken with respect to L.*

Proof: (1) implies (2): By Lemma III-1.7 filtered infs and directed sups are $\lambda(L)$-limits. Hence S is closed under filtered infs, and thus arbitrary infs (see O-1.5), and directed sups.

(2) implies (3): Immediate.

(3) implies (2): Since filtered infs and directed sups are special cases of taking $\underline{\lim}$ of a net, we conclude S is closed with respect to taking directed sups and filtered infs (and hence arbitrary nonempty infs).

(2) implies (1): By Theorem III-1.8 the inclusion mapping from S to L is continuous. Since S is compact by Theorem III-1.9, its image under inclusion is compact and hence closed since L is Hausdorff (Theorem III-1.10). □

Exercises

Exercise III-1.13. Let P be a poset equipped with the lower topology. Show that the order of specialization on P is $\geq$, and that the lower topology is the coarsest topology for which this is true.

Exercise III-1.14. Let L be an algebraic domain. Prove the following.

(i) ΛL has a basis of open–closed sets and hence is a zero dimensional Hausdorff space.
(ii) If L is a semilattice, the Lawson topology has a basis of open–closed subsemilattices.
(iii) If L is an algebraic lattice or a bounded complete algebraic domain, then L is also compact.

Hint. By II-1.15 $\sigma(L)$ has a basis of sets of the form $\uparrow k$ with $k \in K(L)$. But $\uparrow k$ is $\omega(L)$ closed, thus, $\lambda(L)$ closed; whence $\uparrow k$ is $\lambda(L)$ open–closed for any compact k. If L is algebraic, then the sets $\uparrow k \backslash (\uparrow k_1 \cup \cdots \cup \uparrow k_n)$, where $k, k_1, \ldots, k_n \in K(L)$, constitute a basis for $\lambda(L)$. All of these sets are open–closed, and are subsemilattices if L is a semilattice. Compactness follows from III-1.9. $\square$

Counterexample O-4.5(2) shows that there are complete lattices L such that ΛL is compact zero dimensional while L is not algebraic. In the next section we note that in the class of meet continuous lattices this aberration cannot occur.

Exercise III-1.15. Let X be a locally compact topological space. Let $\Gamma(X)$ denote the lattice of closed sets (O-2.7(3)). (Recall $\Gamma(X)^{\mathrm{op}} \cong \mathcal{O}(X)$ is a continuous lattice by I-1.7(5).) Prove the following.

(i) $F \ll G$ in $\Gamma(X)^{\mathrm{op}}$ iff there is a compact set Q such that $F \cup Q = X$ and $G \cap Q = \emptyset$.
(ii) The Scott topology on $\Gamma(X)^{\mathrm{op}}$ has as a basis the sets of the form $\{G \in \Gamma(X) : G \cap Q = \emptyset\}$, where Q is a compact subset of X.
(iii) The lower topology $\omega(\Gamma(X)^{\mathrm{op}})$ has as a subbasis the sets of the form $\{G \in \Gamma(X) : G \cap U \neq \emptyset\}$, where U is an open subset of X.
(iv) The Lawson topology has as a basis the sets of the form

$$\{G \in \Gamma(X) : G \cap Q = \emptyset \text{ and } G \cap U_k \neq \emptyset, \quad k = 1, \ldots, n\},$$

where Q is compact and $U_1, \ldots, U_n$ is a finite collection of open sets.

Hint. Part (i) follows from I-1.4(ii). For (ii) use II-1.14(2) and (i) above. Then (iii) and (iv) are straightforward. $\square$

In the preceding exercise one may restrict oneself to compact saturated sets instead of arbitrary compact sets.

Exercise III-1.16. Show that an order preserving Lawson continuous map $f\colon L \to M$ between **dcpo**s is Scott-continuous.

Hint. Use Lemma III-1.7 and the fact that Scott continuity is characterized by the preservation of directed sups. □

The next exercises relate two traditional lattice topologies, the interval topology and the order topology, to the Lawson topology. The interval topology has for a subbase of closed sets all principal ideals and all principal filters (see O-5.4). In order to define the order topology on L, we say a net $(x_j)_{j\in J}$ *order converges to* x iff $x = \liminf x_j = \limsup x_j$. This notion of convergence defines a topology on L, the *order topology*. (See Section II-1; see also [Birkhoff, B1967] for further details concerning these topologies.)

Exercise III-1.17. Let L be a **dcpo**. Prove the following.

(i) The interval topology is the join (as topologies) of the upper and lower topologies.
(ii) The interval topology is contained in the Lawson topology on L and that on L^{op}.
(iii) If every Scott closed set is closed in the interval topology, then the Lawson topology and the interval topology agree.
(iv) If L is a complete semilattice (lattice) and if the interval topology is Hausdorff, then the interval topology and the Lawson topology on L (and the Lawson topology on L^{op}) agree.

Hint. For (iv), use III-1.9. □

Exercise III-1.18. Let L be a complete lattice. Prove the following.

(i) The open upper sets in the order topology are the Scott open sets (and dually).
(ii) The Lawson topology on L and that on L^{op} are contained in the order topology. □

Exercise III-1.19. Consider a complete lattice L consisting of a countable antichain (all elements incomparable) with a 0 and 1 adjoined. Show that the interval topology is not Hausdorff, that the antichain converges to 0 in the Lawson topology, to 1 in the Lawson topology on L^{op}, and that the order topology is discrete. □

Exercise III-1.20. Show that Lemma III-1.7 remains valid if the Lawson topology is replaced by the interval topology. □

Exercise III-1.21. A *bitopological space* is a set X equipped with two topologies, written (X, τ, ν). A function $f\colon (X, \tau_1, \nu_1) \to (Y, \tau_2, \nu_2)$ is a *bicontinuous function* if it is continuous for both the τ-topologies and the ν-topologies. Show that a semilattice homomorphism $f\colon S \to T$ between complete semilattices is bicontinuous as a function from $(S, \sigma(S), \omega(S))$ to $(T, \sigma(T), \omega(T))$ if and only if it is Lawson continuous.

Hint. Use Lemma III-1.2(i) and Theorem III-1.8. □

Old notes

In describing the history of the Lawson topology, it is best to distinguish two viewpoints: that of topological algebra and that of lattice theory. In topological algebra one studies the structure of algebraic structures such as groups, rings, and semigroups which are already equipped with the topology such that the operations are continuous. In this vein compact topological semilattices and lattices have been studied since the 1950s by A. D. Wallace and the numerous mathematicians following in his footsteps; we will comment on this piece of history in Chapter VI where we concentrate on compact semilattices. However, we will see in the next section of the present chapter how compact semilattice theory and continuous lattice theory relate (III-2.15). In lattice theory, on the other hand, one considers lattices and looks for topologies which are naturally defined in terms of the given order structure. Typical examples are the topologies $\sigma(L)$ (Section II-1), $\omega(L)$, and $\lambda(L)$; there are, of course, others, but these do not interest us here.

The blending of the topological algebra viewpoint and the lattice theoretical viewpoint, as far as continuous lattices were concerned, was accomplished by K. H. Hofmann and A. Stralka in ATLAS [Hofmann and Stralka, 1976]. What was discovered there was that what had been studied by the topological algebraists under the name of compact unital Lawson semilattices was in fact the very same thing as continuous lattices (although in ATLAS this discovery is not phrased quite so explicitly; the first paper in print being explicit about this is Lea's note [Lea, 1976b]).

The explicit definition of the topology $\lambda(L)$ given here has evolved in the SCS Seminar since 1976. The name *Lawson topology* was chosen at the First Workshop on Continuous Lattices in April 1977 at Tulane. However, just as one finds the Scott topology defined for special complete lattices such as those of the form $\mathcal{O}(X)$ before Scott's paper in the work of Day and Kelly, the Lawson topology has its precursors on $\mathcal{O}(X)$. Indeed the Lawson topology was considered as early as 1961 by J. M. G. Fell [Fell, 1962] when X is a locally compact space, although the definition was given in terms of the description of basic open sets which are not on the surface recognizable as yielding the same topology as we see in Exercise III-1.15. In fact the topology was introduced on $\Gamma(X)^{\mathrm{op}}$ and it was shown by Fell that it was compact Hausdorff and that $(A, B) \mapsto A \cup B$ was continuous. These studies were continued later by J. Dixmier [Dixmier, 1968] who provided more information on this topology.

Theorem III-1.8 originated in the *Compendium*. Theorem III-1.9 was first published by Hofmann [Hofmann and Mislove, 1977], with a proof due to D. Scott [SCS 4]. In this line, III-1.10 had been known to the SCS Seminar since 1976.

Because of the properties (iii) and (ii) in Exercise III-1.15 the Lawson topology on the set $\Gamma(X)$ of closed subsets of a locally compact space has also been known under the name *hit-and-miss* topology (see e.g. [Matheron, B1975]).

III-2 Meet Continuity Revisited

For more detailed information on the developments of the previous section we turn to meet continuous semilattices. Here there is rather more to say on the nature of Lawson open sets. In discussing various topologies we will use subscripts to distinguish relative to which the closure, the interior, etc. is to be taken. Recall that, by Definition O-4.1 a meet continuous semilattice always is a **dcpo**.

Perhaps, somewhat surprisingly, we now have at hand tools to develop a meaningful theory of meet continuity for arbitrary **dcpo**s, and we begin our considerations in this general framework.

Definition III-2.1. A **dcpo** L is *meet continuous* if for any $x \in L$ and any directed set D with $x \leq \sup D$, then x is in the Scott closure of $\downarrow D \cap \downarrow x$. □

Remark III-2.2. *For directed complete semilattices the preceding definition of meet continuity is equivalent to the standard one (*O-4.1*).*

Proof: That the standard definition implies the preceding one in the context of semilattices is immediate. Conversely assume the preceding definition, let $d = \sup D$ for a directed set D, and let $x \in L$. If

$$y = \sup xD = \sup(\downarrow x \cap \downarrow D) = \sup(\downarrow xd \cap \downarrow D)$$

is strictly less than xd, then $\downarrow y$ is a Scott closed set containing $\downarrow D \cap \downarrow xd$, but missing xd, a contradiction. □

Proposition III-2.3. *A* **dcpo** *L is meet continuous if and only if for any Scott open set U and any $x \in L$, $\uparrow(U \cap \downarrow x)$ is Scott open.*

Proof: Suppose that $x \in L$, a meet continuous **dcpo**, that U is Scott open, and that $\sup D \in \uparrow(U \cap \downarrow x)$ for some directed set D. Then there exists $z \in U \cap \downarrow x$ such that $z \leq \sup D$. It follows from the hypothesis that $\downarrow D \cap \downarrow z \cap U \neq \emptyset$, so $D \cap \uparrow(U \cap \downarrow x) \supseteq D \cap \uparrow(U \cap \downarrow z) \neq \emptyset$, which establishes that $\uparrow(U \cap \downarrow x)$ is Scott open.

Conversely assume the second condition and let D be a directed set with $x \leq \sup D$. If x is not in the Scott closure of $\downarrow D \cap \downarrow x$, then there exists a Scott open set U containing x but missing $\downarrow D \cap \downarrow x$. By hypothesis $\uparrow(U \cap \downarrow x)$ is Scott open and misses D by construction. But $\sup D$ is in the Scott open set $\uparrow(U \cap \downarrow x)$, and hence some member of D must be, a contradiction. Thus L is meet continuous. □

Meet continuity can simplify checking that a **dcpo** is a domain.

Proposition III-2.4. *Let L be a meet continuous* **dcpo**. *Suppose that for any $x \in X$ and any Scott open set U containing x, there exists $y \in U \cap \downarrow x$ such that $\uparrow y \cap \downarrow x$ is a relative Scott neighborhood of x in $\downarrow x$ (in particular, this holds if each $\downarrow x$ is a domain). Then L is a domain.*

Proof: Let $x \in L$. We consider the family D of all $y \in \downarrow x$ such that $\uparrow y \cap \downarrow x$ is a relative Scott neighborhood of x in $\downarrow x$.

Let $y \in D$. Then there exists a Scott open U such that $\downarrow x \cap U \subseteq \uparrow y$. By the previous proposition we have that $\uparrow(U \cap \downarrow x)$ is Scott open; note that it is also a subset of $\uparrow y$. Thus $\uparrow y$ is a Scott neighborhood of x in L, and hence $y \ll x$. Thus $D \subseteq \twoheaddownarrow x$.

Now suppose that $\uparrow y_i \cap \downarrow x$ is a relative Scott neighborhood of x in $\downarrow x$ for $i = 1, 2$. Then their intersection is also, so there exists V Scott open such that $x \in V \cap \downarrow x \subseteq \uparrow y_1 \cap \uparrow y_2$. By hypothesis there exists $y \in V \cap \downarrow x$ such that $\uparrow y \cap \downarrow x$ is a relative Scott neighborhood of x in $\downarrow x$. Hence D is directed.

Finally let $z < x$. Then $L \backslash \downarrow z$ is a Scott open set containing x, and again by hypothesis we can pick $y \in (L \backslash \downarrow z) \cap \downarrow x$ such that $y \in D$. It follows that $x = \sup D$. Since x was arbitrary, L is continuous.

That the hypotheses are satisfied if each $\downarrow x$ is a domain follows from Proposition I-1.8. □

Meet continuity forces closer relationships between the Scott and Lawson topologies.

Proposition III-2.5. *For a meet continuous* **dcpo** L *we have:*

(i) *if* $U \in \lambda(L)$*, then* $\uparrow U \in \sigma(L)$*;*
(ii) *if* X *is an upper set, then* $\mathrm{int}_\sigma X = \mathrm{int}_\lambda X$*;*
(iii) *if* X *is a lower set,* $\mathrm{cl}_\sigma X = \mathrm{cl}_\lambda X$.

Proof: (i) Let $y \in \uparrow U$, U a Lawson open set. Let $x \in U$ such that $x \leq y$. Then there exists a basic Lawson open set $V \backslash \uparrow F$, where V is Scott open and F is finite, such that $x \in V \backslash \uparrow F \subseteq U$. Then $\uparrow(V \cap \downarrow x) \subseteq \uparrow(V \backslash \uparrow F) \subseteq \uparrow U$. Since the first set is Scott open by the previous proposition, it follows that y is in the Scott-interior of $\uparrow U$. Since y was arbitary in $\uparrow U$, the latter is Scott open.

(ii) Trivially $\mathrm{int}_\sigma X \subseteq \mathrm{int}_\lambda X$. By (i), $\mathrm{int}_\lambda X \subseteq \uparrow \mathrm{int}_\lambda X \subseteq \mathrm{int}_\sigma X$.

The equivalence of (ii) and (iii) is straightforward. □

Proposition III-2.6. *Let* S *and* T *be* **dcpo***s such that* $\sigma(T)$ *is a continuous lattice. Then* $\Lambda(S \times T) = \Lambda S \times \Lambda T$.

Proof: By II-4.13 we have $\Sigma(S \times T) = \Sigma S \times \Sigma T$. From III-1.3 we recall that $\omega(S \times T)$ is the product topology of $\omega(S)$ and $\omega(T)$. Suppose that ξ_1, ξ_2 are topologies on X and η_1, η_2 topologies on Y. Assume $U_k \in \xi_k$ and $V_k \in \eta_k$, $k = 1, 2$. Then the relation $(U_1 \cap U_2) \times (V_1 \cap V_2) = (U_1 \times V_1) \cap (U_2 \times V_2)$ shows that basic open sets of $(\xi_1 \vee \xi_2) \times (\eta_1 \vee \eta_2)$ are basic open in $(\xi_1 \times \eta_1) \vee (\xi_2 \times \eta_2)$ and vice versa. We apply this observation with $X = S$, $\xi_1 = \sigma(S)$, $\xi_2 = \omega(S)$ and $Y = T$, $\eta_1 = \sigma(T)$, $\eta_2 = \omega(T)$ and obtain the assertion. □

Definition III-2.7. A semilattice L endowed with a T_0 topology τ is a *topological semilattice* if $(x, y) \mapsto xy : (L, \tau) \times (L, \tau) \to (L, \tau)$ is continuous. □

Theorem III-2.8. *Let* L *be a meet continuous semilattice. Then all the translations* $x \mapsto ax : \Lambda L \to \Lambda L$ *for* $a \in L$ *are continuous. If also* $\sigma(L)$ *is a continuous lattice, then* $(L, \lambda(L))$ *is a topological semilattice.* □

Proof: The translation function $\lambda_x(y) = xy$ has $(\lambda_x)^{-1}(\uparrow z) = \uparrow z$ if $z \leq x$ and the empty set otherwise; thus λ_x is continuous in the lower topology. By hypothesis it is Scott-continuous, and thus is Lawson continuous.

The function m: $L \times L \to L$ defined by $m(x, y) = xy$ has lower adjoint $\Delta(x) = (x, x)$ and hence is continuous with respect to the lower topologies (Lemma III-1.2(i)); it is Scott-continuous from $(L \times L, \sigma(L \times L)) \to (L, \sigma(L))$ by Theorem O-4.2(6). Hence m: $\Lambda(L \times L) \to \Lambda L$ is continuous. If $\sigma(L)$ is continuous, then III-2.6 applies and gives $\Lambda(L \times L) = \Lambda L \times \Lambda L$, which yields the desired conclusion. □

Corollary III-2.9. *In a meet continuous semilattice L with continuous Scott topology $\sigma(L)$ the Lawson topology is Hausdorff iff the graph of $\leq$ is closed in $\Lambda L \times \Lambda L$.*

Proof: Since ΛL is a topological semilattice by III-2.8, then the function $m\colon \Lambda L \times \Lambda L \to \Lambda L \times \Lambda L$ defined by $m(x, y) = (x, xy)$ is continuous, and the graph of $\leq$ is m^{-1} of the diagonal in $L \times L$. Thus, if the latter is closed, then so is the graph of $\leq$. Since the diagonal is the intersection of the graphs of $\leq$ and of $\leq^{\mathrm{op}}$, the converse is clear. □

Lemma III-2.10. *If F is a finite set in a meet continuous* **dcpo***, then we have*

$$\mathrm{int}_\sigma \uparrow F \subseteq \bigcup \{\Uparrow x \colon x \in F\}.$$

Proof: Suppose $y \in U := \mathrm{int}_\sigma \uparrow F$, but y is not in the right side. If $F = \{x_1, \ldots, x_n\}$, there exists for each i a directed set D_i such that $y \leq \sup D_i$, but $x_i \notin \downarrow D_i$. By finite induction, using meet continuity, we choose $z_i \leq y$ such that $z_1 \in (\downarrow D_1 \cap \downarrow y \cap U)$ and $z_{i+1} \in (\downarrow D_i \cap \downarrow z_i \cap U)$. Then $z_n \in \bigcap_{i=1}^n D_i$, and also $z_n \in \uparrow x_j$ for some j, which contradicts $x_j \notin \downarrow D_j$. □

We now arrive at an important characterization theorem for continuous lattices in the class of meet continuous lattices and generalizations thereof in terms of the Hausdorff separation of the Lawson topology.

Theorem III-2.11. *If L is a domain, then L is meet continuous and the Lawson topology is Hausdorff. Conversely if L is a meet continuous* **dcpo***, if the Lawson topology is Hausdorff, and if each principal ideal $\downarrow x$ is a sup semilattice (in the relative order), then L is a domain.*

Proof: Suppose that L is a domain. By III-1.10 the Lawson topology is Hausdorff. Since for $x \in L$ and any directed set D such that $x \leq \sup D$, we have $\Downarrow x \subseteq \downarrow D$, the meet continuity condition follows.

Conversely, suppose that L is a meet continuous **dcpo** with Hausdorff Lawson topology such that each principal ideal is a sup semilattice. By Proposition III-2.4 it suffices to show that each principal ideal is a domain. Thus we consider some principal domain $\downarrow w$. Consider two points $x, y \in \downarrow w$ with $y < x$. We hope to find a $u \ll x$ such that $u \not\leq y$.

Since L is Hausdorff, there are disjoint open $\lambda(L)$ neighborhoods V and W of y and x, respectively, and we may assume that V is of the form $U \backslash \uparrow F$ with a Scott open neighborhood U of y and a finite set F. But since $y < x$, we may also assume that $W \subseteq U$ (for otherwise we replace W by $U \cap W$). Now we claim that $W \subseteq \uparrow F$. For if not, then there would be a $w \in W \backslash \uparrow F \subseteq U \backslash \uparrow F = V$, which is impossible since V and W are disjoint. But then also $\uparrow W \subseteq \uparrow F$. Since

$\uparrow W$ is Scott open by III-2.5(i), we know that $\mathrm{int}_\sigma \uparrow F$ contains x. By Lemma III-2.10 this implies that there is a $u \in F$ such that $x \in \Uparrow u$; that is, $u \ll x$. Since $y \in U \backslash \uparrow F \subseteq L \backslash \uparrow u$ we have $u \not\leq y$ as was desired.

Now let $x \in \downarrow w$. If x is a minimal element, then by Proposition III-2.3 $\uparrow x$ is Scott open, and hence x is a compact element. Otherwise by the previous paragraph for each $y < x$, we find $u \ll x$ such that $u \not\leq y$. If follows that $x = \sup\{u\colon u \ll x\}$. Now since $\downarrow w$ is a sup semilattice, restricting to $\downarrow w$, we have that x is the directed sup of elements way below it by I-1.2(iii). Thus $\downarrow w$ is a domain. □

Remark. Note that the preceding theorem applies in particular to complete lattices, complete semilattices, and **dcpo**s in which each principal ideal is a complete lattice (which in the presence of continuity means L-domain).

This theorem reveals that in the presence of meet continuity the assumption of continuity is closely related to a separation property (the Hausdorffness of the Lawson topology). Furthermore, III-2.11 together with III-2.9 and II-1.14 shows that the graph of $\leq$ is closed in $\Lambda L \times \Lambda L$ for a continuous lattice L.

Definition III-2.12. We say that a semilattice with a topology *has small (open, resp. compact) semilattices* iff each point has a neighborhood basis of (open, resp. compact) subsemilattices (cf. also VI-3.1 ff.). □

Proposition III-2.13. *Let L be a directed complete semilattice. Then the following are equivalent:*

(1) *each point of L has a Scott neighborhood basis of open filters;*
(2) *L is meet continuous and ΛL has small open semilattices.*

Proof: (1) implies (2): Let D be a directed set with supremum $d \geq x$ and U be a Scott open set containing x. Then there exists a Scott open filter F such that $x \in F \subseteq U$. Eventually the directed set D is in F, so $xD \in F$ eventually, since F is a filter. By Definition III-2.1, L is meet continuous.

Let $x \in W \in \lambda(L)$. Then there are a $V \in \sigma(L)$ and a finite set $F \subseteq L$ with $x \in V \backslash \uparrow F \subseteq W$. By (1) there is a filter $U \in \sigma(L)$ with $x \in U \subseteq V$. Then $x \in U \backslash \uparrow F \in \lambda(L)$, with $U \backslash \uparrow F \subseteq W$. One obtains easily that $U \backslash \uparrow F$ is semilattice.

(2) implies (1): Let $x \in W \in \sigma(L)$. Then there is a subsemilattice $V \in \lambda(L)$ with $x \in V \subseteq W$. Then $x \in \uparrow V \subseteq W$, $\uparrow V$ is a filter, and $\uparrow V \in \sigma(L)$ by III-2.5. □

Lemma III-2.14. *Let L be a complete meet continuous semilattice. Then for each $x \in L$ we have*

$$\sup\{\inf U \colon x \in U \in \lambda(L)\} = \sup\{\inf U \colon x \in U \in \sigma(L)\}.$$

Proof: Since $\sigma(L) \subseteq \lambda(L)$, clearly the left hand side is $\geq$ the right hand side. But if $U \in \lambda(L)$, then $\inf U = \inf \uparrow U$ and $\uparrow U \in \sigma(L)$ by III-2.5, hence the reverse inequality holds. □

We are ready for a crucial theorem:

Theorem III-2.15. *Let L be a complete meet continuous semilattice. The following are equivalent.*

(1) *L is a complete continuous semilattice.*
(2) *ΛL has small open semilattices and $\sigma(L)$ is a continuous lattice.*
(3) *ΛL is a compact Hausdorff topological semilattice with small open semilattices.*
(3^1) *ΛL is a compact Hausdorff topological semilattice with small compact semilattices.*
(4) *$x = \sup\{\inf U \colon x \in U \in \lambda(L)\}$, for all $x \in L$.*

Proof: By III-2.13, condition (2) is equivalent to II-1.14(3), and by III-2.14, condition (4) is equivalent to II-1.14(7). Hence (1) iff (2) iff (4).

(1) implies (3): If L is continuous, then ΛL is compact Hausdorff by III-1.11. By III-2.8 and II-1.14, we know that ΛL is a topological semilattice, and from (2) we know that ΛL has small open semilattices.

That (3) implies (3^1) implies (4) is straightforward. □

In Section VI-3 we will see that any compact Hausdorff topological semilattice with small open semilattices has a complete continuous semilattice as underlying poset, and that its given topology is the Lawson topology. Thus Theorem III-2.15 constitutes an essential portion of a fundamental link between continuous semilattices and compact semilattices.

Exercises

Exercise III-2.16. Let L be a complete meet continuous semilattice. Show that the following statements are equivalent:

(1) L is algebraic;
(2) ΛL has small open semilattices and $\sigma(L)$ is algebraic;

(3) ΛL is a compact zero dimensional Hausdorff topological semilattice with small open–closed semilattices;
(4) ΛL is a zero dimensional Hausdorff topological semilattice.

Hint. For (1) iff (2) refer to III-2.15 and II-1.15, considering II-1.11 and III-2.13. In III-1.14 we showed (1) implies (3), and (3) implies (4) is trivial. Suppose (4) and take $x \in L$ and let U be a neighborhood of x. By (4) find an open compact neighborhood $V \subseteq U$; by continuity of multiplication and the fact that V and its complement are closed find a neighborhood W of x in V such that $WV \subseteq V$. Let $W^* = \bigcup\{W^n : n = 1, 2, \ldots\}$ and note that $W^* \subseteq V$. Now $\inf W^* = \lim W^* \in V^- = V$, whence III-2.15(4) is satisfied. Thus ΛL has small semilattices by III-2.15(2). If W is open–closed in ΛL, then $\uparrow W$ is open–closed, hence is a compact element in $\sigma(L)$; this shows that $\sigma(L)$ is algebraic. Thus we have shown (2). □

Exercise III-2.17. Let L be a continuous semilattice. Show that L equipped with the Lawson topology is a Hausdorff topological semilattice with small open semilattices.

Hint. Adapt the methods of the proof of Theorem III-2.15. □

Exercise III-2.18. Show that a **dcpo** for which the Scott topology has a basis of open filters is meet continuous. □

Exercise III-2.19. Let L be a **dcpo** that is a semilattice. Show that the following are equivalent:

(1) each point of L has a Scott neighborhood basis of open filters and $\sigma(L)$ is a continuous lattice;
(2) ΛL has small open semilattices, L is meet continuous, and $\sigma(L)$ is a continuous lattice;
(3) L is a domain, or equivalently, a continuous semilattice.

Hint. Modify the proof of III-2.13 and use III-2.17 and II-1.14(3). □

Old notes

As the diagram of "hierarchies" at the end of Chapter I indicates (see end of Section I-4), all of the theory directly relevant for continuous lattices takes place within the class of meet continuous lattices. To the discussions on the Lawson topology we have added the hypothesis of meet continuity in the present section and thereby obtained results such as III-2.11 and III-2.15. Theorem III-2.11 is due to Gierz and Lawson [SCS 42], but the proof here is more direct. The equivalence (1) iff (3) in III-2.15 is implicit in [Hofmann and Stralka, 1976] although without identification of the Lawson topology in explicit terms. Theorem III-2.8 originated in the *Compendium* (as did Proposition III-2.6); these results use once again the hypothesis that the Scott topology $\sigma(L)$ is a continuous lattice; the reason for this goes back to II-4.13.

New notes

H. Kou, Y.-M. Liu, and M.-K. Luo have recently extended the theory of meet continuity to general **dcpo**s, and we have modified the earlier treatment of this topic in the *Compendium* to reflect their work. Definition III-2.1 and Proposition III-2.3 are drawn from [Kou *et al.*, 2001].

III-3 Quasicontinuity and Liminf Convergence

In this section we consider those **dcpo**s, and in particular those complete lattices, for which the Lawson topology is Hausdorff. We have seen in the previous section that for meet continuous complete lattices these are precisely the continuous lattices. In this investigation we obtain a class of **dcpo**s called quasicontinuous domains, a generalization of domains, for which substantial portions of domain theory remain valid.

We also resume the theme of Section II-1 where we derived the Scott topology from liminf convergence and then discussed the relations between these notions. Here we refine the concept of liminf convergence and investigate its relationship to the Lawson topology. Since $\sigma(L) \subseteq \lambda(L)$, we expect fewer convergent nets to belong to the finer topology $\lambda(L)$.

Quasicontinuous domains

Definition III-3.1. We order the collection of nonempty subsets of a **dcpo** L by $G \leq H$ if $\uparrow H \subseteq \uparrow G$ (this is only a preorder, not an order, since it is typically not antisymmetric). We say that a family of sets is *directed* if given F_1, F_2 in the family, there exists F in the family such that $F_1, F_2 \leq F$, i.e., $F \subseteq \uparrow F_1 \cap \uparrow F_2$.

We say that G is *way below* H or G *approximates* H and write $G \ll H$ if for every directed set $D \subseteq L$, $\sup D \in \uparrow H$ implies $d \in \uparrow G$ for some $d \in D$. We write $G \ll x$ for $G \ll \{x\}$ and $y \ll H$ for $\{y\} \ll H$. Note that $y \ll x$ is unambiguously defined. Note also that $G \ll H$ iff $G \ll x$ for all $x \in H$. □

Definition III-3.2. A **dcpo** L is called a *quasicontinuous domain* if for each

$x \in L$ the family

$$\operatorname{fin}(x) = \{F\colon F \text{ is finite, } F \ll x\}$$

is a directed family and whenever $x \not\leq y$, then there exists $F \in \operatorname{fin}(x)$ with $y \notin \uparrow F$. □

Remark. Observe that a quasicontinuous domain is generally not actually a domain (domains are a special kind of quasicontinuous domain in which the collection of finite sets $F \ll x$ is replaced by a collection of singleton subsets). Example O-4.5(2) is a typical quasicontinuous domain which is not continuous. However, quasicontinuous domains have so many domain-like features that this terminology seems appropriate. If there is any danger of confusion, we can refer to domains as *continuous domains*.

The following lemma is crucial in the study of quasicontinuous domains.

Lemma III-3.3. (Rudin's Lemma) *Let $\mathcal{F}$ be a directed family of nonempty finite subsets of a partially ordered set P (see* III-3.1*). Then there exists a directed set $D \subseteq \bigcup_{F\in\mathcal{F}} F$ such that $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.*

Proof: Consider the collection of all $E \subseteq \bigcup_{F\in\mathcal{F}} F$ such that (i) $E \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, and (ii) $F,\ G \in \mathcal{F}$ and $G \subseteq \uparrow F$ imply $E \cap G \subseteq \uparrow(E \cap F)$. Such sets exist, specifically the union of all the F. Order all such sets by inclusion. By the Hausdorff Maximality Principle there exists a maximal chain of such subsets. Let D be the intersection. That D meets each F follows from the finiteness of F. The finiteness of members of $\mathcal{F}$ also readily yields that D satisfies (ii). Suppose that some $x \in D$ has the property that $(F \cap D)\backslash\uparrow x \neq \emptyset$ for all $F \in \mathcal{F}$. Then one verifies directly that $D\backslash\uparrow x$ again satisfies (i) and (ii), contradicting the minimality of D. Thus for all $x \in D$, there exists $F_x \in \mathcal{F}$ such that $(F_x \cap D) \subseteq \uparrow x$. Given any $x,\ y \in D$, then there exists $F \in \mathcal{F}$ beyond F_x and F_y, and thus $F \cap D \subseteq \uparrow(F_x \cap D) \cap \uparrow(Fy \cap D) \subseteq \uparrow x \cap \uparrow y$; hence D is directed. □

Corollary III-3.4. *Let $\mathcal{F}$ be a directed family of nonempty finite sets in a* **dcpo**. *If $G \ll H$ and $\bigcap_{F\in\mathcal{F}} \uparrow F \subseteq \uparrow H$, then $F \subseteq \uparrow\mathcal{F}$ for some $F \in \mathcal{F}$.*

Proof: Suppose not. Then the collection $\{F\backslash\uparrow G\colon F \in \mathcal{F}\}$ is a directed family of nonempty finite sets. By Rudin's Lemma there exists a directed set $D \subseteq \bigcup\{F\backslash\uparrow G\colon F \in \mathcal{F}\}$ such that $D \cap (F\backslash\uparrow G) \neq \emptyset$ for all $F \in \mathcal{F}$. Then

$$\sup D \in \bigcap_{d\in D} \uparrow d \subseteq \bigcap_{F\in\mathcal{F}} \uparrow(F\backslash\uparrow G) \subseteq \bigcap_{F\in\mathcal{F}} \uparrow F \subseteq \uparrow H.$$

Since $G \ll H$, there exists $d \in D$ such that $d \in \uparrow G$. But this contradicts $d \in F \backslash \uparrow G$ for some F. □

We now derive the interpolation property for quasicontinuous domains.

Proposition III-3.5. *Let S be a quasicontinuous domain. If $H \ll x$, then there exists a finite set F such that $H \ll F \ll x$.*

Proof: Consider the collection $\mathcal{G} = \{G\colon G$ is finite, there exists F finite such that $G \ll F \ll x\}$. If $x \not\leq z$, then there exists $F \ll x$, F finite, such that $z \notin \uparrow F$. For each $y \in F$, we can pick $F_y \ll y$, F_y finite, such that $z \notin \uparrow F_y$. Set $G = \bigcup_{y \in F} F_y$. It is straightforward to verify that the finite set G satisfies $G \ll F$ and $z \in \uparrow G$. Thus $G \in \mathcal{G}$ and $\bigcap_{G \in \mathcal{G}} \uparrow G \subseteq \uparrow x$.

Now suppose that $G_i \in \mathcal{G}$, $G_i \ll F_i \ll x$, for $i = 1, 2$. Since S is quasicontinuous, there exists $F \ll x$, F finite, such that $F \subseteq \uparrow F_1 \cap \uparrow F_2$. Then $G_i \ll F_i \leq F$ implies $G_i \ll F$ for $i = 1, 2$. Thus $G_i \ll y$ for all $y \in F$ and $i = 1, 2$. Again since S is quasicontinuous, by Corollary III-3.4 there exists $F_y \ll y$, F_y finite, such that $F_y \subseteq \uparrow G_i$ for $i = 1, 2$. Set $E = \bigcup_{y \in F} F_y$. Then $E \ll F \ll x$ and $E \subseteq \uparrow G_i$ for $i = 1, 2$. Thus the family $\mathcal{G}$ is directed. It then follows from Corollary III-3.4 that there exists some $G \in \mathcal{G}$ such that $G \subseteq \uparrow H$. Since $G \ll F \ll x$ for some finite F, we conclude that $H \ll F \ll x$. □

Proposition III-3.6. *Let S be a quasicontinuous domain.*

(i) *A subset U of S is Scott open iff for each $x \in U$ there exists a finite $F \ll x$ such that $\uparrow F \subseteq U$. The sets $\twoheaduparrow F = \{x\colon F$ is finite, $F \ll x\}$ are Scott open and they form a basis for the Scott topology.*

(ii) *For any nonempty set H in S, the set $\twoheaduparrow H$ is equal to the interior of $\uparrow H$ with respect to the Scott topology.*

Proof: Let U be Scott open, $x \in U$. From the definition of the Scott topology we have $U \ll x$, so by Proposition III-3.5 there exists a finite F such that $U \ll F \ll x$. In particular $F \subseteq \uparrow U = U$ and thus $\uparrow F \subseteq U$. Conversely suppose that for each $x \in U$, there exists a finite $F \ll x$ such that $\uparrow F \subseteq U$. Then in particular, $\uparrow x \subseteq U$, so U is an upper set. Let D be a directed set such that $x = \sup D \in U$. Then $d \in \uparrow F \subseteq U$ for some $d \in D$, and hence U is Scott open. The last assertion of part (i) follows from the first, once we know that the sets $\twoheaduparrow F$ are Scott open, and this we now establish as we prove part (ii).

Let H be a nonempty set in S. That $\twoheaduparrow H$ contains the Scott interior of $\uparrow H$ is clear from the definition of a Scott open set. Conversely suppose that $x \in \twoheaduparrow H$, i.e., $H \ll x$. By Proposition III-3.5 there exists a finite set $F \ll x$ such that $H \ll F$. Let $y \in \uparrow F$ and let D be a directed set such that $y \leq \sup D$. Then

$H \ll F$ implies that $d \in \uparrow H$ for some $d \in D$. Thus $\uparrow F \subseteq \twoheaduparrow H$. By the first part of (i), $\twoheaduparrow H$ is Scott open, and hence $\twoheaduparrow H$ is contained in the Scott interior of $\uparrow H$. Thus the two are equal. □

Proposition III-3.7. *Let S be a quasicontinuous domain.*

(i) *Endowed with the Scott topology, S is locally compact and sober.*
(ii) *Endowed with the Lawson topology, S is regular and Hausdorff.*

Proof: (i) Let $x \in U$, where U is Scott open in S. By Proposition III-3.6 there exists a finite set $F \subseteq U$ such that $\twoheaduparrow F$ is a Scott open neighborhood of x. Then $x \in \twoheaduparrow F \subseteq \uparrow F \subseteq U$, and $\uparrow F$ is compact in the Scott topology (since any open cover of F also covers $\uparrow F$). Thus S is locally compact.

Let A be a nonempty closed irreducible set. Consider the collection

$$\mathcal{F} = \{F\colon F \text{is finite, } \twoheaduparrow F \cap A \neq \emptyset\}.$$

Let $F, G \in \mathcal{F}$. Then the Scott open sets $\twoheaduparrow F$ and $\twoheaduparrow G$ both meet A, and since A is irreducible $\twoheaduparrow F \cap \twoheaduparrow G \cap A \neq \emptyset$. Let x be in the intersection; by Proposition III-3.6 there exists a finite set $E \subseteq \twoheaduparrow F \cap \twoheaduparrow G$ such that $E \ll x$. Then $E \in \mathcal{F}$ and $\uparrow E \subseteq \uparrow F \cap \uparrow G$. Thus the collection $\mathcal{F}$ is directed, and hence the collection $\{F \cap A\colon F \in \mathcal{F}\}$ is a directed collection of finite nonempty sets. We apply Rudin's Lemma (III-3.3) to this directed family and let s be the supremum of the resulting directed set D guaranteed by Rudin's Lemma. Since A is Scott closed, $s \in A$. Suppose that $y \in A$, but $y \not\leq s$. Then there exists a finite set $H \ll y$ such that $s \notin \uparrow H$. It follows that $H \in \mathcal{F}$ since $y \in \twoheaduparrow H$. But this is impossible since $s \in \uparrow F$ for all $F \in \mathcal{F}$. Hence $A = \downarrow s$.

(ii) Since a regular T_0 space is Hausdorff, we need only show that a quasicontinuous domain is regular for its Lawson topology. For regularity it suffices to check for subbasic closed sets missing a point x. Let $\uparrow y$ be a subbasic closed set missing x. There exists a finite set F such that $F \ll y$, but $x \notin \uparrow F$. By Proposition III-3.6, $\twoheaduparrow F$ is a Scott open set containing y, hence $\uparrow y$, and $S \backslash \uparrow F$ is a lower open set containing x which is disjoint from $\twoheaduparrow F$.

Now let A be a Scott closed set missing x. By Proposition III-3.6 there exists a finite set $F \ll x$ such that $F \subseteq S \backslash A$. Then $\twoheaduparrow F$ is a Scott open set containing x which is disjoint from the lower open set $S \backslash \uparrow F$, which contains A. Thus S with the Lawson topology is regular. □

Note that Proposition III-3.7(ii) generalizes Theorem III-1.10 to the more general setting of quasicontinuous domains. Note also that III-3.7(i) together with I-1.7(5) implies that the lattice $\sigma(L)$ of Scott open sets in a quasicontinuous domain is continuous.

Definition III-3.8. A lattice L is called a *quasicontinuous lattice* if it is complete and satisfies whenever $x \not\leq y$, then there is a finite $F \subseteq L$ with $y \notin \uparrow F$ and $F \ll x$. □

Counterexample O-4.5(2) provides a simple example of a complete lattice that is a quasicontinuous, but not a continuous, lattice.

Proposition III-3.9. *Let L be a complete lattice. If $X \ll Z$ and $Y \ll Z$, then $X \vee Y \ll Z$, where for short we write $X \vee Y = \{x \vee y : x \in X$ and $y \in Y\}$. Thus a quasicontinuous lattice is a quasicontinuous domain.*

Proof: The first assertion is straightforward and establishes that for a fixed x the set of all finite F such that $F \ll x$ is directed. Since the other condition for a **dcpo** to be a quasicontinuous domain is part of the definition of a quasicontinuous lattice, we are done. □

Proposition III-3.10. *A (continuous) domain is a quasicontinuous domain, and a meet continuous quasicontinuous domain is a domain.*

Proof: The first assertion is immediate from the Definition of a (continuous) domain. Suppose that L is a quasicontinuous domain, $x \in L$. By Proposition III-3.6(i) for each $F \ll x$, x is in the Scott interior of $\uparrow F$. By Lemma III-2.10 there exists at least one $y \in F$ such that $y \ll x$. Let $G_F = \{y \in F : y \ll x\}$. Then it is easy to verify that $\{G_F : F \ll x\}$ satisfies the hypotheses of Rudin's Lemma III-3.3, and it follows from the conclusion thereof that x is the directed supremum of elements way below it (that x is the supremum follows from $\bigcap\{\uparrow F : F \ll x\} = \uparrow x$). Hence L is a (continuous) domain. □

Theorem III-3.11. *For a complete lattice L, the following are equivalent:*

(1) *L is a quasicontinuous domain;*
(2) *L is a quasicontinuous lattice;*
(3) *the Lawson topology is Hausdorff.*

Proof: By Proposition III-3.7 (1) implies (3) and by Proposition III-3.9 (2) implies (1). Thus we show that (3) implies (2).

Let $x \in L$ and suppose that $x \not\leq y$. Let $w \in \uparrow x$. Then $wy < w$, so there exist disjoint basic Lawson open sets $U_i \backslash \uparrow F_i$ for $i = 1, 2$, such that $wy \in U_1 \backslash \uparrow F_1$ and $w \in U_2 \backslash \uparrow F_2$, where U_i is Scott open and F_i is finite for $i = 1, 2$. We may assume that $U_2 \subseteq U_1$ (otherwise replace U_2 with $U_2 \cap U_1$). Then by elementary set theory $U_2 \backslash \uparrow F_2 \subseteq \uparrow F_1$. Let F_w consist of all members z of F_1 such that $z \leq w$. Then $U_2 \backslash (\uparrow F_2 \cup \uparrow (F_1 \backslash F_w))$ is a Lawson open set contained in $\uparrow F_w$ and containing w. Note that $y \notin \uparrow F_w$, for otherwise $wy \in \uparrow F_w$, a contradiction to $wy \in U_1 \backslash \uparrow F_1$. Thus $\uparrow F_w$ contains a Lawson neighborhood of w, but misses

y. Since L is compact and $\uparrow x$ is Lawson closed, there exist finitely many of the $\uparrow F_w$, $x \leq w$, such that the union of their Lawson interiors contains $\uparrow x$. Then the union F of the finite collection of these finite F_w is a finite set, the Lawson interior of $\uparrow F$ contains $\uparrow x$, and $y \notin \uparrow F$. The fact that the Lawson interior of $\uparrow F$ contains $\uparrow x$ and the Lawson open sets have property (S) leads readily to the deduction that $F \ll x$. Thus L is a quasicontinuous lattice. □

Note that Proposition III-3.10 and Theorem III-3.11 provide an alternative proof to the assertion of Theorem III-2.11 that a complete lattice that is meet continuous and Hausdorff in the Lawson topology is a continuous lattice.

The Lawson topology and liminf convergence

In Section II-1 we described the Scott topology in terms of liminfs. We turn now to a similar undertaking for the Lawson topology. Recall that for L a **dcpo**, we say that $x \in L$ is the *liminf* of a net $(x_j)_{j \in J}$, written $x = \underline{\lim}\, x_j$, if (i) x is the supremum of all eventual lower bounds of the net and (ii) $x = \sup D$ for some directed set of eventual lower bounds (see Definition II-1.1).

Our first observation is purely order theoretical in view of the fact that the liminf is a purely order theoretical idea.

Proposition III-3.12. *Let L be a* **dcpo**, *$x \in L$ and $(x_j)_{j \in J}$ a net on L. Then the following statements are equivalent:*

(1) *$x = \underline{\lim}\, y_k$ for all subnets $y_k = x_{f(k)}$ of $(x_j)_{j \in J}$;*
(2) *$x = \underline{\lim}\, x_j$ and $x \geq z$ if z is a cofinal lower bound, i.e., if given $j \in J$, there exists $i \geq j$ such that $z \leq x_i$.*

Proof: (1) implies (2): The first assertion follows by considering the subnet consisting of the original net. For the second, let z be a cofinal lower bound. Consider the subnet of $(x_j)_{j \in J}$ consisting of those indices and elements such that $x_i \geq z$. Then x is the liminf and z is an eventual lower bound of this subnet, so $z \leq x$.

(2) implies (1): Let $(y_k)_{k \in K}$ be a subnet of $(x_j)_{j \in J}$ with $y_k = x_{f(k)}$. Let z be an eventual lower bound for (y_k). Then there exists $k' \in K$ such that $z \leq y_k$ for all $k \geq k'$. The set $\{f(k) : k \geq k'\}$ is cofinal in J by the definition of a subnet and $z \leq x_{f(k)} = y_k$ for each $k \geq k'$. Thus by hypothesis $z \leq x$. Therefore $\downarrow x$ contains all eventual lower bounds of $(y_k)_{k \in K}$.

Consider any eventual lower bound w of $(x_j)_{j \in J}$. Then there exists $j' \in J$ such that $w \leq x_j$ for $j \geq j'$. There exists $k^* \in K$ such that $f(k) \geq j'$ for $k \geq k^*$. Then w is a lower bound for $\{y_k = x_{f(k)} : k \geq k^*\}$. Thus the eventual lower bounds of $(x_j)_{j \in J}$ are also eventual lower bounds for $(y_k)_{k \in K}$. Hence any directed set D in the set of eventual lower bounds of $(x_j)_{j\in J}$ with supremum x is also a directed set of eventual lower bounds of $(y_k)_{k\in K}$ with supremum x. Thus $x = \underline{\lim}\, y_k$. □

Definition III-3.13. We consider the class $\mathcal{LI}$ of all pairs $((x_j)_{j\in J}, x)$ of nets on L and elements in L which satisfy the equivalent conditions of III-3.12. According to the discussion in Section II-1 this convergence notion determines a topology $\mathcal{O}(\mathcal{LI})$. The topology $\mathcal{O}(\mathcal{LI})$ is called the *liminf topology* and is written $\xi(L)$. We abbreviate $(L, \xi(L))$ as ΞL. □

We note immediately that, for any directed set D in a **dcpo** L, the element $x = \sup D$ and the net $(d)_{d\in D}$ satisfy III-3.12(2); whence $((d)_{d\in D}, \sup D) \in \mathcal{LI}$. Thus, if $U \in \xi(L)$ and $\sup D \in U$, then D is eventually in U; that is, U satisfies condition (S) of II-1.3. From II-1.4(v) we then derive immediately that a $\xi(L)$ open upper set is Scott open. Conversely if $((x_j)_{j\in J}, x) \in \mathcal{LI}$ and $x \in U$, a Scott open set, then some eventual lower bound z of $(x_j)_{j\in J}$ is in U, since $x = \underline{\lim}\, x_j$ is the supremum of some *directed* subset of eventual lower bounds. But then $x_j \in U$ for $x_j \geq z$. Thus U is $\xi(L)$ open, and we have the following parallel to III-1.6.

Proposition III-3.14. *Let L be a* **dcpo***.*

(i) *An upper set is $\xi(L)$ open if and only if it is Scott open.*
(ii) *A lower set is $\xi(L)$ closed iff it is closed under sups of directed sets.* □

It is standard that convergence structures given by nets have alternative descriptions as convergence structures given by filters (collections of nonempty subsets closed under taking supersets and finite intersections). In the case of liminf convergence we can say that a filter of sets $\mathcal{F}$ has an *eventual lower bound* z if $\uparrow z \in \mathcal{F}$ and declare x to be the *liminf* of the filter if $\downarrow x$ contains all eventual lower bounds and x is the supremum of a directed set of eventual lower bounds. Recall that an ultrafilter (of subsets) is a maximal filter and is characterized by the property that for any subset either it or its complement belongs to the ultrafilter. For the case of a complete semilattice the liminf always exists and is given by

$$\underline{\lim}\, \mathcal{F} = \sup\nolimits_{F\in\mathcal{F}} \inf F.$$

Lemma III-3.15. *Let L be a* **dcpo** *equipped with the $\xi(L)$-topology. Then every ultrafilter converges to its liminf (provided that it exists). A set $A \subseteq L$ is $\xi(L)$ closed iff for every ultrafilter $\mathcal{F}$ on L with $A \in \mathcal{F}$, one has $\underline{\lim}\, \mathcal{F} \in A$ whenever $\underline{\lim}\, \mathcal{F}$ exists.*

Proof: Let $\mathcal{F}$ be an ultrafilter with $z = \underline{\lim}\,\mathcal{F}$. We define a net on the set J of all pairs (x, F) with $x \in F \in \mathcal{F}$ ordered by $(x, F) \leq (y, G)$ if $G \subseteq F$ by $x_j = x$ for $j = (x, F)$. Then one sees easily that the net $(x_j)_{j\in J}$ has exactly the same set of eventual lower bounds as the ultrafilter $\mathcal{F}$, and hence that $z = \underline{\lim}\, x_j$. Suppose that w is a cofinal lower bound. Then the set $\{x_j\colon w \leq x_j\}$ is cofinal in the net; it follows that $\uparrow w$ meets every member of $\mathcal{F}$, and hence must be in $\mathcal{F}$ since $\mathcal{F}$ is an ultrafilter. Thus w is an eventual lower bound of the ultrafilter, and so $w \leq z$. It follows from Proposition III-3.12(2) that the pair $((x_j)_{j\in J}, z)$ is in $\mathcal{LI}$. Let U be any $\xi(L)$ open set containing z. Then there exists $i \in J$ such that $x_j \in U$ for $j \geq i = (y, F)$. If follows that each $x_j \in U$ for all $x_j = x, x \in F, j = (x, F)$, and hence that $F \subseteq U$. Thus the ultrafilter $\mathcal{F}$ converges to z.

Suppose that A is $\xi(L)$ closed and $\mathcal{F}$ is an ultrafilter on L having A as a member for which $\underline{\lim}\,\mathcal{F}$ exists. By the preceding paragraph the ultrafilter converges to its liminf, and hence the liminf must be in A, since A is closed.

Conversely, suppose that A contains the liminf of every ultrafilter $\mathcal{F}$ with $A \in \mathcal{F}$, provided $\underline{\lim}\,\mathcal{F}$ exists. Consider $((x_j)_{j\in J}, x) \in \mathcal{LI}$, and assume that all x_j are in A. The family $\{G_j\colon j \in J\}$, where we set $G_j = \{x_{j^*}\colon j \leq j^*\}$, is a filter basis on A. Let $\mathcal{F}$ be any ultrafilter containing all the G_j (every filter is contained in an ultrafilter). Clearly $\mathcal{F}$ contains A. If z is an eventual lower bound for $\mathcal{F}$, then $\uparrow z \in \mathcal{F}$. It follows that $\{j \in J : z \leq x_j\}$ is cofinal, and hence z is a cofinal lower bound for $(x_j)_{j\in J}$. By Proposition III-3.12(2), $z \leq x$. Conversely if z is an eventual lower bound for $(x_j)_{j\in J}$, then clearly it is an eventual lower bound for $\mathcal{F}$ (since tails of the net are members of $\mathcal{F}$). Thus any directed set of eventual lower bounds of the net $(x_j)_{j\in J}$ with supremum x is also a directed set of eventual lower bounds of $\mathcal{F}$ with supremum x, and therefore $x = \underline{\lim}\,\mathcal{F}$. By hypothesis $x \in A$, so A is $\xi(L)$ closed. $\square$

Remark. Note that the preceding lemma gives an alternative characterization of the liminf topology on a **dcpo**. One starts with the convergence notion of ultrafilters converging to their liminfs (when these exist), defines the topology from this notion of convergence, and obtains the liminf topology.

Corollary III-3.16. *In a complete lattice or complete semilattice L, the liminf of any filter $\mathcal{F}$ exists and is given by the directed supremum*

$$\underline{\lim}\,\mathcal{F} = \sup\nolimits_{F\in F} \inf F.$$

Hence in the case of a complete lattice or semilattice the $\xi(L)$-topology is compact.

Proof: Clearly for any $F \in \mathcal{F}$, $F \subseteq \uparrow\inf F$. Thus the latter set is in $\mathcal{F}$, and hence $\inf F$ is an eventual lower bound. Conversely if $\uparrow y \in \mathcal{F}$, then $y = \inf F$ for $F = \uparrow y \in \mathcal{F}$. If $\uparrow x, \uparrow y \in \mathcal{F}$, then their intersection is in $\mathcal{F}$, and hence nonempty. Thus the supremum z of x and y exists and is an eventual lower bound for the filter, since $\uparrow z = \uparrow x \cap \uparrow y$ must also be in $\mathcal{F}$. Thus the set of eventual lower bounds is directed, and its supremum is then the liminf and equals $\sup_{F \in \mathcal{F}} \inf F$. It now follows from Lemma III-3.15 that every ultrafilter converges in the liminf topology, and hence that L is compact. □

We are now prepared to consider the question of the relationship between the Lawson topology and liminf convergence.

Theorem III-3.17. *Let L be a* **dcpo***.*

(i) *The Lawson topology is contained in the liminf topology.*
(ii) *The Lawson topology and the liminf topology agree if L is a quasicontinuous domain. In this case an ultrafilter $\mathcal{F}$ converges to x iff $x = \underline{\lim}\,\mathcal{F}$.*
(iii) *In a continuous domain the liminf convergence is topological and agrees with convergence in the Lawson topology.*

Proof: (i) Let $\uparrow x$ be a principal filter. If $(x_j)_{j \in J}$ is a net in $\uparrow x$, then x is an eventual lower bound for the net, and hence the liminf of the net, if it exists, must be in $\uparrow x$. Hence $\uparrow x$ is closed in the liminf topology. From this fact and Proposition III-3.14 it follows that the Scott and lower topologies, and hence the Lawson topology, are contained in the liminf topology.

(ii) Let $\mathcal{F}$ be an ultrafilter converging to x in the Lawson topology. Set $A = \{z \in L : \uparrow z \in \mathcal{F}\}$. Note that each $z \in A$ satisfies $z \le x$ for otherwise $L \setminus \uparrow z$ would be a Lawson open set containing x, but not in the ultrafilter. Let G be a finite set such that $G \ll x$. Then $\twoheaduparrow G$ is a Scott open set containing x by Proposition III-3.6, thus $\uparrow G \in \mathcal{F}$, and hence $\uparrow y \in \mathcal{F}$ for some $y \in G$ (since if a finite union belongs to an ultrafilter, at least one member of the union must belong to it). It follows that $G \cap A \neq \emptyset$ for all finite $G \ll x$. Applying Rudin's Lemma to the collection $\{G \cap A : G \ll x, G \text{ is finite}\}$, we obtain a directed set D of eventual lower bounds whose supremum must be contained in $\bigcap\{\uparrow G : G \ll x, G \text{ is finite}\}$, which is contained in $\uparrow x$ (by Definition III-3.2). But since $z \le x$ for all $z \in A$, we conclude that $x = \sup D$ and therefore $x = \underline{\lim}\,\mathcal{F}$.

Conversely if $x = \underline{\lim}\,\mathcal{F}$, then $\mathcal{F}$ converges to x in the Lawson topology by Lemma III-3.15 and part (i). Thus the second assertion of (ii) is established, from which the first follows readily (since continuity is equivalent to preservation of limits of ultrafilters).

(iii) Suppose that L is a continuous domain and that $(x_j)_{j\in J}$ is a net converging to x in the Lawson topology. If $y \not\leq x$, then $L\backslash\uparrow y$ is a Lawson open set containing x, and hence there exists $j \in J$ such that $x_i \not\geq y$ for $i \geq j$. Thus y is not a cofinal lower bound for $(x_j)_{j\in J}$. Hence $y \leq x$ for every cofinal lower bound. If $z \ll x$, then $\twoheaduparrow z$ is a Scott and hence Lawson open set containing x, and thus z is an eventual lower bound for $(x_j)_{j\in J}$. Since x is the directed supremum of $\twoheaddownarrow x$, $x = \underline{\lim}\, x_j$. Thus x satisfies the two conditions of Proposition III-3.12(2) and hence $((x_j)_{j\in J}, x) \in \mathcal{LI}$. That $((x_j)_{j\in J}, x) \in \mathcal{LI}$ implies the net converges in the Lawson topology follows from part (i). □

We note that Corollary III-3.16 and Theorem III-3.17(i) give an alternative proof of the compactness of the Lawson topology for complete lattices or semilattices (compare III-1.9).

We close this section with another application of liminf convergence.

The Scott topology is robust in the sense that if one takes its join with the lower topology (or any topology of lower sets for that matter), then one recovers the Scott topology as the open upper sets (see III-1.6(i) and its proof). The lower topology is not so robust; see Exercise III-3.32. However if liminfs of ultrafilters exist, then one does have analogous results.

Proposition III-3.18. *Let L be a* **dcpo**.

(i) *The set of $\omega(L)$-cluster (equal convergence) points of an ultrafilter $\mathcal{F}$ is $\uparrow(\underline{\lim}\,\mathcal{F})$, provided $\underline{\lim}\,\mathcal{F}$ exists.*
(ii) *Suppose that every ultrafilter in L has a liminf (for example, L is a complete (semi)lattice). Then an upper set A in L is $\omega(L)$ closed iff it is Lawson closed iff it is $\xi(L)$ closed iff $\underline{\lim}\,\mathcal{F} \in A$ for all ultrafilters $\mathcal{F}$ with $A \in \mathcal{F}$.*

Proof: (i) The ultrafilter $\mathcal{F}$ converges to $y = \underline{\lim}\,\mathcal{F}$ in the $\xi(L)$-topology (Lemma III-3.15), hence in the Lawson topology, and thus in the lower topology. Therefore it converges to every member of $\uparrow y$, the closure of $\{y\}$ in the lower topology. Suppose that $y \not\leq x$, then $L\backslash\downarrow x$ is a Scott open set containing y and hence it must contain an eventual lower bound z. Then $\uparrow z \in \mathcal{F}$. Since $\uparrow z$ is closed for the topology $\omega(L)$, it contains all cluster points of the filter. Thus x is not a cluster point.

(ii) Let A be an upper set. Then A is $\omega(L)$ closed implies A is $\lambda(L)$ closed implies A is $\xi(L)$ closed (from III-3.17(i)). The last condition implies that for every ultrafilter $\mathcal{F}$ with $A \in \mathcal{F}$ we have $\underline{\lim}\,\mathcal{F} \in A$ by Lemma III-3.15. That the last condition implies that A is $\omega(L)$ closed follows from part (i). □

Exercises

Analogous to the case of domains, quasicontinuous domains have an alternative definition that only involves a family of finite sets way below each element, not all such sets.

Exercise III-3.19. Let L be a **dcpo**.

(i) Show that L is a quasicontinuous domain if for each $x \in L$ there exists a directed family $\mathcal{F}_x$ of finite sets such that $F \ll x$ for each $F \in \mathcal{F}_x$ and whenever $x \not\leq y$, then there exists $F \in \mathcal{F}_x$ with $y \notin \uparrow F$.
(ii) Show that L is a quasicontinuous domain if for each $x \in L$ and each Scott open set U containing x, there exists a finite set $F \subseteq U$ such that $x \in \operatorname{int}_\sigma \uparrow F$, the Scott interior of $\uparrow F$.

Hint. (i) First use Corollary III-3.4 to show that for each finite set $G \ll x$, there exists $F \in \mathcal{F}_x$ such that $F \subseteq \uparrow G$. Then use the directedness of $\mathcal{F}_x$ to show that the collection $\{G\colon G \text{ is finite}, G \ll x\}$ is directed.

(ii) For $x \in L$, apply part (i) to the collection of all finite F such that $x \in \operatorname{int}_\sigma \uparrow F$. Note that for $x \not\leq y$, $L \backslash \downarrow y$ is a Scott open set containing x. □

The next exercise may be viewed as a converse of the interpolation property given in Proposition III-3.5.

Exercise III-3.20. Let L be a **dcpo**. Suppose that for any nonempty $H \subseteq L$ and any $x \in L$, $H \ll x$ implies there exists a finite $F \subseteq \uparrow H$ such that $F \ll x$. Show that L is a quasicontinuous domain.

Hint. For $y \not\leq x$, note that $L \backslash \downarrow y \ll x$. Thus there exists $F \subseteq L \backslash \downarrow y$ such that $F \ll x$. If $F_1, F_2 \ll x$, then $\uparrow F_1 \cap \uparrow F_2 \ll x$, so there exists a finite $F \subseteq \uparrow F_1 \cap \uparrow F_2$ such that $F \ll x$. □

The next exercise is useful for verifying that examples such as O-4.5(2) are quasicontinuous lattices.

Exercise III-3.21. Show that any **dcpo** that has no infinite antichain is quasicontinuous.

Hint. Let $H \ll x$ and pick a maximal antichain A in $\uparrow H \backslash \uparrow x$. Then $F = A \cup \{x\}$ is finite, contained in $\uparrow H$, and satisfies $F \ll x$. Apply the previous exercise. To justify $F \ll x$, let D be a directed set with $x \leq \sup D$. Eventually D is in $\uparrow H$, since $H \ll x$. If $d \geq x$ for some d, we are done. Otherwise $D \cap \uparrow H \subseteq (\uparrow A \cup \downarrow A)$ by maximality of A. It can't be the case that $D \subseteq \downarrow A$, since $\sup D \geq x$, and thus $d \in \uparrow A \subseteq \uparrow F$ for some $d \in D$. □

Exercise III-3.22. Let L be the poset obtained by taking infinitely many disjoint copies of the unit interval (with the usual order) and gluing them together in 0 and 1. Show that L is a complete lattice, it has infinite antichains and is not a quasicontinuous lattice. □

Definition III-3.23. A **dcpo** L is a *quasialgebraic domain* if

$$\mathrm{compfin}(x) := \{F\colon F \text{ is finite, } F \ll F\}$$

is a directed family and whenever $x \not\leq y$, there exists $F \in \mathrm{compfin}(x)$ such that $y \notin \uparrow F$. □

Exercise III-3.24. Let L be a quasicontinuous domain. Prove the following.

(i) $\uparrow F$ is open–closed in the Lawson topology whenever F is finite and $F \ll F$.
(ii) A quasialgebraic domain is totally disconnected in the Lawson topology.
(iii) An upper set A is both open and compact in the Scott topology iff $A = \uparrow F$ for some finite $F \ll F$.
(iv) L is quasialgebraic iff the Scott topology has a basis of open sets that are also compact in the Scott topology. □

Exercise III-3.25. Let L be a complete semilattice. Show that the following statements are equivalent:

(1) L is a quasicontinuous domain;
(2) L^1, L with a largest element 1 adjoined, is a quasicontinuous lattice;
(3) the Lawson topology on L is Hausdorff.

Hint. For (3) implies (2), first note that L is Hausdorff implies that L^1 is Hausdorff (since 1 is isolated in the Scott, hence Lawson, topology), and then apply Theorem III-3.11. The other two needed implications follow directly from the text. □

Exercise III-3.26. If L is a complete lattice, or semilattice, with Hausdorff Lawson topology and if S is a subsemilattice of L, show that the following statements are equivalent.

(1) S is closed under directed sups and nonempty infs.
(2) S is Lawson closed.

Hint. First note that L is a quasicontinuous domain by Theorem III-3.11 or the previous exercise with Hausdorff Lawson topology. That (1) implies (2) follows by showing (1) implies S is closed in ΞL and then using III-3.17(ii). That (2) implies (1) follows as in III-1.12. □

Exercise III-3.27. For a complete lattice L show that the following is an equivalent condition to be a quasicontinuous lattice, where $\mathrm{fin}(x) := \{F\colon F$ finite and $F \ll x\}$:

for each $x \in L$ and each choice function $f \in \prod_{F \in \mathrm{fin}(x)} F$ we have
$x \leq \sup\{f(F)\colon F \in \mathrm{fin}(x)\}$. □

In the following exercise we develop for the lower topology $\omega(L)$ analogs of earlier derived results for $\sigma(L)$. A subset X of a **dcpo** L is said to have a property (Ω) if the following condition is satisfied:

(Ω) if $\underline{\lim}\,\mathcal{F} \in X$ for an ultrafilter $\mathcal{F}$, then $X \in \mathcal{F}$.

Compare this with II-1.3(S).

Exercise III-3.28. Let L be a **dcpo** in which every ultrafilter has a liminf, for example, a complete semilattice or complete lattice. Prove the following:

(i) a subset of L is ω open iff it is a lower set satisfying property (Ω) (compare II-1.4(v));
(ii) every Scott open set satisfies property (Ω) (compare II-1.4(vi));
(iii) if $\mathcal{B}$ is a collection of subsets satisfying property (Ω), then every subset in the topology generated by $\mathcal{B}$ also satisfies (Ω) (compare II-1.4(vii));
(iv) in the Lawson topology, the lower open sets are precisely the ω open sets (compare III-1.6).

Hint. Use III-3.18(ii) for part (i). For part (ii), let $x = \underline{\lim}\,\mathcal{F} \in U$, a Scott open set. Then x is the directed supremum of residual lower bounds of $\mathcal{F}$, and thus there exists $z \in U$ such that $\uparrow x \in F$. Hence $U \in F$. The remaining assertions are routine. □

The characterization theorems for the Hausdorff separation of the Lawson topology may be amplified as follows.

Exercise III-3.29. Show that a complete lattice L is quasicontinuous iff the following condition is satisfied:

(UF) for all ultrafilters $\mathcal{F}$ on L the set of $\sigma(L)$-cluster points of $\mathcal{F}$ is $\downarrow(\underline{\lim}\,\mathcal{F})$.

Remark. Note that $\downarrow(\underline{\lim}\,\mathcal{F})$ is always contained in the set of $\sigma(L)$-cluster points of $\mathcal{F}$ (see II-1.1,2,3).

Hint. (UF) implies III-3.11(3): By (UF) and III-3.18(i), the only $\lambda(L)$-cluster point of an ultrafilter $\mathcal{F}$ is $\underline{\lim}\,\mathcal{F}$.

III-3.6(i) implies (UF): If $x \not\leq \underline{\lim}\,\mathcal{F}$, then x has a $\sigma(L)$ open neighborhood U whose $\omega(L)$ closure C does not contain $\underline{\lim}\,\mathcal{F}$. By III-3.18(ii) we know $C \notin \mathcal{F}$, whence $U \notin \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, $L \backslash U \in \mathcal{F}$. Thus x is not a $\sigma(L)$-cluster point of $\mathcal{F}$. □

The next exercise exhibits anew that much of domain theory generalizes to quasicontinuous domains.

Exercise III-3.30. Let L be a quasicontinuous domain. Prove the following.

(i) If A is a Scott closed subset of L, then A is a quasicontinuous domain.
(ii) If M is a **dcpo** and $r\colon L \to M$ and $j\colon M \to L$ are Scott-continuous maps satisfying $rj = 1_M$, then M is a quasicontinuous domain.
(iii) If $f\colon L \to M$ is a Scott-continuous upper adjoint onto a **dcpo** M, then M is a quasicontinuous domain.
(iv) If L is additionally a complete lattice and $M \subseteq L$ is closed under nonempty infs and directed sups, then M is a quasicontinuous domain.
(v) The product $\prod_J L_j$ of quasicontinuous domains is a quasicontinuous domain, provided that at most finitely many fail to have a 0.
(vi) The class of quasicontinuous lattices is closed under the formation of subalgebras, products, and homomorphic images (see I-2.10).

Hint. For (ii), one sees directly that $F \ll j(y)$ in L implies $r(F) \ll y$ and that the family of such $r(F)$ for F finite is directed. Suppose that $y \not\leq w$. Then $r^{-1}(\downarrow w)$ is a Scott closed set missing $j(y)$, and thus there exists a finite F such that $F \ll j(y)$ and $F \cap r^{-1}(\downarrow w) = \emptyset$ (see III-3.6). Then $r(F) \ll y$ and $r(F)$ misses $\downarrow w$. Thus the intersection of all $\uparrow r(F)$ for F finite, $F \ll j(y)$ is contained in $\uparrow y$. Thus M is quasicontinuous.

For the other parts, adapt the techniques of Section I-2 from domains to quasicontinuous domains. □

Exercise III-3.31. Let L be a complete lattice. Prove the following.

(i) The ultrafilter $\mathcal{F}$ converges to x in the interval topology iff $\liminf \mathcal{F} \leq x \leq \limsup \mathcal{F}$.
(ii) The interval topology is Hausdorff iff for every ultrafilter $\mathcal{F}$, $\liminf \mathcal{F} = \limsup \mathcal{F}$. □

Hint. Apply Proposition III-3.18(i) and its order dual to obtain (i); (ii) then follows from (i) and the fact that a space is Hausdorff iff ultrafilters have unique limits. □

Exercise III-3.32. Let L be an infinite **dcpo** with trivial order, i.e., an infinite antichain. Show that the lower topology is the cofinite topology, but that the Lawson open lower sets are all subsets. Conclude that the Lawson topology may, in general, have more open lower sets than those of the lower topology.

Old notes

Theorem III-3.11 appears in [Gierz and Lawson, 1981], where quasicontinuous lattices were called generalized continuous lattices. The basic theory of liminf convergence for the special case of complete lattices is also due to [Gierz and Lawson, 1981]. Exercise III-3.30 is parallel to I-2.11, to O-3.15, O-4.2(1) and I-2.1 ff.

New notes

The preordering on subsets of an ordered sets as defined in III-3.1 is sometimes called the *Smyth order* [Smyth, 1978]. Quasicontinuous domains were introduced and their basic theory developed by Gierz, Lawson and Stralka [Gierz *et al.*, 1983b]. A crucial building block for extending the theory of generalized continuous lattices in the *Compendium* to the theory of quasicontinuous domains given in this section was the lemma of Rudin [Rudin, 1981]. Proposition III-3.10 was proved in the *Compendium* for complete lattices; the extension to domains was first proved in [Kou *et al.*, 2001]. Liminf convergence has been generalized from the case of complete lattices considered in the *Compendium* to the case of general **dcpo**s.

III-4 Bases and Weights

When domains appear in theoretical computer science, one typically wants them to be objects suitable for computation. In particular one is motivated to find a suitable notion of a recursive or recursively enumerable domain. This leads to the notion of a basis.

Definition III-4.1. Let L be a domain. A subset $B \subseteq L$ is called a *basis* of L iff

(i) $\twoheaddownarrow x \cap B$ is directed for all $x \in L$, and
(ii) $x = \sup(\twoheaddownarrow x \cap B)$ for all $x \in L$.

Likewise a subset B of a **dcpo** L is said to be a basis if these two properties are satisfied. $\square$

A **dcpo** has a basis iff it is continuous, that is, iff it is a domain. This follows from the second Remark after Definition I-1.6.

From I-4.2 we recognize immediately that for an algebraic domain L the set $K(L)$ of compact elements is a basis; conversely, if the compact elements form a basis of a **dcpo** L, then L is algebraic.

On the other hand taking $B = L$ always yields a basis if L is a domain. A basis is of course not uniquely determined in general. It is therefore useful to have several ways of recognizing a basis.

Proposition III-4.2. *Let B be a subset of a domain L. Then the following conditions are equivalent:*

(1) *B is a basis of L;*
(2) *for each $x \in L$, there exists a directed set $D \subseteq B \cap \mathord{\downarrow\!\!\downarrow} x$ such that $x = \sup D$;*
(3) *whenever $x \ll y$, there exists $b \in B$ with $x \leq b \ll y$;*
(4) *whenever $x \ll y$, there exists $b \in B$ with $x \ll b \ll y$;*
(5) *every element of L is the supremum of some directed subset of B.*

Proof: It is evident from Definition III-4.1 that (1) implies (2) and clearly (2) implies (5). Assume (5) and suppose $x \ll y$. By the interpolation property (Theorem I-1.9), there exists w such that $x \ll w \ll y$. Since w is the directed supremum of a subset of B, by Theorem I-1.9 again there exists $b \in B$ such that $x \ll b \leq w \ll y$. This establishes (4).

Clearly (4) implies (3). Assume (3) and let $x \in L$. Let $b_1, b_2 \in \mathord{\downarrow\!\!\downarrow} x \cap B$. Then there exists $y \ll x$ such that $b_1, b_2 \leq y$, since $\mathord{\downarrow\!\!\downarrow} x$ is directed. By (3) there exists $b \in B$ such that $y \leq b \ll x$. Thus $\mathord{\downarrow\!\!\downarrow} x \cap B$ is directed. It follows easily from the definition of a domain and property (3) that x is the supremum of $\mathord{\downarrow\!\!\downarrow} x \cap B$. Thus (1) is satisfied. $\square$

It follows from III-4.2(3) that a basis contains all the compact elements of the domain L. As corollary we obtain for algebraic domains L that $K(L)$ is the *unique* smallest basis.

The first part of the following proposition generalizes part of I-1.10.

Proposition III-4.3. *Let L be a domain with a basis B.*

(i) *The function $r_B = (J \mapsto \sup J)\colon \mathrm{Id}\, B \to L$ is a surjective map preserving directed sups whose domain $\mathrm{Id}\, B$ is an algebraic domain with*

$$K(\mathrm{Id}\, B) = \{\downarrow b \colon b \in B\} \cong B.$$

The lower adjoint of r_B is $x \mapsto \;\Downarrow x \cap B$. In the case that L is a continuous lattice and B is a sup subsemilattice, the map r_B preserves arbitrary infs and sups.

(ii) *Let* RId B *denote the set of all* rounded ideals *I of B, that is, those ideals with the property that for any $y \in I$, there exists $x \in I$ such that $y \ll x$. Then* RId B *is the image of the lower adjoint of r_B and the mapping $I \mapsto \sup I : \mathrm{RId}\, B \to L$ is an order isomorphism.*

Proof: (i) The map r_B is surjective by Definition III-4.1. Since the supremum of a directed family of ideals is its union, it is easy to see that r_B preserves directed sups.

Next we show that the function $x \mapsto \;\Downarrow x \cap B : L \to \mathrm{Id}\, B$ is a lower adjoint of r_B. Indeed if $x \in L$ and $J \in \mathrm{Id}\, B$, then $\Downarrow x \cap B \subseteq J$ iff $\Downarrow x \subseteq \downarrow J$ iff $x \leq \sup \downarrow J = \sup J$ by I-1.10, III-4.2 and O-1.5. Thus r_B is an upper adjoint and hence preserves arbitrary existing infs by O-3.3.

In the case that L is a continuous lattice and B is a sup subsemilattice, r_B preserves finite sups, hence arbitrary sups.

(ii) We consider the subposet RId B of Id B consisting of all rounded ideals. By III-4.2(4) $I := \;\Downarrow x \cap B$ is such a rounded ideal with $\sup I = x$ for every $x \in L$. Thus the lower adjoint of r_B maps L into RId B. Suppose that $\sup I = \sup J$ for two rounded ideals I and J. If $b \in I$, then $b \ll x$ for some $x \in I$, and hence $b \ll x \leq \sup I = \sup J$. Since J is directed $b \leq y$ for some $y \in J$, and hence $b \in J$ since J is an ideal. Thus $I \subseteq J$. Similarly $J \subseteq I$, and hence $J = I$. In particular for $x = \sup J = \sup I$, we must have $J = I = \;\Downarrow x \cap B$. Thus the image of the lower adjoint is precisely the set of rounded ideals. It now follows readily that $I \mapsto \sup I$ and $x \mapsto \;\Downarrow x \cap B$ are inverse order preserving isomorphisms between RId B and L. $\square$

Remark. Part (ii) of Proposition III-4.3 is quite important. It tells us that we can recover a domain from any basis and the restriction of $\ll$ to that basis by taking the *rounded ideal completion* of the basis B, that is, the domain RId B of all rounded ideals of B.

For domains which are not algebraic, there are no minimal bases comparable to $K(L)$ in the algebraic case. So there seems little hope of having a "canonical" basis in this case. While there is no minimal basis in general, at least the set of *cardinals* of bases of a domain has a minimum.

Definition III-4.4. Let L be a domain. The cardinal

$$w(L) = \min\{\mathrm{card}\, B \colon B \text{ is a basis of } L\}$$

is called the *weight* of the domain L. If $w(L)$ is countable, that is, if L has a countable basis, then the domain L is said to be *countably based.* □

Remark. As we remarked after III-4.2 an algebraic domain has a smallest basis, the set of all compact elements, and hence $w(L) = \text{card}\, K(L)$. Thus, an algebraic domain is countably based iff it has only countably many compact elements.

The countably based domains are important for applications in theoretical computer science and in computational models. In particular, a countably based domain L is said to be *effectively given* if a basis $B = \{x_n : n \in \mathbb{N}\}$ is specified such that $\{(m, n) \in \mathbb{N} \times \mathbb{N} : x_m \ll x_n\}$ is a recursively enumerable subset of $\mathbb{N} \times \mathbb{N}$. Since by Proposition III-4.3(ii) the basis B and the restriction of $\ll$ to B completely determine the domain (indeed the domain arises as the rounded ideal completion), restricting to such countable bases makes possible the development of a recursive theory for effectively given domains.

Recall that if X is a topological space, then the *weight* $w(X)$ is a cardinal, the smallest cardinality of a basis for the topology:

$$w(X) = \min\{\text{ card}\, \mathcal{B} \colon \mathcal{B} \text{ is a basis of the topology } \mathcal{O}(X)\}.$$

In particular, X satisfies the second axiom of countability iff $w(X) \leq \aleph_0$. (We recall (O-5.8) that a basis $\mathcal{B}$ of the topology is defined in the classical topological sense: given $x \in U$ open, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.)

In the following theorem the weights of ΣL and ΛL are weights in the traditional topological sense, while the other two weights are the weights of domains in the sense of Definition III-4.4.

Theorem III-4.5. *For any domain L one has*

$$w(L) = w(\Sigma L) = w(\Lambda L) = w(\operatorname{Id} B)$$

for any basis B of L with card $B = w(L)$. *In particular, L is countably based iff $(L, \sigma(L))$ is second countable iff $(L, \lambda(L))$ is second countable.*

Proof: In the finite case we must have $B = L$ and L and $\operatorname{Id} B$ are identified via $x \mapsto {\downarrow}x$. All four of the weights reduce to the cardinality of L, since a smallest basis for ΛL consists of all singletons and a smallest basis for ΣL consists of all ${\uparrow}x$, $x \in L$. Thus we restrict our attention to the case that L is infinite in the remainder of the proof.

$w(\Sigma L) \leq w(\Lambda L)$: Let $\mathcal{B}$ be a basis of ΛL of cardinality $w(\Lambda L)$. We claim $\{\operatorname{int}_\sigma({\uparrow}U) \colon U \in \mathcal{B}\}$ is a basis of ΣL. Indeed let $x \in V$, where V is Scott open. Since V is also Lawson open, there exists $U \in \mathcal{B}$ such that $x \in U \subseteq V$. Since U

has property (S), there exists $z \ll x$ such that $z \in U$. Then $x \in \mathord{\uparrow\!\!\uparrow} z \subseteq \uparrow U \subseteq V$. Thus the Scott interior of $\uparrow U$ contains x and is contained in V. Hence,

$$w(\Sigma L) \leq \operatorname{card}\{\operatorname{int}_\sigma(\uparrow U) \colon U \in \mathcal{B}\} \leq w(\Lambda L).$$

$w(\Lambda L) \leq w(L)$: Let B be a basis of L with card $B = w(L)$. For $b_0, b_1, \ldots, b_n \in B$ set

$$W(b_0, b_1, \ldots, b_n) = \mathord{\uparrow\!\!\uparrow} b_0 \backslash (\uparrow b_1 \cup \cdots \cup \uparrow b_n).$$

Then

$$\operatorname{card}\{W(b_0, \ldots, b_n) \colon b_0, \ldots, b_n \in B, n = 0, 1, 2, \ldots\} = \operatorname{card} B = w(L).$$

We claim that the $W(b_0, \ldots, b_n)$ form a basis of ΛL.

Indeed let $U \in \lambda(L)$, and take $x \in U$. Then there is a $\lambda(L)$ neighborhood $\mathord{\uparrow\!\!\uparrow} x_0 \backslash (\uparrow x_1 \cup \cdots \cup \uparrow x_n) \subseteq U$ of x (see II-1.10(i) and the remarks following III-1.5). Thus $x_0 \ll x$, and hence by III-4.2(3) we find a $b_0 \in B$ with $x_0 \leq b_0 \ll x$; that is, $x \in \mathord{\uparrow\!\!\uparrow} b_0 \subseteq \mathord{\uparrow\!\!\uparrow} x_0$. For each $k = 1, \ldots, n$ we have $x_k \not\leq x$; hence, by III-4.1 there are $b_k \ll x_k$ with $b_k \not\leq x$. Therefore

$$\begin{aligned} x \in W(b_0, b_1, \ldots, b_n) &= \mathord{\uparrow\!\!\uparrow} b_0 \backslash (\uparrow b_1 \cup \cdots \cup \uparrow b_n) \\ &\subseteq \mathord{\uparrow\!\!\uparrow} x_0 \backslash (\uparrow x_1 \cup \cdots \cup \uparrow x_n) \subseteq U. \end{aligned}$$

It follows that $w(\Lambda L) \leq w(L)$.

$w(L) \leq w(\Sigma L)$: Let $\mathcal{B}$ be a basis of ΣL of cardinality $w(\Sigma L)$. Let

$$\mathcal{A} := \{(U, V) \in \mathcal{B} \times \mathcal{B} : U \subseteq \uparrow x \subseteq V \text{ for some } x \in L\}.$$

Then card $\mathcal{A} \leq$ card $\mathcal{B}$. For each $\alpha = (U, V) \in \mathcal{A}$, pick some x_α such that $U \subseteq \uparrow x_\alpha \subseteq V$. We claim that $B := \{x_\alpha \colon \alpha \in \mathcal{A}\}$ is a basis for L. Indeed let $z \ll x$. By the interpolation property I-1.9 there exists y such that $z \ll y \ll x$. Pick $U, V \in \mathcal{B}$ such that $y \in V \subseteq \mathord{\uparrow\!\!\uparrow} z$ and $x \in U \subseteq \mathord{\uparrow\!\!\uparrow} y$. Then $(U, V) \in \mathcal{A}$, and hence there exists x_α for $\alpha = (U, V)$ such that $U \subseteq \uparrow x_\alpha \subseteq V$. It follows that $z \ll x_\alpha \ll x$ and thus B is a basis by III-4.2(4). Hence $w(L) \leq w(\Sigma L)$.

$w(L) = w(\operatorname{Id} B)$ for any basis B of L with card $B = w(L)$: We have $w(\operatorname{Id} B) = \operatorname{Card} K(\operatorname{Id} B)$ (since $\operatorname{Id} B$ is algebraic) $=$ card $B = w(L)$ (by III-4.3). $\square$

After Theorem III-4.5 we may say "the weight of a domain" or alternatively "the weight of the Scott topology" or "the weight of the Lawson topology".

Corollary III-4.6. *A domain L is countably based iff ΛL is a separable metrizable space.*

Proof: If L has a countable base, then ΛL has a countable base by the preceding theorem and is regular and Hausdorff by Proposition III-3.7. Then by the Urysohn Metrization Theorem (see e.g. [Kelley, B1955], p. 125) ΛL is separable metrizable. The converse also follows from the preceding theorem since a separable metric space has a countable base of open sets (see e.g. [Kelley, B1955], p. 120). □

Corollary III-4.7. *Every domain is (isomorphic to) the image of an algebraic domain with the same weight under a kernel operator preserving directed suprema. Every continuous lattice is the quotient of an algebraic (and even an arithmetic) lattice of equal weight by a quotient map preserving all sups and infs.* □

Proof: If L is finite, everything is trivial. If L is infinite, let B be a basis of cardinality $w(L)$. In the case that L is a continuous lattice, we may assume that B is a sublattice, since adding all finite meets and joins does not raise the cardinality. Then $\operatorname{Id} B$ is an algebraic domain (and an arithmetic lattice) by I-4.10. By Theorem III-4.5 we have $w(\operatorname{Id} B) = w(L)$; and using III-4.3 completes the proof. □

Lemma III-4.8. *Let X be a topological T_0 space whose topology $\mathcal{O}(X)$ is a continuous lattice. Then $w(X) = w(\mathcal{O}(X))$.*

Remark. See in this connection I-1.7(5), II-4.10 and Section V-5 below.

Proof of lemma: As finite T_0 spaces are nothing but finite posets with the Scott topology, there is nothing to prove in the finite case. Any basis of the continuous lattice $\mathcal{O}(X)$ is clearly a basis of the topological space X; hence $w(X) \le w(\mathcal{O}(X))$. Now let $\mathcal{B}$ be a basis of the topological space X with card $\mathcal{B} = w(X)$. If X is infinite and T_0, then $w(X)$ is infinite since points have distinct closures. We may assume (without changing cardinality) that $\mathcal{B}$ is closed under finite unions and contains $\emptyset$. It is then immediate that $\mathcal{B}$ satisfies III-4.2(5). Hence $w(\mathcal{O}(X)) \le w(X)$. □

We now calculate the weights of function spaces (see II-4.1).

Theorem III-4.9. *Let X and Y be T_0 spaces such that $\mathcal{O}(X)$ is a continuous lattice. If the weight of at least one of X and Y is infinite, then $w([X, Y]) \le \max\{w(X), w(Y)\}$, where $[X, Y]$ is the space of continuous functions equipped with the Isbell topology.* □

Proof: By the preceding lemma the continuous lattice $\mathcal{O}(X)$ has weight equal to $w(X)$, and then by Theorem III-4.5 the Scott topology on $\mathcal{O}(X)$ has a basis $\mathcal{B}$ of cardinality equal to $w(X)$. Let $w = \max\{w(X), w(Y)\}$. Let $g \in N(H \leftarrow V)$, where $H \subseteq \mathcal{O}(X)$ is a Scott open set, V is open in Y, and the continuous function $g\colon X \to Y$ satisfies $g^{-1}(V) \in H$. There exists $B \in \mathcal{B}$ such that $g^{-1}(V) \in B \subseteq H$. Let $\mathcal{C}$ be a basis for Y; we may assume that $\mathcal{C}$ is closed under finite unions and has cardinality less than or equal to w. Then V is a directed union of a collection $\mathcal{D}$ of members of $\mathcal{C}$, and therefore $g^{-1}(V)$ is the directed union of all $g^{-1}(D)$, $D \in \mathcal{D}$. Since B is Scott open and contains $g^{-1}(V) \in B$, there exists $D \in \mathcal{D}$ such that $g^{-1}(D) \in B$, i.e., $g \in N(B \leftarrow D) \subseteq N(H \leftarrow V)$. Thus the sets of the form $N(B \leftarrow D)$, $B \in \mathcal{B}$, $D \in \mathcal{C}$, form a basis for the Isbell topology on $[X, Y]$. Since the cardinality of $\mathcal{B} \times \mathcal{C}$ is equal to $\max\{w(X), w(Y)\}$, the proof is complete. □

Corollary III-4.10. *Let X be a T_0 space with $\mathcal{O}(X)$ continuous and let L be a continuous lattice, resp. a complete continuous semilattice, such that $w = \max\{w(X), w(L)\}$ is infinite. Then the continuous lattice, resp. complete continuous semilattice, $\Omega[X, \Sigma L]$ has Isbell topology equal to the Scott topology and the weight of this topology is equal to the weight of the continuous lattice, resp. complete continuous semilattice, $\Omega[X, \Sigma L]$, which in turn is less than or equal to w. In particular this holds for the case $X = \Sigma S$, where S is a domain.* □

Proof: That $\Omega[X, \Sigma L]$ is in fact a complete continuous semilattice, resp. continuous lattice, and that the Isbell topology on $[X, \Sigma L]$ agrees with the Scott topology on $\Omega[X, \Sigma L]$, was proved in II-4.6. That the weight of the space $[X, \Sigma L]$ is equal to that of the continuous lattice, resp. complete continuous semilattice, $\Omega[X, \Sigma L]$ then follows from Theorem III-4.5, and that this is less than or equal to w follows from the preceding theorem. □

For the case $X = \Sigma S$, where S is a domain, we have the following corollary that follows directly from the preceding one.

Corollary III-4.11. *Let S be an countably based domain and let L be a countably based continuous lattice, resp. complete continuous semilattice. Then $\Omega[\Sigma S, \Sigma L] = [S \to L]$ is a countably based continuous lattice, resp. complete continuous semilattice.* □

From the preceding we see that forming the space of self-maps does not raise weights for infinite continuous lattices. This will become relevant in Sections IV-3 and IV-4. We will now discuss how other basic constructions fare with respect to weights.

Proposition III-4.12.

(i) *Let* $\{L_j\colon j \in J\}$ *be a family of nonsingleton domains with* 0. *Then*

$$w\left(\prod_J L_j\right) = \sum_J w(L_j) = \max\{\text{card } J, \sup\{w(L_j)\colon j \in J\}\},$$

if at least one of J *and* L_j *is infinite, and*

$$w\left(\prod_J L_j\right) = \prod_J w(L_j)$$

if everything in sight is finite.

(ii) *If S, L are domains, if i:* $S \subseteq L$ *is Lawson continuous and injective, and if* ΛS *is compact, then*

$$w(S) \leq w(L).$$

In particular this is the case if L is a continuous lattice or complete continuous semilattice and S a subalgebra of L.

(iii) *If the domain S is the Scott-continuous image of a domain L, then*

$$w(S) \leq w(L).$$

Remark. For the notion of subalgebra see I-2.10.

Proof of proposition: (i) We may assume that one of J and L_j, for $j \in J$, is infinite. Let B_j be a basis of L_j of cardinality $w(L_j)$. Then the set of all $(b_j)_{j\in J}$ with $b_j \in B_j$ and all but a finite number of the b_j equal to 0 is a basis of $\prod_J L_j$ of cardinality $\sum_J w(L_j)$. Hence $w(\prod_J L_j) \leq \sum_J w(L_j)$. Since each factor is a retract with respect to the Scott topologies, say, the reverse inequality is clear from III-4.5.

(ii) We consider the Lawson topologies on S and L. Since the first topology is compact and the second Hausdorff (see III-1.10), the injection i is an embedding. Hence $w(\Lambda S) \leq w(\Lambda L)$. Then $w(S) \leq w(L)$ by III-4.5. For the case that S is a subalgebra, the inclusion map $\Lambda S \to \Lambda L$ is continuous by III-1.8 and ΛS is compact by I-1.11.

(iii) Let $g\colon L \to S$ be a surjective Scott-continuous map and take a basis B of L of cardinality $w(L)$. Then $g(B)$ satisfies Proposition III-4.2(5) and hence is a basis of S. Thus $w(S) \leq \text{card } B = w(L)$. □

Parts (i) and (ii) of III-4.12 together will enable us to calculate the weights of projective limits in the category of continuous lattices and maps preserving infs and directed sups (see IV-5.14).

It is another consequence of the preceding that for countably based continuous (algebraic) lattices L and M, the cartesian product $L \times M$ and the function space $[L \to M]$ are again countably based. Thus, in *DCPO* the full subcategories ω-*CONT* and ω-*ALG* of countably based continuous and algebraic lattices, respectively, are cartesian closed. The same holds for countably based bounded complete (algebraic) domains.

We introduce another standard cardinal invariant for topological spaces and apply it to domains. The density of a topological space X is the minimum of all cardinals card D, where D is a dense subset.

Definition III-4.13. Let L be a domain. Then the *density* of L is the cardinal $d(L) = \min\{\text{card}\, D\colon D \text{ is a dense subset of } \Lambda L\}$. □

Proposition III-4.14. *Let L be a domain. Then $w(L) \leq 2^{d(L)}$, and equality can, but need not, occur, even for continuous lattices.*

Proof: In a regular topological space X with dense subset D, the mapping $A \mapsto \operatorname{int} \bar{A} : 2^D \to \mathcal{O}(X)$ always maps onto a basis of X, and thus $w(\Lambda L) \leq 2^{d(L)}$ by Proposition III-3.7. The asserted inequality follows from Theorem III-4.5.

We remark next that equality may be attained: let X be an arbitrary infinite set, and let βX be the Stone–Čech compactification of the discrete space X. We let $L = \mathcal{O}(\beta X)$ and we know that L is a continuous lattice (see I-1.7(5)); in fact, it is arithmetic with $K(L) =$ lattice of compact open subsets of $\beta X \cong 2^X$. Thus,

$$w(L) = \text{card}\, K(L) \text{ (by III-4.4)} = \text{card}\, 2^X = 2^{\text{card}\, X}.$$

If we let F denote the set of all finite subsets of $X \subseteq \beta X$, then it is a straightforward exercise to show that F is dense in L with respect to $\lambda(L)$. Thus $d(L) \leq \text{card}\, X$; whence,

$$w(L) \leq 2^{d(L)} \text{ (by what was shown above)} \leq 2^{\text{card}\, X} = w(L).$$

Hence $w(L) = 2^{d(L)}$ in this example, and $d(L)$ can be any infinite cardinal. In order to show that the inequality $w(L) < 2^{d(L)}$ can occur, we have many possibilities to choose from. If Ω is the first uncountable ordinal and $L = [1, \Omega]^{\text{op}}$, then L is algebraic and $K(L) = L$. Whence,

$$w(L) = \text{card}\, K(L) = \text{card}\, L = \aleph_1;$$

but since no countable subset of $[1, \Omega]$ can be dense in the interval topology (which agrees with the Lawson topology), we know $\aleph_1 \leq d(L) \leq \text{card}\, L = \aleph_1$. Thus $w(L) = \aleph_1 < 2^{\aleph_1} = 2^{d(L)}$.

One can also show easily that the standard Cantor set C provides an example $L = C$ with

$$w(L) = \text{card}\, K(L) = \aleph_0 < 2^{\aleph_0} = 2^{d(L)}. \qquad \square$$

Exercises

Domains can be described completely in terms of their bases. In a set B with a binary relation $\prec$ we will use the following notation: for every subset F and every element z of B we write

$$F \prec z \quad \text{iff} \quad x \prec z \ \text{ for all } \ x \in F.$$

Definition III-4.15. We define an *abstract basis* to be a nonempty set B together with a binary relation $\prec$ which is transitive and satisfies the following *finite interpolation property*:

(FIP) for every finite subset F and every element $z \in B$ one has

$$F \prec z \Rightarrow (\exists y \in B)\ F \prec y \prec z.$$

The relation $\prec$ is said to satisfy the *separation property* if

(SEP) $x \neq y$ implies there is a z such that $z \prec x$ but $z \not\prec y$ or vice versa.

A subset I of an abstract basis B is called a *rounded ideal* if

(i) I is directed, that is, for every finite subset $F \subseteq I$ there is a $z \in I$ such that $F \prec z$,
(ii) I is a $\prec$-lower set, that is $x \prec y$ and $y \in I$ together imply $x \in I$.

By RId B we denote the set of all rounded ideals of B ordered by inclusion. $\square$

Let us show now, firstly, that every basis of a domain L can viewed as an abstract basis and that the domain L can be recaptured from its basis through the rounded ideals and, secondly, that every abstract basis can be viewed as the basis of a domain, its *rounded ideal completion*. The proofs generalize that of III-4.3(ii).

Exercise III-4.16. Let B be a basis of a domain L.

(i) If we restrict the way-below relation $\ll$ on L to the basis B, show that it satisfies the axioms of an abstract basis.
(ii) For every $x \in L$, show that $I_x = \twoheaddownarrow x \cap B$ is a rounded ideal of B.

(iii) Show that $x \mapsto I_x : L \to \mathrm{RId}\, B$ is an order isomorphism, the inverse map being $I \mapsto \sup I : \mathrm{RId}\, B \to L$. $\square$

Exercise III-4.17. Now let B be any abstract basis.

(i) Show that the set $\mathrm{RId}\, B$ of rounded ideals ordered by inclusion is a **dcpo** (with directed suprema given by union), called the *rounded ideal completion* of B.

(ii) For $b \in B$, show that $I_b = \{a \in B : a \prec b\}$ is a rounded ideal of B, and that $b \mapsto I_b : B \to \mathrm{RId}\, B$ is injective if $\prec$ satisfies the separation property.

(iii) Show that $I \ll J$ holds in $\mathrm{RId}\, B$ iff there are elements $a \prec b$ in B such that $I \subseteq I_a \subseteq I_b \subseteq J$. In particular, $I_a \ll I_b$ iff $a \prec b$.

(iv) Conclude that the rounded ideals form a domain with the ideals $I_b, b \in B$, as a basis. $\square$

Refining the properties of the abstract bases leads to special classes of domains. Every poset B, for example, is an abstract basis; the rounded ideals then coincide with the ideals of the poset and the two preceding exercises reduce to the description of algebraic domains as ideal completions of their posets of compact elements (see I-4.10). The following exercise shows that the rounded ideal completion can be viewed as a possible generalization of the completion of the rationals by Dedekind cuts.

Exercise III-4.18. Consider the set $\mathbb{Q}$ of rationals with the usual strict order $<$. Show that this is an abstract basis and that the rounded ideals correspond bijectively to Dedekind cuts (including the improper Dedekind cut $(\mathbb{Q}, \emptyset)$). Thus the rounded ideal completion of $(\mathbb{Q}, <)$ is $\mathbb{R} \cup \{+\infty\}$. $\square$

Exercise III-4.19. For a compact Hausdorff space X consider the set $C(X, \mathbb{R}_+)$ of all nonnegative continuous real-valued functions with the relation $f \prec g$ iff $f(x) \prec g(x)$ for all $x \in X$. Show that the properties of an abstract basis are satisfied and that the rounded ideal completion of $C(X, \mathbb{R}_+)$ is isomorphic to the continuous lattice $\mathrm{LSC}(X, \mathbb{R}_+^*)$ of all nonnegative lower semicontinuous extended real valued functions. $\square$

Exercise III-4.20. A function $f\colon L \to M$ of **dcpo**s is called *countably continuous* or *ω-continuous* for short, if it is monotone and preserves suprema of ω-chains, that is, if for every monotone increasing sequence $x_0 \leq x_1 \leq x_2 \leq \cdots$ in L one has $f(\sup_n x_n) = \sup_n f(x_n)$.

For countably based domains L and M show the following.

(i) Every element $x \in L$ is the sup of a monotone sequence $b_0 \ll b_1 \ll b_2 \ll \cdots$.

(ii) A function $f\colon L \to M$ is Scott-continuous iff it is countably continuous. $\square$

Exercise III-4.21. Using II-2.15 through II-2.19 on *FS*-domains, show the following.

(i) A **dcpo** L is a countably based *FS*-domain iff there is an ascending sequence $\delta_0 \leq \delta_1 \leq \cdots$ of finitely separating Scott-continuous self-maps of L with $\sup_n \delta_n = 1_L$.
(ii) $L \times M$ and $[L \to M]$ are countably based, if L and M are countably based *FS*-domains.
(iii) Thus the category of countably based *FS*-domains and Scott-continuous maps is cartesian closed. $\square$

Exercise III-4.22. Using II-2.20 through II-2.24 on bifinite domains, show the following.

(i) A **dcpo** L is a countably based bifinite domain iff there is an ascending sequence $\delta_0 \leq \delta_1 \leq \cdots$ of Scott-continuous kernel operators with finite range on L with $\sup_n \delta_n = 1_L$.
(ii) Thus the category of countably based bifinite domains and Scott-continuous maps is cartesian closed. $\square$

Exercise III-4.23. Let L, resp. M, be L-domains consisting of a countable, resp. two element, antichain together with two incomparable lower bounds and a bottom element (thus M is a five element L-domain). Show that the order preserving functions carrying the bottom three elements of L onto the bottom three elements of M and the infinite antichain onto the two element antichain form an uncountable collection of compact elements of $[L \to M]$, and hence the weight of the L-domain $[L \to M]$ is uncountable. Thus the category of countably based L-domains with smallest elements and Scott-continuous maps is not cartesian closed. Note, however, by Theorem III-4.9 that $[L, M]$ with the Isbell topology has countable weight. $\square$

Yu. Eršov gave an early variant of domain theory in a topological, rather than an order theoretic, framework [Eršov, 1973]. The following exercise gives the axioms for the topological spaces he studied.

Exercise III-4.24. Let X be a T_0 space with order of specialization $\leq$. We write $x \prec y$ if $y \in \operatorname{int}\uparrow x$ and set $\uparrow x := \{y\colon x \prec y\}$. We say that X is an *A-space* if there exists $X_0 \subseteq X$ satisfying the following conditions:

(1) if $x, y \in X_0$ have an upper bound z in X, then the supremum $x \vee y$ exists in X and is again in X_0;

(2) if $x \in X_0$, $z \in X$ and $x \prec z$, then there exist $y \in X_0$ such that $x \prec y$ and $y \prec z$;
(3) the collection $\{\Uparrow x \colon x \in X_0\}$ is a basis for the topology of X.

If additionally X has a bottom element 0, then X is called an A_0-*space*.

(i) Show that $\prec$ satisfies the properties (FIP) and (SEP) of Definition III-4.15.
(ii) Show that the rounded ideal completion of an A_0-space $(X, \prec)$ is a bounded complete domain.
(iii) Show that the map $b \mapsto I_b : X \to \mathrm{RId}(X)$ of Exercise III-4.17 is a homeomorphic embedding of X into the rounded ideal completion.

Hint. See [Eršov, 1973]. □

Problem. Develop a theory of weights for arbitrary **dcpo**s and arbitrary complete lattices (cf. III-4.2). □

Old notes

The cardinality results on bases and weights in this section were largely due to Hofmann. Bases for continuous lattices were considered by Scott and by Eršov [Scott, 1972a; Eršov, 1972a,b]. A forerunner of III-4.14 for algebraic lattices was given by [Hofmann *et al.*, B1974].

New notes

Exercises III-4.16 and III-4.17 contain material which – with variations – is due to M. Smyth [Smyth, 1978] (see also [SCS 4], [SCS 12], [SCS 13], [Vickers, B1989], [Abramsky, 1991b], [Abramsky and Jung, 1994]); obviously we have here a variation of the theme of auxiliary relations in I-1.10 ff. These results provide an axiomatic characterization of bases of continuous domains, the objects of study in recursive domain theory.

Substantial portions of domain theory, particularly bounded complete domain theory, were developed at an early stage in [Eršov, 1973] in the context of what were called A-spaces (see Exercise III-4.24); these spaces may be viewed as topological variants of Scott's "information systems". In particular one finds in [Eršov, 1973] cartesian closed categories, rounded ideal completions, existence and continuity of the fixed-point mapping, the fact that a directed complete A_0-space is a bounded complete domain with the Scott topology, and theorems about injectivity and dense injectivity.

III-5 Compact Domains

In the theory of domains, those that are compact in the Lawson topology are of particular interest. We have already seen in III-1.11 that continuous lattices and complete continuous semilattices are included in this class of domains. In this section we derive some of the basic properties satisfied by compact domains and then consider various characterizations of such domains.

We begin with a more general setting that is useful primarily because the symmetry of the situation allows one to derive results about L and L^{op} simultaneously.

Definition III-5.1. Let L be a poset equipped with a topology. The partial order is *semiclosed* if $\downarrow x$ and $\uparrow x$ are both closed for all $x \in L$. □

Remark. A partial order is semiclosed iff the topology refines the lower and the upper topology, hence also the interval topology. A **dcpo** with its Lawson topology has a semiclosed partial order. Note for a fixed topology on a poset L that the partial order is semiclosed if and only if its opposite $\geq$ is semiclosed.

Lemma III-5.2. *Let L be a poset equipped with a compact topology for which the partial order is semiclosed.*

(i) *A directed set (resp. filtered set) converges to its supremum (resp. infimum), and this convergence point is the unique cluster point of the set.*
(ii) *L and* L^{op} *are* **dcpo***s.*
(iii) *For any* $x \in L$*, there exist a maximal element of L above x and a minimal element of L below x.*

Proof: (i) Let D be a directed set. By compactness D (viewed as a net indexed by itself) must have a cluster point y. Any cluster point of D must belong to the closed set $\uparrow d$ for any $d \in D$ (since the net is eventually in $\uparrow d$), and hence y is an upper bound for D. Let z be any other upper bound. Then the closed set $\downarrow z$ contains D and thus all its cluster points. Hence y is the least upper bound. Since any cluster point must be $\sup D$, D has a unique cluster point, and hence by compactness D converges to $\sup D$. The assertions for filtered sets are dual.

(ii) This is immediate from part (i).

(iii) This is a direct application of Zorn's Lemma, since every chain must have a supremum and infimum by part (ii). □

Definition III-5.3. An element $m \in L$, a poset, is a *minimal upper bound* (or "mub" for short) for a subset A if m is an upper bound for A that is minimal in the set of all upper bounds of A. The poset L is *mub-complete* if given any finite subset F of L and any upper bound u of F, there exists a minimal upper bound y of F such that $y \leq u$. $\square$

Proposition III-5.4. *Let L be a poset equipped with a compact topology for which the order is semiclosed. Then L and L^{op} are mub-complete.*

Proof: Let F be a finite set contained in $\downarrow u$. Then $S := \downarrow u \cap \bigcap_{x \in F} \uparrow x$ is closed, hence compact. Thus by Lemma III-5.2(iii) applied to S, there exists a minimal element of S, which is a minimal upper bound of F below u. $\square$

We consider now a variety of conditions that are equivalent to compactness of the Lawson topology in a domain. The first comes directly from the results of Section III-3.

Theorem III-5.5. *In order that a* **dcpo** *L be compact in the Lawson topology, it is sufficient that $\underline{\lim}\,\mathcal{F}$ exist for every ultrafilter $\mathcal{F}$. If L is a quasicontinuous domain, this condition is also necessary.*

Proof: By Lemma III-3.15 and Theorem III-3.17(i) if $\underline{\lim}\,\mathcal{F}$ exists for an ultrafilter $\mathcal{F}$, then $\mathcal{F}$ converges to $\underline{\lim}\,\mathcal{F}$. But if every ultrafilter converges, then L is compact.

Conversely suppose that L is a quasicontinuous domain with a compact Lawson topology. Then every ultrafilter must converge (by compactness) and the point of convergence is $\underline{\lim}\,\mathcal{F}$ by Theorem III-3.17(ii). $\square$

We now derive our most extensive collection of results concerning compactness in domains. Since the first derivations are only marginally more dificult in the setting of quasicontinuous domains, we derive our results in that context. We recall the notation of Section III-3, and in particular Proposition III-3.6, which we use freely.

Lemma III-5.6. *In a* **dcpo** *L, if an upper set A is a directed intersection of finitely generated upper sets, then it is compact in the Scott topology.*

Proof: Let $\{\uparrow F_i : i \in I\}$ be a descending family of finitely generated upper sets with intersection A. Let $\{U : U \in \mathcal{U}\}$ be a cover of A with Scott open sets and let W denote their union. Then W is a Scott open set containing A. If $F_i \setminus W \neq \emptyset$ for all i, then the family $\{\uparrow(F_i \setminus W) : i \in I\}$ satisfies the hypothesis of Rudin's Lemma (III-3.3). Hence there exists a directed subset D of $\bigcup_{i \in I} F_i \setminus W$ which intersects each $F_i \setminus W$. Let d be the supremum of D. Then $d \in \uparrow F_i$ for each i, and hence $d \in A$. But $D \subset L \setminus W$, a Scott closed set, implies d is not in W, a contradiction. Thus $\uparrow F_i \subseteq W$ for some i. Since finitely many of the $U \in \mathcal{U}$ contain F_i, they contain $\uparrow F_i$, and hence A. $\square$

The next lemma provides a converse to the proceeding one.

Lemma III-5.7. *Let L be a quasicontinuous domain. If $A = \uparrow A$ is compact in the Scott topology, then every Scott open neighborhood U of A contains a finite set F such that $A \subseteq \twoheaduparrow F \subseteq \uparrow F \subseteq U$. Furthermore, A is a directed intersection of all finitely generated upper sets that contain A in their Scott interior.*

Proof: Suppose $A = \uparrow A$ is compact in the Scott topology and has U for a Scott open neighborhood. For each $x \in A$, there exists a finite set $F_x \subseteq U$ such that $x \in \twoheaduparrow F_x$. Finitely many of the $\twoheaduparrow F_x$ cover A, and the union F of the finitely many F_x satisfies $\uparrow F \subseteq U$ and $A \subseteq \twoheaduparrow F$.

Consider the family $\mathcal{F}$ of all finite sets F such that A is contained in the Scott interior of $\uparrow F$. Given $F_1, F_2 \in \mathcal{F}$, then U, the intersection of the Scott interiors of $\uparrow F_1$ and $\uparrow F_2$, is a Scott open set containing A. By the preceding paragraph there exists a finite set $F \subseteq U$ such that $A \subseteq \twoheaduparrow F$, and thus A is contained in the Scott interior of $\uparrow F$. Hence the family $\mathcal{F}$ is directed.

For $z \notin A$, the set $U = L \backslash \downarrow z$ is a Scott open set containing A, and one can repeat the preceding argument to find $F \subseteq U$ such that $F \in \mathcal{F}$. Thus the intersection of $\mathcal{F}$ is A. $\square$

We come now to our main theorem on compactness in quasicontinuous domains.

Theorem III-5.8. *Let L be a quasicontinuous domain. The following statements are equivalent.*

(1) *The Lawson topology on L is compact.*
(2) *The sets closed in the lower topology are compact in the Scott topology.*
(3) *Every set closed in the lower topology is a directed intersection of finitely generated upper sets.*
(4) *The Scott compact upper sets are precisely the sets closed in the lower topology.*
(5) *L is a finitely generated upper set in which the intersection of two Scott compact upper sets is again Scott compact.*
(6) *L is a finitely generated upper set and $\uparrow x \cap \uparrow y$ is compact in the Scott topology for all $x, y \in L$.*

Proof: (1) implies (2): Sets closed in the lower topology are closed in the Lawson topology, hence compact in the Lawson topology, and thus compact in the Scott topology.

(2) implies (3): This follows immediately from (2) and Lemma III-5.7.

(3) implies (4): It follows from Lemma III-5.7 that every Scott compact upper set is closed in the lower topology. Conversely Lemma III-5.6 and (3) imply every set closed in the lower topology is Scott compact.

(4) implies (5): Two Scott compact upper sets are also closed in the lower topology, hence their intersection is closed in the lower topology, and thus compact by assumption. Since L is closed in the lower topology, hence Scott compact, it follows from Lemma III-5.7 that L is a finitely generated upper set.

(5) implies (6): This is immediate, since $\uparrow x$ and $\uparrow y$ are clearly Scott compact.

(6) implies (1): Suppose that L satisfies (6). For any finite set $F \subseteq L$ and any $x \in L$, $\uparrow F \cap \uparrow x = \bigcup_{y \in F}(\uparrow y \cap \uparrow x)$ is a finite union of Scott compact sets, hence Scott compact. Since by Lemma III-5.7 any Scott compact upper set A is a filtered intersection of finitely generated upper sets, it follows that $A \cap \uparrow x$ is a filtered intersection of Scott compact upper sets. Since L endowed with the Scott topology is sober (Proposition III-3.7(i)), it follows from Corollary II-1.22 that $A \cap \uparrow x$ is Scott compact. Thus by induction and (6) any finite intersection of principal filters is Scott compact.

Consider any subbasic open cover $\mathcal{U}$ of L in the Lawson topology consisting of open sets which are either Scott open sets U_j or sets of the form $L \backslash \uparrow x_i$, $i \in I$. Let U denote the union of all the Scott open sets U_j. Since L is of the form $\uparrow Z$ for some finite Z, if $U = L$, then finitely many of the Scott open sets U_j contain Z and hence all of L.

If $U \neq L$, then let $A = \bigcap_{i \in I} \uparrow x_i$; note that A is the complement of the union of all sets $L \backslash \uparrow x_i$ in the given cover $\mathcal{U}$. Since $\mathcal{U}$ is a cover, $A \subseteq U$. Let

$$\mathcal{F} := \left\{\uparrow x_{i_1} \cap \cdots \cap \uparrow x_{i_n} : i_k \in I \text{ for } k = 1, \ldots, n\right\}.$$

Then $\mathcal{F}$ consists of Scott compact sets (by the first paragraph in the proof of this implication) and is a filtered family of upper sets. Since L endowed with the Scott topology is sober (Proposition III-3.7(i)), it follows from Proposition II-1.21(3) that some member $\uparrow x_{i_1} \cap \cdots \cap \uparrow x_{i_n}$ of $\mathcal{F}$ is contained in U. Since by hypothesis this finite intersection is Scott compact, finitely many of the Scott open members of $\mathcal{U}$ cover it, and these open sets together with the $L \backslash \uparrow x_{i_k}$, $i = k, \ldots, n$, form a finite subcover. By the Alexander Subbasis Lemma L is compact in the Lawson topology. □

In Theorem II-1.14 we characterized a domain in terms of its lattice of Scott open sets. We carry out an analogous undertaking for compact domains.

Definition III-5.9. The way-below relation $\ll$ in a continuous (semi)lattice L is said to be *multiplicative* if $a \ll b, c$ implies $a \ll b \wedge c$. In this case we call L a *stably continuous* (semi)lattice. □

Remark. Note that $a_i \ll b_i$ for $i = 1, 2$ implies $a_1 \wedge a_2 \ll b_1 \wedge b_2$ if $\ll$ is multiplicative, since $a_1 \wedge a_2 \leq a_i \ll b_i$ implies $a_1 \wedge a_2 \ll b_i$ for $i = 1, 2$, and hence $a_1 \wedge a_2 \ll b_1 \wedge b_2$ by the preceding definition.

Proposition III-5.10. *A quasicontinuous domain L is compact in the Lawson topology iff L is finitely generated as an upper set and the lattice $\sigma(L)$ of Scott open sets is stably continuous.*

Proof: Suppose that L is compact in the Lawson topology. Let $W \ll U$, V in $\sigma(L)$. By Proposition I-1.4(ii) and Proposition III-3.7(i) there exist Scott compact sets K_1, K_2 such that $W \subseteq K_1 \subseteq U$ and $W \subseteq K_2 \subseteq V$. By Theorem III-5.8(5) $K_1 \cap K_2$ is Scott compact, and $W \subseteq K_1 \cap K_2 \subseteq U \cap V$ implies $W \ll U \cap V$ by I-1.4(i). Thus $\sigma(L)$ is a stably continuous lattice. By Theorem III-5.8(6) L is finitely generated.

Conversely suppose that $\sigma(L)$ is a stably continuous lattice and L is finitely generated. Let K_1 and K_2 be two compact saturated sets in ΣL. Let U_i be an open set containing K_i for $i = 1, 2$. Since the open sets containing K_i form a Scott open filter (Lemma II-1.18) and since $\sigma(L)$ is continuous, there exist open sets V_i such that $K_i \subseteq V_i \ll U_i$ for $i = 1, 2$. Then $V_1 \cap V_2 \ll U_1 \cap U_2$ by the hypothesis of stable continuity. It follows that the collection of all $U_1 \cap U_2$ such that U_1 and U_2 are Scott open sets containing K_1 and K_2 respectively form a filter base for a Scott open filter in $\sigma(L)$ (since any member of the filter has another member of the filter way below it). By the Hofmann–Mislove Theorem II-1.20 this filter consists of all open sets containing its intersection, which must be $K_1 \cap K_2$ (since K_1 and K_2 were saturated), and the intersection is compact. It follows from Theorem III-5.8(5) that L is compact in the Lawson topology. □

We introduce now a condition that often facilitates verifying that a domain is compact.

Definition III-5.11. A (continuous) domain L is said to satisfy *property* M with respect to a basis B if for any $x_1, y_1, x_2, y_2 \in B$ with $y_1 \ll x_1$ and $y_2 \ll x_2$, there exists a finite set $F \subseteq B$ such that

$$\uparrow x_1 \cap \uparrow x_2 \subseteq \uparrow F \subseteq \uparrow y_1 \cap \uparrow y_2. \qquad \square$$

Proposition III-5.12. *Let L be a domain. The following are equivalent.*

(1) *The intersection of any two Scott compact upper sets is again Scott compact.*
(2) *L satisfies property* M *with respect to every basis (in particular, with respect to L itself).*
(3) *L satisfies property* M *with respect to some basis.*

Proof: (1) implies (2): Let B be a basis for L. Suppose that we are given given $y_i \ll x_i$ for $x_i, y_i \in B, i = 1, 2$. By hypothesis the set $A := \uparrow x_1 \cap \uparrow x_2$ is compact in the Scott topology. Since the set $\Uparrow y_i$ is Scott open for each i, A is in the Scott open set $W := \Uparrow y_1 \cap \Uparrow y_2$. By Lemma III-5.7, we can obtain a finite set G such that $G \ll A$ and $G \subseteq W$. For each $u \in G$, $y_i \ll u$ for $i = 1, 2$. Pick $v_i(u) \in B$ such that $y_i \ll v_i(u) \ll u$ for $i = 1, 2$. The finite set F consisting of all $v_i(u)$ for $u \in G$, $i = 1, 2$ is then the desired set to establish property M.

(2) implies (3): Immediate.

(3) implies (1): Let $A = \uparrow A$ and $B = \uparrow B$ be compact in the Scott topology. We consider the filter $\mathcal{F}$ generated by the filter base of all $\uparrow G \cap \uparrow H$, where $G \ll A$ and $H \ll B$. Let $\uparrow G \cap \uparrow H$ belong to the filter base. By Lemma III-5.7 and Proposition III-3.6(ii) pick finite sets E_A and E_B with $E_A \ll A$, $E_B \ll B$, $E_A \subseteq \Uparrow G$, and $E_B \subseteq \Uparrow H$. By adjusting each member of E_A and E_B slightly downward with large enough elements way below each of them, we may assume additionally that E_A and E_B consist entirely of elements from the hypothesized basis satisfying property M.

By continuity of L for each $x \in E_A$ and each $y \in E_B$, there exist basis elements $z_x \ll x$ and $z_y \ll y$ such that $z_x \in \uparrow G$ and $z_y \in \uparrow H$. By property M there exists a finite subset $H(x, y)$ such that

$$\uparrow x \cap \uparrow y \subseteq \uparrow H(x, y) \subseteq \uparrow z_x \cap \uparrow z_y \subseteq \uparrow G \cap \uparrow H.$$

Set $F := \bigcup\{H(x, y) \colon x \in E_A, y \in E_B\}$. Then F is finite and $\uparrow E_A \cap \uparrow E_B \subseteq \uparrow F$. Thus $\uparrow F \in \mathcal{F}$ since $\uparrow E_A \cap \uparrow E_B$ is in the filter base, and $\uparrow F \subseteq \uparrow G \cap \uparrow H$. Since we have seen that any member of the filter base for $\mathcal{F}$ contains a finitely generated upper set that is again a member of $\mathcal{F}$, it follows that the finitely generated upper sets in $\mathcal{F}$ also form a filter base for $\mathcal{F}$.

It follows easily from Lemma III-5.7 that $A \cap B$ is the intersection of the filter $\mathcal{F}$, and then by Lemma III-5.6 that $A \cap B$ is Scott compact. □

Corollary III-5.13. *A domain L is compact in the Lawson topology if and only if it is finitely generated (as an upper set) and satisfies property* M *with respect to some (any) basis.*

Proof: The corollary follows immediately from Proposition III-5.12 and Theorem III-5.8(6). □

The Lawson compactness of *FS*-domains (see II-2.15) can be derived directly from III-5.13:

Proposition III-5.14. *Bifinite domains and FS-domains are compact in the Lawson topology.*

Proof: As bifinite domains are the algebraic *FS*-domains by II-2.21, we need only consider an arbitrary *FS*-domain L. We show that the conditions of

Corollary III-5.13 are satisfied for the basis $B = L$. Let (f_i) be an approximate identity for L consisting of finitely separating functions. For every i let M_i denote the finite set separating f_i from the identity. Fix f_i. For every $z \in L$ we have $f_i(z) \leq m \leq z$ for some $m \in M_i$. Hence $L = \uparrow M_i$, that is, L is finitely generated as an upper set.

Let $y_n \ll x_n$ for $n = 1, 2$. As the f_i form an approximate identity, the same holds for $(f_i)^2$ by II-2.14(i). Pick $g_i = (f_i)^2$ such that $y_n \leq g_i(x_n) \leq x_n$ for $i = 1, 2$, and let M_i be a finite separating set for f_i. Pick $u, v \in M_i$ such that $g_i(x_1) \leq u \leq f_i(x_1)$ and $g_i(x_2) \leq v \leq f_i(x_2)$. Let F be the set of upper bounds of $\{u, v\}$ in M_i, i.e., $\uparrow u \cap \uparrow v \cap M_i = F$. Then for $w \in \uparrow x_1 \cap \uparrow x_2$, we have $f_i(w) \leq m \leq w$ for some $m \in M_i$. Then $u \leq f_i(x_1) \leq f_i(w) \leq m$ and similarly $v \leq m$, so $m \in F$. Thus $w \in \uparrow F$. It follows that

$$\uparrow x_1 \cap \uparrow x_2 \subseteq \uparrow F \subseteq \uparrow u \cap \uparrow v \subseteq \uparrow y_1 \cap \uparrow y_2.$$

Thus L satisfies property M. □

Corollary III-5.15. *The following are equivalent in an algebraic domain L:*

(1) *L is Lawson compact;*
(2) *L is finitely generated (as an upper set) and satisfies property* M *with respect to the basis of compact elements;*
(3) *$K(L)$ is mub-complete and every finite set of compact elements has only finitely many minimal upper bounds.*

Proof: The equivalence of (1) and (2) follows from Corollary III-5.13. Assume (1). Then any finite intersection $A = \bigcap_{i=1}^{n} \uparrow k_i$, where each k_i is a compact element, is Scott compact by Theorem III-5.8 and also Scott open. For each $x \in A$, pick a compact element $k \ll x$ such that $k \in A$. Then finitely many of the $\uparrow k$ cover A. If one chooses this set to be of smallest possible cardinality, it follows that each k such that $\uparrow k$ is in the minimal cover is minimal in A, hence a compact element (since A is Scott open). Thus A is the finite union of principal filters of compact elements, each minimal in A, and hence each a mub of the set $\{k_1, \ldots, k_n\}$. Condition (3) now easily follows.

Assume (3). Note that condition (3) applied to the empty set yields that L is a finitely generated upper set. Let $k_1 \ll x_1$ and $k_2 \ll x_2$ for $k_1, k_2, x_1, x_2 \in K(L)$. Then by (3) $\uparrow x_1 \cap \uparrow x_2 \subseteq \uparrow F$, where F is the finite set of minimal upper bounds of $\{k_1, k_2\}$. Thus L satisfies property M with respect to the basis of compact elements, and hence is Lawson compact by Corollary III-5.13. □

Remark. In the theory of algebraic domains, property M has typically been considered only in the context of the poset of compact elements. In this context a partially ordered set P was said to have property M if it is mub-complete and every finite set has only finitely many minimal upper bounds. An alternative characterization is to require that the set of upper bounds for any finite set is a (possibly empty) finitely generated upper set (the minimal elements of the finite generating set being the minimal upper bounds). The preceding corollary shows that our approach to property M is a true generalization of the notion to arbitrary bases.

Lemma III-5.16. *Suppose that $x \in L$ is a* **dcpo**, *U is a Scott open set containing x, and $\bigcap_{y\in A} \uparrow y \subseteq \uparrow x$ for some nonempty $A \subseteq L$ (equivalently x is a lower bound for all upper bounds of A).*

(i) *If L is a sup semilattice, then there exists a finite $F \subseteq A$ such that $\sup F \in U$.*
(ii) *If L is a quasicontinuous domain for which the Lawson topology is compact, then there exists a finite $F \subseteq A$ such that $\bigcap_{y\in F} \uparrow y \subseteq U$.*

Proof: (i) The set of all suprema over all nonempty finite subsets of A is a directed set with supremum in $\uparrow x \subseteq U$. Thus there exists some finite $F \subseteq A$ such that $\sup F \in U$.

(ii) The set of all $\bigcap_{y\in F} \uparrow y$, where F is some nonempty finite subset of A, is a filtered family of nonempty Lawson closed subsets with intersection contained in U. It follows from the fact that the Lawson topology is compact and Hausdorff that $\bigcap_{y\in F} \uparrow y \subseteq U$ for some finite $F \subseteq A$. $\square$

The following theorem is a generalization of Proposition II-4.6.

Theorem III-5.17. *Let X be a space such that $\mathcal{O}(X)$ is a continuous lattice and let L be a domain with least element 0. If $\Omega[X, \Sigma L]$ is a domain for which the Lawson topology is compact, then the Isbell and Scott topologies on $TOP(X, \Sigma L)$ agree. If additionally X is compact, then the Isbell and Scott topologies on $TOP(X, \Sigma(L\backslash\{0\}))$ agree.*

Proof: By Lemma II-4.3(i) $[X, L]$ is a monotone convergence space since ΣL is; it follows that the Scott topology on $\Omega[X, L]$ is finer than the Isbell topology (note, for example, that the topology of a monotone convergence space is order consistent with respect to the order of specialization and apply Exercise II-1.31). The same argument applies to the case of $L\backslash\{0\}$.

Conversely let $f: X \to \Sigma L$ be continuous. Let $f(a) = b$ and let $z \ll b$. By joint continuity of the evaluation map (Proposition II-4.5(ii)), there exist an Isbell open set W containing f and an open set U containing a such that $h(x) \in \twoheaduparrow z$ for all $h \in W$ and $x \in U$. Pick V open containing a such that $V \ll U$. Define $g: X \to \Sigma L$ by $g(x) = z$ if $x \in V$ and $g(x) = 0$ otherwise. Then $g \ll f$

and f is the supremum of such functions by Proposition II-4.20(iii). Note also that $W \subseteq \uparrow g$ since $z \leq h(x)$ for $x \in V$, $h \in W$, and $0 \leq h(x)$ otherwise; hence $\uparrow g$ is a neighborhood of f in the Isbell topology. By the preceding lemma for any Scott open set Q containing f, there exist finitely many such g_i such that $\uparrow g_1 \cap \cdots \cap \uparrow g_n \subseteq Q$. Since each $\uparrow g_i$ is a neighborhood of f in the Isbell topology, it follows that Q is also. Hence the Isbell topology is finer than the Scott topology.

Suppose additionally that X is compact. Note that $\Sigma(L\backslash\{0\}) = \Sigma L\backslash\{0\}$, the latter with the relative topology. The last assertion follows easily from the preceding paragraphs if we know that the Isbell and Scott topologies on $TOP(X, \Sigma(L\backslash\{0\}))$ are the relative ones from TOP. That this is true for the Isbell topology follows from II-4.3(ii) and for the Scott topology from II-3.20. □

Corollary III-5.18. *Let L, M be FS-domains. Then the Isbell topology on* $TOP(\Sigma L, \Sigma M)$ *is equal to the Scott topology on* $[L \to M] = \Omega[\Sigma L, \Sigma M]$. □

Proof: First note that if we adjoin a smallest element 0 to M, then $M_0 = M \cup \{0\}$ is again an *FS*-domain (extend all members of the approximate identity by sending 0 to 0). Since for *FS*-domains the Lawson topology is compact by III-5.14 and the function spaces are again *FS*-domains by II-2.18, we have that $[\Sigma L, \Sigma M_0]$ is a domain with a compact Lawson topology. Also, since L is a domain, the Scott topology $\sigma(L)$ is a continuous lattice, and since L is an *FS*-domain, it is compact in the Lawson and hence in the Scott topology. Thus by the last assertion of Theorem III-5.17 we have that the Isbell and Lawson topologies agree on $TOP(\Sigma L, \Sigma M)$. □

Exercises

Exercise III-5.19. Let A be an upper set in a quasicontinuous domain that is compact in the Lawson topology. Show that the following are equivalent:

(1) A is closed in the Lawson topology;
(2) A is compact in the Scott topology;
(3) A is closed in the lower topology.

Hint. Note that (1) implies (2), that (2) implies (3) by Lemma III-5.7, and that (3) trivially implies (1). □

Exercise III-5.20. Let L be a quasicontinuous domain that is compact in the Lawson topology, M a **dcpo**, and $f : L \to M$ be an order preserving map.

Show that f preserves liminfs of ultrafilters (in the strong sense) iff f is Lawson continuous.

Hint. Suppose that f is Lawson continuous. Let $x = \underline{\lim}\,\mathcal{F}$ for some ultrafilter in L. Then $\mathcal{F}$ converges to x and x alone in the Lawson topology (III-3.17(ii)). Denote the ultrafilter generated by its image by $f(\mathcal{F})$; by continuity this ultrafilter converges to $f(x)$ in the Lawson topology of M. Since x is a directed supremum of eventual lower bounds for $\mathcal{F}$ it follows easily that $f(x)$ is a directed supremum of eventual lower bounds of $f(\mathcal{F})$, recalling that f is Scott-continuous from Exercise III-1.16. Let y be any eventual lower bound of $f(\mathcal{F})$. Then $f(F) \subseteq \uparrow y$ for some $F \in \mathcal{F}$. Thus $f^{-1}(\uparrow y)$ is a Lawson closed subset of L containing F, and hence the convergence point x of the ultrafilter $\mathcal{F}$. Thus $f(x) \geq y$, and therefore, by definition, $f(x)$ is the liminf of $\mathcal{F}$.

The other direction is immediate from III-3.17 and III-3.15. □

Exercise III-5.21. If L is a (Lawson) compact quasicontinuous domain and $f\colon L \to M$ is surjective and Lawson continuous, show that M is a compact quasicontinuous domain.

Hint. The compactness of M is immediate. We use the criterion of III-3.19(ii) to establish quasicontinuity. Let $x \in M$ and let U be a Scott open set containing x. Then $f^{-1}(\uparrow x)$ is a Lawson closed, hence Lawson compact, hence Scott compact subset of L contained in the Lawson open upper set and hence Scott open set $f^{-1}(U)$. By Lemma III-5.7 there exists a finite set F such that

$$f^{-1}(\uparrow x) \subseteq \twoheaduparrow F \subseteq \uparrow F \subseteq f^{-1}(U).$$

Now $B := L \backslash \twoheaduparrow F$ is closed, hence Lawson compact, and thus $f(B)$ is Lawson compact in M and misses $\uparrow x$. Therefore $\downarrow f(B)$ will be a Scott closed set missing $\uparrow x$ (see Proposition VI-1.6(i)), and standard set chasing yields that $M \backslash \downarrow B \subseteq f(\uparrow F) \subseteq \uparrow f(F)$. Thus $f(F)$ is a finite set in U such that x is contained in the Scott interior of $\uparrow f(F)$. By III-3.19(ii) M is quasicontinuous. □

Exercise III-5.22. Let L be the domain $[0, 1] \times |\mathbf{2}|$, where $|\mathbf{2}|$ is the two element antichain. Let M be formed from L by identifying the two maximal elements $(1, 0)$ and $(1, 1)$ with a single point and giving M the smallest induced order making $L \to M$ order preserving. Show that L is a domain, M is a quasicontinuous domain, and the identification map that identifies the two maximal points is Lawson continuous. This shows the result of the preceding exercise fails for domains. □

Exercise III-5.23. A complete lattice L is a quasicontinuous lattice iff the following equivalent conditions are satisfied:

(SM) the sup morphism $(I \mapsto \sup I)$: $\operatorname{Id} L \to L$ is $\omega(L)$ continuous;
(SM′) the sup morphism $(I \mapsto \sup I)$: $\operatorname{Id} L \to L$ is $\lambda(L)$ continuous.

Hint. (SM) implies quasicontinuity: $\operatorname{Id} L$ is algebraic, hence quasicontinuous; thus use the preceding exercise and Theorem III-3.11.

quasicontinuity implies (SM): Show that for each $x \in L$ the set

$$U_x = \{I \in \operatorname{Id} L : x \leq \sup I\}$$

is $\omega(\operatorname{Id} L)$ open. Indeed if $x \not\leq \sup I$, find a finite $F \subseteq L \setminus \downarrow \sup I$ with $\uparrow F \ll \uparrow x$ by quasicontinuity. Set $V = \{J \in \operatorname{Id} L : f \notin J$ for all $f \in F\}$. Then $I \in V$, and $J \in V$ implies $x \leq \sup J$. For $x \leq \sup J$ would imply $\uparrow F \cap J \neq \emptyset$; that is, $\downarrow f \subseteq I$ for some $f \in F$. □

Exercise III-5.24. Let $k\colon L \to L$ be a kernel operator on a **dcpo** L. Show that k is continuous for the lower topology. It follows that a Scott-continuous kernel operator is Lawson continuous. Conclude that the image of a Lawson compact **dcpo** under a Scott-continuous kernel operator is Lawson compact, too. □

New notes

The bulk of the material in this section postdates the *Compendium*. Corollary III-5.15 appears as the “2/3 SFP Theorem” in Plotkin’s Pisa Lecture Notes [Plotkin, B1981]. Property M and Lawson compactness were studied in general domains by A. Jung ([Jung, B1989] and [Jung, 1990b]). In particular one finds there the result that *FS*-domains and bifinite domains are Lawson compact. Theorem III-5.17 generalizes results in [Gierz and Keimel, 1981] and [Lawson, 1987]. [Lawson, 1998b] contains more on the topic of compactness.

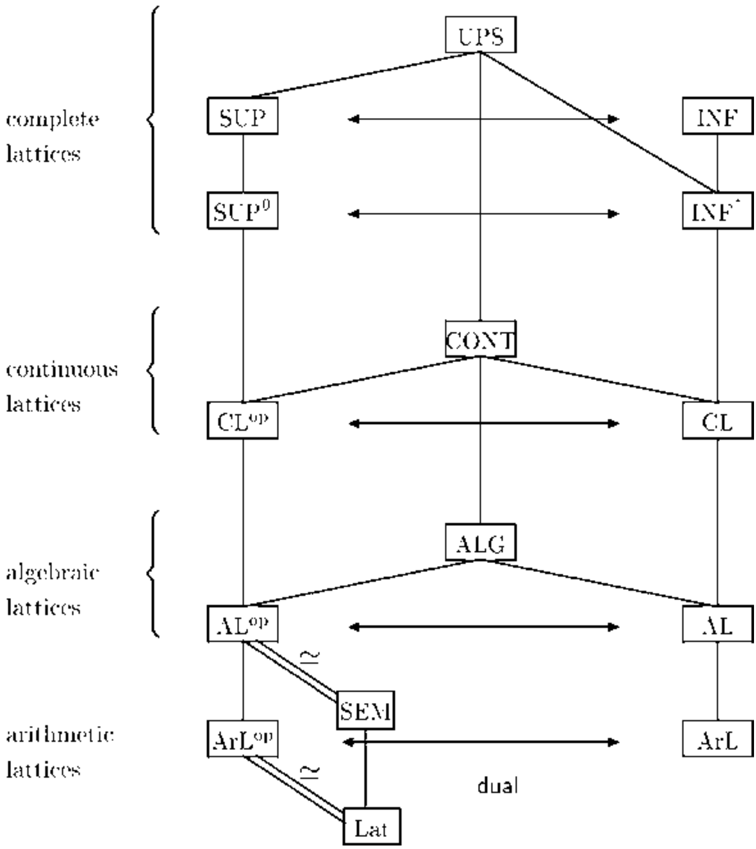
IV

Morphisms and Functors

With the exception of certain developments in Chapter II, notably Sections II-2 and II-4, we largely refrained from using category-theoretic language (even when we used its tools in the context of Galois connections). Inevitably, we have to consider various types of functions between continuous lattices, and this is a natural point in our study to use the framework of category theory.

In Section IV-1 we discuss a duality based on the formalism of Galois connections between the categories $DCPO_G$ and $DCPO_D$ of all **dcpo**s with upper and lower adjoints, respectively, as morphisms. We discuss in particular the categories *INF* and *SUP*, whose objects are complete lattices (in both cases) and whose morphisms are functions preserving arbitrary infs (respectively, sups). These categories are dual (IV-1.3). We saw as early as I-2.10 ff. that maps preserving arbitrary infs and directed sups play an important role in our theory. This leads us to consider the subcategory $INF^{\uparrow}$ of *INF*. Its dual under the *INF–SUP* duality is denoted by SUP^0; its morphisms are precisely characterized in IV-1.4(1)–(2), but as a category in itself, SUP^0 plays a minor role. More important, however, are the full subcategories $AL \subseteq CL \subseteq INF^{\uparrow}$ and $AL^{\mathrm{op}} \subseteq CL^{\mathrm{op}} \subseteq SUP^0$, which consist of algebraic and continuous lattices, respectively. We thus have a duality between *CL* and CL^{op} (IV-1.9) and one between *AL* and AL^{op}; the latter extends to the very useful duality between *AL* and the category *SEM* of semilattices with identity and semilattice morphisms preserving the identity (IV-1.14). Further duality theorems involving distributivity and prime elements will be given at the end of the first section, but this context will not be fully developed before Chapter V. In view of the fact that certain other categories of complete, continuous or algebraic lattices have been introduced in II-2.2, we survey the relevant categories of complete lattices in a diagram on p. 265.

Section IV-2 will introduce a category of domains and present its elegant and important self-duality theory. In order to find elsewhere in mathematics a



similarly satisfactory self-duality theory, one would have to go as far as the category of locally compact abelian groups where the self-duality is provided by Pontryagin duality.

In Section IV-3 we introduce a "character theory" which is appropriate for continuous lattices. Indeed, we show that the homomorphisms (in the sense of I-2.10) of a continuous lattice into the unit interval *separate points*. While this is (modulo results of III-2) a result on compact semilattices, we present here a lattice theoretic approach which was new in the *Compendium* (1980), but which has had related forerunners in the work of Raney on completely distributive lattices in the 1950s. The principal results in the section are more general in that they apply to complete lattices which are not necessarily continuous (IV-3.15, IV-3.19, IV-3.22).

In Section IV-4 we present a general description of a limiting process introduced by Scott which produces continuous lattices L which are naturally isomorphic to their own function spaces $[L \to L]$. We describe the functorial setting to the extent that it is necessary and convenient; we analyze the concept of projective limits in the relevant categories (such as *INF*$^\uparrow$, *CL* and *AL*).

In Section IV-5 we treat systematically the question of which functors preserve projective limits and establish criteria for self-functors on $DCPO_G$, the category of **dcpo**s with morphisms the upper adjoint of an adjoint pair, to preserve projective limits. These criteria lead to not only the result indicated above, but also some results which, for example, give us in a functorial fashion a continuous lattice L which is naturally isomorphic to the lattice of all its Lawson open lower sets.

In Section IV-6 general categorical criteria for fixed-point constructions for functors are worked out, particularly for categories of **dcpo**s. This machinery is applied to the specific study of minimal solutions of domain equations in Section IV-7.

In Section IV-8 we introduce the important topic of powerdomains and give constructions for the basic powerdomains, namely the Hoare, Smyth, and Plotkin powerdomains. The extended probabilistic powerdomain, the domain theoretic version of the space of Borel measures, is studied in Section IV-9.

IV-1 Duality Theory

We recall from Section O-3 the concept of adjoint functions between partially ordered sets. For monotone maps $g\colon S \to T$ and $d\colon T \to S$, we said that d is a lower adjoint of g and that g is an upper adjoint of d iff $g(s) \geq t \Leftrightarrow s \geq d(t)$ for all $s \in S, t \in T$.

Definition IV-1.1. We will consider the following categories.

POSET$_G$ and *POSET*$_D$ have the same class of objects, namely, the class of all posets. The morphisms of *POSET*$_G$ are the order preserving maps g that have a lower adjoint d; the morphisms of *POSET*$_D$ are the order preserving maps d having an upper adjoint g.

INF and *SUP* have the same class of objects, namely, the class of all complete lattices. The morphisms of *INF* preserve arbitrary infs, the morphisms of *SUP* preserve arbitrary sups. □

By O-3.5, a function between complete lattices has a lower adjoint iff it preserves arbitrary infs, and it has an upper adjoint iff it preserves arbitrary sups. Thus *INF* is a full subcategory of *POSET*$_G$ and *SUP* is a full subcategory of *POSET*$_D$. (Recall that, if A is a subcategory of B, one says that A is a *full subcategory* if each B-morphism $f\colon S \to T$ between objects of A is an A-morphism.)

Evidently, $INF \cap SUP$ is the category of *all complete lattices and complete lattice homomorphisms*. It is perhaps noteworthy that the role of $INF \cap SUP$ is relatively secondary in our framework. The supply of morphisms in this category is too restricted. A noteworthy exception is the sup morphism $r = (I \mapsto \sup I)$: $\operatorname{Id} L \to L$, which is in $INF \cap SUP$ for a continuous lattice L (see O-3.15, O-4.2, and I-1.10).

Given a pair of posets S and T, by O-3.2, if a map $g\colon S \to T$ has a lower adjoint $d\colon T \to S$, then this lower adjoint d is uniquely determined by g; we denote it by $d = D(g)\colon T \to S$. And if a map $d\colon T \to S$ has an upper adjoint g, then this upper adjoint is uniquely determined by d. We denote it by $g = G(d)\colon S \to T$. In order to define functors $D\colon POSET_G \to POSET_D$ and $G\colon POSET_D \to POSET_G$, we have to define D and G on objects; thus for any poset S we write simply $D(S) = S$ and $G(S) = S$.

Lemma IV-1.2. *The assignments $D\colon POSET_G \to POSET_D^{\mathrm{op}}$ and $G\colon POSET_D \to POSET_G^{\mathrm{op}}$ are functors (that is, D and G are contravariant functors). They restrict to functors $D\colon INF \to SUP^{\mathrm{op}}$ and $G\colon SUP \to INF^{\mathrm{op}}$.*

Proof: The (lower or upper) adjoint of an identity map of a poset clearly is the identity map. That the composition of upper (resp., lower) adjoints is again the upper (resp., lower) adjoint of the composition is well known in category theory and is immediate from the definition O-3.1; indeed if we have $g_1\colon S_1 \to S_2$ and $g_2\colon S_2 \to S_3$ then $g_2 g_1(s_1) \geq s_3$ iff $g_1(s_1) \geq D(g_2)s_3$ iff $s_1 \geq D(g_1)D(g_2)s_3$ on one hand, but also $g_2 g_1(s_1) \geq s_3$ iff $s_1 \geq D(g_2 g_1)$ on the other. Thus $D(g_2 g_1) = D(g_1)D(g_2)$. The assignment G is treated analogously. Thus D and G are (contravariant) functors. $\square$

Theorem IV-1.3. *The following categories are dual under the functors D and G given through the Galois connection of functions:*

(i) *$POSET_G$ and $POSET_D$;*
(ii) *INF and SUP (INF–SUP* **duality***).*

Specifically, D and G preserve objects (that is, the "dual" of a poset is itself under this duality). Moreover, $GD(g) = g$ and $DG(d) = d$ for all g in $DCPO_G$ for all d in $DCPO_D$.

Proof: This is trivial: by definition D and G preserve objects, and the identities $GD(g) = g$ and $DG(d) = d$ are clear from the adjunction. $\square$

This simple duality nevertheless is quite useful as a basis and a guide to the invention of other duality theories. Examples are the self-dualities of the categories of domains and of continuous semilattices and the duality between unital semilattices and algebraic lattices which we discuss later (see also I-4.10).

Then our first task is to investigate how the functors D and G translate certain preservation properties of morphisms. Note that a lower adjoint map $d\colon T \to S$ between **dcpo**s is always Scott-continuous, as lower adjoints preserve all existing sups, in particular directed sups (O-3.3).

Theorem IV-1.4. *Let S and T be* **dcpo***s and $g\colon S \to T$ the upper adjoint of $d\colon T \to S$. Then the following statements are equivalent:*

(1) *g preserves directed sups (that is, g is Scott-continuous, see* II-2.2*);*
(2) *if $U \subseteq T$ is any Scott open set in T, then $\uparrow d(U)$ is Scott open in S.*

These conditions imply

(3) *d preserves $\ll$, that is, if $t \ll t^*$ in T then $d(t) \ll d(t^*)$ in S,*

and if T is a domain, then all three conditions are equivalent.

Proof: (1) implies (2): Let U be Scott open in T. In order to show that $\uparrow d(U)$ is Scott open in S, we take a directed set $D \subseteq S$ with $\sup D \in \uparrow d(U)$ and we show that $D \cap \uparrow d(U) \neq \emptyset$. Now, $\sup D \in \uparrow d(U)$ implies $d(u) \leq \sup D$ for some $u \in U$. We conclude $u \leq g(\sup D)$ by O-3.1(ii). But $g(\sup D) = \sup g(D)$ by hypothesis (1), and $g(D)$ is directed since g preserves order. Since U is Scott open there is an $x \in D$ with $g(x) \in U$, and thus $dg(x) \in d(U)$. But $dg(x) \leq x$ (by O-3.6(2)), so $x \in \uparrow d(U)$. Thus $D \cap \uparrow d(U) \neq \emptyset$.

(2) implies (1): Let D be a directed set in S with $s = \sup D$. We have always $g(s) \geq \sup g(D)$; hence we must show $g(s) \leq \sup g(D)$. We proceed by contradiction. Assume that $g(s) \notin \downarrow \sup g(D)$. With $U = T \setminus \downarrow \sup g(D)$, we have $g(s) \in U$, $\sup g(D) \notin U$, and U is Scott open. By hypothesis (2) we know that $\uparrow d(U)$ is Scott open in S. We note $dg(s) \in d(U)$ and $dg(s) \leq s$ (O-3.6(2)), hence $s \in \uparrow d(U)$. So, since $\uparrow d(U)$ is Scott open, we have an $x \in D$ with $x \in \uparrow d(U)$, that is, $x \geq d(u)$ for some $u \in U$. But then in view of O-3.1, $g(x) \geq u \in U$, whence $g(x) \neq g(\sup D)$ and this is the desired contradiction.

(1) implies (3): Suppose $t \ll t^*$ in T and let $D \subseteq S$ be directed with $d(t^*) \leq \sup D$. By O-3.1 this means $t^* \leq g(\sup D)$, but $g(\sup D) = \sup g(D)$ by (1). Now there is an $x \in D$ with $t \leq g(x)$ by I-1.1. Thus $d(t) \leq s$ by O-3.1, whence $t \ll t^*$.

(3) implies (1) when T is a domain: Let U be Scott open in T. In order to show that $g^{-1}(U)$ is Scott open in S, take any $s \in g^{-1}(U)$. Then $g(s) \in U$ and, as T is a domain, there is an element $t \in U$ with $t \ll g(s)$ by II-1.10(i). By (3), $d(t) \ll d(g(s)) \leq s$, the latter by O-3.6(2). Thus $d(t) \ll s$. But $g(d(t)) \geq t$ again by O-3.6(2), whence $g(d(t)) \in U$ and $d(t) \in g^{-1}(U)$. Thus, for every

$s \in g^{-1}(U)$ there is $s' = d(t) \in g^{-1}(U)$ with $s' \ll s$ which shows that $g^{-1}(U)$ is Scott open. $\square$

In several of the following results we have occasion to say that a map is *open*, meaning of course that it maps open sets to open sets.

Remark IV-1.5. *Assume the hypotheses of Theorem* IV-1.4. *Then condition* (2) *in* IV-1.4 *implies*

(2′) *d is relatively open onto its image with respect to the Scott topology on T and the topology on $d(T)$ induced by the Scott topology of S.*

Furthermore condition (2′) *implies*

(2″) *the co-restriction $d\colon T \to d(T)$ is open with respect to the Scott topologies.*

Proof: (2) implies (2′): It suffices to observe that $d(U) = d(T) \cap \uparrow d(U)$. The left is always included in the right; so suppose that $s \in d(T) \cap \uparrow d(U)$. Then $s = d(t) \geq d(u)$ for some $t \in T$ and some $u \in U$. Then $g(s) = g(d(t)) \geq g(d(u)) \geq u$ as g is monotone and by O-3.6(2). Hence $g(s) \in U$, as a Scott open set is an upper set. By O-3.6(3), we obtain $d(g(s)) = d(g(d(t))) = d(t) = s$, hence $s \in d(U)$.

(2′) implies (2″): Let $U \subseteq T$ be Scott open and let $D \subseteq d(T)$ be directed with $\sup_{d(T)} D \in d(U)$. Since d is a lower adjoint, d preserves all sups, and so $\sup_S D = \sup_{d(T)} D \in d(U)$. Now, (2′) implies $d(U) = V \cap d(T)$ for some $V \subseteq S$ Scott open, so $\sup_S D \in V$. Hence $D \cap V \neq \emptyset$, and since $D \subseteq d(T)$, we conclude $D \cap d(U) = D \cap (V \cap d(T)) \neq \emptyset$. $\square$

In general, (2′) does not imply (2): If $d\colon \mathbb{I} \to \mathbb{I}^2$ is the embedding $d(0) = (0, 0)$ and $d(t) = (t, 1)$ for $t > 0$ which preserves arbitrary sups, then $V = d(]1/2, 1])$ is relatively Scott open, since it is of the form $d(\mathbb{I}) \cap (]1/2, 1] \times \mathbb{I})$; but $\uparrow V = V$ is not Scott open in $\mathbb{I}^2$.

On the other hand, if d is surjective (that is, g is injective by O-3.7) then $\uparrow d(U) = d(U)$, and thus (2″) implies (2). Hence we have

Corollary IV-1.6. *Let S and T be* **dcpo***s. If $g\colon S \to T$ is upper adjoint to $d\colon T \to S$ and if g is injective (equivalently, d is surjective (see* O-3.7*)), then g is Scott-continuous iff d is Scott open.* $\square$

The following two corollaries constitute a complement to Theorem I-2.2, Corollary I-2.3 and the subsequent Remark; see also I-2.4, I-2.5 and I-2.6.

Corollary IV-1.7. *Let L be a* **dcpo** *and $k\colon L \to L$ a kernel operator (*O-3.8*). Then the following statements are equivalent:*

(1) *k preserves directed sups;*
(2) *for each Scott open set U of $k(L)$ the set $\uparrow U$ is Scott open in L.*

These conditions imply

(3) *for $x, y \in k(L)$, we have $x \ll_{k(L)} y$ iff $x \ll_L y$.*

Moreover, if $k(L)$ is a domain, then all three conditions are equivalent.

Proof: By O-3.10, the co-restriction $k^\circ\colon L \to k(L)$ of k is upper adjoint to the inclusion $k_\circ\colon k(L) \to L$. Since $k_\circ$ preserves sups, then (1) holds iff k° preserves directed sups. Then Theorem IV-1.4 applies to give the equivalence of (1) and (2). We have always that $x \ll_L y$ implies $x \ll_{k(L)} y$. Hence IV-1.4 shows that (3) follows from the other two conditions and is in fact equivalent if $k(L)$ is a domain. □

Corollary IV-1.8. *Let L be a* **dcpo** *and $c\colon L \to L$ a closure operator (*O-3.8*). Then the following statements are equivalent:*

(1) *$c(L)$ is closed in L under directed sups;*
(2) *the co-restriction $c^\circ\colon L \to c(L)$ is Scott open.*

These conditions imply

(3) *$c(x) \ll_{c(L)} c(y)$ for all $x \ll_L y$ in L.*

Moreover, if $c(L)$ is a domain, then all three conditions are equivalent. □

Proof: Condition (1) is equivalent to saying that the inclusion map $c_\circ\colon c(L) \to L$ preserves directed sups. Hence the corollary follows from IV-1.4 and IV-1.5. □

In order to reformulate Theorem IV-1.4 in terms of duality we require suitable categories.

Definition IV-1.9. We introduce the following subcategories of $POSET_G$, INF, $POSET_D$ and SUP.

$DCPO_G$ has as objects **dcpo**s and as morphisms Scott-continuous maps g that have a lower adjoint.

$DCPO_D$ has as objects **dcpo**s and as morphisms maps d that have an upper adjoint and the property that for each Scott open U in the domain of d the set $\uparrow d(U)$ is Scott open in the range. (Note that such maps are Scott-continuous.)

$INF^{\uparrow}$ has as objects all complete lattices and as morphisms maps preserving arbitrary infs and directed sups (that is, *INF*-maps that are also Scott-continuous).
SUP^0 has as objects all complete lattices and as morphisms all *SUP*-morphisms d where for each Scott open U in the domain of d the set $\uparrow d(U)$ is Scott open in the range.
DOM_G has as objects domains and as morphisms Scott-continuous maps that have a lower adjoint.
DOM_D has as objects domains and as morphisms maps that have an upper adjoint and preserve the way-below relation $\ll$.
CL has as objects all continuous lattices and as morphisms all maps preserving directed sups and arbitrary infs.
CL^{op} has as objects all continuous lattices and as morphisms all maps preserving arbitrary sups and the way-below relation $\ll$.

We call *CL* the category of *continuous lattices* and CL^{op} the *dual category of continuous lattices*. □

Evidently $INF^{\uparrow} = INF \cap UPS$. Notice that we view CL^{op} as a concrete category of functions between continuous lattices, and the $^{\mathrm{op}}$-notation is justified by the next theorem.

We have defined two sequences of full subcategories: Firstly, *CL* is full in $INF^{\uparrow}$ and in DOM_G, and the latter two are full in $DCPO_G$. Secondly, CL^{op} is full in SUP^0 and in DOM_D, and the latter two are full in $DCPO_D$, where we make use of IV-1.4.

Theorem IV-1.10. *The following pairs of categories are dual under the adjoint functors D and G:*

(i)	$DCPO_G$	*and*	$DCPO_D$	($DCPO_G$–$DCPO_D$ **duality**),
(ii)	$INF^{\uparrow}$	*and*	SUP^0	($INF^{\uparrow}$–SUP^0 **duality**),
(iii)	DOM_G	*and*	DOM_D	(DOM_G–DOM_D **duality**),
(iv)	CL	*and*	CL^{op}	(CL–CL^{op} **duality**). □

Let us now determine how the morphisms of CL^{op} treat algebraic domains, which, as we know from Section I-4, play an important role in our theory. Our discussion leads us to another important duality theory.

Proposition IV-1.11. *Let $d\colon T \to S$ be a monotone map between* **dcpo***s and consider the following two statements:*

(1) *d preserves $\ll$;*
(2) $d(K(T)) \subseteq K(S)$.

Then (1) *implies* (2)*, and if* T *is an algebraic domain, then the two statements are equivalent.*

Proof: (1) implies (2): If $c \in K(T)$, then $c \ll c$, and thus $d(c) \ll d(c)$ by (1), hence $d(c) \in K(S)$.

Now suppose that T is an algebraic domain. (2) implies (1): Suppose that $t \ll t^*$ in T. By I-4.3 there is a compact c with $t \leq c \leq t^*$. Then $d(t) \leq d(c) \leq d(t^*)$ and $d(c) \in K(S)$ by (2). Hence $d(t) \ll d(t^*)$ (I-4.3). □

Corollary IV-1.12. *Let* $g\colon S \to T$ *be the upper adjoint of a monotone map* $d\colon T \to S$ *between* **dcpo***s and and suppose that* T *is an algebraic domain. Then the following are equivalent:*

(1) *g preserves directed sups;*
(2) *d maps compact elements of T to compact elements of S.*

If these conditions are satisfied, then $K(g) = d|_{K(T)}\colon K(T) \to K(S)$ *has the property that inverse images of ideals are ideals.*

If, moreover, T *is an algebraic lattice, then* $K(g)$ *is a sup semilattice homomorphism.*

Proof: The equivalence of (1) and (2) follows from IV-1.4 and IV-1.11.

Now let I be an ideal of the poset $K(S)$, i.e. a directed lower set in $K(S)$ in the induced order. We claim that the set $K(g)^{-1}(I) = d^{-1}(I) \cap K(T)$ is an ideal of $K(T)$. For this let $c = \sup I \in S$. Note that $I = \{x \in K(S)\colon x \leq c\}$. For a compact element $a \in T$, we have $a \in d^{-1}(I)$ iff $d(a) \leq c$ iff $a \leq g(c)$, as g is the upper adjoint of d. Thus, $d^{-1}(I) \cap K(T) = \downarrow g(c) \cap K(T)$, and this set is an ideal of $K(T)$, as T is an algebraic domain. This proves our claim.

Finally assume that T is an algebraic lattice. Since $K(T)$ is a sup semilattice by I-4.6 and since d preserves sups it follows that d is a sup semilattice homomorphism. □

In order to express this last fact in a systematic way, we introduce further categories (compare, however, II-2.2!). Recall that DOM_G denotes the category of all domains and Scott-continuous morphisms between them having a lower adjoint.

Definition IV-1.13. We define the following categories.

POID is the category of posets and maps under which the inverse image of an ideal is an ideal. (Note that such maps are monotone.)

SEM is the category of all sup semilattices with 0 with maps preserving the semilattice operation and 0.
$ALGDOM_G$ is the full subcategory of DOM_G consisting of all algebraic domains.
$ALGDOM_D$ is the full subcategory of DOM_D consisting of all algebraic domains.
AL and *ArL* are the full subcategories of *CL* consisting of all algebraic and of all arithmetic lattices, respectively.
AL^{op} and ArL^{op} are the full subcategories of CL^{op} consisting of *AL*- and of *ArL*-objects, respectively. (By IV-1.12 the morphisms are the maps preserving arbitrary sups and compact elements.) □

Remark. Let S and T be semilattices with 1. Then a function $f: S \to T$ is a semilattice homomorphism preserving the identity iff $f^{-1}(F)$ is a filter for all filters $F \subseteq T$. Thus *SEM* is a full subcategory of *POID*.

The following is an exercise (cf. Theorem IV-2.1 below).

Corollary IV-1.14.

(i) *The assignment which associates with an algebraic domain L the poset $K(L)$ and with a morphism $g: L \to M$ in $ALGDOM_G$ the map*

$$K(g) = D(g) \mid K(M): K(M) \to K(L)$$

is a functor $ALGDOM_G \to POID^{\mathrm{op}}$, where $D(g)$ is the lower adjoint $d: M \to L$ of g.

(ii) *The assignment which associates with an algebraic lattice L the sup semilattice $K(L)$ and with a morphism $g: L \to M$ in AL the function*

$$K(g) = D(g) \mid K(M): K(M) \to K(L)$$

is a functor $AL \to SEM^{\mathrm{op}}$.

(iii) *The assignment $L \mapsto K(L)$ and $d \mapsto d \mid K(M): K(M) \to K(L)$ for maps $d: M \to L$ in AL^{op} is a functor $AL^{\mathrm{op}} \to SEM$.* □

If S is a poset, then $\mathrm{Id}\, S$ is an algebraic domain, and if S is a sup semilattice with minimal element, then $\mathrm{Id}\, S$ is an algebraic lattice according to I-4.10, and the principal ideal embedding $x \mapsto \downarrow x : S \to \mathrm{Id}\, S$ induces an isomorphism between S and $K(\mathrm{Id}\, S)$. Conversely, we recall from I-4.10 that, if L is an algebraic domain, then each $x \in L$ yields an ideal $\downarrow x \cap K(L)$ of the poset $K(L)$, and that the function $x \mapsto \downarrow x \cap K(L) : L \to \mathrm{Id}\, K(L)$ is an isomorphism. Let us observe that for each morphism $f\colon S \to T$ in *POID* we get a Scott-continuous map $J \mapsto f^{-1}(J)\colon \operatorname{Id} T \to \operatorname{Id} S$ with lower adjoint $J' \mapsto {\downarrow} f(J')\colon \operatorname{Id} S \to \operatorname{Id} T$. Indeed $J_1 \subseteq f^{-1}(J_2)$ iff $f(J_1) \subseteq J_2$ iff ${\downarrow} f(J_1) \subseteq J_2$. Our observations amount to the following theorem.

Theorem IV-1.15. (*$ALGDOM_G$–POID* **duality)** *The category POID of posets with maps under which inverse images of ideals are ideals and the category $ALGDOM_G$ of algebraic domains with Scott-continuous maps having a lower adjoint are dual. The duality is established through the functors* Id: *POID* $\to$ $ALGDOM_G^{\mathrm{op}}$ *with* $(\operatorname{Id} g)(J) = f^{-1}(J)$ *and* K: $ALGDOM_G \to POID^{\mathrm{op}}$, $K(g) = d \mid K(T)\colon K(T)^{\mathrm{op}} \to K(S)^{\mathrm{op}}$ *with the lower adjoint d of the $ALGDOM_G$-morphism $g\colon S \to T$.* □

Let us further note that for each sup semilattice homomorphism $f\colon S \to T$ preserving least elements we induce a morphism $\operatorname{Id} f = (J \mapsto f^{-1}(J))\colon \operatorname{Id} T \to \operatorname{Id} S$ and that this morphism preserves arbitrary intersections and unions of up-directed families and, hence, is an *AL*-morphism. Evidently, Id: $S \to AL^{\mathrm{op}}$ is a functor, and what we have observed amounts to the following theorem.

Theorem IV-1.16. (*AL–SEM* **duality)**

(i) *The categories SEM of sup semilattices with* 0 *(with maps preserving finite sups and* 0*) and AL of algebraic lattices (with maps preserving infs and directed sups) are dual. The duality is established through the functors* Id: *SEM* $\to$ AL^{op} *and* K: *AL* $\to$ SEM^{op}.
(ii) *Under this duality, the full subcategory S_{lat} of lattices with least elements and least element preserving sup semilattice maps is placed into duality with the category ArL of arithmetic lattices.*

Proof: Only the last assertion is not yet proved, but it is immediate from the fact that an algebraic lattice L is arithmetic iff $K(L)$ is a lattice with smallest element (see I-4.7). □

Notice that IV-1.16 also says that the categories *SEM* of sup semilattices and the dual category AL^{op} of algebraic lattices are equivalent.

Definition IV-1.17. Let *LAT* denote the category of all lattices with a least element and all lattice homomorphisms preserving the least elements. □

Theorem IV-1.18. *The category $ArL_{\wedge}^{\mathrm{op}}$ of all arithmetic lattices and all maps preserving finite infs and arbitrary sups and respecting the way-below relation is equivalent to the category LAT.*

Proof: If S and T are algebraic lattices, then for each $d\colon S \to T$ in AL^{op} we have a commutative diagram

$$\begin{tikzcd} S \arrow[r, "d"] & T \\ K(S) \arrow[u, hook] \arrow[r, "d|K(S)"'] & K(T) \arrow[u, hook] \end{tikzcd}$$

and by the Remark following IV-1.13, $d \mapsto d \mid K(S) : AL^{\mathrm{op}}(S, T) \to S(K(S), K(T))$ is a bijection. If now S and T are arithmetic, then $K(S)$ and $K(T)$ are lattices. If further d preserves finite infs, then $d \mid K(S)$ is in *LAT*. Conversely, suppose that $d \mid K(S)$ preserves finite infs. Let $s, s^* \in S$. Then $D = {\downarrow}s \cap K(S)$ and $D^* = {\downarrow}s^* \cap K(S)$ are directed sets in S with $s = \sup D$ and $s^* = \sup D^*$. We have $ss^* = \sup DD^*$, since S is meet-continuous (see O-4.2, I-1.8 and I-4.3). Then $d(ss^*) = d(\sup DD^*) = \sup d(DD^*)$ (since d preserves arbitrary sups) $= \sup d(D) \sup d(D^*)$ (since $d \mid K(S)$ preserves finite infs and T is meet-continuous) $= d(\sup D) d(\sup D^*)$ (since d preserves sups) $= d(s)d(s^*)$. Thus d preserves finite infs. We have shown that $d \mapsto d \mid K(S)$ establishes a bijection between $ArL^{\mathrm{op}}_{\wedge}(S, T)$ and $LAT(K(S), K(T))$. □

We encountered in the proof of the preceding theorem a lower adjoint d which in addition preserves finite infs, that is, which was a lattice homomorphism. It is worth our while to ask systematically whether or not this additional property of a right adjoint is recognizable by looking at its left adjoint. The following propositions provide the tools and are also of independent interest.

Proposition IV-1.19. *Let S and T be posets. Let $g\colon S \to T$ be upper adjoint to $d\colon T \to S$. Then the $ALGDOM_G$-morphism* $\operatorname{Id} g\colon \operatorname{Id} S \to \operatorname{Id} T$ *is given by* $(\operatorname{Id} g)(I) = d^{-1}(I) = {\downarrow}g(I)$*, and we have a commutative diagram (with the principal ideal embeddings as vertical arrows)*

$$\begin{tikzcd} \operatorname{Id} S \arrow[r, "\operatorname{Id} g"] & \operatorname{Id} T \\ S \arrow[u] \arrow[r, "g"'] & T \arrow[u] \end{tikzcd}$$

The lower adjoint $D(\operatorname{Id} g)$ *is given by* $\operatorname{Id} d\colon J \mapsto {\downarrow} d(J)$ *and the following diagram commutes:*

$$\begin{tikzcd}
\operatorname{Id} S & \operatorname{Id} T \arrow[l, "D(\operatorname{Id} g)"'] \\
S \arrow[u] & T \arrow[l, "d"] \arrow[u]
\end{tikzcd}$$

Proof: First we have to show that $d^{-1}(I) = {\downarrow} g(I)$ for all $I \in \operatorname{Id} S$. But $t \in d^{-1}(I)$ means $d(t) \in I$ which is equivalent to the existence of an $x \in I$ with $d(t) \le x$. By O-3.1 this means the existence of an $x \in I$ with $t \le g(x)$ which says precisely that $t \in {\downarrow} g(I)$. The commutativity of the diagram is a consequence of the relation ${\downarrow} g(s) = {\downarrow} g({\downarrow} s)$ for each $s \in S$, which follows from O-1.11.

Next we must identify the lower adjoint of $\operatorname{Id} g$: But $d^{-1}(I) \supseteq J$ means $I \supseteq d(J)$ which is equivalent to $I \supseteq {\downarrow} d(J)$ since I is an ideal. As before, the commutativity of the diagram follows from O-1.11. □

We complement these observations by the following.

Proposition IV-1.20. *Assume that the hypotheses of IV-1.19 are satisfied, and in addition that S and T are complete lattices. Then the following diagram commutes:*

$$\begin{tikzcd}
\operatorname{Id} S \arrow[d, "\sup"'] & \operatorname{Id} T \arrow[l, "D(\operatorname{Id} g)"'] \arrow[d, "\sup"] \\
S & T \arrow[l, "d"]
\end{tikzcd}$$

Moreover $g \in \mathit{INF}^{\uparrow}$ *iff the following diagram commutes also:*

$$\begin{tikzcd}
\operatorname{Id} S \arrow[r, "\operatorname{Id} g"] \arrow[d, "\sup"'] & \operatorname{Id} T \arrow[d, "\sup"] \\
S \arrow[r, "g"'] & T
\end{tikzcd}$$

Proof: See [Hofmann and Stralka, 1976]. □

At this point one should recall that an element p of a poset S is called prime, if $S\backslash{\downarrow}p$ is a filter or is empty (see I-3.11) and that an ideal I of S is called prime, if it is a prime element of $\operatorname{Id} S$ or, equivalently, if $S\backslash I$ is a filter or empty (see I-3.18). Note that for an element p, the principal ideal ${\downarrow}p$ is a prime ideal iff p is a prime element of S.

Now let S be a sup semilattice with a smallest element. By I-4.10, the lattice $\operatorname{Id} S$ is algebraic. We now derive from Theorem I-3.15 and Exercises I-3.30, I-3.31 the following proposition (see also [Hofmann *et al.*, B1974]).

Proposition IV-1.21. *Let S be a sup semilattice with a least element. Then the following statements are equivalent:*

(1) *the lattice* $\operatorname{Id} S$ *of ideals of S is distributive;*
(2) *every ideal of S is the intersection of prime ideals;*
(3) *the sup semilattice S is distributive (see* I-3.11*), that is, whenever $x \le s \vee s^*$ then there exist $t \le s$ and $t^* \le s^*$ with $x = t \vee t^*$.*

Furthermore, if S is a lattice, these conditions are equivalent to

(4) *S is distributive.* □

After this preparation we have the following information on lower adjoints d which are also lattice morphisms.

Theorem IV-1.22. *Let S and T be unital lattices and suppose that $g\colon S \to T$ is an upper adjoint to $d\colon T \to S$. Then each of the following conditions implies the next:*

(1) *d is a lattice morphism (that is, preserves finite infs);*
(2) *for each filter F of S, the set $d^{-1}(F)$ is a filter of T;*
(3) $\operatorname{Id} g$ *(that is, the function $J \mapsto {\downarrow}g(J)$) preserves prime ideals;*
(4) $g(\mathrm{PRIME}\ S) \subseteq \mathrm{PRIME}\ T$*, that is, g preserves primes.*

If S is distributive (3) *implies* (1)*, and if each element of S is an inf of primes, then all four conditions are equivalent.*

Remark. If S is a distributive continuous lattice, then $\mathrm{PRIME}\, S$ order-generates S by I-3.15.

Proof of theorem: Condition (2) is an immediate consequence of (1).

(2) implies (3): If we abbreviate $r = \operatorname{Id} g$, then we have $r(I) = d^{-1}(I)$ by IV-1.19. This is equivalent to $d^{-1}(S\backslash I) = T\backslash r(I)$. If I is a prime ideal, then

$S \backslash I$ is a filter, and then $T \backslash r(I)$ is a filter by (2). This means that $r(I)$ is a prime ideal.

(3) implies (4): Let$p \in$ PRIME S. Then $\downarrow p$ is a prime ideal. Hence $\downarrow g(\downarrow p)$ is a prime ideal by (3). But $\downarrow g(\downarrow p) = \downarrow g(p)$ by O-1.11, and so $g(p)$ is prime.

Now suppose that S is distributive. Then every ideal I of S is an intersection of prime ideals by IV-1.21. In particular every principal ideal is the intersection of prime ideals.

(3) implies (1): Suppose $t, u \in T$; then we always have $d(tu) \leq d(t)d(u)$. To prove the converse, we note that $\downarrow d(tu)$ is the intersection of prime ideals, and so $d(t)d(u) \leq d(tu)$ if $d(t)d(u) \in I$ for every prime ideal of S containing $d(tu)$. Let I be such a prime ideal. Then $tu \in d^{-1}(I) = r(I)$, and $r(I)$ is a prime ideal by (3). Hence $t \in r(I)$ or $u \in r(I)$, whence $d(t) \in d(r(I)) \subseteq I$ or $d(u) \in d(r(I)) \subseteq I$. In either case, $d(t)d(u) \in I$, as we had to show.

Finally suppose that every element of S is an inf of primes. This implies that every principal ideal of S is the intersection of principal ideals generated by primes, and these are prime ideals. Hence the proof of (3) implies (1) applies with $I = \downarrow p$ for some prime p of S, since $d^{-1}(I) = d^{-1}(\downarrow p) = \downarrow g(\downarrow p)$ (by IV-1.19) $= \downarrow g(p)$ (by O-1.11) is prime by (4). □

It appears on the surface that we proved a more general statement in so far as for (3) implies (1) we really used only that the prime ideals separate the points from ideals. But this property in itself is sufficient to make S distributive.

Corollary IV-1.23.

(i) *Let S be a distributive continuous lattice and T a complete lattice. Then the mapping $g \mapsto D(g) : INF(S, T) \to SUP(T, S)$ induces a bijection from the subset of all prime preserving maps onto the subset of all lattice homomorphisms in $SUP(T, S)$.*

(ii) *If T is continuous, then the same map induces a bijection from the set of all prime preserving morphisms in $CL(S, T)$ onto the subset of all lattice homomorphisms in $CL^{\mathrm{op}}(T, S)$.* □

Corollary IV-1.24. *The category of all distributive continuous, resp. algebraic, lattices, with prime preserving CL-maps, resp. AL-maps, is dual to the category of all continuous, resp. algebraic, distributive lattices with morphisms preserving arbitrary sups, finite infs and respecting the way-below relation, resp. compact elements.* □

Exercises

Exercise IV-1.25. If T is a **dcpo** and $S \subseteq T$ such that some function $d\colon T \to S$ is lower adjoint to the inclusion $S \to T$ (that is, S is the image of T of some closure operator of T (see O-3.10)), show that S is closed in T for directed sups iff d is Scott open.

Hint. Check the proof of IV-1.5. □

Exercise IV-1.26. Furnish the proof of IV-1.20. □

Exercise IV-1.27. Let L_1 and L_2 be complete lattices. The poset $SUP(L_1, L_2^{\mathrm{op}})^{\mathrm{op}}$ is called the *tensor product* of L_1 and L_2 and is written $L_1 \otimes L_2$.

(i) Show that $L_1 \otimes L_2 = SUP(L_1, L_2^{\mathrm{op}})^{\mathrm{op}} \cong INF(L_2^{\mathrm{op}}, L_1) = SUP(L_2, L_1^{\mathrm{op}})^{\mathrm{op}} = L_2 \otimes L_1$.

(ii) Is the tensor product associative?

(iii) Show that this tensor product classifies the bimorphisms in SUP in the sense that

$$SUP(L_1 \otimes L_2, L_3) \cong SUP(L_1, SUP(L_2, L_3))$$ □

Exercise IV-1.28. Prove the following.

(i) L_1 and L_2 are complete lattices. For a subset $G \subseteq L_1 \times L_2$ with $(0, 1)$, $(1, 0) \in G$ the following statements are equivalent:
  (1) $G = \{(x_1, x_2) \in L_1 \times L_2 : x_2 \leq f(x_1)\}$ for some $f \in L_1 \otimes L_2$.
  (2) If $X \leq G$ then $(\inf pr_1 X, \sup\ pr_2 X)$, $(\sup\ pr_1 X, \inf pr_2 X) \in G$.
  (3) If $X_1 \times X_2 \subseteq G$, then $\downarrow(\sup X_1, \sup X_2) \subseteq G$.
  (4) G is Scott closed in $L_1 \times L_2$, and if $(x_1, x_2), (y_1, y_2) \in G$, then

$$(x_1 \wedge y_1, x_2 \vee y_2), (x_1 \vee y_1, x_2 \wedge y_2) \in G.$$

(ii) In particular, the poset $\mathcal{G}(L_1, L_2)$ of all Scott closed subsets $G \subseteq L_1 \times L_2$ satisfying (1)–(4) is isomorphic to $L_1 \otimes L_2$.

(iii) The tensor product $L_1 \otimes L_2$ is a continuous lattice iff L_1 and L_2 are continuous lattices.

Hint. For these and more details we refer to [Bandelt, 1980b]. □

Old notes

The general idea of Galois connections is a classical theme in lattice theory almost since its inception, as is exemplified by the paper of O. Ore [Ore, 1944]. In the Notes for Section O-3 we have given some references surveying the background of Theorem IV-1.3. The results of Theorem IV-1.4 (and of IV-1.5, IV-1.6) were new in the *Compendium*, but the equivalence of (1) and (3) was first proved by [Hofmann and Stralka, 1976]. The duality of CL and CL^{op} originated from the same source; this result turned out to be rather useful in dealing with homomorphisms between continuous lattices ([Hofmann and Mislove, 1975; Hofmann and Mislove, 1976; Hofmann and Mislove, 1977] and [Hofmann *et al.*, 1973; Hofmann *et al.*, 1975], [Hofmann and Stralka, 1976]).

The duality described in Theorem IV-1.16 was extensively discussed by [Hofmann *et al.*, 1973], [Hofmann *et al.*, B1974], [Hofmann *et al.*, 1975], although in the form of a Pontryagin duality for semilattices it was introduced by [Austin, 1963]. For further references check [Hofmann *et al.*, B1974]. Theorem IV-1.18 and Proposition IV-1.19 come from [Hofmann and Stralka, 1976]. Except for some embellishments, Theorem IV-1.22 occurs in a paper of [Hofmann and Lawson, 1978]. The material on tensor products in the exercises is due to [Bandelt, 1980b], where one can find further references on the topic of tensor products of continuous lattices.

IV-2 Duality of Domains

This section is devoted to an attractive self-duality theory of a category of all domains equipped with suitable morphisms. This duality assumes a particularly nice form in the subcategory of semilattices with top element. Hence *we assume throughout this section that semilattices all come with a largest element* 1 *and all semilattice homomorphisms preserve* 1.

We have encountered open filters (always assumed to be nonempty) in Chapter I, Definition I-3.1, and in Chapter II, Proposition II-1.11 ff. The concept of open filters on a **dcpo** L and that of the **dcpo**

$$\mathrm{OFilt}(L) = \{U \in \sigma(L) : U \text{ is a filter}\}$$

of all Scott open filters of L under inclusion will be crucial in the following. If $f\colon S \to T$ is a Scott-continuous function between posets, then pulling back Scott open sets, the preimage $f^{-1}(V)$ of every Scott open set $V \subseteq T$ is Scott open in S. But the monotone map $f\colon 2^2 \to 2$ sending $(0, 0)$ to 0 and the remainder to 1 is a Scott-continuous map between domains and $V = \{1\}$ is a filter in 2, but $f^{-1}(V) \neq \emptyset$ fails to be a filter. Thus we do not expect in general that

$$f^{-1}(V) \text{ is an open filter of } S \text{ for every open filter } V \subseteq T. \tag{1}$$

The following proposition shows that functions with the property that inverse images of open filters are open filters arise in a natural way.

Proposition IV-2.1. *Let S and T be continuous semilattices. Then for a function $f\colon S \to T$ such that $f(1) = 1$, the following statements are equivalent:*

(1) *for every open filter V in T, the preimage $f^{-1}(V)$ is an open filter of S;*
(2) *f is a Scott-continuous semilattice homomorphism.*

Proof: (2) implies (1): If V is a Scott open subset of T and f is Scott-continuous, $f^{-1}(V)$ is Scott open. If $x, y \in f^{-1}(V)$ then $f(x), f(y) \in V$. But if V is also a filter, then $f(xy) = f(x)f(y) \in V$. Thus $xy \in f^{-1}(V)$. Therefore $f^{-1}(V)$ is a filter as well.

(1) implies (2): We assume that $f\colon S \to T$ is a function satisfying (1). In particular, $V \mapsto f^{-1}(V)$ maps OFilt(T) into $\sigma(S)$. Since T is a domain, OFilt(T) is a basis of $\sigma(T)$ by Theorem II-1.14(3). We conclude that f is Scott-continuous. Next we claim that $x, y \in S$ implies $f(xy) = f(x)f(y)$. Now $xy \leq x, y$, whence $f(xy) \leq f(x), f(y)$ since Scott-continuous functions are monotone; thus $f(xy) \leq f(x)f(y)$ is always true. Now let $v \ll f(x)f(y)$. Then by I-3.3(i), there is an open filter V in T such that $f(x)f(y) \in V \subseteq \uparrow v$. Then $f(x), f(y) \in V$ and thus $x, y \in f^{-1}(V)$. By (1), $f^{-1}(V)$ is a filter in S. Therefore $xy \in f^{-1}(V)$ and thus $f(xy) \in V$. Hence $v \ll f(xy)$. Since T is a domain we have $f(x)f(y) = \sup\{v\colon v \ll f(x)f(y)\}$. It follows that $f(x)f(y) \leq f(xy)$, and this proves the claim. □

It is clear that the composition of two maps between posets each having the property that the inverse images of open filters are open filters also has this property. We are led to the following definition.

Definition IV-2.2. We define the *category of* **dcpo***s with open filter morphisms DCPOFILT* to be the category whose objects are **dcpo**s and whose morphisms are those functions $f\colon S \to T$ between **dcpo**s which have the property that for each open filter V of T the subset $U = f^{-1}(V)$ of S is an open filter, that is, which satisfy condition (1) of IV-2.1. The *category of domains with open filter morphisms DOMFILT* is defined to be the full subcategory of *DCPOFILT* whose objects are domains.

We let *CSEM* denote the category of all continuous semilattices (with 1) and all Scott-continuous semilattice homomorphisms (preserving 1) and call it the *category of continuous semilattices*. □

In view of IV-2.1(1) and the fact that for a domain T, the subset OFilt(T) of $\sigma(T)$ is a basis by II-1.14(3), we note that *DOMFILT*-morphisms are Scott-continuous, i.e., preserve directed sups.

Proposition IV-2.1 has the following immediate consequence.

Corollary IV-2.3. *The category CSEM is a full subcategory of DOMFILT.* $\square$

There are some fairly immediate observations on *DCPOFILT*- and *DOMFILT*-morphisms which we record first.

Proposition IV-2.4. *Let S and T be two* **dcpo***s and $f \in DCPOFILT(S, T)$.*

(i) *For each $V \in \mathrm{OFilt}(T)$ we have $f(S) \cap V \neq \emptyset$.*
(ii) *If T is a domain, then f has a Scott-dense image.*
(iii) *If T is a domain, for each $t \in T$ we have $\downarrow f(S) \cap \twoheaddownarrow t \neq \emptyset$.*

Proof: (i) Since f is a *DCPOFILT*-morphism, $f^{-1}(V) \in \mathrm{OFilt}(S)$ and thus $f^{-1}(V) \neq \emptyset$, i.e., $f(S) \cap V \neq \emptyset$.

(ii) By Theorem II-1.14(3), the Scott open filters form a basis for the Scott topology on the domain T. Thus, (i) implies that $f(S)$ meets every Scott open subset of T and consequently is Scott-dense.

(iii) Assume that $v \ll t$ in T. Then by Proposition I-3.3 there is a $V \in \mathrm{OFilt}(T)$ such that $t \in V \subseteq \twoheaduparrow v$. By (i) above, there is an $s \in S$ such that $f(s) \in V$. Then $v \ll f(s)$ and thus $v \in \downarrow f(S)$. $\square$

We recall from the paragraph preceding Theorem O-3.4 that a function $f : S \to T$ between posets is called *cofinal* if for every $t \in T$ there is an $s \in S$ such that $t \leq f(s)$, i.e., that $T = \downarrow f(S)$.

Corollary IV-2.5. *Assume that $f \in DCPOFILT(S, T)$ and that T is a domain. Then each of the following conditions is sufficient for f to be cofinal.*

(i) *$\downarrow f(S)$ is Scott closed in T.*
(ii) *S has a top.*
(iii) *Every lower set in T is Scott closed.*

Proof: (i) $\downarrow f(S)$ is Scott closed, then by IV-2.4(ii) above, $\downarrow f(S) = T$.

(ii) If S has a top element 1, then $\downarrow f(S) = \downarrow f(1)$, and this is a Scott closed set. So (i) applies.

Condition (iii) is trivially a sufficient condition. $\square$

In case (ii) we saw that $\downarrow f(1) = T$, and thus $f(1)$ is the top of T. Hence we note

Corollary IV-2.6. *Let S be a* **dcpo** *with a top 1 and T a domain. If $f : S \to T$ is a DCPOFILT-morphism, then T has a top and $f(1) = 1$.* $\square$

Condition (iii) is satisfied whenever T is finite. Another example is a reversely well ordered set. In particular, all *DCPOFILT*-morphisms $\chi : S \to 2$ are cofinal, i.e. satisfy $1 \in \chi(S)$.

Definition IV-2.7. For any **dcpo** S, a *DCPOFILT*-morphism $\chi\colon S \to 2$ will be called a *character*. We set $\hat{S} = DCPOFILT(S, 2)$, the set of all characters with the pointwise order, and call $\hat{S}$ the *character poset* or the *dual* of S. □

If it clarifies the context, we shall also call the dual of a **dcpo** more specifically its *Lawson dual*.

We note that for the constant function $\chi\colon S \to 2$ with value 1 we have $\chi^{-1}(1) = S$; this function is a character if and only if S is filtered. For the constant 0 function $\chi\colon S \to 2$ we have $\chi^{-1}(1) = \emptyset$. Since the empty set is not a filter under any circumstances, this function is not a character.

Proposition IV-2.8. *Let S be a* **dcpo**. *Then for each character χ of S, the set $\chi^{-1}(1)$ is an open filter of S. The function $\varepsilon_S\colon \hat{S} \to \mathrm{OFilt}(S)$, $\varepsilon_S(\chi) = \chi^{-1}(1)$ is an isomorphism of posets. In particular, the character poset of a domain is a domain.*

Proof: Since χ is a *DCPOFILT*-morphism and $\{1\}$ is an open filter in 2, the set $\chi^{-1}(1)$ is an open filter of S. Thus $\varepsilon_S\colon \hat{S} \to \mathrm{OFilt}(S)$ is a well-defined function, and since $\chi_1 \le \chi_2$ in $\hat{S}$ iff $\chi_1^{-1}(1) \subseteq \chi_2^{-1}(1)$ it is order preserving. If $U \in \mathrm{OFilt}(S)$ we let $\chi_U\colon S \to 2$ denote the characteristic function of U, taking the value 1 on U and the value 0 elsewhere. Then χ is at once seen to be a character, and the function $U \mapsto \chi_U : \mathrm{OFilt}(S) \to \hat{S}$ is the inverse of ε_S. Therefore, this latter function is an isomorphism of posets $\hat{S} \to \mathrm{OFilt}(S)$. Since for a domain S the poset $\mathrm{OFilt}(S)$ is a domain by Theorem II-1.17, the character poset $\hat{S}$ is a domain. □

After these remarks we have two ways of looking at the dual of a **dcpo**: one is to consider the dual as the character poset, the other, more geometrical, is to consider it as the open filter poset.

The following lemma will be crucial for the upcoming duality theorem. It is formulated in terms of open filters. For an element s in a **dcpo** S let us write

$$\mathcal{U}(s) = \{U \in \mathrm{OFilt}(S) : s \in U\}.$$

Lemma IV-2.9. *Let S be a* **dcpo**.

(i) *$\mathcal{U}(s)$ is Scott open, and if S is a semilattice, then $\mathcal{U}(s)$ is an open filter on* OFilt*(S) for every $s \in S$.*
(ii) *If S is a domain, then the open filters on* OFilt*(S) are exactly the sets of the form $\mathcal{U}(s)$, $s \in S$.*

Proof: (i) $\mathcal{U}(s)$ is clearly an upper set in $\mathrm{OFilt}(S)$; if a union of any family of sets contains s then one of the members does, too, and thus $\mathcal{U}(s)$ is Scott open in OFilt(S). If S is a semilattice, then the intersection of two filters is a filter and thus $\mathcal{U}(s)$ is a filter.

(ii) By Theorem II-1.17 OFilt(S) is a domain. Let $\mathcal{U}$ be an open filter on OFilt(S). For each $U \in \mathcal{U}$ there is a $\underline{U} \in \mathcal{U}$ such that $\underline{U} \ll U$. By II-1.17(ii) we find an element $s_U \in U$ such that $\underline{U} \subseteq \uparrow s_U$. Thus the set

$$D \stackrel{def}{=} \{s \in S : (\exists\, U, V \in \mathcal{U}) V \subseteq \uparrow s \subseteq U\}$$

is not empty. We claim it is directed: Let $s_1, s_2 \in D$. Then there exist $U_j, V_j \in \mathcal{U}$ such that $V_j \subseteq \uparrow s_j \subseteq U_j$ for $j = 1, 2$. Since $\mathcal{U}$ is a filter, there is a $U \in \mathcal{U}$ such that $U \subseteq V_1 \cap V_2$. Since $\mathcal{U}$ is open, there is a $V \in \mathcal{U}$ such that $V \ll U$; then by II-1.17(ii) on $\mathcal{U}$ again there is an $s \in U$ such that $V \subseteq \uparrow s$. Hence $s \in D$, and since $s \in U \subseteq V_1 \cap V_2 \subseteq \uparrow s_1 \cap \uparrow s_2$ we have $s_1, s_2 \leq s$. This shows that D is directed as asserted. Since S is a **dcpo**, the element $s = \sup D$ is well defined. First we claim that $s \in U$ for all $U \in \mathcal{U}$: If $U \in \mathcal{U}$ we have $\underline{U} \subseteq \uparrow s_U \subseteq U$; then $s_U \in D \cap U$ and thus $s = \sup D \in U$. Secondly we claim that $s \in W \in$ OFilt(S) implies $W \in \mathcal{U}$. Since $\sup D = s \in W$ and W is Scott open, there is a $w \in D$ with $w \in W$. By definition of D, there is a $V \in \mathcal{U}$ such that $V \subseteq \uparrow w \subseteq W$. Since every filter is an upper set, $W \in \mathcal{U}$ follows as asserted. This completes the proof. □

Proposition IV-2.10. *If $f \colon S \to T$ is a DCPOFILT-morphism, then $\hat{f} \colon \hat{T} \to \hat{S}$ defined by $\hat{f}(\chi) = \chi \circ f$ is Scott-continuous. If S is a domain, then $\hat{f}$ is a DCPOFILT-morphism. The assignment $\hat{\ }$: DOMFILT $\to$ DOMFILT is a contravariant functor.*

Proof: Let $f \colon S \to T$ be a *DCPOFILT*-morphism. Then for every character $X \colon T \to 2$ the function $\hat{f}(\chi) = \chi \circ f \colon S \to 2$ is a character of S. Thus $\hat{f}\colon \hat{T} \to \hat{S}$ is a well-defined function. If $V \in$ OFilt(T) then $f^{-1}(V) \in$ OFilt(S). Hence the function OFilt(f): OFilt(T) $\to$ OFilt(S) given by OFilt(f)(V) $= f^{-1}(V)$ is well defined and corresponds to $\hat{f}$ via the commutative diagram

$$\begin{tikzcd}
\hat{T} \arrow[r, "\hat{f}"] \arrow[d, "\varepsilon_T"'] & \hat{S} \arrow[d, "\varepsilon_S"] \\
\mathrm{OFilt}(T) \arrow[r, "\mathrm{OFilt}(f)"'] & \mathrm{OFilt}(S)
\end{tikzcd}$$

Since $V \mapsto f^{-1}(V)$ preserves unions, OFilt(f) is Scott-continuous. Now let $\mathcal{V}$ be an open filter in OFilt(S) and assume that S is a domain. By Lemma

IV-2.9, there is a $u \in S$ such that $\mathcal{V} = \{U \in \mathrm{OFilt}(S) : u \in U\}$. Set $\mathcal{U} \stackrel{def}{=} \mathrm{OFilt}(f)^{-1}(\mathcal{V})$. Now $U \in \mathcal{U}$ iff $f^{-1}(U) = \mathrm{OFilt}(f)(U) \in \mathcal{V}$, i.e., $u \in f^{-1}(U)$, which is tantamount to $f(u) \in U$. By Lemma IV-2.9 again this means that $\mathcal{U}$ is an open filter in OFilt(T). This shows that OFilt(f) and thus also $\hat{f}$ are *DCPOFILT*-morphisms.

The assignment which maps a domain S to its dual $\hat{S}$ and a *DOMFILT*-morphism $f\colon S \to T$ to $\hat{f}\colon \hat{T} \to \hat{S}$ is immediately seen to be a functor satisfying $\widehat{fg} = \hat{g}\hat{f}$. □

Definition IV-2.11. We say that a **dcpo** is *open filter determined* if for each pair of elements $s, t \in S$ such that $t \not\leq s$ there is an open filter $U \in \mathrm{OFilt}(S)$ such that $t \in U$, but $s \notin U$. This is equivalent to saying that the *characters separate the points* of S in the sense that, whenever $s \neq t \in S$, there is a $\chi \in \hat{S}$ such that $\chi(s) \neq \chi(t)$. □

The following is straightforward.

Remark IV-2.12. (i) *A* **dcpo** *S is open filter determined iff for each $t \in S$ we have $\bigcap \mathcal{U}(t) = \uparrow t$.*

(ii) *A domain S is open filter determined or, equivalently, the characters separate the points of S.*

(iii) *Every frame with enough points, i.e., every complete lattice L in which the set PRIME(L) of primes is order generating, is open filter determined. In particular, for a space X, the topology $\mathcal{O}(X)$ is open filter determined.*

Proof: Part (i) is immediate from the definition.

(ii) If S is domain, then $\sigma(S)$ is a T_0 space by II-1.4(iii) and every point of S has a basis of neighborhoods which are open filters by II-1.14(3). This proves the claim.

(iii) Let $a \not\leq b$ in L. As the primes are supposed to be order generating, there is a prime p with $b \leq p$ but $a \not\leq p$. Then $L \backslash \downarrow p$ is an open filter containing a but not b. This establishes the claim. Finally observe that the open sets of a space form a frame with enough points. □

The following is a general categorical setup. For a **dcpo** S which is a semilattice or a domain we define functions $\eta_S\colon S \to \hat{\hat{S}}$ and $\varphi_S\colon S \to$ OFilt (OFilt(S)) by $\eta_S(s)(\chi) = \chi(s)$, and by $\varphi_S(s) = \{U \in \mathrm{OFilt}(S) : s \in U\}$. By Lemma IV-2.9, φ_S is well defined. If $U = \chi^{-1}(1) = \varepsilon_S(\chi)$ then $\chi(s) = 1$

iff $s \in U$. A closer look shows that this makes the following diagram commutative:

$$\begin{tikzcd}
S \arrow[r, "\eta_S"] \arrow[d, "\varphi_S"'] & \hat{\hat{S}} \arrow[d, "\varepsilon_{\hat{S}}"] \\
\mathrm{OFilt}(\mathrm{OFilt}(S)) \arrow[r, "\mathrm{OFilt}(\varepsilon_S)"'] & \mathrm{OFilt}(\hat{S})
\end{tikzcd}$$

Theorem IV-2.13. *Let S be a* **dcpo** *such that* $\mathcal{U}(s)$ *is a filter for all* $s \in S$ *– this is the case if S is a semilattice or a domain. Then we have maps*

$$\eta_S \colon S \to \hat{\hat{S}} \quad \textit{and} \quad \varphi_S \colon S \to \mathrm{OFilt}(\mathrm{OFilt}(S)).$$

(i) *If S is open filter determined, then these functions are injective.*
(ii) *If* OFilt(S) *is a domain and* $U \ll V$ *in* OFilt(S) *implies the existence of a* $v \in V$ *with* $U \subseteq \uparrow v$*, then they are surjective.*
(iii) *If S is a domain,* φ_S *and* η_S *are isomorphisms of domains.*

Proof: By Proposition IV-2.10 it suffices to consider φ_S which is well defined by our first assumption. Part (i) follows directly from the assumption. (ii) By Lemma IV-2.9(ii), φ_S is surjective under the given hypotheses. From Theorem II-1.17(ii) and Remark IV-2.12(ii) we conclude that φ_S is an isomorphism if S is a domain. □

The results IV-2.10 and IV-2.13 yield the following key theorem, reminiscent of the Pontryagin Duality Theorem of Locally Compact Abelian Groups.

Theorem IV-2.14. (The Lawson Duality Theorem of Domains) *The category DOMFILT of domains is self-dual under the contravariant functor* ^. *The dual of a domain S may be considered as the character poset* $\hat{S}$ *as well as the open filter domain* OFilt(S). □

If for a domain S and elements $(\chi, s) \in \hat{S} \times S$ we write $\langle \chi, s \rangle_S = \chi(s)$, then for a *DOMFILT*-morphism $f \colon S \to T$ we have the formula $\langle \chi, f(s) \rangle_T = \langle \hat{f}(\chi), s \rangle_S$. This pairing is symmetric in the sense that S may be regarded as the character poset of $\hat{S}$.

Proposition IV-2.15. *Assume that $f \in DOMFILT(S, T)$. Then*

(i) *f is a monic iff f is injective,*
(ii) *f is an epic iff $\hat{f}$ is injective.*

Proof: An injective morphism is monic and a surjective morphism is epic; it is the reverse implications that have to be proved.

(i) Assume f to be monic in *DOMFILT* and assume $f(s_1) = f(s_2)$. Let $Max(S)$ denote the subset of maximal elements of the **dcpo** S and set $E = Max(S) \cup \{\perp\}$ with a smallest element $\perp$ adjoined to the set $Max(S)$ on which two different elements are incomparable. Then E is a domain, and the functions $\alpha_j \colon E \to S$, $j = 1, 2$, which are the identity on $Max(S)$ and map $\perp$ to s_j, respectively, are *DOMFILT*-morphisms. Moreover, $f \circ \alpha_1 = f \circ \alpha_2$. Since f is a monic we conclude $g_1 = g_2$. This implies $s_1 = g_1(\perp) = g_2(\perp) = s_2$. Hence f is injective.

(ii) Assume that f is an epic. Then $\hat{f}$ is a monic by duality, IV-1.10. By (i) this means that $\hat{f}$ is injective. □

It is not clear at this stage exactly how to characterize epimorphisms in a more explicit way. However, by way of pragmatic experience, it is always substantially more difficult to characterize epimorphisms than to characterize monomorphisms.

The following duality theorem, while, in the face of Theorem IV-2.14, not being the most general one in this area, is the most elegant one from the vantage point of pure mathematical structure.

Theorem IV-2.16. (The Duality Theorem of Continuous Semilattices) *The category CSEM of continuous semilattices and Scott-continuous semilattice morphisms is self-dual under the contravariant functor ˆ. The dual of a continuous semilattice S may be considered as the character poset $\hat{S}$ as well as the open filter poset* OFilt(S).

Proof: By Corollary IV-2.3, the category *CSEM* is a full subcategory of *DOMFILT*. In particular, every (*DOMFILT*-)character $\chi \colon S \to 2$ of a continuous semilattice is a semilattice character. The pointwise product of two of these is again a semilattice character; equivalently, since every filter on S is a sub-semilattice, the intersection of two filters is a semilattice. Hence $\hat{S}$ is not only a domain but a semilattice and is therefore a continuous semilattice. If $f \colon S \to T$ is a *CSEM*-morphism, then $\hat{f}(\chi_1\chi_2) = (\chi_1\chi_2) \circ f = (\chi_1 \circ f)(\chi_2 \circ f) = \hat{f}(\chi_1)\hat{f}(\chi_2)$. Thus $\hat{f} \colon \hat{T} \to \hat{S}$ is a *CSEM*-morphism. Hence the contravariant functor ˆ : *DOMFILT* → *DOMFILT* maps *CSEM* faithfully onto itself and by Theorem IV-2.14 induces a duality. □

One of the attractive features of this duality theorem is that it is completely in accordance with a whole class of duality theorems based on character theories, for which the Pontryagin Duality Theorem for locally compact groups is the most representative one. Here the characters of a continuous semilattice are Scott-continuous semilattice homomorphisms into the continuous semilattice of two elements. Another point is that the category *CL* of continuous lattices is a subcategory of *CSEM*, although not a full one. Moreover, the character semilattice $\hat{L}$ of a continuous lattice L in the sense of Theorem IV-2.16 is not in general a lattice again; a prominent example is the distributive continuous lattice $\mathcal{O}(X)$ of a sober locally compact space, whose dual is $Q(X)$ according to Theorem IV-2.18 below, and $Q(X)$ may well fail to be a lattice. Thus the Lawson duality formalism allows no restriction to continuous lattices, but the category of continuous semilattices appears to be the comfortable and economic supercategory of *CL* in which the duality works perfectly.

The duality theories which we have seen in this section induce other useful duality theorems on smaller subcategories. We record this now.

Theorem IV-2.17. *The Lawson self-duality of the category DOMFILT of domains induces self-dualities of each of the full subcategories ADOMFILT and ASEM of algebraic domains and algebraic semilattices, respectively.*

Proof: Exercise IV-2.25. □

In Chapters I and II we encountered numerous links between order theory and topology. For the following application of the present duality theory to locally compact spaces we refer back to the I-1.24 and II-1.20–II-1.24. In particular recall that the partial order we consider on the set $Q(X)$ of compact saturated subset is $\supseteq$. The first two parts of the Hofmann–Mislove Theorem II-1.20, II-1.24, Corollary II-1.22, and the Duality Theorem of Continuous Semilattices IV-2.16 above yield immediately

Theorem IV-2.18. (Hofmann–Mislove Theorem III) *For a locally compact sober space X the topology $\mathcal{O}(X)$ and the semilattice $Q(X)$ of compact saturated subsets of X are Lawson dual continuous semilattices.* □

In the very early Example I-1.7(5) we saw that the local compactness of a space implies that its topology $\mathcal{O}(X)$ is a continuous lattice. The spectral theory of continuous lattices which we shall develop in the next chapter will show that a sober space X for which $\mathcal{O}(X)$ is a domain is locally compact (see V-5.6). Theorem IV-2.18 above shows that if a sober space X is locally compact, then

$Q(X)$ is a continuous semilattice, but after Example II-1.25 we know that the converse fails in general. Thus the Lawson dual of $Q(X)$ may be properly "bigger" than $\mathcal{O}(X)$.

If $\mathcal{O}(X)$ is a domain then by II-1.20, the **dcpo** OFilt($\mathcal{O}(X)$) is a domain. By II-1.17 this implies that OFilt(L) is a domain with $L = \text{OFilt}(\mathcal{O}(X))$. We recall that $\mathcal{O}(X)$ is open filter determined by Remark IV-2.12(iii) and that $\mathcal{O}(X)$ is a lattice and thus, in particular, a semilattice. Thus Theorem IV-2.13 applies with $L = \mathcal{O}(X)$ and shows that

$$\varphi_{\mathcal{O}(X)}\colon \mathcal{O}(X) \to \text{OFilt}(L) \tag{†}$$

is an isomorphism, provided $Q(X)$ is a domain and for two open filters $\mathcal{U}$ and $\mathcal{V}$ on $\mathcal{O}(X)$ the relation $\mathcal{U} \ll \mathcal{V}$ implies the existence of an open set V in $\mathcal{V}$ contained in all members of $\mathcal{U}$. In view of II-1.19 this means that in $Q(X)$ the relation $K_1 \ll K_2$ holds iff K_2 is contained in the interior of K_1.

Proposition IV-2.19. *Let X be a sober space with the property that in $Q(X)$ the relation $K_1 \ll K_2$ is equivalent to $K_2 \subseteq \text{int}(K_2)$. Then $Q(X)$ is a continuous semilattice if and only if X is locally compact.* □

Exercises

Exercise IV-2.20. Prove the following.

(i) If S is a finite poset, then $\hat{S} \cong S^{\text{op}}$.
(ii) If $S = [0, 1]$, then $\hat{S} \cong (0, 1] \cup \{\top\}$, where $\top$ is an adjoined largest element.

Hint. For (ii) send t to $(1 - t, 1]$ and $\top$ to $[0, 1]$. □

Exercise IV-2.21. Show that a domain S with largest element 1 has a bottom element 0 iff $1 \ll 1$ in $\hat{S}$. □

Exercise IV-2.22. The way-below relation on a continuous domain is said to be *multiplicative* if $a \ll x$ and $a \ll y$ imply that there exists z such that $a \ll z \ll x, y$. Show that a continuous semilattice S with 1 is a lattice iff $\hat{S}$ has a multiplicative way-below relation.

Hint. For open filters, $F \ll G_1, G_2$ iff there exists $z_i \in G_i$ such that $F \subseteq {\uparrow}z_1 \cap {\uparrow}z_2$ (by II-1.17(ii)), which implies $F \subseteq {\uparrow}(z_1 \vee z_2) \subseteq G_1 \cap G_2$ iff $F \ll G_1 \cap G_2$. For the converse, it suffices by duality to assume S has a multiplicative way-below relation and show $\hat{S}$ is a lattice. In this case $F_1, F_2 \in \text{OFilt}(S)$ have least upper bound $\uparrow\{xy \colon x \in F_1, y \in F_2\}$, an open filter by the multiplicative property. $\square$

The next exercise is a corollary of the preceding two.

Exercise IV-2.23. Let S be a continuous semilattice with 1. Show that S is a continuous lattice iff $\hat{S}$ has a multiplicative way-below relation and $1 \ll 1$ in $\hat{S}$. $\square$

Exercise IV-2.24. Let S, T be objects and $f \colon S \to T$ a morphism in *CSEM*. Show that f has a lower adjoint iff $\hat{f} \colon \hat{T} \to \hat{S}$ preserves the relation $\ll$. $\square$

Exercise IV-2.25. Spell out the details of the proof of Theorem IV-2.17. $\square$

New notes

The duality theory of this section is due to [Lawson, 1979]. The theory was sketched in the exercises of the *Compendium*, but has now been upgraded to a fuller treatment. It is one of a large class of dualities in which the dual object arises (or can be viewed as arising) as the hom-set into the two point set equipped with some appropriate structure. Connections of the duality with the Hofmann–Mislove theory provide a nice link with topological ideas. The duality reappears in another guise in Section V-1.

IV-3 Morphisms into Chains

For lattice ordered structures L the morphisms into a complete chain C play a role which is analogous to that of characters in the theory of groups. Along this line it is of great importance to know when the morphisms into the unit interval $\mathbb{I} = [0, 1]$ separate the points. After the developments in Chapters I–III for complete lattices we are primarily interested in $INF^{\uparrow}$-morphisms, that is, morphisms which preserve arbitrary infs and directed sups (see I-2.10ff. and III-1.8). For **dcpo**s in general, we are interested in morphisms that preserve directed sups and have a lower adjoint (IV-1.4).

The duality which we introduced in the preceding sections, notably Theorem IV-1.10, allows us to reduce the question of surjective $INF^{\uparrow}$-morphisms $L \to C$ with C a chain to a question on SUP^0-embeddings $C \to L$; these latter allow by IV-1.4(3) the simpler description of being sup preserving and respecting the relation $\ll$, since complete chains are always continuous lattices. Therefore, we are led to consider subsets in a **dcpo** L which are totally ordered with respect to the way-below relation $\ll$. We first restrict our attention to domains and morphisms into the unit interval $\mathbb{I} = [0, 1]$ with the usual order. We then turn to more general chains. In order to maintain a framework of suitable generality we formulate the discussion in terms of an arbitrary auxiliary order $\prec$ on L (see I-1.11). We soon will notice that the strong interpolation property (I-1.17) plays a central role.

Proposition IV-3.1. *Let L be a domain with a smallest element $\perp$ and let a, b be elements of L such that $a \not\leq b$. Then there is a function $g\colon L \to \mathbb{I}$ preserving directed sups and having a lower adjoint $d\colon \mathbb{I} \to L$ such that $g(a) = 1$ and $g(b) = 0$. In particular, g preserves also all existing infs.*

Proof: We proceed along the lines of the proof of Urysohn's Lemma in general topology (see e.g. [Kelley, B1955], p. 115). As L is a domain and $a \not\leq b$, there is an element $x_0 \ll a$ such that $x_0 \not\leq b$. Let $x_1 = a$. First let us define elements x_p for every p in the set of dyadic rational numbers $D = \{p = m/2^n\colon n \in \mathbb{N}, m = 0, 1, \ldots, 2^n\}$ in such a way that

$$p < q \Rightarrow x_p \ll x_q. \tag{$*$}$$

For this we represent our dyadic rationals $p \neq 0$ in the form $p = m/2^n$ with odd m, and we define x_p recursively over n by successive interpolation. For $p = 0$ and $p = 1$ we choose x_0 and x_1 as above. If the x_q are already defined for all $q = m/2^k$ with $k < n$ and if $p = m/2^n$ is given with any odd $m < 2^n$, then $m - 1$ and $m + 1$ are even and x_{q_1} and x_{q_2} are already defined for $q_1 = (m - 1)/2^n$ and $q_2 = (m + 1)/2^n$ in such a way that $x_{q_1} \ll x_{q_2}$. By the interpolation property I-1.9, there is an element x_p such that $x_{q_1} \ll x_p \ll x_{q_2}$.

As the elements x_p form a chain in the domain L, we may define $d\colon \mathbb{I} \to L$ by $d(r) = \sup\{x_p\colon p < r, p \in D\}$. Note that $d(0) = \perp$. Clearly, d preserves arbitrary sups. Hence d has an upper adjoint $g\colon L \to \mathbb{I}$ defined by $g(x) = \sup\{r \in \mathbb{I} : d(r) \leq x\}$ (see Theorem O-3.4) which preserves all existing infs by O-3.3. As $d(1) \leq x_1 = a$, we have $g(a) = 1$. For real numbers in the unit interval with $r < s$ one may find dyadic rationals p, q with $r < p < q < s$. Then $d(r) \leq x_p \leq x_q \leq d(s)$ by the definition of d and $(*)$, whence $d(r) \ll d(s)$. Hence the upper adjoint g preserves directed sups by IV-1.4 and $g(b) = 0$, as $d(r) \not\leq b$ for all $r > 0$. □

Corollary IV-3.2. *For every domain L with a least element there is an order embedding g from L into some power $\mathbb{I}^X$ of the unit interval $\mathbb{I}$ preserving directed sups and all infs that exist in L.*

Proof: Let X be the set of all pairs (a, b) in $L \times L$ with $a \not\leq b$. By the preceding proposition we can find for every $(a, b) \in X$ a function $g_{a,b}\colon L \to \mathbb{I}$ preserving directed sups and all existing infs such that $g_{a,b}(a) = 1$, $g_{a,b}(b) = 0$. Let $g\colon L \to \mathbb{I}^X$ be the product map $x \mapsto (g_{a,b}(x))_{(a,b)\in X}$. Then g preserves directed sups and existing infs. In order to see that g is an order embedding, take $a \not\leq b$ in L. Then $(a, b) \in X$ and consequently $g_{a,b}(a) = 1 > 0 = g_{a,b}(b)$, whence $g(a) \not\leq g(b)$. □

Powers $\mathbb{I}^X$ of the unit interval $\mathbb{I}$ are also called *cubes*. The previous corollary tells us that domains can be embedded into cubes order theoretically. This is still unsatisfactory as it does not tell us which subsets of a cube are domains. For continuous lattices and, more generally, for bounded complete domains the situation is much better:

Theorem IV-3.3. *For a poset L the following statements are equivalent:*

(1) *L is a continuous lattice, resp. a bounded complete domain;*
(2) *L is isomorphic to a subset of a cube $\mathbb{I}^X$ closed under directed sups and arbitrary infs, resp. nonempty infs.*
(3) *L is a complete lattice and the functions $g\colon L \to \mathbb{I}$ preserving directed sups and arbitrary, resp. nonempty, infs separate the points of L.*

Proof: (1) implies (2): This is a direct consequence of Corollary IV-3.2, as in a continuous lattice, resp. bounded complete domain, all subsets, resp. all nonempty subsets, have an inf.

(2) implies (1): As every cube $\mathbb{I}^X$ is a continuous lattice, a subset closed for directed sups and arbitrary infs, resp. nonempty infs, is a continuous lattice, resp. a bounded complete domain, by I-2.11(ii).

That (2) iff (3) is obvious. □

This theorem – which in the introduction to this book we used as a first, preliminary definition of a continuous lattice – allows us to consider very concrete representations of continuous lattices; but we should recall that the representation of a continuous lattice as a substructure of a cube in itself does not tell us too much about its structure, although for many questions the existence of enough *CL*-morphisms into $\mathbb{I}$ is of vital importance. The elements of $CL(L, \mathbb{I})$ play the role of *characters*.

We now turn to the setting of morphisms into general chains, which we obtain through the dual embedding of the chain. We restrict our attention to complete lattices although there is a straightforward generalization to complete semilattices.

Definition IV-3.4. Let L be a complete lattice. Recall from I-1.11 that a binary relation on L which satisfies the conditions (i) $a \prec b \Rightarrow a \leq b$, (ii) $x \leq a \prec b \leq y \Rightarrow x \prec y$ and (iii) $(\forall x)\ 0 \prec x$ has been called an *auxiliary order* on L.

In analogy to I-1.13 we call an auxiliary order $\prec$ *quasiapproximating* provided that for all $x \in L$ we have $x = \sup\{y \in L : y \prec x\}$. Note that we do not require the set $\{y \in L : y \prec x\}$ to be directed as for an *approximating* auxiliary relation.

A subset $C \subseteq L$ is a *strict chain* (or, more accurately, a $\prec$-*strict chain*, if specification is needed) iff $x, y \in C$ implies $x \prec y$ or $x = y$ or $y \prec x$ in L. □

We now consider complete lattices with an auxiliary relation. All singletons are examples of strict chains, and for each $x \in L$ the set $\{0, x\}$ is a strict chain by IV-3.4(iii). The axiom of choice allows us to find a much greater variety of strict chains, however.

Lemma IV-3.5. *Let L be a complete lattice and C_0 a strict chain. Then the collection Γ of all strict chains C in L with $C_0 \subseteq C$ is inductive with respect to $\subseteq$. Consequently, every strict chain is contained in a maximal one.*

Proof: It is clear that Γ is inductive, and so the result follows by Zorn's Lemma. □

We now concentrate our attention on maximal strict chains.

Lemma IV-3.6. *Let C be a maximal strict chain in a complete lattice L. For a subset $X \subseteq C$, set $s = \sup_L X$. Then $\sup_C X$ exists, and*

(i) $\sup_C X = \min({\uparrow}s \cap C)$,
(ii) *if $s \in C$, then $s = \sup_C X$,*
(iii) *if $s \notin C$, then $s < \sup_C X$.*

Proof: We have the two cases to consider: that $s \in C$ and that $s \notin C$. If $s \in C$, then $s = \sup_C X$ and evidently $s = \min({\uparrow}s \cap C)$. Now suppose that $s \notin C$. Then for any $c \in C$ with $c < s$ there is some $d \in C$ with $c < d < s$. Since C is strict, we have $c \prec d$, whence $c \prec s$ by IV-3.4(ii). Since C is maximal strict, the chain $C \cup \{s\}$ is no longer strict; after what was just said this can only be the case if there is some $c^* \in C$ with $s < c^*$ but not $s \prec c^*$. Suppose for a moment that there were a $d^* \in C$ with $s < d^* < c^*$. Then, since C is strict, we could conclude that $d^* \prec c^*$ and thus $s \prec c^*$ because of IV-3.4(ii), which is not the case. But this means that $c^* = \min({\uparrow}s \cap C)$. □

Proposition IV-3.7. *Every maximal strict chain in a complete lattice is a complete chain (in its own right).*

Remark. Observe that we do not and cannot claim that a maximal strict chain is either sup closed or inf closed in L.

Proof of proposition: By IV-3.6 every subset of a maximal strict chain C has sup in C. By O-2.2 this suffices. $\square$

In the following study of the structure of maximal strict chains we often refer to the *strong interpolation property* for $\prec$, which we recall for reference from I-1.17:

$$x \prec z \text{ and } x \neq z \text{ together imply } (\exists y \in L)(x \prec y \prec z \text{ and } x \neq y). \qquad \text{(SI)}$$

Proposition IV-3.8. *Let C be a maximal strict chain in a complete lattice L.*

(i) $0 \in C$.
(ii) *If* $\max C \prec 1$, *then* $1 \prec 1$ *and* $\max C = 1$.
(iii) *If $\prec$ satisfies the strong interpolation property, then we have for $x, z \in C$*

$$x \prec z \textit{ and } x \neq z \textit{ together imply } (\exists y \in C)(x \prec y \prec z \text{ and } x \neq y). \qquad (\mathrm{SI}_C)$$

Proof: (i) For any $s \in L$ one has $0 \prec s$ by I-1.11(iii); thus, (i) follows from the maximality of C. (ii) Suppose $\max C \prec 1$. If $\max C < 1$, then $C \cup \{1\}$ would be a larger strict chain, which would contradict the maximality of C. Hence, $\max C = 1$, and thus $1 \prec 1$.

(iii) Let $x \prec z$ with $x \neq z$. If $[x, z]_C$ contains an element $y \neq x, z$, then $x \prec y \prec z$, since C is strict. If, however, $[x, z]_C = \{x, z\}$, then we apply (SI) to find a $y \in L$ with $x \neq y$ and $x \prec y \prec z$; then $C \cup \{y\}$ is a strict chain, and by maximality of C we conclude $y \in C$. $\square$

We say that a strict chain C *satisfies the interpolation property* if it satisfies condition (SI_C).

The trouble with maximal strict chains is that, despite their completeness in their own right, they are not in general sup closed in L which is what we need for a SUP^0-embedding. If, for example, L is the square $\mathbb{I}^2$, then the chain $([0, 1[\times \{0\}) \cup \{(1, 1)\}$ is maximal strict but not sup closed. We therefore need a modification procedure. The following lemmas will prove useful in showing that the modified chains we construct have desirable properties.

Lemma IV-3.9. *Let L be a complete lattice equipped with an auxiliary order $\prec$. Let C be a strict chain in L which satisfies the interpolation property.*

If $p, q \in C$ and $p < q$, then there exists a $y \in L$ such that $p < y \prec q$ and $y = \sup_L\{x \in C : x \prec y\}$.

Proof: Since C is strict, we have $p \prec q$. By (SI_C) there exists $w \in C$ such that $p \prec w \prec q$ and $p \neq w$. Again by (SI_C) there exists $x_1 \in C$ such that $p \prec x_1 \prec w$ and $p \neq x_1$. Inductively choose $x_n \in C$ such that $x_{n-1} \prec x_n \prec w$. Let $y = \sup_L\{x_n\}$. Then $x_n \prec x_{n+1} \leq y$ implies $\sup_L\{x \in C : x \prec y\} \geq \sup_L\{x_n\} = y$. Finally $p < x_1 \leq y \leq w \prec q$, that is $p < y \prec q$. □

We call a chain $C \subseteq L$ *sup closed* if $X \subseteq C$ implies $\sup_L X \in C$.

Lemma IV-3.10. *For a sup closed strict chain C in a complete lattice L with an auxiliary order $\prec$ the following statements are equivalent:*

(1) *C satisfies the interpolation property* (SI_C)*;*
(2) *$c = \sup_L\{x \in C : x \prec c\}$, for all $c \in C$.*

Proof: (1) implies (2): Let $c \in C$ and let $d = \sup_L\{x \in C : x \prec c\}$. Then $d \in C$ since C is sup closed. If $d \neq c$, then $d < c$. Since C is strict, we have $d \prec c$. By (SI_C) there exists $y \in C$ such that $d < y \prec c$. This contradicts the definition of d.

(2) implies (1): If $x \prec z$ in C and $x \neq z$ and $z \not\prec z$, then from $z = \sup_L\{y \in C : y \prec z\}$ we can find a $y \in C$ such that $x < y \prec z$. Since C is strict, we conclude $x \prec y$. □

The next proposition involves an important construction which allows one to obtain sup closed chains from given chains.

Proposition IV-3.11. (Chain Modification Lemma) *Let C be a strict chain in a complete lattice L equipped with an auxiliary order $\prec$. Define $D = D(C)$ by*

$$y \in D \textit{ iff } y = \sup_L\{x \in C : x \prec y\}.$$

(i) *D is a strict chain which is sup closed in L.*

(ii) *If C satisfies the interpolation property, then D does also (and hence the equivalent condition* (2) *of* IV-3.10*).*

(iii) *If C satisfies the interpolation property, then $a, b \in C$ and $a < b$ imply that there exists a $d \in D$ such that $a < d \prec b$.*

Proof: (i) Let us first show that D is a strict chain. For this it suffices to show that if $a, b \in D$ and $a \not\leq b$, then $b \prec a$. Since $a = \sup_L\{x \in C : x \prec a\}$, there exists $x \in C$ such that $x \prec a$ and $x \not\leq b$. If $y \in C$ and $y \prec b$ then $y \prec x$

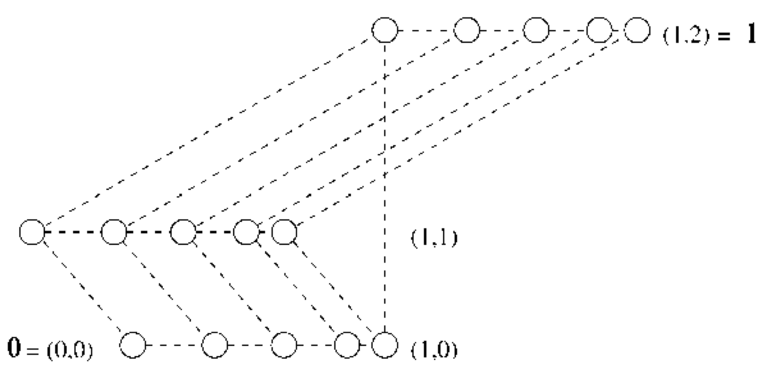
(since C is a strict chain and $x \leq y \prec b$ is impossible). As y was arbitrary, we conclude from $b \in D$ that $b \leq x \prec a$. Thus $b \prec a$. The verification that D is sup closed in L is routine (the sup of sups is a sup).

(ii) Now suppose C satisfies (SI$_C$). Let $p = \sup_L\{b \in D : b \prec d\}$, where $d \in D$. We show the assumption $p < d$ leads to a contradiction. If $p < d$, then for some $c \in C$ we have $c \prec d$ and $c \not\leq p$. If $c \prec c$, then $c = \sup_L\{x \in C : x \prec c\}$. Hence $c \in D$, and thus $c \leq p$, which is impossible. If $c \not\prec c$, then $c < d$. Since $d \in D$, we have $c < c^* \prec d$ for some $c^* \in C$. By Lemma IV-3.9 there exists $y \in L$ such that $c < y \prec c^*$ such that $y \in D$. Then $c < y \leq p$, a contradiction. Hence $p = d$.

(iii) The last assertion follows from IV-3.9. □

For most purposes one can take maximal strict chains and modify them as in IV-3.11 to obtain strict and sup closed chains. One may, however, actually obtain *maximal* strict chains which are sup closed by a little additional work. We take a brief detour to present this construction. The next example shows that if a maximal strict chain is modified as in IV-3.11, it no longer need be maximal strict.

Example IV-3.12. We consider the chain $T = \{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-1}{n}, \ldots, 1\}$ and the chain $3 = \{0, 1, 2\}$ in their natural orders. On the set $S = T \times 3$ we consider the binary relation $\leq$ given by $(t, n) \leq (t^*, n^*)$ iff $(n, n^*) = (0, 2)$ or ($t \leq t^*$ and $n \leq n^*$).



Then it is verified straightforwardly that $\leq$ is the partial order of a continuous lattice. The relations $(1, 0) \ll (1, 1)$ and $(1, 1) \ll (1, 2)$ fail, but $(1, 0) \ll (1, 2)$ holds. Consider the strict chain $C = ((T\backslash\{1\}) \times \{0\}) \cup \{(1, 1)\}$; then C is maximal strict and $D(C) = T \times \{0\}$, but $D(C) \cup \{(1, 2)\}$ is a strict chain; whence, $D(C)$ is not maximal. □

Lemma IV-3.13. *Let L be a complete lattice and C_0 a chain satisfying the following properties*

(a) *C_0 is strict;*
(b) $c = \sup_L\{x \in C_0 : x \prec c\}$ *for all $c \in C_0$.*

Then C_0 is contained in a chain which is maximal with respect to (a) *and* (b)*, and every such chain is sup closed and satisfies the interpolation property* (SI_C)*.*

Proof: We consider the collection $\mathcal{L}$ of all chains C containing C_0 and satisfying (a) and (b) with C in place of C_0. This collection is inductive with respect to $\subseteq$; indeed let $\{C_j : j \in J\}$ be a tower in $\mathcal{L}$ and C its union. Then $c \in C$ implies $c \in C_j$ for some j, and thus

$$c = \sup_L\{x \in C_j : x \prec c\} \leq \sup_L\{x \in C : x \prec c\} \leq c;$$

that C is strict was shown in the proof of IV-3.5. Hence by Zorn's Lemma there are maximal members in $\mathcal{L}$. Let now C be one of them; we claim that C is sup closed in L. Assume not. Then there is a subset $X \subseteq C$ such that if $t = \sup_L X$, then $t \notin C$. Since C contains 0 (by maximality), $X \neq \emptyset$.

We claim that $C \cup \{t\}$ satisfies (a) and (b). If $c \in C$ and $c < t$, then there is an $x \in X$ with $c < x$. Since C is strict, $c \prec x$, and thus $c \prec t$. If $t < c$, then since $C \cup \{t\}$ is a chain and since $c = \sup_L\{y \in C : y \prec c\}$, we have $t < y \prec c$ for some $y \in C$. Hence $t \prec c$. Therefore $C \cup \{t\}$ is a strict chain. We note that $x \in X$ implies the existence of some $x^* \in X$ with $x < x^*$ since $t \notin X$. As $x \prec x^*$, because of strictness of C, we observe

$$X \subseteq \{x \in C : x \prec t\} \subseteq \{x \in C : x \prec t\} \cap (C \cup \{t\}).$$

Hence

$$t = \sup_L X \leq \sup_L(\{x \in C : x \prec t\} \cap (C \cup \{t\})) \leq t.$$

Hence, $C \cup \{t\}$ also satisfies (b) in place of C_0. But C was maximal relative to (a) and (b). Thus $t \in C$. This contradiction establishes the claim by IV-3.10. □

Lemma IV-3.14. *Let L be a complete lattice with an auxiliary order $\prec$ satisfying* (SI) *and C a chain which is maximal with respect to* IV-3.13(a), (b)*. Then C is maximal strict.*

Proof: Suppose not. Then by IV-3.5 there is a maximal strict chain M containing C with $y \in M \backslash C$. Let $t = \sup_L(\downarrow y \cap C)$. By IV-3.10 we have $t \in C$. Since M is a strict chain, $t \neq y$ implies $t \prec y$. By IV-3.8(iii) M satisfies the interpolation property.

We apply the Chain Modification Lemma IV-3.11 to M to obtain $D = D(M)$. Since $C \subseteq M$ and C satisfies IV-3.13(b), we conclude $C \subseteq D$. By IV-3.11, D is a strict chain satisfying IV-3.13(b). The maximality of C implies $C = D$. Again by IV-3.11 there exists $d \in D$ such that $t < d \prec y$. Thus $d \in \downarrow y \cap D = \downarrow y \cap C$, and so $d \leq t$ by definition of t, a contradiction. □

Thus we know that if there are strict chains with IV-3.13(b) at all, then there are maximal strict chains which are sup closed. This is where the construction $D(C)$ for maximal strict chains C comes in. In any case, we have the following result.

Theorem IV-3.15. *Let L be a complete lattice equipped with an auxiliary relation $\prec$ satisfying* (SI). *If $x \neq 0$ in L, then there is a maximal $\prec$-strict chain M which is sup closed in L and contains an element $m \neq 0$ with $m \leq x$.*

Proof: Step 1: $C_1 = \{0, x\}$ is a strict chain by IV-3.4(iii).

Step 2: C_1 is contained in a maximal strict chain C_2 by IV-3.5.

Step 3: Apply the Chain Modification Lemma IV-3.11 to the chain C_2 to obtain $C_3 = D(C_2)$. By IV-3.11, C_3 is a sup closed strict chain satisfying the interpolation property and condition IV-3.13(b) (since C_2 satisfies it by IV-3.8(iii)).

Step 4: By IV-3.13, the chain C_3 is contained in a strict chain C_4 which is maximal with respect to (a) and (b) of IV-3.13, and by IV-3.13, C_4 is sup closed. By IV-3.14, the chain C_4 is maximal strict.

We notice that in Step 3 we have $0 < d \prec x$ for some $d \in C_3$ by IV-3.11. Thus the proof is complete if we set $M = C_4$ and $m = d$. □

For some additional information we need sharper hypotheses:

Proposition IV-3.16. *Assume the hypotheses of Theorem* IV-3.15 *and suppose that y is an arbitrary element of L for which there is a $u \prec x$ with $u \not\leq y$. Then the chain M of* IV-3.15 *can be found so that, in addition, $m \not\leq y$.*

Proof: We only modify Step 1 by setting $C_1 = \{0, u, x\}$, and we proceed for the remainder as in the proof of Theorem IV-3.15. If $u < x$, then again by IV-3.11 the d chosen in the proof of IV-3.15 can be picked larger than u. If $u = x$, then we have $x \in C_3$ and, hence, we choose $m = x$. □

Corollary IV-3.17. *Let L be a complete lattice and $\prec$ an auxiliary relation which satisfies the strong interpolation property and which is quasiapproximating (*IV-3.4*). Then for two elements with $x \not\leq y$ there is a sup closed (maximal) strict chain M with an element m satisfying $y \not\geq m \leq x$.*

Proof: Immediate from IV-3.16. □

The principal application of these results concerns the way-below relation.

Lemma IV-3.18. *Let L be a complete lattice and C a sup closed, $\ll$-strict chain satisfying* (SI_C). *Then $g = (x \mapsto \max(\downarrow x \cap C))$: $L \to C$ is an $INF^{\uparrow}$-morphism from L onto the complete chain C. If $m \in C$ is such that $y \not\geq m \leq x$, then $g(y) < m \leq g(x)$.*

Proof: Since C is sup closed and $\ll$-strict, then the embedding d: $C \to L$ is a SUP^0-map (see IV-1.4 and IV-1.9 and note that $\ll_L$ induces $\ll_C$ by (SI_C)). Hence its upper adjoint g is an $INF^{\uparrow}$-map by IV-1.10. The remainder follows from the definition of g. □

Theorem IV-3.19. *Let L be a complete lattice in which the way-below relation $\ll$ has the strong interpolation property. If $0 \neq x \in L$, then there is an $INF^{\uparrow}$-quotient-map g: $L \to M$ onto a complete chain M such that*

(i) *$g(x) \neq 0$,*
(ii) *for every factorization $L \xrightarrow{h} C \xrightarrow{f} M$ of g in $INF^{\uparrow}$ with C a chain and h surjective, the map f is an isomorphism.*

Remark. We could say more shortly in place of (ii) that g is a maximal chain quotient.

Proof of theorem: By Theorem IV-3.15 we find a sup closed maximal strict chain M and an $m \in M$ with $0 \neq m \leq x$. Then $g = (s \mapsto \max(\downarrow s \cap M))$: $L \to M$ is the desired morphism by IV-3.18. The remainder follows from the maximality of M via the duality IV-1.10. □

Theorem IV-3.19 gives an important characterization of continuous lattices:

Theorem IV-3.20. *For a complete lattice L, the following conditions are equivalent:*

(1) *L is a continuous lattice;*
(2) *there is an $INF^{\uparrow}$-embedding of L into a product of complete chains.*

Proof: (2) implies (1): I-1.7(2), I-2.11(ii).

(1) implies (2): L is continuous iff $\ll$ is approximating (I-1.6). Then, by IV-3.17 and IV-3.18, the $INF^{\uparrow}$-quotients from L onto chains separate the points. The usual method for forming an embedding now applies. □

We round this topic off by observing that, for a complete chain, the CL-morphisms into the unit interval $\mathbb{I} = [0, 1]$ separate the points.

Proposition IV-3.21. *Every chain allows an embedding into a cube (that is, a lattice $\mathbb{I}^X$) which preserves all existing sups and infs.*

Proof: Let C be the given chain. We proceed in steps in order to show that the maps $C \to \mathbb{I}$ preserving arbitrary infs and sups separate the points of C.

Step 1: Every chain allows an embedding into a complete chain which preserves infs and sups. (This is, of course, well known even for lattices; let us briefly indicate a proof for the special case: We let $\mathcal{C} \subseteq \operatorname{Id} C$ for a chain C be the set of all ideals J such that J has a maximum only if $c = \max J$ has a successor c^* in C. Then define $j: C \to \mathcal{C}$ by $j(x) = {\downarrow}x \backslash \{x\}$. Show that $\mathcal{C}$ is closed under arbitrary unions, hence is a complete chain. Clearly j preserves order. Now let $X \subseteq C$ and set $c = \inf X, d = \sup X$ (if they exist). If $c \in X$, then $j(c) = \min j(X)$ and if $d \in X$ then $j(d) = \max j(X)$. Thus assume that $c \notin X, d \notin X$. Show that ${\downarrow}X \in \mathcal{C}$ and ${\downarrow}X = \bigcup\{{\downarrow}x\backslash\{x\}: x \in X\}$ which will show that $j(d) = \sup j(X)$. Show that $\bigcap j(X) = {\downarrow}\inf X = {\downarrow}c \notin \mathcal{C}$ (if $c \notin X$), whence $\inf j(X) = {\downarrow}c\backslash\{c\} = j(c)$.)

Step 2: Assume from now on that C is complete and that $b < a$ in C. By IV-3.1 there is a map $g: C \to \mathbb{I}$ with $g(a) = 1$ and $g(b) = 0$ and which preserves all infs and all directed sups. As all nonempty subsets of a chain are directed, g preserves in fact all sups. (Of course, Step 2 can be proved directly without using IV-3.1, but by a method similar to the one used in IV-3.1.) □

Notice that IV-3.21 says that the *complete* lattice homomorphisms of any chain into $\mathbb{I}$ separate the points. Together with IV-3.17 this gives a second proof for Theorem IV-3.3.

Let us remark also that Theorem IV-3.3 can be sharpened in one direction.

Proposition IV-3.22. *Let L be a complete lattice in which $\ll$ satisfies the strong interpolation property. Suppose that $x \not\leq y$ in L. Then the following statements are equivalent.*

(1) *There is a map $g \in INF^{\uparrow}(L, \mathbb{I})$ with $f(y) < f(x)$.*
(2) *There is a $u \in L$ with $u \ll x$ and $u \not\leq y$.*

Proof: (1) implies (2): Let $d: \mathbb{I} \to L$ be the lower adjoint of g. Then let $v = (g(x) + g(y))/2$, and $u = d(v)$. Now $v < g(x)$ implies $v \ll g(x)$ by I-1.3(1); whence, $u = d(v) \ll d(g(x))$ (by IV-1.4(3)) $\leq x$ (by O-3.6). If we had $u \leq y$, then $v \leq g(d(v))$ (by O-3.6) $= g(u) \leq g(y)$, a contradiction. Thus $u \not\leq y$.

(2) implies (1): By IV-3.17 and IV-3.18 we have an $INF^{\uparrow}$-morphism onto a complete chain separating y and x, then IV-3.21 proves (1). □

Exercises

Exercise IV-3.23. Let L be a complete lattice, $\prec$ an auxiliary relation. Let $\prec\!\bullet$ be the auxiliary relation satisfying the strong interpolation property which is derived from $\prec$ according to I-1.28. If $x \neq 0$ in L, show that there is a maximal $\prec\!\bullet$-strict chain C which is sup closed in L and contains an element $m \neq 0$ with $m \leq x$. □

Problem. Discuss the properties of C with respect to the auxiliary order $\prec$. □

Exercise IV-3.24. Let L be a complete lattice. We recall that for an auxiliary order $\prec$ on L (see I-1.11 ff.) we denote by $\prec\!\bullet$ according to I-1.28 the associated auxiliary relation with the strong interpolation property.

Suppose now that the auxiliary relation satisfies the additional condition

(iv) $x \prec z$ and $y \prec z$ together imply $x \vee y \prec z$.

Prove the following assertions:

The lattice L with the relation $\prec\!\bullet$ is an abstract basis in the sense of III-4.16a. By III-4.18 the set $\mathrm{RId}_{\prec\bullet}L$ of rounded ideals of L is a continuous latttice.

The image L' of the sup map $(I \mapsto \sup I)\colon \mathrm{RId}_{\prec\bullet}L \to L$ is the image of the Scott-continuous kernel operator $k = (x \mapsto \sup\{y \in L : y \prec\!\bullet x\}) : L \to L$.

In the special case $\prec\ =\ \ll$, the sup map $\mathrm{Id}_{\ll\bullet}L \to L$ is injective and L' is a continuous lattice. If $f\colon L \to M$ is any $\mathit{INF}^{\uparrow}$-morphism into a continuous lattice, then $f = (f \mid L')k$.

Hint. The first statements are clear. Consider $\prec\, = \,\ll$: Show $\{x\colon x \ll\!\bullet \sup I\} = I$ for $I \in \mathrm{RId}_{\ll\bullet}L$, which gives injectivity of the sup map. Then L' is a continuous lattice. If f is given, consider the lower adjoint $d\colon M \to L$ and use IV-1.4 to show that d factors through L'. □

Note that this exercise gives an explicit construction of a left reflection of the category $\mathit{INF}^{\uparrow}$ into CL with front adjunction $k\colon L \to L'$.

In the following we discuss some results concerning the lattices of kernel and closure operators. We recall from O-3.13 and the subsequent Remark that the lattice of all closure operators of a complete lattice is isomorphic to the opposite of the lattice of all inf closed subsets, and from I-2.12 that in the case of a continuous lattice the lattice of all Scott-continuous closure operators is isomorphic to the opposite of the lattice of all subalgebras.

Exercise IV-3.25. For any lattice L let $\mathrm{Ker}\ L$ (resp., $\mathrm{Clos}\ L$) denote the poset of all kernel (resp., closure) operators of L. If L is complete, we have two functions $C, N\colon (L \to L) \to (L \to L)$ (cf. I-2.21) given by

$$C(f)(x) = \inf(\uparrow x \cap \{y\colon f(y) \le y\}) \text{ and}$$
$$N(f)(x) = \sup(\downarrow x \cap \{y\colon f(y) \ge y\}).$$

Prove the following:

(i) C is a closure operator and N is a kernel operator;
(ii) $\operatorname{im} C = \operatorname{Clos} L$ and $\operatorname{im} N = \operatorname{Ker} L$;
(iii) $C(f) = \min\{c \in \operatorname{Clos} L : f \le c\}$ and $N(f) = \max\{k \in \operatorname{Ker} L : k \le f\}$;
(iv) $\operatorname{Clos} L$ is inf closed and $\operatorname{Ker} L$ is sup closed (O-3.12);
(v) $\operatorname{Clos} L$ is closed under directed sups and $\operatorname{Ker} L$ under filtered infs;
(vi) if L is a continuous lattice, then $\operatorname{Clos} L$ is a continuous lattice. □

Exercise IV-3.26. For a complete lattice L, let $\ker L \subseteq \operatorname{Ker} L$ be the poset of Scott-continuous kernel operators and $\operatorname{clos} L \subseteq \operatorname{Clos} L$ the poset of Scott-continuous closure operators. Let L be a continuous lattice. Prove the following.

(i) $\operatorname{clos} L$ is a continuous lattice.
(ii) The following statements are equivalent:
  (1) $\ker L$ is a continuous lattice;
  (2) $\ker L$ is an algebraic lattice;
  (3) $\ker L$ is an algebraic lattice all of whose *CL*-quotients are algebraic;
  (4) L is an algebraic lattice all of whose *CL*-quotients are algebraic;
  (5) L is an algebraic lattice and $K(L)$ contains no order dense chains.

Hint. (i) We know that ΛL is a compact semilattice with small semilattices (III-2.15). If X and Y are subalgebras (that is, Lawson closed subsemilattices: see III-1.12) then so is XY, and if S is the semilattice of all subalgebras with respect to $(X, Y) \mapsto XY$, then the partial order of this semilattice is reverse containment, and thus $S \cong \operatorname{clos} L$ by I-2.12. It suffices therefore to show that S is a compact topological semilattice with small semilattices, and that the topology induced on S from the standard topology on the set of compact subsets of the compact space ΛL agrees with the Lawson topology. (See VI-3.4.)

(ii) For a proof we refer to [Hofmann and Mislove, 1977]. □

Exercise IV-3.27. Let L be a complete lattice and $\operatorname{cong}^- L$ the lattice of all congruences on L which are subalgebras of $L \times L$ in the sense of I-2.10, I-2.14, I-2.16. Show that $(\ker L)^{\mathrm{op}} \cong \operatorname{cong}^- L$.

Hint. See I-2.16. □

As a consequence of IV-3.27, the results of IV-3.24(ii) provide information on the lattice of closed congruences on a continuous lattice.

Exercise IV-3.28. Let L be a continuous lattice. Show that the poset $\ker L$ of Scott-continuous kernel operators (see IV-3.26, IV-3.27) is a complete lattice on which the way-below relation has the strong interpolation property.

Hint. We refer to [Hofmann and Mislove, 1977]. □

Exercise IV-3.29. Prove the following.

(i) Let L be a continuous lattice. If R is a congruence of L with more than one class, then there is a surjective map $f\colon \mathrm{cong}^{-}L \to C$ from the lattice of closed congruences on L onto a complete chain C such that $f(R) = \min C < \max C = f(L \times L)$ and that f preserves infs of filtered sets and arbitrary sups.
(ii) There is a morphism $\mathrm{cong}^{-}L \to \mathbb{I}^X$ preserving arbitrary sups and filtered infs into a cube such that the only element mapped to the top is the congruence maximal $L \times L$.

Hint. Apply IV-3.28 and Proposition IV-3.22. □

Exercise IV-3.30. Use Theorem IV-3.3 to give a new proof of the fact that every continuous lattice carries a compact Hausdorff topology such that the inf operation is jointly continuous and that every point has a basis of open semilattice neighborhoods. □

Exercise IV-3.31. Let L be a complete lattice and define $x \lll y$ iff whenever $y \le \sup X$ then $x \le x^*$ for some $x^* \in X$. (Note that for $\mathcal{M} = 2^X$ in I-2.25 we have $\lll\, = \,\dashv$.) Prove the following:

(i) $\lll$ is an auxiliary order;
(ii) $x \lll y$ implies $x \ll y$;
(iii) the relation $\lll$ is quasiapproximating if L is completely distributive;
(iv) under these circumstances, $\lll$ satisfies the interpolation property. □

If $\lll$ is quasiapproximating, then the developments of this section apply and show the existence of maximal $\lll$-strict chains according to Theorem IV-3.15 and Corollary IV-IV-3.18 applies with $\lll$ in place of $\ll$ and yields a morphism $g\colon L \to C$ in *INF* $\cap$ *SUP*. This allows us to deduce a parallel theorem to IV-3.3:

Exercise IV-3.32. For a poset L show that the following statements are equivalent:

(1) L is a completely distributive lattice;

(2) L is isomorphic to a subset of some cube (that is, a lattice $\mathbb{I}^X$) which is closed under *arbitrary* infs and *arbitrary* sups;
(3) L is a complete lattice and the functions $f\colon L \to \mathbb{I}$ preserving arbitrary sups and infs separate the points of L.

Hint. This theorem is due to [Raney, 1952] and was reported in Chapter I following I-3.16. □

Old notes

This section contains two constructions of morphisms into chains, a direct one based on techniques similar to those used to prove Urysohn's Lemma, and a longer one using internal strict chains. The general idea of constructing morphisms into chains by using (maximal) complete strict chains relative to suitable auxiliary relations dates back some forty years; forerunners are to be found in Raney's classical paper on completely distributive lattices [Raney, 1952], and the closest to what we do here is Bruns' treatment of this technique [Bruns, 1961]. However, neither of these papers exactly applies to the situation of continuous lattices which we cover here. In this context a first indication was given by [Hofmann and Stralka, 1976], but the argument was found to contain a gap [SCS 4] which was patched by Scott; a fairly complete elaboration of the techniques presented here was given by Hofmann [scs 5]; supplements were provided by Carruth [SCS 6; SCS 7]. Some new results appearing in the *Compendium* included Example IV-3.12 and the construction of maximal strict chains which are sup closed (Theorem IV-3.15).

Theorem IV-3.3 is a core result of the entire theory of continuous lattices. In a certain form, this theorem is due to Lawson, who showed in [Lawson, 1969] that a compact Hausdorff topological semilattice has enough continuous semilattice homomorphisms into the interval to separate the points if and only if it has small semilattices. Theorem III-1.8, Theorem III-2.15, and Theorem VI-3.4 below show that Lawson's result is equivalent to IV-3.3. The construction of the left reection $INF^{\uparrow} \to CL$ in IV-3.24 was first noted by Gierz, Hofmann, Keimel and Mislove [scs 12]. The results in IV-3.26 are from [Hofmann and Mislove, 1977]; the equivalence of (4) and (5) in IV-3.26(ii) was proved in [Hofmann *et al.*, 1973]. The result in IV-3.29 appeared first in the *Compendium*. Exercise IV-3.31 retrieves the studies of [Raney, 1952], [Bruns, 1961], [Papert, 1959].

We remark that the methods to construct maximal strict chains require the interpolation property in its strong form (SI) (see IV-3.8), and this is the only place where the interpolation property (INT) (see I-1.17) is not sufficient.

IV-4 Projective Limits

One of the original motivations for considering continuous lattices has much to do with the construction of continuous lattices L which are naturally isomorphic to their own function spaces $[L \to L]$ (see II-2.6); indeed, such lattices provide set-theoretical models for the lambda-calculus of Church and Curry. In the next section we discuss the general principle underlying the construction of these lattices; in the present section, however, we prepare for this discussion by providing the main ingredients.

First we present a thorough investigation of projective limits in the category of complete lattices and maps preserving arbitrary infs and directed sups; in fact, the whole theory can be carried through in the category $DCPO_G$ of **dcpo**s and Scott-continuous upper adjoints. It turns out that projective limits in the categories at hand have many features which are not at all apparent from purely categorical considerations.

In a review of projective limits it is just as easy and efficient to recall the concept of a limit in general; in fact, the notation is in many ways simpler if we adopt the category-theoretical conventions from the start. In what follows a *small category* is one whose class of morphisms (and objects) is a *set*. In general, we regard categories like CL as forming a *proper class*, but – as is usually the convention – the hom-sets $CL(S, T)$ are sets and not proper classes.

Definition IV-4.1. Let C be an arbitrary category. A *diagram* in C is simply a functor $D\colon I \to C$ from a small category I.

To each object L of C we can associate a functor $|L|\colon I \to C$ which takes any object i of I to the fixed object L of C and any arrow $i \to j$ to the identity map of L. This is merely a device to introduce in simple terms the concept of a *cone over a diagram* $D\colon I \to C$ *with vertex* L. Such a cone is by definition a natural transformation $g\colon |L| \to D$. Explicitly, this means that for every object i of I we have a C-morphism $g_i\colon L \to D(i)$ such that for any arrow $a\colon i \to j$ in I the diagram

$$
\begin{tikzcd}
 & L \arrow[dl, "g_i"'] \arrow[dr, "g_j"] & \\
D(i) \arrow[rr, "D(a)"'] & & D(j)
\end{tikzcd}
$$

commutes.

A cone $g\colon |L| \to D$ over a diagram D is called a *limit cone* provided it has the following universal property: whenever a cone $h\colon |H| \to D$ over the same diagram D is given, then there is a unique C-morphism $h^*\colon H \to L$ such that $g|h^*| = h$, where $|h^*|\colon |H| \to |L|$ is the constant natural transformation with $|h^*|_i = h^*\colon H \to L$ for all objects i of I. In explicit terms, this means that for each object i of I the diagram

$$\begin{tikzcd} H \arrow[rr, "h^*"] \arrow[dr, "h_i"'] & & L \arrow[dl, "g_i"] \\ & D(i) & \end{tikzcd}$$

commutes.

The vertex L of a limit cone of a diagram D is called the *limit* of D and is denoted by $\lim D$, and the natural transformation $g\colon |\lim D| \to D$ is called the *limit natural transformation*. In the same vein, the maps $g_i\colon \lim D \to D(i)$ are called the *limit maps*. □

Limits – if they exist – are unique up to an isomorphism. Our interest here is in a special kind of limit, called a *projective limit*, but before we make its definition precise we record some special limits to exemplify the concept. Suppose that I is an arbitrary *set*. Then we may consider I as a category, whose objects are the elements of I and whose only maps are the identity maps of these objects. (These morphisms have to be there by the definition of a category, and we allow no others. Such categories are called *discrete categories*.) A diagram $D\colon I \to C$ in a category C is then nothing but a family of objects $\{D(i)\colon i \in I\}$ – indeed each family of objects indexed by a set can be described as a diagram in this fashion. If this diagram has a limit, it is called the *product* of the family, and is written $\prod_{i\in I} D(i)$. The limit maps $\pi_j\colon \prod_{i\in I} D(i) \to D(j)$ are called *projections*.

Another simple type of limit is the *equalizer* of a pair of maps $f, g\colon L \to M$ in C. The equalizer is an object E in C together with a unique map $e\colon E \to L$ such that $fe = ge$ and that for any morphism $h\colon H \to L$ with $fh = gh$ there is a unique map $h^*\colon H \to E$ with $h = eh^*$. It should be clear how this fits into the general scheme of limits: let I be the category

$$\begin{tikzcd} 1 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & 2 \end{tikzcd}$$

(plus the identity maps of 1 and 2), and let $D: I \to C$ be the diagram given by $D(1) = L$, $D(2) = M$, $D(a) = f$ and $D(b) = g$. Then $\lim D = E$, and the limit maps are $g_1 = e$, $g_2 = fe = ge$.

Not only are products and equalizers good examples of limits, they suffice for the construction of arbitrary limits as Freyd has shown (see, e.g., [Mac Lane, B1971], p. 109). Indeed, it follows that in a category every diagram has a limit if every family has a product and every pair of maps with the same domain and codomain has an equalizer. Such categories are called *complete*.

Many of our categories such as *DCPO*, *INF*, *SUP*, *CL*, and *AL* are complete, since the cartesian products are products in the sense of limits and the equalizer of two maps $f, g: L \to M$ is just the subalgebra $E = \{x \in L : f(x) = g(x)\}$ with $e: E \to L$ the inclusion morphism. (In the parlance of category theory, the forgetful functor from these categories into the category *SET* of all sets and functions preserves and creates products and equalizers (hence limits). See, e.g., [Mac Lane, B1971], p. 108.) But some of our categories like the category *DOM* of domains and Scott-continuous maps do not have equalizers.

Now we introduce the concept of a projective limit:

Definition IV-4.2. Let I be a partially ordered set. We may consider I as a small category in the following fashion: the elements of I are the objects, and for two objects i and j there is one and only one arrow $i \to j$ whenever $i \leq j$ (see remarks following O-3.1).

A *projective system* in a category C is a diagram $D: I^{\mathrm{op}} \to C$ whose domain is a poset I which is, in addition, directed. The limit, $\lim D$, of a projective system is called a *projective limit*, and the limit cone over a projective system is called a *projective limit cone*. □

Let us take stock of what a projective system is in terms of objects and maps. It is a family of objects $D(j)$ indexed by the elements j of a directed set I, and a system of maps $g_{ij}: D(j) \to D(i)$ for every pair i, j of elements in I with $i \leq j$ such that the following relations are satisfied for all $i \leq j \leq k$ in I:

(i) $g_{ii} = 1_{D(i)}$;
(ii) $g_{ij} g_{jk} = g_{ik}$.

If the system has a limit, $\lim D$, then there are limit maps $g_i: \lim D \to D_i$ such that, for all $j \leq k$ in I,

(iii) $g_{jk} g_k = g_j$.

The functorial definition of a projective system automatically takes care of all these conditions.

As we have remarked, the category *DCPO* of **dcpo**s and Scott-continuous functions is complete. Hence it has projective limits. Let us indicate a concrete representation of projective limits in this category:

Proposition IV-4.3. *Let D be a projective system of* **dcpo***s L_i over a directed set I with Scott-continuous functions $g_{ij}\colon L_j \to L_i$ for $i \le j$ as above. Then the projective limit is given by*

$$\lim D = L = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} L_i : g_{ij}(x_j) = x_i \text{ whenever } i \le j \text{ in } I \right\};$$

and the restrictions of the canonical projections as limit maps are given by

$$g_j : \lim D \to L_j, (x_i)_{i\in I} \mapsto x_j.$$

Proof: As the maps g_{ij} preserve directed sups, the set L is closed in $\prod_{i\in I} L_i$ for directed sups and hence is a **dcpo**. The maps g_j are the restrictions to L of the canonical projections from the product onto its factors and hence they are Scott-continuous. Whenever $j \le k$, then $g_{jk}(g_k((x_i)_{i\in J})) = g_{jk}(x_k) = x_j = g_j((x_i)_{i\in J})$ by the defining condition on the elements of L, whence $g_{jk}g_k = g_j$. Thus, we have a cone over D.

Now let $h_i\colon S \to L_i$, $i \in I$, be any cone of Scott-continuous functions over our projective system D. Then $g_{jk}h_k = h_j$, whenever $j \le k$. Hence, for every $s \in S$, the family $(h_i(s))_{i\in I}$ is an element of L and we may define a Scott-continuous function $h\colon S \to L$ by $h(s) = (h_i(s))_{i\in I}$ which has the property that $g_j(h(s)) = h_j(s)$, whence $g_jh = h_j$ for every $j \in I$. Moreover, h is the unique function from S to L with this latter property. This proves the universality.

□

Just as in the theory of ordered sets, most of the elementary concepts in category theory have a dual (see O-1.7) or an opposite. The opposite category C^{op} is obtained from a category by reversal of arrows; how this is done formally is explained in any source on category theory, e.g., in [Mac Lane, B1971], p. 33. The introduction of dual concepts is then simple: if any concept is generally introduced, it can be considered in the opposite category and interpreted in the original category; this will give the “co-concept”. *Example*: a *co-cone under a diagram $D\colon I \to C$* in the category C is a natural transformation $g\colon D \to |L|$, i.e., a system of maps $g_i\colon D_i \to L$ with the commuting relations dual to those of the cone. The dual of a limit cone is the *colimit cone*; its *covertex* is called a *colimit* of the system.

Sometimes, for reasons of historical priority, variations to this nomenclature exist. Thus, a co-projective system is called a *direct system*, a co-projective limit is called a *direct limit*. The direct limit of a direct system D will nevertheless be written colim D. The reader who is not already familiar with the practice of dualization should give the explicit definitions of these dual concepts as an exercise.

The formal dualization by reversal of arrows is a convenient routine – but it may not always have much concrete significance in a given category. However, we saw in the discussions of the first section of this chapter that the category *INF* of complete lattices with inf preserving maps was equivalent to the opposite category SUP^{op} of the category *SUP* of all complete lattices with sup preserving maps; thus, SUP^{op} in this case has a concrete meaning. More generally, the dual of the category $POSET_G$ of all posets and monotone functions g having a lower adjoint $d = D(g)$ was seen to be equivalent to the category $POSET_D$ of all posets and monotone maps d having an upper adjoint $g = G(d)$. The functors D and G implementing the equivalence were given by the Galois connection (cf. IV-1.3). For the purpose of our present discussion it will be useful to have some explicit notation.

Notation IV-4.4. If $g\colon S \to T$ is a map in $POSET_G$, that is, a monotone map having a lower adjoint $D(g)$, we write $\hat{g}$ in place of $D(g)$ for the lower adjoint in $POSET_D$. $\square$

For an inf preserving map g of complete lattices, the lower adjoint $\hat{g}$ preserves sups (see remarks following IV-1.1). Thus $\hat{}\colon POSET_G \to POSET_D^{\mathrm{op}}$ or $\hat{}\colon INF \to SUP^{\mathrm{op}}$ is an isomorphism of categories.

It is fairly clear that for any projective system $D\colon J^{\mathrm{op}} \to INF^{\uparrow}$ the limit $\lim D$ serves at the same time as colimit of the direct system $\hat{D}\colon J \to SUP^0$ given by $\hat{D}(i \to j) = (g_{ij})\hat{}$. This is simply a consequence of duality. The limit maps $g_j\colon \lim D \to D(j)$ dualize to colimit maps $\hat{g}_j\colon \hat{D}(j) \to \operatorname{colim}\hat{D} = \lim D$, where we recall $\hat{D}(j) = D(j)$. The same applies for limits of projective systems in $DCPO_G$ and direct systems in the dual category $DCPO_D$. However, a sufficiently careful analysis will reveal more than one would expect from arrow-theoretical generalities. We now undertake such an analysis. It is worth noting that we will not use the concrete representation of projective limits from IV-4.3 but only their existence and uniqueness.

Theorem IV-4.5. **(Limit–Colimit Coincidence)** *Let $D\colon J^{\mathrm{op}} \to DCPO$ be a projective system of* **dcpo***s and Scott-continuous bonding maps $g_{ij}\colon L_j \to L_i$. Let $L = \lim D$ and let $g_j\colon L \to L_j$ denote the limit cone in the category DCPO as in* IV-4.3. *Suppose that all the maps g_{ij} have a lower adjoint $\hat{g}_{ij}\colon L_j \to L_i$, that is, D is in fact a projective system in $DCPO_G$. Then the following conclusions hold:*

(i) *the limit maps $g_j\colon L \to L_j$ have lower adjoints $\hat{g}_j\colon L_j \to L$;*

(ii) *the sets* $\{g_{jk}\hat{g}_{ik}\colon i, j \le k \text{ in } J\}$ *for fixed* $i, j \in J$ *and* $\{\hat{g}_j g_j\colon j \in J\}$ *are directed and*

(A) $$g_j\hat{g}_i = \sup_k\{g_{jk}\hat{g}_{ik}\colon i,\ j \le k \text{ in } J\} \text{ for all } i,\ j \in J;$$

(B) $$\sup_j \hat{g}_j g_j = 1_L;$$

(iii) *given any cone* $\{h_j\colon H \to L_j\colon j \in J\}$ *over the projective system, the unique function* $g\colon H \to L$ *in DCPO such that* $g_j g = h_j$ *for all* j *is given by*

(C) $$g = \sup_j \hat{g}_j h_j, \text{ a directed sup};$$

(iv) $L = \lim D$ *is also the colimit in the category DCPO of the direct system* $\hat{D}$ *given by the lower adjoints* $\hat{g}_{ij}\colon L_i \to L_j$ *for* $i \le j$*; the colimit cone is given by the* $\hat{g}_i\colon L_i \to L$*; more precisely, if*

$$\begin{tikzcd} \cdots & L_i \arrow[rr, "\hat{g}_{ij}"] \arrow[dr, "d_i"'] & & L_j \arrow[dl, "d_j"] & \cdots \\ & & S & & \end{tikzcd}$$

is any co-cone in DCPO under the direct system $\hat{D}$*, then there is a unique function* $d\colon L \to S$ *in DCPO such that* $d_i = d\hat{g}_i$ *for all* $i \in J$*; moreover, the set* $\{d_j g_j\colon j \in J\}$ *is directed and the function* d *is given by the formula*

(D) $$d = \sup_j d_j g_j.$$

Proof: (i) Fix i in J and denote for any j in J the cofinal subset $\{k \in J\colon i, j \le k\}$ by J^{ij}. Then we have a Scott-continuous map $g_{jk}\hat{g}_{ik}\colon L_i \to L_j$ for $k \in J^{ij}$. We claim that $(g_{jk}\hat{g}_{ik})_{k\in J^{ij}}$ is a monotone net in $[L_i \to L_j]$ (cf. O-1.2). Indeed let $k \le k'$ in J^{ij}, then $g_{jk'}\hat{g}_{ik'} = (g_{jk}g_{kk'})(\hat{g}_{kk'}\hat{g}_{ik}) \ge g_{jk}\hat{g}_{ik}$, since $g_{kk'}\hat{g}_{kk'} \ge 1_{L_k}$ by O-3.6. In the **dcpo** $[L_i \to L_j]$ the directed supremum $\sup\{g_{jk}\hat{g}_{ik}\colon k \in J^{ij}\}$ exists, and we call it $h_j\colon L_i \to L_j$.

Suppose $j \le j'$. We claim $h_j = g_{jj'}h_{j'}$. For a proof calculate as follows:

$$\begin{aligned} g_{jj'}h_{j'}(x) &= g_{jj'}(\sup\{g_{j'k}\hat{g}_{ik}(x)\colon k \in J^{ij'}\}) \\ &= \sup\{g_{jj'}g_{j'k}\hat{g}_{ik}(x)\colon k \in J^{ij}\} \\ &= \sup\{g_{jk}\hat{g}_{ik}(x)\colon k \in J^{ij'}\} \\ &= h_j(x). \end{aligned}$$

(The second equation follows because $g_{jj'}$ is Scott-continuous and the net $(g_{j'k}\hat{g}_{ik}(x))_{k\in J^{ij'}}$ is monotone.) Remark then that the infinite diagram

$$\begin{tikzcd}
 & & L_i \arrow[dl, "h_j"'] \arrow[dr, "h_{j'}"] & & \\
\cdots & L_j & & L_{j'} \arrow[ll, "g_{jj'}"] & \cdots
\end{tikzcd}$$

is a cone in *DCPO* over the projective system D: $J^{\mathrm{op}} \to$ *DCPO*.

Now g_k: $L \to L_k$, where $k \in J$, is a limit cone in *DCPO*. Then, by the universal property of the limit in *DCPO* (see IV-4.1 above), there is a unique Scott-continuous map d_i: $L_i \to L$ with $h_j = g_j d_i$ for all $j \in J$.

As the last step we claim $d_i = \hat{g}_i$, that is, d_i is the lower adjoint of g_i. In order to prove the claim we calculate

$$g_i d_i = h_i = \sup\{g_{ik}\hat{g}_{ik}\colon k \in J^{ii}\} \geq 1_{L_i},$$

since $g_{ik}\hat{g}_{ik} \geq 1_{L_i}$ by O-3.6. On the other hand, for all $j \in J$ we have

$$\begin{aligned}
g_j d_i g_i(x) &= f_j g_i(x)\\
&= \sup\{g_{jk}\hat{g}_{ik}g_i(x)\colon k \in J^{ij}\}\\
&= \sup\{g_{jk}\hat{g}_{ik}g_{ik}g_k(x)\colon k \in J^{ij}\}\\
&\leq \sup\{g_{jk}g_k(x)\colon k \in J^{ij}\}\\
&= \sup\{g_j(x)\colon k \in J^{ij}\}\\
&= g_j(x),
\end{aligned}$$

where the inequality in the middle follows because $\hat{g}_{ik}g_{ik} \leq 1_{L_k}$ by O-3.6. Thus $g_j d_i g_i(x) \leq g_j(x)$ for all $j \in J$, and since the limit maps g_j separate the points of L we have $d_i g_i \leq 1_L$. The two relations $g_i d_i \geq 1_{Li}$ and $d_i g_i \leq 1_L$ together show that d_i is the lower adjoint of g_i by O-3.6. Thus $d_i = \hat{g}_i$

(ii) In the proof of (i) we have seen that the set $\{g_{jk}\hat{g}_{ik}\colon k \in J^{ij}\}$ is directed. Since $g_j\hat{g}_i = g_j d_i = f_j = \sup\{g_{jk}\hat{g}_{ik}\colon k \in J^{ij}\}$ by the definition of f_j, we have (A).

We fix an arbitrary $j \in J$ and calculate $g_j(\sup_i \hat{g}_i g_i)$. An argument similar to that in the beginning of this proof shows that $(\hat{g}_i g_i)_{i\in J}$ is directed, and so we may assume that $j \leq i$ for each i we consider. Then the fact that g_j preserves directed sups implies that

$$\begin{aligned}
g_j(\sup_i \hat{g}_i g_i) &= \sup_i g_j \hat{g}_i g_i \\
&= \sup_i \sup_{k\in J^{ij}} g_{jk}\hat{g}_{ik} g_i && \text{(by (A))}\\
&= \sup\{g_{jk}\hat{g}_{ik}g_i \colon i,k\in J,\ j\le i\le k\}\\
&= \sup\{g_{ji}g_{ik}\hat{g}_{ik}g_{ik}g_k \colon i,k\in J, i\le j\le k\}\\
&= \sup\{g_{ji}g_{ik}g_k \colon i,j\in J, i,j\le k\}\\
&= g_j,
\end{aligned}$$

where the next to last equation follows because $g_{ik}\hat{g}_{ik}g_{ik} = g_{ik}$, by O-3.6. The limit maps g_j separate the points of $L = \lim D$, and so we find that $\sup_i \hat{g}_i g_i = 1_L$ as was asserted.

(iii) Since for $i \le j$, $\hat{g}_{ij}g_{ij} \ge 1_{L_i}$, we have $\hat{g}_i h_i = \hat{g}_j \hat{g}_{ij} g_{ij} h_j \le \hat{g}_j h_j$. Thus the family $\hat{g}_i h_i$ is directed. Furthermore,

$$\begin{aligned}
g &= \sup_j \hat{g}_j g_j g \quad \text{(by (B))}\\
&= \sup_j \hat{g}_j h_j .
\end{aligned}$$

(iv) We first note that the functions $d_j g_j$, $j \in J$, form a directed system; indeed for $i \le j$, we have $d_i g_i = d_j \hat{g}_{ij} g_{ij} g_j \le d_j g_j$, since $\hat{g}_{ij} g_{ij} \le 1_{L_j}$ by O-3.6. Define d by equation (D). We then note that d is in *DCPO*, because all d_j and g_j are in *DCPO* and $[L \to S]$ is closed under directed sups.

Now let $i \in J$ and $x \in L_i$. Then we calculate

$$\begin{aligned}
d\hat{g}_i &= \sup\{d_j g_j \hat{g}_i \colon j\in J\} && \text{(by (D))}\\
&= \sup_j \{d_j \sup\{g_{jk}\hat{g}_{ik} \colon k \in J^{ij}\}\} && \text{(by (A))}\\
&= \sup\{d_j g_{jk}\hat{g}_{ik} \colon j\in J, k\in J^{ij}\} && (\text{since } d_j \in DCPO).
\end{aligned}$$

But $j \le k$ implies $d_j = d_k \hat{g}_{jk}$, and so $d_j g_{jk} = d_k \hat{g}_{jk} g_{jk} \le d_k$, since $\hat{g}_{jk} g_{jk} \le 1$ by O- 3.6. Therefore, $d_j g_{jk} \hat{g}_{ik} \le d_k \hat{g}_{ik} = d_i$, whence $d\hat{g}_i \le d_i$.

In order to show the other inequality, we first observe that for a fixed i and any $i \le k$ we always have $d_i = d_k \hat{g}_{ik}$; thus we can write

$$d_i = \sup\{d_k g_{kk} \hat{g}_{ik} \colon k \in J \text{ with } i \le k\}.$$

If we form the sup over the larger index set $\{(j,k) \in J \times J \colon i \le k \text{ and } j \le k\}$ we possibly enlarge the sup; whence

$$\begin{aligned}
d_i(x) &\le \sup\{d_j g_{jk}\hat{g}_{ik} \colon j,k\in J \text{ with } j\le k\}\\
&= \sup_j (d_j \ \sup_{j,i\le k} g_{jk}\hat{g}_{ik})\\
&= \sup_j d_j g_j \hat{g}_i && \text{(by (A))}\\
&= d\hat{g}_i && \text{(by (C))}.
\end{aligned}$$

This shows that $d_i = d\hat{g}_i$ for all $i \in J$.

In order to show the uniqueness of d, we consider a second *DCPO*-mapping $d^*\colon \lim D \to S$ with $d\hat{g}_i = d^*\hat{g}_i$ for all i. Then also $d\hat{g}_i g_i = d^*\hat{g}_i g_i$ for all $i \in J$, and so $d = d \sup_i \hat{g}_i g_i$ (by (B)) $= \sup_i d\hat{g}_i g_i$ (since $d \in DCPO$) $= \sup_i d^*\hat{g}_i g_i = \cdots = d^*$ (for the same reasons). Hence, d is unique, and this completes the proof. $\square$

Some comments are in order. In the first place, the immediate data given in the projective system D are the g_{ij} and then also, by duality, the $\hat{g}_{ij}$. Hence, the right hand side of relation (A) is expressed in terms of the given data. The limit maps g_i are available once the limit is calculated, and they separate the points of $L = \lim D$. Hence for any given $i \in J$, if the left hand side of (A) is known for all $j \in J$, then $\hat{g}_i$ is known. Therefore, (B) gives an explicit way to calculate the maps $\hat{g}_i$ (cf. O-3.7(3)). We know from O-3.6(2) that $\hat{g}_j g_j \le 1_L$; equation (B) tells us that "in the limit" equality holds irrespective of the surjectivity of g_j.

Secondly, the dual maps $\hat{g}_{ij}$ and $\hat{g}_i$ exist whenever a projective system in the bigger category $POSET_G$ is given. However, in the proof we needed to know that all g_{ij} preserved directed sups. Moreover, in order that compositions such as $g_{jk}\hat{g}_{ik}$ or $g_j\hat{g}_i$ are meaningful at all in any of the categories which are of interest to us, the functions g_{jk} and g_j had better be in *DCPO* (since then the compositions in question are still in *DCPO*). It is then clear why – in the calculations of IV-4.5 in particular – we have already left the purely arrow-theoretical domain.

We note that the equation (B) of IV-4.5(ii) is a special case of (D), which one obtains for $d_j = \hat{g}_j$ and $d = 1_L$. However, relation (B) was needed in the proof of IV-4.5(iii) in order to show uniqueness of d. Thus it is not possible to derive (B) as a special case from (D).

There are several consequences to Theorem IV-4.5 which elucidate the nature of projective limits in $DCPO_G$.

Theorem IV-4.6. *Let $D\colon J^{\mathrm{op}} \to DCPO_G$ be a projective system in $DCPO_G$. Then for a cone $g_j\colon L \to L_j$ over D in DCPO, where $L_j = D(j)$, the following statements are equivalent:*

(1) *$g_j\colon L \to L_j$ is a limit cone over D in DCPO;*
(2) *all the g_j have a lower adjoint $\hat{g}_j$ and $\hat{g}_j : L_j \to L$ is a colimit cone under $\hat{D}$ in DCPO;*
(3) *$g_j\colon L \to L_j$ is a limit cone over D in $DCPO_G$;*
(4) *$\hat{g}_j\colon L_j \to L$ is a colimit cone under $\hat{D}$ in $DCPO_D$.*

In this statement, one may replace the dual pair $DCPO_G$–$DCPO_D$ by the dual pair of categories $INF^{\uparrow}$–SUP^0.

Proof: (1) implies (2): This is the content of Theorem IV-4.5.

(2) implies (1): Under the hypotheses of (2), let $h_j\colon M \to L_j$ be a limit cone over D in *DCPO* which exists by IV-4.0. Again by Theorem IV-4.4, the limit maps h_j have lower adjoints and M is also a colimit of the direct system $\hat{D}$ with colimit maps $\hat{h}_j\colon L_j \to M$. By the universal property of colimits, there is a unique isomorphism $f\colon M \to L$ such that $f\hat{h}_j = \hat{g}_j$ for all j. Passing to the adjoints yields $h_j f^{-1} = g_j$ which implies that $g_j\colon L \to L_j$ is also a limit cone over D in DCPO.

(1) and (2) imply (3): Let $h_j\colon S \to L_j$ be any cone over D in $DCPO_G$. As $g_j\colon L \to L_j$ is a limit cone over D in *DCPO* by (1), there is a unique Scott-continuous map $h\colon S \to L$ such that $g_j h = h_j$ for all j. As $\hat{g}_j\colon L_j \to L$ is a colimit cone under $\hat{D}$ in *DCPO*, there is a unique Scott-continuous map $d\colon L \to S$ such that $d\hat{g}_j = \hat{h}_j$. We conclude that $\hat{h}_j h_j = d\hat{g}_j g_j h$ and $h_j\hat{h}_j = g_j h d\hat{g}_j$. As $\hat{h}_j h_j \le 1_S$ and $h_j\hat{h}_j \ge 1_{L_j}$ by O-3.6, we infer $d\hat{g}_j g_j h \le 1_S$ and $g_j h d\hat{g}_j \ge 1_{L_j}$, whence $\hat{g}_j g_j h d\hat{g}_j g_j \ge \hat{g}_j g_j$ by multiplying the last equation by $\hat{g}_j$ on the left and by g_j on the right. With the help of (B) we conclude $dh = d(\sup_j \hat{g}_j g_j)h = \sup_j d\hat{g}_j g_j h \le 1_S$ and $hd = (\sup_j \hat{g}_j g_j)hd(\sup_j \hat{g}_j g_j) \ge \sup_j(\hat{g}_j g_j h d\hat{g}_j g_j) \ge \sup\hat{g}_j g_j = 1_L$. By O-3.6 this suffices to show that d is the lower adjoint of h. Thus L is the limit of the projective system D in the category $DCPO_G$.

Conditions (3) and (4) are equivalent by the duality of the categories $DCPO_G$ and $DCPO_D$.

(3) implies (1): Suppose that the projective system D has a limit in the category $DCPO_G$. It also has a limit in the category *DCPO*. As (1) implies (3), the latter is also a projective limit in $DCPO_G$. By the uniqueness of limits, they agree (up to isomorphism). □

Now let us suppose that we have a projective system $D\colon J^{\mathrm{op}} \to DCPO_G$, and that we are given a morphism $g\colon S \to \lim D$ in $DCPO_G$. We write $h_j = g_j g$, where $g_j\colon \lim D \to L_j$ is the limit map. With the information we have, it is now possible to characterize precisely the circumstances under which g is injective or surjective in terms of the functions h_j alone. In particular, this gives criteria for g to be an isomorphism. This will become important when we explore when a functor preserves projective limits.

Proposition IV-4.7. *The following statements are equivalent:*

(1) *g is injective;*
(2) $\hat{g}g = 1_S$*;*
(3) $\sup_j \hat{h}_j h_j = 1_S$.

Proof: (1) iff (2) by O-3.7.

(2) implies (3): $\sup_j \hat{h}_j h_j = \sup_j \hat{g}\hat{g}_j h_j = \hat{g}(\sup_j \hat{g}_j h_j)$ (since $\hat{g} \in$ *DCPO*) $= \hat{g}g$ (by IV-4.5(C)) $= 1_S$ by (2).

(3) implies (2): $\hat{g}g = (\sup_j \hat{h}_j g_j)g$ (by IV-4.5(D) with $d_j = \hat{h}_j$ and $d = \hat{g}$) $= \sup_j \hat{h}_j g_j g = \sup_j \hat{h}_j h_j = 1_S$ (by (3)). □

Proposition IV-4.8. *The following statements are equivalent:*

(1) g *is surjective;*
(2) $\operatorname{im} g_j \subseteq \operatorname{im} h_j$ *for all* $j \in J$*;*
(3) $h_j\hat{h}_j = g_j\hat{g}_j$ *for all* $j \in J$*;*
(4) $h_j\hat{h}_j = \sup\{g_{jk}\hat{g}_{jk} : j \leq k\}$.

Proof: (1) implies (2): $\operatorname{im} h_j = g_j g(L) = \operatorname{im} g_j$ if g is surjective.

(2) implies (3): Suppose that $y = g_j\hat{g}_j(x)$ with $x, y \in L_j$. Then y is in $\operatorname{im} g_j$, and thus in $\operatorname{im} h_j$ by (2). Thus $y = h_j(z)$ for some $z \in L$, and so $y = h_j(z) = h_j\hat{h}_j h_j(z)$ (by O-3.6(3)) $= h_j\hat{h}_j(y) = h_j\hat{h}_j g_j\hat{g}_j(x) = h_j\hat{g}\hat{g}_j g_j\hat{g}_j(x)$ (since $h_j = g_j g$) $= h_j\hat{g}\hat{g}_j(x)$ (by O-3.6(3) again) $= h_j\hat{h}_j(x)$. Thus (3) is proved since x was arbitrary.

(3) implies (1): Now, $1_{\lim D} = \sup \hat{g}_j g_j$ (by IV-4.5(B)) $= \sup \hat{g}_j g_j \hat{g}_j g_j$ (by O-3.6(3)) $= \sup \hat{g}_j h_j \hat{h}_j g_j$ (by (3)) $= \sup \hat{g}_j g_j g\hat{g}\hat{g}_j g_j = g\hat{g}$, since $\sup \hat{g}_j g_j = 1_{\lim D}$ again. Thus $g\hat{g} = 1_{\lim D}$ which implies that g is surjective.

(3) iff (4) by IV-4.5(A). □

It is not always easy in concrete cases to decide whether the surjectivity of all bonding maps g_{jk} of a projective system entails the surjectivity of the limit maps g_j. The answer is affirmative if the category in question is based on compact spaces and continuous maps. In the situation of the categories $DCPO_G$ and $INF^{\uparrow}$, however, the situation is extremely simple.

Proposition IV-4.9. *Let D be a projective system in $DCPO_G$ such that all the bonding maps g_{jk} of D are surjective. Then the limit maps g_j*: $\lim D \to D(j)$ *are surjective, too. The same holds for projective systems in $INF^{\uparrow}$ and in other full subcategories like CL, AL, etc.*

Proof: If all g_{jk} are surjective, then $g_{jk}\hat{g}_{jk} = 1_{L_j}$ for all $k \geq j$ by O-3.7. But then $g_j\hat{g}_j = 1_{L_j}$ by IV-4.5(A). This implies that g_j is surjective. □

We have discussed projective limits in a category in general and then in the categories *DCPO*, $DCPO_G$ and $INF^{\uparrow}$ in particular. The category *CL* of continuous lattices and *CL*-morphisms, that is, Scott-continuous maps preserving arbitrary infs, is a complete full subcategory of $INF^{\uparrow}$. Thus limits of projective systems in *CL* are again continuous lattices and all of Theorem IV-4.6 and the subsequent propositions apply to the dual pair of categories CL–CL^{op} . But the category DOM_G of domains and Scott-continuous maps having a lower adjoint is not complete. Nevertheless it has projective limits:

Proposition IV-4.10. *Let D be a projective system of domains L_j, $j \in J$, and Scott-continuous maps $g_{ij}\colon L_j \to L_i$ having a lower adjoint $\hat{g}_{ij}$. Form the projective limit* $\lim D$ *as for* **dcpo***s with limit maps $g_j\colon$ $\lim D \to L_j$. Then* $\lim D$ *is a domain. If B_j is a basis for the domain L_j , then $B = \bigcup_j \hat{g}_j(B_j)$ is a basis for* $\lim D$.

Proof: If $s \ll x$ in L_j, then $\hat{g}_j(s) \ll \hat{g}_j(x)$ in $\lim D$ by IV-1.4(3). Now fix $x \in \lim D$. For every index j, the set $C_j := \twoheaddownarrow g_j(x) \cap B_j = \{s \in B_j : s \ll g_j(x)\}$ is directed in L_j and $\sup C_j = g_j(x)$. By the first statement, $\hat{g}_j(s) \ll \hat{g}_j g_j(x) \le x$ for all $s \in C_j$. Moreover, the set $G_j := \hat{g}_j(C_j)$ is directed in $\lim D$. Further $G_j \subseteq \twoheaddownarrow x$ and $\sup\ G_j = \hat{g}_j g_j(x)$.

Let $j \ll k$ and $s \in C_j$. As $s \ll g_j(x)$, we get $\hat{g}_{jk}(s) \ll \hat{g}_{jk} g_j(x) = \hat{g}_{jk} g_{jk} g_k(x) \le g_k(x)$ again by IV-1.4(3). As $g_k(x) = \sup\ C_k$, there is an element $t \in C_k$ such that $\hat{g}_{jk}(s) \le t$. We conclude that $\hat{g}_j(s) = \hat{g}_k \hat{g}_{jk}(s) \le \hat{g}_k(t)$. Thus for every $u \in G_j$ there is a $v \in G_k$ with $u \le v$. This implies that $\bigcup_j G_j$ is directed. By the preceding paragraph, $\bigcup_j G_j \subseteq \twoheaddownarrow x$ and $\sup \bigcup_j G_j = \sup_j \sup G_j = \sup_j \hat{g}_j g_j(x) = x$. Thus, $\lim D$ is a domain and B a basis. $\square$

Corollary IV-4.11. *Let D be a projective system of algebraic domains L_j, $j \in J$, and Scott-continuous maps $g_{ij}\colon Lj \to L_i$ having a lower adjoint $\hat{g}_{ij}$. Form the projective limit* $\lim D$ *as for* **dcpo***s with limit maps $g_j\colon$ $\lim D \to L_j$. Then* $\lim D$ *is an algebraic domain. An element $c \in$* $\lim D$ *is compact iff $c = \hat{g}_j(k)$ for some j and some compact element $k \in L_j$.* $\square$

As a consequence, projective limits exist in all of the full subcategories DOM_G, $ALGDOM_G$, CL, AL of $DCPO_G$. Also, a projective limit of FS-domains is an FS-domain: Indeed let D be a projective system of FS-domains L_j with bonding maps $g_{ij}\colon L_j \to L_i$ that have lower adjoints. For every index j, let $\mathcal{D}_j$ denote an approximate identity for L_j consisting of finitely separating functions (see II-2.15). On $\lim D$ we define finitely separating functions $\hat{g}_j \delta g_j$ for $\delta \in \mathcal{D}_j$ by means of the limit maps g_j: $\lim D \to L_j$ and their adjoints. If we collect these functions for all j, we obtain an approximate identity $\mathcal{D}$ on $\lim D$.

Bifinite domains (see II-2.21) have a nice characterization by means of projective limits, which by the way explains the term *bifinite*:

Proposition IV-4.12. *A* **dcpo** *is a bifinite domain iff it is the limit of a projective system D of finite posets L_j with surjective upper adjoints as bonding maps $g_{ij}\colon L_j \to L_i$ which have a lower adjoint.*

Proof: Suppose first that L is bifinite. By II-2.20(3), L has an approximate identity $\mathcal{D}$ of kernel operators k with finite range $L_k = k(L)$. For kernel operators $h \leq k$, the image L_h is a subset of L_k and the restriction and co-restriction $g_{hk} = h \mid L_k \colon L_k \to L_h$ is surjective and has as its lower adjoint the subset embedding of L_h into L_k. Thus, we have a projective system indexed by $\mathcal{D}$. We show that L is the limit of this projective system, where the limit maps are the co-restrictions $g_k \colon L \to L_k$ of the kernel operators k. Indeed, let $f_k \colon S \to L_k$ be any cone over our projective system. Define $g \colon S \to L$ by $g(s) = \sup_k f_k(s)$. It is straightforward that g is well defined and Scott-continuous, that it satisfies $g_k g = f_k$ and that g is the only continuous map from S to L with this property.

Conversely, if D is a projective system of finite posets L_j with surjective bonding maps $g_{ij} \colon L_j \to L_i$ which have a lower adjoint $\hat{g}_{ij}$, then the limit maps $g_j \colon \lim D \to L_j$ are surjective by IV-4.9, hence their lower adjoints $\hat{g}_j \colon L_j \to \lim D$ are injective. The compositions $k_j = \hat{g}_j g_j$ are kernel operators on $\lim D$, the image of which is isomorphic to the finite poset L_j . Finally $\sup_j k_j = \sup_j \hat{g}_j \hat{g}_j = 1_{\lim D}$ by IV-4.5(B). Thus, we have an approximate identity of kernel operators k_j with finite image, that is, $\lim D$ is bifinite. $\square$

Exercises

Exercise IV-4.13. Show that projective limits of projective systems of bounded complete domains and L-domains, respectively, with Scott-continuous upper adjoints as bonding maps are again bounded complete domains and L-domains, respectively. $\square$

Exercise IV-4.14. Show that every algebraic lattice is the limit of a projective system of finite lattices with inf preserving surjective bonding maps. $\square$

Exercise IV-4.15. Show that the limit of a projective system of Lawson compact domains with Scott-continuous upper adjoints as bonding maps is also Lawson compact. $\square$

Exercise IV-4.16. The full subcategories in $\mathit{INF}^{\uparrow}$ of continuous and algebraic lattices are complete subcategories. What is the situation with arithmetic lattices? $\square$

Old notes

The importance of projective limits for continuous lattices and their applications was first pointed out by D.S. Scott in [Scott, 1972a], where limits of projective systems whose index domain was the natural numbers and whose maps were all surjective were utilized. The results of IV-4.5 in this special case were established in that paper; however, the treatment of arbitrary projective limits given here first appeared in the *Compendium*.

IV-5 Pro-continuous and Locally Continuous Functors

We have discussed projective limits in a category in general and then in the categories $DCPO_G$ and $INF^{\uparrow}$ in particular. Now we discuss the preservation of projective limits by functors between categories in general, and then by self-functors $DCPO_G \to DCPO_G$ in particular. The special characteristics of our categories are responsible for the existence of several relevant functors which do preserve projective limits – while they do not even preserve products in general, let alone arbitrary limits. Our task is to describe manageable criteria which allow us to test concrete functors for the preservation properties vis-à-vis projective limits. We freely use the notation of the previous section.

Definition IV-5.1. Let $F\colon A \to B$ be a functor between complete categories and let $D\colon J \to A$ be a diagram in A. Let $g_j\colon \lim D \to D(j)$ be the limit cone over D in A. Then $Fg_j\colon F(\lim D) \to FD(j)$ is a cone over the diagram $FD\colon J \to B$ in B. Now let $h_j\colon \lim FD \to FD(j)$ be the limit cone over the diagram FD in B. By the universal property of the limit (IV-4.1) there is a unique map $g\colon F(\lim D) \to \lim FD$ such that $h_j g = Fg_j$ for all j. We say that F *preserves the limit of D* iff g is an isomorphism.

In general, F is said to *preserve limits* iff g is an isomorphism for all diagrams D in A and we say that F *preserves projective limits* or simply that F is *pro-continuous* iff g is an isomorphism for all projective systems D in A. □

There are numerous functors occurring in nature which preserve projective limits but do not preserve limits. Čech cohomology on compact spaces with values in the opposite category of graded modules is one of the better-known examples, and we will see that most of the functors which interest us here fall into the same category. We recall (cf. third paragraph following IV-4.1) that a functor which preserves limits must preserve products and equalizers, and by a theorem of Freyd any functor preserving products and equalizers preserves arbitrary limits. But since many of the functors which we will discuss will not preserve products, it will be important to have a criterion for functors preserving projective limits.

Proposition IV-5.2. *Let $F\colon DCPO_G \to DCPO_G$ be a self-functor and let $D\colon J^{\mathrm{op}} \to DCPO_G$ be a projective system in $DCPO_G$. Denote by*

g_j: $\lim D \to D(j)$ *the limit cone of* D *and by* h_j: $\lim FD \to FD(j)$ *the limit cone of* FD. *Then the following statements are equivalent.*

(1) *F preserves the limit of D.*
(2) (i) $(Fg_j)(Fg_j)\hat{} = \sup_j (Fg_{jk})(Fg_{jk})\hat{}$ *for all* $j \in J$, *and*
  (ii) $\sup_j (Fg_j)\hat{}\,(Fg_j) = 1_{F(\lim D)}$.

Proof: We consider the diagram

$$\begin{tikzcd} F(\lim D) \arrow[rr, "g"] \arrow[dr, "Fg_j"'] & & \lim FD \arrow[dl, "h_j"] \\ & FD(j) & j \in J \end{tikzcd}$$

and we find ourselves in the situation discussed in Propositions IV-4.7 and IV-4.8, which gave a characterization for the injectivity and surjectivity of g, respectively. We recognize that condition (2)(ii) is that of injectivity and that of (2)(i) is that of surjectivity of g. Consequently, the theorem follows from IV-4.7 and IV-4.8. □

The conditions of the preceding proposition look quite difficult to verify. They become much nicer if F preserves adjoints, i.e. if $F(\hat{g}_j) = (Fg_j)\hat{}$. Then $(Fg_j)(Fg_j)\hat{} = (Fg_j)(F\hat{g}_j) = F(g_j\hat{g}_j)$, similarly $(Fg_j)\hat{}\,(Fg_j) = F(\hat{g}_j g_j)$ and $(Fg_{jk})(Fg_{jk})\hat{} = (Fg_{jk})(F\hat{g}_{jk}) = F(g_{jk}\hat{g}_{jk})$. Thus, conditions (i) and (ii) then can be rewritten in the following way:

(i′) $F(g_j\hat{g}_j) = \sup_j F(g_{jk}\hat{g}_{jk})$ for all $j \in J$;
(ii) $\sup_j F(\hat{g}_j g_j) = 1_{F(\lim D)}$.

Both of these conditions are automatically satisfied if, in addition, the functor F preserves directed suprema in the sense that, for any directed family of morphisms g_j: $L \to M$, the family of morphisms Fg_j: $FL \to FM$ is also directed and $\sup_j Fg_j = F(\sup g_j)$. We then have indeed $\sup_j F(g_{jk}\hat{g}_{jk}) = F(\sup_j g_{jk}\hat{g}_{jk}) = F((g_j\hat{g}_j)$ by IV-4.5(A) and $\sup_j F(\hat{g}_j g_j) = F(\sup_j \hat{g}_j g_j) = F(1_{\lim D}) = 1_{F(\lim D)}$ by IV-4.5(B).

We are now going to formulate natural conditions on a functor F so that the additional properties just mentioned are satisfied. For this recall that in the category *DCPO* of **dcpo**s and Scott-continuous functions the hom-sets $DCPO(L, M) = [L \to M]$ of all Scott-continuous functions g: $L \to M$ may be considered as **dcpo**s under the pointwise ordering $g \leq h$ iff $g(x) \leq h(x)$ for all $x \in L$.

Definition IV-5.3. A self-functor, or endofunctor, F: *DCPO* $\to$ *DCPO* is called *locally monotone* or *locally order preserving* if it is order preserving on hom-sets, that is, whenever $g \leq h$ in $[L \to M]$ then $Fg \leq Fh$ in $[FL \to FM]$.

An endofunctor F: *DCPO* $\to$ *DCPO* is called *locally continuous* if it is Scott-continuous on hom-sets, that is, it is locally order preserving and whenever g_j, $j \in J$, is a directed family in $[L \to M]$, then $sup_j\ Fg_j = F(sup\ g_j)$ in $[FL \to FM]$.

Of course, the same terminology can be used for endofunctors on any full subcategory of *DCPO*. □

The significance of locally order preserving functors is that they preserve adjointness, and surjectivity and injectivity of maps having an adjoint.

Lemma IV-5.4. *Let F be a locally order preserving endofunctor on DCPO. If a Scott-continuous map g in DCPO has a lower adjoint $\hat{g}$, then Fg has a lower adjoint, too, namely $(Fg)\hat{} = F\hat{g}$. If, moreover, g is surjective, respectively injective, then Fg is surjective, respectively injective, too. In particular, F restricts to a functor $DCPO_G \to DCPO_G$, and likewise to a functor $DCPO_D \to DCPO_D$, that preserves injectivity and surjectivity of maps.*

Proof: For a pair g: $L \to M$ and $\hat{g}$: $M \to L$ of adjoints one has $\hat{g}g \leq 1_L$, $1_M \leq g\hat{g}$ by O-3.6. As the functor F is locally order preserving, we infer $F\hat{g}Fg = F(\hat{g}g) \leq F1_L = 1_{FL}$ and $1_{FM} = F1_M \leq F(g\hat{g}) = FgF\hat{g}$, and this shows that $F\hat{g}$ is the lower adjoint of Fg again by O-3.6. Moreover, g is surjective, respectively injective, iff $1_M = g\hat{g}$, respectively $\hat{g}g = 1_L$, by O-3.7; from this we infer $1_{FM} = F1_M = F(g\hat{g}) = (Fg)(F\hat{g})$, respectively $(F\hat{g}Fg) = F(\hat{g}g) = F1_L = 1_{FL}$, which implies that Fg is surjective, respectively injective, again by O-3.7. □

We now can state a most useful result on preservation of projective limits. The proof is contained in the discussion following Proposition IV-5.2, if one takes into account Lemma IV-5.4.

Theorem IV-5.5. *Let F be a locally continuous endofunctor on the category DCPO. Then F preserves adjoints and restricts to an endofunctor on $DCPO_G$ which preserves limits of projective systems and surjectivity and injectivity of maps in $DCPO_G$. In particular, F preserves limits of projective systems D of* **dcpo***s L_j with bonding maps g_{ij}: $L_j \to L_i$ having a lower adjoint $\hat{g}_{ij}$: $L_i \to L_j$.* □

Similarly, a locally continuous endofunctor on *DCPO* restricts to an endofunctor on $DCPO_D$ which preserves colimits of direct systems.

In the preceding theorem one may replace the category *DCPO* by any full subcategory A. Thus, we may choose for A the category *DOM* of domains, the category *UPS* of complete lattices, the category *CONT* of continuous lattices, etc. with all Scott-continuous maps as morphisms.

It is useful to extend the terminology in the obvious way to contravariant functors and to functors in several arguments. A contravariant functor F: $DCPO^{\mathrm{op}} \to DCPO$ is said to be *locally continuous* if $g \mapsto Fg : [L \to M] \to [FM \to FL]$ is Scott-continuous for all **dcpo**s L and M. A bifunctor B: $DCPO \times DCPO \to DCPO$ is *locally continuous* if it is locally continuous in each of its arguments. It is clear now what we mean by local continuity for a bifunctor B: $DCPO^{\mathrm{op}} \times DCPO \to DCPO$ which is contravariant in its first and covariant in its second argument: for all Scott-continuous functions g: $L' \to L$ and h: $M \to M'$, the map

$$(g, h) \mapsto B(g, h) : [L' \to L] \times [M \to M'] \to [B(L, M) \to B(L', M')]$$

is Scott-continuous in each of its arguments, which is equivalent to saying that it is Scott-continuous in both arguments simultaneously by II-2.8.

Examples IV-5.6. The following functors and bifunctors are locally continuous. (The verification of the local continuity is left to the reader as an exercise. Compare also I-1.31.)

(i) **Lifting.** For any **dcpo** L let $FL = L_\perp$ be the **dcpo** obtained from L by adjoining a new bottom element. For a Scott-continuous function g: $L \to M$ let Fg be defined by $Fg(\perp) = \perp$ and $Fg(x) = g(x)$ for $x \in L$.

(ii) **Disjoint sum.** For two **dcpo**s L and M let $L \sqcup M$ be the disjoint union. This extends in a standard way to a locally continuous functor

$$\sqcup: DCPO \times DCPO \to DCPO.$$

(iii) **Separated sum.** Lifting of the disjoint sum yields the *separated sum* $L + M = (L \sqcup M)_\perp$ which clearly can be extended to a locally continuous bifunctor.

(iv) **Product.** The assignment $(L, M) \mapsto L \times M$ yields a locally continuous functor

$$\times: DCPO \times DCPO \to DCPO.$$

(v) **Function space.** To **dcpo**s L and M we may assign the function space $[L \to M]$ and to continuous maps $g\colon L' \to L$ and $h\colon M \to M'$ the function $[g \to h]\colon [L \to M] \to [L' \to M']$ defined by $\varphi \mapsto h\varphi g$. We have seen in II-2.7 that this yields a functor with mixed variance

$$[. \to .]\colon DCPO^{\mathrm{op}} \times DCPO \to DCPO\colon$$

$$\begin{tikzcd}
L \arrow[r, "\varphi"] & M \arrow[d, "h"] \\
L' \arrow[u, "g"] \arrow[r, "{[g \to h](\varphi)}"'] & M'
\end{tikzcd}$$

(For the local continuity of this functor one uses that composition

$$(g, \varphi, h) \mapsto h\varphi g : [L' \to L] \times [L \to M] \times [M \to M'] \to [L' \to M']$$

is Scott-continuous by II-2.9(ii), whence $(g, h) \mapsto (\varphi \mapsto h\varphi g)$ is a Scott-continuous map $[L' \to L] \times [M \to M'] \to [[L \to M] \to [L' \to M']]$ by II-2.10.) $\square$

From these examples one can obtain further locally continuous functors by composing these functors. Let us specify three natural ways to obtain an endofunctor on $DCPO$ from a bifunctor $B\colon DCPO \times DCPO \to DCPO$ by precomposing B with any of the following three functors:

(a) the diagonal embedding $\Delta = (S \mapsto (S, S))\colon DCPO \to DCPO \times DCPO$;
(b) the functor $L_A = (S \mapsto (S, A))\colon DCPO \to DCPO \times DCPO$, for a fixed **dcpo** A;
(c) the functor $R_A = (S \mapsto (A, S))\colon DCPO \to DCPO \times DCPO$, for a fixed **dcpo** A.

These three functors are clearly locally continuous. Thus, if B is locally continuous, then the three endofunctors $B\Delta\colon L \mapsto B(L, L)$, $BL_A\colon L \mapsto B(L, A)$ and $BR_A\colon L \mapsto B(A, L)$ on $DCPO$ are also locally continuous. The previous results on the preservation of adjoints, their surjectivity and injectivity and on the preservation of limits of projective systems with bonding maps having a lower adjoint (Theorem IV-5.5) can be applied to all covariant endofunctors arising from the above examples.

For contravariant functors and for bifunctors with mixed variance

$$B\colon DCPO^{\mathrm{op}} \times DCPO \to DCPO$$

– as the function space functor, for example – this procedure cannot be applied directly, and we have to introduce a slight modification. Let us begin with a contravariant functor:

Lemma IV-5.7. *Let $F\colon DCPO^{\mathrm{op}} \to DCPO$ be a locally order preserving contravariant endofunctor. If a Scott-continuous function $g\colon L \to M$ has a lower adjoint $\hat{g}\colon M \to L$, then $F\hat{g}\colon FL \to FM$ also has a lower adjoint, namely $(F\hat{g})\hat{} = Fg$. If g is surjective or injective, then $F\hat{g}g$ is also surjective or injective, respectively.*

Proof: The proof is quite similar to that of Lemma IV-5.4. Indeed, as $g\hat{g} \geq 1_M$ by O-3.6 and as F is a locally order preserving contravariant functor, we have $F\hat{g}Fg = F(g\hat{g}) \geq F(1_M) = 1_{FM}$. Similarly, $FgF\hat{g} \leq 1_{FL}$, whence $(F\hat{g})\hat{} = Fg$ by O-3.6. The remaining properties are proved by replacing one of the inequalities by an equality. □

Proposition IV-5.8. *Let $F\colon DCPO^{\mathrm{op}} \to DCPO$ be a locally continuous contravariant endofunctor. Then $g \mapsto F\hat{g}$ defines a covariant endofunctor $\hat{F}\colon DCPO_G \to DCPO_G$ which preserves surjectivity, injectivity of maps and limits of projective systems.*

Proof: By Lemma IV-5.7, $g \mapsto F\hat{g}$ defines indeed a covariant endofunctor on $DCPO_G$ which preserves surjectivity and injectivity of maps. We cannot use that the functor $\hat{F}\colon DCPO_G \to DCPO_G$ is locally continuous. In fact, $g \mapsto \hat{g}$ is order reversing rather than order preserving. It is the local continuity of F that we have to use. We verify IV-5.2(2) with the notation from there:

$$
\begin{array}{llll}
\text{(i)} & \sup_j(\hat{F}g_{jk})(\hat{F}g_{jk})\hat{} & = \sup_j(F\hat{g}_{jk})(F\hat{g}_{jk})\hat{} & \text{by the definition of } \hat{F} \\
 & & = \sup_j(F\hat{g}_{jk})(Fg_{jk}) & \text{by Lemma IV-5.7} \\
 & & = \sup_j F(g_{jk}\hat{g}_{jk}) & \text{as } F \text{ is a contravariant functor} \\
 & & = F(\sup_j g_{jk}\hat{g}_{jk}) & \text{by the local continuity of } F \\
 & & = F(g_j\hat{g}_j) & \text{by IV-4.5(A)} \\
 & & = (F\hat{g}_j)(Fg_j) & \text{as F is a contravariant functor} \\
 & & = (\hat{F}g_j)(\hat{F}g_j)\hat{} & \text{by the definition of } \hat{F}. \\
\text{(ii)} & \sup_j(\hat{F}g_j)(\hat{F}g_j) & = \sup_j(F\hat{g}_j)\hat{}(F\hat{g}) & \text{by the definition of } \hat{F} \\
 & & = \sup_j(Fg_j)(F\hat{g}_j) & \text{by Lemma IV-5.7} \\
 & & = \sup_j F(\hat{g}_j g_j) & \text{as } F \text{ is a contravariant functor} \\
 & & = F(\sup_j \hat{g}_j g_j) & \text{by the local continuity of } F \\
 & & = F(1) = 1 & \text{by IV-4.5(A).}
\end{array}
$$
□

For a locally continuous functor $B\colon DCPO^{\mathrm{op}} \times DCPO \to DCPO$ which is contravariant in the first and covariant in the second argument we may combine the two methods for covariant and contravariant endofunctors. For a **dcpo** L define $FL = B(L, L)$ and for a map $g: L \to M$ having a lower adjoint define $Fg = B(\hat{g}, g)$. As above one then verifies that $B(\hat{g}, g)$ has a lower adjoint, namely $B(g, \hat{g})$. Thus we have defined an endofunctor $F: DCPO_G \to DCPO_G$. As above, one verifies that F satifies condition (2) in Proposition IV-5.2. The reader may do this as an exercise. We conclude as follows.

Proposition IV-5.9. *For a locally continuous functor $B: DCPO^{\mathrm{op}} \times DCPO \to DCPO$ define*

$$FL = B(L, L) \textit{ and } Fg = B(\hat{g}, g) \textit{ for all } \mathbf{dcpo}\textit{s } L \textit{ and all } g \in DCPO_G.$$

Then F becomes a functor $F: DCPO_G \to DCPO_G$ preserving limits of projective systems as well as surjectivity and injectivity of maps. □

One can apply these results on the preservation of limits of projective systems to the functors resulting from the Examples IV-5.6. Let us look at two of these examples in more detail.

Example IV-5.10. (The function space functor) Consider the function space functor B: $DCPO^{\mathrm{op}} \times DCPO \to DCPO$ from Example IV-5.6(v), where $B(L, M) = [L \to M]$ and $B(g, h)(\varphi) = h\varphi g$.

As in Proposition IV-5.9 we form *the function space functor*

$$\mathrm{Funct}: DCPO_G \to DCPO_G$$

which associates with a **dcpo** L the function space $\mathrm{Funct}(L) = [L \to L]$ of all Scott-continuous functions from L to itself. If $g: S \to T$ is a $DCPO_G$-morphism, that is a Scott-continuous function having a lower adjoint $\hat{g}$, then $\mathrm{Funct}(g)(\varphi) = g\varphi\hat{g}$:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "g"] & S \arrow[d, "g"] \\ T \arrow[u, "\hat{g}"] \arrow[r, "\mathrm{Funct}(g)(\varphi)"'] & T \end{tikzcd}$$

If A is any fixed **dcpo**, we form the contravariant functor $[\cdot \to A]$: $DCPO^{\mathrm{op}} \to DCPO$ which assigns to every **dcpo** L the function space $[L \to A]$ and to every Scott-continuous map g: $M \to L$ the function $\varphi \mapsto \varphi g$: $[L \to A] \to [M \to A]$.

As in Lemma IV-5.7 we derive from this functor a covariant endofunctor

$$[? \to A]\colon DCPO_G \to DCPO_G$$

which assigns the function space $[L \to A]$ to every **dcpo** L and the function $\varphi \mapsto \varphi\hat{g}\colon [L \to A] \to [M \to A]$. Then Propositions IV-5.8 and IV-5.9 yield

The function space functor Funct: $DCPO_G \to DCPO_G$ *preserves projective limits, injectivity and surjectivity of morphisms. The functor* $[? \to A]$: $DCPO_G \to DCPO_G$ *has the same property for any* **dcpo** A.

An analogous statement is true for the functor $[A \to ?]$; in the exercises we generalize this case (see IV-5.17).

In this statement the category $DCPO_G$ may be replaced by any of the categories DOM_G, $INF^{\uparrow}$, CL, AL of domains, complete lattices, continuous lattices, algebraic lattices, respectively, with Scott-continuous maps having a lower adjoint, that is, maps preserving arbitrary infs and directed sups in the three last cases. □

A special case of the preceding example is worth noting:

Example IV-5.11. (The Scott topology functor) The functor $\sigma\colon DCPO_G \to DCPO_G$ which associates with a **dcpo** L its Scott topology $\sigma(L)$ and with a $DCPO_G$-morphism $g\colon L \to M$ the function $U \mapsto \hat{g}^{-1}(U) : \sigma(L) \to \sigma(M)$ preserves projective limits, injectivity and surjectivity of morphisms. The same holds if one replaces the category $DCPO_G$ by any of its full subcategories like DOM_G, $INF^{\uparrow}$, CL, AL.

Proof: There is a natural isomorphism

$$f \mapsto f^{-1}(1) : [L \to 2] \to \sigma(L)$$

and the map $B(g, 1_2)$ corresponds to the map $U \mapsto \hat{g}^{-1}(U)$ under this isomorphism. The result is now a special case of the second part of IV-5.10. □

Example IV-5.12. (The ideal functor) As a next example we consider the *ideal functor* Id: $DCPO \to DCPO$ of IV-1.19 and IV-1.20. We recall that for a **dcpo** L we let Id L be the set of ideals of L which is again a **dcpo** ordered by inclusion (even an algebraic domain). And if $g\colon L \to M$ is Scott-continuous, then Id g: Id $L \to$ Id M is defined by $(\mathrm{Id}\, g)(I) = {\downarrow} g(I)$.

The ideal functor Id: $DCPO \to DCPO$ *is locally order preserving*. Indeed if $f \leq g$ for $f, g \in [L \to M]$, then $f(x) \leq g(x)$ for all x and, hence, $f(I) \subseteq {\downarrow} g(I)$ for every ideal I. Thus, by IV-5.4 it preserves adjoints and induces an endofunctor on $DCPO_G$ that preserves surjectivity and injectivity of morphisms in $DCPO_G$.

But the functor Id *is not locally continuous*. The example of $L = \mathbb{N} \cup \{\infty\}$ and $\varphi_n\colon L \to L$, $\varphi_n(x) = \min\{x, n\}$ shows that the ideal functor is not Scott-continuous on $[L \to L]$: indeed, $\sup_n \varphi_n = 1_L$ but for $I = L$ we have $\sup_n (\mathrm{Id}\,\varphi_n)(I) = \sup_n {\downarrow} n = \mathbb{N} \neq L = \mathrm{Id}(1_L)(L)$.

The same example shows that the ideal functor does not preserve projective limits. Indeed, L is the projective limit of the projective sequence of finite chains $L_n = \{0, 1, \dots, n\}$ with bonding maps $\varphi_{nm} = \varphi_n|_{L_m}\colon L_m \to L_n$ for $n < m$. The co-restrictions $\varphi_n\colon L \to L_n$ are the limit maps. All the maps φ_{nm} and φ_n have lower adjoints, namely the canonical embeddings $e_{nm}\colon L_n \to L_M$ and $e_n\colon L_n \to L$. But condition IV-5.2(2)(ii) is violated: for $I = L$ we have $\sup_n(\mathrm{Id}\,\varphi_n)\hat{}\ (\mathrm{Id}\,\varphi_n)\ (L) = \sup_n(\mathrm{Id}\varphi_n)\hat{}\,(L_n) = \sup_n L_n = \mathbb{N} \neq L = 1_{\mathrm{Id}\,L}(L)$, whence $\sup_n(\mathrm{Id}\,\varphi_n)\hat{}\,(\mathrm{Id}\,\varphi_n) \neq 1_{\mathrm{Id}\,L}$. $\square$

The discussion of these self-functors would remain somewhat incomplete without some words of explanation of to what extent each of them "enlarges" the size of an object. In speaking of the size here we make reference to the weight of a complete lattice whose various aspects were analyzed in Section III-4. We also recall that we introduced the concept of weight for domains only. For the function spaces to be domains one needs additional hypotheses.

Proposition IV-5.13. *If L is infinite, then:*

(i) $w(\mathrm{Funct}\ L) = w(L)$ *for every continuous lattice, every bounded complete domain and every FS-domain L;*
(ii) $w([L \to A]) = \max\{w(L), w(A)\}$ *for every domain L and every continuous lattice or bounded complete domain A;*
(iii) $w(\sigma(L)) = w(L)$ *for every domain L;*
(iv) $w(\mathrm{Id}\ L) =$ *card L for every domain L.*

If L is finite, so are $w(\mathrm{Funct}\ L)$, $w(\sigma(L))$ *and* $w(\mathrm{Id}\ L)$.

Proof: Parts (i), (ii) and (iii) are consequences of III-4.10. By the Remark following III-4.4 and I-4.10 we know $w(\mathrm{Id}\,L) = \mathrm{card}(K(\mathrm{Id}\,L)) = \mathrm{card}\ L$, since $K(\mathrm{Id}\ L) \cong L$. This proves (iv). The statement concerning the finite case is clear. $\square$

We note that the functors Funct and σ do not "enlarge" L, while Id does, in general.

If L is a continuous lattice which is the projective limit of continuous lattices L_j, we need to know the weight of L in terms of the weights of the L_j. This is the place to record the relevant information.

Proposition IV-5.14. *Let $D: J^{\mathrm{op}} \to CL$ be a projective system of continuous lattices or of bounded complete domains, and assume that J is infinite. Then*

$$w(\lim D) \le \max\{\operatorname{card} J, \sup\{w(D(j)): j \in J\}\}.$$

Remark. The assumption that J be infinite is no loss of generality: if J is finite, then $k = \max J$ exists and $\lim D = D(k)$.

Proof of proposition: First let $g_j: \lim D \to D(j)$ be the limit maps. Then $x \mapsto (g_j(x))_{j\in J} : \lim D \to \prod_{j\in J} D_{(j)}$ is a subalgebra embedding, whence $w(\lim D) \le w(\prod_{j\in J} D(j))$ by III-4.12(ii). But by III-4.12(i) we have

$$w\left(\prod\nolimits_{j\in J} D(j)\right) = \max\{\operatorname{card} J, \sup\{w(D(j)): j \in J\}\}.$$

This proves the proposition. $\square$

Remark IV-5.15. (dcpo-Enriched categories) All of the developments of this section can be carried through if we replace the category *DCPO* of **dcpo**s and Scott-continuous functions by any full subcategory that has projective limits. We have already seen such sub-categories. Another example of such a category is the category

$$DCPO_{\perp}$$

of **dcpo**s having a smallest element and all Scott-continuous maps as morphisms. But also some non-full subcategories will do. An good example is the category

$$DCPO_{\perp!}$$

of all **dcpo**s having a smallest element and *strict* Scott-continuous functions as morphisms, where the strictness of functions means that they preserve smallest elements. The functors "lifting" and "separated sum" from Example IV-5.6 preserve strictness of maps and induce locally continuous self-functors on $DCPO_{\perp!}$. The functors "disjoint sum", "product" and the functions space functor from Example IV-5.6 do not preserve strictness of maps. One has to modify them:

(i) **Coalesced sum.** For two **dcpo**s L and M with least elements, let $L \oplus M$ be the **dcpo** obtained from the disjoint union $L + M$ by gluing together the least elements of L and M. For maps $g: L \to L'$ and $h: M \to M'$ preserving the least elements, it is clear how to define $g \oplus h: L \oplus M \to L' \oplus M'$.

(ii) **Smash product.** For two **dcpo**s L and M with least elements 0, let $L \otimes M = ((L\backslash\{0\}) \times (M\backslash\{0\}))_\perp$. Alternatively, one may obtain the smash product from the direct product $L \times M$ by identifying all the elements of the form $(x, 0)$ or $(0, y)$ with $(0, 0)$. For maps $g\colon L \to L'$ and $h\colon M \to M'$ preserving the least elements, one defines $g \otimes h$ by $(g \otimes h)(x, y) = (g(x), h(y))$, whenever $x \neq 0$ and $y \neq 0$ and $(g \otimes h)(x, y) = \perp$ otherwise.

(iii) **Strict function space.** For two **dcpo**s L and M with least elements 0, we denote by $[L \overset{!}{\to} M]$ the set of all strict Scott-continuous functions from L to M ordered pointwise. The strict function space is again a **dcpo** with the constant function 0 as least element. As in the case of ordinary function spaces we obtain here a bifunctor which is contravariant in the first and covariant in the second argument.

These functors are locally continuous and the results on projective limits in the previous section IV-4 and on the preservation of projective limits in this section apply accordingly.

An abstract framework for these developments is given by the notion of **dcpo**-*enriched categories*. These are categories A in which every hom-set $A(S, T)$ is endowed with the structure of a **dcpo** in such a way that composition is Scott-continuous. In such categories the notion of adjoints g and d can be defined through the property $dg \leq 1$, $gd \leq 1$ (compare O-3.6). The notion of locally continuous functors is meaningful and the results on projective limits and on preservaton of projective limits can be proved exactly in the same way as we have done it in this and the previous section. □

Remark IV-5.16. **(ω-Complete posets)** In the following section we will use the results of this section on projective limits. But the projective systems that will occur there are of a simple type: They are defined over the set $\omega = \{0, 1, 2, \dots\}$ of nonnegative integers with their natural order. A projective system $D\colon \omega^{\mathrm{op}} \to A$ is then given by a sequence $D(n), n \in \omega$ of objects and morphisms $f n$: $D(n + 1) \to D(n)$:

$$D(0) \xleftarrow[f_0]{} D(1) \xleftarrow[f_1]{} D(2) \xleftarrow[f_2]{} D(3) \xleftarrow[f_3]{} \cdots \tag{1}$$

Such projective systems may be called *projective sequences*. It goes without saying that the bonding maps $g_{mn}\colon D(n) \to D(m)$ for $m < n$ are given by composition $g_{mn} = f_{n-1} \dots f_{m+1} f_m$. In order to obtain the same results as before on limits of projective sequences a simpler setting than that of **dcpo**s is sufficient. Let us outline this setting.

In a poset S, an ascending sequence $x_0 \leq x_1 \leq x_2 \leq \cdots$ may be called an ω-chain. If every ω-chain in S has a sup, then we say that S is an *ω-complete* *poset* or an ω-cpo for short. A function $f: S \to T$ of ω-cpos is said to be ω *continuous*, if f preserves sups of ω-chains, that is, if f is order preserving and if $f(\sup_n x_n) = \sup_n f(x_n)$ for every ascending sequence $x_0 \leq x_1 \leq x_2 \leq \cdots$ in S.

Note that the Least Fixed-Point Theorem (II-2.4) holds for ω-continuous self-maps of ω-cpos with a least element.

We denote by ωCPO the category of ω-cpos and ω-continuous maps. This category has products and equalizers and hence is complete. Projective limits are formed as usual. One also has a function space object $[S \overset{\omega}{\to} T]$ consisting of all ω-continuous functions from S to T with the pointwise order. Thus the notions of locally order preserving and locally ω-continuous functors make sense.

All the proofs and results on projective limits in the previous section IV-4 and in this section remain valid, if we replace the arbitrary directed set J by the set ω of natural numbers and if we restrict our attention to projective sequences in the category ωCPO instead of *DCPO*. In particular, limits of projective sequences with ω-continuous upper adjoints as bonding maps are preserved by locally ω-continuous functors. The various examples of locally continuous functors can easily be adapted to this situation.

For countably based domains, ω-continuity is equivalent to Scott-continuity (see II-2.2) and, thus, the two approaches coincide. □

Exercises

Exercise IV-5.17. Let X be a T_0 space. Then $[X, ?]$: $DCPO \to DCPO$ is a functor given by $[X, L] = TOP(X, \Sigma L)$ with the pointwise order on objects, and for maps by $[X, g]$: $[X, S] \to [X, T]$, where $[X, g](\varphi) = g\varphi$ with $g \in DCPO$. Prove the following.

(i) The functor $[X, ?]$ is locally continuous, that is, the map $\varphi \mapsto [X, \varphi]$: $[S \to T] \to [[X, S] \to [X, T]]$ is Scott-continuous.
(ii) If g: $S \to T$ has a lower adjoint, then $[X, g]$: $[X, S] \to [X, T]$ has a lower adjoint, namely $[X, g]\hat{} = [X, \hat{g}]$.
(iii) The restriction and co-restriction of the functor $[X, ?]$: $DCPO_G \to DCPO_G$ preserve projective limits and injectivity and surjectivity of morphisms.
(iv) If $\mathcal{O}(X)$ is a continuous lattice, the functor maps *CL* into itself, and if $\mathcal{O}(X)$ is an algebraic lattice, it maps *AL* into itself.

Remark. This functor generalizes the functor $BR_A = [A \to ?]$ considered in IV-5.6, since $[A \to L] = [\Sigma A, L]$. □

Exercise IV-5.18. For each infinite cardinal m let CL_m denote the full subcategory of CL with objects L with $w(L) \leq m$. Show that this subcategory is closed under taking limits of diagrams of cardinality at most m and under all finite colimits. What can be said of the formation of function spaces $[L \to M]$?

Hint. Compare IV-5.14. □

Old notes

The question of which functors preserve projective limits was initially treated in the *Compendium*, although several authors recognized earlier how it connects with Scott's constructions. That construction and generalizations of it are the topic of the next section. The preservation of projective limits indexed by the natural numbers and sufficient conditions for such preservation were considered by [Smyth and Plotkin, 1978], [Smyth and Plotkin, 1982].

New notes

The *Compendium* contained an error in claiming that the ideal functor preserves projective limits. This also led to the false statement that every continuous lattice is a retract of a continuous lattice which is isomorphic to its lattice of ideals. The error has been corrected in IV-5.12. The error had been discovered by M. Erné [Erné, 1985], who showed that a poset is isomorphic to the poset of its ideals if and only if it satisfies the ascending chain condition.

IV-6 Fixed-Point Constructions for Functors

Scott's construction of continuous lattices L which are isomorphic to their own function space $[L \to L]$ is a special case of the general construction to be discussed in this section; the first part is entirely functorial, and the second part, in which we study applications to categories of **dcpo**s, relies on the results prepared in the previous section.

In order to illustrate the basic idea, we return briefly to a fixed-point theorem for posets and summarize some of the ideas going into its well-known proof (compare II-2.4 and II-2.30). To this end suppose that A is a **dcpo** and F: $A \to A$ a self-map. We require that F is Scott-continuous, that is,

(i) F is monotone,
(ii) F preserves directed sups.

(Note that actually (ii) implies (i).) Now look at the subset A_F of all $x \in A$ such that $x \leq Fx$. We assume that A_F is nonempty (if A has a smallest element 0, then it is necessarily in A_F). For $x \in A_F$ inductively construct the sequence

$$x \leq Fx \leq F^2x \leq F^3x \leq \cdots,$$

where the monotonicity of the sequence follows from property (i) of F. Since A is directed complete, we can form $\sup_n F^n x$, which we denote by $\tilde{F}x$.

From property (ii) we calculate

$$F\tilde{F}x = F(\sup_n F^n x) = \sup_n F^{n+1}x = \sup_n F^n x = \tilde{F}x.$$

Thus, $\tilde{F}x$ is in fact a fixed-point of F and is of course contained in that subset $A_F^\circ \subseteq A_F$ which consists of all the fixed-points. Note that the restriction and co-restriction $\tilde{F}\colon A_F \to A_F^\circ$ is a retraction. It is straightforward that $\tilde{F}x$ is in fact the least fixed-point above x. If A has a smallest element 0, then $\tilde{F}0$ is the least fixed-point of F (compare II-2.4).

In order to strike the proper analogy, remember that every poset A may be considered a category with $x \geq y$ being tantamount to $x \to y$. In this reading, a directed net $(x_j)_{j\in J}$ is just a projective system, and the existence of sups of directed sets means the existence of projective limits. A function $F\colon A \to A$ satisfying (i) is simply a *functor*, and, if (ii) is satisfied, then the functor *preserves projective limits*. This, then, is the way we want to generalize the fixed-point construction to arbitrary categories and later apply it to categories like $DCPO_G$ or CL with the kind of functors preserving projective limits we saw in the previous section. It remains to be seen, however, how we should generalize the definitions of the subsets A_F and A_F°.

Definition IV-6.1. We will say that a category A is *pro-complete* iff every projective system (see IV-4.2) has a limit – in short iff projective limits exist. A functor between pro-complete categories will be called *pro-continuous* iff it preserves projective limits (see IV-5.1). □

Remark. The attentive reader may have noticed that we did not use the full strength of Scott-continuity for F in the preliminary considerations. We only used that F preserves sups of increasing sequences, i.e., that F is ω-continuous. Likewise, in all of the following we will not use the full strength of pro-continuity of functors. We only use that our functors F are ω-continuous in the sense that they preserve limits of projective sequences.

Similarly we did not use the full strength of directed completeness in **dcpo**s. We only used that increasing sequences have sups, i.e., that A is ω-complete. Analogously in the following we only need that our category A has limits of projective sequences and not nessessarily limits of arbitrary projective systems.

Construction IV-6.2. Let $F: A \to A$ be a self-functor of a pro-complete category A. (We will eventually assume that F is pro-continuous.) We assume that $p: FL \to L$ is an arbitrary morphism from FL to L for some object L. We denote with $\tilde{F}L$ the projective limit of the following inverse system in A:

$$\begin{tikzcd} L & FL \arrow[l, "p"'] & F^2L \arrow[l, "Fp"'] & F^3L \arrow[l, "F^2p"'] & \cdots \arrow[l, "F^3p"'] \end{tikzcd} \tag{1}$$

It is to be understood that the full inverse system contains all finite compositions of the morphisms listed in diagram (1).

Let $p' : \tilde{F}L \to L$ be the limit map from the limit to the first term of the sequence. We apply the functor F to diagram (1) together with its limit cone. By IV-4.1 there is a unique natural map

$$\tilde{p}: F\tilde{F}L \to \tilde{F}L \tag{2}$$

such that the following diagram commutes:

$$\begin{tikzcd} & FL \arrow[d, equal] & F(FL) \arrow[l, "Fp"'] \arrow[d, equal] & F(F^2L) \arrow[l, "F(Fp)"'] \arrow[d, equal] & \cdots \arrow[l, "F(F^2p)"'] & F\tilde{F}L \arrow[d, "\tilde{p}"] \\ L & FL \arrow[l, "p"] & F^2L \arrow[l, "Fp"] & F^3L \arrow[l, "F^2p"] & \cdots \arrow[l, "F^3p"] & \tilde{F}L \end{tikzcd} \tag{3}$$

We have $p(Fp') = p'\tilde{p}$, which means that the following diagram commutes:

$$\begin{tikzcd} FL \arrow[d, "p"'] & F\tilde{F}L \arrow[l, "Fp'"'] \arrow[d, "\tilde{p}"] \\ L & \tilde{F}L \arrow[l, "p'"] \end{tikzcd} \tag{4}$$

If F is pro-continuous (i.e., preserves projective limits), then $\tilde{p}$ is an isomorphism. We may summarize some of this information:

Theorem. *Let $F: A \to A$ be a pro-continuous self-functor on a pro-complete category A. Then the "equation"*

$$X \cong FX$$

has a solution in A whenever there is a morphism p: $FL \to L$ for some object L; indeed the limit of the projective sequence (1) *is a solution.* □

We have now associated with a morphism p: $FL \to L$ a new morphism $\tilde{p}$: $F\tilde{F}L \to \tilde{F}L$ in a natural fashion. We must discuss in what way this process is functorial. The guiding idea is to consider p: $FL \to L$ as an "algebra" (which could be an object of a suitable category of algebras, called *comma categories* in the literature, but whose formalism we do not need to enter into here) to which we associate a new "algebra" $\tilde{p}$: $F\tilde{F}L \to \tilde{F}L$. This new algebra is more special if F is pro-continuous, as then $\tilde{p}$ is an isomorphism. This now is the idea that generalizes the formation of the subset A_F of a poset A.

Definition IV-6.3. Let F: $A \to A$ be a self-functor of a category A. An *F-algebra* is a pair (L, p) consisting of an object L of A together with an A-morphism p: $FL \to L$. If (S, p) and (T, q) are F-algebras, then a *morphism of F-algebras* $f: (S, p) \to (T, q)$ is an A-map $f: S \to T$ such that the following diagram commutes:

$$\begin{tikzcd} FS \arrow[r, "Ff"] \arrow[d, "p"'] & FT \arrow[d, "q"] \\ S \arrow[r, "f"'] & T \end{tikzcd}$$

The class of all F-algebras together with the class of F-algebra morphisms clearly forms a category which we call the *category of F-algebras* and denote by A_F. The full subcategory of all F-algebras (L, p) for which p is an isomorphism will be denoted A_F°. □

At this point A_F° may very well be an empty category. In Construction IV-6.2 we have associated with each F-algebra (L, p) an F-algebra $(\tilde{F}L, \tilde{p})$ which in fact is in A_F° if F is pro-continuous. We would like to know, of course, to what extent the assignment $(L, p) \mapsto (\tilde{F}L, \tilde{p})$ is functorial.

Lemma IV-6.4. *Let A be a pro-complete category and F a self-functor on A. Then for each F-algebra morphism f: $(S, p) \to (T, q)$, there is an F-algebra* *morphism* $\tilde{F}f\colon (\tilde{F}S, \tilde{p}) \to (\tilde{F}T, \tilde{q})$. *Furthermore, the map* p' *of* IV-6.2 *is an* F*-algebra morphism* $p'\colon (\tilde{F}L, \tilde{p}) \to (L, p)$ *such that the following diagram commutes:*

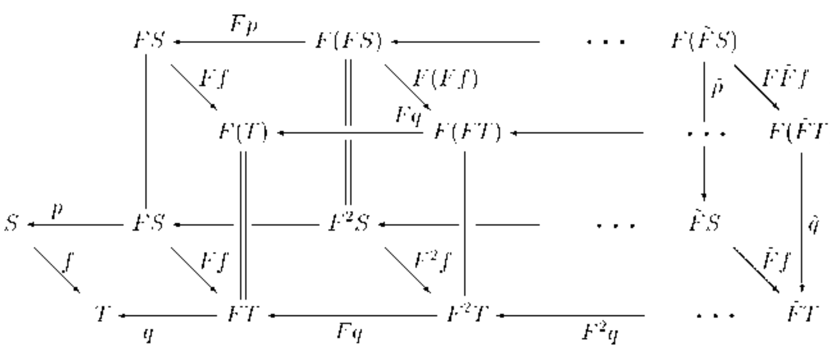
$$\begin{tikzcd}
(\tilde{F}S, \tilde{p}) \arrow[r, "\tilde{F}f"] \arrow[d, "p'"'] & (\tilde{F}T, \tilde{q}) \arrow[d, "q'"] \\
(S, p) \arrow[r, "f"'] & (T, q)
\end{tikzcd} \tag{1}$$

Proof: The existence of $\tilde{F}f$ follows immediately from the properties of the limit; it is the unique map which makes the following diagram commute:

$$\begin{tikzcd}
S \arrow[d, "f"'] & FS \arrow[l, "p"'] \arrow[d, "Ff"'] & F^2S \arrow[l, "Fp"'] \arrow[d, "F^2f"'] & \cdots \arrow[l, "F^2p"'] & \tilde{F}S \arrow[d, "\tilde{F}f"] \\
T & FT \arrow[l, "q"] & F^2T \arrow[l, "Fq"] & \cdots \arrow[l, "F^2q"] & \tilde{F}T
\end{tikzcd} \tag{2}$$

In particular, this proves $fp' = q'(\tilde{F}f)$, which shows the commutativity of (1).

Applying F to diagram (2) gives rise to a three dimensional commutative diagram



(3)

The commutativity of the right-most facet is the statement that $\tilde{F}f$ is an F-algebra morphism. The commutativity of diagram (4) in IV-6.2 shows that p' is an F-algebra morphism. □

Corollary IV-6.5. *The assignments* $(L, p) \mapsto (\tilde{F}L, \tilde{p})$ *and* $f \mapsto \tilde{F}f$ *determine a self-functor* $\Phi\colon A_F \to A_F$ *of the category of* F*-algebras.* □

If F preserves projective limits, then each $\Phi(p)$ is contained in the subcategory A_F°. Let $\Phi^\circ\colon A_F \to A_F^\circ$ denote the co-restriction of this functor. We now settle the question to what extent our construction is universal.

Theorem IV-6.6. *Let $F\colon A \to A$ be a pro-continuous self-functor of a pro-complete category. Then the functor $\Phi^\circ\colon A_F \to A_F^\circ$ is right adjoint to the inclusion functor.*

Remark. We reformulate in explicit terms what the assertion means: Suppose $f\colon (S, q) \to (L, p)$ is a morphism in A_F where $q\colon FS \to S$ is an isomorphism and where $p\colon FL \to L$ is arbitrary. Then there is a unique $f_\circ\colon (S, q) \to (\tilde{F}L, \tilde{p})$ such that $f = p' f_\circ$; that is, there is a commutative diagram

$$\begin{tikzcd}
& F\tilde{F}L \arrow[dd, "\tilde{p}"] \arrow[dr, "Fp'"] & \\
FS \arrow[ur, "Ff_\circ"] \arrow[rr, "Ff"] \arrow[dd, "q"'] & & FL \arrow[dd, "p"] \\
& \tilde{F}L \arrow[dr, "p'"] & \\
S \arrow[ur, "f_\circ"] \arrow[rr, "f"'] & & L
\end{tikzcd} \tag{1}$$

Proof of theorem: If $q\colon FS \to S$ is an isomorphism, then so is $F^n q\colon F^{n+1}S \to F^n S$. As a consequence $q'\colon \tilde{F}S \to S$ is an isomorphism. We take $f_\circ = (\tilde{F}f)q'^{-1}$ and observe the following commutative diagram:

$$\begin{tikzcd}
F\tilde{F}S \arrow[rr, "F\tilde{F}f"] \arrow[dd, "\tilde{q}"'] \arrow[dr, "Fq'"] & & F\tilde{F}L \arrow[dd, "\tilde{p}"] \arrow[dr, "Fp'"] & \\
& FS \arrow[rr, "Ff"] \arrow[dd, "q"] & & FL \arrow[dd, "p"] \\
\tilde{F}S \arrow[rr, "\tilde{F}f"] \arrow[dr, "q'"'] & & \tilde{F}L \arrow[dr, "p'"] & \\
& S \arrow[rr, "f"'] & & L
\end{tikzcd} \tag{2}$$

This proves the *existence* of the required morphism $f_\circ$.

In order to establish *uniqueness*, we assume that $f = p'g$ in A_F with a map $g\colon (S, q) \to (\tilde{F}L, p)$. Thus we have a commutative diagram

$$\begin{tikzcd}
FS \arrow[r, shift left, "Fg"] \arrow[r, shift right, "Ff_\circ"'] \arrow[d, "q"] & F\tilde{F}L \arrow[r, "Fp^{(n)}"] \arrow[d, "\tilde{p}"] & F^{n+1}L \arrow[r] \arrow[d] & \cdots \arrow[r] & F^3L \arrow[r, "F^2p"] \arrow[d, "F^2p"] & F^2L \arrow[r, "Fp"] \arrow[d, "Fp"] & FL \arrow[d, "p"] \\
S \arrow[r, shift left, "g"] \arrow[r, shift right, "f_\circ"'] & \tilde{F}L \arrow[r, "p^{(n)}"'] & F^nL \arrow[r] & \cdots \arrow[r] & F^2L \arrow[r] & FL \arrow[r, "p"'] & L
\end{tikzcd} \tag{3}$$

where $p^{(n)}\colon \tilde{F}L \to F^nL$ is the limit map. Then $p^{(1)}g = p^{(1)}f_\circ$. Since $\tilde{p}$ is an isomorphism, we observe

$$p^{(2)} = (F^2p)F\big(p^{(2)}\big)\tilde{p}^{-1} = F\big(p^{(1)}\big)\tilde{p}^{-1}.$$

Thus

$$\begin{aligned}
p^{(2)}f_\circ &= F\big(p^{(1)}\big)\tilde{p}^{-1}f_\circ = F\big(p^{(1)}\big)(Ff_\circ)q^{-1} \\
&= F\big(p^{(1)}f_\circ\big)q^{-1} = F\big(p^{(1)}g\big)q^{-1} \\
&= \cdots = p^{(2)}g.
\end{aligned}$$

Now we attack $p^{(3)} = (F^3p)\big(Fp^{(3)}\big)\tilde{p}^{-1} = F\big(p^{(2)}\big)p^{-1}$ and calculate

$$p^{(3)}f_\circ = \cdots = F\big(p^{(2)}f_\circ\big) = F\big(p^{(2)}g\big) = \cdots = p^{(3)}g.$$

Continuing by induction, this diagram chasing yields the information

$$p^{(n)}f_\circ = p^{(n)}g \text{ for } n = 1, 2, 3, \ldots.$$

By the uniqueness in the universal property of the limit $\tilde{F}L = \lim F^nL$ we now conclude $f_\circ = g$. This completes the proof of the theorem. □

Before we apply the general construction to the special categories we are working with, we observe that for some functors $F\colon A \to A$ there is in fact at least one functor from A into the category A_F of F-algebras. This together with the functor of IV-6.5 and IV-6.6 gives a functorial method to associate with any A-object L an F-algebra $(\tilde{F}L, \tilde{p})$ for which p is an isomorphism.

Observation IV-6.7. *Let $F\colon A \to A$ be a self-functor of a category and suppose that there is a natural transformation $p\colon F \to 1_A$. Then the assignment $L \mapsto (L, p_L)$ is a functor $A \to A_F$.*

Proof: We only need to recall that by the definition of a natural transformation, for each A-morphism $f\colon S \to T$, the diagram

$$\begin{tikzcd} FS \arrow[r, "Ff"] \arrow[d, "p_S"'] & FT \arrow[d, "p_T"] \\ S \arrow[r, "f"'] & T \end{tikzcd}$$

commutes. The rest is clear. □

We proved that the functor $\Phi\colon A_F \to A_F^\circ$ was universal. If the functor $L \mapsto (L, p_L)$ which we just noted were also universal, then in fact we could speak of a universal construction $L \mapsto (\tilde{F}L, \tilde{p}_L)$. However, the universality of Φ is that of a right adjoint; thus $L \mapsto (L, p_L)$ would have to be a right adjoint in order to compose. A quick inspection of what this means will identify this as a rare occurrence unless F preserves arbitrary limits – which is not the case for the functors we consider here. This observation is independent of the particular nature of the natural transformation p. It is therefore not to be expected that the construction $L \mapsto (\tilde{F}L, \tilde{p}_L)$ is universal in the sense of being an adjoint functor. We simply record

Observation IV-6.8. *If F is a pro-continuous self-functor of a pro-complete category A and if there is a natural transformation $p\colon F \to 1_A$, then there is a functor $L \mapsto (\tilde{F}L, \tilde{p}_L)\colon A \to A_F^\circ$ from A to the category of F-algebras (S, q) with q an isomorphism.* □

At this point we specialize to the categories $DCPO_G$, DOM_G, $INF^\uparrow$, CL and AL which we have treated in Sections IV-4 and IV-5. We consider a pro-complete subcategory A of $DCPO_G$ and a pro-continuous self-functor $F\colon A \to A$. Proposition IV-5.2 gives necessary and sufficient conditions for F to be pro-continuous. A sufficient condition is the local continuity of F by Theorem IV-5.5. For any **dcpo** L in A and each morphism $p\colon FL \to L$ – that is, for each F-algebra (L, p) – we create an F-algebra $(\tilde{F}L, \tilde{p})$ and an F-algebra morphism $p'\colon (\tilde{F}L;\ p) \to (L; p)$ as in IV-6.2. Recall that this is a morphism $p'\colon \tilde{F}L \to L$ compatible with $\tilde{p}$ and p. If $p\colon FL \to L$ is surjective, and if F preserves surjectivity (as is guaranteed by local continuity – IV-5.5), then all maps of the projective system (1) in IV-6.2 are surjective. From IV-4.9 we know that the limit maps are surjective, and this says in particular that p' is surjective. We can then say that $p'\colon (\tilde{F}L, \tilde{p}) \to (L, p)$ is a *quotient* of *F-algebras*. It will serve a good purpose for the applications to summarize:

Scholium IV-6.9. *Let A be a pro-complete subcategory of $DCPO_G$ (such as DOM_G, $INF^{\uparrow}$, CL and AL). Let $F\colon A \to A$ be a pro-continuous self-functor preserving surjectivity. (Any locally continuous functor has these properties by* IV-5.5*). Then we have the following conclusions.*

(i) *There is a functorial retraction from the category A_F of F-algebras in A to the full subcategory A_F° of algebras (S, q) on which q is an isomorphism, and this retraction is a right reflection. Associated with a given surjective F-algebra (L, p) is an F-algebra $(\tilde{F}L; \tilde{p})$ with a natural quotient map $p'\colon (\tilde{F}L, \tilde{p}) \to (L, p)$.*

(ii) *If $p\colon F \to 1_A$ is a surjective natural transformation, then there is a* functorial *construction whereby every object L* of *A is a quotient of the underlying A-object $\tilde{F}L$ of an F-algebra $(\tilde{F}L, \tilde{p})$* with *p an isomorphism.*

□

Now we apply this scholium to the following functors:

(a) Funct: $DCPO_{\perp G} \to DCPO_{\perp G}$, the function space functor (IV-5.10) restricted to the category of **dcpo**s with a smallest element 0 and Scott-continuous upper adjoints;

(b) $\sigma\colon DCPO_G \to DCPO_G$, the Scott-topology functor (IV-5.11).

In the case of (a) there are natural surjective transformations $FL \to L$.

Lemma IV-6.10. *For a* **dcpo** *L with a smallest element* 0 *let $z_L = (f \mapsto f(0))\colon [L \to L] \to L$. Then z_L is a natural surjective Scott-continuous map that has a lower adjoint, namely the map $x \mapsto \mathrm{const}_x$, where const_x is the constant function with value x.*

Proof: For $f \in [L \to L]$ and $x \in L$ we have $\mathrm{const}_x \le f$ iff $x \le f(0)$ since f is monotone; hence $x \mapsto \mathrm{const}_x$ is lower adjoint to $z_L = (f \mapsto f(0))$ by O-3.1. Surjectivity and naturality are clear. □

For the Scott topology functor σ there is no natural transformation $\sigma(L) \to L$.

We notice that the lower adjoint $x \mapsto \mathrm{const}_x$ of z_L likewise preserves arbitrary sups and infs to the extent they exist. By IV-1.4 it preserves the way-below relation provided that L is a domain. In this case it is also true (see IV-6.13 below) that z_L preserves the way-below relation.

We now summarize the scholium in the particular case of the functors Funct and σ. The list, of course, is in no way exhaustive. It is, however, somewhat representative of the situation. In each case we record to what extent the functor F "increases" L in terms of weight.

Theorem IV-6.11.

(i) *If* $(DCPO_{\perp G})_{\mathrm{Funct}}$ *denotes the category of all function space algebras* (L, p), $p\colon [L \to L] \to L$ *in* $DCPO_{\perp G}$ *and* $(DCPO_{\perp G})_{\mathrm{Funct}^\circ}$ *the full subcategory of algebras* (L, p) *with* p *an isomorphism, then there is a functorial retraction (indeed right reflection)* $(L, p) \mapsto (\mathrm{Funct}^\sim L, \tilde{p})$ *from the former to the latter category such that there is a natural quotient of* Funct-*algebras* $(\mathrm{Funct}^\sim L, \tilde{p}) \to (L, p)$.

(ii) *There is a functorial construction associating with any* **dcpo** L *with bottom a* **dcpo** $\mathrm{Funct}^\sim L$ *such that* L *is a quotient of* $\mathrm{Funct}^\sim L$ *and such that* $D = \mathrm{Funct}^\sim L$ *is naturally isomorphic to its own function space* $[D \to D]$. *If* L *is a continuous or algebraic lattice, so is* $\mathrm{Funct}^\sim L$.

(iii) *For continuous lattices, bounded complete domains and FS-domains with* card $L > 1$ *we have* $w(\mathrm{Funct}^\sim L) = \max\{\aleph_0, w(L)\}$.

Proof: The scholium together with IV-5.10 proves everything with the exception of the statement on the weights. By induction, from IV-5.13(i) we derive $w(\mathrm{Funct}^n L) = w(L)$ and so $w(\mathrm{Funct}^\sim L) = w(L)$ by IV-5.14, if L is infinite. If L is finite, then $w(\mathrm{Funct}^n L) < \aleph_0$ by induction and IV-5.13(i). The assertion $w(\mathrm{Funct}^\sim L) = \aleph_0$ then follows from IV-5.14 and card $L <$ card Funct L for $1 <$ card $L < \aleph_0$. □

Proposition IV-6.12.

(i) *There is a right reflection* $(L, p) \mapsto (\tilde{\sigma}(L), \tilde{p})$ *from the category* $INF^\uparrow$ *of Scott-topology algebras* (L, p) *with* $p\colon \sigma(L) \to L$ *to the full subcategory of all algebras* (L, p) *for which* p *is an isomorphism.*

(ii) *There is a natural map* $(\tilde{\sigma}(L), \tilde{p}) \to (L, p)$ *in* $INF^\uparrow\sigma$.

(iii) *For continuous* L *we have* $w(\tilde{\sigma}(L)) = \max\{\aleph_0, w(L)\}$.

Proof: The proof is immediate from Scholium IV-6.9, IV-5.11, IV-5.13 and IV-5.14. □

The significance of function space algebras (L, p) with $p\colon [L \to L] \to L$ an isomorphism lies in the fact that every element in such an algebra may be identified with a Scott-continuous function $L \to L$, and every Scott-continuous self-function of L is so obtained. Theorem IV-6.11 shows that such algebras exist in abundance; in fact, every continuous lattice is a quotient of one of these.

Of course one may apply these constructions to the other examples of locally continuous functors that we have seen in IV-5.6 and IV-5.10. The powerdomain functors treated in Section 8 of this chapter are also important examples of locally continuous functors to which these constructions apply.

Exercises

Exercise IV-6.13. Let L be a **dcpo** with a least element 0 and a greatest element 1. Show that the natural map $z_L\colon [L \to L] \to L$ given by $z_L(f) = f(0)$ has a Scott-continuous upper adjoint. In particular, z_L preserves $\ll$, provided that L is a domain.

Hint. For $a \in L$ define $u_L(a) \in [L \to L]$ by

$$u_L(a)(x) = \begin{cases} a, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Then $f \le u_L(a)$ for $f \in \operatorname{Funct} L$ iff $f(0) \le a$. Scott continuity is straightforward. □

We generalize this last result in the next exercise.

Exercise IV-6.14. For $x \in L$ we define the evaluation map $ev_x\colon [L \to A] \to A$ by $ev_x(f) = f(x)$ for continuous lattices L and A. Show that the following are equivalent:

(1) ev_x has a lower adjoint;
(2) $\inf f_j(x) = (\inf f_j)(x)$ for any family f_j in $[L \to A]$;
(3) $x \in K(L)$.

Show moreover that, if these conditions are satisfied, the lower adjoint $m_x\colon A \to [L \to A]$ is given by $m_x(a) = (\!| x \Rightarrow a |\!)$ (see II-2.31).

Hint. Conditions (1) and (2) are equivalent by O-3.5. The implication (3) $\Rightarrow$ (1) is readily verified: $m_x(a) \le f$ iff $m_x(a)(y) \le f(y)$ for all y iff $a \le f(y)$ for all y with $x \ll y$, and since $x \ll x$ by (3) and f is monotone, this is the case iff $a \le f(x)$, that is, $a \le ev_x(f)$. There remains (1) implies (3). Because of the presence of constant functions in $[L \to A]$, the function ev is clearly surjective, and thus its lower adjoint m_x is given by $m_x(a) = \min\ ev_x^{-1}(\{a\}) = \min\{g \in [L \to A] : g(x) = a\}$ (O-3.7). If $z \ll x$, define $c_z = (\!| z \Rightarrow a |\!)$ (II-2.31). Then $c_z \in [L \to A]$ with $c_z(x) = a$, and thus $m_x \le c_z$. Since L is continuous $x = \sup\{z\colon z \ll x\}$ and thus, since $m_x(a)(x) = a$ and m_x is monotone, $m_x(a) = \inf\{c_z\colon z \ll x\}$. We conclude from this relation that

$$m_x(a)(y) = \begin{cases} a, & \text{if } x \le y, \\ 0, & \text{otherwise.} \end{cases}$$

Since $m_x(a)$ is Scott-continuous, this implies that $\uparrow x$ is open, i.e., $x \in K(L)$. □

Exercise IV-6.15. Let L be a **dcpo** that is isomorphic to its own function space, that is, there is an isomorphism $p: [L \to L] \to L$. Prove the following.

(i) There is a binary operation $(x, y) \mapsto x(y) : L \times L \to L$ which is given by

$$x(y) = p^{-1}(x)(y) = (ev_y \circ p^{-1})(x)$$

which is Scott-continuous in both arguments.

(ii) If L is a continuous lattice, $x \mapsto x(y)$ is Lawson continuous for a given fixed y iff $y \in K(L)$.

(iii) If L is a continuous lattice, $y \mapsto x(y)$ is Lawson continuous for a given x iff there is an element $x^\wedge$ in L such that

$$x^\wedge\ (x(y)) \vee y = y = x(x^\wedge(y)) \wedge y \text{ for all } y \in L.$$

Hint. Since p is an isomorphism and the function $(f, x) \mapsto f(x) : [L \to L] \times L \to L$ is Scott-continuous by II-2.9(i), then the binary operation is Scott-continuous. For fixed y it is Lawson continuous in x iff ev_y is Lawson continuous, since p is Lawson continuous as is p^{-1}. This is the case iff y is compact by IV-6.14 above. For fixed x it is Lawson continuous in y iff $p^{-1}(x)$ preserves arbitrary infs. This is the case iff it has a lower adjoint by O-3.3 and O-3.4. This is the case iff there is an $x \in L$ such that $x^\wedge(x(y)) \leq y$ and $x(x^\wedge(y)) \geq y$ for all y by O-3.6 and the definition of the binary operation. □

Exercise IV-6.16. Under the hypotheses of IV-6.15, show that the space L with the Scott topology is a topological monoid relative to the operation $(x, y) \mapsto x \circ y$ that is given by $x \circ y = p(p^{-1}(x) \circ p^{-1}(y))$, and that moreover one has the identity

$$(x \circ y)(z) = x(y(z)).$$

Hint. The space $[L \to L]$ is a topological monoid under composition by II-4.25(ii). Since p is an isomorphism and the operation "$\circ$" on L is just transported composition, the first assertion follows. The second is straightforward from the definitions. □

The two preceding exercises show that every function space algebra (L, p) in which p is an isomorphism is a topological monoid relative to the Scott topology in such a way that it acts on itself (although not generally by translation!) in a Scott-continuous fashion; the action is such that every Scott-continuous self-function is realized by the action of precisely one monoid element. Let us express this in a definition:

Definition IV-6.17. A continuous lattice L will be called a *self-acting monoid* if there are a topological monoid multiplication $(x, y) \mapsto x \circ y$ on L and a monoid action of L on itself $(x, y) \mapsto x(y)$ which is Scott-continuous in both arguments such that for each $f \in [L \to L]$ there is a unique $x \in L$ such that $f(y) = x(y)$ for all y. $\square$

We recall that the term monoid action means the validity of the equations $e(x) = x$ for the identity e of the monoid and of $(x \circ y)(z) = x(y(z))$. We have observed the following result; its converse is also true.

Exercise IV-6.18. Show that there is a bijection between the objects of $CL^{\circ}_{\mathrm{Funct}}$ (function space algebras (L, p) with p an isomorphism) and self-acting monoids. $\square$

Exercise IV-6.19. A morphism $f\colon S \to T$ *of self-acting monoids* is a Scott-continuous map which satisfies $f(x(y)) = f(x)(f(y))$ and $f(x \circ y) = f(x) \circ f(y)$. Prove that the second relation will follow from the first. $\square$

Exercise IV-6.20. If $f\colon (S, p) \to (T, q)$ is a morphism in $CL^{\circ}_{\mathrm{Funct}}$ and if f is injective (that is, if (S, p) is a "subalgebra" of (T, q)), show that $f\colon S \to T$ is an injective morphism of self-acting monoids (that is, S is a self-acting submonoid of T).

Hint. We have to verify the relation $f\varphi = f\varphi f^{\wedge} f$ for all $\varphi \in [L \to L]$; if f is injective, $f^{\wedge} f = 1$ by O-3.7. $\square$

In more general terms, the proof for the preceding exercise shows that we need the relation $f\varphi = f\varphi f^{\wedge} f$ for all $\varphi \in [L \to L]$ in order to conclude that a $CL^{\circ}_{\mathrm{Funct}}$-morphism f induces a morphism of self-acting monoids. Since every $\varphi \in [L \to L]$ is the sup of functions of the form $(\!|x \Rightarrow a|\!)$ (II-2.31), it suffices that $f(\!|x \Rightarrow a|\!) = f(\!|x \Rightarrow a|\!)f^{\wedge} f$ holds for all a and x. If f is nonconstant, this holds for all a and x iff $x \ll y$ is equivalent to $x \ll f^{\wedge} f y$. If we fix y and remember that $y = \{\sup x\colon x \ll y\}$, then this condition evidently implies $f^{\wedge} f \geq 1$. The converse is always true, thus $f^{\wedge} f = 1$, that is, the injectivity of f is a necessary and sufficient condition for f to induce a morphism of self-acting monoids.

New notes

The fundamental construction of IV-6.2 was introduced in the special case of the function space functor $F = \mathrm{Funct}$ in [Scott, 1972a]. The objective was to construct the function space algebras which we obtained in IV-6.11; they serve as set-theoretical models for the lambda-calculus of Church and Curry. IV-6.15 through IV-6.20 contribute in this direction. The investigation of the functorial framework of the fixed-point construction for functors, independently of the development here, has also been discussed in [Smyth and Plotkin, 1978], [Smyth and Plotkin, 1982]. They have the concept of F-algebras in their setup, and instead of arbitrary projective limits and their preservation through functors they concentrate on limits of systems indexed by the natural numbers. The universality theorem IV-6.6 was new in the *Compendium* as was, in its particular form, Scholium IV-6.9, which we apply to special situations. The function space functor and the construction of continuous lattices which are isomorphic to their self-functions are due to Scott.

The results of IV-6.12 concerning the Scott-topology functor were new in the *Compendium*. There is a slight difference as far as the latter is concerned: there is no canonical construction starting from a continuous lattice and producing a continuous lattice which is naturally isomorphic to its own Scott topology. If, instead of the Scott open sets, one takes the Scott closed sets, one obtains a locally continuous functor $L \mapsto \Gamma(L)$ together with a natural transformation $q_L\colon L \to \Gamma(L)$ defined by $q_L(x) = \downarrow x$. This functor is one of the powerdomain functors discussed in Section IV-8.

IV-7 Domain Equations and Recursive Data Types

In the previous section we saw how to solve the "equation" $X \cong FX$ for any pro-continuous self-functor on any pro-complete category A and we applied the results to self-functors on certain categories of domains. In this section we concentrate on features of the solutions of domain equations that not only are specific to categories of domains, but are essential for semantics.

Throughout this section we work in the categories $DCPO_\perp$ and $DCPO_{\perp!}$. The objects are the **dcpo**s with a least element $\perp$ in both cases; the morphisms are all Scott-continuous functions in the first case, whilst in the second case one only admits strict functions, that is, functions that map $\perp$ to $\perp$ (compare Remark IV-5.15).

For two **dcpo**s with bottom, the **dcpo** of strict Scott-continuous functions $f\colon L \to M$ will be denoted by $[L \circ\!\!\to M]$ in contrast to the **dcpo** $[L \to M]$ of all Scott-continuous functions. Both have a bottom element

$$\perp_{L,M},$$

the constant function with value $\perp_M$. One may notice that $[L \circ\!\!\to M]$ is a Scott closed subset of $[L \to M]$, that is, a lower set closed for directed sups.

The singleton domain $\mathbf{1} = \{\perp\}$ is a final object in both categories, it is also initial in the category $DCPO_{\perp!}$. For every object L, there are obvious maps

$$p\colon L \to \mathbf{1}, \quad \hat{p}\colon \mathbf{1} \to L;$$

$\hat{p}$ maps $\perp$ to $\perp_L$, it is injective and is the lower adjoint of p which is surjective.

In applications it may occur that one wants to replace the categories $DCPO_\perp$ and $DCPO_{\perp!}$ by appropriate subcategories. There is no obstacle to that under quite weak hypotheses.

Domain equations for covariant functors

Let us consider in this subsection a fixed locally continuous self-functor

$$F\colon DCPO_\perp \to DCPO_\perp$$

on the category $DCPO_\perp$. Occasionally we will have to require that F restricts to a self-functor of the category $DCPO_{\perp!}$, that is F *preserves strictness of maps* in the sense that Ff is strict whenever f is strict.

We are interested in "minimal" solutions of the "domain equation"

$$X \cong FX,$$

i.e., we look for domains L with bottom such that L is isomorphic to FL and which are minimal in a sense that will be made precise. (We use the term "equation" although we only mean "equality up to isomorphism".) For this, we specialize Construction IV-6.2 to the very special case where $L = \mathbf{1}$:

Construction IV-7.1. (Minimal solutions of domain equations) We define recursively **dcpo**s

$$L_0 = \mathbf{1}, L_{n+1} = F^{n+1}\mathbf{1} = FL_n \tag{1}$$

and maps

$$p_n\colon L_{n+1} \to L_n, \quad \widehat{p_n} : L_n \to L_{n+1},$$

$$p_0 = p, p_{n+1} = F^{n+1}p = Fp_n, \quad \widehat{p_0} = \hat{p}, \widehat{p_{n+1}} = F^{n+1}\hat{p} = F\widehat{p_n}, \tag{2}$$

where p and $\hat{p}$ are the obvious maps as above. As F is locally continuous, it preserves adjoints and their surjectivity and injectivity, respectively, by IV-5.4; so $\widehat{p_n}$ is the lower adjoint of p_n, the former is injective, the latter surjective. (In the semantics community, $(p_n, \widehat{p_n})$ is called a *projection–embedding pair*, a terminology that we have not adopted in this work.)

Let L_∞ be the projective limit of the projective sequence

$$L_0 = \mathbf{1} \underset{p_0}{\longleftarrow} L_1 = F\mathbf{1} \underset{p_1}{\longleftarrow} L_2 = F^2\mathbf{1} \underset{p_2}{\longleftarrow} L_3 = F^3\mathbf{1} \underset{p_3}{\longleftarrow} \cdots \tag{3}$$

with limit maps $g_n\colon L_\infty \to L_n$. By IV-4.5(i), the g_n have lower adjoints $\widehat{g_n}\colon L_n \to L_\infty$. By IV-4.9, the limit maps g_n are surjective; thus the lower adjoints $\widehat{g_n}$ are injective. From IV-4.5(A) we have

$$\sup_n \widehat{g_n} g_n = 1_{L_\infty}. \tag{4}$$

As in IV-6.2 we apply the functor F to the diagram (3) together with its limit cone, and we obtain a cone again:

$$\begin{tikzcd}
 & L_\infty \arrow[dl, "g_{n-1}"'] \arrow[dr, "g_{n-2}"] & \\
L_{n+1} \arrow[d, dash] & & L_{n+2} \arrow[ll, "p_{n+1}"'] \arrow[d, equal] \\
FL_n & & FL_{n+1} \arrow[ll, "Fp_n"] \\
 & FL_\infty \arrow[ul, "Fg_n"] \arrow[ur, "Fg_{n-1}"'] &
\end{tikzcd} \tag{5}$$

By the universal property of the limit cone, there is a unique morphism

$$p_\infty\colon FL_\infty \to L_\infty \tag{6}$$

such that

$$Fg_n = g_{n+1} p_\infty \text{ for all } n. \tag{7}$$

As F is locally continuous, it preserves projective limits; hence p_∞ is an isomorphism, that is,

$$L_\infty \cong FL_\infty \text{ via } p_\infty.$$

Thus we have found a solution to our domain equation. As lower adjoints and surjective upper adjoints are always strict maps, all maps occuring in this construction are strict, so that everything is indeed happening in the category $\mathit{DCPO}_{\perp!}$. $\square$

There may be many other solutions to the domain equation $X \cong FX$. From IV-6.6 we know that the above solution is minimal in the sense that it is a quotient of any other solution of the domain equation $X \cong FX$. But L_∞ has even more spectacular properties. In order to express them, we need some auxiliary considerations. We first claim

$$p_\infty = \sup_n \widehat{g_{n+1}} F g_n \text{ and } {p_\infty}^{-1} = \sup_n (F\widehat{g_n}) g_{n+1}. \tag{8}$$

Proof of claim: From (7) we obtain $\widehat{g_{n+1}} g_{n+1} p_\infty = \widehat{g_{n+1}} F g_n$. Hence,

$$p_\infty = 1_{L\infty} p_\infty \overset{(4)}{=} \sup_n \widehat{g_{n+1}} g_{n+1} p_\infty \overset{(7)}{=} \sup_n \widehat{g_{n+1}} F g_n.$$

This proves the first equation. The second follows by a dual argument passing to the adjoints. □

In the following we shall see that L_∞ is at the same time initial and final in a precise sense. For this we recall the notions of F-algebra and F-coalgebra:

Definition IV-7.2. In IV-6.3, every pair (L, p), where L is a **dcpo** with bottom and p: $FL \to L$ a Scott-continuous map, had been called an *F-algebra*. Dually, an *F-coalgebra* is defined to be a pair (L, p), where p is a Scott-continuous map p: $L \to FL$. If p is, in addition, a strict map, then we say that we have a *strict F-algebra* or *F-coalgebra* respectively.

A *morphism of F-algebras*, respectively *F-coalgebras*, h: $(L, p) \to (M, q)$ is given by a Scott-continuous map h: $L \to M$ such that

$$hp = q(Fh), \quad \text{respectively} \quad qh = (Fh)p, \tag{1}$$

which means that the following diagrams commute:

$$\begin{tikzcd} FL \arrow[r, "Fh"] \arrow[d, "p"'] & FM \arrow[d, "q"] \\ L \arrow[r, "h"'] & M \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, "h"] \arrow[d, "p"'] & M \arrow[d, "q"] \\ FL \arrow[r, "Fh"'] & FM \end{tikzcd}$$

A morphism h is called strict, if it preserves the bottom element. A morphism of an F-algebra (or F-coalgebra) into itself is called an *endomorphism*. □

Those F-algebras (L, p) for which p: $FL \to L$ is an isomorphism are of particular interest. For these algebras, we can rewrite condition IV-7.2(i) for algebra morphisms. The dual remarks apply to the F-coalgebra (L, p^{-1}).

Remark IV-7.3. We consider F-algebras (L, p) and (M, q), where $p: FL \to L$ is an isomorphim and $q: FM \to M$ an arbitrary Scott-continuous map.

(i) From the definition we readily see that, for a Scott-continuous map $h: L \to M$, the following conditions are equivalent:
(1) h is an algebra morphism from (L, p) to (M, q);
(2) $h = q(Fh)p^{-1}$;
(3) h is a fixed-point of the map $\varphi = (f \mapsto q(Ff)p^{-1}): [L \to M] \to [L \to M]$.

The function space $[L \to M]$ is a **dcpo** with bottom element $\perp_{L,M}$. The map φ defined above is Scott-continuous, as F is locally continuous by hypothesis and as composition of maps is Scott-continuous. Hence, φ has a least fixed-point h (see II-2.4). As the fixed-points of φ are precisely the F-algebra morphisms, we conclude:

(ii) There is a least F-algebra morphism $h: (L, p) \to (M, q)$, namely the least fixed-point of the map φ defined in (i). According to II-2.4(ii), h can be constructed as the sup

$$h = sup_n h_n \tag{1}$$

of the increasing sequence of maps defined recursively by

$$h_0 = \perp_{L,M}, \quad h_{n+1} = \varphi(h_n) = q(Fh_n)p^{-1}. \tag{2}$$

If the functor F preserves strictness of maps and if (M, q) is a strict F-algebra, then the least morphism h is also strict, as all the approximating functions h_n are strict.

In order to consider endomorphisms of the F-algebra (L, p) we specialize these observations to the case $L = M$, $p = q$ and we obtain

(iii) For a Scott-continuous map $e: L \to L$ the following conditions are equivalent:
(1) e is an algebra endomorphism of (L, p);
(2) $e = p(Fe)p^{-1}$;
(3) e is a fixed-point of the map $\psi = (f \mapsto p(Ff)p^{-1}): [L \to L] \to [L \to L]$.
There is a least F-algebra endomorphism e of (L, p), namely the least fixed-point of the map ψ above, and e can be constructed as the sup

$$e = sup_n e_n \tag{1}$$

of the increasing sequence of maps defined recursively by

$$e_0 = \bot_{L,L}, \quad e_{n+1} = \varphi(e_n) = p(Fe_n)p^{-1}. \tag{2}$$

As the identity map is an endomorphism, we have $e \leq 1_L$, which implies that e is strict. □

The preceding remarks are used in the following remarkable lemma.

Lemma IV-7.4. (Least *F*-Algebra Morphism Lemma) *Let (L', p'), (L, p) and (M, q) be F-algebras such that $p\colon FL \to L$ and $p'\colon FL' \to L'$ are isomorphisms, while $q\colon FM \to M$ may be arbitrary. From* IV-7.3(ii) *we know that there are least morphisms from the F-algebras (L, p) and (L', p') to any other F-algebra.*

(i) *If k is the least F-algebra morphism $k\colon (L', p') \to (L, p)$ and $h\colon (L, p) \to (M, q)$ an arbitrary strict F-algebra morphism, then hk is the least F-algebra morphism from (L', p') to (M, q).*
(ii) *If $k\colon (L', p') \to (L, p)$ is an arbitrary F-algebra morphism and h the least F-algebra morphism $h\colon (L, p) \to (M, q)$, then hk is the least F-algebra morphism from (L', p') to (M, q).*

Proof: We just prove the first assertion. The proof of the second one is similar, but one does not need the strictness assumption there. Let h' be the least F-algebra morphism from (L', p') to (M, q). From IV-7.3(ii) we know that $h' = \sup_n h'_n$ with $h'_0 = \bot_{L',M}$, $h'_{n+1} = q(Fh'_n)p'^{-1}$. Similarly, for the least F-algebra morphism $k\colon (L', p') \to (L, p)$ one has $k = \sup_n k_n$, where $k_0 = \bot_{L',L}$, $k_{n+1} = p(Fk_n)p'^{-1}$ For proving $h' = hk$, it suffices to show – inductively – that $h'_n = hk_n$ for all n: For $n = 0$ we have $hk_0 = h\bot_{L',L} = \bot_{L',M} = h'_0$, because h is supposed to be strict. By the recursive definition of k_n and because h is an F-algebra morphism, we have $hk_{n+1} = (q(Fh)p^{-1})(p(Fk_n)p'^{-1}) = qF(hk_n)p'^{-1} = q(Fh'_n)p'^{-1} = h'_{n+1}$, where we have used the induction hypothesis $hk_n = h'_n$ for the third equality. □

Definition IV-7.5. The F-algebras, and similarly the F-coalgebras, together with their morphisms form a category. As in any category, an *initial F-algebra* is an *F-algebra* (L, p) such that for an arbitrary F-algebra (M, q) there is one and only one F-algebra morphism $h\colon (L, p) \to (M, q)$.

Dually, a *final F-coalgebra* is an *F-coalgebra* (M, q) such that for an arbitrary F-coalgebra (L, p) there is one and only one F-coalgebra morphism $h\colon (L, p) \to (M, q)$. One may restrict these notions of initiality and finality to the subcategories of strict F-algebras and strict F-coalgebras. □

Initial and final algebras exist for locally continuous self-functors on the category $DCPO_{\perp!}$ and they agree in the sense stated in the following theorem. This is a strong result. It will be an immediate consequence of Theorem IV-7.9 and Proposition IV-7.7.

Theorem IV-7.6. *Let F be a locally continuous self-functor on the category $DCPO_\perp$ which preserves strictness of maps and let $p_\infty \colon FL_\infty \to L_\infty$ be the "minimal" solution of the domain equation $FX \cong X$ from* IV-7.1. *Then (L_∞, p_∞) is initial in the category of strict F-algebras and $(L_\infty, {p_\infty}^{-1})$ is final in the category of all F-coalgebras.* □

Clearly, an initial F-algebra, and similarly a final F-coalgebra, are unique up to isomorphism, and each allows only one endomorphism, namely the identity. It is surprising that the converse holds in the setup of the preceding theorem, as we shall see. An isomorphism $p \colon FL \to L$ with the property that the F-algebra (L, p) allows no endomorphism different from the identity is sometimes called *F-invariant*. We first show that the isomorphism $p_\infty \colon FL_\infty \to L_\infty$ constructed in IV-7.1 is F-invariant:

Proposition IV-7.7. *Let F be a locally continuous self-functor on the category $DCPO_\perp$. Then the identity function is the only endomorphism of the F-algebra (L_∞, p_∞) and the only endomorphism of the F-coalgebra $(L_\infty, p_\infty^{-1})$.*

Proof: Let $h \colon L_\infty \to L_\infty$ be any endomorphism of the F-algebra (L_∞, p_∞) or of the F-coalgebra $(L_\infty, p_\infty^{-1})$. In both cases $h = p_\infty(Fh)p_\infty^{-1}$. In order to show that h is the identity map, we first show by induction that $g_n h = g_n$ for all n, where the $g_n \colon L_\infty \to L_n$ are the limit maps from IV-7.1 and $\widehat{g_n} \colon L_n \to L_\infty$ their lower adjoints.

As g_0 maps everything to $\perp$, clearly $g_0 h = g_0$. This takes care of the case $n = 0$. Now we calculate

$$\begin{aligned}
g_{n+1}h &= g_{n+1}p_\infty(Fh){p_\infty}^{-1} & \\
&= (Fg_n)(Fh){p_\infty}^{-1} & \text{by equation (7) in IV-7.1}\\
&= F(g_n h){p_\infty}^{-1} & \text{as } F \text{ is a functor}\\
&= (Fg_n){p_\infty}^{-1} & \text{by induction hypothesis}\\
&= g_{n+1} & \text{by equation (7) in IV-7.1.}
\end{aligned}$$

Using equation (4) from IV-7.1, we finally get $h = 1_{L_\infty} h = (\sup_n \widehat{g_n} g_n)h = \sup_n \widehat{g_n}(g_n h) = \sup_n \widehat{g_n} = 1_{L_\infty}$ as desired. □

The preceding proposition allows us to say exactly in what sense L_∞ is the minimal solution of the domain equation $X \cong FX$:

Corollary IV-7.8. *Whenever $p\colon FL \to L$ is an isomorphism, there is a surjective Scott-continuous map $h\colon L \to L_\infty$ which has a lower adjoint $\hat{h}\colon L_\infty \to L$ (which then is injective). Moreover, h and $\hat{h}$ are F-algebra morphisms between (L, p) and (L_∞, p_∞).*

Proof: From IV-7.3(ii) we know that there are a least F-algebra morphism $h\colon L \to L_\infty$ and a least F-algebra morphism $\hat{h}\colon L_\infty \to L$. Then $h\hat{h}$ is an endomorphism of the F-algebra (L_∞, p_∞). By Proposition IV-7.7, $h\hat{h} = 1_{L_\infty}$. By IV-7.4, $\hat{h}h$ is the least endomorphism of the F-algebra (L, p). As the identity map is an endomorphism, we conclude that $\hat{h}h \leq 1_L$. The assertion now follows from O-3.6, O-3.7. □

By the preceding proposition and corollary, the equivalent conditions of the following theorem are all satisfied for the canonical isomorphism $p_\infty\colon FL_\infty \to L_\infty$, which finishes the proof of Theorem IV-7.6.

Theorem IV-7.9. *Let $F\colon DCPO_\perp \to DCPO_\perp$ be a locally continuous functor which preserves strictness of maps. Let L be a* **dcpo** *with bottom and $p\colon FL \to L$ an isomorphism. Then the following conditions are equivalent.*

(1) *(L, p) is initial in the category of strict F-algebras.*
(2) *(L, p^{-1}) is final in the category of all F-coalgebras.*
(3) *1_L is the only endomorphism of the F-algebra (L, p).*
(4) *1_L is the only endomorphism e of the F-algebra (L, p) with $e \leq 1_L$.*

Proof: We have seen that (2) implies (3) and, as every map below the identity is strict, that (1) implies (4) for trivial reasons. Clearly (3) implies (4). We only prove that (4) implies (1). The proof that (4) implies (2) is dual to it and it does not use the strictness of the F-coalgebras.

So we suppose that the F-algebra $p\colon FL \to L$ does not admit any endomorphism strictly below the identity. By IV-7.3(iii), this means that the identity is the least endomorphism of (L, p).

To prove initiality, we consider an arbitrary **dcpo** M with bottom and a strict Scott-continuous map $q\colon FM \to M$. We have to show that there is a unique strict F-algebra morphim $h\colon (L, p) \to (M, q)$.

The existence of such an h has been shown in IV-7.3(ii). There is indeed a least F-algebra morphism $h' : (L, p) \to (M, q)$, and h' is strict. For the uniqueness, let $h\colon (L, p) \to (M, q)$ be an arbitrary strict F-algebra morphim. If e is the least F-algebra endomorphism of (L, p), we deduce $he = h'$ from IV-7.4 with $L = L'$, $p = p'$. By our hypothesis, $e = 1_L$, whence $h = h'$. □

Domain equations for mixed variance functors

There is no obstacle to extending the above results to functors in several arguments, as long as they are covariant in every argument. For functors with mixed variance, things become more complicated. Let us consider in this subsection a fixed locally continuous functor

$$B\colon DCPO_{\perp}^{\mathrm{op}} \times DCPO_{\perp} \to DCPO_{\perp},$$

that is, a bifunctor which is contravariant in its first and covariant in its second argument. First let us solve the domain equation

$$X \cong B(X, X).$$

From the bifunctor B we can derive a locally continuous covariant functor F on the subcategory of all **dcpo**s L with bottom and all Scott-continuous maps g having a lower adjoint $\hat{g}$ by

$$FL = B(L, L) \quad \text{and} \quad Fg = B(\hat{g}, g)$$

as in IV-5.9. We now can proceed exactly as in Construction IV-7.1 and define recursively domains L_n and maps $p_n\colon L_{n+1} \to L_n$ and $\widehat{p_n}\colon L_n \to L_{n+1}$ by

$$L_0 = 1, \quad L_{n+1} = F(L_n) = B(L_n, L_n), \tag{7.1$'$}$$

$$\begin{aligned} p_0 = p,\ \widehat{p_0} = \hat{p}, \quad p_{n+1} &= F(p_n) = B(\widehat{p_n}, p_n), \\ \widehat{p_{n+1}} &= F(\widehat{p_n}) = B(p_n, \widehat{p_n}), \end{aligned} \tag{7.2$'$}$$

where $p\colon B(\mathbf{1}, \mathbf{1}) \to \mathbf{1}$ is the obvious map and $\hat{p}\colon \mathbf{1} \to B(\mathbf{1}, \mathbf{1})$ its lower adjoint (mapping $\perp$ to $\perp$). We obtain a projective limit

$$L_\infty = \lim_n (L_n, p_n)$$

and an isomorphism $p_\infty\colon FL_\infty \to L_\infty$, i.e., an isomorphism

$$p_\infty\colon B(L_\infty, L_\infty) \to L_\infty.$$

Thus we have a solution for our domain equation which is minimal in the same sense as in IV-7.1. We are now going to give an analogue to the universal properties IV-7.6 and IV-7.9. But we will have to restrict ourselves everywhere to strict Scott-continuous functions. We henceforward suppose that B is a bifunctor of mixed variance

$$B\colon DCPO_{\perp!}^{\mathrm{op}} \times DCPO_{\perp!} \to DCPO_{\perp!}.$$

Definition IV-7.10. An isomorphism p: $B(L, L) \to L$ is B-*invariant*, if the identity map 1_L is the only strict Scott-continuous map h: $L \to L$ satisfying

$$h = pB(h, h)p^{-1}, \tag{1}$$

which is tantamount to saying that the diagram

$$\begin{tikzcd} B(L, L) \arrow[r, "B(h, h)"] \arrow[d, "p"'] & B(L, L) \arrow[d, "p"] \\ L \arrow[r, "h"'] & L \end{tikzcd}$$

commutes for $h = 1_L$ only. □

A slight modification of the proof of Proposition IV-7.7 yields

Lemma IV-7.11. *The isomorphism* p_∞: $B(L_\infty, L_\infty) \to L_\infty$ *constructed above is* B*-invariant.*

Proof: Suppose that h: $L_\infty \to L_\infty$ is a strict Scott-continuous map satisfying

$$h = p_\infty B(h, h){p_\infty}^{-1}. \tag{1}$$

We want to show that h is the identity map.

We first show by induction that $g_n h = g_n$ and $h\widehat{g_n} = \widehat{g_n}$ for all n, where the g_n: $L_\infty \to L_n$ are the limit maps from IV-7.1 and $\widehat{g_n}$: $L_n \to L_\infty$ their lower adjoints. As g_0 maps everything to $\perp$, clearly $g_0 h = g_0$. As h is strict, $h\widehat{g_0}(\perp) = h(\perp) = \perp = \widehat{g_0}(\perp)$, whence $h\widehat{g_0} = \widehat{g_0}$. This takes care of the case $n = 0$. Now we calculate

$$\begin{aligned}
g_{n+1}h &= g_{n+1}p_\infty B(h, h){p_\infty}^{-1} && \text{by (1)} \\
&= (Fg_n)B(h, h){p_\infty}^{-1} && \text{by equation (7) in IV-7.1} \\
&= B(\widehat{g_n}, g_n)B(h, h){p_\infty}^{-1} && \text{by the above definition of } F \\
&= B(h\widehat{g_n}, g_n h){p_\infty}^{-1} && \text{as } B \text{ is a functor of mixed variance} \\
&= B(\widehat{g_n}, g_n){p_\infty}^{-1} && \text{by induction hypothesis} \\
&= (Fg_n){p_\infty}^{-1} && \text{by the above definition of } F \\
&= g_{n+1} && \text{by equation (7) in IV-7.1.}
\end{aligned}$$

The inductive step for the second equations works similarly. Using equation (4) from IV-7.1, we finally get $h = 1_{L_\infty} h = (\sup_n \widehat{g_n} g_n)h = \sup_n \widehat{g_n}(g_n h) = \sup_n \widehat{g_n} g_n = 1_{L_\infty}$ as desired. □

We merge the concepts of initial F-algebra and final F-coalgebra into a unifying concept:

Definition IV-7.12. An isomorphism $p: B(L, L) \to L$ is called B-*bifree* if, for every **dcpo** M with bottom and for every pair of strict Scott-continuous functions $q: B(M, M) \to M$ and $r: M \to B(M, M)$, there is one and only one pair of strict Scott-continuous functions $h: L \to M$ and $k: M \to L$ such that

$$h = qB(k, h)p^{-1} \text{and } k = pB(h, k)r \tag{1}$$

which is tantamount to saying that the following diagrams commute:

$$\begin{tikzcd} B(L,L) \arrow[r, "B(k,h)"] \arrow[d, "p"'] & B(M,M) \arrow[d, "q"] \\ L \arrow[r, "h"'] & M \end{tikzcd} \qquad \begin{tikzcd} B(L,L) \arrow[d, "p"'] & B(M,M) \arrow[l, "B(h,k)"'] \\ L & M \arrow[l, "k"] \arrow[u, "r"'] \end{tikzcd}$$

□

A standard argument shows that, if a bifree object exists, then it is unique up to isomorphism.

The following is the extension of Theorem IV-7.9 to functors with mixed variance.

Theorem IV-7.13. *An isomorphism $p: B(L, L) \to L$ is B-bifree if and only if it is B-invariant.*

Proof: Suppose p to be B-bifree. Choose $L = M, q = p$ and $r = p^{-1}$. As p is B-bifree, there is a unique pair of strict Scott-continuous maps h, k from L into L such that

$$h = pB(k, h)p^{-1} \quad \text{and} \quad k = pB(h; k)p^{-1}. \tag{1}$$

As these two equations are satisfied for $h = 1_L, k = 1_L$, we conclude $h = 1_L, k = 1_L$ by uniqueness. In order to prove that p is B-invariant, let $g: L \to L$ be any strict Scott-continuous map satisfying $g = pB(g, g)p^{-1}$. Then the pair $h = g, k = g$ also satisfies the equations (1). From the uniqueness of the solutions we conclude that $g = 1_L$ which shows that p is B-invariant.

For the converse, we first notice that the map

$$\psi = (f \mapsto pB(f, f)p^{-1}): [L \circ\!\!\to L] \to [L \circ\!\!\to L]$$

is Scott-continuous, as B is locally continuous; hence ψ has a least fixed-point e. As e satisfies $e = pB(e, e)p^{-1}$ and as p is B-invariant, we conclude that $e = 1_L$. The least fixed-point e is the sup of the recursively defined sequence (cf. IV-7.3(iii)), equation(2))

$$e_0 = \bot_{L,L}, \quad e_{n+1} = pB(e_n, e_n)p^{-1}. \tag{2}$$

Thus, $1_L = \sup_n e_n$.

In order to prove that p is B-bifree, take a **dcpo** M with bottom and a pair of strict Scott-continuous maps $q\colon B(M, M) \to M$, $r\colon M \to B(M, M)$. Then

$$\begin{aligned}(f, g) \mapsto (qB(g, f)p^{-1}, pB(f, g)r) : [L \circ\!\!\to M] \\ \times [M \circ\!\!\to L] \longrightarrow [L \circ\!\!\to M] \times [M \circ\!\!\to L]\end{aligned}$$

is Scott-continuous, as B is locally continuous, and hence has a least fixed-point (h, k), i.e., the maps h and k satisfy equation IV-7.12(I). We have to show their uniqueness. For this we recall first that h and k can be approximated by the following recursively defined sequences in the sense that $h = \sup_n h_n$, $k = \sup_n k_n$:

$$\begin{aligned} h_0 &= \bot_{L,M}, \qquad & h_{n+1} &= qB(k_n, h_n)p^{-1}, \\ k_0 &= \bot_{M,L}, \qquad & k_{n+1} &= pB(h_n, k_n)r. \end{aligned}$$

Now let $h'\colon L \to M$ and $k'\colon M \to L$ be another pair of strict Scott-continuous maps satisfying IV-7.12(1). We show by induction that

$$h'e_n = h_n \text{ and } e_nk' = k_n.$$

For $n = 0$ this is obvious using that h' is strict. For the induction step we calculate

$$\begin{aligned} h'e_{n+1} &= h'pB(e_n, e_n)p^{-1} && \text{by (2)} \\ &= qB(k', h')p^{-1}pB(e_n, e_n)p^{-1} && \text{by (IV-7.12(1))} \\ &= qB(k', h')B(e_n, e_n)p^{-1} && \\ &= qB(e_nk', h'e_n)p^{-1} && \text{as } B \text{ is a functor of mixed variance} \\ &= qB(k_n, h_n)p^{-1} && \text{by induction hypothesis} \\ &= h_{n+1} && \text{by the definition of } h_{n+1} \end{aligned}$$

and similarly for the second equation.

Now we can conlude that $h' = h'1_L = h' \sup_n e_n = \sup_n h'e_n = \sup_n h_n = h$ and similarly $k' = k$ which shows the uniqueness. □

From IV-7.11 and IV-7.13 we conclude:

Theorem IV-7.14. *For a locally continuous bifunctor*

$$B\colon \mathit{DCPO}_{\bot!}^{\mathrm{op}} \times \mathit{DCPO}_{\bot!} \to \mathit{DCPO}_{\bot!}$$

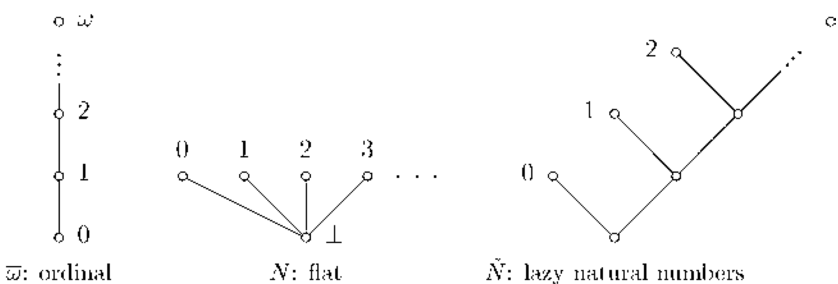
which is contravariant in the first and covariant in the second argument, the isomorphism $p_\infty\colon B(L_\infty, L_\infty) \to L_\infty$ *which is the "minimal" solution of the domain equation* $X \cong B(X, X)$ *is* B*-bifree.* □

Examples of domain equations

In the following we present a number of domain equations which are actually used in semantics. We recall some examples of locally continuous functors from IV-5.6 and IV-5.15: *lifting* $FL = L_\perp$ – adding a new bottom element; *separated sum* $F(L, M) = L + M$ – the disjoint union with a new bottom element added; *coalesced sum* $F(L, M) = L \oplus M$ – the disjoint union with the two bottom elements identified; *smash product* $F(L, M) = L \oplus M$ – the direct product with all elements $(a, \perp)$ and $(\perp, b)$ identified with $(\perp, \perp)$. The reader is invited to exhibit the functors F that are involved in the respective domain equations and to "construct" the solutions of the domain equations as in IV-7.1 step by step. He also should visualize explicitly the isomorphism $p_\infty\colon FL_\infty \to L_\infty$.

Example IV-7.15. (Domains of natural numbers) Verify that the three domain theoretical versions of the natural numbers depicted below are minimal solutions of the domain equations

$$X \cong X_\perp, \quad X = 1_\perp \oplus X, \quad X \cong 1 + X.$$



□

Example IV-7.16. (The partial Cantor space) The minimal solution C of the domain equation

$$X \cong X + X$$

is a binary tree with infinite branches only. The space of maximal elements with the induced Scott topology is homeomorphic to the Cantor set. The maximal elements correspond to infinite sequences of zeros and ones. The nonmaximal elements correspond to finite sequences of zeros and ones; they may be viewed as partial elements of the Cantor space. Every finite binary tree is a retract of the Cantor tree. □

Example IV-7.17. (Streams and lists) (i) The domain Str(N) of streams of natural numbers is the flat domain consisting of all infinite sequences of natural numbers with a bottom adjoined. Alternatively, it is the minimal solution of the domain equation

$$X \cong N \otimes X_\perp.$$

Informally the domain equation can be read in the following way: *a stream is a pair consisting of a natural number and a stream, or undefined.*

(ii) The domain L(N) of lists of natural numbers is the minimal solution of the domain equation

$$X \cong 1_\perp \oplus (N \otimes X).$$

L(N) is indeed a flat domain the nonbottom elements of which can be identified with all finite and infinite lists including the empty list. The domain equation can be read in the following way: *a list is either empty or a pair consisting of a natural number and a list, or undefined.*

In the above, N denotes the domain of at natural numbers from IV-7.15. One may replace it by any other flat domain, for example by the flat Booleans $B = (\mathbf{1} + \mathbf{1})_\perp$. The domain of streams of Booleans is isomorphic to the partial Cantor space in the previous example. □

We remark that the minimal solution of the domain equation

$$X \cong [X \to X]$$

is the trivial singleton domain $\mathbf{1}$. Here, the relevant functor is the function space functor $B(L, M) = [L \to M]$, and starting with $L_0 = \mathbf{1}$ yields $L_1 = [\mathbf{1} \to \mathbf{1}] = \mathbf{1}$, etc. Thus, we do not obtain a nontrivial model of untyped-calculus as a B-bifree B-algebra. Of course, as we have seen in Section IV-6, there are lots of nontrivial solutions of this domain equation. Surprisingly, the situtation changes by a tiny modification of the domain equation:

Example IV-7.18. (Lazy lambda-calculus) The minimal solution D of the domain equation

$$X \cong [X \to X]_\perp$$

is highly nontrivial. It is a model of the lazy lambda-calculus (see [Abramsky and Ong, 1993]).

Example IV-7.19. (Lambda-calculus with arithmetic) The minimal solution D of the domain equation

$$X \cong N \oplus [X \to X]_\perp$$

is a model for untyped lambda-calculus with arithmetic: *an element of D is either a natural number or a function, or undefined*; the undefined natural number is identified with the undefined element of D in this model, but the nowhere defined function is not identified with the undefined element of D. □

In the preceding two examples we met bifunctors $B(L, M) = [L \to M]_\perp$ and $B(L, M) = N \oplus [L \to M]_\perp$ of mixed variance. Note that these bifunctors like all of the covariant functors that we have met in the previous examples preserve strictness of maps so that all of our results of this section on initiality, finality and bifreeness apply.

Example IV-7.20. (Continuations) Let R be a fixed-pointed **dcpo** of "responses". The minimal solution of the domain equation

$$X \cong [X \to R] \times X$$

is used as a semantic domain for the interpretation of *continuations* for untyped lambda-calculus.

Exercises

Exercise IV-7.21. Reformulate Remark IV-7.3 for morphisms of F-coalgebras. □

Exercise IV-7.22. Prove thoroughly that (4) implies (2) in Theorem IV-7.9 and check that you do not need to suppose that the F-coalgebras are strict. □

Exercise IV-7.23. Let F be a locally continuous self-functor on the category *DCPO* preserving strictness of functions. Let $p_\infty\colon FL_\infty \to L_\infty$ be the minimal solution of the domain equation $FX \cong X$ as in IV-7.1. Let $q\colon FM \to M$ be any Scott-continuous map.

Show that two F-algebra morphisms g and h from (L_∞, p_∞) to (M, q) agree if and only if $g(\perp) = h(\perp)$.

Hint. There are different ways to attack this exercise. One possiblity is to reduce the situation to that of strict morphisms: Let $a = g(\perp) = h(\perp)$ and consider the **dcpo** $M' := \uparrow a \subseteq M$, which has a as its smallest element. As the image of h is contained in M', we have a factorization $h = ih^\circ$ into its corestriction $h^\circ: L_\infty \to M'$ and the injection $i: \uparrow a \to M$. Then $Fg = (Fi)(Fh^\circ)$ and $q' = q(Fi)$ maps FM' into M' and is strict. As h° and likewise g° are two strict F-algebra morphisms from (L_∞, p_∞) to (M', q'), they agree by Theorem IV-7.9. □

The following justifies calling L_∞ the "minimal" solution of the domain equation $L \cong B(L, L)$.

Exercise IV-7.24. Let B be a bifunctor as in Theorem IV-7.14. Show that for every isomorphism $p: B(L, L) \to L$ there is a surjective Scott-continuous map $h: L \to L_\infty$ which has a lower adjoint $\hat{h}: L_\infty \to L$ (which then is injective).

Hint. Compare IV-7.8. □

Exercise IV-7.25. Let C be the minimal solution of the domain equation $X \cong [X \to R] \times X$ in IV-7.20. Show the following.

(i) The domain $D = [C \to R]$ is a (nontrivial) solution of the domain equation $X \cong [X \to X]$.
(ii) The domain $D = [C \to R]$ is isomorphic to the solution R_∞ of the domain equation $X \cong [X \to X]$ obtained as the limit of the projective sequence as in Construction IV-6.2 and Theorem IV-6.11 starting with

$$R \longleftarrow FR = [R \to R] \longleftarrow F^2R = [FR \to FR] \longleftarrow \cdots .$$

Hint. See [Streicher and Reus, 1998]. □

New notes

Data types can be viewed as algebras, that is, sets endowed with finitary operations. They have been studied under the name of *initial algebra semantics* by Goguen, Thatcher, Wagner and Wright (see [Goguen *et al.*, 1977]), for example; see also [Scott, 1976]. Coalgebras appeared in semantics shortly afterwards (see e.g. [Rutten, 2000] for references). The book [Reynolds, B1998] is an excellent source of information about the use of domains in semantics for programming languages and in particular about the use of solutions of domain equations as treated in this section. Peter Freyd based his axiomatic approach to domain theory in the language of category theory on these developments (see [Freyd, 1991], [Freyd, 1992], [Barr, 1992]). He was the first to see how to treat the case of functors with mixed variance adequately by using a notion that unifies algebras and coalgebras. A. Pitts has shown how to apply these ideas to domains and how to use them to reason about induction and coinduction principles for data types [Pitts, 1993; Pitts, 1994; Pitts, 1996].

There have been lots of papers dealing with solutions of domain equations in categories slightly more general than categories of **dcpo**s. Examples are **dcpo**-enriched categories, categories in which the hom-sets are **dcpo**s (see e.g. [Edalat and Smyth, 1993a], [Adámek, 1997]).

An alternative approach to solving domain equations involves creating a domain whose members consist of countably based pointed domains ordered by inclusion and containing representatives of all countably based pointed domains [Winskel and Larsen, 1984]. One then views the right side of a domain equation as a continuous function on this domain and seeks the least fixed-point as a solution of the equation. Here a domain equation is an *equation* and not *equality up to an isomorphism as* in our approach.

IV-8 Powerdomains

Powerset constructions, for example the powerset of a set or the lattice of closed sets of a topological space, play important roles in various mathematical categories. In this section we examine some basic powerset constructions in domain theory; the resulting domains are called *powerdomains*. They have been important in the modeling of nondeterministic choice and of parallel processing in theoretical computer science.

Recall that an *algebra* (or *universal algebra*) consists of a set together with a family of finitary operations (functions from some finitary power of the set into itself). The family of sizes or arities of the operations that one considers is called the *signature*; for example, a group has signature consisting of a nullary operation (the identity element), a unary operation (inversion), and a binary operation (multiplication).

Definition IV-8.1. A *directed complete partially ordered algebra*, or **dcpo**-*algebra* for short, is a **dcpo** that is also an algebra for which all the operations are Scott-continuous (from the appropriate products endowed with the Scott topology). A **dcpo**-*morphism* or *homomorphism* is a function between **dcpo**-algebras of the same signature that is Scott-continuous and a homomorphism for each of the corresponding operations. □

Given any variable set X and any signature Σ, there is a free (universal) algebra over X, $T_\Sigma(X)$, for the signature Σ, called the *term algebra*, consisting of expressions or terms that can be built up recursively from X by formally applying the various operations in the signature. Any map from X into a Σ-algebra extends uniquely to an algebra homomorphism of $T_\Sigma(X)$ into the algebra. A family of **dcpo**-algebras with signature Σ is said to satisfy an equality $\tau_1 = \tau_2$ or an inequality $\tau_1 \leq \tau_2$, where τ_1 and τ_2 are members of the term algebra, if equality or inequality respectively holds for all homomorphic extensions of all possible functions of X into members of the family. For a family of equalities and inequalities $\mathcal{E}$, we denote by $DCPO(\Sigma, \mathcal{E})$ the category of **dcpo**-algebras with signature Σ that satisfy all members of $\mathcal{E}$.

Definition IV-8.2. Let X be a **dcpo** and let $DCPO(\Sigma, \mathcal{E})$ be the category of **dcpo**-algebras with signature Σ satisfying $\mathcal{E}$. A pair (A, j) is the *free algebra over* X with respect to $DCPO(\Sigma, \mathcal{E})$ if A is an object of $DCPO(\Sigma, \mathcal{E})$, $j\colon X \to A$ is Scott-continuous, and any Scott-continuous map $f\colon X \to B$, an object of $DCPO(\Sigma, \mathcal{E})$, extends uniquely to a **dcpo**-morphism $h\colon A \to B$ such that $hj = f$. □

Free objects always exist. One method of establishing the general existence is via the adjoint functor theorem. One ingredient that is needed is a complete category $DCPO(\Sigma, \mathcal{E})$ and a (forgetful) functor into $DCPO$ that preserves all limits. This follows from the fact that $DCPO(\Sigma, \mathcal{E})$ is closed under products and equalizers, both defined as in the ordinary case. The other necessary ingredient is that each **dcpo** can generate only some cardinal number of (nonisomorphic) **dcpo**-algebras (sometimes called the solution set condition). We leave the discussion of this condition to Exercise IV-8.21. In the case of the power domains that we consider, we shall seek concrete constructions however.

We consider **dcpo**-algebras equipped with a binary operation, called *formal union* and denoted by $\uplus$, that satisfies the commutative, associative, and idempotency laws (i.e., the equations $x \uplus y = y \uplus x$, $x \uplus (y \uplus z) = (x \uplus y) \uplus z$, and $x \uplus x = x$). Thus these **dcpo**-algebras are commutative idempotent semigroups, i.e., semilattices. We refer to them as **dcpo**-*semilattices*. Note that the semilattice operation $\uplus$ is supposed to be Scott-continuous, but it is not supposed that $x \uplus y$ is the least upper or the greatest lower bound of x and y with respect to the order. These are two special cases that will be treated first.

Throughout this section we adopt the notation

$$\uparrow A := \bigcup\{\uparrow x\colon x \in A\} \text{ and } \downarrow A := \bigcup\{\downarrow x\colon x \in A\}$$

for all nonempty $A \subseteq L$ for L a domain.

The Hoare powerdomain

The *Hoare* or *lower power theory* arises from adjoining to the three equations for a **dcpo**-semilattice the additional inequality

$$x \le x \uplus y$$

and considering the corresponding **dcpo**-algebras.

Definition IV-8.3. The objects of $DCPO(\Sigma, \mathcal{E})$, where Σ consists of a single binary operation $\uplus$ and $\mathcal{E}$ consists of the three equations for commutativity, associativity, and idempotency together with the inequality $x \le x \uplus y$, are called *inflationary semilattices*. The free inflationary semilattice over a domain L is called the *Hoare* or *lower powerdomain* of L and denoted by $\mathbf{P}^H(L)$. □

Lemma IV-8.4. *In an inflationary semilattice we have $x \uplus y = \sup\{x, y\}$. Thus inflationary semilattices are* **dcpo***s that are (complete) sup semilattices in the given order.*

Proof: Let L be an inflationary semilattice. The binary operation $(x, y) \mapsto x \uplus y : L \times L \to L$ is Scott-continuous, hence order preserving. Thus if $x, y \in {\downarrow}z, x \uplus y \le z \uplus z = z$. Since $x \le x \uplus y$ and $y \le y \uplus x = x \uplus y$ by hypothesis, it follows that $x \uplus y = x \vee y$, the least upper bound of x and y. Since any arbitrary nonempty sup may be written as the directed sup of all finite sups in the set, we conclude that L is a complete sup semilattice.

Conversely let L be a **dcpo** that is a sup semilattice. Then L is a complete sup semilattice, as noted in the previous paragraph. Since only sup operations are involved, one verifies directly that $(x, y) \mapsto \sup\{x, y\} : L \times L \to L$ is Scott continuous, and thus $x \uplus y := \sup\{x, y\}$ gives an inflationary semilattice. □

We turn now to the construction of inflationary semilattices with freeness properties.

Proposition IV-8.5. *Let X be a topological space, let $\Psi(X)$ denote the complete sup semilattice of nonempty closed sets ordered by inclusion, and let $j\colon X \to \Psi(X)$ defined by $j(x) = \{x\}^-$ be the embedding of X into $\Psi(X)$. If $f\colon X \to L$ is a continuous function into any inflationary semilattice L (equipped with the Scott topology), then there exists a unique* **dcpo***-morphism $h\colon \Psi(X) \to L$ such that $hj = f$.*

Proof: By Lemma IV-8.4 $x \uplus y = x \vee y = \sup\{x, y\}$ in L. For any nonempty closed subset $A \in \Psi(X)$, define $h(A) = \sup f(A)$ (the supremum exists by Lemma IV-8.4). Since $\sup(f(A \cup B)) = \sup(f(A) \cup f(B)) = \sup(f(A)) \vee \sup(f(B))$, f is a sup semilattice homomorphism. We establish the Scott continuity of f. Let $\mathcal{D}$ be a directed family of closed sets in $\Psi(X)$. Then $\sup \mathcal{D} = (\bigcup \mathcal{D})^-$, the closure of the union. Since f is a sup semilattice homomorphism, it is order preserving, and thus $b := \sup\{f(D)\colon D \in \mathcal{D}\} \le f(\sup \mathcal{D})$. Since f is continuous, $f^{-1}(\downarrow b) = f^{-1}(\{b\}^-)$ is a closed set containing every $D \in \mathcal{D}$. It follows that $f^{-1}(\downarrow b)$ contains $\sup \mathcal{D} = (\bigcup \mathcal{D})^-$. Thus $f(\sup \mathcal{D}) \le \sup\{f(D)\colon D \in \mathcal{D}\}$, and equality is established.

For any $x \in X$, $hj(x) = h(\{x\}^-) = h(\downarrow x) = \sup f(\downarrow x)$, where $\downarrow x$ is taken in the order of specialization. Since the continuous map f is monotone with respect to the orders of specialization, it follows that $\sup f(\downarrow x) = f(x)$, and thus $hj(x) = f(x)$. If $g\colon \Psi(X) \to L$ is any other **dcpo**-morphism such that $gj = f$, then $g(\downarrow F) = h(\downarrow F)$ for all finite subsets F of X by the sup semilattice homomorphism property of g, and then $g = f$ by Scott continuity of both, since every closed set is a directed supremum of all $\downarrow F$ over its finite subsets. □

We can in particular restrict our attention to those spaces X that are **dcpo**s or domains equipped with the Scott topology and apply Proposition IV-8.5 in this setting.

Corollary IV-8.6. *Let L be a* **dcpo**. *Then the free inflationary semilattice over L consists (up to isomorphism) of all nonempty Scott closed subsets with operation $\uplus$ given by binary union, order given by the standard inclusion relation, and the embedding of L given by $j(x) = \downarrow x$.*

Proof: Since the binary sup operation on any **dcpo** that is a sup semilattice is always Scott-continuous in each coordinate, it is continuous for the Scott topology on the product. Hence the construction of the corollary yields an inflationary semilattice. Its freeness follows from the preceding proposition. □

Corollary IV-8.7. *If L is a domain, the Hoare powerdomain $\mathbf{P}^H(L)$ is again a domain. For nonempty sets A, B we have $B^- \ll A^-$ iff there exists a finite set F such that $B \subseteq \downarrow F \subseteq \;\twoheaddownarrow A$.* □

Proof: By II-1.14 the lattice of Scott open sets is a completely distributive lattice, and thus the same applies to the Scott closed sets. In particular, the lattice of Scott closed sets ordered by inclusion is a continuous lattice. Thus throwing away the bottom element Ø still yields a domain. Suppose that there exists a finite set F such that $B \subseteq \downarrow F \subseteq \;\twoheaddownarrow A$. Let $\mathcal{D}$ be a directed family of Scott closed sets such that the Scott closure of the union contains A^- or equivalently A. Let $I = \bigcup \mathcal{D}$; note $I = \downarrow I$. Let x be in the Scott closure of I. If $z \ll x$, then we must have $z \in I$, for otherwise $\twoheaduparrow z$ is a Scott open set containing x but missing I. Hence $x = \sup\{z\colon z \ll x\}$ is the sup of a directed subset of I. Since each member of A is contained in the Scott closure of I, it follows that each member of F is in I, and hence that $F \subseteq D$ for some $D \in \mathcal{D}$, since $\mathcal{D}$ is directed. Thus $B \subseteq \downarrow F \subseteq D$, and thus $B^- \subseteq D$. This establishes $B^- \ll A^-$. Since A^- is the directed supremum of all $\downarrow F$, F finite, $F \subseteq \mathop{\downarrow\!\!\!\!\downarrow} A$, the converse also follows. $\square$

The Smyth powerdomain

The *Smyth* or *upper power theory* arises from adjoining to the three equations for a **dcpo**-semilattice the additional inequality

$$x \geq x \uplus y$$

and considering the corresponding **dcpo**-algebras.

Definition IV-8.8. The objects of $DCPO(\Sigma, \mathcal{E})$, where Σ consists of a single binary operation $\uplus$ and $\mathcal{E}$ consists of the three equations for commutativity, associativity, and idempotency together with the inequality $x \geq x \uplus y$, are called *deflationary semilattices*. The free deflationary semilattice over a domain L is called the *Smyth* or *upper powerdomain* of L and denoted by $\mathbf{P}^S(L)$. $\square$

Lemma IV-8.9. *In a deflationary semilattice we have $x \uplus y = \inf\{x, y\}$. Thus deflationary semilattices are* **dcpo***s that are meet continuous inf semilattices in the given order equipped with the binary operation of meet.*

Proof: Let L be a deflationary semilattice. The binary operation $(x, y) \mapsto x \uplus y : L \times L \to L$ is Scott-continuous, hence order preserving. Thus if $x, y \in \uparrow z$, $x \uplus y \geq z \uplus z = z$. Since $x \geq x \uplus y$ and $y \geq y \uplus x = x \uplus y$ by hypothesis, it follows that $x \uplus y = \inf\{x, y\}$. Since $(x, y) \mapsto x \uplus y : L \times L \to L$ is Scott-continuous, it follows directly that L is a meet continuous semilattice.

Conversely let L be a **dcpo** that is a meet continuous semilattice in its given order. Then the meet operation is Scott-continuous in each coordinate, and hence Scott-continuous on the product endowed with the Scott topology. Thus L equipped with the binary meet operation is a deflationary semilattice. $\square$

Theorem IV-8.10. *The Smyth powerdomain over a domain L may be realized as the set $Q^*(L)$ of nonempty subsets of L that are compact and saturated in the Scott topology ordered by reverse inclusion. The embedding j of L into $Q^*(L)$ is given by $j(x) = \uparrow x$.*

Proof: Since L equipped with its Scott topology is locally compact and sober (II-1.13), by II-1.22 the set $Q^*(L)$ of all nonempty compact saturated sets ordered by reverse inclusion is a continuous semilattice, hence meet continuous (with binary operation $(K_1, K_2) \mapsto K_1 \cup K_2$). Thus for $\uplus = \cup$, $Q^*(L)$ is a deflationary semilattice by Lemma IV-8.9 and a domain. It is straightforward to see that $j(x) = \uparrow x$ is a Scott-continuous embedding of L into $Q^*(L)$.

Let $f\colon L \to S$ be a Scott-continuous map, where S is a deflationary semilattice. By Lemma IV-8.9 the binary operation in S is the meet operation with respect to its order. Let $K \in Q^*(L)$. We define $h(K)$ as follows. Consider the set $D_K = \{y \in S : K \subseteq \operatorname{int}_\sigma f^{-1}(\uparrow y)\}$. Clearly each member of D_K is a lower bound for $f(K)$. Let $y_1, y_2 \in D_K$ and set $U = \operatorname{int}_\sigma f^{-1}(\uparrow y_1) \cap \operatorname{int}_\sigma f^{-1}(\uparrow y_2)$, a Scott open set containing K. For each $x \in K$ pick $z_x \ll x, z_x \in U$. Then finitely many of the $\Uparrow z_x$ cover K, say $K \subseteq \Uparrow z_1 \cup \cdots \cup \Uparrow z_n$. Since $y_1 \le f(z_i)$ for each i, we have $y_1 \le w := \inf\{f(z_i)\colon 1 \le i \le n\}$. Similarly $y_2 \le w$. It follows from the definition of w that $w \in D_K$, and we conclude that D_K is directed. Hence $b = \sup D_K$ exists. Since each member of D_K is a lower bound for $f(K)$, we conclude that b is a lower bound for $f(K)$. We define $h(K) = b = \sup D_K$.

Since for $J, K \in Q^*(L)$, $J \subseteq K$ implies $D_J \supseteq D_K$, we conclude that h is monotone. Hence $h(J \cup K) \le h(J)h(K)$ for any $J, K \in Q^*(L)$. Conversely if $y \in D_J$ and $z \in D_K$, then it follows easily that $yz \in D_{J\cup K}$. Since by meet continuity of S, we have that $h(J)h(K)$ is the directed sup of all yz, $y \in D_J$, $z \in D_K$, we conclude that $h(J)h(K) \le h(J \cup K)$. Thus h is a semilattice homomorphism.

We next establish Scott continuity of h. Let $\mathcal{K}$ be a filtered family in $Q^*(L)$ with intersection J. For each $y \in D_J$, $f^{-1}(\uparrow y)$ contains J in its Scott interior. By II-1.21 there exists some $K \in \mathcal{K}$ such that K is contained in the Scott interior of $f^{-1}(\uparrow y)$, and thus $h(K) \ge y$. It follows that $\sup\{h(K)\colon K \in \mathcal{K}\} \ge h(J)$; the other inequality follows from monotonicity.

Let $x \in L$. Then $hj(x) = \sup D_{\uparrow x}$. Since for each $z \ll x$, $f(z)$ is in $D_{\uparrow x}$, we conclude from the Scott continuity of f that $f(x) \le hj(x)$. On the other hand, since each member of $D_{\uparrow x}$ is less than or equal to $f(x)$, we have $hj(x) = f(x)$. Let $g\colon Q^*(X) \to S$ be another morphism of deflationary semilattices such that $gj = f$. Then g agrees with f on $j(L)$, hence on the semilattice of all finitely generated upper sets, and hence on its Scott closure $Q^*(L)$ (see III-5.7). $\square$

The Plotkin powerdomain

The *Plotkin* or *convex power theory* arises from considering the **dcpo**-algebras corresponding to the three equations for a **dcpo**-semilattice.

Definition IV-8.11. The objects of *DCPO* $(\Sigma, \mathcal{E})$, where Σ consists of a single binary operation $\uplus$ and $\mathcal{E}$ consists of the three equations for commutativity, associativity, and idempotency, are called **dcpo**-*semilattices*. The free

dcpo-semilattice over a domain L is called the *Plotkin powerdomain* of L and denoted by $\mathbf{P}^P(L)$. □

There is a general construction via rounded ideal completions (see III-4.15, III-4.17) for obtaining the free **dcpo**-algebra over any domain L with respect to $DCPO(\Sigma, \mathcal{E})$ for any signature Σ and set of equalities and inequalities $\mathcal{E}$. We illustrate this technique for the case of the Plotkin powerdomain, i.e., the case that Σ is a binary operation and $\mathcal{E}$ consists of the three equations for the associative, commutative, and idempotent laws.

Let L be a domain and let B be a basis for L. We implement the associativity law, which allows us to drop all parentheses in the term algebra over B, so that it reduces to the free semigroup of all finite strings that can be formed from the alphabet B with operation juxtaposition. For any string $w = b_1 \dots b_n$ we define its alphabet by $\alpha(w) = \{b_1, \dots, b_n\}$, the set of letters appearing in w. Note that α is a function into the set FB of nonempty finite subsets of B. We say that two strings w_1 and w_2 are equivalent, written $w_1 \sim w_2$, if $\alpha(w_1) = \alpha(w_2)$. This is equivalent to saying that the two strings can be transformed one to the other by using the substitutions $xy = yx$ and $xx = x$, the other two laws of $\mathcal{E}$. Since the set of finite strings equipped with the operation of juxtaposition gives the free semigroup over B, the congruence classes with respect to $\sim$ give the free semilattice. These classes may be identified with their images under α, the semigroup of finite subsets of B with operation union. We define a relation $\ll$ on the set of strings by $a_1 \dots a_n \ll b_1 \dots b_n$ if $a_i \ll b_i$ for $i = 1, \dots, n$. Note that since B is a basis of a domain and hence satisfies the interpolation property, if $w_i \ll w$ for $i = 1, 2$ then there exists w' such that $w_i \ll w' \ll w$, again for $i = 1, 2$.

We define a relation $\prec$ on the set of nonempty finite subsets FB of B by $F_1 \prec F_2$ if there exist strings w_1, w_2 such that $w_1 \ll w_2, \alpha(w_1) = F_1$, and $\alpha(w_2) = F_2$. This implies that $F_1 \subseteq \mathord{\Downarrow} F2$ and $F_2 \subseteq \mathord{\Uparrow} F_1$. Conversely assume the latter containments hold for $F_1 = \{b_1, \dots, b_m\}$ and $F_2 = \{a_1, \dots, a_n\}$. For each b_i, choose $\hat{a}_i \in F_2$ such that $b_i \ll \hat{a}_i$ and for each a_i, choose $\hat{b}_i \in F_1$ such that $\hat{b}_i \ll a_i$. Then for $w = b_1 \dots b_m \hat{b}_1 \dots \hat{b}_n$ and $w' = \hat{a}_1 \dots \hat{a}_m a_1 \dots a_n$ we have that $w \ll w'$ and $\alpha(w) = F_1, \alpha(w') = F_2$. Thus

$$F_1 \prec F_2 \text{ iff } F_1 \subseteq \mathord{\Downarrow} F_2 \text{ and } F_2 \subseteq \mathord{\Uparrow} F_1.$$

We use the last characterization to show that $\prec$ satisfies the appropriate interpolation property and is transitive. Suppose that $F_1, \dots, F_n \prec F$. By the interpolation property of B, for each $a_i \in F$, pick $b_i \in B$ such that $b_i \ll a_i$ and $c \ll b_i$ for each $c \in (F_1 \cup \dots \cup F_n) \cap \mathord{\Downarrow} a_i$. Set F' equal to the set consisting of all the b_i. Then it is straightforward to verify that $F_1, \ldots, F_n \prec F' \prec F$. The transitivity of $\prec$ is straightforward to verify.

Thus $\prec$ is a transitive relation on the set FB of nonempty finite subsets of B that satisfies the finite interpolation property of Definition III-4.15. We may thus take the rounded ideal completion RId FB of $(FB, \prec)$ (see III-4.15 and III-4.17); by III-4.17 RId FB is a domain. We have a map $j\colon FB \to \mathrm{RId}\, FB$ given by $j(F) = \{G \in FB : G \prec F\}$ and a related map $i\colon L \to \mathrm{RId}\, FB$ given by $i(x) = \{F \in FB : F \prec \{y\} \text{ for some } y \in B, y \ll x\}$.

We wish to establish that $i\colon L \to \mathrm{RId}\, FB$ can be identified with the Plotkin powerdomain. First we need a semilattice operation on RId FB. For two rounded ideals I_1, I_2, we define

$$I_1 \uplus I_2 = \{F\colon F \prec F_1 \cup F_2 \text{ for some } F_1 \in I_1, F_2 \in I_2\}.$$

If $F_1 \prec G_1$ and $F_2 \prec G_2$, then one sees directly that $F_1 \cup F_2 \prec G_1 \cup G_2$. It follows that $I_1 \uplus I_2$ is a rounded ideal that contains $F_1 \cup F_2$ for each $F_1 \in I_1$, $F_2 \in I_2$. Using the fact that directed sups are unions in RId FB, one obtains in a straightforward fashion that RId FB is a **dcpo**-semilattice (i.e., the appropriate Scott continuity of the operation holds). Finally note that $G \prec F_1 \cup F_2$ iff $G = G_1 \cup G_2$ for $G_i = \twoheaddownarrow F_i \cap G$, and $G_i \prec F_i$ for $i = 1, 2$. From this observation it follows that $j(F_1 \cup F_2) = j(F_1) \uplus j(F_2)$, i.e., $j\colon FB \to \mathrm{RId}\, FB$ is a semilattice homomorphism.

We establish the universal property. Let $f\colon L \to A$ be a Scott-continuous map into a **dcpo**-semilattice A. For any string $w = b_1 \ldots b_n$, we set $\xi(w) = f(b_1) \uplus \cdots \uplus f(b_n)$. Since f and the operation $\uplus$ in A are both order preserving, $w \ll w'$ implies $\xi(w) \le (w')$. Since the set of finite subsets under union is the free semilattice on the set L, there exists a unique homomorphism $\hat{f}$ from $(FB, \cup)$ to A given by $\hat{f}(\{b_1, \ldots, b_n\}) = \hat{f}(b_1) \uplus \cdots \uplus f(b_n)$ that extends $f \mid B$. Since $(A, \uplus)$ is a semilattice, it follows that $\xi = \hat{f} \circ \alpha$. Thus if $F \prec G$, then $F = \alpha(w)$ and $G = \alpha(w')$ for some $w \ll w'$, and hence $\hat{f}(F) = \xi(w) \le \xi(w') = \hat{f}(G)$. It follows that the image under $\hat{f}$ of any rounded ideal of FB is a directed subset of A, and hence we may define $\psi(I)$ for any $I \in \mathrm{RId}\,(FB)$ by $\psi(I) = \sup\{\hat{f}(F)\colon F \in I\}$. It now follows directly from the calculations of the preceding paragraph and the Scott continuity of the operation $\uplus$ in A that ψ is a semilattice homomorphism. That it is Scott-continuous follows from the fact that the sup of the union of a family of increasingly larger directed sets is the directed sup of the set of sups of each member of the family.

Finally we establish the extension and uniqueness property of ψ. Let $x \in L$. We consider $\psi(i(x))$. By definition of $i(x)$, we have for $F \in i(x)$ that there exists $b \in B \cap \twoheaddownarrow x$ such that $F \prec \{b\}$ and thus $\hat{f}(F) \le \hat{f}(\{b\}) = f(b) \le f(x)$; we conclude that $\psi(i(x)) \le f(x)$. Conversely for $b \in B$ with $b \ll x$, there exists

$c \in B$ such that $b \ll c \ll x$. Then $\{b\} \prec \{c\}$ implies $\hat{f}(\{b\}) = f(b) \le \psi(i(x))$. Since f is Scott-continuous and $x = \sup\{b \in B : b \ll x\}$, a directed sup, we have $f(x) \le \psi(i(x))$. Thus $\psi(i(x)) = f(x)$. It follows that for $F = \{b_1, \dots, b_n\}$ (since ξ and j are semilattice homomorphisms)

$$\xi(j(F)) = \xi(j(b_1)) \uplus \cdots \uplus \xi(j(b_n)) = f(b_1) \uplus \cdots \uplus f(b_n).$$

Thus ξ is uniquely determined from f on $j(FB)$. Since ξ is Scott-continuous and since $j(FB)$ is a basis for RId FB (III-4.17), we conclude that ξ is uniquely determined.

We have thus verified in the preceding the validity of the following construction for the Plotkin powerdomain.

Construction IV-8.12. Let L be a domain with basis B. Let FB denote the set of nonempty finite sets equipped with the transitive relation $\prec$ defined by $F_1 \prec F_2$ iff $F_1 \subseteq \Downarrow F_2$ and $F_2 \subseteq \Uparrow F_1$. Then $(FB, \prec)$ satisfies the finite interpolation property, and hence the rounded ideal completion RId FB is a domain. The domain RId FB becomes the free **dcpo**-semilattice, i.e., the Plotkin powerdomain, when equipped with the operation

$$I_1 \uplus I_2 = \{F\colon F \prec F_1 \cup F_2 \text{ for some } F_1 \in I_1, F_2 \in I_2\}.$$

The map $i\colon L \to$ RId FB is given by $i(x) = \{F \in FB : F \prec \{y\}$ for some $y \in B, y \ll x\}$. □

We turn now to more concrete realizations of the Plotkin power domain.

Proposition IV-8.13. *Let L be a domain equipped with the Scott topology, and let A, B be nonempty subsets. If A and B are compact, then the following are equivalent:*

(1) $\uparrow B \ll \uparrow A$ *in* $Q^*(L) = \mathbf{P}^S(L)$*;*
(2) $A \subseteq \mathrm{int}_\sigma(\uparrow B)$*;*
(3) $A \subseteq \Uparrow B$.

Similarly if A, B are nonempty, then the following are equivalent:

(1′) $B^- \ll A^-$ *in* $\mathbf{P}^H(L)$*;*
(2′) $B \subseteq \downarrow F \subseteq \Downarrow A$ *for some finite F.*

For a finite subset F and any compact G, we have $(\downarrow F, \uparrow F) \ll (G^-, \uparrow G)$ *in* $\mathbf{P}^H(L) \times \mathbf{P}^S(L)$ *iff* $F \prec G$*, i.e.,* $F \subseteq \Downarrow G$ *and* $G \subseteq \Uparrow F$.

Proof: The equivalence of (1) and (2) follows from I-1.24.2(ii), II-1.21, and II-1.13. If $A \subseteq \Uparrow B = \bigcup\{\Uparrow b\colon b \in B\}$, then $A \subseteq \mathrm{int}_\sigma(\uparrow B)$ since each $\Uparrow b$ is open (II-1.10). Conversely suppose that $A \subseteq \mathrm{int}_\sigma(\uparrow B)$. Then for each $a \in A$, there exists $y \ll a$ such that $y \in \mathrm{int}_\sigma(\uparrow B)$, since a is the directed sup of $\twoheaddownarrow a$. Thus $a \in \twoheaduparrow B$, since $b \leq y$ for some $b \in B$. The equivalence of (1′) and (2′) follows from Corollary IV-8.7.

The last assertion follows from the previous ones and the fact that the relation $\ll$ in $\mathbf{P}^H(L) \times \mathbf{P}^S(L)$ is the product of the $\ll$ relation in the components. $\square$

By Construction IV-8.12 we may identify the Plotkin powerdomain with the rounded ideal completion RId FB for a domain L with basis B. The map $f(x) = (\downarrow x, \uparrow x)\colon L \to \mathbf{P}^H(L) \times \mathbf{P}^S(L)$ extends (by freeness) to a continuous semilattice homomorphism ψ from the Plotkin powerdomain, since $\mathbf{P}^H(L) \times \mathbf{P}^S(L)$ is a **dcpo**-semilattice. By the construction outlined above, this homomorphism carries a rounded ideal I of $(FB, \prec)$ to the directed sup over all images $\hat{f}(F) = (\downarrow F, \uparrow F)$, $F \in I$. By the preceding proposition the collection $\{(\downarrow F, \uparrow F)\colon F \in I\}$ is a $\ll$-rounded ideal in $\hat{f}(FB)$ and the assignment $I \to \{(\downarrow F, \uparrow F)\colon F \in I\}$ is an order isomorphism between RId FB and RId $\hat{f}(FB)$. By a mild extension of Exercise III-4.16 (see Exercise IV-8.25 below), we have that RId $\hat{f}(FB)$ may be identified with the smallest directed complete subset of $\mathbf{P}^H(L) \times \mathbf{P}^S(L)$ containing all $(\downarrow F, \uparrow F)$, $F \in FB$. We have thus established the following result.

Theorem IV-8.14. *Let L be a domain with basis B. We may identify $\mathbf{P}^P(L)$ with the* **dcpo**-*subsemilattice of $\mathbf{P}^H(L) \times \mathbf{P}^S(L)$ consisting of all directed sups taken from $\{(\downarrow F, \uparrow F)\colon F$ is a finite subset of $B\}$. The inclusion is given by $x \mapsto (\downarrow x, \uparrow x) : L \mapsto \mathbf{P}^P(L)$.* $\square$

Remark. Note that since $\mathbf{P}^P(L)$ is a **dcpo**-subsemilattice of $\mathbf{P}^H(L) \times \mathbf{P}^S(L)$ containing all $(\downarrow x, \uparrow x)$, $x \in L$, it must also contain their semilattice products, namely all $(\downarrow F, \uparrow F)$, F a nonempty finite subset of L, and hence the set consisting of all directed sups of these. By the preceding theorem (applied to $L = B$), the latter set is $\mathbf{P}^P(L)$.

We wish to identify both some sufficient and some necessary conditions for a pair $(A, B) \in \mathbf{P}^H(L) \times \mathbf{P}^S(L)$ to be in $\mathbf{P}^P(L)$.

Definition IV-8.15. Let L be a domain equipped with the Scott topology. A nonempty subset A is a *lens* if A can be written as the intersection of a closed set and a compact saturated set. A lens $A = C \cap K$ has a *canonical* representation of the form $A^- \cap \uparrow A$ (since $A \subseteq A^- \cap \uparrow A \subseteq C \cap K = A$); note that $\uparrow A$ is compact since A is. A pair $(C, K) \in \mathbf{P}^H(L) \times \mathbf{P}^S(L)$ is called a *lens factorization* if $C = A^-$ and $K = \uparrow A$ for some lens A; it is then immediate that $A = C \cap K$ and $C \cap K$ is the canonical representation of A. We denote by Lens L the set of all lenses of L ordered by the *topological Egli–Milner* *order*: $A \leq B$ iff $A \subseteq B^-$ and $B \subseteq {\uparrow}A$. (The usual Egli–Milner order is given by $A \leq B$ if $A \subseteq {\downarrow}B$ and $B \subseteq {\uparrow}A$.) $\square$

Remark. *For any nonempty compact set A, $(A^-, {\uparrow}A)$ is a lens factorization for the lens $A^- \cap {\uparrow}A$, and the latter is the smallest lens containing A.* $\square$

Remark IV-8.16. *We can embed* Lens L *into* $\mathbf{P}^H(L) \times \mathbf{P}^S(L)$ *by sending A to $(A^-, {\uparrow}A)$. Since $A = A^- \cap {\uparrow}A$, the function is injective. We have $A^- \leq B^-$ in $\mathbf{P}^H(L)$ iff $A^- \subseteq B^-$ iff $A \subseteq B^-$. Similarly ${\uparrow}A \leq {\uparrow}B \in \mathbf{P}^S(L)$ iff $B \subseteq {\uparrow}A$ (since the order is reverse inclusion). Thus the embedding $A \mapsto (A^-, {\uparrow}A)$ is an order isomorphism between* Lens L *equipped with the topological Egli–Milner order and the set of pairs in $\mathbf{P}^H(L) \times \mathbf{P}^S(L)$ that are lens factorizations equipped with the relative order.* $\square$

We show that lens factorizations are closely related to members of the Plotkin powerdomain.

Proposition IV-8.17. *For a domain L endowed with the Scott topology, if A is a lens, then the lens factorization $(A^-, {\uparrow}A)$ is in $\mathbf{P}^P(L)$. Conversely if $(C, D) \in \mathbf{P}^P(L)$, then $C \cap D$ is a lens and $D = {\uparrow}(C \cap D)$. Furthermore if L is coherent (i.e., the intersection of any two compact saturated sets is again compact), then $C = {\downarrow}(C \cap D)$ and if L is countably based, then $C = (C \cap D)^-$.*

Proof: For any lens A, consider $I_A = \{F \in FL : F \prec A\}$. If $F, G \prec A$, then ${\uparrow}A$ is contained in the Scott open set $U = {\twoheaduparrow}F \cap {\twoheaduparrow}G$. For each $x \in A$, pick $y_x \ll x$ such that $y_x \in U$. Then finitely many ${\twoheaduparrow}y_x$, say $y_1, \ldots, y_n$, cover A. We augment the set $\{y_1, \ldots, y_n\}$ by adjoining for each $x \in F \cup G$ an element $z_x \in U$ such that $x \ll z_x \ll a$ for some $a \in A$ (this utilizes the fact that $F, G \subseteq {\twoheaddownarrow}A$). If H consists of all y_i, $i = 1, \ldots, n$, and all $\{z_x : x \in F \cup G\}$, then H is finite and $F, G \prec H \prec A$. Thus I_A is a rounded ideal (clearly it is lower closed under $\prec$).

The argument of the preceding paragraph allows us to construct within any Scott open set U containing ${\uparrow}A$ a finite set $H \prec A$ such that $H \subseteq U$. We conclude that ${\uparrow}A = \bigcap\{{\uparrow}F : F \in FL, F \prec A\}$, a filtered intersection. Since we can augment any $F \prec A$ with any finite set of points out of ${\twoheaddownarrow}A$ and get a new set in I_A, we conclude also that A^- is equal to the Scott closure of $\bigcup\{{\downarrow}F : F \in FL, F \prec A\} = {\twoheaddownarrow}A$, where the union is a directed union. Thus $(A^-, {\uparrow}A)$ is the directed sup of $\{({\downarrow}F, {\uparrow}F) : F \in I_A\}$ in $\mathbf{P}^H(L) \times \mathbf{P}^S(L)$. This shows for any lens A that $(A^-, {\uparrow}A)$ is in $\mathbf{P}^P(L)$.

Conversely suppose that $\{({\downarrow}F, {\uparrow}F) : F \text{ finite}, F \in I\}$ is a $\ll$-directed set in $\mathbf{P}^P(L)$ with sup (C, D), where C is the closure of $\bigcup\{{\downarrow}F : F \in I\}$ and $D = \bigcap\{{\uparrow}F : F \in I\}$. Applying II-1.22 and II-1.12 to the domain C, we conclude that the filtered intersection $\bigcap\{\uparrow F \cap C : F \in I\} = C \cap D$ is compact and nonempty. Similarly for any $y \in D$, applying II-1.22 to the domain $C \cap \downarrow y$, we conclude that the filtered intersection $\bigcap\{\uparrow F \cap C \cap \downarrow y : F \in I\} = C \cap D \cap \downarrow y$ is compact and nonempty. Thus $y \in \uparrow(C \cap D)$. Hence $D \subseteq \uparrow(C \cap D) \subseteq \uparrow D = D$, so $D = \uparrow(C \cap D)$.

Suppose now that L is coherent. Let E be a directed set in $\downarrow(C \cap D)$. The $\{\uparrow e \cap D : e \in E\}$ is a filtered family of compact saturated sets (by coherence) and hence $\{\uparrow e \cap C \cap D : e \in E\}$ is a filtered family of nonempty compact sets in the domain C. Applying II-1.22 and II-1.12 to the domain C, we conclude that there exists $x \in \bigcap\{\uparrow e \cap C \cap D : e \in E\}$, and thus $\sup E \leq x \in C \cap D$. It follows from this argument that $\downarrow(C \cap D)$ is closed.

Let $y \in C$ and let $z \ll y$. Then since C is the closure of the set $\bigcup\{\uparrow F : F \in I\}$, we conclude that some $F \in I$ meets the open set $\Uparrow z$. Hence for all $G \in I$ beyond F, we have $\uparrow z \cap G \neq \emptyset$ and hence $\uparrow z \cap \uparrow G \cap C$ is a nonempty saturated subset of C which is compact (since C is closed and $\uparrow z \cap \uparrow G$ is compact by coherence). Applying II-1.22 again to the filtered collection $\uparrow z \cap G \cap C$, we conclude that there exists $w \in \uparrow z \cap C \cap D$, i.e., $z \in \downarrow(C \cap D)$. Since z was arbitrary in $\Downarrow y$, we conclude that $y \in (C \cap D)^- = \downarrow(C \cap D)$, by the previous paragraph. Thus $C \subseteq \downarrow(C \cap D)$, and the reverse containment always holds.

Finally suppose that L has a countable basis BI. Assume that $(C, D) \in \mathbf{P}^P(L)$. By Theorem IV-8.14 (C, D) is the supremum of a directed set of $(\downarrow F, \uparrow F)$, where each F is a finite subset of B. Since FB is countable, the directed set will be countable. Thus we can inductively pick a countable chain $F_1 \leq F_2 \leq \cdots$ in the directed set that is cofinal in the directed set, and hence has supremum (C, D). As in the preceding paragraph, let $y \in C$, $z \ll y$, and pick F_n such that $z \leq x_n$ for some $x_n \in F_n$. Since $F_n \leq F_{n+1} \leq \cdots$, we inductively choose $x_{n+k} \in F_{n+k}$ such that $\{x_{n+k} : k \geq 0\}$ is an increasing chain in C. Thus the supremum x of the chain is in $C \cap \bigcap\{\uparrow F_{n+k} : k \geq 0\} = C \cap D$. We conclude that $z \in \downarrow(C \cap D)$, and since z was arbitrary in $\Downarrow y$, that $y \in (C \cap D)^-$. □

Theorem IV-8.18. *Let L be a domain equipped with the Scott topology. If L is coherent, resp. countably based, then the Plotkin powerdomain $\mathbf{P}^P(L)$ may be identified with* Lens L *equipped with the Egli–Milner order, resp. topological Egli–Milner order, the embedding $x \mapsto \{x\} : L \to$* Lens L, *and semilattice operation $A \uplus B = \downarrow(A \cup B) \cap \uparrow(A \cup B)$ resp. $A \uplus B = (A \cup B)^- \cap \uparrow(A \cup B)$.*

Proof: If follows from the preceding proposition IV-8.17 that $(C, D) \mapsto C \cap D : \mathbf{P}^P(L) \to$ Lens L is a bijection with inverse $A \mapsto (\downarrow A, \uparrow A)$ (resp.

$A \mapsto (A^-, \uparrow A)$). It follows from Remark IV-8.16 that it is then an order isomorphism with respect to the topological Egli–Milner order, which collapses to the Egli–Milner order by IV-8.17 if L is coherent. Since the semilattice operation in $\mathbf{P}^P(L)$ is set-theoretic union in both coordinates, one verifies by direct computation that the given bijection is a semilattice homomorphism from Lens L equipped with the semilattice operation given in the statement of the theorem to $\mathbf{P}^P(L)$. Finally the embedding $x \mapsto \{x\}$ is clearly the composition $x \mapsto (\downarrow x, \uparrow x) \mapsto \downarrow x \cap \uparrow x$. □

We close this section with some remarks on the functorial properties of the powerdomain constructions for $\mathbf{P}^* \in \{\mathbf{P}^H, \mathbf{P}^S, \mathbf{P}^P\}$, which we now view as functors from the category of **dcpo**s and Scott-continuous maps to the categories of **dcpo**-algebras that are inflationary semilattices, deflationary semilattices or semilattices. For $f: L \mapsto M$ in *DCPO*, the freeness of the construction gives rise to $f^* = \mathbf{P}^*(f) : \mathbf{P}^*(L) \to \mathbf{P}^*(M)$ such that

$$\begin{tikzcd} \mathbf{P}^*(L) \arrow[r, "f^*"] & \mathbf{P}^*(M) \\ L \arrow[u, "j_L"] \arrow[r, "f"'] & M \arrow[u, "j_M"'] \end{tikzcd}$$

Indeed, as was remarked at the beginning of this section and as one can easily see from the freeness, $\mathbf{P}^*$ is the adjoint functor to the forgetful functor from **dcpo**-algebras to **dcpo**s.

Proposition IV-8.19. *The functors* $\mathbf{P}^H$, $\mathbf{P}^S$, $\mathbf{P}^P$ *are locally continuous and hence pro-continuous functors when restricted to* $DCPO_G$.

Proof: Suppose that $f \leq g : L \mapsto M$. Then by commutativity of the above diagram one sees that $f^* = \mathbf{P}^*(f) \leq g^* = \mathbf{P}^*(g)$ at all points of $j_L(L)$. By the Scott continuity of the semilattice operations on $\mathbf{P}^*(L)$ and $\mathbf{P}^*(M)$ and the fact f^* and g^* are Scott-continuous homomorphisms, one sees directly that the set of elements of $\mathbf{P}^*(L)$ where $f^* \leq g^*$ is a subalgebra of L closed under directed sups and containing the generating set $j_L(L)$, and hence must be all of L. A similar argument yields that if $f: L \mapsto M$ is the directed (and hence pointwise) sup of a family $\{f_\alpha\}$, then the set of points in $\mathbf{P}^*(L)$, where f^* is the pointwise sup of the f^*_α is a subalgebra closed under directed sups and containing the generating set. Thus $f \mapsto \mathbf{P}^*(f)$ is Scott-continuous, and the functor is locally continuous. The last assertion follows from Theorem IV-5.5.

Exercises

Exercise IV-8.20. Let X be a nonempty subset of a **dcpo** L, and let Y be the smallest subset of L containing X that is closed with respect to taking directed sups, i.e., is a sub**dcpo**. Show that the cardinality of Y is less than or equal to the cardinality $2^{|X|}$ of the powerset of X.

Hint. Let Z be the set of all existing suprema over all subsets of X. Observe that Z contains X and is closed with respect to taking sups of any of its subsets that exist, in particular directed subsets, and hence must contain Y. Thus the cardinality of Y is bounded by that of Z, which in turn is bounded by $2^{|X|}$. $\square$

Exercise IV-8.21. Let X be a **dcpo** and let $DCPO(\Sigma, \mathcal{E})$ be the category of **dcpo**-algebras with signature Σ satisfying $\mathcal{E}$. A pair (A, j) is a **dcpo**-*algebra generated by X with respect to DCPO* $(\Sigma, \mathcal{E})$ if A is an object of $DCPO$ $(\Sigma, \mathcal{E})$, $j\colon X \to A$ is Scott-continuous, and there is no proper **dcpo**-subalgebra of A containing $j(X)$. Two such are *isomorphic* if there is an order and algebra isomorphism between them commuting with the respective embeddings of X. Show that one can pick a set of representatives of all isomorphism classes of **dcpo**-algebras generated by X of bounded cardinality.

Hint. Given a **dcpo** X and a continuous map $j\colon X \to A$, then the **dcpo**-subalgebra generated by $j(X)$ is constructed in two stages. First let F be the ordinary subalgebra of A which is generated (algebraically) from $j(X)$. Its cardinality is bounded by an expression depending on the cardinality of X and the cardinality of the number of operations. Then let $\bar{F}$ denote the smallest subset of A containing F closed under directed sups. Because the operations on A are Scott-continuous, $\bar{F}$ remains a subalgebra, so is a **dcpo**-subalgebra, the smallest contining $j(X)$. By the previous exercise its cardinality is bounded by $2^{|F|}$. Thus there is some cardinal bounding the cardinality of all **dcpo**-algebras generated by X. Standard arguments then finish the exercise. $\square$

For algebraic domains it is convenient to have descriptions of the powerdomains that can be carried out at the level of the compact elements.

Exercise IV-8.22. Let A be an algebraic domain with poset of compact elements $K(A)$.

(i) If M is a nonempty lower set of A, show that its Scott closure is obtained by adding on the suprema of all directed sets (or ideals) contained in M.

(ii) Order the set $\Psi(A)$ of all nonempty Scott closed lower sets by inclusion. Show that the compact elements of $\Psi(A)$ are precisely sets of the form $\downarrow F$, where F is a finite subset of compact elements.

(iii) Order the set of finite antichains of $K(A)$ by the *Hoare order*: $F_1 \leq F_2$ *iff* $F_1 \subseteq \downarrow(F_2)$. Show that the mapping $F \mapsto \downarrow F$ is an order isomorphism from the Hoare ordered poset of finite antichains onto the compact elements of $\Psi(A)$. Conclude (using IV-8.6) that $\mathbf{P}^H(A)$ can be realized as the ideal completion of the Hoare ordered poset of finite antichains of A. □

Exercise IV-8.23. Let A be an algebraic domain with poset of compact elements $K(A)$.

(i) If K is a nonempty compact saturated set and U is a Scott open set containing K, show that there exists a finite set $F \subseteq K(A)$ such that $K \subseteq \uparrow F \subseteq U$. Conclude that K is the filtered intersection of all $\uparrow F$, F finite and contained in $K(A)$.
(ii) Show that $A \in \mathbf{P}^S(A)$ is a compact element iff $A = \uparrow F$ for some finite set $F \subseteq K(A)$.
(iii) Order the set of finite antichains of $K(A)$ by the *Smyth order*: $F_1 \leq F_2$ iff $F_2 \subseteq \uparrow(F_1)$. Show that the mapping $F \mapsto \uparrow F$ is an order isomorphism from the Smyth ordered poset of finite antichains onto the compact elements of $Q^*(A)$. Conclude (using IV- 8.10) that $\mathbf{P}^S(A)$ can be realized as the ideal completion of the Smyth ordered poset of finite antichains of A. □

Exercise IV-8.24. Let L be an algebraic domain with poset of compact elements $K(L)$.

(i) Deduce from IV-8.14 that every $(C, D) \in \mathbf{P}^P(L) \subseteq \mathbf{P}^H(L) \times \mathbf{P}^S(L)$ is the directed sup of the collection $\{(\downarrow F, \uparrow F)\colon F \text{ finite}, F \subseteq C \cap K(L), D \subseteq \uparrow F\}$.
(ii) Show that the compact elements of $\mathbf{P}^P(L)$ are precisely those of the form $(\downarrow F, \uparrow F)$ for F finite contained in $K(L)$. (Note that the compactness of these elements follows from the preceding two exercises.)
(iii) Call a nonempty finite subset of L an *extreme set* if it can be written as the union of two antichains. Order the set of extreme sets of $K(L)$ by the *Egli–Milner order*: $F_1 \leq F_2$ iff $F_2 \subseteq \uparrow F_1$ and $F_1 \subseteq \downarrow F_2$. Show that the mapping $F \mapsto (\downarrow F, \uparrow F)$ is an order isomorphism from the Egli–Milner ordered poset of finite extreme sets onto the compact elements of $\mathbf{P}^P(L)$. Conclude that $\mathbf{P}^P(A)$ can be realized as the ideal completion of the Egli–Milner ordered poset of extreme sets of L.

Hint. Observe that the inverse mapping in part (iii) sends $(\downarrow F, \uparrow F)$ to the union of the maximal and minimal elements of F, which is the union of two antichains. □

Exercise IV-8.25. Let L be a **dcpo** and let B be a nonempty subset of L such that b is the directed sup of $\twoheaddownarrow b \cap B$ for each $b \in B$. Show that $(B, \ll)$ is an abstract basis and $I \mapsto \sup I : \mathrm{RId}\, B \to M$ is an order isomorphism of domains, where M is the smallest subset of L containing B that is closed under directed sups.

Hint. Define M to be the image of the map $I \mapsto \sup I$ from $\mathrm{RId}\, B$; by hypothesis $B \subseteq M$. Let D be a directed set in M, and pick a rounded ideal I_d for each $d \in D$ such that $d = \sup I_d$. Using the roundedness of the ideals, one sees that $\bigcup_{d \in D} I_d$ is again a rounded ideal, and one also sees directly that its sup is $\sup D$. Thus M is closed under directed sups. It is clearly the smallest such set containing B. Since $I \mapsto \sup I$ has image M, it follows that B is a basis for M. The exercise now follows by applying Exercise III-4.16. □

Exercise IV-8.26. Show that, if L is a countably based domain, then so are the Hoare, Smyth, and Plotkin powerdomains.

Hint. Bases for these powerdomains can be constructed using the finite subsets of a given basis in L. □

New notes

Powerdomains are domain-theoretic versions of powersets in set theory and of hyperspaces in topology. Our approach has been to consider them first as free **dcpo**-algebras in certain equational theories (which also include inequalities) and then derive them as concrete objects represented by certain subsets of the domain. This free algebra approach was originally suggested in [Hennessy and Plotkin, 1979]. However, there are certainly other interesting powerdomains that have arisen by considering directly certain subsets of domains that again form a domain or other constructions; see, for example, the work of R. Heckmann [Heckmann, 1991a; Heckmann, 1991b; Heckmann, 1992a; Heckmann, 1992b; Heckmann, 1993a; Heckmann, 1993b]. One approach of Heckmann is to define abstractly what constitutes a powerdomain construction and build a theory upon the definition. Powerdomain theory can also be treated via domain theory in logical form; see, for example, Section 7.3 of [Abramsky and Jung, 1994].

IV-9 The Extended Probabilistic Power Domain

The extended probabilistic powerdomain may be viewed as a domain analogue of the space of regular Borel measures on a topological space that, among other things, provides a constructive framework for measure theory. One feature that distinguishes it from classical measure theory is its order theoretic structure. The principal goal of this section is to derive its existence. In our approach we replace measures by their "skeletons", valuations. We begin with their study.

We first consider a *lattice of subsets* of a given set X, a collection that is closed under finite unions and intersections and contains the empty set.

Definition IV-9.1. A *valuation* (on $\mathcal{L}$) is a function $\mu\colon \mathcal{L} \to [0, \infty]$ from a lattice of subsets $\mathcal{L}$ of X to the additive semigroup $[0, \infty]$ (with $\infty + t = \infty = t + \infty$) that is

(i) *strict* – $\mu(\emptyset) = 0$;
(ii) *monotone* – $V \subseteq U$ implies $\mu(V) \leq \mu(U)$;
(iii) *modular* – $\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V)$.

Property (iii) is also called the inclusion–exclusion principle. The valuation is called *finite* if $\mu(\mathcal{L}) \subseteq [0, \infty)$. □

Clearly, modularity together with strictness implies that a valuation is *finitely additive*:

$$\mu(U + V) = \mu(U) + \mu(V) \text{ whenever } U \cap V = \emptyset.$$

Conversely let $\mathcal{R}$ be a *Boolean ring of sets* (or *ring of sets* for short), i.e., a lattice of subsets of the powerset of X closed under relative complements:

$$U, V \in \mathcal{R} \Rightarrow U \backslash V \in \mathcal{R}.$$

Every strict and finitely additive function $\nu\colon \mathcal{R} \to [0, \infty]$ is a valuation; such functions are called *finitely additive measures*.

Given a lattice $\mathcal{L}$ of subsets of X, there is a smallest ring of sets containing $\mathcal{L}$:

Lemma IV-9.2. *Let L be a lattice of subsets of X.*

(i) *The ring $\mathcal{R}(\mathcal{L})$ generated by the lattice $\mathcal{L}$ consists of all finite unions*

$$R = \bigcup_{i=1}^{n} (U_i \backslash V_i)\,, \quad \text{with} \quad U_i, V_i \in \mathcal{L} \text{ for } i = 1, \ldots, n.$$

Moreover, one may suppose that $V_i \subseteq U_i$ for each i and that the sets $U_i \backslash V_i$ are pairwise disjoint.

(ii) *If $\mathcal{L}$ contains X as a member, then the collection of subsets $U \backslash V$, $U, V \in \mathcal{L}$, forms a semialgebra, that is, a collection closed under finite intersection, and the complement of any member is a disjoint union* *of finitely many members of the family. In this case the ring of sets is an algebra (closed under finite unions, finite intersections, and complements).*

Proof: The (straightforward and standard) proof we leave to the reader (see Exercise IV-9.26). □

It is a standard result, sometimes called the Smiley–Horn–Tarski Theorem, that a finite valuation on $\mathcal{L}$ extends uniquely to a finitely additive measure on the ring of sets $\mathcal{R}(\mathcal{L})$.

Proposition IV-9.3. *Let $\mathcal{L}$ be a lattice of subsets of X, and let μ be a finite valuation on $\mathcal{L}$. Then μ has a unique extension to a finitely additive finite measure on the ring of subsets generated by $\mathcal{L}$.*

Proof: We proceed with the extension in stages. First consider a set that can be represented in the form $U \backslash V$ for $U,\ V \in \mathcal{L},\ V \subseteq U$. We then define $\hat{\mu}(U \backslash V) = \mu(U) - \mu(V)$. Suppose that $U \backslash V = U_1 \backslash V_1$, where $U \subseteq U_1,\ V \subseteq V_1$. Then $U \cup V_1 = U_1$ and $U \cap V_1 = V$ and by modularity

$$\mu(U_1) + \mu(V) = \mu(U \cup V_1) + \mu(U \cap V_1) = \mu(U) + \mu(V_1),$$

and thus $\mu(U_1) - \mu(V_1) = \mu(U) - \mu(V)$. Thus $\hat{\mu}(U \backslash V)$ is well defined in this case. In general for $U' \backslash V' = U \backslash V$, note by the argument just given that $\mu(U') - \mu(V') = \mu(U_1) - \mu(V_1) = \mu(U) - \mu(V)$ for $U_1 = U \cup U'$ and $V_1 = V \cup V'$. As we want $\hat{\mu}$ to be additive and U is the disjoint union of V and $U \backslash V$, note that our choice of $\hat{\mu}$ in this case is the only one possible.

We next consider the case that the lattice $\mathcal{L}$ is finite. In this case for each $x \in \bigcup \mathcal{L}$, define $U_x = \bigcap\{U \in \mathcal{L} : x \in U\}$, $V_x = \bigcup\{V \in \mathcal{L} : x \notin V\}$ and $A_x = U_x \backslash V_x$. We note that A_x is the equivalence class of x with respect to the relation $x \sim y$ iff $x \in U \Leftrightarrow y \in U$ for all $U \in \mathcal{L}$, that these equivalence classes partition all members of $\mathcal{L}$, and that $\mathcal{R}(\mathcal{L})$ consists of all, necessarily disjoint, unions of these equivalence classes. (In lattice language the A_x form the atoms of the Boolean ring $\mathcal{R}(\mathcal{L})$.) By the preceding paragraph $\hat{\mu}$ is (uniquely) defined on all A_x by $\hat{\mu}(U_x \backslash V_x) = \mu(U_x) - \mu(U_x \cap V_x)$. There is then only one possible way to extend $\hat{\mu}$ to all of $\mathcal{R}(\mathcal{L})$ so that it is additive: namely $\hat{\mu}(A) = \sum\{\hat{\mu}(A_x)\colon A_x \subseteq A\}$, where the empty sum is taken to be 0. This clearly defines a finitely additive finite measure on $\mathcal{R}(\mathcal{L})$.

We show by induction on the number of equivalence classes contained in $U \in \mathcal{L}$ that $\hat{\mu}(U) = \mu(U)$ on members of $\mathcal{L}$. If U contains 0 equivalence classes, then $U = \emptyset$, which has value 0 for μ and $\hat{\mu}$. Suppose the equality is true for $n < k+1$ and that U contains $k+1$ equivalence classes. If $U = V \cup W$, where $V,\ W \in \mathcal{L}$ have strictly fewer equivalence classes, then the inductive hypothesis and the modularity yield the result. Otherwise there is a largest proper subset $V \in \mathcal{L}$ of U. It follows that $U \backslash V = A_x$ for any $x \in U \backslash V$, and thus that

$$\hat{\mu}(U) = \hat{\mu}(V) + \hat{\mu}(A_x) = \mu(V) + (\mu(U) - \mu(V)) = \mu(U).$$

For the general case note that the ring of subsets generated by any lattice $\mathcal{L}$ of subsets consists of the union of all the rings of subsets over the finite sublattices of $\mathcal{L}$, and by uniqueness of the extension in the finite case, we get a well-defined and unique extension on the union. □

Proposition IV-9.4. *Let* $\mu: \mathcal{L} \to [0, \infty]$ *be a valuation on a lattice* $\mathcal{L}$ *of subsets of* X *with* $X \in \mathcal{L}$. *Then* μ *has an extension to a finitely additive measure* $\nu: \mathcal{R}(\mathcal{L}) \to [0, \infty]$, *where* $\mathcal{R}(\mathcal{L})$ *is the smallest algebra of sets containing* $\mathcal{L}$.

Proof: We first consider the lattice $\mathcal{L}_0 = \{U \in \mathcal{L} : \mu(U) < \infty\}$ (note that it is a sublattice by modularity). By the previous proposition we can uniquely extend μ to a finitely additive finite measure $\hat{\mu}$ on the ring of subsets $\mathcal{R}_0$ generated by $\mathcal{L}_0$. Note that if there exists a μ-finite O in $\mathcal{L}$ such that $U \backslash V \subseteq O$, then $U \backslash V = (O \cap U) \backslash (O \cap U \cap V)$, which is in the ring $\mathcal{R}_0$. Furthermore, $\hat{\mu}(U \backslash V)$ must then equal $\mu(O \cap U) - \mu(O \cap U \cap V)$. In particular if $\mu(U) < \infty$, then for any $V \in \mathcal{L}$, $\hat{\mu}(U \backslash V) = \mu(U) - \mu(U \cap V)$.

We first define ν on all sets of the form $A = U \backslash V$. If A contains a *bad point*, a point for which every member of $\mathcal{L}$ containing the point is μ-infinite, we define $\nu(A) = \infty$. If A consists of only *good points*, points that have μ-finite members of $\mathcal{L}$ containing them, we define ν by

$$\nu(A) = \sup\{\hat{\mu}(W_1 \backslash V_1) \colon W_1, V_1 \in \mathcal{L}, (W_1 \backslash V_1) \subseteq A, \mu(W_1) < \infty\}.$$

If $W_1 \backslash V_1 \subseteq A = U \backslash V$, then $W_1 \backslash V_1 \subseteq W_1 \cap U \backslash V$. It thus follows (by taking $W = W_1 \cap U$) that we can alternatively define $\nu(A)$ by

$$\nu(U \backslash V) = \sup\{\hat{\mu}(W \backslash V) \colon W \subseteq U, \mu(W) < \infty\}. \tag{1}$$

The fact the $\hat{\mu}$ is independent of the representation of A follows from the first equality, while the second will be the more useful for our purposes.

Note that if $U \backslash V \in \mathcal{R}_0$, then, as above, we may assume without loss of generality that $V \subseteq U$ and $\mu(U) < \infty$. In this case it is easy to see that the supremum defining $\nu(U \backslash V)$ occurs and is given by $\hat{\mu}(U \backslash V) = \mu(U) - \mu(V)$. Thus ν extends $\hat{\mu}$ for these sets.

We next show additivity of ν on all sets of the form $U \backslash V$. Consider $A = U \backslash V = \bigvee_{i=1}^{n} (U_i \backslash V_i)$, a disjoint union, where without loss of generality $V, U, V_i, U_i \in \mathcal{L}, V \subset U$, and $V_i \subset U_i \subseteq U$ for each i. In the case where A contains a bad point, then some $U_i \backslash V_i$ must contain it and so

$\nu(U\backslash V) = \infty = \sum_{i=1}^{n} \nu(U_i\backslash V_i)$. Hence we assume that A contains only good points and note that

$$\begin{aligned}\nu(A) &= \sum_{i=1}^{n} \sup\{\hat{\mu}(W\backslash V_i)\colon W \subseteq U_i,\, \mu(W) < \infty\}\\ &= \sup\left\{\sum_{i=1}^{n} \hat{\mu}(W_i\backslash V_i)\colon W_i \subseteq U_i,\, \mu(W_i) < \infty\right\}\\ &= \sup\left\{\hat{\mu}\left(\bigcup_{i=1}^{n} W_i\backslash V_i\right)\colon W_i \subseteq U_i,\, \mu(W_i) < \infty\right\},\end{aligned}$$

where the last equality follows from the additivity of $\hat{\mu}$. We need to show that the right side is equal to $\nu(U\backslash V)$. To that end, let $W \subseteq U$, $\mu(W) < \infty$. For $W_i = W \cap U_i$, we have

$$W\backslash V = W \cap (U\backslash V) = W \cap \bigcup_{i=1}^{n}(U_i\backslash V_i) = \bigcup_{i=1}^{n}(W_i\backslash V_i).$$

Conversely if $W_1, \ldots, W_n$ are given such that $W_i \subseteq U_i$, $\mu(W_i) < \infty$ for each i, then for $W = \bigcup_{i=1}^{n} W_i \subseteq U$, we have

$$\bigcup_{i=1}^{n}(W_i\backslash V_i) = \bigcup_{i=1}^{n}(W_i \cap (U_i\backslash V_i)) = W \cap (U\backslash V) = W\backslash V.$$

Therefore we can conclude that

$$\begin{aligned}&\sup\left\{\hat{\mu}\left(\bigcup_{i=1}^{n}(W_i\backslash V_i)\right)\colon W \subseteq U_i,\, \mu(W) < \infty\right\}\\ &\qquad = \sup\{\hat{\mu}(W\backslash V)\colon W \subseteq U,\, \mu(W) < \infty\}.\end{aligned}$$

To complete the proof, one partitions a member of the algebra $\mathcal{R}(\mathcal{L})$ as a disjoint union of sets of the form $U\backslash V$ in two different ways, takes the refining partition of the two partitions consisting of all nonempty pairwise intersections, and then uses the preceding case to see that summing over either partition equals summing over the common refinement. Thus ν defined as the sum of the ν-measures of members of the partition is well defined. For additivity of ν, for disjoint sets A and B in the algebra $\mathcal{R}(\mathcal{L})$, one sees directly from the definition of ν that $\nu(A) + \nu(B) = \nu(A + B)$ by first partitioning A and B and combining the partitions to partition $A \cup B$. (More generally, the sets of the form $U\backslash V$, U, $V \in \mathcal{L}$, form a semialgebra. It is a standard elementary measure-theoretic fact that since ν is finitely additive on the semialgebra, which we established, it is finitely additive on the algebra it generates.)

We have already observed that ν extends $\hat{\mu}$ on sets of the form $U\backslash V$, and the finite additivity of both on $\mathcal{R}_0$ ensures that they agree on it also. $\square$

Remark. It follows from the uniqueness of $\hat{\mu}$, the definition of ν, and monotonicity that ν must be the smallest possible extension of μ to $\mathcal{R}(\mathcal{L})$ such that ν is infinite on sets containing bad points. We point out that for valuations with infinite value, there may in general be more than one extension to the generated algebra, but we will use exclusively the one defined in the preceding proposition.

Since we have now established that ν is an extension of $\hat{\mu}$, we henceforth label the extension ν by $\hat{\mu}$ also and no longer use the notation ν. □

Definition IV-9.5. A valuation $\mu\colon \mathcal{O}(X) \to [0, \infty]$ from the lattice of open sets of a topological space X is *continuous* if for any directed family $\mathcal{D}$ of open sets with union $U = \bigcup \mathcal{D}$, we have $\mu(U) = \sup\{\mu(V)\colon V \in \mathcal{D}\}$. □

Every regular Borel measure restricted to the lattice of open sets is a continuous valuation. Indeed for any open set U, by regularity its measure $\mu(U)$ can be approximated arbitrarily closely from below by $\mu(K)$ for compact sets $K \subseteq U$. Since the members of any directed family of open sets with union U will eventually contain any such K, their measures must converge to $\mu(U)$ from below. In the converse direction, although it is not always true that a continuous valuation μ on $\mathcal{O}(X)$ extends to a regular Borel measure, this can be established under quite general hypotheses.

For open sets U, V in a topological space X the set $U\backslash V = U \cap (X\backslash V)$ is called a *locally closed set*. The family of all locally closed sets forms a semialgebra (see Exercise IV-9.26), and their set of finite unions forms an algebra called the *ℓc-algebra*. (Motivated by considerations of algebraic geometry, members of the ℓc-algebra are called "globally quasiconstructible" subsets in [Grothendieck and Dieudonné, B1971].)

We have seen above that any valuation on $\mathcal{O}(X)$ can be extended to a finitely additive measure on the ℓc-algebra. In the next lemma we assume the setting of Proposition IV-9.4.

Lemma IV-9.6. *Let $\mu\colon \mathcal{O}(X) \to [0, \infty]$ be a continuous valuation on the lattice of open sets of a topological space X. Let $\mathcal{D}$ be a directed family of open sets with union U. Then for any locally closed set $U\backslash V$,*

$$\hat{\mu}(U\backslash V) = \sup\{\hat{\mu}(D\backslash V)\colon D \in \mathcal{D}\}.$$

Proof: The case that $U\backslash V$ contains a bad point is trivial. Thus we assume that $U\backslash V$ consists of good points. In this case by definition of the extension $\hat{\mu}$ to the ℓc-algebra, we have

$$\hat{\mu}(U\backslash V) = \sup\{\hat{\mu}(W\backslash V)\colon W \subseteq U, \mu(W) < \infty\}.$$

Consider any $W\backslash V$, where $\mu(W) < \infty$. Let $\varepsilon > 0$. Since μ is continuous and W is the directed union of $\{W \cap D\colon D \in \mathcal{D}\}$, we have for some $D \in \mathcal{D}$, $\mu(W) - \varepsilon < \mu(W \cap D)$. It follows that

$$\begin{aligned}\hat{\mu}(W\backslash V) - \varepsilon = \mu(W) - \mu(W \cap V) - \varepsilon &< \mu(W \cap D) - \mu(W \cap V)\\ &\le \mu(W \cap D) - \mu(W \cap V \cap D) = \hat{\mu}(W \cap D\backslash V) \le \hat{\mu}(D\backslash V).\end{aligned}$$

Since ε was arbitrary, we conclude that $\sup\{\hat{\mu}(D\backslash V)\colon D \in \mathcal{D}\} \ge \hat{\mu}(W \cap V)$. Since W was arbitrary as a μ-finite open subset of U, we conclude that

$$\hat{\mu}(U\backslash V) \le \sup\{\hat{\mu}(D\backslash V)\colon D \in \mathcal{D}\}.$$

Since the reverse inequality is obvious from monotonicity, we are done. □

Definition IV-9.7. For a topological space X, the *valuation powerdomain* (also called the *extended probabilistic powerdomain*) $\mathcal{V}(X)$ of X is the set of all continuous valuations on $\mathcal{O}(X)$ with the *pointwise order*, sometimes called the *stochastic order*: $\mu \le \nu$ iff $\mu(U) \le \nu(U)$ for all open sets U. The *probabilistic powerdomain* $\mathcal{P}(X)$ consists only of those continuous valuations μ with $\mu(X) \le 1$. □

Lemma IV-9.8. *Let X be a topological space. Then the valuation powerdomain and the probabilistic powerdomain are* **dcpo***s.*

Proof: It is straightforward to verify that for a directed family of continuous valuations (with $\mu(X) \le 1$), the pointwise sup is another such. □

The following elementary valuations play a basic role in the theory.

Definition IV-9.9. Let X be a topological space. A *point valuation* for $x \in X$ is defined by $\eta_x(U) = 1$ if $x \in U$ and $\eta_x(U) = 0$ if $x \notin U$. A finite linear sum $\xi = \sum_{b\in B} r_b\eta_b$, with $0 < r_b < \infty$ and $|B| < \infty$, and defined by $\xi(U) = \sum_{b\in U} r_b$, is called a *simple valuation*, and the set B is called its *support*. □

Remark. (i) One observes that point valuations and hence simple valuations are continuous, thus members of the valuation powerdomain. A simple valuation $\sum_{b\in B} r_b\eta_b$ is a member of the probabilistic powerdomain iff $\sum_{b\in B} r_b \le 1$. Also observe that a simple valuation admits an extension to a (finitely) additive measure on the powerset of X given by $\xi(A) = \sum_{b\in A\cap B} r_b$ for $\xi = \sum_{b\in B} r_b\eta_b$.

(ii) It is sometimes notationally convenient to allow "phantom" summands of the form $r_b\eta_b$ where $r_b = 0$. We identify any such finite linear sum with the simple valuation consisting of its nonzero terms. We identify all expressions with all phantom terms with the constant 0 valuation, which we also regard as a degenerate simple valuation with support Ø.

We now begin to work directly toward the principal goal of this section: the valuation domain of a domain is indeed a domain. We henceforth assume that X is a domain endowed with the Scott topology $\sigma(X)$, that $\mu\colon \sigma(X) \to [0, \infty]$ is some fixed continuous valuation, and will eventually show that μ is the directed supremum of a family of simple valuations each way below μ.

Definition IV-9.10. For a simple valuation $\xi = \sum_{b\in B} r_b\eta_b$ and a continuous valuation μ on X, a domain equipped with the Scott topology, we set $\xi \prec \mu$ if for all nonempty $K \subseteq B$, we have $\sum_{b\in K} r_b < \mu(\bigcup_{b\in K} \uparrow b)$. □

Lemma IV-9.11. *Let ξ and ζ be simple valuations and μ a continuous valuation on a domain X, and suppose $\zeta \le \xi \prec \mu$. Then*

(i) $\zeta \prec \mu$,
(ii) $\xi \ll \mu$,
(iii) *there exists $t > 1$ such that $t\xi \prec \mu$ (equivalently $\xi \prec (1/t)\mu$).*

Proof: Let $\zeta = \sum_{c\in C} s_c\eta_c$ and $\xi = \sum_{b\in B} r_b\eta_b$.

(i) Let $\phi \neq K \subseteq C$, and let $K' := B \cap \uparrow K$. Since $\uparrow K$ is saturated, hence an intersection of open sets, there exists some open set U containing $\uparrow K$ and missing the finite set $B\backslash\uparrow K = B\backslash K'$. Then

$$0 < \sum_{c\in K} s_c \le \zeta(U) \le \xi(U) = \sum_{b\in U\cap B} r_b$$
$$= \sum_{b\in K'} r_b < \mu\left(\bigcup_{b\in K'} \uparrow b\right) \le \mu\left(\bigcup_{c\in K} \uparrow c\right);$$

it follows that $\zeta \prec \mu$.

(ii) Let $D \subseteq \mathcal{V}(X)$ be directed with $\mu \le \sup D$, and let $K = U_0 \cap B \neq$ Ø for some open U_0. Then for each open set U with $U \cap B = K$,

$$\xi(U) = \sum_{b\in K} r_b < \mu\left(\bigcup_{b\in K} \uparrow b\right).$$

For $V := \bigcup_{b\in K} \uparrow b$, we have that there exists a $\sigma_K \in D$ such that $\sum_{b\in B} r_b < \sigma_K(V)$. It follows that $\xi(U) < \sigma_K(V) \le \sigma_K(U)$ for all U with $U \cap B = K$. Since B is finite, it has only finitely many nonempty subsets K, and thus in the directed set D there exists an upper bound σ of all the σ_K. We then have $\xi \le \sigma$ since for any open U with $K = B \cap U$,

$$\xi(U) = \sum_{b\in K} r_b \le \sigma_K(U) \le \sigma_(U).$$

(iii) Set

$$s := \min \left\{ \frac{\mu\left(\bigcup_{b\in K} \Uparrow b\right) - \sum_{b\in K} r_b}{\sum_{b\in K} r_b} : \emptyset \neq K \subseteq B \right\}$$

and $t = 1 + s/2$. A straightforward verification establishes (iii). $\square$

To construct a rich supply of simple valuations, we shall use ℓc-partitions, finite partitions of X where each member has the form $U \backslash V$ for some U, $V \in \sigma(X)$, the Scott topology on the domain X. Note that we will always assume that ℓc-partitions are *finite*.

Remark IV-9.12. *Any locally closed set belongs to an ℓc-partition, for example the one consisting of it and the finitely many disjoint locally closed sets making up its complement. Since locally closed sets are closed under finite intersections, two ℓc-partitions have a largest common refinement consisting of all pairwise nonempty intersections of members of the two partitions. Combining these two remarks, one sees that given any finite number of locally closed sets (in particular, open sets), there is an ℓc-partition such that each of the given locally closed sets is a disjoint union of members of the partition.* $\square$

As suggested by the Remark we can manipulate ℓc-partitions in a manner reminiscent of their use in elementary Riemann integration theory. And indeed our construction of simple valuations approximating a given continuous valuation has certain analogies to the construction of lower Darboux sums.

Definition IV-9.13. Let $\mathcal{P}$ be a partition of a domain X by locally closed sets, $0 < s < 1$, $M > 0$, and $\mu \in \mathcal{V}(X)$. We define a $(\mathcal{P}, s, M, \mu)$-*valuation* to be a simple valuation $\xi \prec \mu$ such that for all $A \in \mathcal{P}$,

$$\begin{aligned} s\hat{\mu}(A) &\leq \xi(A) \quad \text{in case } \hat{\mu}(A) < \infty, \\ M &\leq \xi(A) \quad \text{if } \hat{\mu}(A) = \infty. \end{aligned}$$ $\square$

We want both to compare and to construct simple valuations for given ℓc-partitions. We assume throughout that we are now working in a fixed domain X.

Lemma IV-9.14. *Suppose that ζ is a simple valuation with support B, $\zeta \prec s\mu$ for $0 < s < 1$, and $\zeta(X) \leq M$, $M > 0$. If ξ is a $(\mathcal{P}, s, M, \mu)$-valuation such that each $\Uparrow b$, $b \in B$, is a union of members of $\mathcal{P}$, then $\zeta \leq \xi$.*

Proof: We note for any nonempty $F \subseteq B$ that by hypothesis for $V := \bigcup_{b\in F} \Uparrow b$,

$$s\mu(V) = s\hat{\mu}\left(\bigcup\{P \in \mathcal{P}: P \subseteq V\}\right) = \sum_{\substack{P\in\mathcal{P}\\ P\subseteq V}} s\hat{\mu}(P) \leq \sum_{\substack{P\in\mathcal{P}\\ P\subseteq V}} \xi(P) = \xi(V),$$

provided V is μ-finite. Let $\zeta = \sum_{b \in B} r_b \eta_b$. For any open set $U \in \sigma(X)$, set $K = U \cap B$. Then

$$\zeta(U) = \sum_{b \in K} r_b < s\mu\left(\bigcup_{b \in K} \uparrow b\right) \leq \xi\left(\bigcup_{b \in K} \uparrow b\right) \leq \xi(U)$$

in case $\mu(\bigcup_{b \in B} \uparrow b)$ is finite; the other case is trivial. □

Next we give a key lemma concerning the existence of simple valuations with desired properties.

Lemma IV-9.15. *Let X be a domain, $\sigma(X)$ the lattice of Scott open sets, $\mu\colon \sigma(X) \to [0, \infty]$ a continuous valuation, $\mathcal{P}$ an ℓc-partition, $0 < s < 1$, and $M > 0$. Then there exists a $(\mathcal{P}, s, M, \mu)$-valuation ξ.*

Proof: Let $\hat{\mu}$ be the extension given by Proposition IV-9.4 of μ to a finitely additive measure on the whole ℓc-algebra.

We assume $\mathcal{P} = \{T_1, \ldots, T_n\}$ and temporarily fix some locally closed set $T_j = U \backslash V$. If $\hat{\mu}(T_j) = 0$, then we will choose no support points of ξ in T_j. If $U \backslash V$ contains a bad point y, then there exists $b \ll y$ such that $b \in U$ and then necessarily $b \notin V$. Let $b_j = b$ be a support point of ξ, the only one in T_j, with weight M. Note that $\mu(\uparrow b_j) = \infty$ since $y \in \uparrow b_j$.

The remaining case is that where all points of $T_j = U \backslash V$ are good points and $0 < \hat{\mu}(T_j)$. Pick r such that $s < r < 1$. By equation (1) in the proof of Proposition IV-9.4 there exists an open set $W \subseteq U$ such that $\mu(W) < \infty$ and

$$s\hat{\mu}(U \backslash V) < r\hat{\mu}(W \backslash V) \text{ if } \hat{\mu}(U \backslash V) < \infty,$$
$$M < r\hat{\mu}(W \backslash V) \text{ if } \hat{\mu}(U \backslash V) = \infty.$$

Since

$$W = \bigcup \left\{ \uparrow F \left(= \bigcup_{x \in F} \uparrow x \right) : F \subseteq W, |F| < \infty \right\}$$

and the union is directed, we have from Lemma IV-9.6 that

$$\hat{\mu}(W \backslash V) = \sup\{\hat{\mu}(\uparrow F \backslash V) \colon F \subseteq W, |F| < \infty\}.$$

Thus there exists a finite set $F \subseteq W$ such that

$$s\hat{\mu}(U \backslash V) < r\hat{\mu}(\uparrow F \backslash V) \text{ if } \hat{\mu}(U \backslash V) < \infty,$$
$$M < r\hat{\mu}(\uparrow F \backslash V) \text{ if } \hat{\mu}(U \backslash V) = \infty.$$

In either case for $F = \{x_1, \ldots, x_{m_j}\}$, let $b_{j,i} = x_i$, $1 \leq i \leq m_j$, be the support points of ξ in T_j with weight

$$r_{j,i} := r\hat{\mu}\left(\Uparrow x_i \Big\backslash \left(\bigcup_{k<i} \Uparrow x_k \cup V \right) \right).$$

(Note that some phantom points may arise at this stage.) Since the sets used to define the $r_{j,i}$ are pairwise disjoint and $\hat{\mu}$ is additive, we have

$$\xi(T_j) = \sum_{i=1}^{m_j} r_{j,i} = r\hat{\mu}\left(\bigcup_{i=1}^{m_j} \Uparrow x_i \Big\backslash V \right). \tag{1}$$

We carry out the outlined construction of support points as above for all T_j and suppose that T_j contains m_j support points. The simple valuation ξ is now defined by

$$\xi = \sum_{j=1}^{n} \sum_{i=1}^{m_j} r_{j,i} \eta_{b_{j,i}}.$$

If $0 < \hat{\mu}(T_j) < \infty$, we have

$$\begin{aligned} s\hat{\mu}(T_j) < r\hat{\mu}\left(\bigcup_{i=1}^{m_j} \Uparrow x_i \Big\backslash V \right) &= \sum_{i=1}^{m_j} r\hat{\mu}\left(\Uparrow x_i \Big\backslash \left(\bigcup_{k<i} \Uparrow x_k \cup V \right) \right) \\ &= \sum_{i=1}^{m_j} r_{j,i} = \xi(T_j). \end{aligned}$$

If $\hat{\mu}(T_j) = 0$, then trivially $s\hat{\mu}(T_j) \leq \xi(T_j)$.

In the case that all points of T_j are good and $\hat{\mu}(T_j) = \infty$, by the choice of the x_i and equation (1) above

$$M \leq r\hat{\mu}\left(\bigcup_{i=1}^{m_j} \Uparrow x_i \Big\backslash V \right) = \xi(T_j).$$

If T_k contains a bad point, then $M = \xi(T_k)$.

Finally we must verify that $\xi \prec \mu$. Let B denote the support of ξ. Let $K \subseteq B$ be nonempty and set $Q := \cup_{b \in K} \Uparrow b$. If $\mu(Q) = \infty$, then trivially $\sum_{b \in K} r_b < \mu(Q)$. So we assume $\mu(Q) < \infty$. Suppose that $\uparrow K \cap T_j \cap B \neq \emptyset$, where $T_j = U \backslash V$. Then

$$\begin{aligned} 0 < \sum_{b_{j,i} \in \uparrow K} r_{j,i} &= \sum_{x_i \in \uparrow K \cap T_j} r\hat{\mu}\left(\Uparrow x_i \Big\backslash \left(\bigcup_{j<i} \Uparrow x_j \cup V \right) \right) \\ &\leq r\hat{\mu}\left(\bigcup_{x_i \in \uparrow K \cap T_j} \Uparrow x_i \Big\backslash V \right) \leq r\hat{\mu}\left(\bigcup_{b \in K} \Uparrow b \cap T_j \right) = r\hat{\mu}(Q \cap T_j) < \hat{\mu}(Q \cap T_j). \end{aligned}$$

Using this inequality, we obtain

$$0 < \sum_{b\in K} r_b \le \sum_{x\in B\cap\uparrow K} r_x \le \sum_{j=1}^{n} \sum_{x\in B\cap\uparrow K\cap T_j} r_x < \sum_{k=1}^{n} \hat{\mu}(Q\cap T_j)$$
$$= \hat{\mu}(Q) = \mu(Q).$$

This completes the proof. □

We now have the machinery in place for our main theorem.

Theorem IV-9.16. *For a domain X the valuation powerdomain $\mathcal{V}$(X) is a domain. Each continuous valution μ is the directed supremum of the simple valuations way below it, and for simple valuations ξ, one has $\xi \ll \mu$ iff $\xi \prec \mu$.*

Proof: Let μ be a continuous valuation, let U be an open set, and let $\xi \prec \mu$ be a $(\mathcal{P}, s, M, \mu)$-valuation where U is a union of members of $\mathcal{P}$ (Lemma IV-9.15 and Remark IV-9.12). If $\mu(U)$ is μ-finite, then each of the members of $\mathcal{P}$ contained in U is μ-finite. Then $s\mu(U) = s\hat{\mu}(U) \le \xi(U)$ since this inequality holds for each partition member contained in U, and hence for U by finite additivity of $\hat{\mu}$ and ξ. If U is μ-infinite, then by finite additivity of $\hat{\mu}$ some partition element contained in U must be $\hat{\mu}$-infinite. Thus ξ applied to this partition member must have value greater than or equal to M and hence $\xi(U) \ge M$. Therefore we can approximate $\mu(U)$ as closely as desired by $\xi(U)$ by choosing s sufficiently close to 1 and M sufficiently large.

We next show that the collection $\{\xi\colon \xi \prec \mu\}$ is directed. Let $\xi_1, \xi_2 \prec \mu$. By Lemma IV-9.11(iii), there exist $0 < s_1, s_2 < 1$ such that $\xi_i \prec s_i\mu$ for $i = 1, 2$. Let $s = \max\{s_1, s_2\}$. Pick an ℓc-partition $\mathcal{P}$ such that each $\uparrow b$ is a finite union of partition members (Remark IV-9.11a), as b ranges over all support points for ξ_1 and ξ_2. Pick $M = \max\{\xi_1(X), \xi_2(X)\}$. Then by Lemma IV-9.14 any $(\mathcal{P}, s, M, \mu)$-valuation will be an upper bound for ξ_1 and ξ_2, and such valuations exist by Lemma IV-9.15.

The preceding two paragraphs establish that the set $\{\xi\colon \xi \prec \mu\}$ is a directed set with supremum μ. Since $\xi \ll \mu$ for each ξ in the set (Lemma IV-9.11(ii)), we conclude that the valuation domain is continuous, i.e., is a domain, and that each continuous valuation is a directed supremum of simple valuations way below it. Now suppose that ζ is a simple valuation and $\zeta \ll \mu$. Then $\zeta \le \xi$ for some $\xi \prec \mu$, since the latter form a directed set with supremum μ. This establishes the converse assertion and shows the equivalence of $\ll$ and $\prec$ for simple valuations. □

Corollary IV-9.17. *The probabilistic powerdomain of a domain X is a continuous domain, and each element is a directed supremum of simple valuations way below it. Furthermore, for a simple valuation $\xi = \sum_{b\in B} r_b\eta_b$, $\xi \ll \mu$ iff $\sum_{b\in K} r_b < \mu(\bigcup_{b\in K} \uparrow b)$ for all nonempty subsets $K \subseteq B$.* □

Proof: This follows immediately from the preceding theorem, since the probabilistic powerdomain is a Scott closed subset of the valuation powerdomain. □

We turn now to a precise characterization of the stochastic partial order and the way-below order on the simple valuations.

Proposition IV-9.18. (Splitting Lemma) *For two simple valuations in the valuation powerdomain $\mathcal{V}(X)$, X a T_0 space, we have $\zeta = \sum_{b\in B} r_b\eta_b \le \sum_{c\in C} s_c\eta_c = \xi$ if and only if there exist $\{t_{b,c} \in [0,\infty) : b \in B, c \in C\}$ such that for each $b \in B, c \in C$,*

$$\sum_{c\in C} t_{b,c} = r_b, \quad \sum_{b\in B} t_{b,c} \le s_c$$

and $t_{b,c} \ne 0$ implies $b \le c$.

Proof: Suppose the "splitting" condition holds. Let $U \in \mathcal{O}(X)$. Then $\zeta \le \xi$ since for $K := U \cap B$,

$$\begin{aligned}\zeta(U) = \sum_{b\in K} r_b = \sum_{b\in K}\sum_{c\in C} t_{b,c} = \sum_{b\in K}\sum_{c\in C\cap\uparrow K} t_{b,c}\\ \le \sum_{c\in C\cap\uparrow K}\sum_{b\le c} t_{b,c} \le \sum_{c\in C\cap\uparrow K} s_c = \mu(U).\end{aligned}$$

The proof of the converse depends on an application of a directed graph version of the Max-Flow, Min-Cut Theorem, which we recall. Let G be a finite directed graph with two distinguished vertices, a source α with only outgoing arrows and a sink ω with only incoming arrows. We associate with all (directed) edges a number belonging to $[0,\infty)$, called the *capacity* of the edge. (We may think of α as a water provider, ω as a water consumer, the edges as pipes that can carry up to their individual capacities of water, and the direction of the edge giving the direction of flow.) A *cut* of the directed, labeled graph G is a subset T containing the source α, but not the sink ω. We assign to each cut a number, namely the sum of all those capacities of edges that have one vertex in T and the other in its complement (we may geometrically visualize the cut as slicing through these edges). *A minimal cut* is one having a minimal value assigned to it. A *flow* is an assignment to each edge of a value from $[0, \infty)$ that (i) does not exceed the capacity of that edge, and (ii) has the property that the sum of the values flowing into any vertex, excluding the source and sink, is equal to the sum of the values flowing out. (We visualize the flow values as giving the amount of water flowing through the pipes for some stable water delivery scheme.) We assign to any flow a value, namely the sum of the values flowing out from the source (which will equal the sum flowing into the sink, a fact we don't need). The Max-Flow, Min-Cut Theorem then asserts that the value of the max-flow is equal to the value of the min-cut.

For the problem at hand, we construct a graph with vertices $B \cup C$, together with an added source α and sink ω. The directed edges are given by arrows from all points of α to all points $b \in B$ with capacity r_b, arrows from all points of $c \in C$ to ω with capacity s_c, and arrows from $b \in B$ to $c \in C$ iff $b \leq c$, each with capacity larger than $\sum_{b \in B} r_b$ (note that it is possible to have arrows that begin and end at the same point whenever $B \cap C \neq \emptyset$). Now a flow through the network represents a splitting of the $\{r_b \colon b \in B\}$, that is the flow along the arrow from b to c for $b \leq c$ gives the value $t_{b,c}$. If there is a flow that has value r_b along each arrow from α to b, $b \in B$, then this flow will give values $t_{b,c}$ which satisfy the conditions of the theorem, and this flow will clearly have maximal value.

Thus by the Max-Flow, Min-Cut Theorem it remains to show that the value of any cut is greater than or equal to the value of such a flow (which would have value $\sum_{b \in B} r_b$). Let T be any cut. We want to show that unless T consists of the source alone, we can find a cut of smaller value. If T were such that for some $b \in B$ and $c \in C$, $b \leq c$, and $b \in T$, but $c \notin T$, then the value of T would include the capacity of the edge joining b and c and would therefore be larger than $\sum_{b \in B} r_b$. So suppose that T is such a cut that has no such pairs b and c. Let $K := B \cap T$ and $K' = B \backslash K$. From our assumption we deduce that for all $b \in K$ and all $c \in \uparrow b \cap C$, we must have $c \in T$. But since ω is not in the cut T, we then have that the value of the cut is

$$\sum_{b \in K'} r_b + \sum_{c \in T} s_c.$$

As we have already seen, we can pick an open set U containing $\uparrow K$ such that $U \cap (B \cup C) \subseteq \uparrow K$. Then

$$\sum_{b \in K} r_b \leq \sum_{b \in \uparrow K} r_b = \zeta(U) \leq \xi(U) = \sum_{c \in \uparrow K \cap C} s_c \leq \sum_{c \in T} s_c.$$

Thus

$$\sum_{b \in B} r_b = \sum_{b \in K} r_b + \sum_{b \in K'} r_b \leq \sum_{c \in T} s_c + \sum_{b \in K'} r_b,$$

and so the value of the cut T is greater than or equal to $\sum_{b \in B} r_b$. Hence there is a flow with value $\sum_{b \in B} r_b$. $\square$

There is a modified version of the Splitting Lemma that characterizes the way-below relation.

Proposition IV-9.19. *For two simple valuations, we have*

$$\zeta = \sum_{b \in B} r_b \eta_b \ll \xi = \sum_{c \in C} s_c \eta_c$$

iff there exist $t_{b,c} \geq 0$ *such that*

$$\sum_{c \in C} t_{b,c} = r_b, \quad \sum_{b \in B} t_{b,c} < s_c$$

and $t_{b,c} \neq 0$ *implies* $b \ll c$*.*

Proof: Suppose $\zeta \ll \xi$. We consider the directed set $\sum_{c \in C}(s_c - \varepsilon)\eta_{c'}$ for any $c' \ll c$ and $\varepsilon > 0$ such that $\min_{c \in C} s_c > \varepsilon$; that it is directed and has supremum ξ follows directly from the definition of the stochastic order. Therefore we can find $c' \ll c$ and $\varepsilon > 0$ such that $\sum_{b \in B} r_b \eta_b \leq \sum_{c \in C}(s_c - \varepsilon)\eta_{c'}$. By the Splitting Lemma there exist $t_{b,c}$ such that

$$\sum_{c \in C} t_{b,c} = r_b, \quad \sum_{b \in B} t_{b,c} \leq s_c - \varepsilon$$

and such that $t_{b,c} \neq 0$ implies $b \leq c' \ll c$.

Conversely assume the conditions of the proposition for ζ and ξ. Let $A \subseteq B$. Then

$$\sum_{b \in A} r_b = \sum_{b \in A} \sum_{b \ll c} t_{b,c} \leq \sum_{c \in C \cap \Uparrow A} \sum_{b \ll c} t_{b,c} < \sum_{c \in C \cap \Uparrow A} s_c = \xi(\Uparrow A).$$

Thus $\zeta \prec \xi$ and by Theorem IV-9.16 $\zeta \ll \xi$. □

In order to present the universal property of the valuation domains, we introduce the concept of a **dcpo**-cone.

Definition IV-9.20. A **dcpo**-*cone* is a **dcpo** C equipped with a distinguished element $0 \in C$, an addition $+\colon C \times C \to C$, and a scalar multiplication $\cdot\colon \mathbb{R}^+ \times C \to C$ such that the usual axioms of a vector space hold, except for the existence of an additive inverse (in this case, one must also postulate that $0 \cdot a = 0$ for all $a \in C$). We assume further that addition and scalar multiplication are Scott-continuous functions from the Scott topology on the product into C. □

Lemma IV-9.21. *For X a domain, $\mathcal{V}(X)$ is a* **dcpo**-*cone. Furthermore, the function $\eta_X\colon X \to \mathcal{V}(X)$ defined by $\eta_X(x) = \eta_x$ is Scott-continuous (even an embedding, see Exercise* IV-9.28*).*

Proof: One verifies directly that sums and scalar products of continuous valuations are again continuous valuations. Since the operations are defined pointwise, all the appropriate algebraic laws of a cone are satisfied.

Let $\{r_\alpha\}$ be a directed net in $\mathbb{R}^+$ with supremum r and $\{\mu_\alpha\}$ and $\{\nu_\alpha\}$ be directed nets in $\mathcal{V}(X)$ with suprema μ and ν respectively. Then for any open set U in X, $(\mu_\alpha + \nu_\alpha)(U)$ and $r_\alpha\mu_\alpha(U)$ are directed sets with suprema $(\mu+\nu)(U)$ and $r\mu(U)$ respectively. Since the pointwise supremum of a directed set of continuous valuations is again a continuous valuation and the supremum of the collection, we have that addition and scalar multiplication are Scott-continuous.

That $x \mapsto \eta_x$ is Scott-continuous follows directly from the definition of the stochastic order. □

Lemma IV-9.22. *Let $\zeta = \sum_{b\in B} r_b\eta_b$ and $\xi = \sum_{c\in C} s_c\eta_c$ be two simple valuations on $\mathcal{O}(X)$, where X is a T_0-space. If ξ and ζ are distinct as linear combinations, then they are distinct as valuations.*

Proof: Suppose that there exists some $b \in B$ such that either $b \notin C$ or if $b = c \in C$, then $r_b \neq s_c$. If $\zeta = \xi\colon \mathcal{O}(X) \to \mathbb{R}^+$, then their extensions $\hat{\mu}$ and $\hat{\xi}$ to the ℓc-algebra will be equal, since by the Smiley–Horn–Tarski Theorem IV-9.3, the extensions exist and are unique. By the T_0-separation there exists a locally closed set A that contains b, but no other member of $B \cup C$ (for example, take $A = U\backslash V$, where $U = X\backslash{\downarrow}F$, where F consists of all members of $B\cup C$ strictly below x, and $V = X\backslash{\downarrow}x$). Then $\hat{\zeta}(A) = r_b \neq \hat{\xi}(A)$. □

We now characterize the valuation cone as the free **dcpo**-cone over X. We begin with a useful general lemma.

Lemma IV-9.23. *Let X be a domain, B a basis for X, Y a* **dcpo**, *and $f\colon B \to Y$ an order preserving function. Then there exists a Scott-continuus function $f_*\colon X \to Y$ that is the largest continuous function such that $f_*(b) \le f(b)$ for all $b \in B$. If for each $b \in B$, there exists a directed set $D_b \subseteq {\twoheaddownarrow}b \cap B$ with supremum b such that $f(b) = \sup\{f(d)\colon d \in D_b\}$, then f_* is a continuous extension of f.*

Proof: Define $f_*(x) = \sup\{f(b)\colon b \in B \cap {\twoheaddownarrow}x\}$. Since B is a basis the set $B \cap {\twoheaddownarrow}x$ is directed and has supremum x, and thus the supremum on the right is a directed supremum. Clearly $f_*(b) \le f(b)$ for all $b \in B$ and f_* is order preserving.

Let D be a directed set in X with supremum x. Then for any $b \in {\twoheaddownarrow}x \cap B$, there exists $d \in D$ such that $b \le d$. Now $f_*(x)$ is, by definition, the directed supremum of $f(B \cap {\twoheaddownarrow}x)$, and it follows that $f_*(x)$ must be the directed supremum of $f_*(D)$ as well. Thus f_* is Scott-continuous.

Let $g: X \to Y$ be Scott-continuous and $g(b) \leq f(b)$ for all $b \in B$. Then for any $x \in X$,

$$g(x) = \sup\{g(b) : b \in \ \twoheaddownarrow x \cap B\} \leq \sup\{f(b) : b \in \ \twoheaddownarrow x \cap B\} = f_*(x).$$

Suppose the last hypothesis of the lemma is satisfied. Then

$$f(b) = \sup\{f(d) : d \in D_b\} \leq \sup\{f(c) : c \in \ \twoheaddownarrow b \cap B\} = f_*(b).$$

Since we have seen that the reverse inequality always holds, we have $f(b) = f_*(b)$. Thus f_* extends f. □

Theorem IV-9.24. *Given any* **dcpo***-cone C and a continuous function $f: X \to C$, where X is a domain equipped with the Scott topology, there exists a unique continuous linear map $f^* : \mathcal{V}(X) \to C$ such that $f^* \eta_X = f$.*

Proof: For any simple valuation $\xi = \sum_{b \in B} r_b \eta_b$, we define $f^*(\xi) = \sum_{b \in B} r_b f(b)$. By Lemma IV-9.22 f^* is well defined, and it readily follows that it is linear on the subcone of simple valuations and satisfies $f^* \eta_X = f$.

We show that f^* is order preserving on the subcone of simple valuations. Let $\zeta = \sum_{b \in B} r_b \eta_b \leq \sum_{c \in C} s_c \eta_c = \xi$. Then by the Splitting Lemma there exist $\{t_{b,c} \in [0, \infty) : b \in B, c \in C\}$ such that for each $b \in B, c \in C$,

$$\zeta = \sum_{c \in C} t_{b,c} = r_b, \quad \xi = \sum_{b \in B} t_{b,c} \leq s_c$$

and $t_{b,c} \neq 0$ implies $b \leq c$. From the definition of f^*, the monotonicity of f, and the monotonicity of addition and scalar multiplication in the codomain, we have

$$\begin{aligned} f^*(\zeta) &= \sum_{b \in B} r_b f(b) = \sum_{b \in B} \sum_{c \in C} t_{b,c} f(b) \\ &\leq \sum_{c \in C} \sum_{b \in B} t_{b,c} f(c) \leq \sum_{c \in C} s_c f(c) = f^*(\xi). \end{aligned}$$

We now extend f^* to all of $\mathcal{V}(X)$ by

$$f^*(\mu) := \sup\{f^*(\xi) : \xi \ll \mu, \xi \text{ is simple}\}.$$

Any $\mu \in \mathcal{V}(X)$ is a directed supremum of the set D_μ of simple valuations way below it by Theorem IV-9.16, so the set used to define $f^*(\mu)$ is directed, and hence the supremum exists. By the preceding lemma f^* is continuous. We verify that it is an extension. Let $\xi = \sum_{c \in C} s_c \eta_c$ be a simple valuation. We consider the directed set $\sum_{c \in C} (s_c - \varepsilon) \eta_{c'}$ (constructed in IV-9.19) for any $c' \ll c$ and $\varepsilon > 0$ such that $\min_{c \in C} s_c > \varepsilon$; that it is directed and has supremum ξ follows directly from the definition of the stochastic order. Then $f^*(\sum_{c\in C}(s_c - \varepsilon)\eta_{c'}) = \sum_{c\in C}(s_c - \varepsilon)f(c')$, and the latter has directed supremum $f(\xi) = \sum_{c\in C} s_c f(c)$ by the Scott continuity of f and the Scott continuity of the addition and scalar multiplication operators in C. Thus by the preceding lemma the extension of f^* to all of $\mathcal{V}(X)$ is continuous and agrees with the old f^* on the subcone of simple valuations. Then finally the Scott continuity of f^* and of the addition and scalar multiplication on C ensures that the extension of f^* to all of $\mathcal{V}(X)$ is linear with respect to the operations of addition and scalar multiplication. □

Exercises

Exercise IV-9.25. Let $\mu : \mathcal{L} \to [0, \infty)$ be a valuation on a lattice of subsets of X. Show for $A, B, C \in \mathcal{L}$ that

$$\begin{aligned}\mu(A \cup B \cup C) = \mu(A) + \mu(B) + \mu(C) - \mu(A \cap B) - \mu(A \cap C)\\ -\mu(B \cap C) + \mu(A \cap B \cap C)\end{aligned}$$

and that higher order versions of the inclusion–exclusion principle are true. Show that if the minus terms in the preceding equation are moved to the left, then the equation also holds for $\mu: \mathcal{L} \to [0, \infty]$. □

Exercise IV-9.26. Let X be a topological space. Verify that the locally closed sets $U\backslash V, U, V \in \mathcal{O}(X)$ form a semialgebra and that the smallest algebra of sets containing the open sets is the family of all sets that can be written as a finite (disjoint) union of locally closed sets.

Hint. Note that $(U_1\backslash V_1) \cap (U_2 \cap V_2) = (U_1 \cap U_2)\backslash(V_1 \cup V_2)$ and $X\backslash(U\backslash V) = (X\backslash(U \cup V)) \cup (V\backslash U)$. Also $\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n (A_i\backslash(\bigcup_{j<i} A_j))$, where the A_i are members of the ℓc-algebra. □

Exercise IV-9.27. (i) Show that the construction of the valuation powerdomain is functorial on the category of T_0-spaces and continuous maps, where $\mathcal{V}f: \mathcal{V}(X) \to \mathcal{V}(Y)$ is defined by $\mathcal{V}f(\mu)(U) = \mu(f^{-1}(U))$ for an open set U.

(ii) Show that the functor $\mathcal{V}$ restricted to the category *DCPO* (with each **dcpo** equipped with the Scott topology) is locally continuous.

Hint. For part (ii), use the continuity of μ to show for a directed family of continuous functions $\{f_i\}$ with supremum f that $\mathcal{V}(\sup\{(f_i)\}(\mu)(U) = \sup\{\mathcal{V}(f_i)\}(\mu)(U)$. □

Exercise IV-9.28. Show that the map $x \mapsto \eta_x$ from a domain $(X, \sigma(X))$ to $(\mathcal{V}(X), \sigma(\mathcal{V}(X)))$ is a homeomorphism onto its image. □

Exercise IV-9.29. Show that if X is a countably based domain, then the valuation domain $\mathcal{V}(X)$ is also.

Hint. Let B be a countable base for X. Show that the simple valutions with support contained in B and range contained in the rationals form a countable base for $\mathcal{V}(X)$. □

Exercise IV-9.30. For a **dcpo** $X_\perp$ with bottom element, we define the *normalized probabilistic powerdomain* to be all continuous valuations that have the value 1 on the whole **dcpo**. This powerdomain is closely connected with the probabilistic powerdomain on X, the **dcpo** with the bottom element removed. For any continuous valuation $\mu \leq 1$ on X, extend it to one on $X_\perp$ by defining $\mu(X_\perp) = 1$, and conversely simply restrict a probabilistic valuation on the open sets of $X_\perp$ to the open sets of X. Show that this correspondence gives an order isomorphism of powerdomains. Conclude that the normalized probabilistic powerdomain of a domain with bottom is a domain. Characterize the orders $\leq$ and $\ll$ in the normalized probabilistic powerdomain (see [Edalat, 1995b]). □

New notes

The probabilistic powerdomain was introduced by C. Jones [Jones, D1989] and Jones and Plotkin [Jones and Plotkin, 1989]. It was generalized to the extended probabilistic powerdomain or valuation powerdomain by O. Kirch [Kirch, D1993]. Jones proved the Splitting Lemma and showed that the probabilistic powerdomain of a domain is again a (continuous) domain, and Kirch made the nontrivial extension of these results to the extended probabilistic power domain. We have borrowed heavily from their methods in this section. Besides its semantic applications, the probabilistic powerdomain has been used in [Edalat, 1995b] as a computational approach to a general Riemann integral.

The universal property (see IV-9.24) for the probabilistic powerdomain is due to [Jones and Plotkin, 1989], that for the extended probabilistic powerdomain to [Tix, D1995]. The notion of a **dcpo**-cone has been investigated in [Tix, D1999], [Tix, 2001].

It has been shown by [Jung and Tix, 1998] that, for Lawson compact domains, the probabilistic powerdomain is Lawson compact. Even for finite domains it is not known whether the probabilistic powerdomain is an *FS*-domain except for very special domains like trees, or trees turned upside down (see [Jung and Tix, 1998]).

We have not included any of the numerous results on extending continuous valuations to Borel measures; see [Lawson, 1982], [Norberg, 1989], [Alvarez-Manilla *et al.*, 1998], [Alvarez-Manilla, D2000].

V

Spectral Theory of Continuous Lattices

Spectral theory plays an important and well-known role in such areas as the theory of commutative rings, lattices, and of C^*-algebras, for example. The general idea is to define a notion of "prime element" (more often: ideal element) and then to endow the set of these primes with a topology. This topological space is called the "spectrum" of the structure. One then seeks to find how algebraic properties of the original structure are reflected in the topological properties of the spectrum; in addition, it is often possible to obtain a representation of the given structure in a concrete and natural fashion from the spectrum.

By means of the spectral theory of this chapter we associate with every complete lattice L a topological space, denoted by Spec L, and a representation $L \to \mathcal{O}(\mathrm{Spec}\ L)$ of the given lattice into the lattice of open subsets of the spectrum. Frequently one reduces the spectral theory in other mathematical contexts (such as those listed above) to this lattice theoretical spectral theory by considering a distinguished lattice of subobjects and identifying the spectrum of this lattice with the spectrum of the original structure in a natural way. Since the lattice of open sets of a topological space is a frame, it should be noted that a spectral representation can be an *isomorphism* only if L itself is a frame.

The chapter begins with an important lemma (frequently referred to as "The Lemma") which plays a vital role in the spectral theory of continuous lattices. It states that a "finitely prime" element of a continuous lattice is also "compactly prime" with respect to the Lawson topology. Section V-2 then resumes the theme of order generation begun in Section I-3, where it was shown that in a continuous lattice the set of irreducible elements is order generating in the sense that every element is an inf of irreducibles (see I-3.9 ff.). The investigation is expanded here to *topologically generating* sets – subsets for which the whole lattice is the smallest *closed* subsemilattice containing the set. We show that the closure of the set of nonidentity irreducibles is the unique smallest closed order generating subset of a continuous lattice as well as being the unique smallest closed topologically generating subset of the lattice. In particular, for a distributive continuous lattice, the closure of the set of nonidentity primes is the unique smallest closed order generating subset.

This line is further pursued in Section V-3, where we identify the closure of the primes in a distributive continuous lattice as being exactly the pseudoprimes of I-3, or, as they are also recognized here, the set of weak primes. Analogs results are also obtained for the weak irreducibles in a general continuous lattice.

In Section V-4 begins the principal topic of the chapter, the spectral theory of frames in which the primes order generate. We give the set of nonidentity primes the hull–kernel topology and call this space the spectrum; the given lattice is isomorphic to the lattice of open subsets of this space. In this fashion we record the duality between the category of frames with points and lattice morphisms preserving arbitrary sups on one hand, and sober spaces and continuous maps on the other. This prepares the way for the specific spectral theory of continuous lattices discussed in Section V-5: the spectrum of a continuous lattice is locally compact and sober, and all locally compact sober spaces are so obtained. The category of continuous frames is dual to the category of locally compact sober spaces and continuous maps.

Unlike the previous chapters, lattices are predominant in this chapter on primes and spectral theory. More general **dcpo**s play a marginal role only.

We collect a good deal of supplementary information in the exercises. Several important applications of the spectral theory of continuous lattices will be given in Section VI-7 and Chapter VII, when more information on the topological algebra of continuous lattices will be available.

V-1 The Lemma

Let us recall that an element p of a lattice L is called *irreducible*, if the relation $a \wedge b = p$ always implies $a = p$ or $b = p$. The element p is called *prime*, if $a \wedge b \leq p$ always implies $a \leq p$ or $b \leq p$. (See Section I-3, especially Definitions I-3.5 and I-3.11.)

These definitions can be rephrased in the following way: p is irreducible, resp. prime, if $\inf F = p$, resp. $\inf F \leq p$, implies $p \in F$, resp. $p \in \uparrow F$, for every finite nonempty subset F of L. In the presence of a topology on L, one could define p to be *strongly irreducible*, resp. *strongly prime*, if $\inf K = p$, resp. $\inf K \leq p$, implies $p \in K$, resp. $p \in \uparrow K$, for every nonempty *compact* subset K of L. Since compactness is often a kind of substitute for finiteness, one may conjecture that these strengthened notions of irreducibility and primality are in reality identical with the first ones. We shall prove this conjecture in the case of *continuous* lattices. The following lemma is crucial, and we give two versions.

Theorem V-1.1. (The Lemma) *Let L and M be complete lattices and let i: $L \to M$ be Scott-continuous.*

(i) *Suppose that p is a prime element of M and that $A \subseteq L$ is such that* $\inf i(A) \leq p$. *Then there is an ultrafilter $\mathcal{U}$ on A with $i(\underline{\lim}\,\mathcal{U}) \leq p$.*
(ii) *Suppose that the map i also preserves arbitrary infs. Suppose that p is an irreducible element of M and that $A \subseteq L$ is such that* $\inf i(A) = p$. *Then there is an ultrafilter $\mathcal{U}$ on A with $i(\underline{\lim}\,\mathcal{U}) = p$.*

Proof: (i) Let A and p satisfy the hypotheses of (i) and define $\mathcal{I}$ to be the set of all subsets B of A with $\inf i(B) \not\leq p$. Then we can assert:

(a) $A \notin \mathcal{I}$;
(b) if $C \subseteq B \in \mathcal{I}$, then $C \in \mathcal{I}$;
(c) if $B \in \mathcal{I}$ and $C \in \mathcal{I}$, then $B \cup C \in \mathcal{I}$.

Clauses (a) and (b) are immediate. For (c), note that if $B \in \mathcal{I}$ and $C \in \mathcal{I}$, then $\inf i(B) \not\leq p$ and $\inf i(C) \not\leq p$, whence we see that $\inf i(B \cup C) = \inf i(B) \wedge \inf i(C) \not\leq p$ by the primality of p. Thus, $\mathcal{I}$ is a proper ideal of subsets of A. But then there is an ultrafilter $\mathcal{U}$ on A disjoint from $\mathcal{I}$ (by I-3.20 and its preceding Remark). The latter means that $\inf i(B) \leq p$ for all $B \in U$. We conclude that

$$\begin{aligned} i(\underline{\lim}\,\mathcal{U}) &= i(\sup\{\inf B \colon B \in \mathcal{U}\}) = \sup\{i(\inf B) \colon B \in U\} \\ &\leq \sup\{\inf i(B) \colon B \in \mathcal{U}\} \leq p, \end{aligned}$$

where we have only used the fact that i preserves directed sups.

(ii) The proof is the same as that of (i), if one replaces everywhere $\not\leq$ by $>$, and $\leq$ *by* $=$. (One needs that i preserves arbitrary infs to replace the first $\leq$ by $=$ in the last equation.) □

If the set A contains $\underline{\lim}\,\mathcal{U}$ for every ultrafilter $\mathcal{U}$ on A, we may conclude under the hypotheses of V-1.1(i), resp. (ii), that $\inf i(A) \leq p$, resp. $\inf i(A) = p$, implies $p \in \uparrow i(A)$, resp. $p \in i(A)$. Since in a continuous lattice $\underline{\lim}\,\mathcal{U}$ is the topological limit of $\mathcal{U}$ with respect to the Lawson topology (III-3.17), we have proved

Corollary V-1.2. *Let L be continuous and M be complete. If $i \colon L \to M$ is Scott-continuous and $K \subseteq L$ is compact in the Lawson topology, then*

(i) *if p is prime in M, then* $\inf i(K) \leq p$ *implies $p \in \uparrow i(K)$;*
(ii) *if p is irreducible in M and, in addition, i preserves arbitrary infs, then* $\inf i(K) = p$ *implies $p \in i(K)$.* □

If we specialize V-1.1 to the case where $L = M$ and i is the identity map, then we obtain

Corollary V-1.3. *Let L be a complete lattice.*

(i) *If p is prime in L, then on every subset A of L with* $\inf A \le p$ *there is an ultrafilter* $\mathcal{U}$ *such that* $\underline{\lim}\,\mathcal{U} \le p$.
(ii) *If p is irreducible in L, then on every subset A of L with* $\inf A = p$ *there is an ultrafilter* $\mathcal{U}$ *such that* $\underline{\lim}\,\mathcal{U} = p$. □

If L is a continuous lattice, then the set of all liminfs of ultrafilters on A is just the closure A^- of A with respect to the Lawson topology (III-3.17) . Thus V-1.3 implies

Corollary V-1.4. *Let L be a continuous lattice.*

(i) *If p is prime in L, then* $\inf A \le p$ *implies* $p \in \uparrow(A^-)$ *for every nonempty subset A of L.*
(ii) *If p is irreducible in L, then* $\inf A = p$ *implies* $p \in A^-$ *for every nonempty subset A of L.* □

We can now state the result that we promised.

Theorem V-1.5. *Let L be a continuous lattice. Then for every nonempty subset K of L which is compact in the Lawson topology we have:*

(i) *if p is prime in L, then* $\inf K \le p$ *implies* $p \in \uparrow K$*;*
(ii) *if p is irreducible in L, then* $\inf K = p$ *implies* $p \in K$. □

Applying The Lemma on primes, we next complete a theme begun in Exercise I-3.39, where we started to investigate the relationship between completely distributive lattices (I-2.8, I-2.9, I-3.16, I-3.39) and domains.

Lemma V-1.6. *Let L be a completely distributive lattice and* $p \neq 0$ *a co-prime. Then* $\twoheaddownarrow p \cap P$ *is directed, where P denotes the set of nonzero co-primes.*

Proof: Let q, r be two nonzero co-primes with $q \ll p$ and $r \ll p$. Then $\twoheaduparrow q \cap \twoheaduparrow r$ is a Scott (hence, Lawson) open neighborhood of $\uparrow p$. But in a completely distributive lattice we have $\lambda(L) = \lambda(L^{\mathrm{op}})$. (This follows immediately from the fact that L can be embedded into a cube $[0, 1]^X$ for some X under a map preserving arbitrary sups and infs: see IV-3.32; alternatively refer to Proposition VII-2.10.) From I-3.39 we know then that $p = \sup(\twoheaddownarrow p \cap P)$, whence $(\twoheaddownarrow p \cap P)^- \cap \uparrow p \neq \emptyset$, with closure taken with respect to $\lambda(L^{\mathrm{op}}) = \lambda(L)$ by an application of V-1.4 to L^{op}. Then we find $(\twoheaddownarrow p \cap P) \cap \twoheaduparrow q \cap \twoheaduparrow r \neq \emptyset$, since $\twoheaduparrow q \cap \twoheaduparrow r$ is an open neighborhood of $\uparrow p$. □

We now are ready for the following theorem.

Theorem V-1.7. *If L is a completely distributive lattice, then the poset P of nonzero co-primes in the induced order is a domain. Dually, if Q is the set of primes $p < 1$, then Q^{op} is a domain.* □

Proof: Take I-3.39 and note that Lemma V-1.6 above shows that $\Downarrow_P p$ is directed for $p \in P$. For the dual statement apply the preceding result to L^{op}. □

Lemma V-1.8. *If L is completely distributive and P the poset of nonzero co-primes, and if U is an open filter in P, then $\uparrow U$ is an open prime filter in L.*

Proof: Let $v \in \uparrow U$; then $u \le v$ with $u \in U$; then by I-3.39 and V-1.6 above, there is a $u^* \in \Downarrow u \cap P$ with $u^* \in U$; but then $u^* \ll v$. Moreover, $\uparrow U$ is a prime filter. For if $a \vee b \in \uparrow U$, then $a \vee b \ge p$ for some $p \in U$, whence $a \ge p$ or $b \ge p$, as p is co-prime. □

We can now throw additional light on the Lawson duality which we discussed in Section IV-2.

Proposition V-1.9. *Let L be a completely distributive lattice. Let Q be the domain of primes different from* 1 *with the partial order induced from $\ge$ and let P be the domain of nonzero co-primes with the partial order induced from $\le$. Then P and Q are duals of each other in the sense of Lawson duality of* IV-2.14.

Proof: Let U be an open filter of P. Then $\uparrow U$ is an open prime filter in L (see V-1.8). Let $q_U = \max(L \setminus \uparrow U)$. Then $q_U \in Q$. Conversely if $q \in Q$, let $U_q = P \setminus \downarrow q$; then U_q is an open filter in P. The maps $U \mapsto q_U : \mathrm{OFilt}\, P \to Q$ and $q \mapsto U_q : Q \to \mathrm{OFilt}\, P$ are inverses of each other. Hence $Q \cong \mathrm{OFilt}\, P$. In the light of Lawson duality, this gives the assertion. □

The preceding proposition sheds a new light on Lawson duality for domains: For a domain L the Scott open sets form a completely distributive lattice $\sigma(L)$. Thus, also the Scott closed sets form a completely distributive lattice $\gamma(L)$ ordered by inclusion. There is an obvious order embedding $a \mapsto \downarrow a$ of the domain L into $\gamma(L)$ mapping L onto the set of co-primes of $\gamma(L)$, i.e., L may be viewed as the domain of co-primes of the completely distributive lattice $\gamma(L)$. The Lawson dual of L may be viewed as the domain of prime elements of $\gamma(L)$, i.e., the set of Scott closed subsets B of L, whose complement is a filter, ordered by $\supseteq$.

Exercises

Exercise V-1.10. Give an independent proof of Corollary V-1.2(i) under the following weaker primality condition on p: For every nonempty finite subset F of L, $\inf i(F) \leq p$ implies $p \in {\uparrow} i(F)$. $\square$

Exercise V-1.11. Let K be a compact convex subset of a locally convex topological vector space, and denote by $\mathrm{Con}(K)$ the lattice of all closed convex subsets of K. Recall that $\mathrm{Con}(K)^{\mathrm{op}}$ is a continuous lattice (see Example I-1.23). Prove that if $A \subseteq K$ has the property that its closed convex hull is equal to K, then A^- contains all extreme points of K (see I-3.36). $\square$

Exercise V-1.12. Let X be a compact Hausdorff space. Show that for every closed prime ideal I of the ring $C(X)$ of real- or complex-valued continuous functions on X, there is an element $x \in X$ such that $I = \{f \in C(X) : f(x) = 0\}$.

Hint. Consider X as a subset of $\mathcal{O}(X)$ via the embedding $x \mapsto X\backslash\{x\}$ and use for i the map $U \mapsto \{f \in C(X) : f(x) = 0 \text{ for all } x \notin U\}$ from the lattice $L = \mathcal{O}(X)$ of all open subsets of X to the lattice M of all closed ideals of $C(X)$. $\square$

Exercise V-1.13. Prove the following.

(i) Let $i\colon L \to M$ be a morphism in *INF*$^{\uparrow}$. If $p < 1$ is a prime in M and $x \in L$ satisfies $i(x) \leq p$, then there is an irreducible $q \in L$ with $x \leq q$ and $i(q) \leq p$.

(ii) The conclusion also holds for L a **dcpo** semilattice and M any poset (where $p \in M$ is prime iff $M\backslash{\downarrow}p$ is a filter in M), provided that $i^{-1}(U)$ is an open filter for any open filter U of M.

Hint. Pick q maximal in ${\uparrow}x\backslash i^{-1}(M\backslash{\downarrow}p)$. $\square$

Old notes

The Lemma (V-1.1) is an abstract version of the so-called Jónsson Lemma which plays an important role in universal algebra [Jónsson, 1967]. Its relevance in continuous lattices in the form of V-1.2, V-1.4, V-1.5 was discovered by Gierz and Keimel [Gierz and Keimel, 1976]. From that latter paper we have also drawn the exercises; V-1.11 and V-1.12 are well-known theorems in analysis. The same paper contains more material on the uses of *The Lemma*; in particular one obtains a characterization of the extreme points in the dual unit ball of semicontinuous function spaces originally due to [Cunningham and Roy, 1974]. The results on irreducibles contained in V-1.4 and V-1.5 are from [Hofmann and Lawson, 1976]. The duality of domains and completely distributive lattices in V-1.6–V-1.9 is due to [Lawson, 1979]. Finally, Exercise V-1.13 is from Hofmann, Keimel, and Watkins [scs 51; scs 52].

V-2 Order Generation and Topological Generation

A subset X of a lattice L has been called *order generating* (I-3.8), if every element of L is the inf of a subset of X. It has been proved (I-3.10) that, in a continuous lattice L, the set IRR L of all irreducible elements of L is order generating. In an algebraic lattice, the set Irr L of completely irreducible elements is the (unique) smallest order generating subset (I-4.26). But in general a continuous lattice does not have any minimal order generating subset. In the unit interval [0, 1] every order dense subset is order generating, but there is no minimal subset of this type. For this reason we restrict our attention for the moment to order generating sets which are closed with respect to the Lawson topology. As a consequence of The Lemma of Section V-1 we obtain

Theorem V-2.1. *Among the order generating subsets of a continuous lattice L which are closed with respect to the Lawson topology there is a unique smallest one: the closure* $(\mathrm{IRR}\, L\backslash\{1\})^-$ *of the set of irreducible elements* <1 *in L.*

Proof: By I-3.10, $(\mathrm{IRR}\, L\backslash\{1\})^-$ is order generating. Let X be any Lawson closed order generating subset of L. Then V-1.5(ii) implies that X contains every irreducible element; whence, $(\mathrm{IRR}\, L\backslash\{1\})^- \subseteq X$. □

In a topological semilattice another notion of generation is natural:

Definition V-2.2. A subset X of a topological semilattice L is said to be *topologically generating* if the smallest closed subsemilattice of L containing X and 1 is L itself. □

As a matter of convention in a continuous lattice topological generation is always understood with respect to the Lawson topology. In compact semilattices, and in particular in continuous lattices, topological generation is weaker than order generation:

Proposition V-2.3. *Let L be a continuous lattice. Then every order generating subset of L is topologically generating with respect to the Lawson topology.*

Proof: Let X be an order generating subset and T the smallest closed subsemilattice of L containing X and 1. Then T is closed under arbitrary infs by virtue of III-1.12(2). Thus for all $x \in L$, we have $x = \inf(\uparrow x \cap X) \in T$, and consequently $L = T$. □

The preceding proposition remains true for arbitrary compact semilattices (see VI-2.9 below) and, in fact, for a complete lattice with a compact topology in which $\uparrow x$ and $\downarrow x$ are always closed (cf. O-4.4 and VI-1.3 below). The converse of Proposition V-2.3, however, is false. In the unit square $[0, 1] \times [0, 1]$ with the usual order, the set $[0, 1[\times [0, 1[$ is topologically but not order generating.

How does topological generation work in continuous lattices? Let X be any subset of a continuous lattice L. If we assign to every filter $\mathcal{F}$ on X the element $\underline{\lim}\,\mathcal{F}$ of L, we obtain a map preserving arbitrary infs and directed sups from the lattice $F = \mathrm{Filt}\, 2^X$ of all filters on X into L. (This is nothing but the theorem that the lattice F is the free continuous lattice on the set X (I-4.19).) By III-1.8 and III-3.17, the image of this map is the smallest *closed* subsemilattice containing X. In other words, we obtain the smallest closed subsemilattice containing X by taking the set of all directed sups in the set of all infs of subsets of X. In particular, L is topologically generated by X if every element of L is a directed sup of infs of subsets of X. This remark leads us to

Proposition V-2.4. *Let L be a continuous lattice.*

(i) *A subset X is topologically generating if and only if its closure X^- with respect to the Lawson topology is order generating.*
(ii) *Among the Lawson closed topologically generating subsets of L there is a (unique) smallest one: the closure* $(\mathrm{IRR}\, L\backslash\{1\})^-$ *of the set of irreducible elements* <1.

Proof: (i) From V-2.3 and V-2.2, we deduce "if". In order to prove "only if" we let Y be the set of all infs of subsets of X^-. We use III-1.12 to show that Y is a closed subsemilattice. Firstly, Y is clearly closed under the formation of all infs. We must show that for every directed subset $D \subseteq Y$, we have $\sup D \in Y$. For this purpose set $s = \sup D$ and $t = \inf(\uparrow s \cap X^-) \in Y$; then $s \le t$, and we must now show $t \le s$. In view of I-1.6 it suffices to show $w \le s$ for any $w \ll t$: Now $d = \inf(\uparrow d \cap X^-)$ since $d \in Y$, and so

$$\bigcap_{d\in D}(\uparrow d \cap X^-) = \left(\bigcap_{d\in D} \uparrow d\right) \cap X^- = \uparrow s \cap X^- \subseteq \uparrow t.$$

Thus the intersection of the filter basis of the $\lambda(L)$ compact sets $\uparrow d \cap X^-$ is contained in the open neighborhood $\twoheaduparrow w$ of $\uparrow t$. Hence there is some $c \in D$ with

$\uparrow c \cap X^- \subseteq \uparrow w$, whence $c = \inf(\uparrow c \cap X^-) \geq w$, and since $c \leq s$ we have indeed $w \leq s$.

If we now assume that X is topologically generating, then $Y \cup \{1\} = L$. This means that X^- is order generating.

(ii) Use (i) and V-2.1 above. □

Exercises

Exercise V-2.5. Let L be an algebraic lattice endowed with its Lawson topology.

(i) Show that $(\mathrm{Irr}\, L)^-$ is the smallest closed order generating and topologically generating set, where Irr L denotes the set of completely irreducible elements (see I-4.21).
(ii) If Y is a topologically generating subset of L, show that every compact element of L is the inf of a subset of Y. □

Exercise V-2.6. Let L be a continuous lattice and Id L its ideal lattice. Consider the adjunction $i = (x \mapsto \downarrow x)$ and $r = (I \mapsto \sup I)$ between L and Id L as in O-3.15.

(i) If X is order generating in L, show that $i(X)$ is topologically generating in Id L.
(ii) If Y is topologically generating in Id L, show that $r(Y)$ is order generating in L.

Hint. Use I-1.10 and V-2.5(ii) above. □

Exercise V-2.7. Let L be a complete lattice for which IRR L is order generating. Show that among the subsets of L which are order generating and closed with respect to the liminf topology (see Section III-3) there is a smallest one. (Compare Theorem V-2.1.) □

Hint. Use III-3.15 and V-1.3(ii) above. □

Exercise V-2.8. Let L and M be complete lattices and $f\colon L \to M$ a surjective *UPS*-map preserving finite infs. Show that IRR $M \subseteq f(\mathrm{IRR}\, L)$.

Hint. For $p \in \mathrm{IRR}\, M$ pick q maximal in $f^{-1}(p)$. Show that q is irreducible. □

Old notes

R. Jamison [Jamison, D1974] gave a slightly restricted version of Theorem V-2.1 in his dissertation. His interest was in abstract theories of convexities. In his context subsemilattices were "convex" subsets, and from this viewpoint irreducibles become "extreme points". Theorem V-2.1 then becomes an analogue to the Krein–Milman theorem: the closed convex hull of the extreme points (that is, the smallest closed subsemilattice containing the irreducibles) is all of L, and conversely any closed set whose closed convex hull is all of L contains the extreme points.

The treatment given in this section is essentially that of [Hofmann and Lawson, 1976].

V-3 Weak Irreducibles and Weakly Prime Elements

Throughout this section L denotes a continuous lattice in the Lawson topology. In the preceding section we characterized the closure of the set of irreducible elements as being the smallest closed order generating subset of L. In this section we characterize the individual elements of $(\mathrm{IRR}\ L)^-$. The distributive case is of particular interest in this regard; here $(\mathrm{IRR}\ L)^-(= (\mathrm{PRIME}\ L)^-)$ consists precisely of the pseudoprime elements already introduced in Chapter I. (Indeed the notion of a pseudoprime element arose in the context of characterizing those distributive continuous lattices for which the set of prime elements is closed.)

Definition V-3.1. An element p of L is called *weakly irreducible* if for any finite family $X_1, X_2, \dots, X_n$ of subsets of L, the relation $p \in \mathrm{int}(X_1X_2\dots X_n)$ implies $p \in X_k^-$ for some k. We call *p weakly prime* if for any finite family $X_1, X_2, \dots, X_n$ of subsets of L, the relation $p \in \mathrm{int}{\uparrow}(X_1X_2\dots X_n)$ implies $p \in ({\uparrow}X_k)^-$ for some k. We denote by WIRR L and WPRIME L the sets of weakly irreducible and weakly prime elements of L, respectively. □

In the definition, the notation $X_1X_2\dots X_n$ stands for the pointwise product, that is, the pointwise inf, of the sets. Recall that an element p of L is called *pseudoprime* if p is the sup of a prime ideal of L (see Definition I-3.24). The set of pseudoprimes is denoted by ΨPRIME L. One easily verifies that every irreducible element is weakly irreducible and that every prime element is weakly prime; that is, we have IRR $L \subseteq$ WIRR L, and PRIME $L \subseteq$ WPRIME L. We already have seen that PRIME $L \subseteq \Psi$PRIME L in the remarks following I-3.24.

Lemma V-3.2. *The sets* WIRR L *and* WPRIME L *are closed.*

Proof: As the proofs in the two cases are similar, we only consider $W = \mathrm{WIRR}\, L$. Let $p \in W^-$. We want to show $p \in W$. So we take an arbitrary finite family $X_1, \ldots, X_n$ of subsets of L such that $p \in \operatorname{int} X_1 \ldots Xn$. Pick any net $(p_j)_{j \in J}$ in W such that $p = \lim_j p_j$. Then we can find a j_0 such that $p_j \in \operatorname{int} X_1 \ldots X_n$ for all $j \geq j_0$. As all the p_j are weakly irreducible, we conclude that $p_j \in X_1^- \cup \cdots \cup X_n^-$ for all $j \geq j_0$. But this implies $p = \lim_j p_j \in X_1^- \cup \cdots \cup X_n^-$; that is, p is weakly irreducible. □

Definition V-3.1 makes sense in every semilattice endowed with a topology and Lemma V-3.2 remains true in general. But in order to show that the elements in the closure of IRR L are *exactly* the weakly irreducible elements, we have to use our standing hypothesis that L is a continuous lattice:

Proposition V-3.3. *In a continuous lattice L, we have* $(\mathrm{IRR}\, L)^- = \mathrm{WIRR}\, L$.

Proof: Suppose that there is an element $p \in L \backslash (\mathrm{IRR}\, L)^-$. Then every $x \in (\mathrm{IRR}\, L)^-$ has a neighborhood $U(x)$ not containing p. By III-2.15, we may suppose that all the $U(x)$ are closed subsemilattices of L. As the interiors of the $U(x)$ cover the compact space $(\mathrm{IRR}\, L)^-$, we find a finite subcover of compact subsemilattices $X_1, \ldots, X_n$ which do not contain p but where $\mathrm{IRR}\, L \subseteq X_1 \cup \cdots \cup X_n$.

By adjoining the identity to each of the X_k if necessary, we can assert that the pointwise product $X_1 \ldots X_n$ is a closed subsemilattice of L containing IRR L. As IRR L is order generating in L (I-3.10), we conclude that $L = X_1 \ldots X_n$. In particular, $p \in \operatorname{int} X_1 \ldots X_n$. As $p \notin X_k$ for each k, we conclude that $p \notin \mathrm{WIRR}\, L$. □

We do not have an order theoretical characterization of weakly irreducible elements. However, for weakly prime elements such a characterization is possible. Note that condition (2) in the following theorem is the primality condition of I-3.25(2), and that condition (3) is analogs to V-1.5.

Proposition V-3.4. *For an element p of a continuous lattice L the following conditions are equivalent:*

(1) *p is weakly prime;*
(2) *for any nonempty finite set of elements $x_1, \ldots, x_n$ in L, the relation $x_1 \wedge \cdots \wedge x_n \ll p$ implies $x_k \leq p$ for some k;*
(3) *for any nonempty compact subset K of L, the relation* $\inf K \ll p$ *implies $p \in \uparrow K$.*

Proof: (3) implies (2): Immediate.

(2) implies (1): Suppose that $p \in \operatorname{int}\uparrow(X_1 \dots X_n)$. Then there is an $x \ll p$ with $x \in \uparrow(X_1 \dots X_n)$ by III-1.6(i) and II-1.10(i). Thus $x \geq x_1 \wedge \cdots \wedge x_n$ with suitable $x_k \in X_k$. Now (2) implies $x_k \leq p$ for some k; whence, we find $p \in \uparrow X_k \subseteq (\uparrow X_k)^-$.

(1) implies (3): Assume that $x \not\leq p$ for all $x \in K$. As L is a continuous lattice, for each $x \in K$ we find an element $u(x) \ll x$ with $u(x) \not\leq p$. As $\uparrow u(x)$ is a neighborhood of x, the compactness of K implies the existence of finitely many elements $x_1, \dots, x_n$ in K such that $K \subseteq \uparrow u(x_1) \cup \cdots \cup \uparrow u(x_n)$. From the relation $\inf K \ll p$ we conclude $p \in \operatorname{int}\uparrow\inf K \subseteq \operatorname{int}\uparrow(u(x_1) \wedge \cdots \wedge u(x_n))$; whence, $p \in (\uparrow u(x_k))^- = \uparrow u(x_k)$ for some k by (1). But this is a contradiction to the choice of $u(x) \not\leq p$. □

In the preceding proposition condition (2) cannot be strengthened to $x_1 \wedge \cdots \wedge x_n \ll p$ implies $x_k \ll p$ for some k. Indeed, in the unit square $L = [0, 1] \times [0, 1]$ the element $p = (1, 1)$ is prime; but for $x_1 = (1, 0)$ and $x_2 = (0, 1)$, we have $x_1 \wedge x_2 \ll p$ and neither $x_1 \ll p$ nor $x_2 \ll p$.

In I-3.25 it had been shown if a continuous lattice that every pseudoprime element satisfies condition (2) of the preceding proposition and that (2) characterizes pseudoprimes for distributive continuous lattices. Thus:

Corollary V-3.5. $\Psi\mathrm{PRIME}\,L \subseteq \mathrm{WPRIME}\,L$ *and if L is distributive, then equality holds.* □

Condition (3) in Proposition V-3.4 allows us to conclude that every weakly prime element is weakly irreducible:

Proposition V-3.6. *For a continuous lattice,* $\mathrm{WPRIME}\,L \subseteq \mathrm{WIRR}\,L$.

Proof: Let p be weakly prime. For every $t \ll p$, we have

$$t = \inf(\uparrow t \cap \mathrm{WIRR}\,L),$$

as $\mathrm{IRR}\,L \subseteq \mathrm{WIRR}\,L$ and as $\mathrm{IRR}\,L$ is order generating (I-3.10). Because we know $\uparrow t \cap \mathrm{WIRR}\,L$ is compact by V-3.2, we can find an element $p_t \in \mathrm{WIRR}\,L$ such that $t \leq p_t \leq p$ by V-3.4(3). Because in the Lawson topology $p = \lim(t)_{t \ll p}$, we conclude that $p = \lim(p_t)_{t \ll p}$. As $p_t \in \mathrm{WIRR}\,L$ and as $\mathrm{WIRR}\,L$ is closed, we get $p \in \mathrm{WIRR}\,L$. □

The containment relations between the different kinds of irreducible and prime elements in a *continuous lattice* are summarized in the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Irr}\,L \subseteq & \mathrm{IRR}\,L & \subseteq & (\mathrm{IRR}\,L)^- & = & \mathrm{WIRR}\,L \\
& \cup\| & & & & \cup\| \\
\{1\} \subseteq & \mathrm{PRIME}\,L & \subseteq & \Psi\mathrm{PRIME}\,L & \subseteq & \mathrm{WPRIME}\,L
\end{array}$$

In the distributive case one has IRR L = PRIME L by I-3.12 and ΨPRIME L = WPRIME L by I-3.25 and V-3.4. Thus, the containment diagram simplifies considerably for *distributive continuous lattices* L:

$$\begin{array}{ccccccc} \mathrm{Irr}\,L \subseteq \mathrm{IRR}\,L & \subseteq & (\mathrm{IRR}\,L)^- & = & & & \mathrm{WIRR}\,L \\ \| & & \| & & & & \| \\ \{1\} \subseteq \mathrm{PRIME}\,L & \subseteq & (\mathrm{PRIME}\,L)^- & = & \Psi\mathrm{PRIME}\,L & = & \mathrm{WPRIME}\,L \end{array}$$

In particular, the pseudoprimes are exactly the elements in the closure of the set of primes. As we had characterized in I-3.28 and I-4.8 the distributive continuous lattices and, in particular, the algebraic lattices for which every pseudoprime element is prime, we obtain the following criterion for the closedness of PRIME L.

Proposition V-3.7.

(i) *In a distributive continuous lattice L the set* PRIME L *of prime elements of L is closed iff the relation* $\ll$ *on L is multiplicative, that is, iff L is stably continuous.*

(ii) *In a distributive algebraic lattice L the set* PRIME L *of prime elements is closed iff L is arithmetic (that is, if the compact elements form a sublattice of L).* □

In the nondistributive case we do not know similar characterizations of those continuous lattices L in which IRR L or PRIME L is closed.

Example V-3.8.

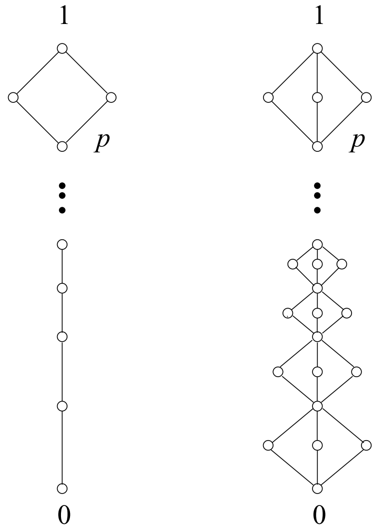
(1) The left hand figure below indicates a distributive continuous lattice L with a pseudoprime element p which is not prime. Thus, PRIME L is not closed; indeed, PRIME L is just $L\backslash\{p\}$ in this example. The prime ideal P with $p = \sup P$ is $P = \{x\colon x < p\}$.

(2) The right hand figure above indicates a nondistributive continuous lattice with no prime element except 1 but with a pseudoprime element p different from 1. Again $P = \{x\colon x < p\}$ is a prime ideal with $\sup P = p$.

Exercises

Exercise V-3.9. Let L be a continuous lattice in which the set PRIME L of prime elements is topologically generating. Show that L is distributive if and only if the set $T = \{x \in L : x = \inf(\uparrow x \cap \mathrm{PRIME}\,L)\}$ is a sublattice of L. □

Hint. See [Hofmann and Lawson, 1976]. □



Exercise V-3.10. Let K be a compact convex set in a locally convex topological vector space with the property that the set of extreme points is dense in K. (Such K exist.) Show that in the lattice $L = \mathrm{Con}\,(K)^{\mathrm{op}}$ the set PRIME L is topologically generating but that L is not distributive except in the case where K is a singleton.

Hint. Cf. I-1.23, I-3.36 and Exercise V-1.11. □

Exercise V-3.11. Show that for a continuous lattice L the following condition is equivalent to the conditions (1)–(4) of I-3.38:
(5) WIRR L = WPRIME L. □

Note that the conditions in Exercises V-3.11 and I-3.38 represent a kind of weak distributivity; indeed, a lattice is distributive if and only if $a_1 \wedge a_2 \leq x$ implies $(a_1 \vee x) \wedge (a_2 \vee x) = x$. Note also that the equivalent conditions (1)–(5) in Exercise V-3.11 and I-3.38 are satisfied, whenever the set PRIME L of prime elements is topologically generating. Thus Exercise V-3.10 gives an example of a nondistributive continuous lattice which is weakly distributive in the sense of V-3.11.

Old notes

Most of the material contained in this section is due to [Hofmann and Lawson, 1976]. Pseudoprimes appear in a second paper of [Hofmann and Lawson, 1978], sec. 8, after having been motivated in an SCS Memo of Keimel and

Mislove [scs 19], where Proposition V-3.7 appears. More on the subject of Exercise V-3.11 can be found in the 1976 paper of Hofmann and Lawson mentioned above. The same paper also contains results on the so-called spanning dimension of a continuous lattice L: this is the least cardinal α such that L is topologically generated by a set which is the union of α nonempty chains. The relations of this and similar dimensional concepts with width and breadth of L and IRR L are discussed there in detail. These notions of breadth and width in compact semilattices have also been studied in [Baker and Stralka, 1970] and [Lea, 1972].

V-4 Sober Spaces and Complete Lattices

In this section we assign to every complete lattice a sober topological space and to every topological space a certain distributive complete lattice. These assignments are functorial and establish a dual equivalence between the category of all sober spaces and a certain category of distributive complete lattices, namely primally generated frames. In the next section we specialize to distributive continuous lattices, that is, continuous frames.

On the level of objects, the essential results can be summarized as follows: a complete lattice L can be represented as the lattice of open subsets of a topological space iff L is order generated by its prime elements. In particular, these lattices are frames. A topological space X is homeomorphic to the space Spec L of nonunit prime elements endowed with the hull–kernel topology of some lattice L iff X is sober.

Definition V-4.1. For a complete lattice L we denote by Spec L the set of all prime elements of L different from 1; that is,

$$\operatorname{Spec} L = \operatorname{PRIME} L \backslash \{1\}.$$

For every $a \in L$, let

$$\nabla_L(a) = \{p \in \operatorname{Spec} L : a \leq p\} = \uparrow a \cap \operatorname{Spec} L$$

be the *hull* of a, and let

$$\Delta_L(a) = \operatorname{Spec} L \backslash \nabla_L(a) = \operatorname{Spec} L \backslash \uparrow a$$

be the complement of the hull of a in Spec L. □

Proposition V-4.2. *We have for all $X \subseteq L$ and all finite $F \subseteq L$:*

(i) $\Delta_L(0) = \emptyset$, $\qquad\nabla_L(0) = \operatorname{Spec} L$;
(ii) $\Delta_L(1) = \operatorname{Spec} L$, $\qquad\nabla_L(1) = \emptyset$;

(iii) $\bigcup\{\Delta_L(a)\colon a \in X\} = \Delta_L(\sup X)$, $\quad\bigcap\{\nabla_L(a)\colon a \in X\} = \nabla_L(\sup X)$;
(iv) $\bigcap\{\Delta_L(a)\colon a \in F\} = \Delta_L(\inf F)$, $\quad\bigcup\{\nabla_L(a)\colon a \in F\} = \nabla_L(\inf F)$.

Proof: Parts (i) and (ii) are clear. For (iii) it suffices to remark that the condition $p \in \bigcap\{\nabla_L(a)\colon a \in X\}$ means $p \geq a$ for all $a \in X$; this in turn is equivalent to saying that $p \geq \sup X$; that is, $p \in \nabla_L(\sup X)$. The first equation is proved similarly. For (iv) we note that $p \in \nabla_L(\inf F)$ means $p \geq \inf F$; as F is finite and as p is prime, this is equivalent to $p \geq a$ for some $a \in F$, that is, $p \in \bigcup\{\nabla_L(a)\colon a \in F\}$. The first equation is similar. $\square$

From V-4.2 we conclude that the sets of the form $\nabla_L(a)$ for $a \in L$ are exactly the closed sets and the sets of the form $\Delta_L(a)$ for $a \in L$ are exactly the open sets of one and the same topology on Spec L.

Definition V-4.3. The *hull–kernel topology* on Spec L is defined to be the topology the open sets of which are the sets of the form $\Delta_L(a)$ for $a \in L$. $\square$

From now on, Spec L will always be endowed with this topology; that is, $\mathcal{O}(\text{Spec } L) = \{\Delta_L(a)\colon a \in L\}$. Recall that the lower topology $\omega(L)$ on L is defined to have the principal filters $\uparrow a$ for $a \in L$ as subbasic closed sets (III-1.1). As $\nabla_L(a) = \uparrow a \cap \text{Spec } L$ and as these sets are the closed sets of the hull–kernel topology, we conclude that the hull–kernel topology on Spec L is the subspace topology induced from the lower topology $\omega(L)$ on L. It follows that the hull–kernel topology is coarser than the topology induced from the Lawson topology on L (see III-1.5).

Remark. The assignment $a \mapsto \Delta_L(a) : L \to \mathcal{O}(\text{Spec } L)$ is surjective and preserves arbitrary sups and finite infs (V-4.2); the map is bijective if and only if Spec L is order generating in L. Indeed, since $X = \nabla_L(\inf X)$ iff X is a hull–kernel closed subset of Spec L, then ∇_L is injective iff $a = \inf \nabla_L(a)$ for all $a \in L$ iff Spec L is order generating. In this case, the inverse of Δ_L is given by

$$U \mapsto \inf(\text{Spec } L \backslash U) : \mathcal{O}(\text{Spec } L) \to L.$$

It is important to recall at this point the notion of a sober space and that of an irreducible closed set (see O-5.6 and O-5.6).

Proposition V-4.4. *For every complete lattice L, the space* Spec L *is sober.*

Proof: It is obvious that $\{p\}^- = \nabla_L(p)$ for every $p \in \text{Spec } L$. Now, if p, q are elements in Spec L with $\{p\}^- = \{q\}^-$, then $p \leq q$ and $q \leq p$. Thus, we have proved Spec L is a T_0-space.

Now let A be any nonempty irreducible closed subset of Spec L. We show that $A = \{p\}^-$ for some $p \in \operatorname{Spec} L$. Since A is closed, $A = \nabla_L(a)$ for some $a \in L$. Let $p = \inf \nabla_L(a)$. Then $p \neq 1$ and $\nabla_L(a) = \nabla_L(p)$. We wish to show that p is prime.

Suppose that $b \wedge c \leq p$. If $x \in A$, then $b \wedge c \leq p \leq x$. Since x is prime either $b \leq x$ or $c \leq x$; that is, $x \in \uparrow b \cup \uparrow c$. Thus, $A \subseteq \nabla_L(b) \cup \Delta_L(c)$. Since A is irreducible, either $A \subseteq \uparrow b$ or $A \subseteq \uparrow c$. Hence, either $b \leq p$ or $c \leq p$. $\square$

We now have assigned to every complete lattice L a topological space Spec L. In order to make this assignment functorial, we consider a map $\varphi\colon L \to M$ of complete lattices *preserving arbitrary sups and finite infs* (cf. O-3.24). (Note that the preservation of finite infs includes the property $\varphi(1) = 1$.) The upper adjoint (recall O-3.1, O-3.5) $\tau = (y \mapsto \max \varphi^{-1}(\downarrow y)) : M \to L$ has the fundamental property

$$\tau(b) \geq a \text{ iff } b \geq \varphi(a) \text{ for all } a \in L \text{ and all } b \in M,$$

which is equivalent to

$$\tau^{-1}(\uparrow a) = \uparrow \varphi(a) \text{ for all } a \in L.$$

We note

Lemma V-4.5. *The adjoint τ maps* Spec M *into* Spec L.

Proof: By IV-1.22, $\tau(\text{PRIME } M) \subset \text{PRIME } L$. Let $p \in \operatorname{Spec} M$. We have $\tau(p) \neq 1$; indeed $\tau(p) = 1$ would imply $p \geq \varphi(1) = 1$; that is, $p = 1$. Hence $\tau(\operatorname{Spec} M) \subset \operatorname{Spec} L$. $\square$

We therefore denote by

$$\operatorname{Spec} \varphi\colon \operatorname{Spec} M \to \operatorname{Spec} L$$

the restriction and co-restriction of $\tau\colon M \to L$. The following formula clearly implies that Spec φ is continuous with respect to the hull–kernel topologies on Spec M and Spec L.

Lemma V-4.6. $(\operatorname{Spec} \varphi)^{-1}(\Delta_L(a)) = \Delta_M(\varphi(a))$ *for all $a \in L$.*

Proof:

$$\begin{aligned}(\operatorname{Spec} \varphi)^{-1}(\Delta_L(a)) &= \tau^{-1}(\operatorname{Spec} L \backslash \uparrow a) \cap \operatorname{Spec} M \\ &= \operatorname{Spec} M \backslash \tau^{-1}(\uparrow a) = \operatorname{Spec} M \backslash \uparrow \varphi(a) = \Delta_M(\varphi(a)).\end{aligned}$$ $\square$

If φ has an upper adjoint τ and φ' an upper adjoint τ', then $\tau' \circ \tau$ is the upper adjoint of $\varphi \circ \varphi'$. Thus $\operatorname{Spec}(\varphi \circ \varphi') = \operatorname{Spec} \varphi' \circ \operatorname{Spec} \varphi$. This and the preceding remarks show that we have indeed a functor

$$\operatorname{Spec}\colon SUP^{\wedge} \to TOP^{\mathrm{op}}$$

Here *TOP* is the category of all topological T_0-spaces and all continuous maps (cf. O-5.1) and $SUP^{\wedge}$ is the category of all complete lattices and all maps between them preserving arbitrary sups and finite infs (see IV-1.23).

There is an obvious functor the other way around,

$$\mathcal{O}\colon TOP^{\mathrm{op}} \to SUP^{\wedge},$$

where to every topological space X we assign its lattice $\mathcal{O}(X)$ of open subsets, and for every continuous function $f\colon X \to Y$ we assign the function

$$\mathcal{O}(f) = (U \mapsto f^{-1}(U))\colon \mathcal{O}(Y) \to \mathcal{O}(X),$$

which clearly preserves arbitrary unions and finite intersections and, hence, is a morphism in the category $SUP^{\wedge}$.

For an arbitrary topological space X, we now consider the set Spec $\mathcal{O}(X)$ of all prime elements $U \neq X$ of the lattice $\mathcal{O}(X)$ endowed with the hull–kernel topology. Since for every $x \in X$ the open set $X\backslash\{x\}^-$ is prime in $\mathcal{O}(X)$, we may define a function

$$\xi_X = (x \mapsto X\backslash\{x\}^-)\colon X \to \operatorname{Spec} \mathcal{O}(X).$$

Note that $U = \xi_X^{-1}(\Delta_{\mathcal{O}(X)}(U))$ for every open subset U of X; in particular, we find that $\xi_X\colon X \to \operatorname{Spec} \mathcal{O}(X)$ is continuous. Indeed,

$$\begin{aligned} x \in \xi_X^{-1}\big(\Delta_{\mathcal{O}(X)}(U)\big) \quad &\text{iff} \quad \xi_X(x) \in \Delta_{\mathcal{O}(X)}(U) \\ &\text{iff} \quad U \not\subseteq \xi_X(x) = X\backslash\{x\}^- \\ &\text{iff} \quad x \in U. \end{aligned}$$

Note also that $\xi_X(U) = \Delta_{\mathcal{O}(X)}(U) \cap \xi_X(X)$ for every open subset U of X; in particular, ξ_X is an open map onto its image. Furthermore ξ_X is injective iff $\{x\}^- = \{y\}^-$ implies $x = y$; that is, iff X is a T_0-space. And in this case ξ_X is an embedding. Finally, ξ_X is surjective iff every prime element of $\mathcal{O}(X)$ can be written in the form $X\backslash\{x\}^-$ for some $x \in X$, which is equivalent to saying that every irreducible closed subset of X is of the form $\{x\}^-$ for some $x \in X$. We thus conclude that ξ_X is bijective iff X is a sober space, and that in this case ξ_X is a homeomorphism.

Now we formulate the main result of this section. In the following *SOB* denotes the full subcategory of *TOP* whose objects are the sober topological spaces, and FRM_0 the full subcategory of $SUP^{\wedge}$ whose objects are the complete lattices in which the prime elements are order generating. Note that the category FRM_0 is a full subcategory of *FRM*, the category of all frames (see remarks preceding II-2.24). As prime elements of a frame are often called “points”, one also says that a frame has “enough points” if the primes are order generating.

Proposition V-4.7.

(i) *The functor*

$$\mathrm{Spec}\colon SUP^{\wedge} \to TOP^{\mathrm{op}}$$

is left adjoint to the functor

$$\mathcal{O}\colon TOP^{\mathrm{op}} \to SUP^{\wedge}.$$

Front and back adjunctions are given by

$$\Delta_L\colon L \to \mathcal{O}(\mathrm{Spec}\ L) \textit{ and } \xi_X\colon X \to \mathrm{Spec}\ \mathcal{O}(X)$$

where $\Delta_L(a) = \mathrm{Spec} L \backslash {\uparrow} a$ *and* $\xi_X(x) = X \backslash \{x\}^-$ *for any complete lattice* L *and any topological space* X*. Moreover,*

$$\Delta_{\mathcal{O}(X)}\colon \mathcal{O}(X) \to \mathcal{O}(\mathrm{Spec}\ \mathcal{O}(X)) \textit{ and } \xi_{\mathrm{Spec}\ L}\colon \mathrm{Spec}\ L \to \mathrm{Spec}\ \mathcal{O}(\mathrm{Spec}\ L)$$

are isomorphisms.

(ii) *The categories* SOB *and* FRM_0 *are dual under the restrictions of the functors* Spec *and* $\mathcal{O}$*.*

(iii) *The functors*

$$\mathcal{O}\ \mathrm{Spec}\colon SUP^{\wedge} \to FRM_0 \quad \textit{and} \quad \mathrm{Spec}\ \mathcal{O}\colon \mathrm{TOP} \to \mathrm{SOB}$$

are reflections. In particular, Δ_L *is an isomorphism iff* $L \in FRM_0$ *and* ξ_X *a homeomorphism iff* X *is sober.*

Remark. For any space X, its sober reflection $X^S = \mathrm{Spec}\ \mathcal{O}(X)$ is called the *sobrification* of X and the natural map $\xi_X\colon X \to X^S$ is called the *sobrification map*.

Proof of proposition: We first prove the naturality of Δ_L and ξ_X. The commutativity of the diagram

$$\begin{tikzcd}
L \arrow[r, "\Delta_L"] \arrow[d, "\varphi"'] & \mathcal{O}(\mathrm{Spec}\ L) \arrow[d, "\mathcal{O}(\mathrm{Spec}\ \varphi)"] \\
M \arrow[r, "\Delta_M"'] & \mathcal{O}(\mathrm{Spec}\ M)
\end{tikzcd}$$

follows from the fact that

$$(\mathcal{O}\mathrm{Spec}\ \varphi)(\Delta_L(a)) = (\mathrm{Spec}\ \varphi)^{-1}(\Delta_L(a)) = \Delta_M(\varphi(a))$$

for every $a \in L$ (see V-4.6). The commutativity of

$$\begin{tikzcd}
X \arrow[r, "\xi_X"] \arrow[d, "f"'] & \mathrm{Spec}\, \mathcal{O}(X) \arrow[d, "\mathrm{Spec}\, \mathcal{O}(f)"] \\
Y \arrow[r, "\xi_Y"'] & \mathrm{Spec}\, \mathcal{O}(Y)
\end{tikzcd}$$

can be seen in the following way:

$$\begin{aligned}
(\mathrm{Spec}\ \mathcal{O}(f))(\xi_X(x)) &= (\mathrm{Spec}\ \mathcal{O}(f))(X\backslash\{x\}^-) \\
&= \bigcup \mathcal{O}(f)^{-1}(\downarrow(X\backslash\{x\}^-)) \\
&= \bigcup\{U \in \mathcal{O}(Y) : \mathcal{O}(f)(U) \subseteq X\backslash\{x\}^-\} \\
&= \bigcup\{U \in \mathcal{O}(Y) : x \notin f^{-1}(U)\} \\
&= \bigcup\{U \in \mathcal{O}(Y) : f(x) \notin U\} \\
&= Y\backslash\{f(x)\}^- \\
&= \xi_Y(f(x)),
\end{aligned}$$

for all $x \in X$.

The fact that $\Delta_{\mathcal{O}(X)}: \mathcal{O}(X) \to \mathcal{O}(\mathrm{Spec}\ \mathcal{O}(X))$ is a lattice isomorphism follows from the comments before V-4.4 and before V-4.7 as does the fact that the mapping $\xi_{\mathrm{Spec}\ L}$: Spec $L \to$ Spec $\mathcal{O}$(Spec L) is a homeomorphism.

In order to show the adjointness of the functors Spec and $\mathcal{O}$, we prove the commutativity of the following diagrams:

$$\begin{tikzcd}
\mathrm{Spec}\, L \arrow[rr, equal] \arrow[dr, "\xi_{\mathrm{Spec}\, L}"'] & & \mathrm{Spec}\, L \\
& \mathrm{Spec}\, \mathcal{O}(\mathrm{Spec}\, L) \arrow[ur, "\mathrm{Spec}\, (\Delta_L)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}(X) \arrow[rr, equal] \arrow[dr, "\Delta_{\mathcal{O}(X)}"'] & & \mathcal{O}(X) \\
& \mathcal{O}(\mathrm{Spec}\, \mathcal{O}(X)) \arrow[ur, "\mathcal{O}(\xi_X)"'] &
\end{tikzcd}$$

This can be done in the following way:

$$\begin{aligned}
(\mathrm{Spec}\ \Delta_L)(\xi_{\mathrm{Spec}\ L}(p)) &= (\mathrm{Spec}\ \Delta_L)(\mathrm{Spec}\ L\backslash\{p\}^-) \\
&= (\mathrm{Spec}\ \Delta_L)(\Delta_L(p)) \\
&= \sup \Delta_L^{-1}(\downarrow\Delta_L(p)) \\
&= p
\end{aligned}$$

for every $p \in \operatorname{Spec} L$, and

$$\mathcal{O}(\xi_X)(\Delta_{\mathcal{O}(X)}(U)) = \xi_X^{-1}\big(\Delta_{\mathcal{O}(X)}(U)\big) = U$$

for every $U \in \Delta(S)$.

The remaining assertions now follow in a routine manner. □

Exercises

Exercise V-4.8.

(i) Let L be a complete lattice, $X \subseteq L$. Show that the following statements are equivalent.
 (1) X is sober with respect to the relative lower topology.
 (2) For $x \in L$, if $x = \underline{\lim}\,\mathcal{F}$ for some ultrafilter $\mathcal{F}$ on $\uparrow x \cap X$, then $x \in X$.

(ii) Show that the set $\operatorname{Spec} L$ satisfies condition (2) in any complete lattice. This gives an alternative argument that $\operatorname{Spec} L$ is sober.

 Hint. Use Proposition III-3.18. □

The following gives an alternative approach to constructing the sobrification of a space.

Exercise V-4.9. Let X be a topological space.

(i) Define a set X^S by

$$X^S = \{A \subseteq X : A \text{ is closed, irreducible, and nonempty}\}.$$

 Topologize X^S by open sets $U^S = \{A \in X^S : A \cap U \neq \emptyset\}$ for each open set U of X. If we let $j\colon X \to X^S$ be the map $x \mapsto \{x\}^-$, show that (X^S, j) is – up to commuting homeomorphism – the sobrification of X.

(ii) Show that if $f\colon X \to Y$ is continuous and Y is sober, then there exists a unique continuous $h\colon X^S \to Y$ such that $f = hj$.

(iii) Show that the universal property given in (ii) characterizes the sobrification – up to commuting homeomorphism. □

Exercise V-4.10. Let L be a complete lattice. A filter $F \subseteq L$ is said to be *completely prime* if $\sup A \in F$ and $A \neq \emptyset$ always imply $A \cap F \neq \emptyset$. Show that the following statements are equivalent for a proper filter F.

(1) F is completely prime.
(2) F is Scott open and prime.
(3) There exists a prime $p \neq 1$ such that $F = L \backslash {\downarrow} p$. □

The following exercise gives an alternative approach to defining Spec L.

Exercise V-4.11. Let L be a complete lattice. Define

$$\mathrm{Spec}\, L = \{F \subseteq L : F \text{ is a proper completely prime filter}\}.$$

Topologize Spec L by taking as open sets all sets of the form

$$\$(x) = \{F \in \mathrm{Spec}\, L : x \in F\}.$$

Show that all sets of the form $\$(x)$ for $x \in L$ form a topology on Spec L, and show that this definition is equivalent to the one given in the text.

Hint. Use part (3) of V-4.10 to set up the equivalence. □

Old notes

The material contained in this section is standard. The theme of representing lattices by suitable topologies – usually hull–kernel topologies – goes back to M.H. Stone's famous papers [Stone, 1936; Stone, 1937] on the topological representation of Boolean algebras and distributive lattices. Of the authors who have pursued this theme we only quote [Büchi, 1952], [Papert, 1959], [Bruns, 1962a], [Thron, 1962], and [Drake and Thron, 1965]. In these papers one finds the duality between sober spaces and complete lattices order generated by their prime elements, at least on the object level. Explicit formulations of this and other dualities in the language of category theory have been collected in the memoir [Hofmann and Keimel, B1972].

V-5 Duality for Distributive Continuous Lattices

In this section it is our aim to show that there is a one-to-one correspondence between distributive continuous lattices (that is, continuous frames) and locally compact sober spaces in the sense of a duality of categories. It will take some development, however, to specify the maps of the desired categories precisely.

If X is a locally compact space (meaning that every point in X has a neighborhood basis of compact sets), it is easy to see that the lattice $\mathcal{O}(X)$ of open subsets is a distributive continuous lattice (see I-1.7(5)). Conversely, if L is a distributive continuous lattice, then the space Spec L of all prime elements $p \neq 1$ endowed with the hull–kernel topology is sober by V-4.4; we have to prove that Spec L is locally compact, too. For this we first need a general criterion for compactness of sets in Spec L.

Lemma V-5.1. *Let L be a complete lattice. A subset Q of* Spec L *is compact for the hull–kernel topology iff $\downarrow Q$ is Scott closed in L.*

Proof: Suppose that $Q \subseteq \operatorname{Spec} L$ is compact, and let D be a directed set in $\downarrow Q$. Then $\{\nabla_L(d) \cap Q\colon d \in D\}$ is a filter base of (nonempty) closed subsets of Q. As Q is compact, we have

$$\bigcap\{\nabla_L(d) \cap Q\colon d \in D\} = \bigcap\{\nabla_L(d)\colon d \in D\} \cap Q$$
$$= \nabla_L(\sup D) \cap Q \neq \emptyset.$$

But this means that $\sup D \in \downarrow Q$. Thus, $\downarrow Q$ is Scott closed.

Suppose conversely that $\downarrow Q$ is Scott closed. In order to show that Q is compact, let $\mathcal{F}$ be a filter base of closed subsets F of Spec L with $F \cap Q \neq \emptyset$ for all $F \in \mathcal{F}$. Then $F = \nabla_L(\inf F)$ for all $F \in \mathcal{F}$, and the set $\{\inf F\colon F \in \mathcal{F}\}$ is directed and contained in $\downarrow Q$ as $F \cap Q \neq \emptyset$. As $\downarrow Q$ is Scott closed, we see that $\sup\{\inf F\colon F \in \mathcal{F}\} \in \downarrow Q$. Thus, an element $q \in Q$ exists such that $q \geq \sup\{\inf F\colon F \in \mathcal{F}\}$; that is, $q \in \nabla_L(\inf F) = F$ for all $F \in \mathcal{F}$. □

For a more specific criterion of compactness we recall the following notion (see O-5.3).

Definition V-5.2. A subset Q of a topological space X is called *saturated* if $\{x\}^- \cap Q \neq \emptyset$ always implies $x \in Q$. □

Using the specialization order on X, where $x \leq y$ iff $x \in \{y\}^-$ (see II-3.6), we can say that Q is saturated in X iff $q \in Q$ and $q \leq x \in X$ always imply $x \in Q$ iff $Q = \uparrow Q$ in ΩX (cf. II-3.6). For more information on saturation, see Exercise O-5.14. In the case $X = \operatorname{Spec} L$ with the hull–kernel topology, where L is a complete lattice, we have $\{x\}^- = \nabla_L(x)$; thus, Q is saturated in Spec L iff $q \in Q$ and $q \geq x \in \operatorname{Spec} L$ always imply $x \in Q$ iff Q is a lower subset of Spec L. The reader should note that on Spec L the specialization order is *opposite* to the order induced from L.

Lemma V-5.3. *Let L be a complete lattice. A subset Q of* Spec L *is saturated and compact for the hull–kernel topology iff there is a Scott open filter $F \subseteq L$ such that $Q = \operatorname{Spec} L \backslash F$ and $\downarrow Q = L \backslash F$.*

Proof: First let Q be saturated and compact. Then $\downarrow Q$ is Scott closed by V-5.1. Thus $F = L \backslash \downarrow Q$ is Scott open. As $L \backslash \downarrow Q = \bigcap\{L \backslash \downarrow p\colon p \in Q\}$ and as all $p \in Q$ are prime, $L \backslash \downarrow Q$ is a filter. Clearly, because Q is saturated, we have $\downarrow Q = L \backslash F$ and $Q = \operatorname{Spec} L \backslash F$.

Conversely, if there is a Scott open filter F with both $Q = \operatorname{Spec} L \backslash F$ and $\downarrow Q = L \backslash F$, then Q is saturated and compact by V-5.1. □

Corollary V-5.4. *Let L be a distributive complete lattice. Then a subset Q of* Spec L *is saturated and compact for the hull–kernel topology iff there is a Scott open filter $F \subseteq L$ such that $Q =$* Spec $L \backslash F$.

Proof: By V-5.3 we only have to show that if F is a Scott open filter and if $Q = \operatorname{Spec} L \backslash F$, then $\downarrow Q = L \backslash F$. Indeed, if F is Scott open, then $L \backslash F$ is Scott closed. Thus, every element in $L \backslash F$ is dominated by a maximal element of that set. As F is a filter, these maximal elements are prime (cf. I-3.12(4)) by the distributivity of L. Hence, they belong to Q, and so $\downarrow Q = L \backslash F$. □

Now we prove the two principal results of this section:

Theorem V-5.5. *Let L be a distributive continuous lattice. The space* Spec L *is sober and locally compact and $\nabla_L \colon L \to \mathcal{O}(\operatorname{Spec} L)$ is an isomorphism.*

Remark. As a consequence of I-3.40.9, all these spaces are Baire spaces. If L is infinite, one has for the weights $w(\operatorname{Spec} L) = w(L)$ by III-4.8.

Proof of theorem: By V-4.4, Spec L is sober. Let U be a neighborhood of a point p in Spec L. We want to find a compact neighborhood Q of p contained in U. We may suppose that $U = \Delta_L(a) = \operatorname{Spec} L \backslash \uparrow a$ for some a in L. As L is continuous, there is an element $b \ll a$ with $b \not\leq p$. By I-3.3 there is a Scott open filter F with $a \in F \subseteq \uparrow b$. Let $Q = \operatorname{Spec} L \backslash F$. By V-5.4 Q is compact. Further, $a \in F \subseteq \uparrow b$ implies $\Delta_L(b) \subseteq Q \subseteq \Delta_L(a) = U$. As $b \not\leq p$, we have $p \in \Delta_L(b)$, and Q is a compact neighborhood of p contained in U. Finally, V-4.7 shows that Δ_L is an isomorphism. □

Theorem V-5.6. *For a sober space X, the lattice $\mathcal{O}(X)$ of open subsets is continuous iff X is locally compact.*

Proof: In I-1.7(5) it has been shown that $\mathcal{O}(X)$ is continuous for every locally compact space. Finally, let X be sober. Then X is homeomorphic to Spec $\mathcal{O}(X)$ by V-4.7. If, in addition, $\mathcal{O}(X)$ is continuous, then Spec $\mathcal{O}(X)$, and hence X, is locally compact by V-5.5. □

As every Hausdorff space is sober, Theorem V-5.6 yields

Corollary V-5.7. *For a Hausdorff space X, the lattice $\mathcal{O}(X)$ of open subsets is continuous iff X is locally compact.* □

In general, that is, for nonsober spaces X, we do not have any characterization of the continuity of $\mathcal{O}(X)$ that is as satisfactory as Theorem V-5.6. Of course, we can say that $\mathcal{O}(X)$ is continuous if and only if the sobrification X^S of X is locally compact. For T_0-spaces this can be made more specific:

Definition V-5.8. A function $f\colon X \to Y$ is called a *quasihomeomorphism* if the map $U \mapsto f^{-1}(U)\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is bijective and, hence, an isomorphism of lattices. A topological embedding $i\colon X \to Y$ of topological spaces is called *strict* if i is a quasihomeomorphism. □

We observe that the image of a strict embedding is always dense. The sobrification map $\xi_X\colon X \to X^S = \operatorname{Spec} \mathcal{O}(X)$ is a strict embedding if X is a T_0-space by the results of Section V-4.

Lemma V-5.9. *Let L be a complete lattice in which* Spec L *is order generating. For a subset $\Sigma \subseteq$* Spec L *the inclusion map $\Sigma \to$* Spec L *is a strict embedding iff Σ is also order generating in L.*

Proof: The inclusion $\Sigma \to \operatorname{Spec} L$ is a strict embedding iff

$$\nabla_L(s) \cap \Sigma = \nabla_L(t) \cap \Sigma \text{ implies } \nabla_L(s) = \nabla_L(t)$$

for all s, t in L. As Spec L is order generating, this is equivalent to saying that $\nabla_L(s) \cap \Sigma = \nabla_L(t) \cap \Sigma$ implies $s = t$. As in any case

$$\nabla_L(s) \cap \Sigma = \nabla_L(\inf(\nabla_L(s) \cap \Sigma)) \cap \Sigma,$$

the foregoing statement implies that $s = \inf(\nabla_L(s) \cap \Sigma)$ for all s in L; that is, Σ is order generating in L. Clearly, if Σ is order generating, then we find that $\nabla_L(s) \cap \Sigma = \nabla_L(t) \cap \Sigma$ implies $s = \inf(\nabla_L(s) \cap \Sigma = \inf(\nabla_L(t) \cap \Sigma) = t$. □

Proposition V-5.10. *For a T_0-space X, the following statements are equivalent:*

(1) *$\mathcal{O}(X)$ is a continuous lattice;*
(2) *X allows a strict embedding into a locally compact space;*
(3) *X allows a strict embedding into a locally compact sober space;*
(4) *X is homeomorphic to an order generating subspace of* Spec L *for some distributive continuous lattice L;*
(5) *the sobrification X^S of X is locally compact.*

Proof: (1) iff (5): This follows from Theorem V-5.6 and from the fact that $\mathcal{O}(X) \cong \mathcal{O}(\operatorname{Spec}(\mathcal{O}(X))) = \mathcal{O}(X^S)$ by V-4.7.

(5) implies (4): As X is a T_0-space, the sobrification map $\xi_X\colon X \to X^S = \operatorname{Spec}(\mathcal{O}(X))$ is a strict embedding. Hence $\xi_X(X)$ is order generating in $\mathcal{O}(X)$

by V-5.9, and $\mathcal{O}(X) \cong \mathcal{O}(X^S)$ is distributive and continuous.

(4) implies (3): By V-5.9.

(3) implies (2) implies (1): Clear. □

The reader should recall that the spaces characterized by the equivalent conditions in the preceding proposition have been considered in Section II-4 because of their good behavior with respect to *function spaces*.

We have seen that the hull–kernel topology of Spec L is induced from the lower topology $\omega(L)$ of L. We now show that the topology induced from the Lawson topology $\lambda(L)$ of L on Spec L can be characterized – in the case of a continuous distributive lattice L – in terms of the hull–kernel topology of Spec L. The procedure we are using associates with every topological space a refinement of its topology called the patch topology already introduced in O-5.10.

Definition V-5.11. Let X be a topological space. The *co-compact topology* is the topology generated by the complements $X \setminus Q$ of arbitrary compact saturated subsets Q of X. We define the *patch topology* on X to be that generated by the original topology $\mathcal{O}(X)$ together with the co-compact topology. □

We now consider a complete lattice L and the patch topology for $X =$ Spec L with the hull–kernel topology, which is the topology induced by the lower topology on L. In parallel, we consider PRIME $L =$ Spec $L \cup \{1\}$ also topologized by the topology induced by the lower topology on L.

We wish next to compare the patch topology on Spec L and PRIME L with the topology induced from the Lawson topology on L. Recall from III-1.5 that the Lawson topology is generated by the lower topology together with the Scott topology. But the lower topology induces on Spec L and on PRIME L exactly the hull–kernel topology. By V-5.3, every set of the form Spec $L \setminus Q$, for a compact saturated subset Q of Spec L can be written as $F \cap$ Spec L for some Scott open filter F of L. Thus, the patch topology on Spec L is coarser than the topology induced from the Lawson topology on L. The same holds for PRIME L.

If L is distributive, then by V-5.4 the sets of the form Spec $L \setminus Q$ with Q compact and saturated are exactly the sets of the form $F \cap$ Spec L with F a Scott open filter on L. Since on a continuous lattice the Scott topology is generated by the Scott open filters (II-1.14), we have

Proposition V-5.12. (i) *In an arbitrary complete lattice L, the patch topology on* Spec L *and on* PRIME $L =$ Spec $L \cup \{1\}$ *is coarser than the topology induced by the Lawson topology on L.*

(ii) *In a distributive continuous lattice L, the two topologies agree on* Spec L *and on PRIME $L =$* Spec $L \cup \{1\}$. *In particular, the patch topology is Hausdorff.* □

The criteria in V-3.7 for PRIME L to be closed with respect to the Lawson topology together with V-5.12 yield the following.

Corollary V-5.13. (i) *In a distributive continuous lattice L, the patch topology on* PRIME L *is compact iff L is stably continuous.*

(ii) *In a distributive algebraic lattice L, the patch topology on* PRIME L *is compact iff L is arithmetic.* □

In a countably based domain L the Lawson topology has a countable base and it is metrizable (see III-4.5, III-4.6). We shall see now that it is even completely metrizable, that is, every countably based domain with its Lawson topology is a Polish space. Recall (O-5.13) that a space is called *Polish* if it is separable and its topology is completely metrizable. As any compact metric space is complete, a countably based continuous lattice with its Lawson topology is always Polish. We shall use the result that a G_δ-subset, i.e., an intersection of countably many open sets, of a Polish space is also a Polish space (cf. remarks after O-5.13).

Lemma V-5.14. *In a countably based continuous lattice L, the set* IRR L *of irreducible elements is a G_δ-set for the Lawson topology and hence a Polish space.*

Proof: Consider L with its Lawson topology. In $L \times L$ the set C of all pairs (x, y) of comparable elements is closed, hence compact, with respect to the Lawson topology. In fact, the graph G of the order relation $\leq$ is closed, and C is the union of G and the graph G^{-1} of the opposite order $\geq$. The set $L\backslash\mathrm{IRR}\,L$ is the image of the complement $(L \times L)\backslash C$ under the map $(x, y) \mapsto x \wedge y$ (see Exercise I-3.37). As the Lawson topology on L has a countable basis, the compact set C is the intersection of countably many open subsets U_n of $L \times L$. Their complements $A_n = (L \times L)\backslash U_n$ are compact, so their images $B_n \subseteq L$ under the inf map, which is continuous, are compact and hence closed, and the B_n do not meet IRR L. Thus IRR L is the intersection of the countable family of open sets $V_n = L\backslash B_n$, which shows that it is a G_δ-set. □

Lemma V-5.15. *On every quasicontinuous domain, in particular on every continuous domain, the Lawson topology agrees with the patch topology associated to the Scott topology.*

Proof: It suffices to show that the lower topology agrees with the co-compact topology associated to the Scott topology: The finitely generated upper sets form a basis for the closed sets of the lower topology, and these sets are Scott compact and saturated. Thus the co-compact topology is finer than the lower topology. On the other hand, by III-5.7 every Scott compact saturated set is an intersection of finitely generated upper sets, hence closed for the lower topology. Thus, the lower topology is finer than the co-compact topology. □

Lemma V-5.16. *For a quasicontinuous domain S, the map $\xi_S\colon x \mapsto (S\backslash{\downarrow}x) : S \to \operatorname{Spec}\sigma(S)$ is a homeomorphism for the Lawson toplogy on S and the topology on $\operatorname{Spec}\sigma(S)$ induced by the Lawson topology on the continuous lattice $\sigma(S)$.*

Proof: Let L denote the continuous lattice $\sigma(S)$ of Scott open subsets of S. By V-5.12(ii) the topology induced on $\operatorname{Spec} L$ by the Lawson topology on L coincides with the patch topology of $\operatorname{Spec} L$. As every quasicontinuous domain is sober in its Scott topology by III- 3.7, the spectrum of $\operatorname{Spec} L$ is homeomorphic to S with its Scott topology by V-4.7(iii). As the patch topology associated with the Scott topology on S is equal to the Lawson topology on S by V-5.15, we see that S with its Lawson topology is homeomorphic to $\operatorname{Spec} L$ endowed with the topology induced by the Lawson topology of L. $\square$

Proposition V-5.17. *A countably based domain with its Lawson topology is a Polish space.*

Proof: Consider a countably based domain S with its Scott topology. The Scott open sets form a distributive continuous lattice $L := \sigma(S)$ which is countably based by III-4.5. By V-5.14 the spectrum $\operatorname{Spec} L$ is a Polish space when endowed with the topology induced by the Lawson topology on L. As this space is homeomorphic to S with its Lawson topology by V-5.16, S is also a Polish space for its Lawson topology. $\square$

Let us turn now to the functorial aspects of the correspondence between distributive continuous lattices and locally compact sober spaces. Recall the duality between the categories *SOB* and FRM_0 of the last section. This duality restricts, by what we have just shown, to a duality between the full subcategory of *SOB* consisting of the locally compact sober spaces and the full subcategory of *FRM* consisting of the distributive continuous lattices, or, equivalently, the continuous frames. Indeed, $\mathcal{O}(X)$ is a continuous distributive lattice for every locally compact sober space, and $\operatorname{Spec} L$ is a locally compact sober space for every distributive continuous lattice L by V-5.5.

Unfortunately, the morphisms in $\mathit{SUP}^{\wedge}$ and in particular in *FRM* (that is, the maps preserving arbitrary sups and finite infs) are of no particular significance for continuous lattices. What we want to do is to restrict attention to maps which, in addition, preserve the relation $\ll$. This is motivated by the fact that a map $\varphi\colon L \to M$ between continuous lattices preserves arbitrary sups and $\ll$ if and only if its upper adjoint $\tau\colon M \to L$ preserves arbitrary infs and directed sups (see IV-1.4). (That is, τ is a $\wedge$-semilattice homomorphism which is continuous for the respective Lawson topologies; see III-1.8.) Now by V-4.5, the upper adjoint maps $\operatorname{Spec} M$ into $\operatorname{Spec} L$, and the restriction and co-restriction

$\operatorname{Spec}\varphi\colon \operatorname{Spec} M \to \operatorname{Spec} L$ is continuous for the respective hull–kernel topologies. What is new in the discussion is that we can verify an additional property for these maps, as we show in the next two lemmas.

Lemma V-5.18. *Let L and M be distributive continuous lattices and $\varphi\colon L \to M$ a map preserving arbitrary sups, finite infs and the relation $\ll$. Then for every saturated compact subset Q of* $\operatorname{Spec} L$*, the preimage* $(\operatorname{Spec}\varphi)^{-1}(Q)$ *is also saturated and compact.*

Proof: Let Q be saturated and compact in $\operatorname{Spec} L$. By V-5.4, we can write $Q = \operatorname{Spec} L \backslash F$ for some Scott open filter F of L. Since the upper adjoint $\tau\colon M \to L$ is a $\wedge$-homomorphism which is continuous for the respective Lawson tologies, $\tau^{-1}(F)$ is a Scott open filter of M. As $\operatorname{Spec}\varphi$ is the restriction of τ, we conclude from V-5.4 that the set

$$(\operatorname{Spec}\varphi)^{-1}(Q) = \tau^{-1}(\operatorname{Spec} L\backslash F) = \operatorname{Spec} M\backslash\tau^{-1}(F)$$

is a saturated compact subset of $\operatorname{Spec} M$. □

Lemma V-5.19. *Let X and Y be locally compact spaces and $f\colon X \to Y$ a continuous map with the property that $f^{-1}(Q)$ is compact in X for every saturated compact subset Q of Y. Then the map $\mathcal{O}(f)\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ preserves the relation $\ll$.*

Proof: Let $U \ll V$ in $\mathcal{O}(Y)$. We want to show that $f^{-1}(U) \ll f^{-1}(V)$. By I-1.4(ii) there is a compact set Q with $U \subseteq Q \subseteq V$. Let P be the saturation of Q, that is, $P = \{y \in Y\colon \{y\}^{-} \cap Q \neq \emptyset\}$. Since every open set is saturated, the open coverings of Q and of P are the same. Thus P is also compact, and we still have $U \subseteq P \subseteq V$. By hypothesis, $f^{-1}(P)$ is also compact. Because $f^{-1}(U) \subseteq f^{-1}(P) \subseteq f^{-1}(V)$, we conclude that $f^{-1}(U) \ll f^{-1}(V)$. □

A continuous map $f\colon X \to Y$ is called *proper* if it is closed (for any A closed in X, $\downarrow f(A)$ is closed in Y) and $f^{-1}(B)$ is compact for all B compact and saturated in Y; see Definition VI-6.20. By Lemma VI-6.21 the maps of the preceding two lemmas are precisely the proper maps if the spaces X and Y belong to the class of locally compact sober spaces, the class we consider in the following.

Consider now the following categories:

LCSOB which has as objects the locally compact sober spaces and as morphisms the proper maps,

CL$^{\mathrm{op}}$ $\cap$ *FRM* which has as objects continuous distributive lattices and as maps the morphisms $\varphi\colon L \to M$ preserving arbitrary sups, finite infs and the relation $\ll$, *DL* which has as objects continuous distributive lattices and *CL*-maps preserving spectra.

Then *DL* and CL^{op} *FRM* are dual by IV-1.24.

Proposition V-5.20. (i) *A dual equivalence of categories is given by the functors*

$$\mathrm{Spec}\colon CL^{\mathrm{op}} \cap FRM \to LCSOB \quad and \quad \mathcal{O}\colon LCSOB \to CL^{\mathrm{op}} \cap FRM.$$

(ii) *The categories DL and LCSOB are equivalent.* □

Exercises

Exercise V-5.21. (i) Let L be a distributive algebraic lattice. Show that the open set $\Delta_L(k) \subseteq \mathrm{Spec}\, L$ is compact for the hull–kernel topology whenever k is a compact element of L, and that conversely every compact open subset of $\mathrm{Spec}\, L$ is of this form. Show that, moreover, $\mathrm{Spec}\, L$ has a basis of compact open sets (cf. also I-4.24).

(ii) Using V-5.19 show that the functors Spec and $\mathcal{O}$ establish a dual equivalence between the following categories:

$AL^{\mathrm{op}} \cap FRM$, whose objects are the distributive algebraic lattices and whose morphisms are the maps $\varphi\colon L \to M$ which preserve arbitrary sups, finite infs and which map compact elements of L to compact elements of M;

BCSOB, whose objects are the sober spaces having a basis of compact open sets and whose maps are the continuous functions $f\colon X \to Y$ such that $f^{-1}(Q)$ is compact for every compact open subset Q of Y. □

Exercise V-5.22. Consider the following categories.

The full subcategory *DAR* of $ArL^{\mathrm{op}} \cap FRM$ the objects of which are the distributive arithmetic lattices with 1 a compact element.

The full subcategory *CCSOB* of *BCSOB* the objects of which are the compact sober spaces with a basis of compact open sets closed under finite intersections.

The category *DLat* of distributive lattices with 0 and 1 and all 0 and 1 preserving lattice homomorphisms.

Show that the categories *DAR* and *CCSOB* are dually equivalent under the functors Spec and $\mathcal{O}$, and that similarly, the categories *DLat* and *CCSOB* are dually equivalent.

Hint. The categories *DLat* and *DAR* are equivalent. One may use the functor Id: $DLat \to DAR$ which associates with every distributive lattice its ideal lattice, and the functor $K: DAR \to DLat$ which associates with each arithmetic distributive lattice its lattice of compact elements (cf. IV-1.16). □

Exercise V-5.23. (i) Let X be a sober space and L a continuous distributive lattice. Show that $\operatorname{Spec}[X, \Sigma L] \cong X \times \operatorname{Spec} L$ (see II-4.23, II-4.24).

Hint. We know from II-4.19 that $[X, \Sigma L]$ is a frame if L is one. In II-4.24, for a sober space X and an arbitrary continuous lattice L we constructed a bijection $\beta\colon X \times (\operatorname{IRR} L\backslash\{1\}) \to \operatorname{IRR}[X, \Sigma L]\backslash\{1\}$, which was given by $\beta(x, p) = \chi_{X\backslash\{x\}^-} \vee \operatorname{const}_p$. As in distributive lattices irreducible elements are prime, we have indeed a bijection $\beta\colon X \times \operatorname{Spec} L \to \operatorname{Spec}[X, \Sigma L]$.

We show that β is a homeomorphism. The generic closed sets of $S = \operatorname{Spec}[X, \Sigma L]$ are of the form $\uparrow f \cap S$, $f \in [X, \Sigma L]$. Now $\beta^{-1}(\uparrow f \cap S) = \{(x, p)\colon x \in X, p \in \operatorname{Spec} L, f(x) \leq p\}$. We claim that the complement of this set is open in $X \times \operatorname{Spec} L$. Indeed suppose $f(x) \not\leq p$. Pick an $s \in L$ with $s \not\leq p$ and $s \ll f(x)$. Then $U = f^{-1}(\twoheaduparrow s)$ is an open neighborhood of x in X, and $V = (\operatorname{Spec} L)\backslash \uparrow s$ is an open neighborhood of p in $\operatorname{Spec} L$. If $u \in U$ and $v \in V$, then $f(u) \not\leq v$, since otherwise $s \ll f(u) \leq v$ implies $v \in \uparrow s$. This proves the claim. Conversely, let A be closed in X and s in L so that $\uparrow s \cap \operatorname{Spec} L$ is a generic closed set of $\operatorname{Spec} L$. Define $f\colon X \to L$ by

$$f(x) = \begin{cases} s, & \text{if } x \in A, \\ 1, & \text{otherwise.} \end{cases}$$

Then f is continuous, that is, $f \in [X; \sigma L]$. Moreover, $f \leq \beta(x, p)$ iff $f(x) \leq p$ iff $s \leq p$ for $x \in A$ and $1 \leq p$ for $x \notin A$; but $p < 1$, hence $a \leq \beta(x, p)$ iff $x \in A$ and $p \in \uparrow s$. Thus $A \times (\uparrow s \cap \operatorname{Spec} L) = \beta(\uparrow f \cap S)$ is the image of a generic closed set in $\operatorname{Spec}[X, \Sigma L]$.

Remark. We have in fact proved a stronger statement, since in the proof we did not use the distributivity of L.

(ii) With $L = \mathcal{O}(Y)$ for a locally compact sober space, and in view of the duality between these spaces and continuous distributive lattices, retrieve the following corollary (cf. II-4.10): for two locally compact sober spaces X and Y, one has $[X, \Sigma\mathcal{O}(Y)] \cong \mathcal{O}(X \times Y)$. □

Exercise V-5.24. Let L_1 and L_2 be two continuous distributive lattices. Prove the following.

(i) $\operatorname{Spec}(L_1 \otimes L_2) \cong \operatorname{Spec} L_1 \times \operatorname{Spec} L_2$(cf. IV-1.27 f.).

(ii) Dually, for two locally compact sober spaces X and Y, one has $\mathcal{O}(X \times Y) \cong \mathcal{O}(X) \otimes \mathcal{O}(Y)$. (See [Bandelt, 1980b].) □

The following exercise gives an indication of an example of a T_0-space X for which $\mathcal{O}(X)$ is a continuous lattice but which fails to be locally compact so badly that every compact subset of X has *empty* interior.

Exercise V-5.25. Let $\mathbb{I} = [0, 1]$ be the unit interval and L the continuous lattice $[\mathbb{I} \to \mathbb{I}] = \mathrm{LSC}(\mathbb{I}, \mathbb{I})$ of lower semicontinuous functions from $\mathbb{I}$ into $\mathbb{I}$. Then $Y = \operatorname{Spec} L$ may be identified with $\mathbb{I} \times (\mathbb{I}\backslash\{1\})$ via V-5.23 above, where we transport the hull–kernel topology of the spectrum to the topless unit square. By heavy use of the axiom of choice we pick a dense subset A of $\mathbb{I}\backslash\{1\}$ such that $A \cap U$ is not a Borel set (or, if one prefers, not even Lebesgue measurable) for every nonempty open subset U of the unit interval. Let Q be the set of rational points of $\mathbb{I}$. Now define

$$X = (A \times Q\backslash\{1\}) \cup (\mathbb{I}\backslash A) \times (\mathbb{I}\backslash Q),$$

and give X the topology inherited from the spectrum of L. Prove the following:

(i) X is a T_0-space for which $\mathcal{O}(X) = L$; in particular, $\mathcal{O}(X)$ is a continuous lattice;
(ii) every compact subset of X has empty interior. (For details see [Hofmann and Lawson, 1978].) □

Exercise V-5.26. Let X be a sober topological space and ΩX the set X considered as a poset with the specialization order (see O-5.2). Show that the following statements are equivalent:

(1) $\mathcal{O}(X)$ is completely distributive;
(2) X is locally compact, $\sigma(\Omega X) \subseteq \mathcal{O}(X)$ and ΩX is a domain.

Moreover, show that, if these conditions are satisfied, then $\sigma(\Omega(X) = \mathcal{O}(X)$.

Hint. (1) implies (2): If $\mathcal{O}(X)$ is completely distributive, we may identify X with $\operatorname{Spec} \mathcal{O}(X)$ and ΩX with $(\operatorname{Spec} \mathcal{O}(X), \supseteq)$. Then V-1.7 shows that ΩX is a domain.

(2) implies (1): We identify X with $\operatorname{Spec} L$ for the lattice $L = \mathcal{O}(X)$, and ΩX with $(X, \geq)$. Since X is locally compact, L is continuous (V-5.6), and the given topology on X is identified with the hull–kernel topology on X. Thus $U \in \mathcal{O}(X)$ means $U = X\backslash\uparrow x$ for some $x \in L$. But then clearly $U \in \sigma(\Omega X)$. Hence $\mathcal{O}(X) \subseteq \sigma(\Omega X)$. Thus $\mathcal{O}(X) = \sigma(\Omega X)$. But $\sigma(\Omega X)$ is completely distributive by II-1.14. □

We utilize this information to give a detailed analysis of the subcategories of the category *FRM* of frames with morphisms preserving arbitrary sups and finite infs and their duals. This is best depicted in diagram form.

Exercise V-5.27.

(i) Consider the categories shown in the two tables at the end of this section. Show that the corresponding categories are dual under the duality

$$FRM_0 \underset{\mathcal{O}}{\overset{\text{Spec}}{\rightleftarrows}} SOB$$

(ii) The category *FRM* has no dual based on topological spaces. (This has given rise to the study of "pointless topology" by passing to the formal dual (with objects sometimes called "locales").) Show that by Theorem IV-1.22, FRM^{op}, however, can be realized as a concrete category: namely the category of all frames ("locales") together with all maps g preserving arbitrary infs and prime ideals (that is, $\downarrow g(P)$ is prime for any prime ideal P). □

Exercise V-5.28. Let CL_d be the full subcategory in CL of all distributive continuous lattices. For a CL-morphism $f: L \to M$ between distributive continuous lattices L and M, let Spec f be the multivalued function Spec $L \to$ Spec M which associates with a $p \in$ Spec L the subset $\nabla_L(f(p)) \subseteq$ Spec M. Prove the following.

(i) Spec $fg =$ Spec $f \circ$ Spec g (with composition of binary relations on the right hand side).
(ii) Spec f is a multivalued function $F: X \to Y$ between locally compact sober spaces satisfying the following properties:
 (a) The image of a point is closed.
 (b) $F(A^-) = F(A)^-$ for all $A \subseteq X$.
 (c) $F^{-1}(Q)$ is compact and saturated whenever Q is compact and saturated in Y.
(iii) Spec is a functor from the category CL_d to the category of all multivalued maps between locally compact sober spaces satisfying (a), (b) and (c) above, and this functor gives an equivalence of categories. (This result generalizes V-5.20.)

Hint. (i) Use V-1.13.

(ii) Condition (a) is clear from the definition. For (b), $F(A)^{-1} \subseteq F(A^-)$, use V-1.13. For (c) recall V-5.4. (iii) Find an inverse functor for Spec by associating with a space X the lattice $\Gamma(X)^{\text{op}}$ (O-2.7(3)) and with a multivalued map $F: X \to Y$ satisfying (a), (b) and (c) the function $A \mapsto F(A): \Gamma(X)^{\text{op}} \to \Gamma(Y)^{\text{op}}$. □

Exercise V-5.29. Let X be a locally compact sober space. The co-compact topology generated by the set of all compact saturated sets as a subbasis for the closed sets is precisely the one making the function $x \mapsto X\backslash\{x\}^- : X \to \Sigma\mathcal{O}(X)$ an embedding.

Hint. Consider X as Spec L for $L = \mathcal{O}(X)$ and modify the proof of V-5.12(ii). □

Exercise V-5.30. Prove the following.

(i) Let X be a T_0-space such that $\mathcal{O}(X)$ is a continuous lattice (cf. V-5.10 above) and that its sobrification X^S (see Remark after V-4.7, and V-4.9) is first countable. Then the following conditions are equivalent:
   (1) X is sober;
   (2) all closed subspaces of X are Baire spaces (cf. O-5.13);
   (3) all closed irreducible subspaces of X are Baire spaces.

(ii) If X is a second countable T_0-space such that $\mathcal{O}(X)$ is continuous, then X is sober iff all closed subspaces are Baire spaces.

Remark. It is known that the primitive ideal spectrum Prim $\mathcal{A}$ of a C^*-algebra $\mathcal{A}$ is a locally compact Baire space and that it is second countable if $\mathcal{A}$ is separable. Every closed subset is a primitive ideal spectrum. Hence (ii) above shows that Prim $\mathcal{A}$ is sober in this case. This means that Prim $\mathcal{A}$ = Spec L, where L is the lattice of closed ideals of $\mathcal{A}$, in the case of a separable C^*-algebra (cf. I-3.34).

Hint. Part (ii) is a consequence of (i). In order to prove (i) we consider X as an order generating subset of Spec L for a continuous distributive lattice L (see V-5.10). First show that if $t \in (\text{Spec}L)\backslash X$ and $\omega(L)$ has a countable basis at t, then X is not a Baire space. For this purpose it is useful to argue that for $s \neq t$, the set $\uparrow s \cap X$ is nowhere dense in X.

Next show that if each $p \in \text{Spec}L$ has a countable $\omega(L)$ neighborhood basis in $\uparrow p$, then for each $p \in (\text{Spec}L)\backslash X$ the set $\uparrow p \cap X$ is a closed irreducible subset of X which is not a Baire space.

In order to complete the proof, observe that (1) is equivalent to X = Spec L (see discussion of ξ_X preceding V-4.7). Since closed subspaces of sober spaces are sober, (1) implies (2). That (2) implies (3) is clear, and not (1) implies not (3) follows now from the statement in the last paragraph. □

In the next exercises we expand on ideas related to strict embeddings (see Definition V-5.8).

Exercise V-5.31. Let X be a T_0-space. Show that the following refinement topologies are all equal:

(i) the topology with basis all locally closed sets, where a set is *locally closed* if it is the intersection of an open and a closed subset;
(ii) the topology with subbasis for the open sets given by the union of the family of all open sets and the family of all closed sets;
(iii) the topology with a basis of neighborhoods at each point x given by $\downarrow x \cap U$, where $\downarrow x$ is taken in the order of specialization and U is an open set;
(iv) the join of the original topology with the topology of all lower sets (in the specialization order); the second is just the order dual Alexandroff or A-discrete topology.

The common topology that arises from the preceding definitions will be called the *strong* topology. □

Exercise V-5.32. For a continuous map $j\colon X \to Y$ between T_0-spaces, show that the following are equivalent:

(1) j is a strict embedding;
(2) j is a quasihomeomorphism;
(3) j is a topological embedding and $j(X)$ is dense in the strong topology of Y.

Hint. Note that (2) is just condition (1) with the requirement that j is an embedding dropped. If (2) is assumed, then the open set lattice isomorphism induces a homeomorphism of the spectra, which are the sobrifications of X and Y, and the map j is then the restriction and co-restriction of this homeomorphism to the embedded images of X and Y. If $j(X)$ misses some $\downarrow y \cap V$, where $y \in V$ open, then $j^{-1}(Y \backslash {\downarrow} y) = j^{-1}((Y \backslash {\downarrow} y) \cup V)$, a contradiction. Assume (3) and suppose that $j^{-1}(U) = j^{-1}(V)$ for $U \neq V$; then there exists $x \in U \backslash V$, say. Thus $\downarrow x \cap U$ misses V, since $x \notin V$ and V is an upper set. It then follows from $j^{-1}(U) = j^{-1}(V)$ that $j(X)$ misses $\downarrow x \cap U$, a contradiction. □

Exercise V-5.33. Let $f\colon X \to Y$ be a continuous map between T_0-spaces. Show that the following are equivalent:

(1) f is an epimorphism in the category of T_0-spaces;
(2) the image $f(X)$ is strongly dense, that is, dense in the strong topology of Y;

(3) the inclusion of $f(X)$ into Y is a quasihomeomorphism, or a strict embedding.

Hint. The equivalence of (2) and (3) follows from the preceding exercise. Assume (1) and suppose that $f(X)$ is not dense in the strong topology of Y. Then there exist $y \in Y$ and U open containing y such that $f(X)$ misses $\downarrow y \backslash U$. Define $g, h\colon Y \to \{0, 1\}$, the Sierpinski space, by $g(\downarrow y) = 0$, $h(\downarrow y \backslash U) = 0$, otherwise 1. Then $gf = hf$, but $g \neq h$, a contradiction. Conversely assume (2). Since the continuous functions into the Sierpinski space separate points, it suffices to show that if $g, h\colon Y \to \{0, 1\}$ satisfy $gf = hf$, then $g = h$. Suppose not; then $U = g^{-1}(1) \neq h^{-1}(1) = V$ for some g, h. Then $U \backslash V \neq \emptyset$, say. It follows from $gf = hf$ that the image of f misses $U \backslash V = U \cap (Y \backslash V)$, an open set in the strong topology. $\square$

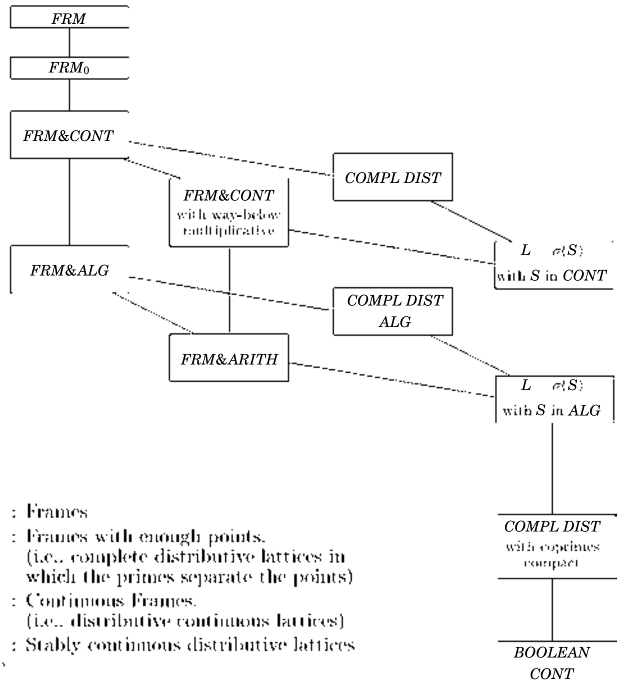
Exercise V-5.34. Show that a mapping $j\colon X \to \widehat{X}$ between T_0 spaces is the sobrification of X (up to an X-homeomorphism) iff $\widehat{X}$ is sober and j is a strict embedding.

Hint. If $\widehat{X}$ is sober and j is a strict embedding, then the induced isomomorphism between their lattices of open sets induces a homeomorphism between their spectra. Compose this homeomorphism with the one that identifies $\widehat{X}$ with the spectrum of its open set lattice. $\square$

Old notes

The results in the body of this section are to be found in the paper [Hofmann and Lawson, 1978]. Corollary V-5.7 was already found in [Day and Kelly, 1970] as well as in [Isbell, 1975a]. Exercise V-5.21 shows how a duality theorem on algebraic lattices in Hofmann and Keimel's memoir [Hofmann and Keimel, B1972] fits into the framework of this section, and Exercise V-5.22 establishes the link with M. H. Stone's original representation theorem for distributive lattices [Stone, 1937]. For remarks on the spaces X for which $\mathcal{O}(X)$ is continuous (V-5.10) we refer to the Old Notes of Section II-4. The result in V-5.23 is due to Hofmann and Scott [SCS 41], and the example in V-5.25 to [Hofmann and Lawson, 1978]; a similar example had been given by [Isbell, 1975b]. The result of V-5.26 is due to Lawson. Exercise V-5.29 is from Hofmann and Watkins [SCS 51], while Exercise V-5.30 is from Hofmann [SCS 43]. Some of the material in Exercises V-5.32 through V-5.34 is closely related to results in [Grothendieck and Dieudonné, B1971]; the terminology "quasihomeomorphism" and "strongly dense", for example, comes from that source.

Table of Frames



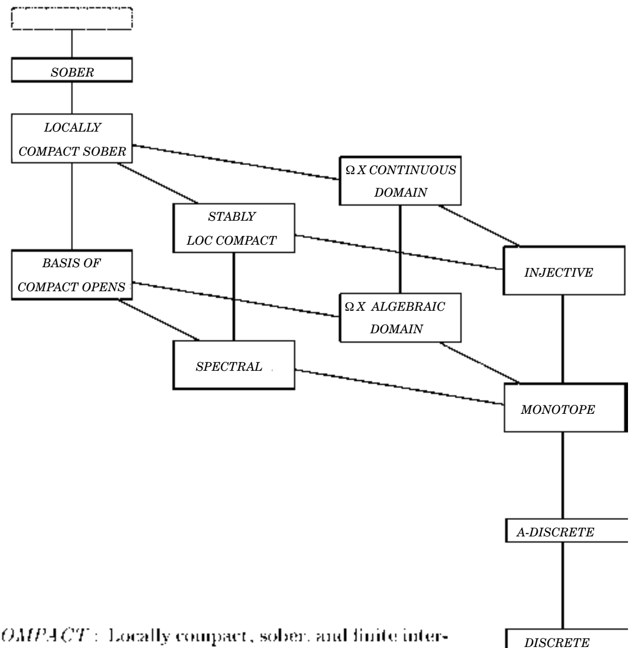
FRM	: Frames
FRM_0	: Frames with enough points. (i.e., complete distributive lattices in which the primes separate the points)
FRM & CONT	: Continuous Frames. (i.e., distributive continuous lattices)
FRM & CONT with way-below multiplicative	: Stably continuous distributive lattices
FRM & ARITH	: Arithmetic distributive lattices
FRM & ALG	: Algebraic distributive lattices
COMPL DIST	: Completely distributive lattices
COMP DIST ALG	: Completely distributive algebraic lattices
$L = \sigma(S)$ with S in *CONT*	: Scott topologies of continuous lattices
$L = \sigma(S)$ with S in *ALG*	: Scott topologies of algebraic lattices
COMPL DIST with coprimes compact	: Completely distributive lattices in which all coprimes are compact
BOOLEAN CONT	: Continuous Boolean algebras, (i.e., complete and atomic Boolean algebras)

All morphisms preserve arbitrary sups and finite infs.
All subcategories are full.

Let us note, in passing, that the fact that for most topological spaces, all the information on the space is already contained in its lattice of open subsets has led to the idea of considering certain types of lattices as substitutes for topological spaces (see, e.g., [Dowker and Papert, 1966], [Isbell, 1972]). This suggests that continuous lattices are in a sense substitutes for locally compact spaces.

Lemma V-5.14 is a lattice theoretical version of a result of [Dixmier, 1968].

TABLE OF SOBER SPACES



STABLY LOCALLY COMPACT : Locally compact, sober, and finite intersections of saturated compact sets are compact (See VI-6.16 for an alternative description.)

SPECTRAL : Has basis of compact open sets which is closed under finite intersection

MONOTOPE : Has a basis of monogeneric open sets which is closed under finite intersection (monogeneric open: minimal neighborhood of one of its points) (These spaces are also called Batbedat spaces.)

A-DISCRETE : Alexandroff discrete: each point has a smallest open neighborhood (These spaces are also called Kripke models.)

ΩX DOMAIN : Order of specialization makes X a domain (These spaces are also called projective, see [Hoffmann, 1979b].)

All morphisms are continuous functions.
All subcategories are full.

V-6 Domain Environments

Domain theory can be useful for providing appropriate partially ordered structures for the study, the modeling and the computation of computable functions. A basic illustrative example of such an ordered structure is the set of all closed subintervals of a closed real interval (viewed as a partially ordered set ordered by reverse inclusion), which is a useful computational model for the study of continuous and computable functions on the closed interval.

Consider, for example, a standard basic problem of numerically approximating a zero of a continuous computable function. Suppose that the continuous function f is defined on [0, 1], is strictly increasing, and satisfies $f(0) < 0$ and $f(1) > 0$. If we trisect the interval by $0 < 1/3 < 2/3 < 1$, then since f must be nonzero at one of the trisection points, we can determine at some finite stage whether $0 < f(2/3)$, in which case we pass to the subinterval $[0, 2/3]$, or $f(1/3) < 0$, in which case we pass to the subinterval $[1/3, 1]$. (The motivation for the trisection scheme instead of a bisection scheme is that it may be impossible in general to determine whether $f(x) = 0$ with a finite computation.) We repeatedly apply this trisection scheme to the subintervals obtained at each stage. By this procedure we obtain a sequence of nested intervals having a one point intersection, and it follows from continuity that this point must be a zero of f. This procedure may be terminated at any finite stage at which the interval obtained lies within a predetermined range of accuracy.

An important goal of domain theory is to identify essential mathematical features of such computational examples, abstract and axiomatize these features, and develop general constructions that provide an appropriate computational framework for the space [0, 1] and a much wider class of spaces besides. The construction of these computational frameworks will typically involve "powerset- or hyperspace-like" constructions. We model the trisection algorithm and other computational algorithms on $\mathbb{I} = [0, 1]$ in the "approximate unit interval"

$$\mathbf{P}\mathbb{I} := \{[a, b] : 0 \leq a \leq b \leq 1\}.$$

Points a of $\mathbb{I}$ are identified with the degenerate closed intervals $[a, a]$. Since successful algorithms for computing some real number compute smaller and smaller intervals ("approximate reals") containing that number, we order $\mathbf{P}\mathbb{I}$ with the "information ordering" (i.e., smaller intervals give more information about the point in question):

$$[a, b] \leq [c, d] \Leftrightarrow [a, b] \supseteq [c, d].$$

The computation of the trisection algorithm may be represented in the model $\mathbf{P}\mathbb{I}$ by an increasing sequence $[a_n, b_n]$ ($m \leq n$ implies $[a_m, b_m] \leq [a_n, b_n]$) with intersection point $\bar{x} = \bigcap_n [a_n, b_n]$, the directed sup. Since typical computational algorithms anvolve infinitely many iterations of increasing accuracy, in actual computations one must settle for approximate solutions of $a = [a, a]$ given by an interval $[a - \varepsilon, a + \varepsilon]$.

With this motivation we introduce the fundamental concept of this section. We abstain from the terminology "computational model" since we introduce a much more general mathematical structure.

Definition V-6.1. A *domain environment* for a topological space X is an embedding

$$j\colon X \leftrightarrow \operatorname{Max} P \hookrightarrow P$$

such that the double arrow represents a homeomorphism onto the space Max P of maximal elements of a domain P equipped with relative Scott topology and the hooked arrow is the inclusion map.

We often consider domains P satisfying the condition

(†) for all $p \in P$, ($\exists A$ Scott closed in P) $\uparrow p \cap \operatorname{Max} P = A \cap \operatorname{Max} P$,

where again Max P is the set of elements in P which are maximal in the partial order. Alternatively

(‡) Scott topology|Max $P =$ Lawson topology|Max P,

i.e., the Scott and Lawson topologies restricted to the set of maximal elements agree. □

Indeed the subbasic closed sets in the Lawson topology on P are either Scott closed or of the form $\uparrow p$ for some $p \in P$, and from this it follows easily that (†) and (‡) are equivalent. In this case, Max P is a regular Hausdorff space and a separable metric space in the case that P is countably based, since these assertions are true for the Lawson topology on P (see Corollary III-4.6 and Proposition III-3.7) and the properties are hereditary.

Example V-6.2. The approximate unit interval $\mathbf{P}\mathbb{I}$ is a domain environment satisfying (†) for $\mathbb{I}$, where the inclusion is the obvious one, $x \mapsto [x, x] : \mathbb{I} \to \mathbf{P}\mathbb{I}$. □

Example V-6.3. (The upper space) For a locally compact Hausdorff space X, there is a standard domain environment for X called the *upper space UX*, which consists of the set of all nonempty compact subsets of X ordered by reverse inclusion. Since X is locally compact, UX is a domain and the homeomorphic injection $x \mapsto \{x\} : X \to UX$ is a domain environment for X. The Scott topology on UX is the topology with basis

$$(U) := \{K \in UX : K \subseteq U\}, \text{ for all } U \text{ open in } X.$$

The Lawson topology is the usual Vietoris topology, or equivalently the topology on UX induced by the Hausdorff metric in case X is metrizable. This example also satisfies (†). □

The next example is a typical one arising in computer science settings.

Example V-6.4. **(The Cantor tree)** Consider the set P consisting of all finite and infinite strings of $\{0, 1\}$ (including the empty string $\perp$) ordered by the prefix order, i.e., one string is less than or equal to a second string if and only if it is a prefix of the second. The set of maximal elements Max P consists of all infinite strings. The restriction of the Scott (or Lawson) topology to Max P gives a space homeomorphic to the usual Cantor set. Hence the Cantor tree is a domain environment for the Cantor set. $\square$

Recall that a *Polish space* is a separable metric space for which the topology is given by a complete metric. Polish spaces have an important alternative characterization involving domain environments.

Lemma V-6.5. *A domain P satisfies condition* (†) *for the set X of maximal points iff given any $x \in X$ and any Lawson open set U containing x, there exists a Scott open set V such that $x \in V \subseteq U$.*

Proof: Suppose that (†) is satisfied. We may assume that U is a basic Lawson open set, i.e., is of the form $W \setminus {\uparrow} F$, where W is Scott open and F is finite. By condition (†) ${\uparrow} F \cap X$ is relatively Scott closed in X; hence there exists a Scott closed set A such that $A \cap X = {\uparrow} F \cap X$. Set $V = W \cap (P \setminus A)$. Then $x \in V$ and V is Scott open. We claim that $V \subseteq U$. Suppose there exists $y \in V \setminus U$. Then there exists a maximal element x^* above y, and since $y \notin U$, but $y \in W$, we conclude that $y \in {\uparrow} F$ and hence $x^* \in {\uparrow} F \cap X = A \cap X$. But $y \in V = {\uparrow} V$ also implies $x^* \in V \subseteq P \setminus A$, a contradiction.

Conversely suppose the second condition holds, and let $y \in P, x \in X \cap (P \setminus {\uparrow} y)$. By hypothesis there exists a Scott open set V such that $x \in V \subseteq P \setminus {\uparrow} y$. This shows that $P \setminus {\uparrow} y$ is relatively Scott open in X, and hence that (†) is satisfied. $\square$

Theorem V-6.6. *A topological space X has a countably based domain environment satisfying* (†) *if and only if it is a Polish space.*

Proof: Suppose that $j \colon X \to P$ is a domain environment satisfying (†), where P is a countably based domain. We identify X with its homeomorphic image in P. By Corollary III-4.6 the domain P is separable metrizable in the Lawson topology and by Proposition V-5.17 it is a Polish space. As a G_δ-subset of a Polish space is again a Polish space (see remarks after O-5.13), we need only show that X is a G_δ-subset of P.

Let d be a metric on P that gives the Lawson topology. Define A_n for each n by $y \in A_n$ iff $d(x, y) < 1/n$ for all $x \in X \cap {\uparrow} y$. Note that $X \subseteq A_n$ for all n. For each m and each $x \in X$ pick $x_m \ll x$ in P such that $\twoheaduparrow x_m \subseteq B_{1/m}(x)$;

this is possible by Lemma V-6.5 since the sets $\uparrow y$ form a basis for the Scott topology and $B_{1/m}(x)$ is Lawson open. For fixed n, pick $m = 2n$. Then for $x \in X$, $y \in \Uparrow x_m$, and $z \in X \cap \uparrow y$, we have $z \in \uparrow y \subseteq \Uparrow x_m \subseteq B_{1/m}(x)$. Thus $d(y, z) \le d(y, x) + d(x, z) < 1/m + 1/m = 1/n$. This shows that X is contained in the Scott interior of each A_n. Since every point not in X has a maximal point strictly above it, it follows that the intersection of the A_n is just X. Hence X is the countable intersection of their interiors. This complete the first half of the proof.

To prove the reverse direction we give a general construction of domain environments for metric spaces due to A. Edalat and R. Heckmann [Edalat and Heckmann, 1998] culminating in Proposition V-6.9.

First, however, we note a corollary of the proof.

Corollary V-6.7. *Let P be a countably based domain satisfying* (†). *Then the space of maximal points is a countable intersection of Scott open sets.*

Example V-6.8. (The domain of closed formal balls) Let (X, d) be a metric space. The set of *closed formal balls* is given by

$$\mathbf{B}X := X \times \mathbb{R}^+, \text{ where } \mathbb{R}^+ = [0, \infty).$$

Intuitively the pair (x, r) represents the closed formal ball of radius r around x. A partial order $\le$ of formal reverse inclusion is defined on $\mathbf{B}X$ by

$$(x, r) \le (y, s) \quad \text{if} \quad d(x, y) \le r - s.$$

If X be a normed linear space, then the ordered set of closed formal balls is order isomorphic to the set of closed balls ordered by reverse inclusion,

$$(\mathbf{B}X, \le) \approx (\{B_{\le\varepsilon}(x) \colon x \in X, \varepsilon \ge 0\}, \supseteq),$$

where the order isomorphism is the obvious one taking (x, r) to the closed ball of radius r around x. However, for more general metric spaces, this function need no longer be an order isomorphism. □

It follows immediately from the definition of the order that if $(x, r) \le (y, s)$, then $r \ge s$. Thus for any directed set (x_i, r_i) in $\mathbf{B}X$, it must be the case that the net $\{r_i\}$ is a decreasing net on $\mathbb{R}^+$, and hence must converge to its inf $r \ge 0$. Since $d(x_i, x_j) \le |r_i - r_j|$, we conclude that the net $\{x_i\}$ is a Cauchy net. If the metric is complete, then this net converges to some x, so

$$d(x, x_j) = \lim_i d(x_i, x_j) \le \lim_i (r_j - r_i) = r_j - r$$

and thus $(x_j, r_j) \le (x, r)$ for all j. Morever, if $(x_i, r_i) \le (y, s)$ for all i, then

$$d(x, y) = \lim_i d(x_i, y) \le \lim_i (r_i - s) = r - s$$

and thus $(x, r) \le (y, s)$, that is, (x, r) is the supremum of the directed set. Hence $\mathbf{B}X$ is a **dcpo** if X is a complete metric space.

There is a very nice connection between completeness for metric spaces and directed compdeteness for domains, namely, *a metric space is complete iff the domain of closed formal balls is directed complete*. We have just established one direction and leave the analogs reverse implication to the exercises.

But even more is true: *for a complete metric space X, the* **dcpo** $\mathbf{B}X$ *is a domain with*

$$(x, r) \ll (y, s) \Leftrightarrow d(x, y) < r - s.$$

Indeed suppose the condition holds and (u_i, t_i) is a directed set with supremum $(u, t) \ge (y, s)$. Then from our preceding computation $t_i \to t$ and $u_i \to u$, so

$$d(x, u) \le d(x, y) + d(y, u) < r - s + s - t = r + \lim_i t_i.$$

Since $d(x, u_i) \to d(x, u)$, we conclude that $d(x, u_i) < r + t_i$ for large i, i.e., $(x, r) \le (u_i, t_i)$. Thus $(x, r) \ll (y, s)$. We again leave the converse as an exercise. Since $(y, s + 1/n) \ll (y, s)$ for each n and the former is an ω-chain with supremum (y, s), we conclude that $\mathbf{B}X$ is a domain.

The domain $\mathbf{B}X$ *has maximal elements* $\operatorname{Max}(\mathbf{B}X)$ *consisting of all* $(x, 0)$, $x \in X$, *and satifies condition* (†). Indeed the maximality assertion follows directly from the definition of the order. We verify the validity of condition (†). Suppose that $(y, 0) \notin \uparrow(x, r)$. Then $\varepsilon := d(y, x) - r > 0$. Hence $\twoheaduparrow(y, \varepsilon)$ is a Scott open set containing $(y, 0)$ and missing $\uparrow(x, r)$, and thus we conclude that the complement of $\uparrow(x, r)$ intersected with $\operatorname{Max}(\mathbf{B}X)$ is relatively Scott open.

Since $(y, \varepsilon) \ll (x, 0)$ iff $d(x, y) < \varepsilon$, we have $\twoheaduparrow(y, \varepsilon) \cap X \times \{0\} = B_\varepsilon(y) \times \{0\}$. We obtain our final observation: *the embedding* $x \mapsto (x, 0) : X \to \mathbf{B}X$ *is an embedding from X onto the maximal points of* $\mathbf{B}X$. Hence *for a complete metric space X,* $\mathbf{B}X$ *gives a domain environment of X that satisfies condition* (†).

Via the domain of closed formal balls, one has available an order theoretic approach to the theory of metric spaces. Standard properties of metric spaces typically have very natural order theoretic counterparts in the corresponding domain of closed formal balls. For example *the metric space X is separable iff* $\mathbf{B}X$ *is countably based*. Indeed it is not difficult to show, given our preceding results, that if $\{x_n\}$ is a countable dense subset for X, then $\{(x_n, r) \colon r > 0$ and r is rational$\}$ is a countable base for $\mathbf{B}X$. Conversely given a countable base for $\mathbf{B}X$, then the Scott topology is second countable (III-4.5), and hence the subspace $X = X \times \{0\}$ is second countable, thus separable.

Combining all the preceding results together, we conclude as follows.

Proposition V-6.9. *Let (X, d) be a complete separable metric space, and let $\mathbf{B}X$ denote the ordered set of closed formal balls. Then $x \mapsto (x, 0)\colon X \to \mathbf{B}X$ is a countably based domain environment satisfying $(\dagger)$ for X.*

Let $\mathcal{L}$ denote the category with objects complete metric spaces and morphisms Lipschitz maps (f, c) where $f\colon X \to Y$ satisfies $d(f(x), f(x')) \leq cd(x, x')$ for all $x, x' \in X$. Let *DOM* denote the category with objects domains and morphisms Scott-continuous functions. Then there is a functor $\mathbf{B}\colon \mathcal{L} \to DOM$ which sends a space X to $\mathbf{B}X$, the domain of closed formal balls, and sends $(f, c)\colon X \to Y$ to $\mathbf{B}(f, c)\colon \mathbf{B}X \to \mathbf{B}Y$ defined by $\mathbf{B}(f, c)(x, r) = (f(x), cr)$. Thus the construction of the domain of closed formal balls is functorial on $\mathcal{L}$, and hence on any subcategory. In particular, it carries the complete separable metric spaces to domain environments for those spaces. $\square$

Exercises

Exercise V-6.10. Show that if the poset of closed formal balls of a metric space (X, d) is a **dcpo**, then the metric space is complete. $\square$

Exercise V-6.11. For the poset of closed formal balls of a metric space (X, d), prove that $(x, r) \ll (y, s)$ implies $d(x, y) < r - s$. $\square$

New notes

Dana Scott suggested in the early days of domain theory the possibility of using domains to study computability on metric spaces [Scott, 1970], although notions of computability on metric spaces had certainly surfaced much earlier. An early study of a variant notion of a maximal point space is found in [Weihrauch and Schreiber, 1981]. Maximal point spaces were studied by Kamimura and Tang [Kamimura and Tang, 1984] for the case that the domain environments were bounded complete domains. They called such spaces "total spaces".

The notion of a domain environment in the form presented in this section is due to [Lawson, 1997], [Lawson, 1998a]. The domain of closed formal balls was introduced and studied in [Edalat and Heckmann, 1998]. A more general concept of domain representations of classical spaces than that of a domain environment has been investigated in [Blanck, 1997], [Blanck, 1998], [Blanck, 1999].

Given some computational structure, one can seek to embed it in a larger structure that allows one to model computational algorithms and study computational questions. In this case one might seek a domain environment of the original structure that was especially adapted for such purposes, and in this case call the specific domain environment a “computational model”. The survey of Edalat [Edalat, 1997a] provides a variety of specific examples of this type.

K. Martin has introduced the notion of a “measurement” in the study of domain environments [Martin, D2000]. These are special functions from a domain into the nonnegative reals that “measure” how far an element is from the maximal elements. His work represents one important contribution to what one might refer to as “quantitative domain theory”, in which the domain is enriched with some appropriate numerical function that gives some measurement of how far one element is below another.

VI

Compact Posets and Semilattices

As the title of the chapter indicates, we now turn our attention from the principally algebraic properties of continuous lattices to the position these lattices hold in topological algebra as certain compact semilattices. Indeed, as the Fundamental Theorem VI-3.4 shows, complete continuous semilattices are exactly the compact semilattices with small semilattices in the Lawson topology. Thus, complete continuous semilattices not only comprise an intrinsically important subcategory of the category of compact semilattices but also form the most well-understood category of compact semilattices. In fact, there are only two known examples of compact semilattices which are *not* complete continuous semilattices; these are presented in Section VI-4. The paucity of such examples attests to the unknown nature of compact semilattices in general.

We begin the chapter with some background remarks on compact pospaces and topological semilattices. This is followed by a order theoretic description of the topology of a compact semilattice in Section VI-2. Starting from any compact topological semilattice whatsoever, we deduce that the topology may be derived from the order. Indeed the topology is a variant "liminf" topology, one considerably more complicated than earlier ones we have considered. This allows order theoretic descriptions of continuous semilattice morphims, closed subsemilattices, etc., much in the spirit that we have already encountered for the Lawson topology is Section III-1. While this section is not in the mainstream of our development of continuous lattice theory, Theorem VI-2.7, Lemma VI-2.8 and their corollaries VI-2.9 and VI-2.10 will prove invaluable in our further developments in Chapter VII.

Section VI-3, the principal section of the chapter, contains the Fundamental Theorem together with numerous other useful results about continuous lattices as compact semilattices. It identifies compact topological semilattices with a basis of subsemilattices as complete continuous semilattices equipped with the Lawson topology. Section VI-4 is devoted to the examples alluded to above, compact topological semilattices which do not have a basis of subsemilattices and are hence not domains. Section VI-5 considers chains and order arcs in compact semilattices.

Section VI-6 develops the important theory of stably compact spaces, which may be viewed as T_0-variants of compact pospaces. This provides a more general topological setting to the relationship between the Lawson and Scott topologies. Section VI-7 continues the theme of Chapter V by developing the spectral theory of stably compact spaces. This leads to a duality between the category of compact pospaces and continuous monotone maps, on the one hand, and a category of stably continuous frames, on the other.

VI-1 Pospaces and Topological Semilattices

There are several ways of interrelating a topology and a partial order. Our first definition singles out some of the topological properties of a relation that we often meet. We have already met one of these properties in Chapter III (see III-5.1).

Definition VI-1.1. Let X be a topological space. A partial order $\leq$ is said to be *lower semiclosed* if $\downarrow x$ is closed for each $x \in X$; *upper semiclosed* if each $\uparrow x$ is closed; *semiclosed* if it is both lower and upper semiclosed. The relation $\leq$ is said to be *closed* or have a *closed graph* if the relation $\leq$ is a closed subset of $X \times X$ in the product topology. In that case $(X, \leq)$ is called a *pospace*. □

Note that the concepts of semicontinuity and pospaces are symmetric with respect to the partial order; hence, all of the theorems and properties concerning them have order duals which are also valid. Note also that all pospaces are semiclosed.

Recall from O-1.2 that a net $(x_j)_{j \in J}$ in a poset is *directed* if for $i, j \in J$ there exists $k \in J$ such that if $k \leq m$, then $x_i \leq x_m$ and $x_j \leq x_m$. If the net is directed and the set $\{x_j \colon j \in J\}$ has a supremum x, then x is called the *directed sup of the net*. Filtered nets and filtered infs of nets are defined dually. The notions of directed sups and filtered infs give an "algebraic" notion of convergence in a poset – in general we desire that such algebraic convergence imply topological convergence.

Definition VI-1.2. Let $(X, \leq)$ be a poset equipped with a topology. The topology is said to be *compatible* if whenever x is the directed sup or filtered inf of a net $(x_j)_{j \in J}$, then the net converges to x topologically. □

Proposition VI-1.3. *Let $(X, \leq)$ be a poset equipped with a topology. Let $(x_j)_{j \in J}$ be a directed net in X.*

(i) *If $\leq$ is upper (resp. lower) semiclosed, x is the directed sup of the net, and if the net clusters to y in the topological space X, then $x \leq y$ (resp., $y \leq x$). Hence, if the relation is semiclosed, then $x = y$.*
(ii) *If X is compact and $\leq$ is semiclosed, then $(x_j)_{j \in J}$ has a directed sup to which it converges topologically. Hence, in this case the topology is compatible.*

Proof: (i) Suppose $\leq$ is upper semiclosed. Then for each j, $\uparrow x_j$ is closed. Since the net is directed, there exists an index k_0 such that $x_k \in \uparrow x_j$ for $k \geq k_0$. Hence, $y \in \uparrow x_j$; that is, $x_j \leq y$ for each $j \in J$. Thus $x = \sup x_j \leq y$.

Suppose now that $\leq$ is lower semiclosed. Then $\downarrow x$ is closed, and of course it contains $(x_j)_{j \in J}$. Since the net clusters to y, we have $y \in \downarrow x$; that is $y \leq x$.

(ii) If X is compact, then the net has a cluster point x. Since each $\uparrow x_j$ is closed and the net is eventually in this set, we have $x_j \leq x$ for each j. Thus, x is an upper bound. Suppose y is also an upper bound. Then $\downarrow y$ is closed and contains the net. Hence $x \in \downarrow y$, that is $x \leq y$. Therefore, x is a least upper bound. Since, as we have just seen, *any* cluster point is the directed sup of the net, there is a *unique* cluster point. But, as X is compact,this implies that the net converges to x. By what we have just proved and its dual, it follows that the topology on X is compatible. □

The next, obvious, proposition gives a straightforward equivalent form of the definition of a pospace in terms of open sets.

Proposition VI-1.4. *Let $(X, \leq)$ be a poset with a topology. The relation $\leq$ is closed iff whenever $a \not\leq b$, there exist open sets U and V with $a \in U$ and $b \in V$ such that if $x \in U$ and $y \in V$, then $x \not\leq y$. Hence, a pospace is Hausdorff.*

□

Definition VI-1.5. A subset A of a poset X is *order convex* (or simply *convex*) if $p \leq q \leq r$ and $p, r \in A$ always imply $q \in A$. For an arbitrary set $A \subseteq X$, the *order convex hull* $[A]$ of A is defined to be $\uparrow A \cap \downarrow A$; it is the smallest convex set containing A. A pospace X is *locally order convex* if X has a basis of open sets each of which is order convex. □

Proposition VI-1.6.

(i) *Let X be a topological space with an upper semiclosed partial order. If A is a compact subset of X, then $\downarrow A$ is Scott closed.*
(ii) *Let X be a pospace. If A is a compact subset, then $\downarrow A$, $\uparrow A$, and $[A]$ are closed subsets of X. Hence, in particular, $\leq$ is semiclosed.*

Proof: (i) Let D be a directed subset of $\downarrow A$. For each $d \in D$, $\uparrow d \cap A$ is nonempty and closed in A. Since A is compact, there exists an $a \in \bigcap\{\uparrow d \cap A: d \in D\}$. Thus, a is an upper bound for D, and hence $\sup D \in \downarrow A$.

(ii) Let π_1 denote projection into the first coordinate from $X \times A$ into X. Since $\downarrow A = \pi_1((\text{graph} \leq) \cap (X \times A))$, and since projection into the noncompact factor is a closed mapping, $\downarrow A$ is closed. Dually, $\uparrow A$ is closed, and hence $[A] = \downarrow A \cap \uparrow A$ is closed. □

Definition VI-1.7. A pospace X is said to be *monotone normal* if given two closed sets $A = \downarrow A$ and $B = \uparrow B$ such that $A \cap B = \emptyset$, then there exist open sets $U = \downarrow U$ and $V = \uparrow V$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$. □

Proposition VI-1.8. *If X is a compact pospace, then X is monotone normal.*

Proof: Let $A = \downarrow A$ and $B = \uparrow B$ be disjoint closed sets. Since X is compact Hausdorff, there exist open sets P and Q such that $A \subseteq P$, $B \subseteq Q$, and $P \cap Q = \emptyset$. Let $U = X \backslash \uparrow(X \backslash P)$ and $V = X \backslash \downarrow(X \backslash Q)$. By VI-1.6 U and V are open. Since $A = \downarrow A$, we have $A \subseteq U$; similarly $B \subseteq V$. Also $U \subseteq P$ and $V \subseteq Q$. Hence $U \cap V = \emptyset$. □

Corollary VI-1.9. *A compact pospace has a subbasis of open upper and open lower sets. Hence, it is locally order convex.*

Proof: Let X be a compact pospace with topology $\mathcal{U}$. Let $\mathcal{V}$ be the topology generated by the open upper and open lower sets. By definition $\mathcal{U}$ is finer than $\mathcal{V}$. Hence, if $\mathcal{V}$ is Hausdorff, then $\mathcal{U} = \mathcal{V}$.

Let $x, y \in X$, $x \neq y$. Then $x \not\leq y$ or $y \not\leq x$. Assume $x \not\leq y$. Then $\uparrow x \cap \downarrow y = \emptyset$. By VI-1.8 there exist open sets $U = \uparrow U$, $V = \downarrow V$ such that $x \in U$, $y \in V$ and also $U \cap V = \emptyset$. Thus $(X, \mathcal{V})$ is Hausdorff.

Since open upper and open lower sets are order convex and the intersection of order convex sets is order convex, the space is locally order convex. □

Proposition VI-1.10. *Let X be a compact pospace. Then X has a basis of order convex compact neighborhoods at each point.*

Proof: Let $p \in X$ and let U be an open neighborhood of p. By VI-1.9 there exists an open order convex set V such that $p \in V \subseteq U$. Since X is compact Hausdorff, there exists a compact neighborhood A of p such that $p \in A \subseteq V$. Then, by VI-1.6, the set $[A]$ is compact, and since it is the order convex hull of A, we find $[A] \subseteq V$. Thus, $[A]$ is a compact order convex neighborhood of p contained in U. □

We now specialize our considerations from pospaces to topological semilattices, but first we recall the definition.

Definition VI-1.11. Let S be a semilattice endowed with a topology. The meet operation is said to be *separately continuous* on S if for each $y \in S$, the function $x \mapsto xy$ from S to S is continuous. In this case S is called a *semitopological semilattice*. The meet operation is said to be *jointly continuous* (or *continuous*) if the function $(x, y) \mapsto xy$ is continuous from $S \times S$ into S, and in this case S is called a *topological semilattice*. If S is a lattice and the join operation is also continuous, then S is called a *topological lattice*.

Topological semilattices are not necessarily Hausdorff; however, we use the term "compact semilattice" as a shorthand notation for "compact Hausdorff topological semilattice". $\square$

Remark VI-1.12.

(i) *The meet operation is separately continuous iff given $x, y \in S$ and an open set U containing xy, there exists an open set V containing x such that $Vy \subseteq U$.*
(ii) *The meet operation is jointly continuous iff given $x, y \in S$ and an open set U containing xy, there exist open sets V and W such that $x \in V$, $y \in W$, and $VW \subseteq U$.* $\square$

Dual definitions and remarks can be made for the join operation in a sup semilattice or lattice. The next proposition lists several of the elementary properties of semitopological semilattices. Some of these results have appeared earlier, but we collect them here for convenient reference.

Proposition VI-1.13. *Let S be a Hausdorff semitopological semilattice.*

(i) *For each $x \in S$, $\downarrow x = Sx$ is a retract of S and hence closed;*
(ii) *the relation $\leq$ is semiclosed;*
(iii) *if U is open, then $\uparrow U$ is open (this holds without Hausdorffness).*

If further S is compact, then

(iv) *the topology of S is compatible;*
(v) *S is a complete semilattice, and hence has a least element* 0*. Also if S has a* 1*, it is a complete lattice;*
(vi) *for all $A \subseteq S$, we have $\downarrow A^- \subseteq (\downarrow A)^-$;*
(vii) *the semilattice S is meet continuous.*

Proof: (i) The mapping $y \mapsto xy$ is a retraction of S onto $\downarrow x$. A retract of a Hausdorff space is closed.

(ii) Let $\lambda_x \colon S \to S$ be defined by $\lambda_x(y) = xy$. Then $\uparrow x = (\lambda_x)^{-1}(\{x\})$, and hence is closed. Thus $\leq$ is semiclosed.

(iii) For an open set U, we have $\uparrow U = \bigcup\{(\lambda_x)^{-1}(U): x \in U\}$, and hence $\uparrow U$ is open.

(iv) and (v): By Proposition VI-1.3 filtered and directed sets converge to their greatest lower bounds and least upper bounds, respectively. Hence, by O-2.14, S is a complete semilattice. A complete semilattice with a 1 is a complete lattice (O-2.12).

(vi) Let $y \leq x$ for some $x \in A^-$. Then there exists a net $(x_j) \subseteq A$ converging to x. Then yx_j converges to $yx = y$ and $yx_j \in \downarrow A$ for each j. Hence, $y \in (\downarrow A)^-$.

(vii) This was shown in O-4.4. □

Proposition VI-1.14. *Let S be a Hausdorff topological semilattice. The partial order $\leq$ is closed, and hence S is a pospace.*

Proof: Define $f: S \times S \to S \times S$ by $f(x, y) = (x, xy)$. Then graph $\leq = f^{-1}(\Delta)$, where $\Delta = \{(x, x): x \in S\}$. But Δ is closed since S is Hausdorff. □

Note that the proof of VI-1.14 was used in proving III-2.9. The next result connects pospaces with domain theory.

Proposition VI-1.15. *A quasicontinuous domain, in particular a domain, is a pospace with respect to its Lawson topology.*

Proof: Suppose that $x \not\leq y$. By Lemma III-5.7 there exists a finite set F such that $x \in \Uparrow F \subseteq \uparrow F \subseteq L\backslash\downarrow y$ and $\Uparrow F$ is Scott open. Then $x \in U = \Uparrow F$ and $y \in V = L\backslash\uparrow F$ satisfy the conditions of Proposition VI-1.4, and thus the partial order is closed. □

Exercises

Exercise VI-1.16. (The Urysohn–Nachbin Lemma) Let X be a monotone normal pospace. If A is a closed upper set, B is a closed lower set, and $A \cap B = \emptyset$, show that there exists a continuous order preserving function $f: X \to \mathbb{I}$ such that $f(B) = 0$, $f(A) = 1$.

Hint. Construct inductively a collection of open sets U_r, where r is a dyadic rational between 0 and 1, with $U_1 = X$ and such that each U_r is lower and $r < s$ always implies $B \subseteq U_r \subseteq U_r^- \subseteq U_s \subseteq X\backslash A$ (by a process analogous to that employed in Urysohn's Lemma).

Define $f: X \to \mathbb{I}$ by $f(x) = \inf\{r \in \mathbb{I} : x \in U_r\}$. As in Urysohn's Lemma, f is continuous. It follows easily that f is order preserving and that we have $f(A) = 1$ and $f(B) = 0$. □

Definition VI-1.17. A metric p on a poset X is *radially convex* if $x \leq y \leq z$ implies that $p(x, y) + p(y, z) = p(x, z)$. □

Exercise VI-1.18. (The Urysohn–Carruth Metrization Theorem) Let X be a compact metrizable pospace. Show that there exists a radially convex metric giving rise to the original topology.

Hint. If g is a continuous order preserving function from X into $\mathbb{I} = [0, 1]$, then $W_g = \{(x, y) \in X \times X : g(x) < 1/2 < g(y)\}$ is an open subset of $X \times X$. By Exercise VI-1.16 if $y \not\leq x$, there exists a continuous order preserving function $g: X \to \mathbb{I}$ such that $g(x) = 0, g(y) = 1$. Hence $(x, y) \in W_g$. Since X is a compact metric space, there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ of continuous order preserving functions such that if $y \not\leq x$, then $(x, y) \in W_{gn}$, for some n. Then the mapping $g = (x \mapsto (g_n(x))_{n\in\mathbb{N}}): X \to \mathbb{I}^{\mathbb{N}}$ is a topological and order isomorphism. The metric $p(x, y) = \sum_n (|x_n - y_n|/2^n)$ is a radially convex metric on $\mathbb{I}^{\mathbb{N}}$; when restricted to $g(X)$, it gives a radially convex metric on X. □

Old notes

The notion of a pospace has proved useful in topological algebra and in some aspects of functional analysis, and, in particular, in the study of topological semilattices. Although there were certainly important forerunners to his work, apparently L. Nachbin was the first to explicitly define and to investigate the notion of a pospace ([Nachbin, B1965], originally published in 1960). Most of the important results of this section are due to him (Propositions VI-1.6(ii), VI-1.8, Definition VI-1.7, Corollary VI-1.9). J. H. Carruth's work on metrization appears in [Carruth, 1968], where a slightly weaker version of Exercise VI-1.18 appears.

Topological semilattices appear as early as [Nachbin, B1965]. They were also studied by workers in topological algebra as a special and important class of topological semigroups (see, e.g., [Anderson and Ward, 1961], [Koch, 1959], [Brown, 1965] for some of the earlier work on topological semilattices). This section gives some of the most basic results concerning topological semilattices from the folklore of the subject.

VI-2 Compact Topological Semilattices

In this section we extend some of the theory of continuous lattices to compact semilattices. Principally we show that the topology of a compact topological semilattice must arise as a certain liminf topology, and hence be definable from the order itself. This allows standard order theoretic descriptions of closed lower sets, closed subsemilattices, and continuous semilattice homomorphisms. Since this development goes beyond the main scope of the book, the reader may wish only to skim this material at first reading. Although such generalizations are usually more difficult and tedious than the corresponding results for continuous lattices (as we saw, for example, in the presentation of quasicontinuous lattices in Chapter III), we obtain such important results as an algebraic characterization of the topology of a compact semilattice (VI-2.6) and its consequences (VI-2.7, VI-2.8, VI-2.9).

We recall the standard topological notion of a pseudometric: it fails to be a metric only to the extent that $d(x, y) = 0$ does not necessarily imply that $x = y$.

Definition VI-2.1. Let S be a semilattice. A pseudometric d is said to be *subinvariant* if $d(ax, ay) \leq d(x, y)$ for all $a, x, y \in S$. □

Remark VI-2.2. *If d is a subinvariant pseudometric on S, then*

$$d(ax, by) \leq d(ax, ay) + d(ay, by) \leq d(x, y) + d(a, b).$$

Hence, by induction,

$$d(a_1 \dots a_n, b) = d(a_1 \dots a_n, b_n) \leq d(a_1, b) + \cdots + d(a_n, b). \qquad \square$$

The topology of any compact semigroup is defined by a set of subinvariant pseudometrics. This fact is one of the early theorems about compact semigroups, first proved by S. Eilenberg around 1938. Alternative proofs have been given by [Hofmann and Mostert, B1966], [Hofmann, 1970] and [Friedberg, 1972]. The proof is deferred until the exercises.

Proposition VI-2.3. *Let S be a compact topological semilattice, and let* $\mathbf{P}$ *be the set of continuous subinvariant pseudometrics on S. If $\mathcal{C}$ is an open cover of S, then there exist $d \in \mathbf{P}$ and an $\varepsilon > 0$ such that the set of all neighborhoods $N_\varepsilon(x) = \{y : d(x, y) < \varepsilon\}$ for $x \in S$ refines $\mathcal{C}$.* □

We next give a lattice theoretical characterization of convergence in a compact semilattice.

Lemma VI-2.4. *Let S be a compact semilattice and d a continuous subinvariant pseudometric on S. Let $(x_j)_{j \in J}$ be a net in S with limit x. Then for each sequence $f : \mathbb{N} \to J$, there exists a (monotone) sequence $f_0 : \mathbb{N} \to J$ with $f \leq f_0$ such that whenever $f_0 \leq g$ then $d(\underline{\lim}\, x_{g(n)}, x) = 0$.*

Remark. Recall that $\underline{\lim}\, x_{g(n)} = \sup_n \inf_{m \geq n} x_{g(m)}$.

Proof of lemma: Let $n \in \mathbb{N}$. Since $x = \lim x_j$, there is an index $f_0(n) \geq f(n)$ (and also $f_0(n) \geq f_0(n-1)$) such that $d(x_j, x) \leq 1/2^{n+1}$ for all $j \geq f_0(n)$. Let $g \geq f_0$. For each n, let $y_n = \inf_{m \geq n} x_{g(m)}$. Since a filtered net converges to its infimum (by the dual of Proposition VI-1.3), we have

$$y_n = \lim{}_k x_{g(n)} x_{g(n+1)} \ldots x_{g(n+k)}.$$

Employing the earlier Remark VI-2.2, we calculate

$$d(x_{g(n)} \ldots x_{g(n+k)}, x) \leq \sum_{0 \leq i \leq k} d(x_{g(n+i)}, x) \leq \sum_{1 \leq i \leq k+1} \frac{1}{2^{n+i}} \leq \frac{1}{2^n}$$

for all $k \in \mathbb{N}$. Thus, we conclude $d(y_n, x) \leq 1/2^n$ for all n. Since, by definition, $\underline{\lim}\, x_{g(n)} = \sup y_n$, and since this is a directed supremum, we find that $\underline{\lim}\, x_{g(n)} = \lim y_n$. Because y_n converges to both x and $\underline{\lim}\, x_{g(n)}$, we have $d(\underline{\lim}\, x_{g(n)}, x) = 0$. $\square$

Proposition VI-2.5. *If $(x_j)_{j \in J}$ is a net in a compact semilattice S converging to x, then $x = \underline{\lim}_f\ \underline{\lim}_n\ x_{f(n)}$, where f ranges in $J^{\mathbb{N}}$ and that set has the pointwise ordering.*

Remark. The proposition remains valid if only monotone f are considered.

Proof of proposition: Let d be an arbitrary subinvariant pseudometric on S. For each $g \in J^{\mathbb{N}}$, set $y_g = \underline{\lim}\, x_{g(n)}$. By Lemma VI-2.4 above, for every $f \in J^{\mathbb{N}}$ there exists an f_0 such that $d(y_g, x) = 0$ for all $g \geq f_0$. If F is any finite set in $\{g \in J^{\mathbb{N}} : g \geq f_0\}$, then $d(\inf_{g \in F} y_g, x) = 0$ by Remark VI-2.2. Since

$$\inf{}_{g \geq f_0} y_g = \inf{}_{F \text{ finite}} \inf{}_{g \in F} y_g = \lim{}_{F \text{ finite}} \inf{}_{g \in F} y_g,$$

because the first infimum is filtered, we conclude that $d(\inf_{g \geq f_0} y_g, x) = 0$. Once again, since

$$\underline{\lim}\, y_f = \sup{}_f \inf{}_{g \geq f}\, y_g = \lim{}_f \inf{}_{g \geq f}\, y_g,$$

because the supremum is directed, we conclude that $d(\underline{\lim}_f y_f, x) = 0$. Now this holds for any continuous subinvariant pseudometric d, and these generate the topology, thus we conclude that $x = \underline{\lim}_f y_f = \underline{\lim}_f \underline{\lim}_n x_{f(n)}$. $\square$

Theorem VI-2.6. *Let $(x_j)_{j \in J}$ be a net in a compact semilattice. Then the following are equivalent:*

(1) $x = \lim x_j$,
(2) $x = \underline{\lim}_f\ \underline{\lim}_n\ y_{f(n)}$ *where f ranges in $K^{\mathbb{N}}$ for all subnets $(y_k)_{k \in K}$ of the given net.*

Proof: If $x = \lim x_j$, then the same is true for any subnet. Hence (2) follows from Proposition VI-2.5.

Conversely suppose (2) holds. If x is not equal to $\lim x_j$, then there exists a subnet converging to some $z \neq x$ since S is compact. Again by Proposition VI-2.5 we have $z = \underline{\lim}_f \underline{\lim}_n y_{f(n)}$ for this subnet. But, by hypothesis, this double limit must be x, a contradiction. Thus, $x = \lim x_j$. $\square$

Theorem VI-2.7. *Let $f: S \to T$ be a semilattice homomorphism between compact semilattices. Then the following are equivalent:*

(1) *f is continuous;*
(2) *f preserves infs of arbitrary nonempty sets and directed sups;*
(3) *f preserves liminfs of nets.*

If, in addition, T is a continuous lattice, then these conditions are equivalent to

(4) *the lower adjoint $d: T \to S$ preserves $\ll$.*

Proof: The equivalence of (2) and (3) follows from the proof of Theorem III-1.8. The equivalence of (1) and (3) is a straightforward consequence of Theorem VI-2.6. Now suppose that T is a continuous lattice. Then (4) is equivalent to (2) by IV-1.4. $\square$

Note that in view of Theorems III-1.8 and VI-2.7 a semilattice morphism between compact unital semilattices is continuous iff it is Lawson-continuous.

Lemma VI-2.8. *Let S be a complete semilattice and T a subsemilattice. We denote by $\bigwedge(T)$ the set of infima of nonempty subsets of T. Let $\nearrow(T)$ denote the set of all suprema of directed subsets of T. We let $\bigwedge_\omega(T)$ and $\nearrow_\omega(T)$ denote the corresponding notions employing only countable subsets of T.*

(i) *If S is a compact topological semilattice, then the topological closure can be written as $T^- = \nearrow \bigwedge \nearrow_\omega \bigwedge_\omega(T)$.*
(ii) *If $T = \downarrow T$, then $T^- = \nearrow \nearrow_\omega (T)$.*

Proof: Since T is a subsemilattice, $\bigwedge_\omega(T)$ consists of meets of down-directed sequences in T. Since S is compact, these sequences converge to their infima. Hence $\bigwedge_\omega(T) \subseteq T^-$. It is easily verified $\bigwedge_\omega(T)$ is again a subsemilattice. Again since S is compact, upward directed sequences in $\bigwedge_\omega(T)$ converge to their suprema. Thus $\nearrow_\omega \bigwedge_\omega(T) \subseteq T^-$. Continuity of the meet operation implies $\nearrow_\omega \bigwedge_\omega(T)$ is a subsemilattice. By an argument which is essentially a repetition of what we have just done, one concludes that $\nearrow \bigwedge \nearrow_\omega \bigwedge_\omega(T) \subseteq T^-$.

Conversely if $x \in T^-$, then there exists a net in T converging to x. Now employ Theorem VI-2.6. $\square$

The next two results follow easily from the preceding lemma.

Proposition VI-2.9. *Let T be a subsemilattice of a compact semilattice S. The following statements are equivalent:*

(1) *T is closed (topologically);*
(2) *T is closed with respect to infs of arbitrary nonempty sets and directed sups (see* O-2.1*);*
(3) *T is closed with respect to taking liminfs.* □

Proposition VI-2.10. *Let $A = \downarrow A$ be a subset of a compact semilattice S. The following statements are equivalent:*

(1) *A is closed (topologically);*
(2) *A is Scott closed (that is, $A = \nearrow(A)$).*

Thus the closed lower sets are precisely the Scott closed sets. □

Exercises

Exercise VI-2.11. Prove Proposition VI-2.3.

Hint. By Urysohn's Lemma S can be topologically embedded in a product of intervals, $\mathbb{I}^A$. For each finite subset $F \subseteq A$ one defines a pseudometric on the product (and hence on S) by using the Euclidean metric d_F on the coordinates of F. Define a new pseudometric p_F in terms of d_F by

$$p_F(u, v) = \sup\{d_F(xu, xv): x \in S \cup \{1\}\}.$$

Continuity from (S, p_F) to (S, d_F) is clear. Using the compactness of S and continuity of the meet operation, one obtains continuity in the other direction. Hence p_F and d_F give rise to the same topology. It is easily verified that p_F is subinvariant. All such pseudometrics generate the topology of S; the compactness of S allows one to complete the proof. □

Exercise VI-2.12. Let S be a compact topological semilattice. Show that if $k \in S$ is a compact element, then $\uparrow k$ is open in S.

Hint. If a net (x_j) in $S \backslash \uparrow k$ converges to $x \in \uparrow k$, then Proposition VI-2.5 and the compactness of k imply $x_j \in \uparrow k$ for some index j. □

Old notes

The ideas behind the results of this section originate in the work of Lawson [Lawson, 1973]. The ideas are only implicit there although Theorem VI-2.7 and slightly weaker versions of VI-2.8, VI-2.9 and VI-2.10 do appear explicitly. H. Bauer and G. Gierz pointed toward the explicit characterization of convergence in compact semilattices given here; the pattern of proof was suggested by Gierz and Hofmann [SCS 34] (replacing a somewhat more technical version given by Lawson).

VI-3 The Fundamental Theorem of Compact Semilattices

The class of topological semilattices which, at each point, possess a basis of neighborhoods which are subsemilattices was early singled out as an extremely important class of semilattices – because of both its widespread occurrence and its greater theoretical tractability. In their study of the algebraic properties of these semilattices, Hofmann and Stralka [Hofmann and Stralka, 1976] discovered that the *compact* members of this class possessing a 1 are precisely the continuous lattices, in a sense to be made explicit shortly. (This identification was actually only implicit in the paper and explicitly pointed out shortly thereafter by Stralka.) The consequences of this realization have been far reaching for both the theory of topological semilattices and that of continuous lattices.

We repeat, for the sake of easy reference, Definition III-2.12:

Definition VI-3.1. A topological semilattice S is said to have *small semilattices at x* if the point x has a basis of neighborhoods which are subsemilattices of S. The semilattice S has *small semilattices* iff it has small semilattices at every point. □

Note that if S is regular and has small semilattices at x, then x has a basis of *closed* neighborhoods which are subsemilattices – because the closure of a subsemilattice is a subsemilattice. The next proposition is an easy consequence of the definition.

Proposition VI-3.2.

(i) *Let S be a topological semilattice with small semilattices, and let T be a subsemilattice (equipped with the relative topology). Then T has small semilattices.*

(ii) *Let $\{S_j : j \in J\}$ be a collection of topological semilattices with small semilattices. Then $\prod_J S_j$ endowed with coordinatewise operations and the product topology has small semilattices.* □

There are some rather useful reformulations of the property of having small semilattices at a point. (Compare II-1.14(3) and III-2.13, III-2.15.)

Proposition VI-3.3. *Let S be a locally compact Hausdorff topological semilattice. For $x \in S$ the following statements are equivalent:*

(1) *S has small semilattices at x;*
(2) *the semilattice $\downarrow x$ has small semilattices at x;*
(3) *if $U = \uparrow U$ is open and $x \in U$, then there exists a filter F such that $x \in \mathrm{int}(F) \subseteq F \subseteq U$;*
(4) *if V is open and $x \in V$, then there exists a $y \in V$ such that $x \in \mathrm{int}(\uparrow y)$.*

Proof: (1) implies (2): Straightforward.

(2) implies (4): Let V be open, $x \in V$. By regularity there exists an open set U such that $x \in U$ and the closure of U is compact and a subset of V. Let N be a neighborhood of x in $\downarrow x$ which is both a subsemilattice and a subset of $U \cap \downarrow x$. Then N^- is a compact semilattice and, hence, has a least element y. Since N is a neighborhood of x in $\downarrow x$, and since translation by x from S to $\downarrow x$ is continuous, we have $\{w \in S : xw \in N\}$ is a neighborhood of x in S. Since this set is contained in $\uparrow y$, then $\uparrow y$ is a neighborhood of x. Clearly $y \in V$.

(4) implies (3): Let $x \in U = \uparrow U$. Then there exists an element $y \in U$ such that $x \in \mathrm{int}(\uparrow y)$. Let $F = \uparrow y$.

(3) implies (1): Let U be an open set, $x \in U$. Without loss of generality we may assume that U^- is compact.

Since as a partially ordered space U^- is locally order convex, there exist open sets $V, W \subseteq U$ such that $x \in W$, $WW \subseteq V$, $VV \subseteq U$, and V is order convex in U^-. Choose a filter F such that $x \in \mathrm{int}(F) \subseteq F \subseteq \uparrow W$. Let $N = V \cap F$. Then $x \in \mathrm{int}(N) \subseteq N \subseteq U$. If $p, q \in N$, then there exist $u, v \in W$ such that $u \leq p$, $v \leq q$ (since $N \subseteq \uparrow W$). Then $uv \in WW \subseteq V$, $uv \leq pq \leq p \in V$, and $pq \in VV \subseteq W$ imply $pq \in V$ (since V is order convex in W). Also $pq \in F$, and thus $pq \in N$. Hence, N is a subsemilattice. □

We come now to the main theorem of this section and chapter.

Theorem VI-3.4. (The Fundamental Theorem of Compact Semilattices)

(i) *Let L be a complete continuous semilattice, resp. continuous lattice. Then with respect to the Lawson topology L is a compact, resp. unital, topological semilattice with small semilattices.*
(ii) *Conversely, if S is a compact, resp. unital, topological semilattice with small semilattices, then with respect to its semilattice structure S is a complete continuous semilattice, resp. continuous lattice. Furthermore, the topology of S is equal to the Lawson topology.*

(iii) *Let L and M be compact, resp. unital, topological semilattices with small semilattices, and let $f: L \to M$ be a semilattice homomorphism. The following are equivalent:*
(1) *f is continuous;*
(2) *f preserves directed sups and arbitrary nonempty infs.*

If, moreover, $f(1) = 1$, then (1) *and* (2) *are equivalent to*

(3) *the lower adjoint d of f exists and preserves the relation $\ll$.*

Remark. The functor Λ which assigns to a complete continuous semilattice, resp. continuous lattice, the semilattice endowed with its Lawson topology and is the identity on homomorphisms is an isomorphism from the category of complete continuous semilattices and morphisms preserving directed sups and arbitrary nonempty infs, resp. *CL* (see IV-1.7), to the category of compact, resp. unital, topological semilattices with small semilattices and continuous, resp. semilattice, homomorphisms preserving units. One may therefore identify the two categories.

Proof of theorem: It suffices to carry out the proof in the lattice (equivalently complete semilattice with 1) case, since the general case then follows by adjoining discrete 1s to all complete semilattices and taking identity preserving mappings.

(i) By Theorem III-2.15.

(ii) Conversely suppose S is a compact unital topological semilattice with 1 and with small semilattices. By Proposition VI-1.13, S is a complete lattice.

Let $x \in S$. By VI-3.3(4), $x = \sup\{y \in S : x \in \operatorname{int}(\uparrow y)\}$, since if $x \not\leq w$, $S \setminus \downarrow w$ is an open set around x. Suppose $x \in \operatorname{int}(\uparrow y)$, that is, there exists an open set U such that $x \in U \subseteq \uparrow y$. Then $x \in \uparrow U \subseteq \uparrow y$ and $\uparrow U$ is open by VI-1.13(iii). If D is directed and $\sup D \geq x$, then since D converges to $\sup D \in \uparrow U$, there exists $d \in \uparrow U \subseteq \uparrow y$. Hence $y \ll x$.Thus S is a continuous lattice.

To complete the proof, we must argue that the topology of S is the Lawson topology. It follows from Proposition VI-1.13 that each set of the form $S \setminus \downarrow x$ is open in S. Let V be a Scott open set, $x \in V$. By the preceding paragraph we have that $x = \sup\{z \in S : x \in \operatorname{int}(\uparrow z)\}$. This set is easily verified to be directed. Thus, there exists $z \in V$ such that $x \in \operatorname{int}(\uparrow z) \subseteq \uparrow z \subseteq V$. Hence, the identity function on S is continuous from S with the given topology to S with the Lawson topology. Since the given topology is compact and the Lawson topology is Hausdorff (see III-1.10), we conclude that they agree.

(iii) This part is a consequence of part (i), III-1.8 and IV-1.4 (where, if necessary, discrete (isolated) identities are adjoined to L and M and f is extended by sending the new identity of L to that of M). □

Theorem VI-3.4 is a powerful tool for the study of continuous lattices and compact semilattices. It allows an algebraic treatment of topological problems and vice versa. We illustrate this with the following important proposition.

Proposition VI-3.5. *Let S and T be compact topological semilattices and let f be a continuous homomorphism from S onto T. If S has small semilattices, then T has small semilattices, too.*

Proof: By adjoining discrete identities to S and T and extending f to be identity preserving, we may assume without loss of generality that S and T have identities and f is identity preserving. The proposition now becomes a corollary to Theorem VI-3.4 and I-2.11(iii). □

We close this section with three important examples. In general these examples seem better suited to a free-flowing exposition rather than a formal series of propositions and proofs, and hence are presented in this fashion (although many of the properties displayed actually could easily be presented as propositions).

Example VI-3.6. (The role of the unit interval) Let $\mathbb{I} = [0, 1]$ be the unit interval equipped with its usual topology and order. Then $\mathbb{I}$ is a compact connected topological semilattice (in fact a topological lattice) which has small subsemilattices. Furthermore any topologically closed subsemilattice of a product of copies of $\mathbb{I}$ with the relative topology is again a compact semilattice. Furthermore, this example is exhaustive in a way the following proposition makes precise.

Proposition VI-3.7. *Let S be a compact topological semilattice. The following statements are equivalent:*

(1) *S is a complete continuous semilattice with respect to its order structure;*
(2) *S has small semilattices;*
(3) *$Hom(S, \mathbb{I})$ separates points, where $Hom(S, \mathbb{I})$ denotes the set of all continuous semilattice homomorphisms;*
(4) *S is topologically isomorphic to a closed subsemilattice of a product of copies of $\mathbb{I}$.*

Proof: The equivalence of (1) and (2) follows easily from the Fundamental Theorem, and that of (1) and (3) was established in IV-3.2 (where we use the characterization of continuous homomorphisms as those preserving arbitrary meets and directed joins). To see that (3) implies (4), consider $T = \mathbb{I}^{\mathrm{Hom}(S,\mathbb{I})}$. Define $F\colon S \to T$ by $\pi_f(F(x)) = f(x)$. Then F is a continuous isomorphism; since S is compact, F is a homeomorphism. Finally, (4) implies (2) by Proposition VI-3.2 (or (4) implies (1) by IV-3.20). □

Example VI-3.8. (The Vietoris topology)

(i) $\Gamma(\cdot)^{\mathrm{op}}$ **as a functor**: Let X be a compact Hausdorff space. Let $\Gamma(X)$ denote the set of closed subsets ordered by inclusion. The lattice $\Gamma(X)^{\mathrm{op}}$ is isomorphic (via complementation) to the lattice of open sets and hence is a continuous lattice (see Example I-1.7(5)). The set $\Gamma(X)$ is an object of great interest to topologists, and is standardly endowed with the Vietoris topology in order to make it a topological space. The Vietoris topology has as subbasis sets of the forms

$$N(U) = \{A \in \Gamma(X) : A \subseteq U\} \quad \text{and} \quad D(V) = \{A \in \Gamma(X) : V \cap A \neq \emptyset\},$$

where U and V are open sets in X. Note that sets of the form $N(U)$ are open filters in the lattice $\Gamma(X)^{\mathrm{op}}$, and hence they generate the Scott topology since $\Gamma(X)^{\mathrm{op}}$ is a continuous lattice. Also $D(V)$ is the complement of the principal filter generated by $X\backslash V$, hence, these sets generate the lower topology on $\Gamma(X)^{\mathrm{op}}$. *Thus, the Vietoris topology is precisely the Lawson topology of the lattice* $\Gamma(X)^{\mathrm{op}}$ *and so is compact and Hausdorff.* Since in a continuous lattice convergence in the Lawson topology is liminf convergence, it follows that if a net of closed sets A_j converges to A, then A is indeed the limit (in the technical topological sense) of the A_j.

The assignment of $\Gamma(X)^{\mathrm{op}}$ to X extends to a functor from the category of compact Hausdorff spaces to the category *CL* of continuous lattices and Lawson-continuous identity preserving homomorphisms. If $f\colon X \to Y$ is a continuous function between compact Hausdorff spaces, define

$$\Gamma(f) = (A \mapsto f(A)) : \Gamma(X)^{\mathrm{op}} \to \Gamma(Y)^{\mathrm{op}}.$$

It is easily verified that

$$[f^{-1}] = (B \mapsto f^{-1}(B)) : \Gamma(Y)^{\mathrm{op}} \to \Gamma(X)^{\mathrm{op}}$$

is a lower adjoint for $\Gamma(f)$. From Proposition I-1.4(ii), it follows that $A \ll B$ if and only if $B \subseteq \mathrm{int}(A)$. Thus $[f^{-1}]$ preserves $\ll$, and therefore $\Gamma(f)$ is a *CL*-morphism.

(ii) **The free continuous lattice generated by a compact Hausdorff space:** We next observe that Γ is the "free" functor on the category of compact Hausdorff spaces to the category *CL* in the sense that if $f\colon X \to L$ is a continuous mapping from a compact Hausdorff space X into a continuous lattice L (equipped with the Lawson topology), then there exists a unique, continuous, identity preserving homomorphism F from $\Gamma(X)^{\mathrm{op}} \to L$ such that $F \circ i = f$ where $i = (x \mapsto \{x\}) : X \to \Gamma(X)^{\mathrm{op}}$. By standard categorical arguments, this is equivalent to saying that the functor Γ and the "forgetful" functor from *CL* to the category of compact Hausdorff spaces are adjoint functors.

To prove the assertion, let $f: X \to L$; then we have a continuous homomorphism $\Gamma(f)$: $\Gamma(X)^{\text{op}} \to \Gamma(L)^{\text{op}}$. By applying Proposition VI-3.9 below, the mapping $(A \mapsto \inf A)$: $\Gamma(L)^{\text{op}} \to L$ is a continuous homomorphism. Hence, we may let F: $\Gamma(X)^{\text{op}} \to L$ be the composition, which is continuous, identity preserving, and a semilattice homomorphism. Moreover

$$F \circ i(x) = F(\{x\}) = \inf\{f(x)\} = f(x).$$

The uniqueness of F follows from the fact $i(X)$ order generates $\Gamma(X)^{\text{op}}$. □

Proposition VI-3.9. *Let S be a compact unital topological semilattice. The function $A \mapsto \inf A : \Gamma(S)^{\text{op}} \to S$ is continuous iff S is a continuous lattice.*

Proof: Since $\{x\}$ goes to x, the function is onto. Hence, by Proposition VI-3.5, S is a continuous lattice if the function is continuous.

Conversely suppose S is a continuous lattice. It is easily verified that $x \mapsto \uparrow x : S \to \Gamma(S)^{\text{op}}$ is a lower adjoint. If $x \ll y$, then $\uparrow y \subseteq \text{int}(\uparrow x)$. Whence, $\uparrow x \ll \uparrow y$ in $\Gamma(S)^{\text{op}}$. Thus, the way-below relation is preserved; and therefore we have shown that $A \mapsto \inf A$ is continuous. □

The preceding discussion has given us a free continuous lattice over any compact Hausdorff space. But in I-4.19 we have shown that the free continuous lattice over a *set* is the lattice of filters on X. We may rederive this result now as follows: recall that the free compact Hausdorff space over the set X is the Stone–Čech compactification $\beta(X)$ of X as a discrete space. The free continuous lattice over $\beta(X)$ is $\Gamma(\beta(X))^{\text{op}}$. Since the composition of free functors is a free functor, it follows that $\Gamma(\beta(X))^{\text{op}}$ is the free continuous lattice over the set X. This latter turns out to be isomorphic to the lattice of filters on X. We suggest a detailed verification in the exercises (see VI-3.23) which can also be regarded as an alternative proof of I-4.19.

Example VI-3.10. (Another free construction) Let X be a compact pospace. Let $\Xi(X)$ denote the closed *upper* subsets of X ordered by inclusion. Then $\Xi(X)^{\text{op}}$ is a continuous lattice.

By way of proof note that if $A, B \in \Xi(X)^{\text{op}}$, and $A \subseteq \text{int}(B)$, then $B \ll A$. Since X is monotone normal (by VI-1.8), we have $A = \sup \twoheaddownarrow A$. Hence, $\Xi(X)^{\text{op}}$ is a continuous lattice.

The Lawson topology on $\Xi(X)^{\text{op}}$ is a modified version of the Vietoris topology on $\Gamma(X)^{\text{op}}$. Take for a subbase of open sets, sets of the form $N(U)$ and $D(V)$ where U is open in X and V is open in X and satisfies $V = \downarrow V$. In $\Xi(X)$ arbitrary meets are just intersections, finite joins are unions, and the join of $\{A_i: i \in I\}$ is given by $\uparrow A$ where $A = (\bigcup_{i \in I} A_i)^-$.

The lattice $\Xi(X)^{\mathrm{op}}$ is the "free" continuous lattice on the compact pospace X. Alternatively Ξ is the adjoint to the forgetful functor from the category *CL* to the category of compact pospaces and continuous order preserving mappings. The arguments are similar to those presented in Example VI-3.8 and are deferred to the exercises (see VI-3.20, VI-3.21). □

Remark. Since the opposite of a pospace is a pospace, the lattice $\Upsilon(X)$ of closed *lower* subsets is also a continuous lattice; however, it is not the "free" continuous lattice on X but on X^{op}.

Example VI-3.10 and the following Proposition VI-3.11 should be viewed in the context of Section VI-6. The hypothesis of VI-3.11 is slightly different from that of VI-3.10.

Proposition VI-3.11. *Let S be a compact semilattice.*

(i) $\Upsilon(S)$*, the lattice of closed lower sets, is the lattice of Scott closed sets.*
(ii) $\Upsilon(S)^{\mathrm{op}}$ *forms a continuous lattice.*
(iii) $\Upsilon(S)^{\mathrm{op}}$ *is a closed subspace of* $\Gamma(S)^{\mathrm{op}}$ *with the Lawson topology being the relative Vietoris topology.*

Proof: We saw in VI-2.10 that the Scott closed sets are precisely the topologically closed sets A such that $A = {\downarrow}A$. Another proof will be given in Section VII-1. This proves (i). The remarks in VI-3.10 establish (ii).

(iii) Let $A = {\downarrow}A$, let $x \in A^-$, and let $y \leq x$. We show $y \in A^-$. Let (x_j) be a net in A converging to x; by continuity $x_j y$ converges to $xy = y$. Since we have $A = {\downarrow}A$, $x_j y \in A$ for all j, and so $y \in A^-$. Thus $A^- = {\downarrow}A^-$.

If $(A_j)_{j\in J}$ is a collection of closed lower sets, then in $\Gamma(S)$ the meet of the collection is $\bigcap_{j\in J} A_j$ and the join is $(\bigcup_{j\in J} A_j)^-$. Since both of these are again closed lower sets, $\Upsilon(S)$ is closed in $\Gamma(S)$ by III-1.12. It follows from the Fundamental Theorem that the relative Vietoris topology must be the Lawson topology, since the latter is the only one making $\Upsilon(S)$ into a compact topological semilattice with small semilattices. □

In earlier chapters we have seen that algebraic lattices form a basic subcategory of the category of continuous lattices. We close this section with that version of the Fundamental Theorem which applies to them.

Lemma VI-3.12. *Let S be a compact topological semilattice. If the connected component of x in S is contained in ${\uparrow}x$, then S has small semilattices at x. Furthermore, x is the sup of compact elements.*

Proof: Let U be an open convex set in ${\downarrow}x$ containing x. Then ${\uparrow}U$ is open containing ${\uparrow}x$. Since every component in a compact Hausdorff space is the directed intersection of open and closed (clopen) sets, there exists a clopen set V such that $C \subseteq V \subseteq \uparrow U$, where C is the component of x in S. Let $Q = V \cap \uparrow x$. Then Q is clopen in $\downarrow x$. Let $W = \{y \in Q : yQ \subseteq Q\}$. By continuity of the meet operation W is clopen in $\downarrow x$. Clearly $x \in Q$ since x is an identity for $\downarrow x$. If $y, z \in W$, then $yzQ \subseteq yQ \subseteq Q$ and $yz = yzx \in yzQ \subseteq Q$. Thus $yz \in W$, that is, W is a subsemilattice. Now $W \cap \downarrow x \subseteq V \cap \downarrow x \subseteq \uparrow U \cap \downarrow x \subseteq U$ as U is convex.

By VI-1.10, the arguments of the preceding paragraph show that $\downarrow x$ has small semilattices at x, and hence by Proposition VI-3.3 so also does S.

Now W is a compact semilattice and by VI-1.13(v) has a least element z. Let D be a directed set with $z = \sup D$. Then by VI-1.13(iv) D converges to z. Since $D \subseteq \downarrow x$ and W is open in $\downarrow x$, there exists $d \in D$ such that $d \in W$. Hence $z \leq d$. Since S is meet continuous, z is a compact element. As U was an arbitrary open convex set around x and $z \in U$, we conclude x is the supremum of the compact elements below it. □

Theorem VI-3.13. (The Fundamental Theorem for Compact Totally Disconnected Semilattices)

(i) *Let L be an algebraic lattice. Then with respect to the Lawson topology L is a compact totally disconnected topological semilattice with unit.*
(ii) *Conversely if L is a compact totally disconnected topological semilattice with unit, then L has small semilattices and with respect to its semilattice structure is an algebraic lattice. Furthermore the topology of L is the Lawson topology.*

Proof: The proof follows from III-2.16, the Fundamental Theorem of Compact Semilattices, and VI-3.12. □

Note that we did not assume the existence of small semilattices in the preceding theorem. For compact totally disconnected topological semilattices this is guaranteed.

Exercises

Exercise VI-3.14. (i) Formulate and prove in the context of VI-3.13 the proper analogue to VI-3.4(iii).

(ii) Establish the analogue of VI-3.4 for compact semilattices for the totally disconnected case. □

Definition VI-3.15. A metric ρ on a semilattice S is called an *ultrametric* if

$$\rho(ax, by) \leq \max\{\rho(a, b), \rho(x, y)\}$$

holds for all $a, b, x, y \in S$. $\square$

Exercise VI-3.16. (i) If ρ is an ultrametric on a semilattice, then prove that each open and each closed ε-ball around a point is a semilattice.

(ii) Establish in addition the formula

$$\rho(xy, p) \leq \max\{\rho(x, p), \rho(y, p)\}. \qquad \square$$

Exercise VI-3.17. Let S be a compact metric topological semilattice. Show that the following are equivalent.

(1) S has small semilattices.
(2) The topology of S is given by an ultrametric.

Hint. Condition (2) implies (1) by VI-3.16. Conversely by VI-3.7 $\mathrm{Hom}(S, \mathbb{I})$ separates points. Since S is compact metric, countably many members of $\mathrm{Hom}(S, \mathbb{I})$ separate points. Embed S in $\mathbb{I}^{\mathbb{N}}$ with these homomorphisms. The metric $\rho((x_i), (y_i)) = \max_i\{|x_i - y_i|/2^i\}$ is an ultrametric on $\mathbb{I}^{\mathbb{N}}$, and, hence, is also one when restricted to the image of S. $\square$

The next exercise is a restatement of Example VI-3.8 for complete continuous semilattices.

Exercise VI-3.18. Let X be a compact Hausdorff space and let $\Gamma_0(X)$ denote the set of *nonempty* closed subsets ordered by inclusion. Prove the following.

(i) The Vietoris topology on $\Gamma_0(X)^{\mathrm{op}}$ is the Lawson topology; with respect to this topology $\Gamma_0(X)^{\mathrm{op}}$ is a compact topological semilattice with small semilattices.
(ii) Furthermore if $f\colon X \to S$ is a continuous mapping into a compact topological semilattice with small semilattices, then there exists a unique continuous homomorphism $F\colon \Gamma_0(X)^{\mathrm{op}} \to S$ such that $F \circ i = f$ where we define $i\colon X \to \Gamma_0(X)^{\mathrm{op}}$ by $i(x) = \{x\}$.

Hint. The proof follows from Example VI-3.8 by throwing in empty sets and identities where needed. $\square$

Exercise VI-3.19. Let S be a compact topological semilattice. Show that the function $A \mapsto \inf A : \Gamma_0(S)^{\mathrm{op}} \to S$ is continuous iff S has small semilattices.

Hint. Adjoin identities and use VI-3.9. $\square$

If S is a compact topological semilattice with small semilattices, then the mapping $A \mapsto \inf A$ is a retraction of $\Gamma_0(S)^{\mathrm{op}}$ onto S. It is an old result that if S is a Peano continuum, then $\Gamma_0(S)$ is an absolute retract; indeed it has been proved that $\Gamma_0(S)$ is topologically the Hilbert cube; that is, it is a countable product of intervals (see [Wojdysławski, 1939], [Curtis, 1974] and [Curtis and Schori, 1974]). Thus, in this case S is also an absolute retract.

Exercise VI-3.20. Let X be a compact pospace. Show that the mapping from X to $\Xi(X)^{\mathrm{op}}$, the lattice of closed upper subsets endowed with the Lawson topology, which sends x to $\uparrow x$ is continuous.

Hint. Suppose $B \in \Xi(X)$, $B \ll \uparrow x$. Then $\uparrow x \subseteq \operatorname{int}(B)$. If (x_α) is a net converging to x, then eventually $x_\alpha \in B$. Since $B = \uparrow B$, eventually $\uparrow x_\alpha \subseteq B$, that is, $B \leq \uparrow x_\alpha$. Suppose $D \in \Xi(X)$, $\uparrow x \not\subseteq D$. Since $D = \uparrow D$, $x \notin D$. Since D is closed, eventually $x_\alpha \notin D$. Thus for sets of the form $\twoheaduparrow B$ and $\Xi(X)\backslash \uparrow D$ which contain $\uparrow x$ we have eventually that $\uparrow x_\alpha$ belongs to such sets. Since such sets form a subbase for the Lawson topology for continuous lattices, we conclude that the injection of X into $\Xi(X)^{\mathrm{op}}$ is continuous. □

Exercise VI-3.21. Let P be a compact pospace and let $i\colon P \to \Xi(P)^{\mathrm{op}}$ be the embedding $x \mapsto \uparrow x$. If $f\colon P \to L$ is a continuous order preserving function from P into a compact topological semilattice with small semilattices, then there exists a unique continuous homomorphism $F\colon \Xi(P)^{\mathrm{op}} \to L$ such that $F \circ i = f$.

Remark. There are actually two versions of this exercise depending on whether one includes the empty set as a closed descending set or not. If it is included then L must have an identity, that is, be a continuous lattice; the empty set is then mapped to this identity.

Hint. Define $F\colon \Xi(P)^{\mathrm{op}} \to L$ by $F(A) = \inf f(A)$ for a closed upper set A. Let $M = \downarrow(f(P))$; note that $f(P)$ is compact and hence also M is. Also we note that $F(\Xi(P)) \subseteq M$. Define $G\colon M \to \Xi(P)^{\mathrm{op}}$ by $G(y) = f^{-1}(\uparrow y)$. Since f is continuous and order preserving $G(y) \in \Xi(P)$ (and is nonempty). A straightforward calculation gives that G is a lower adjoint for F (with codomain M). If $z \ll y$ in M, then by continuity of f, $f^{-1}(\uparrow y) \subseteq \operatorname{int}(f^{-1}(\uparrow z))$. Thus G preserves $\ll$. Therefore F is continuous from $\Xi(P)^{\mathrm{op}}$ to M and hence also to L. □

Exercise VI-3.22. Let S be compact topological semilattice. If $A, B \in \Upsilon(S)$, show that $B \ll A$ in $\Upsilon(S)^{\mathrm{op}}$ iff $A \subseteq \operatorname{int} B$.

Hint. Suppose $A \subseteq \operatorname{int} B$. If $\mathcal{D}$ is a descending family of closed lower sets and $\bigcap \mathcal{D} \subseteq A$, then by compactness $D \subseteq B$ for some $D \in \mathcal{D}$. Thus $B \ll A$ in $\Upsilon(S)^{\mathrm{op}}$.

Conversely suppose $B \ll A$. Define $\mathcal{D}$ by

$$\mathcal{D} = \{D \subseteq S : D \text{ is closed}, \downarrow D = D,\ A \subseteq \operatorname{int} D\}.$$

Then $\mathcal{D}$ is descending and using the monotone normality of S, one sees that in fact $A = \bigcap \mathcal{D}$. Since $B \ll A$ in $\Upsilon(S)^{\mathrm{op}}$, there exists $D \in \mathcal{D}$ such that $D \subseteq B$. Hence $A \subseteq \operatorname{int} B$. $\square$

Exercise VI-3.23. Prove the following.

(i) Let X be a set and let $(\beta(X), i)$ be the Stone–Čech compactification of the discrete X. Then $(\Gamma(\beta(X))^{\mathrm{op}}, j)$ is the free continuous lattice over X where $j(x) = \{i(x)\}$ for $x \in X$; that is, if $f\colon X \to L$ is a function into a continuous lattice L, then there exists a unique continuous homomorphism $F\colon \Gamma(\beta(X))^{\mathrm{op}} \to L$ such that $F \circ j = f$.

(ii) Alternatively let $\mathrm{Filt}\, 2^X$ denote the lattice of all set-theoretic filters on X ordered by inclusion (with the powerset of X included as the largest element of $\mathrm{Filt}\, 2^X$) and define $j\colon X \to \mathrm{Filt}\, 2^X$ by $j(x) = \{A \subseteq X : x \in A\}$. Then $(\mathrm{Filt}\, 2^X, j)$ forms the free continuous lattice over X. (See I-4.19).

(iii) Therefore, the two constructions are isomorphic.

Hint. (i) Let $f\colon X \to L$ where L is a continuous lattice. If L is equipped with the Lawson topology, then by the universal properties of $\beta(X)$, there exists an unique $f^{-}\colon \beta(X) \to L$ such that $f^{-} \circ i = f$. By Example VI-3.8 there exists a unique continuous homomorphism $F\colon \Gamma(\beta(X))^{\mathrm{op}} \to L$ such that $F \circ k = f$ where $k\colon \beta(X) \to \Gamma(\beta(X))^{\mathrm{op}}$ is defined by $k(y) = \{y\}$. Then $F \circ j = F \circ k \circ i = f \circ i = f$. The uniqueness follows from the uniqueness of f^{-} and F.

(ii) Since $F = \mathrm{Filt}\, 2^X$ is the set of all filters of the lattice of all subsets of X, we know that it is an algebraic lattice. If $f\colon X \to L$ is a function into a continuous lattice L, define $G\colon L \to F$ by

$$G(y) = \{A \subseteq X : f^{-1}(\uparrow z) \subseteq A \text{ for some } z \ll y\}.$$

Since $\{z\colon z \ll y\}$ is directed, it follows that $G(y) \in F$ for each $y \in L$.

We claim G is a lower adjoint for the function $F(\mathcal{F}) = \underline{\lim}\, \mathcal{F}$. Indeed suppose $F(\mathcal{F}) \geq y$. If $A \in G(y)$ then $f^{-1}(\uparrow z) \subseteq A$ for some $z \ll y$. By definition of F, there exists $B \in \mathcal{F}$ such that $z \leq \inf B$; that is, $B \subseteq f^{-1}(\uparrow z)$. Thus, $A \in \mathcal{F}$ since $\mathcal{F}$ is a filter. Hence $\mathcal{F} \supseteq G(y)$. Conversely, suppose $G(y) \subseteq \mathcal{F}$.

Then $f^{-1}(\uparrow z) \in \mathcal{F}$ for all $z \ll y$. Hence, $\underline{\lim}\,\mathcal{F} \geq \underline{\lim}\{z\colon z \ll y\} = y$, that is, $F(\mathcal{F}) \geq y$.

To show F is continuous it suffices by IV-1.4 to show that G preserves $\ll$. Let $z \ll y$. Then there exists $w \in L$ such that $z \ll w \ll y$. Then $\mathcal{F} = \{A \subseteq X : f^{-1}(\uparrow w) \subseteq A\}$ is a principal filter on the lattice of all subsets of X, and hence is a compact element of F. Since $G(z) \subseteq \mathcal{F} \subseteq G(y)$, we conclude $G(z) \ll G(y)$. Thus F is a continuous homomorphism.

Clearly $F \circ j = f$. Now every principal filter is the intersection of principal ultrafilters, every filter is the directed union of principal filters, and $j(X)$ is the set of principal ultrafilters. Hence, there exists at most one F such that $F \circ j = f$. Since we have seen that one does exist, it is unique. Therefore (F, j) is free over X.

(iii) The equivalence between the two preceding constructions for the free object is obviously obtained by using (i) and (ii) to get maps in both directions whose compositions are both identity functions. However, the conclusion can also be obtained by assigning to a filter all the ultrafilters containing it (and identifying $\beta(X)$ with the set of all ultrafilters on X). It turns out this association defines a lattice isomorphism from Filt 2^X to $\Gamma(\beta(X))^{\mathrm{op}}$. □

If we denote by *CS* the category of compact semilattices with identities and continuous semilattice morphisms preserving identities, and if we consider the category *CL* as a full subcategory of *CS* according to VI-3.4, then we can reinterpret Proposition VI-3.7 by saying that *CL* is the full subcategory in *CS* cogenerated by $\mathbb{I} = [0, 1]$. This allows us to apply Freyd's existence theorem to obtain a left reflection of *CS* into *CL*. However, the construction of the reflection of a compact semilattice S with identity into the category *CL* can easily be given explicitly.

Exercise VI-3.24. Let S be a compact semilattice with identity. Show that there is a universal continuous lattice quotient $q\colon S \to T$ such that all compact semilattice morphisms $S \to L$ into a continuous lattice L factor through q in a unique fashion.

Hint. Consider $H = \mathrm{Hom}(S, \mathbb{I})$ as in VI-3.7(3). Let $q^*\colon S \to \mathbb{I}^H$ be the evaluation map. Then $T = q^*(S)$ and the co-restriction q of q^* to its image satisfy the requirements. □

Exercise VI-3.25. Let $q\colon S \to T$ be as in Exercise VI-3.24. If s and t are in distinct connected components of S, show that $q(s) \neq q(t)$.

Hint. Consider the morphism $S \to S/R$ where R is the connectivity relation. Then S/R is a zero dimensional compact semilattice which is a continuous lattice by VI-3.13. Apply VI-3.24. □

Old notes

The notion of a topological semilattice with small semilattices was introduced and studied in the 1967 University of Tennessee dissertation of J. Lawson [Lawson, D1967]. The major results appeared in [Lawson, 1969]. The same idea appeared independently and simultaneously in a paper of M. McWaters [McWaters, 1969].

The problem of whether every compact topological semilattice has small semilattices attracted attention to this class of semilattices. In [Lawson, 1973] it was shown that the topology of any compact semilattice was an "intrinsic" topology, one that can be defined from the semilattice structure. This result indicated that these semilattices were some class of semilattices that could be defined in terms of the semilattice structure. Hofmann and Stralka [Hofmann and Stralka, 1976] addressed themselves to this problem and showed that a complete lattice L admitted a topology for which it was a compact topological semilattice with small semilattices (what they called a "Lawson semilattice") if and only if for every $x \in L$ there existed a smallest ideal I with $\sup I \geq x$. (Of course by I-2.1 this is equivalent to L being a continuous lattice.)

The first explicit version of the Fundamental Theorem to appear in print was given by J. Lea [Lea, 1976b].

Example VI-3.8 is a composite from numerous sources. Lawson observed in his dissertation that $\Gamma(X)^{\mathrm{op}}$ forms a topological semilattice with small semilattices for a compact Hausdorff space X. Hofmann (unpublished notes) recognized the "freeness" of the construction. A detailed treatment from a categorical viewpoint has been given by O. Wyler [Wyler, 1981a; Wyler, 1985].

A thorough treatment of algebraic lattices and compact totally disconnected topological semilattices appears in [Hofmann *et al.*, B1974]. Additional results related to Example VI-3.10 may be found in [Gierz and Keimel, 1977].

For a treatment of ultrametrics and an alternative approach to Exercise VI-3.17, see [Hofmann, 1970].

Several of the results of this section were discovered much earlier than the Fundamental Theorem and hence were proved without the machinery of continuous lattices. For example it was shown that a compact semilattice with small semilattices could be embedded in a product of intervals by techniques similar to those employed in the proof of Urysohn's Lemma. This type of approach appears in [Lawson, 1969].

VI-4 Some Important Examples

We give in this section two examples of unital compact topological semilattices which have no basis of subsemilattices and are thus *not* continuous lattices.

The examples are important because they show that the theory of compact topological semilattices stretches strictly beyond domain theory (although the latter is our chief interest in this work). The first example is topologically contained in the product of the unit interval and the Cantor set and, hence, is one dimensional and metric. The second example is constructed in terms of a space of closed convex subsets of a topological vector space and its topological structure, therefore, is not quite so immediate.

As in the previous section, a compact semilattice is tacitly assumed to be Hausdorff.

For the first example we develop a general method of construction and then apply it to a specific situation.

Proposition VI-4.1. *Let $(T, \wedge)$ and $(S, \cdot)$ be semilattices, and let $f\colon S \to T$ be any order preserving function. Set $W = \{(t, s) \in T \times S : t \leq f(s)\}$. Then W is a semilattice with respect to the operation*

$$(t, s) \wedge (u, v) = (t \wedge u \wedge f(sv), sv).$$

Proof: The set W is a partially ordered set with respect to the order inherited from $T \times S$. Clearly W is closed under this product, and a product is a lower bound to any pair of arguments. To show that it is the greatest lower bound, suppose that $(p, q) \leq (t, s)$ and $(p, q) \leq (u, v)$ hold for three pairs in W. Then $q \leq sv$, and hence $p \leq f(q) \leq f(sv)$. Thus, it follows that $p \leq t \wedge u \wedge f(sv)$. Therefore, $(p, q) \leq (t, s) \wedge (u, v)$, as we wished to show. □

Another way to regard this construction is to think of $(t, s) \mapsto (t \wedge f(s), s)$ as a kernel operator k on $T \times S$, which is a semilattice under the pointwise operation. The set W is just the range of k, which is easily proved to be a semilattice under the operation of composing k with the product in $T \times S$.

Let now T be a *continuous* lattice and $\{S_j\colon j \in J\}$ be a collection of *compact topological* semilattices. For each j let $f_j\colon S_j \to T$ be a given *continuous* order preserving function. Let $S = \prod_{j \in J} S_j$, and let $\pi_j : S \to S_j$ be the projection onto the jth coordinate. Define $f\colon S \to T$ by $f(s) = \bigwedge_{j \in J} f_j \pi_j(s)$. This function is order preserving. Define the semilattice W as in VI-4.1. In view of the topological assumptions, T, S and $T \times S$ are compact pospaces, and, since the maps $f_j \pi_j$ are continuous, W is closed and therefore compact – because we can write $W = \{(t, s) \in T \times S : t \leq f_j \pi_j(s)$ for all $j \in J\}$. To prove that W is a topological semilattice with respect to the operation of VI-4.1 and the relative topology, we require a further assumption.

Proposition VI-4.2. *W is a compact topological semilattice provided that for all $x, y \in T$ with $x \ll y$, there exists a finite $F \subseteq J$ such that $y \leq f(u) \wedge f(v)$ always implies $x \leq f_j \pi_j(uv)$ for all $j \notin F$.*

Proof: All that is really required to be shown is that the semilattice operation in W is continuous. To this end let (a_α, r_α) and $(b_\alpha s_\alpha)$ be two nets in W converging to (a, r) and (b, s) respectively. Clearly $r_\alpha s_\alpha$ converges to rs by hypothesis on S, so what remains to be shown is that $z_\alpha = a_\alpha \wedge b_\alpha \wedge f(r_\alpha s_\alpha)$ converges to $z = a \wedge b \wedge f(rs)$. (This would be easy if f were continuous, but that is not quite our assumption.) We wish to show that for any subbasic neighborhood U of z, eventually $z_\alpha \in U$.

Let $x \ll z$ and pick y such that $x \ll y \ll z$. Let F be the finite set promised by assumption. For an arbitrary i, we have $f_i\pi_i(r_\alpha s_\alpha)$ converges to $f_i\pi_i(rs)$. Since $z \leq f(rs) \leq f_i\pi_i(rs)$ and $y \ll z$, we have eventually that the net $f_i\pi_i(r_\alpha s_\alpha)$ is in $\uparrow y \subseteq \uparrow x$, because $\uparrow y$ is a neighborhood of z. Thus, because F is finite, there is a β_0 such that $x \leq f_i\pi_i(r_\alpha s_\alpha)$ for all $\alpha \geq \beta_0$ and $i \in F$.

Since $z \leq a \wedge b$, there exists a β_1 such that $\alpha \geq \beta_1$ implies $a_\alpha, b_\alpha \in \uparrow y$, by the same style of argument. By reference to the definition of W and since the nets are in W, we conclude that also $f(r_\alpha), f(s_\alpha) \in \uparrow y$ for $\alpha \geq \beta_1$. Hence, by our special assumption on F, we see that $x \leq f_j\pi_j(r_\alpha s_\alpha)$ for all $\alpha \geq \beta_1$, $j \notin F$.

Putting the two cases together, we have shown that eventually for *all* j *simultaneously* we have $x \leq f_j\pi_j(r_\alpha s_\alpha)$. Hence, eventually it is the case that $x \leq z_\alpha = a_\alpha \wedge b_\alpha \wedge f(r_\alpha s_\alpha)$. Since given any subbasic open set of the form $U = \{p\colon q \ll p\}$ for some $q \ll z$, we can find $x \ll z$ such that $q \ll x$, it follows from this argument that the net z_α is eventually in U.

The other type of subbasic open set is one of the form $T \backslash \uparrow d$ where $z \notin \uparrow d$. Hence either $a \notin \uparrow d, b \notin \uparrow d$ or $f(rs) \notin \uparrow d$. If $a \notin \uparrow d$ or $b \notin \uparrow d$, then eventually $a_\alpha \notin \uparrow d$ or $b_\alpha \notin \uparrow d$ respectively. Thus $a_\alpha \wedge b_\alpha \wedge f(r_\alpha s_\alpha) \notin \uparrow d$ eventually. If $f(rs) \notin \uparrow d$, then $f_j\pi_j(rs) \notin \uparrow d$ for some j. Since $f_j\pi_j$ is continuous, eventually $f_j\pi_j(r_\alpha s_\alpha) \notin \uparrow d$. Therefore eventually $a_\alpha \wedge b_\alpha \wedge f(r_\alpha s_\alpha) \notin \uparrow d$. Either way we eventually have $z_\alpha \in T \backslash \uparrow d$. □

We now specialize further by letting T denote $[0, \infty]$, the extended non-negative reals, which is a topological lattice with respect to its natural order. For each positive integer i, we will choose an integer $s(i) \geq 2$ and set $S_i = \{0, 1\}^{s(i)}$, a finite lattice with respect to the coordinatewise order. We require a lemma to give us a suitably divergent series governing the choice of the $s(i)$.

For each positive integer n larger than 1, we set $\alpha_n = 1/m2^{m-1}$ where $2^{m-1} < n \leq 2^m$. The series $\sum_{n \geq 2} \alpha_n$ may be thought of as the result of dividing the m^{th} term of the harmonic series into 2^{m-1} parts. Hence this series is divergent. The rate of growth is slow, however:

Lemma VI-4.3. *For any $\varepsilon > 0$, there is an integer $p \geq 1$ such that if $k \geq p$, then*

$$\sum_{n=2}^{k} \alpha_n + \varepsilon > \sum_{n=2}^{2k} \alpha_n.$$

Proof: We first note that $\sum_{n \in A} \alpha_n = 1/m$ if $A = \{n\colon 2^{m-1} < n \leq 2^m\}$. Choose q and p such that $2/\varepsilon < q$ and $2^{q-1} < p$. If $k \geq p$, there exists a unique m such that $2^{m-1} < k \leq 2^m$. Then

$$\sum_{n=2}^{2k} \alpha_n \leq \sum_{n=2}^{2m+1} \alpha_n = \sum_{n=2}^{2m-1} \alpha_n + (1/m + 1/(m+1)) \leq \sum_{n=2}^{k} \alpha_n + 2/m.$$

Since $m \geq q$, we have $2/m \leq 2/q < \varepsilon$; this completes the proof. □

For each positive integer i, let $s(i)$ be the least integer where

$$i \leq \sum_{n=2}^{s(i)} \alpha_n.$$

Such an integer exists since $\sum \alpha_n$ is divergent. For $x \in S_i$, let $\theta(x) \leq s(i)$ denote the number of zero entries of x. We define $f_i\colon S_i \to T$ by

$$f_i(x) = \begin{cases} \infty & \text{if } \theta(x) = 0, \\ i & \text{if } \theta(x) = 1, \\ 0 & \text{if } \theta(x) = s(i), \\ i - \sum_{n=2}^{\theta(x)} \alpha_n & \text{for all other cases.} \end{cases}$$

Lemma VI-4.4. (i) *Each f_i is a continuous order preserving function from S_i into T.*

(ii) *If $\tau > \varepsilon > 0$, there exists a positive integer q such that, for all $i \geq q$ and all $u, v \in S_i$, if $f_i(u) > \tau$ and $f_i(v) > \tau$, then $f_i(uv) > \tau - \varepsilon$.*

(iii) *Consequently, f satisfies the assumption of VI-4.2.*

Proof: (i) That each f_i is order preserving is a straightforward consequence of its definition. Continuity is trivial since the lattice S_i is finite.

(ii) Assume that $\tau > \varepsilon > 0$. Choose the p guaranteed by Lemma VI-4.3 which corresponds to ε. Choose q larger than $\tau + \sum_{2 \leq n \leq 2p} \alpha_n$.

We suppose that $i \geq q, u, v \in S_i$, $f_i(u) > \tau$ and $f_i(v) > \tau$. If $z = uv$, then either $\theta(z) \leq 2\theta(u)$ or $\theta(z) \leq 2\theta(v)$ obtains; we arbitrarily assume $\theta(z) \leq 2\theta(u)$. (The reason one of the inequalities prevails is that uv can have at most twice as many zero entries as either u or v.) We note from the definition of f_i

that in all cases

$$f_i(z) \geq i - \sum_{n=2}^{\theta(z)} \alpha_n,$$

if the summation is interpreted to be 0 for $\theta(z)$ equal to 0 or 1. If $\theta(u) \leq p$, then

$$f_i(z) \geq i - \sum_{n=2}^{\theta(z)} \alpha_n \geq q - \sum_{n=2}^{2\theta(u)} \alpha_n \geq q - \sum_{n=2}^{2p} \alpha_n \geq \tau.$$

The last inequality follows from the choice of q. Hence $f_i(z) > \tau - \varepsilon$ if $\theta(u) \leq p$.

If $p < \theta(u)$, then

$$f_i(z) \geq i - \sum_{n=2}^{\theta(z)} \alpha_n \geq i - \sum_{n=2}^{2\theta(u)} \alpha_n \geq i - \left(\sum_{n=2}^{\theta(u)} \alpha_n + \varepsilon\right) = f_i(u) - \varepsilon > \tau - \varepsilon.$$

Hence $f_i(z) > \tau - \varepsilon$ for both cases.

(iii) Suppose now that $x \ll y$ holds in T. The case $x = 0$ is trivial, so we suppose that the element is positive. We then interpolate τ and ε so that $x < \tau - \varepsilon < \tau < y$. We choose q as in (ii) and let F be the set of indices below q. If we then had $y \leq f(u) \wedge f(v)$ for $u, v \in S$, this would imply $f_i\pi_i(u) > \tau$ and $f_i\pi_i(v) > \tau$ for all indices i. But then by (ii), we would have $x \leq f_j\pi_j(uv)$ for all $j \notin F$ as desired. □

Theorem VI-4.5. *W is a unital compact topological semilattice without a basis of subsemilattices.*

Proof: Note that $1 = (\infty, (u_i))$ where each u_i has entries all 1, and that $1 \in W$. (For simplicity we are using the subscript notation rather than the projection notation on S.) All that remains to check is the basis assertion; in fact, we show that if A is a subsemilattice and $1 \in \mathrm{int}(A)$, then $A \cap (0 \times S) \neq \emptyset$.

There exists at 1 a basis of open sets of the form $U = \{(t, (u_i)) \in W : n < t$, u_i has entries all 1 for $i \leq n\}$, where n is a positive integer. We assume n is chosen so that $U \subseteq \mathrm{int}(A)$. Let B be the set of all elements of the form $(n+1, (u_i))$ such that u_i has entries all 1 for $i \neq n+1$ and u_{n+1} has one zero entry. Then B has $s(n+1)$ elements. For each element of B, $\inf\{f_i(u_i) : 1 \leq i\} = f_{n+1}(u_{n+1}) = n+1$; hence $B \subseteq W$ and thus $B \subseteq U$. Let $(t, (z_i))$ be the greatest lower bound in W of B. Then $(t, (z_i)) \in A$, since A is a subsemilattice. As $(t, (z_i)) \in W$, $t \leq f_{n+1}(z_{n+1}) = 0$, since z_{n+1} has entries all 0. Hence $t = 0$. This completes the proof. □

The reader should consult Exercise VII-2.13 for an important further development of this example.

Example VI-4.6. For the second example let V be any Hausdorff topological vector space and let K be a compact, convex subset of V. As in I-1.23 let $\mathrm{Con}(K)$ denote the set of closed convex subsets of K (including the empty set). Define the function $f\colon K \times K \times [0, 1] \to K$ by $f(x, y, t) = tx + (1 - t)y$. Then f is continuous since V is a topological vector space. The function f induces a continuous mapping $F\colon \Gamma(K) \times \Gamma(K) \times \Gamma([0, 1]) \to \Gamma(K)$ by

$$F(A, B, M) = \{ta + (1 - t)b\colon a \in A, b \in B, t \in M\}.$$

If $A, B \in \mathrm{Con}(K)$, then $F(A, B, [0, 1])$ is the closed convex hull of A and B. Hence $(A, B) \mapsto$ closed convex hull of $A \cup B$ is a continuous function on $\mathrm{Con}(K)$, and with respect to this operation $\mathrm{Con}(K)^{\mathrm{op}}$ is a compact topological semilattice.

Now if V is a locally convex vector space, then a closed convex subset of K has a basis of neighborhoods in K which are closed and convex. It is easily verified that if $K_1 \subseteq \mathrm{int}(K_2)$, then $K_2 \ll K_1$. Hence in this case $\mathrm{Con}(K)^{\mathrm{op}}$ is a continuous lattice (cf. I-1.23). Conversely, it is shown in [Lawson, 1976b] that if $\mathrm{Con}(K)^{\mathrm{op}}$ is a continuous lattice, then K can be embedded in a locally convex separated topological vector space by an affine homomorphism. J.W. Roberts [Roberts, 1977] has obtained examples of compact convex sets which have no extreme points and hence admit no such embedding; thus for such a K, $\mathrm{Con}(K)^{\mathrm{op}}$ is not a continuous lattice, although it is a compact unital topological semilattice. However, Roberts' constructions are at least as complicated as the one given for VI-4.5. □

Old notes

After the notion of a compact topological semilattice with small semilattices was introduced, it remained an open question for several years whether every compact topological semilattice had small semilattices. Lawson solved the problem in the negative with the first counterexample of this section which appeared in [Lawson, 1970].

An interesting topological question is to find topological properties which insure that a compact topological semilattice will have small semilattices. Theorem VI-3.13 states that total disconnectedness is such a condition. [Lawson, 1969] showed that this conclusion remains true for finite dimensional Peano continua. The most general class of spaces so far discovered appear in [Lawson, 1977]; this class includes spaces that locally are homeomorphic to a product of a totally disconnected space and a finite dimensional Peano continuum.

VI-5 Chains in Compact Pospaces and Semilattices

In this section we investigate the nature of chains (totally ordered sets) in pospaces, topological semilattices and lattices. Maximal chains are a particularly useful tool, and we invoke freely the axiom – equivalent to the Axiom of Choice – that in a poset every chain is contained in a maximal chain, the well-known Hausdorff Maximality Principle. Theorems VI-5.11 and VI-5.15 employ chains to give a criterion for connectedness.

The first proposition is quite straightforward.

Proposition VI-5.1. *If M is a maximal chain in a poset equipped with a topology, then we have*

$$M = \bigcap\{{\downarrow}x \cup {\uparrow}x \colon x \in M\}.$$

Hence, if $\leq$ is semiclosed, M is closed. □

Corollary VI-5.2. *If $\leq$ is semiclosed, then the closure of a chain is a chain.*

Proof: Any chain is contained in a maximal chain, which is closed and contains the closure of the given chain. Thus, the closure being contained in a chain is itself a chain. □

The next proposition is due to A. D. Wallace [Wallace, 1945] and is one of the oldest results in the theory of topological ordered spaces.

Proposition VI-5.3. *Consider a poset equipped with a compact topology for which the order is lower semiclosed. Then any element has a minimal element below it.*

Proof: Let M be a maximal chain containing a given element q. Then $\{{\downarrow}x \colon x \in M\}$ is a tower of closed sets whose intersection is nonempty since the space is compact. Let p be in the intersection. Then $\{p\} \cup M$ is a chain, and hence $p = \inf M \in M$. Then p is minimal in the whole poset, for otherwise the chain M could be extended. □

Proposition VI-5.4. *If $\leq$ is semiclosed and C is a compact chain, then the relative topology of C is the order topology. Moreover, if C is nonempty, then C is complete.*

Proof: Since $\{x \in C : a < x < b\} = C \backslash ({\downarrow}a \cup {\uparrow}b)$, the relative topology is finer than the order topology. Since the order topology is Hausdorff, the two agree.

In case C is nonempty, then by VI-5.3 it has a minimum and a maximum element; hence, in particular, every subset is bounded from below. If $S \subseteq C$ is nonempty, let L be the set of lower bounds of S. The family of closed intervals $[x, y]$ for $x \in L$ and $y \in S$ must have a nonempty intersection in C; it is easy to argue that the intersection is in fact $\{\inf S\}$. $\square$

Definition VI-5.5. Let X be a pospace. We say $A \subseteq X$ is an *arc chain* iff A is a nontrivial, compact, connected chain. $\square$

Since we have just seen that the relative topology on an arc chain A is the order topology, it follows that, topologically, A is an arc, that is, a continuum with exactly two noncutpoints.

Proposition VI-5.6. *Let X be a pospace, and let $A \subseteq X$.*

(i) *If A is an order dense compact chain, then A is an arc chain.*
(ii) *If X is compact and order dense and A is a maximal chain, then A is either an arc chain or a point.*
(iii) *If X has a 0 and 1 and A is a connected chain containing 0 and 1, then A is a maximal chain.*

Proof: (i) The proof that A is connected is analogous to the proof that the unit interval is connected and can be left to the reader.

(ii) It follows easily from hypothesis that every maximal chain is order dense; hence, the conclusion follows from part (i) and VI-5.1.

(iii) Suppose A is not maximal. Then there exists $p \in X \backslash A$ such that $A \subseteq {\uparrow} p \cup {\downarrow} p$. Then $A \cap {\uparrow} p$ and $A \cap {\downarrow} p$ are closed, nonempty, disjoint subsets of A, which contradicts the assumption that A is connected. $\square$

Proposition VI-5.7. *Let X be a compact pospace. Every convergent net of arc chains in X converges in the space of closed subsets of X endowed with the Vietoris topology to an arc chain or a point.*

Proof: Let A be the limit of such a net. It is well known that the limit of continua is a continuum. Also a set K is a chain if and only if $K \times K \subseteq (\leq \cup \geq)$, and the latter is a closed set, since X is a pospace. Hence, in the Vietoris topology, the family of all closed subsets whose squares are contained in $(\leq \cup \geq)$ is closed; thus, A is a chain. $\square$

Definition VI-5.8. Let X be a pospace. A point $p \in X$ is a *local minimum* iff there exists an open set U with ${\downarrow} p \cap U = \{p\}$; that is, $\{p\}$ is open in ${\downarrow} p$. $\square$

The next result was the discovery of R.J. Koch [Koch, 1959] and is one of the principal results in the theory of pospaces.

Theorem VI-5.9. (Koch's Arc Theorem) *Let U be an open subset with compact closure in a pospace X. If U contains no local minimum, then every point of U lies on an arc chain which meets the boundary of U.* □

The proof of this theorem is rather lengthy and is deferred to the exercises. The idea of the proof is to employ the nonexistence of local minima to construct, for each neighborhood $\mathcal{U}$ of the diagonal $\Delta = \{(x, x) \in X \times X\}$, a chain in U^- from p to $X \setminus U$ such that the chain is $\mathcal{U}$-connected (that is, if the chain is written as the disjoint union of two nonempty sets P and Q, then there exist $a \in P$, $b \in Q$ such that $(a, b) \in \mathcal{U}$). One then takes a limit of these chains over all neighborhoods $\mathcal{U}$ of the diagonal in the compact space of closed subsets of U^-. This limit is the desired chain.

We turn now to the topic of the existence of arc chains in topological semilattices.

Lemma VI-5.10. *Let S be a semitopological semilattice.*

(i) *Then $k \in S$ is a local minimum iff $\uparrow k$ is open.*
(ii) *If S is a compact topological semilattice, then $k \in S$ is a local minimum iff $k \in K(S)$.*

Proof: (i) If $\uparrow k$ is open, clearly k is a local minimum. Conversely, if $\{k\}$ is open in $\downarrow k = Sk$, then $\uparrow k$ is open as the inverse image of $\{k\}$ under the continuous map $x \mapsto xk : S \to \downarrow k$.

(ii) If $\uparrow k$ is open, then $k \in K(S)$ as long as the topology is compatible (VI-1.2). Conversely if $k \in K(S)$, then $\uparrow k$ is open by Exercise VI-2.12. □

Theorem VI-5.11. *Let S be a compact semilattice. The following statements are equivalent:*

(1) *S is connected;*
(2) 0 *is the only compact element of S;*
(3) 0 *is the only local minimum in S;*
(4) *each point of S lies on an arc chain containing* 0.

Proof: The equivalence of (2) and (3) follows from Lemma VI-5.10. That (3) implies (4) follows easily from Theorem VI-5.9 applied to the open set $U = S \setminus \{0\}$. If (4) holds, then S is arcwise connected and hence connected. Finally if S is connected, then for $x \in S$, we have $\downarrow x$ is connected. If x is a local minimum, then $\{x\}$ is open and closed in $\downarrow x$, so $\downarrow x = \{x\}$, whence $x = 0$. Hence, (1) implies (3). □

It is not necessarily the case that there exists an arc chain between x and y whenever $x \leq y$ in a compact connected topological semilattice. It becomes of interest to consider those pairs which are so connected, and we write $x \dashv y$ for this relationship (including the case $x = y$ and always implying $x \leq y$).

Proposition VI-5.12. *Let S be compact semilattice. Then the following hold:*

(i) $\dashv$ *is a partial order having a closed graph;*
(ii) $x \dashv y$ *iff* $x \leq y$ *and* $[x, y]$ *is connected.*

Proof: (i) Everything is immediate except that $\dashv$ is closed. Let nets (x_α) converge to x, (y_α) converge to y, and $x_\alpha \dashv y_\alpha$. For each α, let A_α be an arc chain between x_α and y_α. Then by Proposition VI-5.7 a subnet of the A_αs converges to an arc chain A containing x and y. Hence, $x \dashv y$.

(ii) Suppose $x \dashv y$. Then there exists an arc chain A between x and y. Then $[x, y] = \bigcup\{Az \colon x \leq z \leq y\}$. Since each Az is connected and contains x, their union is connected; hence $[x, y]$ is connected.

Conversely, suppose $[x, y]$ is connected. Then $[x, y]$ is a compact connected topological semilattice, and, by Theorem VI-5.11, there exists an arc chain between x and y. □

Definition VI-5.13. A topological semilattice S is *order connected* iff $[x, y]$ is connected for all $x, y \in S$ with $x \leq y$. □

Proposition VI-5.14. *The following statements are equivalent in a compact topological semilattice S:*

(1) *S is order connected;*
(2) *the relations* $\leq$ *and* $\dashv$ *agree;*
(3) $\uparrow x$ *is connected for all* $x \in S$*;*
(4) *S is order dense.*

If S is unital, the above are also equivalent to

(5) *for all* $x \in S$*, there exists an arc chain from* 1 *to* x*.*

Proof: The equivalence of (1) and (2) follows from Theorem VI-5.11.

Since $\uparrow x = \bigcup_{x \leq y}[x, y]$, (1) implies (3). Conversely if $\uparrow x$ is connected, then $[x, y] = (\uparrow x)y$ is connected.

Clearly (2) implies (4). Conversely, if $x < y$, let A be a maximal chain between x and y. Then A is closed and order dense and hence an arc chain (see Proposition VI-5.6).

To conclude the proof, note that if S has a 1, then (2) implies (5). Assume (5) and let $x, y \in S$, $x < y$. If A is an arc chain from 1 to x, then yA is a connected set containing x and y. Hence, $\uparrow x$ is connected. Thus, (5) implies (3). □

We consider now the existence of arc chains in topological lattices. The third condition gives an algebraic characterization.

Proposition VI-5.15. *Let L be compact topological lattice. The following statements are equivalent:*

(1) *L is connected;*
(2) *if $x < y$, then there exists an arc chain between x and y;*
(3) *L is order dense.*

Proof: (1) implies (2): Since L is compact and connected and since $[x, y] = y(L \vee x)$, we have that $[x, y]$ is a compact connected topological lattice. Hence by VI-5.11 (applied to $[x, y]$) there exists an arc chain between x and y.

(2) implies (3): Trivial.

(3) implies (1): Let x and y be elements of L, and let A and B be maximal chains containing $\{x, xy\}$ and $\{y, xy\}$ respectively. By VI-5.6, A and B are arc chains which both contain xy. Thus, x and y lie in the same component of L. Hence, L is connected. $\square$

Exercises

Exercise VI-5.16. Let X be a compact pospace, let $x < y$, and let U and V be disjoint open sets containing x and y respectively. Define a new relation $\preceq$ by

$p \preceq q$ iff $p \leq q$ and either $(p, q) \notin U \times V$ or there exists $t \notin U \cup V$ such that $p \leq t$ and $t \leq q$.

(i) Show $\preceq$ is a closed partial order contained in the original order.
(ii) Show that X has the same set of local minima for both orders.

Hint. (i) Reflexivity and antisymmetry of $\preceq$ are immediate. Suppose $x \preceq y$ and $y \preceq z$. Then $x \leq y \leq z$. If $(x, z) \notin U \times V$, then $x \preceq z$. If $x \in U$, $z \in V$ and $y \notin U \cup V$, then $x \preceq z$. If $y \in U$, then $y \preceq z$ implies there exists $w \notin U \cup V$ such that $y \leq w$ and $w \leq z$. Then $x \leq w$ and $w \leq z$ imply $x \preceq z$. Similarly if $y \in V$. Hence $\preceq$ is transitive.

Let (x_α, y_α) be a net in $X \times X$ converging to (x, y) such that $x_\alpha \preceq y_\alpha$ for each α. Then $x_\alpha \leq y_\alpha$ for each α, and hence $x \leq y$. If $(x, y) \notin U \times V$ then $x \preceq y$. Otherwise suppose $x \in U$, $y \in V$. Then eventually $x_\alpha \in U$ and $y_\alpha \in V$. Thus, eventually there exists $w_\alpha \notin U \cup V$ and then clearly $x_\alpha \leq w_\alpha \leq y_\alpha$. The net

w_α clusters to some $w \notin U \cup V$ such that $x \leq w \leq y$. Hence $x \preceq y$, and $\preceq$ is a closed relation.

(ii) A local minimum for $\leq$ is clearly one for $\preceq$. Conversely let p be a local minimum for $\preceq$. If $p \notin V$, then the lower set of p is the same for $\leq$ and $\preceq$. Hence p is a local minimum for $\leq$. If $p \in V$, then since $U \cap V = \emptyset$ the lower set of p intersected with V is the same for $\preceq$ and $\leq$. □

Exercise VI-5.17. Let X be a compact Hausdorff space and let U be an open subset. If $\mathcal{P}$ is a descending family of closed partial orders on X each of which has no local minimumin U, show that the intersection is also a closed partial order with no local minimum in U.

Hint. The intersection is easily seen to be a closed partial order. Let $x \in U$ and let W be an open set containing x. Pick open sets N and V such that $x \in N \subseteq N^- \subseteq V \subseteq V^- \subseteq W \cap U$.

Let $\leq$ be a partial order in $\mathcal{P}$. By Proposition VI-5.3 there exists an element y minimal (with respect to $\leq$) in N^- such that $y \leq x$. Since y is not a local minimum, there exists $z \in V$ such that $z < y$. Then we must have $z \in V \backslash N^-$. Hence for each partial order $(\leq)_\alpha \in \mathcal{P}$, there exists $z_\alpha \in V \backslash N^-$ such that $z_\alpha < x$. Since each z_α lies in the compact set $V^- \backslash N$, they cluster to some $z \in V^- \backslash N$. One argues that the pair (z, x) is in each partial order and hence in the intersection. Since $z < x$ and $z \in W$, we conclude x is not a local minimum. □

Exercise VI-5.18. Prove Theorem VI-5.9.

Hint. We may assume that X itself is a compact pospace. By picking a maximal chain of closed partial orders containing the given order which have no local minimum in U and taking the intersection, one obtains (VI-5.17) a minimal such partial order. In this new order pick a maximal chain M containing p, and let $q = \sup(M \cap {\downarrow} p) \backslash U$. Since M is closed, $q \in M \backslash U$. To complete the proof, we need only show that $A = [q, p] \cap M$ is an arc chain.

If A is order dense, then by Proposition VI-5.6 it is an arc chain. But if $x, y \in A$, $x < y$, and $[x, y] \cap A = \{x, y\}$, then $[x, y] = \{x, y\}$ (otherwise one extends the maximal chain M by adding some element between x and y). But then (VI-5.16) one eliminates thepair (x, y) from the order, contradicting the fact the order is minimal. Hence, A is order dense. □

Old notes

Koch's Arc Theorem was one of the major early advances in the theory of pospaces and topological semilattices and has continued to be an important tool. The proof given in Exercises VI-5.16 through VI-5.18 is Ward's [Ward, 1965a]. Koch's work led to a detailed study of the existence of arc chains in topological semilattices and lattices (see, e.g., [Anderson and Ward, 1961], [Brown, 1965], [Lawson, 1969]).

VI-6 Stably Compact Spaces

Compactness alone is a rather weak hypothesis for T_0 spaces; for example, compact spaces need not be locally compact. However, there is a particularly well-behaved and important class of spaces that has emerged which does appear to be a suitable generalization of compact Hausdorff spaces to the T_0-setting. In this section we develop the basic properties of these spaces, called stably compact spaces, and their close connections with compact pospaces. In relation to domain theory, stably compact spaces are analogous to compact domains equipped with the Scott topology, while the associated patch topology that we study in this section is analogous to the Lawson topology.

We begin with a strengthening of certain conclusions in the machinery of the Hofmann–Mislove Theorem (II-1.20).

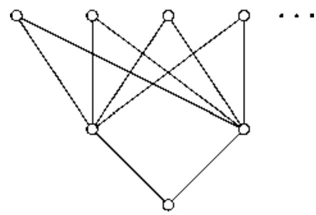
Lemma VI-6.1. *Let X be a sober space. Let $\mathcal{K}$ be a nonempty filtered family of compact saturated sets and let $\mathcal{C}$ be a filtered family of closed sets. If for each $C \in \mathcal{C}$ and $K \in \mathcal{K}$, we have $C \cap K \neq \emptyset$, then the intersection $(\bigcap \mathcal{C}) \cap \bigcap \mathcal{K}$ is compact and nonempty. Furthermore, every open set containing the intersection contains some $C \cap K$.*

Proof: Set $A = (\bigcap \mathcal{C}) \cap \bigcap \mathcal{K}$ and $K_0 = \bigcap \mathcal{K}$. If $K_0 \cap C = \emptyset$ for some $C \in \mathcal{C}$, then $\bigcap \mathcal{K} \subseteq X \backslash C$, and by Theorem II-1.21(3), there exists $K \in \mathcal{K}$ such that $K \subseteq X \backslash C$, i.e., $C \cap K = \emptyset$, a contradiction. Thus $\{K_0 \cap C \colon C \in \mathcal{C}\}$ is a filtered family of nonempty sets closed in the compact space K_0. By compactness such a family must have a nonempty intersection (equal to A) that is compact in K_0 and hence in X.

Now let U open contain A. We modify the family $\mathcal{C}$ of the preceding paragraph to $\mathcal{C}' = \{C \cap X \backslash U \colon C \in \mathcal{C}\}$. If no $C \cap K$ is contained in U, then every $(C \cap X \backslash U) \cap K$ is nonempty. By the argument of the preceding paragraph applied to $\mathcal{K}$ and $\mathcal{C}'$ we conclude that the intersection $\mathcal{K} \cap \mathcal{C}'$ is nonempty. But this contradicts $\mathcal{K} \cap \mathcal{C} \subseteq U$. □

Definition VI-6.2. A space X is *coherent* if the intersection of any two compact saturated sets is again compact. □

Example VI-6.3. The following is a simple example of a domain, actually an L-domain, that is compact and locally compact, but not coherent, in its Scott topology.



□

Lemma VI-6.4. *In a sober coherent X space the intersection of any nonempty family of compact saturated sets is again compact and saturated. The intersection is nonempty if every finite intersection is nonempty.*

Proof: Any intersection of saturated (equivalently, upper) sets is again saturated, and by induction any finite intersection is compact. The remaining assertions follow from the machinery of the Hofmann–Mislove Theorem, or from Lemma VI-6.1 with the collection $\mathcal{C} = \{X\}$, applied to the collection of finite intersections of the given family. □

Recall that the *patch topology* arises by taking all closed sets together with all compact saturated sets as a subbasis for the closed sets of the patch topology (O-5.10).

Lemma VI-6.5. *Let X be a compact coherent sober space. Then the patch topology is compact.*

Proof: By the Alexander Lemma for compactness, showing that any subbasis of closed sets with the finite intersection property (fip) has nonempty intersection establishes compactness. Thus let $\mathcal{K}$ be a family of compact saturated sets and let $\mathcal{C}$ be a family of closed sets such that any finite intersection of sets from $\mathcal{K} \cup \mathcal{C}$ is nonempty. Note that we can assume without loss of generality that $\mathcal{K}$ is a nonempty family by adding the compact saturated set X if necessary; doing so will not change the intersection. Without loss of generality we can augment $\mathcal{K}$ by adding all finite intersections of its members (since X is coherent) and $\mathcal{C}$ by adding all finite intersections of its members. We denote the new collections again by $\mathcal{K}$ and $\mathcal{C}$ respectively; note they are filtered families, and note that by hypothesis $K \cap C$ is nonempty for any $K \in \mathcal{K}$ and any $C \in \mathcal{C}$. Then by

Lemma VI-6.1 the intersection $\bigcap \mathcal{K} \cap \bigcap \mathcal{C}$ is nonempty, and thus X is compact in the patch topology. □

Lemma VI-6.6. *A locally compact space* (X, τ) *is a pospace with respect to its order of specialization and the patch topology.*

Proof: Let $x \not\leq y$ in X in the order of specialization. Then $U := X \setminus \downarrow y$ is a τ open set containing x, and thus there exist a τ open set V and a compact set K such that $x \in V \subseteq K \subseteq U$. Without loss of generality we may assume that K is saturated, since its saturation remains compact and the same containments hold. Then V contains x and is an open upper set in the order of specialization, $X \setminus K$ contains y and is a lower set open set in the patch topology, and $V \cap (X \setminus K) = \emptyset$. Thus the order of specialization is a closed set in the product $X \times X$ equipped with the product of the patch topology. □

Definition VI-6.7. A space is *stably compact* if it is compact, locally compact, coherent, and sober. It is *stably locally compact* if it is locally compact, coherent, and sober. □

Proposition VI-6.8. *Let* (X, τ) *be a stably compact space. Then with respect to the patch topology and the order of specialization X is a compact pospace. Furthermore the patch open upper sets are precisely the members of* τ.

Proof: The first assertions follow immediately from Lemmas VI-6.5 and VI-6.6. Since τ open sets are upper sets in the order of specialization, they are patch open upper sets. Conversely, let U be a patch open upper set and let $x \in U$. Since (X, τ) is locally compact, the set $\mathcal{K}$ of compact saturated neighborhoods of x in the τ-topology is filtered with intersection $\uparrow x$. By definition each member of $\mathcal{K}$ is closed in the patch topology, hence compact since the patch topology is compact. Since $\uparrow x \subseteq \uparrow U = U$, we conclude that some member of $\mathcal{K}$ must be in U. Thus U is a neighborhood of x in the τ-topology. Since x was arbitrary in U we conclude that U is τ open. □

Thus stably compact spaces give rise to compact pospaces. We show that there is a reverse construction.

Definition VI-6.9. Let $(X, \pi, \leq)$ be a pospace. We define $\pi^\sharp$ to be the topology of all upper open sets, called the *open upper set topology* and $\pi^\flat$ to be the topology of all open lower sets, the *open lower set topology*. □

Lemma VI-6.10. *Let* $(X, \pi, \leq)$ *be a compact pospace. An upper set A is compact in the* $\pi^\sharp$*-topology iff it is compact in the* π*-topology.*

Proof: We establish the nontrivial direction of proof. Let A be an upper set compact in the $\pi^\sharp$-topology and let $y \notin A$. Note that for each $x \in A$, $\uparrow x$ and $\downarrow y$ are disjoint closed sets. By Proposition VI-1.8 for each $x \in A$, there exist disjoint π open sets $U_x = \uparrow U_x$ and $V_x = \downarrow V_x$ such that $\uparrow x \subseteq U_x$ and $\downarrow y \subseteq V_x$. Since each U_x is $\pi^\sharp$ open, finitely many, say $U1, \ldots, U_n$, cover A. Set $V = \bigcap_{i=1}^n V_i$, the corresponding intersection. This shows that the closure of A in the π-topology misses y. Since y was arbitrary outside A, we conclude that A is closed in the π-topology, hence compact. □

Proposition VI-6.11. *Let $(X, \pi, \leq)$ be a compact pospace. Then $(X, \pi^\sharp)$, resp. $(X, \pi^\flat)$, is a stably compact space with order of specialization the original order, resp. the reverse $\geq$ of the original order.*

Proof: Since X is compact in the π-topology, it is compact in the coarser $\pi^\sharp$-topology. Given two $\pi^\sharp$ compact saturated sets A, B, they are π compact by the preceding lemma, and hence their intersection is π- and thus $\pi^\sharp$ compact. Thus $(X, \pi^\sharp)$ is coherent.

Suppose that $x \leq y$. Then every open upper set containing x contains y, and thus x is below y in the order of specialization for $\pi^\sharp$. Conversely suppose that every open upper set that contains x contains y. Then $\downarrow y$ must contain x, for otherwise its complement would be an open set containing x, but missing y. Thus we obtain $x \leq y$. Hence the order of specialization for $\pi^\sharp$ and the original order $\leq$ agree.

For local compactness let $x \in U = \uparrow U$. By Proposition VI-1.8 there exist disjoint open sets V, W such that $\uparrow x \subseteq V = \uparrow V$ and $X \backslash U \subseteq W = \downarrow W$. Then $x \in V \subseteq X \backslash W \subseteq U$ and $X \backslash W$ is compact in (X, π), hence in $(X, \pi^\sharp)$.

To show X is sober we apply Theorem II-1.21(3), since we now have local compactness. But then any filter base of compact saturated sets in $(X, \pi^\sharp)$ is also a filtered family of compact sets in (X, π) (by the preceding lemma), and hence the conditions of (3) are easily seen to be true in the Hausdorff space (X, π).

The assertions about $(X, \pi^\flat)$ follow by applying the preceding results to the compact pospace $(X, \pi, \geq)$. □

We derive an alternative topological characterization of stably compact spaces.

Definition VI-6.12. A topological space is called *strongly sober* if for every ultrafilter the set of limit points is nonempty and consists of the closure of a unique singleton set. The space is *locally strongly sober* if every ultrafilter has either no limit points or the set of limit points consists of the closure of a unique singleton set. □

Lemma VI-6.13. *Locally strongly sober spaces are sober. Furthermore, a space is strongly sober iff it is locally strongly sober and compact.*

Proof: Let A be an irreducible closed set in a locally strongly sober space X. Then every subset of A which is open in A is dense in A; hence, the open subsets of A form a filter base. Extend this filter base to an ultrafilter $\mathcal{F}$ on X. Since A is closed and in $\mathcal{F}$, the set of points of convergence of $\mathcal{F}$ is a subset of A. Conversely let $x \in A$, and let U be an open neighborhood of x. Since $U \cap A \in \mathcal{F}$ by definition of $\mathcal{F}$, we conclude $\mathcal{F}$ converges to x. We see then that A is precisely the set of limit points of $\mathcal{F}$. Thus, A is the closure of a unique singleton, and we have proved that X is sober.

A space X is both compact and locally strongly sober iff for every ultrafilter the set of limit points is nonempty and hence the closure of a unique singleton iff X is strongly sober. □

Lemma VI-6.14. *A locally strongly sober space is coherent.*

Proof: Let A, B be compact saturated sets and let $\mathcal{F}$ be an ultrafilter having $A \cap B$ as a member. Then by compactness of A and B the set of limit points of $\mathcal{F}$ exists and meets both A and B. Then by local strong sobriety there exists a limit point y of $\mathcal{F}$ such that the set of all its limit points is $\downarrow y$. Since $\downarrow y$ must meet both A and B, we have $y \in A \cap B$ (by saturation). Thus any ultrafilter containing $A \cap B$ converges to some point of $A \cap B$, so it is compact. □

Proposition VI-6.15. *A space is stably compact iff it is locally compact and strongly sober.*

Proof: Suppose that X is locally compact and strongly sober. By Lemma VI-6.13 it is compact and sober. Coherence follows from Lemma VI-6.14.

Conversely suppose that (X, τ) is stably compact. Then it is locally compact and we must show it is strongly sober. Let $\mathcal{F}$ be an ultrafilter. By VI-6.8 the patch topology gives a compact pospace, in particular a compact Hausdorff space, and hence the ultrafilter $\mathcal{F}$ converges to some unique point x in the patch topology. Hence $\mathcal{F} \to x$ in the τ-topology. Since the set of limit points is always closed, it follows that the set of limit points contains $\downarrow x$. Suppose $y \not\leq x$. Then by Proposition VI-1.8 we find an open upper set U containing $\uparrow y$ and an open lower set V containing $\downarrow x$ disjoint from U. Then U is a patch open upper set, hence τ open by Proposition VI-6.8. Since the ultrafilter converges to x in the patch topology, we have $V \in \mathcal{F}$, which implies that U is not in $\mathcal{F}$. We conclude that y is not a limit point of $\mathcal{F}$ in (X, τ). Thus $\mathcal{F} \to \downarrow x$. □

Corollary VI-6.16. *A space X is stably locally compact iff it is locally compact and locally strongly sober.*

Proof: Let X be stably locally compact. Let $X_\perp$ denote X with a bottom element attached and one new open set, $X_\perp$ itself. It is straightforward to verify that if X is stably locally compact, then $X_\perp$ is stably compact, hence strongly sober. But it is easy to see that an upper set of a strongly sober space is locally strongly sober; hence X is locally strongly sober. The converse follows from VI-6.13 and VI-6.14. □

Definition VI-6.17. Let (X, τ) be a topological space. The *co-compact topology* τ^k on a space has as a subbasis for the closed sets all compact saturated sets in the original topology (see O-5.10). The *patch topology* π, as already mentioned, is defined as $\tau \vee \tau^k$, the join of the two topologies. A topology τ_d on X is called a *dual topology* if the order of specialization is $\geq$, the reverse or opposite of the order of specialization for τ. A *separating dual topology* is a dual topology τ_d such that if $x \not\leq y$, then there exist a τ open set U containing x and a τ_d open set V containing y such that $U \cap V = \emptyset$. □

We now gather together our results.

Theorem VI-6.18. *The following are equivalent for a T_0 space (X, τ) with order of specialization $\leq$.*

(1) *X is a stably compact space.*
(2) *X is strongly sober and locally compact.*
(3) *The patch topology π makes $(X, \leq)$ a compact pospace with $x^\sharp = \tau$ and $\pi^\flat = \tau^k$, the co-compact topology.*
(4) *There exists a compact pospace topology on $(X, \leq)$ such that τ is the collection of open upper sets.*
(5) *There exists a separating dual topology τ_d such that the join of this topology with τ makes X a compact space.*
(6) *There exists a compact Hausdorff topology ω on X containing τ such that given $y \not\leq x$, there exists disjoint $U \in \tau$ containing y and $V \in \omega$ containing x.*
(7) *X is locally compact and the patch topology is compact.*

Suppose that (X, τ) satisfies any, and hence all, of the equivalent (1)–(7). *Then the compact topology in* (4) *and* (6) *is unique and is equal to the patch topology. In part* (5) *the topology predicated to exist is unique and is equal to the co-compact topology, which is also* (patch)$^\flat$.

Proof: (1)⇔(2): Proposition VI-6.15.

(2)⇒(3): Proposition VI-6.8 yields the first two assertions. Let V be a lower open set. Then $X \backslash V$ is a π compact upper set, hence a π compact saturated set. Thus the topology $\pi^\flat$ is contained in τ^k. Conversely let K be a τ compact saturated set. Then K is an upperset and $\pi^\sharp$ compact, hence π compact by Lemma VI-6.10, hence π closed. Thus $X \backslash K$ is a π open lower set. Hence $\pi^\flat = \tau^k$.

(3)⇒(4): Immediate.

(4)⇒(1): Proposition VI-6.11.

uniqueness in (4): Assume that the equivalent conditions (1) through (4) are satisfied. Let ω be any compact pospace topology on $(X, \leq)$ with $\tau = \omega^\sharp$. Then by Lemma VI-6.10 the ω compact upper sets are precisely the τ compact saturated sets, and thus the ω open lower sets are their complements. By Corollary VI-1.9 these sets together with τ are a subbasis for the ω open sets. Hence ω must be the patch topology of τ.

(1)⇒(5): Consider the co-compact topology τ^k. Since principal filters are compact and compact saturated sets are upper sets, we conclude that the co-compact topology is a dual topology. If $x \not\leq y$, then there exist by local compactness an open set V and a compact set K such that $x \in V \subseteq K \subseteq X \backslash {\downarrow} y$. By passing to the saturation of K, we may assume that K is saturated. Then the pair V and $X \backslash K$ show that τ and τ^k are separated. Since (1) implies (3), we conclude their join, the patch topology, yields a compact pospace.

(5)⇒(4): Let τ_d be a separating dual topology for which the join $\omega = \tau \vee \tau_d$ is compact. Suppose that $x \not\leq y$. By hypothesis there exist a τ open set U containing x and a τ_d open set containing y such that $U \cap V = \emptyset$. Then with respect to $\leq$, U is an upper set and V is a lower set. It follows that $\leq$ is closed in $(X, \omega) \times (X, \omega)$, and hence $(X, \omega, \leq)$ is a compact pospace.

To complete the implication, we show that the open upper sets U of with respect to ω are τ open (note that τ open sets are automatically ω open upper sets). Let $x \in U$. For each $y \in X \backslash U$, pick $U_y \in \tau$ containing x and V_y in τ_d containing y such that $U_y \cap V_y = \emptyset$ (possible since the topologies are separated). Let $K_y = X \backslash V_y$. Then each K_y is compact in the patch topology, misses y, and is a τ neighborhood of x. Since their intersection is contained in U, by compactness there exist finitely many whose intersection is inside U. This finite intersection is a τ neighborhood of x inside U. Since x was arbitrary in U, we conclude that U is τ open. Thus the open upper sets in the compact pospace $(X, \omega, \leq)$ are precisely the τ open sets.

uniqueness in (5): The topologies τ and τ_d stand in symmetric relationship to one another. Thus reversing the roles of them in the argument of the preceding paragraph, we conclude that the open lower sets in $(X, \omega, \leq)$ are precisely the τ_d open sets. By the uniqueness of the compact pospace in (4), the topology ω must be π, the patch topology, and hence $\tau_d = \pi^\flat$, which by (3) is also the co-compact topology.

(4)⇔(6): Property (4) easily implies (6) using the monotone normal property of compact pospaces (VI-1.8). The converse arises by showing that X with the postulated compact Hausdorff topology is a pospace, and then that the open upper sets are precisely the τ open sets. The proof mimics the techniques used in (5) $\Rightarrow$ (4) and is left as an exercise. The uniqueness of the topology in (6) then follows from the uniqueness in (4).

(3)⇒(7): (3) implies (2) and the two together immediately imply (7).

(7)⇒(5): We saw in the preceding proof of (1) implies (5) that the co-compact topology is a separating dual topology for a locally compact space. Since the patch topology is assumed compact, we are done. □

Remark. Property (5) of Theorem VI-6.18 asserts that a stably compact space is one for which there exists a (unique) suitable partner topology with which it can produce a compact Hausdorff space. Property (6) says that the original topology can be strengthened (uniquely) in an appropriate way to get a compact Hausdorff space.

Corollary VI-6.19. *Let (X, τ) be a stably compact space. Then (X, τ^k) is a stably compact space, where τ^k is the co-compact topology. Furthermore, τ is the co-compact topology for τ^k.*

Proof: The first assertion follows from Proposition VI-6.11 and Theorem VI-6.18(3). The last assertion follows from the uniqueness in Theorem VI-6.18(5). □

Definition VI-6.20. A continuous map $f: X \to Y$ between topological spaces is *proper* if (i) for any closed set $A \subseteq X$, $\downarrow f(A)$ is closed in Y, and (ii) the inverse image of a compact saturated set is compact and saturated. □

Lemma VI-6.21. *A continuous $f: X \to Y$ is proper*

(i) *if $\downarrow f(A)$ is closed for all A closed in X and $f^{-1}(\uparrow y)$ is compact for all $y \in Y$, or*
(ii) *if X is sober, Y is locally compact, and inverse images of compact saturated sets are compact.*

Proof: (i) Since a continuous map is order preserving with respect to the orders of specialization, the inverse image of saturated sets is again saturated. Let $\mathcal{U}$ be a collection of open sets in X which cover $f^{-1}(K)$, where K is compact saturated in Y. For $y \in K$, there exist finitely many that cover $f^{-1}(\uparrow y)$. Let W_y be their union. Then $V_y = Y \setminus \downarrow f(X \setminus W_y)$ is an open set containing $\uparrow y$ such that $f^{-1}(V_y) \subseteq W_y$. Then finitely many V_y cover K, so finitely many W_y cover $f^{-1}(K)$ and thus finitely many members of $\mathcal{U}$ also cover.

(ii) We must show that $\downarrow f(A)$ is closed for A closed in X. Let $y \in Y \backslash \downarrow f(A)$. Since Y is locally compact, there exists a filter base $\mathcal{K}$ of compact saturated neighborhoods of Y such that $\bigcap \mathcal{K} = \uparrow y$. Then $f^{-1}(\mathcal{K}) = \{f^{-1}(K)\colon K \in \mathcal{K}\}$ is a filtered collection of compact saturated sets in X with intersection $f^{-1}(\uparrow y)$, and thus the intersection is contained in $X \backslash A$. By the Hofmann–Mislove machinery (II-1.21) there exists $K \in \mathcal{K}$ such that $f^{-1}(K) \subseteq X \backslash A$. We conclude that the neighborhood K of y misses $\downarrow f(A)$. Since y was arbitrary in the complement, we are done. □

Definition VI-6.22. Let *SCTOP* denote the category with objects stably compact spaces and morphisms proper maps, and let *CPOSP* denote the category of compact pospaces with morphisms continuous monotone maps. □

Proposition VI-6.23. *The categories SCTOP and CPOSP are functorially equivalent via the functors F and G, where F assigns to a stably compact space* (X, τ) *the compact pospace* $(X, \pi, \leq)$ *equipped with the patch topology and the order of specialization, G assigns to a compact pospace* $(X, \omega, \leq)$ *the stably compact space* $(X, \omega^\sharp)$*, and F and G carry functions to themselves. These functors are inverse functors, the composition either way being the identity.*

Proof: We check first that morphisms go to morphisms. Suppose that $f\colon X \to Y$ is a proper map between stably compact spaces. Then f is continuous implies that it is monotone. Also since the inverse of a compact saturated set is again compact, f is continuous for the patch topologies.

Conversely suppose that $f\colon X \to Y$ is a monotone continuous map between compact pospaces. If W is an open upper set in Y then $f^{-1}(W)$ is open and an upper set (by monotonicity). Thus f is continuous for the topologies of open upper sets. Let K be a compact saturated set in the open upper set topology of Y. Then by Lemma VI-6.10 K is compact, hence closed, in the given topology of Y, hence its inverse image is closed, hence compact in the given topology of X, and thus compact in the open upper set topology.

By Proposition VI-6.8 $G \circ F$ is the identity on objects (and it is trivially so on morphisms). Let X be a compact pospace. Then the open upper topology gives a stably compact space by Proposition VI-6.11. This topology is the open upper topology for the original compact pospace and for the pospace obtained by applying F to it by Theorem VI-6.18(3). But by part (4) of the same theorem the pospace topology is unique, and hence $G \circ F$ is the identity. □

We close this section by specializing our results to the case of domains.

Proposition VI-6.24. *Let L be a domain (or even a quasicontinuous domain). Then the co-compact topology for the Scott topology is the lower topology, and* *the patch topology is the Lawson topology. If further the domain is compact, then the Scott and lower topologies are each stably compact, are co-compact duals for each other, and the patch topology is again the Lawson topology.*

Proof: Let A be a compact saturated set with respect to the Scott topology. Then A is an upper set, and by Lemma III-5.7 A is a directed intersection of finitely generated upper sets. It follows that A is closed in the lower topology. Conversely any principal filter is certainly compact in the Scott topology, so the lower topology is contained in the co-compact topology. Hence the co-compact topology and the lower topology agree. It is then immediate that the Lawson topology and the patch topology agree.

Assume that L is compact in the Lawson topology. By VI-1.15 L is a pospace with respect to the Lawson topology. Since the closed lower sets are precisely the Scott closed sets (III-1.6), it follows from Proposition VI-6.11 that L is stably compact with respect to the Scott topology. By the first part the lower topology is the co-compact topology for the Scott topology, and thus by Corollary VI-6.19 stably compact with co-compact topology equal to the Scott topology. □

The preceding material gives an alternative approach to the study of the topology of an arbitrary compact topological semilattice.

Proposition VI-6.25. *Let S be a compact topological semilattice. Then with respect to the Scott topology S is a stably compact space, and the original topology on S arises as the patch topology associated to the Scott topology. This topology will be the Lawson topology iff S has small semilattices.*

Proof: By Proposition VI-2.10 the closed lower sets of S are the Scott closed sets. Hence the open upper sets are precisely the Scott open sets. Hence by Theorem VI-6.18 S equipped with the Scott topology is a stably compact space, since S is a pospace by Proposition VI-1.14. By the uniqueness of the pospace topology (VI-6.18(4)), we conclude that the original topology is the patch topology for the Scott topology.

By the Fundamental Theorem VI-3.4 if the semilattice has small semilattices then its topology is the Lawson topology. Conversely, if the Lawson topology is equal to the original topology, then by III-3.11 S^1 is a quasicontinuous lattice. Since by Proposition VI-1.13(vii) S is meet-continuous, it follows from III-3.10 that S^1 is a continuous lattice. Hence S is a complete continuous semilattice endowed with the Lawson topology and thus has small semilattices by III-2.15.

□

Condition (4) of Theorem VI-6.18 can be strengthened for the case of the Scott topology. Thus the following condition is equivalent to the others of Theorem VI-6.18 for the Scott topology.

Proposition VI-6.26. *Let $(P, \leq)$ be a poset endowed with a compact Hausdorff topology such that the order is semiclosed and the open upper sets contain the Scott open sets. Then the given compact Haudorff topology makes P a compact pospace, and the open upper sets are precisely the Scott open sets.*

Proof: We need to show under the given hypotheses that P has a closed partial order. Suppose that $y \not\leq x$. Then $\uparrow y$ is a closed set missing x, so there exist disjoint open sets U and V such that $\uparrow y \subseteq U$ and $x \in V$. By VI-1.6(i) $A = \downarrow(X \backslash U)$ is Scott closed, hence closed by hypothesis. Thus $(X \backslash A) \times V$ is an open set containing (y, x) that misses the partial order $\leq$. Hence the order is closed. That every open upper set is Scott open follows from Proposition VI-1.3(ii) □

Proposition VI-6.27. *Let S and T be domains endowed with the Scott topology and let $f\colon S \to T$ be a Scott-continuous upper adjoint with lower adjoint d. Then f is a proper map, and hence continuous for the Lawson topologies.*

Proof: For $y \in Y$, we have

$$x \in f^{-1}(\uparrow y) \Leftrightarrow y \leq f(x) \Leftrightarrow d(y) \leq x \Leftrightarrow x \in \uparrow d(y).$$

Thus $f^{-1}(\uparrow y) = \uparrow d(y)$, and the latter is compact in the Scott topology. Also for Scott closed $A = \downarrow A \subseteq X$, we have

$$y \in \downarrow f(A) \Leftrightarrow \exists x \in A, y \leq f(x) \Leftrightarrow \exists x \in A, d(y) \leq x \Leftrightarrow d(y) \in A.$$

Thus $\downarrow f(A) = d^{-1}(A)$. The latter is Scott closed, since d as a lower adjoint preserves all existing sups, in particular, directed sups, and thus $\downarrow f(A)$ is closed. Hence f is proper, and thus patch continuous since the inverse of any compact saturated set is again compact. But by VI-6.24 the patch topology is the Lawson topology.

Exercises

Exercise VI-6.28. Provide the details of the proof that (6)⇒(5) in Theorem VI-6.18. Proceed along the lines of (5)⇒(4). □

Exercise VI-6.29. Show that if S is a compact topological semilattice for which the Lawson topology is Hausdorff, then its topology is the Lawson topology and S has small semilattices.

Hint. Adapt the methods in the proof of VI-6.25. □

Exercise VI-6.30. Let X be a poset with an order consistent topology. Prove the following:

(i) the saturated sets are precisely the upper sets;
(ii) each set of the form $\uparrow x$ is a saturated compact set;
(iii) each saturated compact set is Scott closed in X^{op};
(iv) the co-compact topology is order consistent on X^{op};
(v) if X is locally compact, then under the patch topology X is a pospace (and hence Hausdorff).

Hint. (i) Each open set is an upper set and hence the same is true for the intersection. Conversely if $A = \uparrow A$, then $A = \bigcap\{X \backslash \downarrow x \colon x \notin A\}$. Hence A is saturated.

(ii) Any open set containing x contains $\uparrow x$. Hence $\uparrow x$ is compact.

(iii) Apply VI-1.6(i) to X^{op}.

(iv) It follows immediately from (ii) and (iii) that the co-compact topology contains the upper topology and is contained in the Scott topology on X^{op}. Hence it is order consistent.

(v) Let $x, y \in X, x \not\leq y$. Since $X \backslash \downarrow y$ is an open set containing x, there exists a compact neighborhood Q of x contained in $X \backslash \downarrow y$. It is easily verified that $\uparrow Q$ is also compact and $\uparrow Q \subseteq X \backslash \downarrow y$. The interior of Q is an open upper set containing x in the given, and hence in the patch, topology. The complement of $\uparrow Q$ is a lower set which is a neighborhood of y in the co-compact and, hence, in the patch topology. Clearly the two are disjoint. Therefore, we may apply VI-1.4. □

In the next exercise we connect the considerations of this section with related work of Ralph Kopperman [Kopperman, 1995].

Exercise VI-6.31. A bitopological space (X, τ, σ) is an *asymmetric space* if σ and τ are T_0-topologies with orders of specializations being the order dual of each other. We order X with the order of specialization of the first topology. The asymmetric space X is *pseudo-Hausdorff* if given $x \not\leq y$, there exists $U \in \tau$ and $V \in \sigma$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$, and *join compact* if $\tau \vee \sigma$ is a compact topology. Prove the following.

(i) If (X, τ) is a stably compact space, then (X, τ, τ^k) is a join compact pseudo-Hausdorff asymmetric space.
(ii) If (X, τ, σ) is a join compact pseudo-Hausdorff asymmetric space, then (X, τ) and (X, σ) are stably compact and $\sigma = \tau^k$.

Hint. Use Theorem VI-6.18. □

New notes

This material represents an expanded version of material originally formulated by Gierz and Lawson that appeared in the Exercises to Section VII-1 of the *Compendium*. Various papers have elaborated on these notions including [Hofmann, 1984b], [Smyth, 1991], [Smyth, 1992a], [Kegelmann, D1999], and [Kopperman, 1995]. These references indicate the increasingly important role that stably compact spaces have played since the publication of the *Compendium*. We have opted for the terminology "stably compact", although a variety of other terminology has been suggested and can be found in the literature ([Kopperman, 1995], for example, uses the terminology "skew compact").

VI-7 Spectral Theory for Stably Compact Spaces

In this section we consider the spectral theory of the stably compact and stably locally compact spaces studied in Section VI-6. Since their theory is intimately connected with that of compact pospaces, the latter are also included in the current discussion. We recall from Section V-5, particularly Proposition V-5.20, the duality between the categories *LCSOB*, which has as objects the locally compact sober spaces and as its maps the continuous functions $f: X \to Y$ with the property that $f^{-1}(Q)$ is compact for every saturated compact subset Q of Y (equivalently the proper maps by Lemma VI-6.21(ii)), and the category $CL^{\mathrm{op}} \cap FRM$, which has as objects continuous distributive lattices and as maps the morphisms $\varphi: L \to M$ preserving arbitrary sups, finite infs and the relation $\ll$. The duality is given by the functors $\mathcal{O}$, assigning the lattice of open sets to a space, and Spec, which assigns to a distributive continuous lattice its spectrum with the hull–kernel topology. In this section we consider the full subcategories of stably locally compact and stably compact spaces and their dual continuous frames.

On a continuous distributive lattice L, we consider $X = \operatorname{Spec} L$ with the hull–kernel topology, which is also the topology induced by the lower topology on L, and the corresponding patch topology, which is also the relative Lawson topology (V-5.12). In parallel, we consider PRIME $L = \operatorname{Spec} L \cup \{1\}$ also topologized by its hull–kernel topology induced by the lower topology on L, and the corresponding patch topology, which again by V-5.12 is the relative Lawson topology. However, it is convenient to distinguish the orders: with the Lawson topology we associate the order of L restricted to the spectrum, and with the patch topology we associate the order of specialization from the hull–kernel topology, which is the opposite order of the order inherited from L.

Now let L be a distributive continuous lattice in which the set PRIME L of prime elements is closed with respect to the Lawson topology. From V-3.7 we know that the latter is the case iff the way-below relation is multiplicative on L. If we alternatively assume that Spec L is closed with respect to the Lawson topology, then PRIME $L = \text{Spec } L \cup \{1\}$ is also closed, hence the way-below relation is again multiplicative. But additionally, since L is primally generated, $\{1\} = L\setminus\downarrow\text{Spec } L$ is Lawson open, hence Scott open, hence a compact element. Conversely if PRIME L is Lawson closed and 1 is compact element, then $\{1\}$ is Lawson open, and hence Spec $L = \text{PRIME } L\setminus\{1\}$ is Lawson closed. Since L is continuous, Spec L and PRIME L are locally compact, and thus by Theorem VI-6.18(7) we know that Spec L, resp. PRIME L, is compact in the Lawson topology iff Spec L, resp. PRIME L, is a stably compact space in the relative lower topology.

Proposition VI-7.1. *Let L be a distributive continuous lattice. The following are equivalent:*

(1) *the way-below relation is multiplicative;*
(2) PRIME *L endowed with the hull–kernel topology is stably compact;*
(3) PRIME *L is closed in the Lawson topology of L;*
(4) Spec *L (with the hull–kernel topology) is stably locally compact.*

Also the following are equivalent:

(1) *the way-below relation is multiplicative and* 1 *is a compact element;*
(2) Spec *L endowed with the hull–kernel topology is stably compact;*
(3) Spec *L is closed in the Lawson topology of L.*

In each case the patch topology (for the hull-kernel topology) of PRIME *L or* Spec *L agrees with the relative Lawson topology from L. The function* $\Delta_L(a) :=$ PRIME $L\setminus\uparrow a$*, resp.* $\Delta_L(a) :=$ Spec$L\setminus\uparrow a$*, is a lattice isomorphism from L onto the nonempty, resp. all lower, sets in the inherited order from L that are relatively Lawson open.*

Proof: The proof that items (1)–(3) are equivalent in both settings follows from the comments preceding the lemma. The equivalence of (2) and (4) follows from the elementary observation that a space is stably locally compact (locally compact, coherent, and sober) iff adding on a bottom element (in the order of specialization) makes it stably compact.

By V-5.12 the Lawson and patch topologies agree on Spec L and PRIME L. Since these are stably compact in the hull-kernel topologies, by Proposition VI-6.8 the patch open upper sets in the order of specialization are precisely the hull–kernel open sets; these are then the relative Lawson open lower sets, since the order of specialization is the order dual of the relative order from L. □

Consider now the following categories:

SCTOP, which has as objects the stably compact spaces and as its maps the proper maps $f: X \to Y$;
SLCTOP, which has as objects the stably locally compact spaces and as its maps the proper maps $f: X \to Y$;
SCFRM, which has as objects continuous distributive lattices L with multiplicative way-below relation (equivalently stably continuous frames) and as maps the morphisms $\varphi: L \to M$ preserving arbitrary sups, finite infs and the relation $\ll$;
SCFRM$_1$, which has as objects stably continuous frames with 1 compact, and as maps the morphisms $\varphi: L \to M$ preserving arbitrary sups, finite infs and the relation $\ll$.

Corollary VI-7.2. *The functor* Spec *is a functor from the category SCFRM, resp. SCFRM*$_1$*, to the category SLCTOP, resp. SCTOP.*

Proof: This follows immediately from the preceding proposition and Proposition V-5.20. □

We consider now the functor $\mathcal{O}$.

Proposition VI-7.3. *Let X be a* T_0 *space. The following are equivalent:*

(1) *X is stably locally compact;*
(2) $X_\perp = X \cup \{\perp\}$ *with the patch topology and order of specialization is a compact pospace;*
(3) $\mathcal{O}(X)$ *is a stably continuous frame.*

Also the following are equivalent:

(1) *X is stably compact;*
(2) *X with the patch topology and order of specialization is a compact pospace;*
(3) $\mathcal{O}(X)$ *is a stably continuous frame with compact* 1.

Proof: That $X_\perp$ (with the only neighborhood of $\perp$ being the whole space) is locally compact, coherent, sober, and compact is straightforward. Hence (1) implies (2) by Proposition VI-6.8 and the reverse implication is straightforward.

Now by spectral duality X is naturally homeomorphic to Spec $(\mathcal{O}(X))$, and it then follows from Proposition VI-7.1 that if X is stably locally compact, then the distributive continuous lattice $\mathcal{O}(X)$ has multiplicative way-below relation and conversely. Hence (1) is equivalent to (3). The arguments for the second group of equivalences are similar. □

Theorem VI-7.4. *The functors* Spec: *SCFRM* $\to$ *SLCTOP, resp.* Spec: *SCFRM*$_1$ $\to$ *SCTOP, and* $\mathcal{O}$: *SLCTOP* $\to$ *SCFRM, resp.* $\mathcal{O}$: *SCTOP* $\to$ *SCFRM*$_1$*, define a duality of categories.*

Proof: This theorem follows readily from the previous results of this section and Proposition V-5.20, since the asserted duality of the theorem is just a restriction of that given in V-5.20. □

In Section VI-6 we have established the functorial equivalence of the categories *SCTOP* consisting of stably compact spaces and proper maps and *CPOSP* consisting of compact pospaces and continuous monotone maps. We can compose this equivalence with the duality functors of this section to obtain a duality of *CPOSP* and *SCFRM*$_1$.

Remark VI-7.5. *For a compact pospace X, let $\mathcal{O}^{\uparrow}(X)$ denote the lattice of open upper sets. Since X endowed with the topology of open upper sets is a stably compact space (*VI-6.11*), it follows from* VI-7.3 *above that $\mathcal{O}^{\uparrow}(X)$ is a distributive continuous lattice with multiplicative way-below and compact unit* 1*, i.e., an object in SCFRM*$_1$*. Since the functor of passing to the topology of open upper sets and viewing a continuous monotone map as a proper map is an equivalence of categories from CPOSP to SCTOP (*VI-6.23*) and since $\mathcal{O}$ yields a duality of the categories SCTOP and SCFRM*$_1$ *(*VI-7.4*), we conclude that the composition of sending X to $\mathcal{O}^{\uparrow}(X)$ and a continuous monotone map $f\colon X \to Y$ to $\mathcal{O}^{\uparrow}(f)$: $\mathcal{O}^{\uparrow}(Y) \to \mathcal{O}^{\uparrow}(X)$ defined by $\mathcal{O}^{\uparrow}(f)(V) = f^{-1}(V)$ for $V = \uparrow V$ open in Y is one of a pair of unctors that defines a duality between CPOSP and SCFRM*$_1$*. The other functor is the spectrum functor* Spec_{λ} *that assigns to a continuous distributive lattice L with multiplicative way-below relation and compact* 1*, the spectrum* Spec L *equipped with the order of specialization (the reverse of the inherited order from L) and the patch topology of the hull-kernel topology, which is the same as the relative Lawson topology. At the morphism level* $\mathrm{Spec}_{\lambda}(f)$ *for $f\colon L \to M$ is the restriction of the upper adjoint of f to* Spec (M)*, and is a continuous monotone map (by* VI-7.4 *and* VI-6.23*). Thus the categories CPOSP and SCFRM*$_1$ *are dual with duality given by the functors $\mathcal{O}^{\uparrow}$ and* Spec_{λ}. □

Exercises

Exercise VI-7.6. (Functoriality of the patch topology) Let L and M be complete lattices and $\tau\colon M \to L$ a map preserving arbitrary infs and directed sups. Suppose, in addition, that the lower adjoint $\varphi\colon L \to M$ of τ preserves finite infs. Show that $\tau(\mathrm{PRIME}\ M) \subseteq \mathrm{PRIME}\ L$ and that the restriction of τ to PRIME M is continuous with respect to the patch topologies on PRIME M and PRIME L.

Hint. Use V-4.5 and V-5.3. □

Exercise VI-7.7. Let X be a stably locally compact space. By Proposition V-4.7, the mapping $\xi_X\colon X \to \mathrm{Spec}(\mathcal{O}(X))$ is a homeomorphism. Let $X_\perp$ denote X with a bottom element adjoined whose only neighborhood is all of $X_\perp$. Extend ξ_X by $\xi_X(\perp) = X$, the largest element of $\mathcal{O}(X)$.

(i) Show that the extended $\xi_X\colon X_\perp \to \mathrm{PRIME}\ \mathcal{O}(X)$ is a homemorphism from $X_\perp$ to PRIME $\mathcal{O}(X)$ with the relative lower topology.
(ii) Show that ξ_X from $X_\perp$ with the patch topology to PRIME $\mathcal{O}(X)$ with the relative Lawson topology is a homeomorphism. In particular, the relative Lawson topology and patch topology agree on PRIME $\mathcal{O}(X)$ (by letting $X = \mathrm{Spec}\ \mathcal{O}(X)$).
(iii) Show that both spaces $X_\perp$ and PRIME $\mathcal{O}(X)$ are stably compact. □

Exercise VI-7.8. Show that the compact open subsets form a basis for a stably compact space iff it is a *totally order disconnected compact pospace* in the patch topology, that is, a compact pospace such that for any two elements x and y with $x \not\leq y$ there is an open–closed upper set containing x but not y. □

Exercise VI-7.9. (i) Let L be a distributive arithmetic lattice with compact 1. Show that Spec L is closed with respect to the Lawson topology; endowed with the topology induced from the Lawson topology on L, Spec L is a totally order disconnected compact pospace; moreover, L is isomorphic to the lattice $\mathcal{O}^\uparrow(\mathrm{Spec}\ L)$ of all patch open upper sets in Spec L, which are the relative Lawson open lower sets; the compact elements of L correspond to the open–closed lower sets in Spec L.

(ii) Let X be a totally order disconnected compact pospace. Show that $\mathcal{O}^\uparrow(X)$ is a distributive arithmetic lattice such that X is homeomorphic to the space Spec $\mathcal{O}^\uparrow(X)$ endowed with the topology induced from the Lawson topology on $\mathcal{O}^\uparrow(X)$.

(iii) Let M be a distributive lattice with 0 and 1. Let L be the lattice of all ideals of M. Show that L is a distributive arithmetic lattice (see V-5.22) and that M is isomorphic to the lattice $\mathcal{O}\Upsilon(\mathrm{Spec}\ L)$ of all clopen lower sets of Spec L endowed with the topology induced from the Lawson topology on L. □

Exercise VI-7.10. (Priestley duality) Consider the following categories:

DAR has as objects all distributive arithmetic lattices with compact 1 and as morphisms all maps preserving arbitrary sups, finite infs and compact elements (cf. V-5.22);
TCPOSP has as objects all totally order disconnected compact pospaces and as morphisms all continuous order preserving maps;
DLat is the category of all distributive lattices with 0 and 1 and all 0 and 1 preserving lattice homomorphisms.

Show that the categories *DAR* and *TCPOSP* and likewise the categories *DLat* and *TCPOSP* are dually equivalent. □

Old notes

To the best of our knowledge, the first to consider the patch topology was [Hochster, 1969] who defined this topology for the spectrum of commutative rings. In a general setting, the patch topology has been introduced by Hofmann and Lawson [Hofmann and Lawson, 1978]. In Section 6 of that paper one finds most of the material treated here from the perspective of compact pospaces (but no stable compactness). Duality and representation theorems similar to ones appearing here are already contained in a paper of Gierz and Keimel [Gierz and Keimel, 1977], although in the latter paper other morphisms are used. The exercises VI-7.9 and VI-7.10 show how our results are related to Priestley's representation theorems for distributive lattices by compact totally order disconnected partially ordered spaces, and related duality theorems [Priestley, 1970; Priestley, 1972]. The representation theorems for distributive lattices of Priestley are more appealing to the average mathematical intuition than the original ones of M.H. Stone, since the representing spacesare Hausdorff spaces.

VII

Topological Algebra and Lattice Theory: Applications

Our final chapter is devoted to exploring further links between topological algebra and continuous lattice and domain theory. This theme has already played an important role: the Fundamental Theorem of Compact Semilattices (VI-3.4) is just one example. In this chapter, however, the methods of topological algebra occupy a more central role, while the methods of continuous lattices are somewhat less prominent.

Section VII-1 is devoted to somewhat technical results about certain non-Housdorff topological semilattices; they are included primarily to facilitate the proof of later results concerning separate continuity of semilattice and lattice operations implying joint continuity. Section VII-2 makes various observations about topological lattices and their topologies, with a particular focus on completely distributive lattices.

Section VII-3 introduces the class of continuous lattices for which the Lawson topology is equal to the interval topology: the hypercontinuous lattices. The distributive ones are paired with the quasicontinuous domains via the spectral theory of Chapter V. Thus several earlier themes are nicely rounded out.

Section VII-4 characterizes those meet continuous complete lattices which admit a compact semilattice topology as being exactly those lattices whose lattice of Scott open sets forms a continuous lattice; this augments II-1.14, which shows that the continuous lattices are exactly those complete lattices whose Scott open sets form a completely distributive lattice. The final part of Section VII-4 is devoted to a proof that a compact semitopological semilattice is in fact topological. This is a particularly appropriate note on which to end this treatise, since the proof we present utilizes those aspects of the theory of continuous lattices which we have sought to stress: namely, the algebraic theory and its utility in applications to related areas of mathematics.

VII-1 One-Sided Topological Semilattices

So far, our consideration of topological semilattices has assumed that they are Hausdorff; however, in II-4.15 we had occasion to consider those non-Hausdorff topological semilattices that were topological semilattices with respect to the Scott topology. The purpose of this section is to study non-Hausdorff semilattices in more detail; not only is this class of interest in itself, but also we shall find useful applications of the theory to the Hausdorff setting.

We introduce first some convenient notation and af easy lemma.

Definition VII-1.1. Let S be a semilattice, $A \subseteq S$. If $x \in S$, we define

$$x^{[-1]}A = \{y \in S : xy \in A\}. \qquad \square$$

Lemma VII-1.2. *A semilattice S is a semitopological semilattice under a topology $\mathcal{U}$ iff $x \in S$ and $U \in \mathcal{U}$ always imply $x^{[-1]}U \in \mathcal{U}$.* $\square$

Every semitopological semilattice gives rise to a "one-sided" one, in the sense of having a new topology where all open sets are upper sets as was the case in the Scott topology.

Proposition VII-1.3. *Let $(S, \mathcal{U})$ be a semitopological semilattice. Set*

$$\mathcal{V} = \{U \in \mathcal{U} : U = \uparrow U\}.$$

Then $(S, \mathcal{V})$ is a semitopological semilattice.

Proof: Let $V \in \mathcal{V}$. Then $V \in \mathcal{U}$ and $V = \uparrow V$. If $x \in S$, then by Lemma VII-1.2 $x^{[-1]}V \in \mathcal{U}$. If $y \in x^{[-1]}V$ and $z \geq y$, then $xz \geq xy \in V$. Thus $xz \in V$, that is, $z \in x^{[-1]}V$. Thus $x^{[-1]}V$ is also an upper set and hence in $\mathcal{V}$. By Lemma VII-1.2 again, $(S, \mathcal{V})$ is a semitopological semilattice.

Definition VII-1.4. For a topology $\mathcal{U}$ on a semilattice S, define $\mathcal{U}^\wedge$ to be the topology generated by the subbase

$$\left\{x^{[-1]}U : x \in S, U \in \mathcal{U}\right\} \cup \mathcal{U}. \qquad \square$$

Proposition VII-1.5. *For any topology $\mathcal{U}$ on a semilattice S the topology $\mathcal{U}^\wedge$ is the weakest topology on S containing $\mathcal{U}$ for which S is a semitopological semilattice.*

Proof: To show $(S, \mathcal{U}^\wedge)$ is a semitopological semilattice, we apply Lemma VII-1.2 to the subbasis of $\mathcal{U}^\wedge$. If $U \in \mathcal{U}$, clearly $x^{[-1]}U \in \mathcal{U}^\wedge$. If $x \in S$ and $y^{[-1]}U \in \mathcal{U}^\wedge$, then $x^{[-1]}(y^{[-1]}U) = (yx)^{[-1]}U \in \mathcal{U}^\wedge$.

If $(S, \mathcal{V})$ is a semitopological semilattice and $\mathcal{U} \subseteq \mathcal{V}$, then by Lemma VII-1.2 each $x^{[-1]}U$ is in $\mathcal{V}$; hence, $\mathcal{U}^\wedge \subseteq \mathcal{V}$. $\square$

Recall from Definition II-1.30 that a topology $\mathcal{U}$ on a **dcpo** X is called *order consistent* if the lower set of a point is the closure of that point and if directed nets converge to their sups. Equivalently we could say that $\mathcal{U}$ contains the upper topology and is contained in the Scott topology. The next definition introduces some other connections between order and topology and is somewhat technical, but it includes notions which will prove quite useful in the development of this section. We recall from Definition VI-1.2 that a topology on a poset is called *compatible* if directed nets converge to their sups, and dually filtered nets converge to their infs. For a subset $V \subseteq X$, we write $V^\diamond$ for the set of all such directed sups and filtered infs of nets in V, and we remark that in the compatible case we have $V \subseteq V^\diamond \subseteq V^-$.

Definition VII-1.6. A topology on a poset X is called *order regular* (or *o-regular*) if for every open neighborhood $U \in \mathcal{U}$ of a point there exists another neighborhood $V \in \mathcal{U}$ of the point such that $V^\diamond \subseteq U$ and filtered sets in V have infs.

A point in X is called an *ω-point* if there exists a countable collection of open neighborhoods of the point, $\{U_n : n \in \mathbb{N}\}$, such that $U_{n+1}^\diamond \subseteq U_n$, and the point is a minimal element of the set $\bigcap\{U_n : n \in \mathbb{N}\}$. □

Proposition VII-1.7. *Let X be a poset equipped with a topology.*

(i) *If the relation $\leq$ is lower semiclosed and the topology is compatible, then the set of open upper sets is an order consistent topology on X.*
(ii) *If X is a compact pospace, then the set of open upper sets forms an o-regular order consistent topology on X.*
(iii) *If the relation $\leq$ is lower semiclosed and the topology is o-regular, then each point in X is the supremum of the ω-points below it.*

Proof: (i) It is immediate that the collection of open upper sets is closed under arbitrary unions, finite intersections, and contains X and $\emptyset$; hence it forms a topology.

For $x \in X$ with respect to this topology $\{x\}^-$ is a lower set since its complement is open; thus $\downarrow x \subseteq \{x\}^-$. Since the order is lower semiclosed, $\downarrow x$ is closed in both topologies; thus $\{x\}^- \subseteq \downarrow x$. Hence, the two sets are equal, and the first condition for being order consistent is satisfied. The second condition follows from the compatability of the topology.

(ii) By part (i) the topology is order consistent. Suppose $x \in X$ and U is an open upper set containing x. Then $\uparrow x$ and $X \setminus U$ are disjoint, $\uparrow x$ is a closed upper set and $X \setminus U$ is a closed lower set. Since X is monotone normal (see VI-1.8), there exist an open upper set P and an open lower set Q such that $\uparrow x \subseteq P$, $X \setminus U \subseteq Q$, and $P \cap Q = \emptyset$. Let F be a filtered set in P. Since X

has a compatible topology, $\inf F \in P^- \subseteq X\backslash Q \subseteq U$. Clearly, since $P = \uparrow P$, if D is a directed subset of P then $\sup D \in P \subseteq U$. Thus the open upper sets form an o-regular topology.

(iii) Let X be a lower semiclosed space equipped with an o-regular topology, and let $x \in X$. Suppose y is an upper bound for all ω-points below x. We show then that $x \leq y$ and hence x is the supremum of all ω-points below it.

Suppose on the contrary that $x \not\leq y$. Then $U_1 = X\backslash\downarrow y$ is an open neighborhood of x. Pick U_2 such that U_2 is open, $x \in U_2$, and $U_2^\lozenge \subseteq U_1$. Continue this procedure inductively. Let $P = \bigcap\{U_n : n \in \mathbb{N}\}$.

Let M be a maximal chain in P containing x. Then since $M \subseteq U_{n+1}$, $\inf M$ exists and is a member of U_n for all n. Hence $\inf M \in P$. Clearly $\inf M \leq x$. Also $\inf M$ is minimal in P (otherwise the maximal chain M could be extended). Thus $\inf M$ is an ω-point. Since $\inf M \in P \subseteq U_1 \subseteq X\backslash\downarrow y$, we have $\inf M \not\leq y$, a contradiction. Thus $x \leq y$. □

We come now to a key (and difficult) lemma.

Lemma VII-1.8. *Let L be a complete lattice with a topology $\mathcal{U}$ and $a_1, \ldots, a_n \in L$ be a finite set of ω-points. If an ultrafilter on L converges to each of the points $a_1, \ldots, a_n$ in the topology $\mathcal{U}^\wedge$, then it converges to $s = a_1 \vee \cdots \vee a_n$ in the Scott topology.*

Proof: The proof for any finite number of points is essentially the same as that for two. We restrict our attention to the latter case in order to simplify the bookkeeping. Suppose then that a and b are ω-points and the ultrafilter $\mathcal{F}$ converges to both a and b in the $\mathcal{U}^\wedge$ topology. Let M be a Scott open set around $s = a \vee b$. Let $\{U_n\}$ and $\{V_n\}$ be the sequences of open neighborhoods of a and b guaranteed by the definition of ω-points. Let $F \in \mathcal{F}$.

We proceed recursively to obtain three sequences $\{a_n\}$, $\{b_n\}$, and $\{x_n\}$ such that for all n and all $j < n$ we have

(i) $a_n = ax_n \in U_n$ and $b_n = bx_n \in V_n$,
(ii) $a_j a_{j+1} \ldots a_n = a_j a_{j+1} \ldots x_n \in U_j$ and $b_j b_{j+1} \ldots b_n = b_j b_{j+1} \ldots x_n \in V_j$,
(iii) $x_n \in F$.

Since $a \in a^{[-1]}U_1 \in \mathcal{U}^\wedge$ and $b \in b^{[-1]}V_1 \in \mathcal{U}^\wedge$, we have $a^{[-1]}U_1 \in \mathcal{F}$ and $b^{[-1]}V_1 \in \mathcal{F}$; thus $a^{[-1]}U_1 \cap b^{[-1]}V_1 \cap F \in \mathcal{F}$. Let x_1 be a point in this intersection. Then $a_1 = ax_1 \in U_1$ and $a_1 \leq a$; similarly $b_1 = bx_1 \in V_1$ and $b_1 \leq b$.

Next $a \in a^{[-1]}U_2$ and $a \in a_1^{[-1]}U_1$, because $aa_1 = a_1$. Similarly we find $b \in b^{[-1]}V_2 \cap b_1^{[-1]}V_1 \in \mathcal{U}^\wedge$. Thus there exists an element

$$x_2 \in a^{[-1]}U_2 \cap a_1^{[-1]}U_1 \cap b^{[-1]}V_2 \cap b_1^{[-1]}V_1 \cap F.$$

Let $a_2 = ax_2$ and $b_2 = bx_2$. Then $a_2 \le a$, $a_2 \in U_2$, and $a_1a_2 = a_1x_2 \in U_1$; analogs statements hold for b_2. The recursive procedure should now be clear.

We set $c_n = \bigwedge\{a_j : j \ge n\}$, $d_n = \bigwedge\{b_j : j \ge n\}$, and $y_n = \bigwedge\{x_j : j \ge n\}$. Then the sequences $\{c_n\}$, $\{d_n\}$, and $\{y_n\}$ are all directed. Let $y = \bigvee_n y_n$. By part (ii) of the recursive definition, for each k, $\bigwedge_{i \le k} a_{n+i} \in U_n$. Since c_n is the filtered inf of $\{\bigwedge_{i \le k} a_{n+i} : k \ge 1\}$, we have $c_n \in U_{n-1}$ (from the definition of ω-point). Similarly $d_n \in V_{n-1}$ for all $n > 1$.

Let $c = \bigvee\{c_n : n \ge 1\}$. Since the set $\{U_n\}$ is towered, the preceding paragraph implies $c_m \in U_{n+1}$ for $m > n+1$. Since $\{c_n\}$ is a directed set, $c \in U_n$ (from the definition of an ω-point). Since n was arbitrary, $c \in \bigcap_n U_n$. Similarly $d \in \bigcap_n V_n$. For each n, $c_n \le a_n \le a$; hence $c \le a$. Since a is minimal in $\bigcap_n U_n$, we conclude $c = a$. Similarly $d = b$.

From the definition of c_n, d_n and y_n we conclude $c_n \le y_n$ and $d_n \le y_n$. Hence $c \le y$ and $d \le y$, that is, $s = a \vee b = c \vee d \le y$. Since we began with a Scott open set M around s and $s \le y$, we conclude $y \in M$. Since y is the sup of a directed set, there exists n such that $y_n \in M$. Since $y_n \le x_n$, we conclude $x_n \in M$. Finally, $x_n \in F$ (part (iii) of the recursive definition) implies $F \cap M \ne \emptyset$.

Now F was an arbitrary member of $\mathcal{F}$, so we conclude $F \cap M \ne \emptyset$ for every $F \in \mathcal{F}$. As $\mathcal{F}$ is an ultrafilter, this implies $M \in \mathcal{F}$. Since M was an arbitrary Scott open set around s, we conclude that $\mathcal{F}$ converges to s in the Scott topology. □

We come now to a major theorem.

Theorem VII-1.9. *Let L be a complete lattice with a topology $\mathcal{U}$ in which every open set is an upper set.*

(i) *If each point is a supremum of ω-points, then $\mathcal{U}^\wedge$ contains the Scott topology.*
(ii) *If $\mathcal{U}$ is order consistent and o-regular and L is meet continuous, then $\mathcal{U}^\wedge$ is the Scott topology.*
(iii) *If $\mathcal{U}$ is order consistent and o-regular and if L is a semitopological semilattice with respect to the meet operation, then $\mathcal{U}$ is the Scott topology.*

Proof: (i) Showing $\mathcal{U}^\wedge$ contains the Scott topology is equivalent to showing the identity function is continuous from $(L, \mathcal{U}^\wedge)$ to $(L, \sigma(L))$. For this it suffices to show that if an ultrafilter $\mathcal{F}$ converges to x in $(L, \mathcal{U}^\wedge)$, then the same obtains in $(L, \sigma(L))$.

Let M be a Scott open set containing x. Since M is Scott open and x is the sup of all ω-points below it, we conclude there exist ω-points $x_1, \ldots, x_n$ such that $y = x_1 \vee \cdots \vee x_n \in M$ and $y \leq x$. Since each open set in $\mathcal{U}$, and hence in $\mathcal{U}^\wedge$, is an upper set, the ultrafilter $\mathcal{F}$ also converges to each of the points $x_1, \ldots, x_n$ in $(L, \mathcal{U}^\wedge)$. By Lemma VII-1.8 $\mathcal{F}$ converges to y in the Scott topology. Since $y \in M$, we conclude $M \in \mathcal{F}$. Since M was an arbitrary Scott open set containing x, we conclude $\mathcal{F}$ converges to x in $\sigma(L)$.

(ii) Since L is meet-continuous, $(L, \sigma(L))$ is a semitopological semilattice. Hence by Proposition VII-1.5 $\mathcal{U}^\wedge \subseteq \sigma(L)$.

Conversely by Proposition VII-1.7(iii), each point in L is a supremum of ω-points. Hence by part (i), we have $\sigma(L) \subseteq \mathcal{U}^\wedge$. Thus $\sigma(L) = \mathcal{U}^\wedge$.

(iii) Since $\mathcal{U}$ is order consistent, $\mathcal{U} \subseteq \sigma(L)$. Conversely by VII-1.7(iii) each point is a supremum of ω-points, and hence $\sigma(L) \subseteq \mathcal{U}^\wedge$ by part (i). But since $(L, \mathcal{U})$ is a semitopological semilattice, we have $\mathcal{U}^\wedge = \mathcal{U}$. □

Lattices of the types appearing in Theorem VII-1.9 possess other interesting properties; in the following we make a rather brief allusion to some of these.

Theorem VII-1.10. *Let L be a meet continuous complete lattice with an order consistent o-regular topology $\mathcal{U}$. Then $(L, \sigma(L))$ is strongly sober; that is, every ultrafilter in L has a largest point of convergence in the Scott topology.*

Proof: Let $\mathcal{F}$ be an ultrafilter which converges to points x and y in the Scott topology. Let $A = \{z\colon z \text{ is an } \omega\text{-point}, z \in \downarrow x \cup \downarrow y\}$. Since x and y are each the supremum of the ω-points below them by VII-1.7(iii), we conclude $\sup A = x \vee y$.

Since $\mathcal{F}$ converges to each of x and y, we have $\mathcal{F}$ converges to every member of A in the Scott topology. By Theorem VII-1.9 $\mathcal{U}^\wedge = \sigma(L)$; thus, by Lemma VII-1.8, the ultrafilter $\mathcal{F}$ Scott-converges to $\sup F$ for every finite set $F \subseteq A$. Since the set $\{\sup F\colon F \text{ is finite}, F \subseteq A\}$ is directed, $\mathcal{F}$ converges to its supremum, $x \vee y$.

Since x and y were arbitrary points of convergence for $\mathcal{F}$, we conclude the set of convergence points of $\mathcal{F}$ is directed. Then $\mathcal{F}$ will also converge to the supremum of the convergence points, since the set of all convergence points of $\mathcal{F}$ must be Scott closed. Finally note that any ultrafilter converges to 0 in the Scott topology, so $(L, \sigma L)$ must be strongly sober. □

Proposition VII-1.11. *Let L be a meet continuous complete lattice such that the Scott topology on $L \times L$ is strongly sober.*

(i) *The Scott topology on $L \times L$ is the square of the Scott topology on L.*
(ii) $(L, \sigma(L))$ *is a topological lattice.*

Proof: (i) Since the product of two Scott open sets is Scott open, we have always that $\sigma(L) \times \sigma(L) \subseteq \sigma(L \times L)$. To show containment in the other direction, we must show that any ultrafilter $\mathcal{F}$ which converges to (x, y) in the product topology also converges to the same point in the Scott topology on $L \times L$.

Since each open set in the product topology is an upper set, $\mathcal{F}$ also converges to $(x, 0)$ in the product topology. Let U be a Scott open set around $(x, 0)$. Let $A = \{p \in L\colon (p, 0) \in U\}$. Since U is Scott open, it follows that A is Scott open in L. Since U is an upper set, $A \times L \subseteq U$. But $A \times L$ is open in the product topology; hence $A \times L \in \mathcal{F}$ and thus $U \in \mathcal{F}$. Since U was arbitrary, $\mathcal{F}$ converges to $(x, 0)$ in the Scott topology. Similarly $\mathcal{F}$ converges to $(0, y)$ in the Scott topology. Since $\mathcal{F}$ has a largest point of convergence, we conclude that $\mathcal{F}$ converges to (x, y) in the Scott topology.

(ii) It is easily verified that the inverse image of a Scott closed set in L is Scott closed in $L \times L$ for the mapping $(x, y) \to x \vee y$; a similar conclusion holds for $(x, y) \to xy$ if L is meet continuous. Since by part (i) the Scott topology is the product topology we conclude both mappings are continuous. □

Corollary VII-1.12. *Let L be a complete lattice that is a semitopological semilattice with an order consistent o-regular topology. Then the topology is the Scott topology, and L is a strongly sober topological lattice.*

Proof: Use VII-1.9, VII-1.10 and VII-1.11. (To apply VII-1.11 note that the product topology on $L \times L$ is order consistent and o-regular and that $L \times L$ is also meet continuous since L is; thus, by Theorem VII-1.10, $L \times L$ is strongly sober with respect to the Scott topology.) □

Exercises

Exercise VII-1.13. For each $\alpha \in A$, let L_α be a complete lattice with an order consistent o-regular topology. Show the Scott-topology on $\prod_\alpha L_\alpha$ is the product of the Scott topologies. □

Exercise VII-1.14. Assume that X is a **dcpo** which is a sup semilattice and has an order consistent topology. Show that X is locally strongly sober implies that X is a (T_0) topological semilattice, and that the latter in turn implies coherence.

Conclude that the three conditions are equivalent if X is locally compact and sober.

Hint. Let $\mathcal{F}$ be an ultrafilter on $X \times X$ converging to (x, y) in the product topology. Let $\mathcal{G}$ be the ultrasfilter generated by the image of $\mathcal{F}$ under the sup mapping. One verifies easily that if $\pi_1: X \times X \to X$ is the projection into the first coordinate, then $\uparrow(\pi_1(F)) \in \mathcal{G}$ for each $F \in \mathcal{F}$. Since the topology on X is order consistent and since the first projection of $\mathcal{F}$ converges to x, it follows that $\mathcal{G}$ converges to x. Similarly $\mathcal{G}$ converges to y. By local strong sobriety $\mathcal{G}$ converges to something greater than or equal to $x \vee y$. Thus, the sup mapping is continuous.

Since the intersection of the two upper sets is just their image under the sup mapping, sup continuity implies coherence. Use Corollary VI-6.16 for the last assertion. □

Problem. Characterize those complete lattices for which the Scott topology is strongly sober. (This is true for lattices that are compact topological semilattices by VI-6.25 and for quasicontinuous lattices by VI-6.24.) □

Problem. Let L be a meet continuous complete lattice. If L can be given some topology making it a compact pospace, is the Scott topology locally compact? □

Old notes

The techniques and results of this section constitute previously unpublished work of Gierz and Lawson. Traditionally topological algebraists have restricted their attention to the Hausdorff setting, but with the advent of non-Hausdorff topologies one is motivated to consider other assumptions. Certainly much remains to be done in this area.

VII-2 Topological Lattices

There are many examples of topological lattices (always assumed Hausdorff unless this is clearly not the case from context). The unit interval $\mathbb{I} = [0, 1]$, as everyone knows, is a topological lattice with respect to its usual order and topology. (In fact, more generally any chain under the order topology is a topological lattice.) Moreover, any cartesian product of the interval $\prod_\alpha \mathbb{I}_\alpha$ of any number of copies of $\mathbb{I}$ is a distributive topological lattice with respect to the product topology and the coordinatewise order. We shall return to a characterization of certain sublattices of such products at the end of this section.

Examples of a different kind can be extracted from "hyperspaces" of closed subsets of suitable semilattices. Recall the discussion of the Vietoris topology in VI-3.8 and VI-3.10. Let S be a compact topological semilattice, hence, a compact pospace. In the notation adopted in the remark preceding VI-3.11, $\Upsilon(S)$ is the lattice of all closed lower sets in S ordered by inclusion and equipped with the relative Vietoris topology. We have already remarked that $\Upsilon(S)^{\mathrm{op}}$ is a continuous lattice (VI-3.11 and VI-3.22), but we can say more:

Proposition VII-2.1. *Let S be a compact topological semilattice. Then $\Upsilon(S)$ is a distributive topological lattice. Furthermore, the embedding $x \mapsto \downarrow x : S \to \Upsilon(S)$ is a topological and semilattice monomorphism.*

Proof: Recall that $\Upsilon(S)$ is a closed subset of $\Gamma(S)$. It is standard fare in point-set topology that continuous mappings on compact Hausdorff spaces induce continuous mappings between their hyperspaces (see Example VI-3.8); thus, the mapping $(A, B) \mapsto AB = \{ab\colon a \in A, b \in B\}$ is continuous from $\Gamma(S) \times \Gamma(S) \to \Gamma(S)$ and also when restricted to $\Upsilon(S)$. Since the lattice operations are just union and intersection, $\Upsilon(S)$ is distributive. The mapping $x \mapsto \downarrow x$ is easily verified to be a semilattice monomorphism from S to $\Upsilon(S)$. It is also continuous (see Exercise VI-3.20). □

In the example just given the join operation is very quickly shown to be jointly continuous (indeed $\Upsilon(S)$ is a closed subsemilattice of the continuous lattice of closed sets of S under reverse inclusion). But it is often enough to check separate continuity. In this regard the next proposition demonstrates the power of the theory developed in the preceding section.

Proposition VII-2.2. *Let L be a complete lattice endowed with a Hausdorff topology making L into a compact topological semilattice. If the join operation is separately continuous, then L is a topological lattice.*

Proof: By VI-1.14 we know that L is a compact pospace. By VII-1.3 applied to L^{op}, we find L endowed with the topology of all open lower sets is a semitopological semilattice with respect to the join operation, and by VII-1.7(ii) this topology is order consistent and o-regular. Corollary VII-1.12 then implies L is a topological lattice with respect to this topology. A similar argument implies that L is a topological lattice when endowed with the topology of all open upper sets. Since VI-1.9 shows L has a subbasis of open increasing and open decreasing sets, we conclude that L with its original topology is a topological lattice. □

Corollary VII-2.3. *A compact topological lattice L has for its topology the topology generated by the Scott open sets and their duals (the Scott open subsets of L^{op}).*

Remark. The topology just mentioned will sometimes be referred to as the *bi-Scott topology*. See Exercise VII-2.12 for a general condition implying the Hausdorff property.

Proof of corollary: The proof follows immediately from the proof of VII-2.2 if one notes that by VII-1.12 the increasing open sets are the Scott open sets and the decreasing open sets the dual Scott open sets. □

Corollary VII-2.4. *Let L be a distributive continuous lattice. The following statements are equivalent:*

(1) *L is join continuous;*
(2) *L is a topological lattice with respect to the Lawson topology.*

Proof: (2) implies (1): Use O-4.4.

(1) implies (2): We need only show the join operation is separately continuous. But this follows from the Fundamental Theorem VI-3.4 since the mapping $x \to x \vee y$ for a fixed $y \in L$ preserves arbitrary nonempty sups and meets. □

Definition VII-2.5. A complete lattice L is a *bicontinuous* lattice if L is a continuous lattice with respect to both the meet and join operations (that is, both L and L^{op} are continuous). If further the two Lawson topologies on L and L^{op} agree, the lattice is called a *linked bicontinuous lattice*. □

Definition VII-2.6. Let L be a complete lattice. The *interval topology* on L is the join of the lower topology and its order dual, the upper topology. Hence, the set of principal filters and principal ideals form a subbasis for the closed sets for the interval topology. □

Lemma VII-2.7. *Let L be a complete lattice. The Lawson and the dual Lawson topologies agree (that is, the lattice is linked) iff they are both the interval topology. This is the case if the interval topology is Hausdorff.*

Proof: Assume the lattice is linked. By III-3.18(ii) the closed upper sets in the Lawson topology are closed sets in the lower topology and hence are closed in the interval topology. Dually the closed lower sets in the dual Lawson topology (equal to the Lawson topology) are closed sets in the upper topology; this means the Scott topology is equal to the upper topology. Since the two generate the Lawson topology we conclude the Lawson topology is contained in the interval topology, and the reverse inclusion always holds.

Since the Lawson topology on a complete lattice is compact (III-1.9), we conclude that it must agree with the interval topology, if the latter is Hausdorff.

We now formulate a version of the Fundamental Theorem VI-3.4 appropriate to topological lattices.

Proposition VII-2.8. (i) *Let L be a linked bicontinuous lattice. Then with respect to the Lawson topology L is a compact topological lattice which at each point has a basis of neighborhoods which are sublattices.*

(ii) *Conversely, given a compact topological lattice L which has small semilattices for both operations, then L is a linked bicontinuous lattice, and the topology on L is the Lawson topology.*

Proof: (i) Applying the Fundamental Theorem VI-3.4 to L endowed with the meet and join operations we conclude that L is a topological lattice. Let x be a member of an open set $U = \uparrow U$. Since L is continuous there exists an open filter F such that $x \in F \subseteq U$. Dually if $x \in V$, V is open, and $\downarrow V = V$, there exists an open ideal M such that $x \in M \subseteq V$. Since the intersection of open filters (resp., open ideals) is again an open filter (resp., open ideal) and since the intersection of an open filter and an open ideal is an open sublattice, it follows from VI-1.9 and VI-1.14 that L has a basis of open sublattices.

(ii) Conversely suppose L is a compact topological lattice which has small semilattices for both operations. Then by the Fundamental Theorem VI-3.4 L is a continuous lattice with respect to both the meet and join operations and the topology of L has to be the same as the Lawson topology and the dual Lawson topology. Thus, L is linked bicontinuous. □

The next proposition gives another important characterization of linked bicontinuous lattices.

Proposition VII-2.9. *Let L be a complete lattice. The following statements are equivalent:*

(1) *L is a linked bicontinuous lattice;*
(2) *L is a meet continuous and join continuous lattice and the interval topology is Hausdorff;*
(3) *L admits the structure of a compact topological lattice (with respect to the interval topology) with a basis of sublattices.*

Proof: (1) $\Rightarrow$ (2): If L is a linked bicontinuous lattice, then the Lawson and dual Lawson topologies agree with the interval topology by Lemma VII-2.7. That L is meet and join continuous follows from I-1.8.

(2) $\Rightarrow$ (1): Suppose L is both meet and join continuous and that the interval topology is Hausdorff. By Lemma VII-2.7 L is linked, and the Lawson

topologies agree with the Hausdorff interval topology. By III-2.11, L is a continuous lattice and dually is also cocontinuous. Therefore, L is bicontinuous and linked.

(1) $\Rightarrow$ (3): By VII-2.8 L with the Lawson topology is a compact topological lattice with a basis of open sublattices with respect to the Lawson topology. But since also (1) implies (2) the coarser interval topology is Hausdorff, and hence must agree with the Lawson. Dually it agrees with the dual Lawson topology.

(3) $\Rightarrow$ (1): Follows from Proposition VII-2.8. □

Completely distributive lattices have already made their appearances in our study (see I-2.8, I-2.9, I-3.16, I-3.39 and II-1.13). The next proposition relates them to the considerations of this section.

Proposition VII-2.10. *Let L be a distributive complete lattice. The following statements are equivalent:*

(1) *L is completely distributive;*
(2) *L is linked bicontinuous;*
(3) *the set of lattice homomorphisms into $\mathbb{I}$ preserving arbitrary meets and joins separates points;*
(4) *L admits a topology making it a compact topological lattice for which the set of continuous lattice homomorphisms into $\mathbb{I}$ separates points;*
(5) *L is bicontinuous;*
(6) *L is order isomorphic to the lattice of Scott open sets of some domain, or equivalently L has spectrum a domain with the Scott topology;*
(7) *L is a continuous lattice in which every element is a sup of co-primes;*
(8) *L is meet continuous, join continuous, and has Hausdorff interval topology;*
(9) *L admits the structure of a compact topological lattice with a basis of sublattices.*

Remark. The conditions of this theorem therefore completely characterize those lattices representable as sublattices of direct powers of the unit interval closed under arbitrary inf and sup.

Proof of proposition: The equivalence of (1), (5), and (7) follows from I-3.16, and the further equivalence with (3) follows from the note following its proof. (See also Exercise IV-3.30.) The equivalence of (2), (8), and (9) follows from Proposition VII-2.9.

(3) implies (4): Embed L in a product of intervals with lattice homomorphisms preserving arbitrary joins and meets. Then the image is closed under arbitrary joins and meets. Thus the image is closed under arbitrary infs and

directed sups, and hence is closed in the product topology by III-1.12. Thus, with respect to the relative topology, it is a compact topological lattice. The projections into the coordinates show there are enough continuous lattice homomorphisms to separate points.

(4) implies (2): The hypothesis implies that L can be embedded in a product of intervals by a topological lattice isomorphism. Since a closed sublattice of a product of intervals is linked bicontinuous, (2) follows.

(2) implies (5): Immediate.

(1) iff (6): That (1) implies (6) follows from II-1.14 and the converse follows from the spectral theory of continuous lattices, since the spectrum of a completely distributive lattice is a domain endowed with its Scott topology (see Sections V-1 and V-4). □

Without assuming (1) implies (3), one can argue directly that (2) implies (4) and hence (3) by an Urysohn-type argument (employing Lemma I-3.20). These Urysohn-type arguments appear in [Davies, 1968] and [Lawson, D1967].

We have just seen that a distributive lattice for which L and L^{op} are both continuous lattices is linked bicontinuous. This conclusion can fail if the lattices are *not* assumed to be distributive. Indeed, let C be a countably infinite set with the trivial partial ordering (just the equality relation). Let $L = \{0, 1\} \cup C$ be the resulting lattice obtained by adjoining a zero and a unit. Then L and L^{op} are (isomorphic) algebraic lattices. However, in the Lawson topology on L, 0 is the one point compactification of C, while 1 is the one point compactification of C in L^{op}. (That is to say, $\{0\} \cup C$ is closed but not open in the Lawson topology of L; whereas $\{1\} \cup C$ is open but not closed.) Thus, the two topologies are different.

Exercises

Exercise VII-2.11. Consider the following already established statements.

(i) $\mathrm{Hom}(L, \mathbb{I})$, the set of continuous lattice homomorphisms, separates points if L is a completely distributive complete lattice (Proposition VII-2.9).

(ii) $\mathrm{Hom}(S, \mathbb{I})$, the set of continuous semilattice homomorphisms, separates points if S is a continuous lattice with the Lawson topology (Proposition VI-3.7).

(iii) $\mathrm{Hom}(X, \mathbb{I})$, the set of continuous order preserving functions, separates points if X is a compact pospace (Exercise VI-1.16).

Show that (i) implies (ii) implies (iii).

Hint. (i) implies (ii): If S is as hypothesized, then $\Upsilon(S)$, the set of closed lower sets, is dual to the Scott topology, and hence completely distributive (II-1.14). Furthermore S can be embedded in $\Upsilon(S)$ by a topological semilattice isomorphism by sending s to $\downarrow s$. Compose the members of $\mathrm{Hom}(\Upsilon(S), \mathbb{I})$ with the embedding.

(ii) implies (iii): $\Upsilon(X)$, the set of closed lower sets, is a continuous lattice (see Example VI-3.10). Embed X in $\Upsilon(X)$ by sending x to $\downarrow x$. Compose members of $\mathrm{Hom}(\Upsilon(X), I)$ with the embedding. □

Exercise VII-2.12. Let L be the lattice of regular open sets of the unit interval ordered by inclusion (an open set is regular if it is equal to the interior of its closure). Prove the following.

(i) For any collection $\mathcal{A}$ of regular open sets, the interior of $\bigcap \mathcal{A}$ is regular and open. Hence L is a subset of the open set lattice of the unit interval which is closed under arbitrary infs, and therefore a complete lattice in its own order (note, however, that it is not a sublattice of the full open set lattice).
(ii) The lattice L is a Boolean lattice, where complement is given by taking the interior of the set complement.
(iii) L is meet and join continuous.
(iv) Any dual Scott open set of which the empty set is a member contains a monotone increasing sequence with supremum equal to $1 = [0, 1]$.
(v) Any order consistent topology on L is not o-regular, and the bi-Scott topology is not Hausdorff.

Hint. Items (i) and (ii) are straightforward. By Lemma O-3.16 and Exercise O-3.19 a Boolean lattices is meet continuous. Since complementation in L is an order anti-isomorphism, L is also join continuous. For item (iv), we refer the reader to [Floyd, 1955]. Again by complementarity every Scott open set containing 1 must contain a monotone decreasing sequence with infimum $0 = \emptyset$. This shows that L cannot be o-regular, nor can one separate 0 and 1 in the bi-Scott topology. □

Similar pathologies are proved for the lattice of closed congruences on [0, 1] in [Clinkenbeard, 1981].

Exercise VII-2.13. Show that Corollary VII-2.4 cannot be sharpened to conclude that L^{op} is a continuous lattice.

Hint. Let S be the compact unital topological semilattice constructed in Section VI-4 which is not a continuous lattice by Theorem VI-4.5. Then $\Upsilon(S)$, the set of all (topologically) closed lower sets, is a compact distributive topological lattice, that is dually a continuous lattice with respect to the operation of union, and that the embedding $s \mapsto \downarrow s$ from S to $\Upsilon(S)$ is a topological and semilattice monomorphism. Hence, since S fails to have a basis of subsemilattices, the same must be true of $\Upsilon(S)$ with respect to the intersection operation. We have the peculiar phenomenon that a compact distributive topological lattice can be a continuous lattice with respect to one operation, but may fail to be one with respect to the other. □

Exercise VII-2.14. Let L and L^{op} be **dcpo**s. Show that the Lawson topologies are linked iff they are both equal to the bi-Scott topology. Conclude that the Lawson topologies on a complete lattice are linked iff the bi-Scott topology is equal to the interval topology.

Hint. Using III-1.6, proceed as in the proof of Lemma VII-2.7. The second assertion follows from the first and VII-2.7. □

In the next exercise we examine equivalent conditions for the interval topology to be Hausdorff on a complete lattice. Conditions (4), (5), and (6) have equivalent dual conditions which we do not bother to list.

Exercise VII-2.15. Let L be a complete lattice. Show that the following are equivalent:

(1) the interval topology is Hausdorff;
(2) for every ultrafilter $\mathcal{F}$, $\liminf\mathcal{F} = \limsup\mathcal{F}$;
(3) L and L^{op} are quasicontinuous and the Lawson topologies are linked;
(4) L is quasicontinuous and the interval topology is equal to the bi-Scott topology.
(5) L is quasicontinuous and the Scott topology is contained in the dual Lawson topology.
(6) L is quasicontinuous and the Scott topology is equal to the upper topology.

Hint. For the equivalence of (1) and (2), see Exercise III-5.23. That (1) implies (3) follows from Lemma VII-2.7 and Theorem III-3.11. If L is quasicontinuous and the Lawson topologies are linked, then they are equal to the interval topology by VII-2.7 again, and then all are Hausdorff by III-3.7(ii). Thus (1), (2), and (3) are equivalent.

If (4), then since the Lawson topology is trapped between the bi-Scott and interval topologies, it must agree with them. Again by III-3.7(ii) the interval topology must be Hausdorff. That (3) implies (4) follows by the previous exercise.

Assume (5). Then the bi-Scott topology is contained in and hence equal to the dual Lawson topology (since the reverse containment always holds). Thus the bi-Scott topology is compact (III-1.9). Since the Lawson topology is Hausdorff (III-3.7(ii)), we conclude that the bi-Scott and Lawson also agree, and thus the Lawson topologies are linked. It follows from VII-2.7 that they are equal to the interval topology, and thus the interval topology is Hausdorff. Conversely if (3) is assumed, then the Scott topology is contained in the Lawson topology, which is the dual Lawson topology.

Condition (6) easily implies (5). That (3) implies (6) follows from III-1.6 applied to the Lawson topology and III-3.18(ii) applied to the dual Lawson topology. □

There is an interesting connection between the interval topology and the topological notion of a supercompact space. A compact Hausdorff space is said to be *supercompact* if it admits a subbasis of open sets for which every covering of the space by members of the subbasis admits a subcovering of cardinality 2.

Exercise VII-2.16. Let L be a complete lattice endowed with a Hausdorff interval topology. Show that L is supercompact with respect to the interval topology.

Hint. Choose for the subbasis of L all complements of principal ideals and principal filters, the standard subbase for the interval topology. Take some cover of L by subbasic open sets and let x be the supremum of all x_i such that $L \backslash \uparrow x_i$ is in the given cover. Then the complement of $\uparrow x$ is equal to the union of all $L \backslash \uparrow x_i$. It must be the case that $x \in L \backslash \downarrow y$ for some member of the cover, and thus $x_i \not\leq y$ for some i. Then $L \backslash \downarrow y$ and $L \backslash \uparrow x_i$ form a subcover. □

Old notes

Topological lattices have a rather long-standing history. The idea of a topological lattice is implicit in the work of G. Birkhoff on the order topology in the late 1930s and a short time later in the work of O. Frink on the interval topology in a lattice. The theory of topological lattices was first explicitly studied by L.W. Anderson in his 1954 thesis directed by A.D. Wallace. The early work on topological lattices actually preceded the investigation of topological semilattices and was instrumental in shaping the direction of the latter research, although the study of semilattices has surpassed that of topological lattices in recent years.

The equivalence of (1) and (3) in Proposition VII-2.10 is an old result of Raney's [Raney, 1953]. The construction of $\Upsilon(S)$ was early recognized by D.R. Brown and others as an important one in the investigation of topological semilattices. Proposition VII-2.2 is a result of Lawson's. The original proof employed the rather difficult theorem that a separately continuous compact Hausdorff topological semilattice is jointly continuous (a result we obtain in Section VII-4). The present proof is new and employs the machinery built up in Section VII-1 (rather than the intricate topological machinery involved in the other route). For further related details concerning intrinsic lattice topologies, topological lattices, and completely distributive lattices, see [Strauss, 1968], [Lawson, 1973], [Gierz and Lawson, 1981].

New notes

Gierz and Stralka have rather extensively investigated sublattices of Euclidean space, in particular what they call homogeneous sublattices [Gierz and Stralka, 1995]. A particular class of linked bicontinuous lattices that are lattice analogs of FS-domains has been employed by Huth, Jung, and Keimel as objects for their model of linear logic [Huth *et al.*, 2000]. A rich class of linked bicontinuous algebraic lattices has recently been discovered – see [Adaricheva *et al.*, 2001]. These lattices are not supposed to be complete, but they can be completed and are linked bicontinuous. These lattices are not distributive.

VII-3 Hypercontinuity and Quasicontinuity

We have seen in Section VII-2 circumstances in which the interval topology on a complete lattice is Hausdorff and hence agrees with the Lawson topology and dual Lawson topology, that is the topologies are linked. We here explore the interesting class of continuous lattices which have this property. In the distributive case we will find that they stand in duality with quasicontinuous domains through spectral theory.

The next three items develop basic properties of hypercontinuous lattices, analogs of continuous lattices.

Let L be a **dcpo**. We define a relation $\prec$ on L by $x \prec y$ iff whenever the intersection of a nonempty collection of upper sets is contained in $\uparrow y$, then the intersection of finitely many is contained in $\uparrow x$.

Lemma VII-3.1. *Let L be a* **dcpo**.

(i) *The following statements are equivalent:*
  (1) *$x \prec y$;*
  (2) *if the intersection of a nonempty collection of sets open in the upper topology $\nu(L)$ (see* O-5.4*) is contained in $\uparrow y$, then the intersection of finitely many of them is contained in $\uparrow x$;*

(3) *if* $T \subseteq L$, $T \neq \emptyset$, *and* $\bigcap\{L \setminus \downarrow t : t \in T\} \subseteq \uparrow y$, *then there exists a finite subset* $F \subseteq T$ *such that* $\bigcap\{L \setminus \downarrow t : t \in F\} \subseteq \uparrow x$;

(4) $y \in \mathrm{int}_\nu \uparrow x$, *where the interior is taken in the upper topology.*

(ii) *If* $x \prec y$, *then* $x \ll y$.

(iii) *The relation* $\prec$ *is an auxiliary relation such that* $u \prec y$ *and* $v \prec y$ *together imply* $u \vee v \prec y$, *whenever* $u \vee v$ *exists in* L.

Proof: (i) It is straightforward that (1) implies (2) implies (3) implies (4). (4) implies (1): There exists a basic open set $U = \bigcap_{i=1}^{n}(L \setminus \downarrow x_i)$ in the upper topology such that $y \in U \subseteq \uparrow x$. Then for any collection of upper sets with intersection contained in $\uparrow y$, pick for each i one missing x_i. This finite subcollection will intersect inside of $\uparrow x$.

(ii) For a directed set D with $y \leq \sup D$, consider the family of principal filters generated by members of D. Their intersection is contained in $\uparrow y$, and hence some $\uparrow d \subseteq \uparrow x$. The verification that $\prec$ is an auxiliary relation satisfying (iii) is straightforward. □

Definition VII-3.2. Let L be a complete lattice. The lattice L is called a *hypercontinuous lattice* if the auxiliary relation $\prec$ is approximating, that is, if for all $y \in L$, we have $y = \sup\{x : x \prec y\}$. (Note that for every element y in a complete lattice L, the set $\{x : x \prec y\}$ is directed by VII-3.1(iii).) □

Proposition VII-3.3. *Let* L *be a complete lattice. The following statements are equivalent:*

(1) L *is a hypercontinuous lattice;*

(2) L *is a continuous lattice in which* $x \ll y$ *implies* $x \prec y$;

(3) *for all* $y \in L$,

$$y = \sup\{\inf U : y \in U \text{ and } U \text{ is open in the upper topology}\}.$$

Proof: (1) iff (2): If L is a hypercontinuous lattices, then the relation $\prec$ is approximating. By I-1.15, $\ll$ is contained in every approximating auxiliary relation, that is $z \ll y$ implies $z \prec y$. From Lemma VII-3.1(ii) the relation $\ll$ is approximating since $\prec$ is. Thus L is continuous.

Conversely if (2) holds, then $\prec$ is approximating since $\ll$ is.

(1) iff (3): This equivalence follows readily from the equivalence of (1) and (4) in VII-3.1(i). □

The next proposition gives various equivalent formulations for the notion of a hypercontinuous lattice in the context of continuous lattices.

Theorem VII-3.4. *Let L be a continuous lattice. The following statements are equivalent:*

(1) *L is hypercontinuous;*
(2) *for all $x, y \in L$, $x \prec y$ iff $x \ll y$;*
(3) *the Scott topology is the upper topology ($\sigma(L) = \nu(L)$);*
(4) *the Lawson topology is the interval topology ($\lambda(L) = \omega(L) \vee \nu(L)$);*
(5) *the Lawson topologies on L and L^{op} are linked;*
(6) *L^{op} is a quasicontinuous lattice and the Lawson topologies are linked;*
(7) *the interval topology is Hausdorff;*
(8) *for every ultrafilter $\mathcal{F}$ on L, $\sup\{\inf F : F \in \mathcal{F}\} = \inf\{\sup F : F \in \mathcal{F}\}$.*

$\square$

Proof: (1) implies (2): Lemma VII-3.1(ii) and Proposition VII-3.3(2).

(2) implies (3): Since the upper topology is always coarser than the Scott topology, we need to show that a Scott open set U is open in the upper topology. Let $y \in U$. Since y is the directed sup of $x \prec y$ (by continuity of L and (2)), there exists $x \in U$ such that $x \prec y$. By (4) of Lemma VII-3.1(i), $y \in \operatorname{int}_\nu \uparrow x$, where the interior is taken in the upper topology, and this interior is contained an $\uparrow x$, a subset of U.

(3) implies (4): This is immediate since the Lawson topology is the join of the Scott topology and the lower topology, and the interval is the join of the upper and lower.

(4) implies (5): Recall from Theorem III-1.10 that the Lawson topology is Hausdorff, and hence the interval topology is Hausdorff. Now apply Lemma VII-2.7.

(5) implies (6): That L^{op} is quasicontinuous follows from the Hausdorffness of the dual Lawson topology by III-3.11.

(6) implies (7): This follows immediately from Lemma VII-2.7 and Theorem III-1.10

(7) implies (4): This is immediate from Lemma VII-2.7.

(6) implies (8): This follows immediately from Theorem III-3.17(ii) and its order dual.

(8) implies (1): Suppose that $x \ll y$ and that there exists a family $\mathcal{A}$ of upper sets such that $\bigcap \mathcal{A}$ is contained in $\uparrow y$, but no finite intersection of members of $\mathcal{A}$ is contained in $\uparrow x$. Let $\mathcal{F}$ be the filter generated by the filter base $(A_1 \cap \cdots \cap A_n) \setminus \uparrow x$ over all finite collections $A_1, \ldots, A_n \in \mathcal{A}$. Extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$.

Consider $u = \sup U$ for some $U \in \mathcal{U}$. For any $A \in \mathcal{A}$, $A \cap U \neq \emptyset$, so $u \in \uparrow A = A$. Since A was arbitrary, we conclude that $u \in \bigcap \mathcal{A} \subseteq \uparrow y$. Thus $y \leq \inf\{\sup U : U \in \mathcal{U}\}$. Thus $y \leq \sup\{\inf U : U \in \mathcal{U}\}$, and hence $x \leq \inf U$ for small enough $U \in \mathcal{U}$ since the family of anfima is directed. But this contradicts the fact that $U \cap \uparrow x = \emptyset$ for small enough U.

We conclude that $x \ll y$ implies that $x \prec y$, and hence that L is hypercontinuous. $\square$

As we have seen, even a bicontinuous lattice need not necessarily be linked, but this is the case for distributive lattices. We see that this generalizes to the current setting.

Proposition VII-3.5. *Let L be a distributive continuous lattice. Then L is a hypercontinuous lattice iff L^{op} is a quasicontinuous lattice.*

Proof: One implication is immediate from part (6) of the preceding theorem.

Suppose that L^{op} is a quasicontinuous lattice. Suppose that there exists an ultrafilter $\mathcal{F}$ such that its liminf x is strictly smaller than its limsup y. Then there exists a finite set $H = \{h_1, \ldots, h_n\}$ such that $H \ll x$ in L^{op} and $\downarrow H \cap \uparrow y = \emptyset$, since L^{op} is quasicontinuous. Since L is primally generated (I-3.15), there exist primes $p_1, \ldots, p_n$ such that $h_i \le p_i$ and $y \not\le p_i$ for each i. Let $K = \bigcup_{i=1}^{n} \downarrow p_i$. The $L \setminus K = \bigcap_{i=1}^{n} (L \setminus \downarrow p_i)$, an open filter containing y. If $K \in \mathcal{F}$, then one of the sets $\downarrow p_i$ in the finite union must be in $\mathcal{F}$; but then the limsup x is less than or equal to p_i, a contradiction. We conclude that the open filter $L \setminus K \in \mathcal{F}$. Since $x = \sup\{\inf F \colon F \in \mathcal{F}\}$ and $H \ll x$ in L^{op}, we conclude that $\inf(L \setminus K) \le x$, and hence $\inf G \in \downarrow H$ for some finite subset $G \subseteq L \setminus K$. But this contradicts the fact $L \setminus K$ is a filter. Hence by condition (8) of the preceding theorem, L is hypercontinuous. $\square$

We turn now to the spectral theory of hypercontinuous lattices.

Lemma VII-3.6. *Let L and M be continuous lattices, and let $f \colon L \to M$ be a surjective order preserving map that preserves directed sups and filtered infs. If L is hypercontinuous, then so is M.*

Proof: Since the image of the liminf is less than or equal to the liminf of the image and the image of the limsup is greater than or equal to the limsup of the image for any order preserving map, it follows from the hypothesis for any ultrafilter in L that condition (8) of VII-3.4 is preserved by f. Since any ultrafilter in M is the image of one in L, the lemma follows. $\square$

Proposition VII-3.7. *Let P be a quasicontinuous domain, and let* $\mathbf{U}(P)$ *be the completely distributive lattice of all upper sets ordered by inclusion. Then the function* $\mathrm{int}_\sigma \colon \mathbf{U}(P) \to \sigma(P)$ *that sends an upper set to its Scott interior perserves arbitrary infs and directed sups. Hence $\sigma(P)$ is a hypercontinuous lattice.*

Proof: The inclusion mapping from $\sigma(P)$ to $\mathbf{U}(P)$ is easily seen to be a lower adjoint to the map $A \mapsto \operatorname{int}_\sigma(A)$. Thus the latter preserves arbitrary infs (I-3.5). Let $\mathcal{A}$ be a directed family of upper sets in $\mathbf{U}(P)$ and let $B = \bigcap \mathcal{A}$. We need to show that $\operatorname{int}_\sigma(B) = \bigcup\{\operatorname{int}_\sigma(A)\colon A \in \mathcal{A}\}$. Since taking interior is order preserving, $\operatorname{int}_\sigma(B) \supseteq \bigcup\{\operatorname{int}_\sigma(A)\colon A \in \mathcal{A}\}$. Suppose that $x \in \operatorname{int}_\sigma(B)$. By III-3.6(i), there exists a finite set $F \subseteq \operatorname{int}_\sigma(B)$ such that $F \ll x$. Since $F \subseteq B$ and $\mathcal{A}$ is directed, there exists $A \in \mathcal{A}$ such that $F \subseteq A$. Then we have (again by III-3.6) that $x \in \operatorname{int}_\sigma(\uparrow F) \subseteq \operatorname{int}_\sigma(\uparrow A) = \operatorname{int}_\sigma(A)$. Since x was arbitrary in $\operatorname{int}_\sigma(B)$, we conclude that $\operatorname{int}_\sigma(B) \subseteq \bigcup\{\operatorname{int}_\sigma(A)\colon A \in \mathcal{A}\}$.

Now $\mathbf{U}(P)$ is completely distributive, being a complete sublattice of the powerset lattice, hence has Hausdorff interval topology by VII-2.10, and thus is hypercontinuous by VII-3.4. Hence $\sigma(P)$ is hypercontinuous by the preceding lemma. □

Proposition VII-3.8. *Let L be a distributive hypercontinuous lattice. Then its spectrum is a quasicontinuous domain equipped with the Scott topology. Hence L is isomorphic to the lattice of Scott open sets of a quasicontinuous domain.*

Proof: We show that $\operatorname{Spec} L$ (equipped with the reverse order) is a quasicontinuous domain and that its hull–kernel topology is the Scott topology. Since the filtered inf of primes is again a prime (the order dual of I-3.39(i)), we have that $\operatorname{Spec} L$ is a **dcpo** (always with respect to the reverse order).

Let $p \in \operatorname{Spec} L$, and let $\mathcal{F}(p) = \{F \subseteq \operatorname{Spec} L\colon F \text{ is finite}, \downarrow p \subseteq \operatorname{int}(\downarrow F)\}$, where $\downarrow F$ is taken in L and the interior is taken in the dual Scott topology. We claim for each $F \in \mathcal{F}(p)$, $F \ll p$ in $(\operatorname{Spec} L, \geq)$. Indeed suppose that D is a filtered set in $\operatorname{Spec} L$ with $q = \inf D \leq p$. Since filtered sets converge to their infima in the dual Scott topology, we have $d \in \operatorname{int}(\downarrow F)$ for some $d \in D$, and thus $F \ll p$.

Next we show that the family $\mathcal{F}(p)$ is a descending family in $\operatorname{Spec} L$. Let $F_1, F_2 \in \mathcal{F}(p)$. Then $U := \operatorname{int}(F_1) \cap \operatorname{int}(F_2)$ is a dual Scott open set containing p. By Theorem VII-3.4(5), we have that U is open in the Lawson topology of L, and hence $A = L\backslash U$ is a compact upper set in the Lawson topology of L. Since $A \cap \downarrow p = \emptyset$, it follows from the Lemma on Primes (V-1.5) that $y = \inf A \not\leq p$. Then $V = L\backslash\uparrow y$ is a dual Scott open lower set containing $\downarrow p$ and $V \subseteq U$. Since L equipped with the dual order is a quasicontinuous domain by VII-3.4(6), there exists a finite set $G \subseteq V$ such that $G \ll p$ in L^{op}, and thus $p \in \operatorname{int}(\downarrow G)$ (Proposition III-3.6(i)). Since L is primally generated (see I-3.10 and I-3.12), for each $x \in G$, there exists $q_x \in \operatorname{Spec} L$ such that $x \leq q_x$, but $y \not\leq q_x$. Then $F := \{q_x\colon x \in G\}$ is a finite set contained in V and thus $\downarrow F \subseteq V \subseteq U$. Also $p \in \operatorname{int}\downarrow G \subseteq \operatorname{int}\downarrow(F)$. The argument just given applies to any $L\backslash\uparrow y$ for $y \not\leq p$, and thus the intersection of all $\downarrow F$, $F \in \mathcal{F}(p)$, is contained in $\downarrow p$. Thus Spec L is a quasicontinuous domain (actually we have verified the alternative definition given in Exercise III-3.19).

Finally we establish that the spectral topology on Spec L agrees with the Scott topology on (Spec $L, \geq$). Since Spec L is closed with respect to filtered infs, it follows easily that any $\uparrow x \cap$ Spec L is Scott closed in (Spec $L, \geq$).

Conversely let A be Scott closed in Spec L, i.e., A is closed with respect to filtered infs and $\uparrow A \cap$ Spec $L = A$. Let $p \in$ Spec $L \backslash A$. Let $\mathcal{G} = \{F \cap A \colon F \in \mathcal{F}(p)\}$. If each of these sets is nonempty, it follows from Rudin's Lemma III-3.3 applied to (Spec $L, \geq$) that there is a filtered set $D \subseteq \bigcap\{F \cap A \colon F \in \mathcal{F}(p)\}$ such that $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}(p)$. Then $\inf D \in A$ since A is Scott closed. On the other hand,

$$\inf D \in \bigcap\{\downarrow d \colon d \in D\} \subseteq \bigcap\{\downarrow F \colon F \in \mathcal{F}(p)\} \subseteq \downarrow p.$$

This contradiction implies $A \cap F = \emptyset$ for some $F \in \mathcal{F}(p)$, and thus $\uparrow A \cap \operatorname{int}(\downarrow F) = \emptyset$. The Lemma on Primes (Corollary V-1.4(i)) implies that for $z := \inf A$, $z \not\leq p$. Since p was arbitrary in Spec $L \backslash A$, we conclude $A = \uparrow z \cap A$. Thus A is closed in the spectral topology.

Finally from the duality between sober spaces and primally generated lattices (Section V-4), we know that L is isomorphic to the lattice of open sets of its spectrum. □

Theorem VII-3.9. *Let P be a* **dcpo**. *The following statements are equivalent:*

(1) *P is a quasicontinuous domain;*
(2) *the mapping $A \mapsto \operatorname{int}(A)$ from $\mathbf{U}(P)$ to $\sigma(P)$ preserves arbitrary infs and directed sups;*
(3) *the lattice $\sigma(P)$ of Scott open sets is hypercontinuous.*

Proof: That (1) implies (2) implies (3) follows from VII-3.7. Thus we need only show that (3) implies (1).

Now P embeds in $\sigma(P)$ (with the order reversed) by $x \mapsto P \backslash \downarrow x = P \backslash \{x\}^-$. Since for every upper set A, $A = \bigcap\{P \backslash \downarrow x \colon x \notin A\}$, we have that the image of P in $\sigma(P)$ is order generating. Furthermore, each $P \backslash \downarrow x$ is prime in $\sigma(P)$. Thus the image of P in $\sigma(P)$ is an order generating subset of primes. We identify P with its image; note that the order of P is the restricted reverse order of $\sigma(P)$.

Let $p \in$ Spec $\sigma(P)$. The proof of the preceding proposition carries over to the embedded image of P in Spec $\sigma(P)$ (since the image is order generating) to show that $\{F \subseteq P \colon F \text{ is finite}, \downarrow p \subseteq \operatorname{int}(\downarrow F)\}$ is a directed family in Spec $\sigma(L)$ with intersection contained in $\downarrow p$. We apply Rudin's Lemma III-3.3 to the family of all $\uparrow p \cap F$ to obtain a directed set in P with infimum p.

Since P is a **dcpo** endowed with the Scott topology and the embedding into the spectrum is a homeomorphic embedding (see Section V-4), it follows that p is in P. Thus P maps onto $\operatorname{Spec}\sigma(P)$, and hence is a quasicontinuous domain. $\square$

We close with an alternative characterization of a quasicontinuous domain in terms of the liminf and Lawson topologies (cf. III-3.11).

Proposition VII-3.10. *Let L be a* **dcpo** *for which every ultrafilter has a liminf, which will be the case if L is a complete lattice or semilattice. If L is Hausdorff in the Lawson or liminf topology, then L is a quasicontinuous domain. Conversely if L is a quasicontinuous domain, then the Lawson and liminf topologies agree and L is a compact pospace for this topology.*

Proof: Suppose first that the liminf topology is Hausdorff. By Lemma III-3.15 each ultrafilter converges to its liminf in the liminf topodogy, and thus the liminf topology is compact. Since this topology is Hausdorff, the liminf of an ultrafiler must be its unique limit point.

By Proposition III-3.14 the closed lower sets in the liminf topology are precisely the Scott closed sets. Since by III-3.17 the Lawson topology is contained in the liminf topology we conclude that principal filters are also closed in the liminf topology. Then by Proposition VI-6.26 we conclude that L is a compact pospace in the liminf topology.

To show that L is a quasicontinuous domain, we show for any $x \in L$ and Scott open set U containing x, there exists a finite set $F \subseteq U$ such that $x \in \operatorname{int}_\sigma(\uparrow F)$ (see III-3.19(ii)). On the contrary suppose that for all finite $F \subseteq U$, x is not in the Scott interior of $\uparrow F$. Then the collection of all $V \setminus \uparrow F$ such that V is a Scott open set containing x and F is a finite set contained in U is a filter base of nonempty sets. Extend it to an ultrafilter $\mathcal{F}$. Then $\mathcal{F}$ converges to its liminf y. It follows from Proposition VI-1.8 that $y \in \uparrow x$ (since any Scott open set containing x is in the ultrafilter). By definition of the liminf, there exists a directed set D such that $\uparrow d \in \mathcal{F}$ for each $d \in D$ and $y = \sup D$. Then $d \in U$ for some d, and thus $U \setminus \uparrow d \in \mathcal{F}$, a contradiction.

If the Lawson topology is Hausdorff, then the finer liminf topology as Hausdorff, so the former case reduces to the latter.

Conversely suppose that L is a quasicontinuous domain. Then by Proposition VI-1.15 L is a pospace with respect to its Lawson topology, in particular Hausdorff. By Lemma III- 3.15 L is compact with respect to the liminf topology (since ultrafilers converge), and thus the weaker Hausdorff Lawson topology agrees with the stronger compact liminf topology. $\square$

Exercises

Exercise VII-3.11. Define the notion of a hypercontinuous domain and determine how many of the equivalences in Theorem VII-3.4 remain valid in this setting. □

Exercise VII-3.12. Show that a completely distributive lattice is hypercontinuous.

Hint. A completely distributive lattice has a Hausdorff interval topology (see VII-2.10). □

Exercise VII-3.13. A complete lattice L is said to be *lean* if every upper set closed in the dual Scott topology is compact in the Scott topology. Show that a lean continuous lattice L for which the dual Lawson topology is Hausdorff is hypercontinuous.

Hint. It follows from Proposition VI-6.24 that the dual Scott topology is contained in the lower topology, and hence that the two are equal, since the reverse containment always holds. Hence the Lawson topology and the bi-Scott topology agree. Since the dual Lawson topology is Hausdorff and coarser than the compact bi-Scott topology, they also are equal, and thus the Lawson topologies are linked. Now apply Theorem VII-3.4. □

New notes

For more details on hypercontinuous lattices, see [Gierz and Lawson, 1981], where the concept was first introduced, and [Gierz *et al.*, 1983b]. The notion of leanness (Exercise VII-3.13) was introduced in [Huth *et al.*, 2000].

VII-4 Lattices with Continuous Scott Topology

In Chapter II we have provided much information on **dcpo**s and on complete lattices L for which the lattice $\sigma(L)$ of Scott open subsets is continuous (II-1.13, II-4.13, II-4.15, II-4.16, II-4.18). In this section, we give a characterization of these complete lattices. Remarkably, the fact that $\sigma(L)$ is a continuous lattice relates to the existence of a compact semilattice topology on L itself.

The key idea is the following: recall in V-4.7 the use made of the natural map $\xi_X = (x \mapsto X\backslash\{x\}^-)\colon X \to \mathcal{O}(X)$ for an arbitrary topological space X. For each $x \in X$, we have $\xi(x) \in \operatorname{Spec}\mathcal{O}(X)$, the set of all prime elements of $\mathcal{O}(X)$ other than X itself, and this map is continuous if $\operatorname{Spec}\mathcal{O}(X)$ is endowed with the hull–kernel topology.

We now assume that $X = (L, \sigma(L))$ where L is an arbitrary complete lattice with its Scott topology $\sigma(L)$. In this topology $\{x\}^- = \downarrow x$, so we have the natural map

$$\xi_L = (x \mapsto L\backslash\downarrow x) \colon L \to \operatorname{Spec} \sigma(L)$$

which is continuous on L under the Scott topology to $\operatorname{Spec} \sigma(L)$ with the hull–kernel topology. We also may consider $\operatorname{Spec} \sigma(L) \subseteq \sigma(L)$ as a set ordered by inclusion. One then has for every subset $A \subseteq L$ and for every directed $D \subseteq L$

$$\xi_L(\inf A) = \bigcup\{\xi_L(a) \colon a \in A\} \text{ and } \xi_L(\sup D) = \operatorname{int}\left(\bigcap\{\xi_L(d) \colon d \in D\}\right).$$

This means that the image of L under ξ_L is closed in $\sigma(L)$ with respect to arbitrary sups and filtered infs. Moreover, L^{op} is order isomorphic to its image.

Note that the image of ξ_L is *not* closed in $\sigma(L)$ under finite infs. Indeed, if x and y are incomparable elements of L, then $\xi_L(x) \cap \xi_L(y) = L\backslash(\downarrow x \cup \downarrow y) \neq L\backslash\downarrow z = \xi_L(z)$ for all $z \in L$.

Under the assumption that $\sigma(L)$ is continuous we can say rather more.

Proposition VII-4.1. *Let L be a complete lattice such that $\sigma(L)$ is continuous.*

(i) $\xi_L \colon L \to \operatorname{Spec} \sigma(L)$ *is a homeomorphism, if L is endowed with the Scott topology and* $\operatorname{Spec} \sigma(L)$ *with the hull–kernel topology.*
(ii) $\xi_L \colon L \to \operatorname{Spec} \sigma(L)$ *is an order anti-isomorphism, and* $\operatorname{Spec} \sigma(L)$ *is closed in $\sigma(L)$ with respect to arbitrary sups and filtered infs.*
(iii) *With respect to the Scott topology, L is a locally compact sober space.*
(iv) $\operatorname{Spec} \sigma(L)$ *is closed in $\sigma(L)$ with respect to the Lawson topology.*

Proof: (i) By II-4.16, L is sober with respect to the Scott topology. Thus ξ_L is a homeomorphism by the remarks preceding V-4.7.

(ii) Use (i) and the remarks preceding this proposition.

(iii) As $\sigma(L)$ is supposed to be a continuous lattice, $(L, \sigma(L))$ is locally compact by V-5.6.

(iv) From I-1.4(ii) we see that two Scott open sets U and V satisfy the relation $U \ll V$ iff there is a compact set Q such that $U \subseteq Q \subseteq V$. As all Scott open sets are upper sets, Q is compact iff $\uparrow Q$ is compact, and we may restrict our attention to compact upper sets. Let U, V, W be Scott open sets such that $U \ll V$ and $U \ll W$. There are compact upper sets Q_1 and Q_2 such that $U \subseteq Q_1 \subseteq V$ and $U \subseteq Q_2 \subseteq W$. Then $U \subseteq Q_1 \cap Q_2 \subseteq V \cap W$, and $Q_1 \cap Q_2$ is also compact; indeed, $Q_1 \cap Q_2 = Q_1 \vee Q_2$ as Q_1 and Q_2 are upper sets, and $Q_1 \vee Q_2$ is the image of $Q_1 \times Q_2$ under the map $(x, y) \mapsto x \vee y \colon (L, \sigma(L)) \times (L, \sigma(L)) \to (L, \sigma(L))$, which is continuous by II-4.15. Thus, $U \ll V \cap W$, that is, the relation $\ll$ on $\sigma(L)$ is multiplicative. By V-3.7, $\operatorname{PRIME} \sigma(L)$ is then closed with respect to the Lawson topology, and as L (as an element of $\sigma(L)$) is isolated, we have the desired result that $\operatorname{Spec}\sigma(L) = \operatorname{PRIME}\sigma(L)\backslash\{L\}$ is also closed. $\square$

We can now characterize those lattices L for which $\sigma(L)$ is continuous.

Theorem VII-4.2. *For a semilattice L with 1, the following properties are equivalent.*

(1) *L is a complete lattice for which the lattice $\sigma(L)$ of Scott open subsets is continuous.*
(2) *L admits a topology τ finer than the Scott topology such that (L, τ) is a compact pospace.*
(3) *L admits a topology τ such that (L, τ) is a compact pospace and such that a lower subset of L is τ closed iff it is Scott closed.*

Proof: (1) implies (2): By VII-4.1(iv) $\operatorname{Spec}\sigma(L)$ is Lawson closed in the continuous lattice $\sigma(L)$. Thus, the restriction of the Lawson topology to $\operatorname{Spec}\sigma(L)$ yields a compact pospace topology τ' on $\operatorname{Spec}\sigma(L)$. This topology is finer than the hull–kernel topology on $\operatorname{Spec}\sigma(L)$ (cf. remarks following Definition V-5.11). Let τ be the inverse image of the topology τ' under the map $\xi_L\colon L \to \operatorname{Spec}\sigma(L)$. As ξ_L is an order anti-isomorphism by VII-4.1(ii), τ is a compact pospace topology on L. From VII-4.1(i) we conclude that τ is finer than the Scott topology.

(2) implies (3): Indeed a closed lower set in a compact pospace is closed for directed sups, that is, Scott closed (see VI-1.3).

(3) implies (1): Under the assumption (3), the lattice $\sigma(L)$ of Scott open sets in L is the opposite of the lattice of all τ closed lower sets of the compact pospace (L, τ) and, hence, continuous by VI-3.10. Moreover, a compact pospace has filtered infs; if in addition it has finite infs and a greatest element, then it is a complete lattice. $\square$

We insert at this point an interesting addendum to VII-4.2 concerning the existence of the compact topology.

Proposition VII-4.3. *If a complete lattice L has a topology satisfying* (2) *or* (3) *in* VII-4.2*, then this topology is unique.*

Proof: This follows readily from Theorem VI-6.18. $\square$

The following theorem characterizes those lattices that admit a compact semilattice topology.

Theorem VII-4.4. *For a semilattice L with 1, the following properties are equivalent.*

(1) *L admits a compact $\wedge$-semilattice topology; that is, a compact Hausdorff topology such that the operation $(x, y) \mapsto x \wedge y \colon L \times L \to L$ is continuous.*
(2) *L is a meet continuous lattice which admits a compact pospace topology finer than the Scott topology.*
(3) *L is a meet continuous lattice such that $\sigma(L)$ is continuous.*
(4) *L is a complete lattice such that $\sigma(L)$ is continuous and join continuous.*

Proof: Let L be a compact semilattice with 1. Then L is meet continuous by VI-1.13(vii); the topology of L is a pospace topology by VI-1.14 and finer than the Scott topology by VI-2.10. Thus, (1) implies (2). The implication (2) implies (3) follows from VII-4.2, and (3) implies (4) from II-4.17. Let us show finally that (4) implies (1). By VII-2.4 every continuous and join continuous lattice endowed with the Lawson topology is a compact lattice; in particular, the join operation is continuous on the lattice $\sigma(L)$ with respect to the Lawson topology on $\sigma(L)$ if (4) is fulfilled. By VII-4.1 L is order anti-isomorphic to a subset of $\sigma(L)$ which is closed with respect to joins and closed with respect to the Lawson topology. Thus, L can be endowed with a compact $\wedge$-semilattice topology by VI-2.9. □

In the following table we collect the information contained in this volume about the transfer of properties from a complete lattice L to its lattice $\sigma(L)$ of Scott open sets and vice versa:

L	$\sigma(L)$	Reference
Meet continuous	Join continuous	II-4.15
Admits a (unique) compact pospace topology τ finer than the Scott topology	Continuous	VII-4.2
Admits a (unique) compact $\wedge$-semilattice topology	Continuous and join continuous	VII-4.4
Continuous	Completely distributive	II-1.14
Algebraic	Algebraic and completely distributive	II-1.15

Our next goal is to show that the category of compact semilattices is (dually) equivalent to a certain subcategory of distributive continuous lattices (which itself is a subcategory of the category of compact semilattices). More precisely, we shall consider the following categories.

CS is the category of compact semilattices with 1 and all continuous semilattice homomorphisms.
H is the category whose objects are the lattices L with the following properties:

(1) L is distributive, continuous and join continuous;
(2) Spec L is closed in L with respect to arbitrary sups.

The morphisms are the maps $\varphi: L \to M$ which

(1) preserve arbitrary sups, finite infs and the relation $\ll$,
(2) have an adjoint $\tau: M \to L$ preserving finite sups.

Clearly, *CS* is a subcategory of the category *CPOSP* of compact pospaces considered in Section VI-7. The following lemma shows that *H* is a subcategory of the category $SCFRM_1$ also introduced in Section VI-7.

Lemma VII-4.5. *If L is in H, then* Spec L *is closed in L with respect to the Lawson topology on L and also is a compact $\vee$-semilattice with respect to this topology.*

Proof: By definition, Spec L is closed in L with respect to arbitrary sups; as the inf of a filtered set of prime elements is also prime, Spec L is also closed with respect to filtered infs. As L is distributive continuous and join continuous, L is a compact lattice with respect to the Lawson topology by VII-2.4 and in particular, a compact $\vee$-semilattice. From VI-2.9 we conclude that Spec L is closed with respect to the Lawson topology on L, and hence a compact $\vee$-subsemilattice of L. □

In VI-7.5 we have seen that the category of compact pospaces is dually equivalent to the category of continuous distributive lattices L for which Spec L is Lawson closed with appropriate morphisms; the duality is given by the functors Spec_λ and $\mathcal{O}^\uparrow$. We now show that the restrictions of these functors establish a dual equivalence between the categories H and *CS*.

Proposition VII-4.6. *The categories CS and H are dually equivalent.*

Proof: With every object $L \in H$ we associate $\mathrm{Spec}_\lambda\, L$, that is, the set Spec L with the topology induced from the Lawson topology on L and the order *opposite* to the order induced from the order on L. By VII-4.5 $\mathrm{Spec}_\lambda L$ is a compact $\wedge$-semilattice. For every H-morphism $\varphi: L \to M$, $\mathrm{Spec}_\lambda \varphi$ is the upper adjoint of φ restricted to $\operatorname{Spec}_\lambda M$; as in Section VII-3, $\operatorname{Spec}_\lambda \varphi$ is continuous and order preserving. As, by hypothesis, τ also preserves finite sups, $\operatorname{Spec}_\lambda \varphi$ preserves finite infs and is a *CS*-morphism. Thus $\operatorname{Spec}_\lambda$ is indeed a functor from H to *CS*.

Conversely, we associate with every compact semilattice S its Scott topology $\sigma(S)$ which is an H-object by VII-4.1 and VII-4.4. For every continuous semilattice homomorphism $f\colon S \to T$ we define $\sigma(f)\colon \sigma(T) \to \sigma(S)$ as usual by $U \mapsto f^{-1}(U)$. As in Section VII-3, $\sigma(f)$ preserves arbitrary sups, finite infs and the relation $\ll$. Let us show that the upper adjoint τ of $\sigma(f)$ preserves finite sups: a straightforward calculation shows that τ is given by $\tau(V) = T\backslash{\downarrow} f(S\backslash V)$ for all $V \in \sigma(S)$. We want to show that $\tau(V \cup W) = \tau(V) \cup \tau(W)$ for arbitrary $V, W \in \sigma(S)$. This is equivalent to saying that ${\downarrow} f(A \cap B) = {\downarrow} f(A) \cap {\downarrow} f(B)$ for arbitrary Scott closed sets A and B. Thus equality holds, as $A \cap B = A \wedge B$ for all lower sets and as $f(A \wedge B) = f(A) \wedge f(B)$ by the hypotheses on f. Thus, σ is indeed a functor from *CS* to H.

Now, $\operatorname{Spec}_\lambda$ is the functor of VI-7.5 and σ is nothing but the functor $\mathcal{O}^\uparrow$, as the Scott open upper sets are exactly the open upper sets for the original topology on a compact semilattice (VI-2.10). Thus VI-7.5 yields the desired duality result. $\square$

We note that the preceding duality result means in particular that the objects of H, that is, the distributive, continuous, and join continuous lattices L in which $\operatorname{Spec} L$ is closed with respect to arbitrary sups, are exactly the lattices which arise as Scott topologies of compact semilattices.

We close our work with an interestang application of the theory we have built up to compact semilattices. We first recall that a compact semitopological semilattice with identity is a meet continuous lattice (VI-1.13(vii)). We next identify the open upper sets in S.

Proposition VII-4.7. *The family of open upper sets in a compact semitopological semilattice is the family of Scott open subsets of S. Hence, the graph of $\leq$ is closed in $S \times S$.*

Proof: We want to apply Theorem VII-1.9, and so we show that the family $\mathcal{U}$ of open upper sets in S is an order consistent, o-regular topology and $(S, \mathcal{U})$ is a semitopological semilattice.

If $x \in S$, then $xS = {\downarrow} x$ is closed in S by VI-1.13(i). Moreover, directed subsets of S converge to their sups by VI-1.13(iv), and so they do also in the topology $\mathcal{U}$. Thus, $\mathcal{U}$ is order consistent.

To show $\mathcal{U}$ is o-regular, assume that $x \in U \in \mathcal{U}$, and then choose an open subset V of S with $x \in V \subset V^- \subset U$. Then ${\uparrow} V \subseteq {\uparrow} V^- \subseteq U$ and ${\uparrow} V$ is open.

By VI-1.13(ii) and the dual of VI-1.6(i) $\uparrow V^-$ is closed with respect to taking filtered infs. This shows that $\mathcal{U}$ is o-regular. Now $(S, \mathcal{U})$ is semitopological, since S is; therefore, we have satisfied the hypotheses of Theorem VII-1.9(iii). We conclude that $\mathcal{U} = \sigma(S)$, the Scott topology on S.

Corollary VII-1.12 now shows that the graph of $\leq$ is closed in $S \times S$. □

Theorem VII-4.8. *A compact semitopological semilattice is in fact topological.*

Proof: By Proposition VII-4.7, $\sigma(S)$ is the family of open upper sets in S and S is a pospace. Hence by VII-4.2, $\sigma(S)$ is a continuous lattice.

Now, each compact semitopological semilattice is a meet-continuous lattice, and so Theorem VII-4.4 implies that S is a compact semilattice in the patch topology S inherits as the spectrum of $\sigma(S)$. By VII-4.3 the latter topology coincides with the original one. □

Exercises

Exercise VII-4.9. Show that the morphisms $\varphi\colon L \to M$ in the category H are characterized by the property that they are lower adjoints of maps $\tau\colon M \to L$ preserving arbitrary sups and infs, as well as primes. □

Exercise VII-4.10. Show that the category of continuous (algebraic) lattices is dually equivalent to a subcategory of the category of completely distributive lattices; more exactly, to the full subcategory of H whose objects are completely distributive (algebraic) lattices L for which the set $\operatorname{Spec} L$ is closed under arbitrary sups.

Hint. Use VII-4.6, II-1.13 and II-1.14. □

The next exercise is an alternative proof to Theorem VII-4.8.

Exercise VII-4.11. Show that each compact semitopological semilattice is topological.

Hint. Note that $S \times S$ in the product topology is also a compact semitopological semilattice, and so Proposition VII-4.7 applies to both S and $S \times S$ to show that the Scott topology on each is the family of open upper sets on each and that each is a pospace. Since the products of open upper sets in S form a base for the open upper sets in $S \times S$, it follows that the Scott topology on $S \times S$ is the product of the Scott topology on S with itself. Since S is a meet continuous lattice, we easily conclude that the semilattice map $(x, y) \mapsto xy : S \times S \to S$ is continuous with respect to the Scott topologies.

Thus, the result is proved if we show that this map is also continuous when S and $S \times S$ are equipped with the topologies consisting of open lower sets, since both S and $S \times S$ are monotone normal by Nachbin's Lemma VI-1.8. Thus, we need to show that $xy \in U = \downarrow U$ and U open in S imply that there are open lower sets V and W containing x and y, respectively, with $VW \subseteq U$. But, $VW = V \cap W$ since V and W are lower sets. Finally, $\downarrow x$ and $\downarrow y$ are the intersection of those compact neighborhoods which are lower sets, and $\downarrow xy$ is then the intersection of these compact sets. Since $\downarrow xy \subseteq U$ and U is open, it follows that there are indeed some compact neighborhood V of $\downarrow x$ and some compact neighborhood of W of $\downarrow y$ with $V \cap W \subseteq U$. This proves the results. $\square$

Old notes

The material of this section through VII-4.6 is due to Gierz and Hofmann [SCS 34]. Let us indicate the following consequence. There are compact distributive lattices that are not completely distributive or, equivalently, there are compact distributive lattices which are continuous, but the opposites of which are not continuous. Take any compact semilattice S with 1 which is not a continuous lattice (see Section VI-4 for examples). Then the lattice $\sigma(L)$ of all Scott open subsets of S is distributive, continuous and join continuous, hence a compact lattice by VII-2.4; but S is not completely distributive by II-1.13.

Theorem VII-4.8 was first shown by [Lawson, 1976a], but the proof contained therein utilizes some rather technical results from topology. An alternative proof was announced by Mislove [SCS 40], but a gap in the proof of what appears here as Corollary VII-2.4 was found by Harvey Carruth. The results of Section VII-1 serve to patch that gap, and also provide a much simpler proof of the theorem. The alternative proof presented as Exercise VII-4.11 was first noticed by Gerhard Gierz.

Bibliography

The general form of citations is [*author, year*]. A 'B' before the year indicates that the citation refers to a book, a monograph, or a collection, a 'D' indicates a dissertation or Master's thesis. If the letter is missing, it is an article published in a journal.

Books, Monographs, and Collections

[Abbott, B1969] J.C. Abbott. *Sets, Lattices and Boolean Algebras*. Allyn and Bacon, 1969. xiii+282 pp. [MR 39:4052].

[Abramsky *et al.*, B1992 ff.] S. Abramsky, D.M. Gabbay, and T.S.E. Maibaum, editors. *Handbook of Logic in Computer Science*. Oxford University Press, 1992 ff. Vol. 3, *Semantic Structures*, 1994. Vol. 4, *Semantic Modelling*, 1995.

[Amadio and Curien, B1998] R.M. Amadio and P.-L. Curien. *Domains and Lambda-Calculi*, volume 46 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1998.

[Asperti and Longo, B1991] A. Asperti and G. Longo. *Categories, Types, and Structures: an Introduction to Category Theory for the Working Computer Scientist*. MIT Press, 1991.

[Balbes and Dwinger, B1974] R. Balbes and Ph. Dwinger. *Distributive Lattices*. University of Missouri Press, 1974. xiii+294 pp. [MR 51:10185].

[Barendregt, B1981] H. Barendregt. *The Lambda-Calculus: Its Syntax and Semantics*. North-Holland, 1981. xiv+615 pp. [MR 83b:03016]. Revised edition 1984.

[Barwise, B1977] J. Barwise, editor. *Handbook of Mathematical Logic*. North-Holland, 1977.

[Birkhoff, B1967] G. Birkhoff. *Lattice Theory*, volume 25 of *AMS Colloquium Publications*. American Mathematical Society, revised edition, 1967. vi+418 pp. [MR 37:2638].

[Blyth and Janowitz, B1972] T.S. Blyth and M.F. Janowitz. *Residuation Theory*. Pergamon Press, 1972. ix+379 pp. [MR 53:226].

[Bourbaki, B1966] N. Bourbaki. *General Topology, Parts 1 and 2*. Hermann and Addison-Wesley, 1966.

[Crawley and Dilworth, B1973] P. Crawley and R.P. Dilworth. *Algebraic Theory of Lattices*. Prentice-Hall, 1973. vi+201 pp.

[Curien, B1993] P.-L. Curien. *Categorical Combinators, Sequential Algorithms and Functional Programming*. Progress in Theoretical Computer Science. Birkhäuser, second edition, 1993.

[Davey and Priestley, B1990] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 1990. (Second edition 2002.)

[Dubreil-Jacotin *et al.*, B1953] M.-L. Dubreil-Jacotin, L. Lesieur, and R. Croisot. *Leçons sur la théorie des treillis, des structures algébriques ordonnées et des treillis géométriques*. Gauthier-Villars, 1953.

[Engelking, B1989] R. Engelking. *General Topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, revised and completed edition, 1989.

[Erné, B1982] M. Erné. *Einführung in die Ordnungstheorie*. Bibliographisches Institut, Mannheim, 1982. v+296 pp. [MR 84b:06001].

[Fiore, B1996] M.P. Fiore. *Axiomatic Domain Theory in Categories of Partial Maps*. Cambridge University Press Distinguished Dissertations in Computer Science, 1996.

[Fletcher and Lindgren, B1982] P. Fletcher and W.F. Lindgren. *Quasi-uniform Spaces*, volume 77 of *Lecture Notes in Pure and Applied Mathematics*. Marcel Dekker, 1982. viii+216 pp. [MR 84h:54026].

[Freyd and Scedrov, B1990] P.J. Freyd and A. Scedrov. *Categories, Allegories*. North-Holland, 1990.

[Gericke, B1963] H. Gericke. *Theorie der Verbände*, volume 38 of *BI-Hochschultaschenbücher*. Bibliographisches Institut, Mannheim, 1963. (Second edition 1967.)

[Gierz *et al.*, B1980] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, 1980. xx+371 pp. [MR 82h:06005].

[Girard, B1989] J.-Y. Girard. *Proofs and Types*, volume 7 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1989. Translated and with appendices by Y. Lafont and P. Taylor.

[Gordon, B1979] M.J.C. Gordon. *The Denotational Description of Programming Languages*. Springer-Verlag, 1979.

[Grätzer, B1978] G. Grätzer. *General Lattice Theory*. Birkhäuser, 1978. xiii+381 pp. [MR 80c:06001], 2nd ed. 1998, xiii+63 pp.

[Grothendieck and Dieudonné, B1971] A. Grothendieck and J.A. Dieudonné. *Eléments de géométrie algébrique I*, volume 166 of *Grundlehren*. Springer-Verlag, 1971.

[Gunter, B1992] C. Gunter. *Semantics of Programming Languages: Structures and Techniques*. Foundations of Computing. MIT Press, 1992.

[Halmos, B1963] P.R. Halmos. *Lectures on Boolean Algebras*. D. Van Nostrand, 1963. v+147 pp. [MR 29:4713]. Reprinted: Springer-Verlag, 1974.

[Hermes, B1967] H. Hermes. *Einführung in die Verbandstheorie*. Springer-Verlag, 1967. 2. erw. Auflage.

[Herrlich and Strecker, B1973] H. Herrlich and G.E. Strecker. *Category Theory*. Allyn and Bacon, 1973.

[Hindley and Seldin, B1980] J.R. Hindley and J.P. Seldin, editors. *To H.B. Curry: Essays in Combinatory Logic, Lambda Calculus and Formalism*. Academic Press, 1980.

[Hindley and Seldin, B1986] J.R. Hindley and J.P. Seldin. *Introduction to Combinators and λ-Calculus*, volume 1 of *London Mathematical Society Student Texts*. Cambridge University Press, 1986.

[Hofmann and Keimel, B1972] K.H. Hofmann and K. Keimel. *A General Character Theory for Partially Ordered Sets and Lattices*, volume 122 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 1972. iv+121 pp. [MR 49:4885].

[Hofmann and Mostert, B1966] K.H. Hofmann and P.S. Mostert. *Elements of Compact Semigroups*, volume 5. Merrill, 1966. xiii+384 pp. [MR 35:285]. (See also J.H. Carruth, K.H. Hofmann, M.W. Mislove, Errors in "Elements of Compact Semigroups", *Semigroup Forum*, 1972, pp. 285–322 [MR 47:8757].)

[Hofmann *et al.*, B1974] K.H. Hofmann, M.W. Mislove, and A.R. Stralka. *The Pontryagin Duality of Compact* 0*-Dimensional Semilattices and Its Applications*, volume 396 of *Lecture Notes in Mathematics*. Springer-Verlag, 1974. xvi+122 pp. [MR 50:7398].

[Johnstone, B1982] P.T. Johnstone. *Stone Spaces*. Cambridge University Press, 1982. xxi+370 pp.

[Jung, B1989] A. Jung. *Cartesian Closed Categories of Domains*, volume 66 of *CWI Tracts*. Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

[Kelley, B1955] J.L. Kelley. *General Topology*. D. Van Nostrand, 1955. xiv+298 pp. [MR 16-1136]. Reprinted: volume 27 of *Graduate Texts in Mathematics*, Springer-Verlag, 1975.

[Krivine, B1990] J.L. Krivine. *Lambda-calcul. Types et modèles*. Masson, 1990.

[Lambek and Scott, B1986] J. Lambek and P.J. Scott. *Introduction to Higher-Order Categorical Logic*. Cambridge University Press, 1986.

[Mac Lane, B1971] S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, 1971. ix+262 pp. [MR 50:7275].

[Manes and Arbib, B1986] E.G. Manes and M.A. Arbib. *Algebraic Approaches to Program Semantics*. Springer-Verlag, 1986.

[Matheron, B1975] G. Matheron. *Random Sets and Integral Geometry*. Wiley, 1975.

[Milne and Strachey, B1976] R.E. Milne and C. Strachey. *A Theory of Programming Language Semantics*. Chapman and Hall, 1976.

[Mulmuley, B1987] K. Mulmuley. *Full Abstraction and Semantic Equivalence*. MIT Press, 1987.

[Murdeshwar and Naimpally, B1966] M.C. Murdeshwar and S.A. Naimpally. *Quasiuniform Topological Spaces*. Noordhoff, 1966. vi+73 pp. [MR 35:2267].

[Nachbin, B1965] L. Nachbin. *Topology and Order*. D. Van Nostrand, 1965. vi+122 pp. [MR 36:2125].

[Normann, B1980] D. Normann. *Recursion on the Countable Functionals*, volume 811 of *Lecture Notes in Mathematics*. Springer-Verlag, 1980.

[Plotkin, B1985] G. Plotkin. *Denotational Semantics with Partial Functions*. CSLI Lecture Notes. Stanford University, 1985.

[Plotkin, B1981] G.D. Plotkin. Post-graduate lecture notes in advanced domain theory (incorporating the "Pisa Notes"). Department of Computer Science, University of Edinburgh, 1981.

[Reynolds, B1998] J.C. Reynolds. *Theories of Programming Languages*. Cambridge University Press, 1998.

[Rosenthal, B1990] K.I. Rosenthal. *Quantales and Their Applications*. Research Notes in Mathematics. Pitman, 1990.

[Schmidt, B1986] D.A. Schmidt. *Denotational Semantics: a Methodology for Language Development*. Allyn and Bacon, 1986.

[Scott, B1981] D.S. Scott. *Lectures on a Mathematical Theory of Computation*. Oxford University Computing Laboratory, 1981. Technical Report, no. PRG-19, iv+148 pp.

[Sikorski, B1964] R. Sikorski. *Boolean Algebras*. Springer-Verlag, 1964. ix+237 pp. [MR 31:2178].

[Stoltenberg-Hansen *et al.*, B1994] V. Stoltenberg-Hansen, I. Lindström, and E.R. Griffor. *Mathematical Theory of Domains*. Cambridge University Press, 1994.

[Stoughton, B1988] A. Stoughton. *Fully Abstract Models of Programming Languages*. Research Notes in Theoretical Computer Science. Pitman/Wiley, 1988.

[Stoy, B1977] J.E. Stoy. *Denotational Semantics – the Scott–Strachey Approach to Programming Language Theory*. MIT Press, 1977. xxxi+414 pp. [MR 58:8460].

[Taylor, B1998] P. Taylor. *Practical Foundations of Mathematics* volume 59 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1998. xi+572 pp.

[Tennent, B1991] R.D. Tennent. *Semantics of Programming Languages*. Prentice-Hall, 1991.

[van Breugel, B1997] F. van Breugel. *Comparative Metric Semantics of Programming Languages: Nondeterminism and Recursion*. Progress in Theoretical Computer Science. Birkhäuser, 1997.

[van de Vel, B1993] M. van de Vel. *Theory of Convex Structures*, volume 50 of *North-Holland Mathematical Libray*. North-Holland, Amsterdam, 1993. xvi+540pp.

[van Leeuwen, B1990] J. van Leeuwen, editor. *Handbook of Theoretical Computer Science, Vol. B: Formal Models and Semantics*. Elsevier/MIT Press, 1990.

[Vickers, B1989] S.J. Vickers. *Topology via Logic,* volume 5 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1989. xii+200pp.

[Weihrauch, B1980] K. Weihrauch. Berechenbarkeit auf cpo's. Eine Vorlesung ausgearbeitet von Th. Deil. Schriften Inf. Angew. Math., Bericht 63, RWTH Aachen, 1980. (Lecture notes) 101 pp.

[Weihrauch, B1987] K. Weihrauch. *Computability*. Springer-Verlag, 1987.

[Winskel, B1993] G. Winskel. *The Formal Semantics of Programming Languages. An Introduction*. MIT Press, 1993.

[Zhang, B1991] G.-Q. Zhang. *Logic of Domains*. Progress in Theoretical Computer Science. Birkhäuser, 1991.

Conference Proceedings

[1973, Houston] S. Fajtlowicz and K. Kaiser, editors. *Proceedings of the University of Houston Lattice Theory Conference, Houston, Texas, March 22–24, 1973*. University of Houston, 1973.

[1975, Rome] C. Böhm, editor. *λ-Calculus and Computer Science Theory, Proceedings of the Symposium Held in Rome, March 25–27, 1975*, volume 37 of *Lecture Notes in Computer Science*. Springer-Verlag, 1975.

[1978, Durham] M.P. Fourman, C.J. Mulvey, and D.S. Scott, editors. *Applications of Sheaves, Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis, Durham, U.K., July 9–21, 1978*, volume 753 of *Lecture Notes in Mathematics*. Springer-Verlag, 1979. [MR 82d:03061].

[1979, Bremen] B. Banaschewski and R.-E. Hoffmann, editors. *Continuous Lattices, Proceedings of the Conference on Topological and Categorical Aspects of Continuous Lattices (Workshop IV), University of Bremen, Germany, November 9–11, 1979*, volume 871 of *Lecture Notes in Mathematics*. Springer-Verlag, 1981. x+413 pp. [MR 83d:06001].

[1981, Banff] I. Rival, editor. *Ordered Sets, Proceedings of the NATO Advanced Study Institutute, Banff, Canada, August 28 to September 12, 1981*, volume 83 of *Nato Advanced Study Institute, Series C*. D. Reidel, 1982.

[1981, Bremen] R.-E. Hoffmann, editor. *Continuous Lattices and Related Topics, Proceedings of the Conference on Topological and Categorical Aspects of Continuous Lattices (Workshop V), Universität Bremen, Germany, May 8–10, 1981*, volume 27 of *Mathematik-Arbeitspapiere*. Universität Bremen, 1982. vii+314 pp.

[1982, Bremen] R.-E. Hoffmann and K.H. Hofmann, editors. *Continuous Lattices and Their Applications. Proceedings of the Conference on Topological and Categorical Aspects of Continuous Lattices (Workshop VI), University of Bremen, Germany, July 2–3, 1982*, Lecture Notes in Pure and Applied Mathematics. Marcel Dekker, 1985.

[1984, Sophia Antipolis] G. Kahn, D.B. MacQueen, and G.D. Plotkin. *Semantics of Data Types*, volume 173 of *Lecture Notes in Computer Science*. Springer-Verlag, 1984. 391 pp.

[1985, Manhattan] A. Melton, editor. *Mathematical Foundation of Programming Semantics. International Conference, Manhattan, Kansas, April 11–12, 1985*, volume 239 of *Lecture Notes in Computer Science*. Springer-Verlag, 1986.

[1987, New Orleans] M. Main, A. Melton, M. Mislove, and D. Schmidt, editors. *Mathematical Foundations of Programming Language Semantics. Third Workshop Tulane University, New Orleans, Louisiana, USA, April 8–10, 1987*, volume 298 of *Lecture Notes in Computer Science*. Springer-Verlag, 1988.

[1989, New Orleans] M. Main, A. Melton, M. Mislove, and D. Schmidt, editors. *Mathematical Foundations of Programming Semantics. Fifth International Conference, Tulane University, New Orleans, Louisiana, USA, March 29–April 1, 1989*, volume 442 of *Lecture Notes in Computer Science*. Springer-Verlag, 1990.

[1991, Dagstuhl] M. Droste and Y. Gurevich, editors. *Semantics of Programming Languages and Model Theory,* volume 5 of *Algebra, Logic, and Applications*. Gordon and Breach, 1993.

[1991, Pittsburgh] S. Brookes, M. Main, A. Melton, M. Mislove, and D. Schmidt, editors. *Mathematical Foundations of Programming Semantics. Seventh International Conference, Pittsburgh, PA, USA, March 25–28, 1991*, volume 598 of *Lecture Notes in Computer Science*. Springer-Verlag, 1992.

[1993, New Orleans] S. Brookes, M. Main, A. Melton, M. Mislove, and D. Schmidt, editors. *Mathematical Foundations of Programming Semantics. Ninth International Conference, New Orleans, LA, USA, April 7–10, 1993*, volume 802 of *Lecture Notes in Computer Science*. Springer-Verlag, 1994.

[1993, Noordwijkerhout] J.W. de Bakker, W.-P. de Roever, and G. Rozenberg, editors. *A Decade of Concurrency. Reflections and Perspectives. REX School/Symposium Noordwijkerhout, The Netherlands, June 1–4, 1993*, volume 803 of *Lecture Notes in Computer Science*. Springer-Verlag, 1994.

[1994, Darmstadt] M. Huth, A. Jung, and K. Keimel, editors. *Workshop Domains*. Technische Hochschule Darmstadt, June 10–12 1994. 133 pp.

[1995, Darmstadt] M. Huth, A. Jung, and K. Keimel, editors. *Logic, Domains, and Programming Languages, Technische Hochschule Darmstadt, May 24–27, 1995*. Special Issue of *Mathematical Structures in Computer Science*, 7(4):399–618, 1997.

[1995, New Orleans] S. Brookes, M. Main, A. Melton, and M. Mislove, editors. *Mathematical Foundations of Programming Semantics, Eleventh Annual Conference. Tulane University, New Orleans, LA, March 29–April 1, 1995*, volume 1 of *Electronic Notes in Theoretical Computer Science*. Elsevier, 1995. ⟨http://www.elsevier.nl/locate/entcs/⟩.

[1996, Braunschweig] J. Adámek, J. Koslowski, V. Pollara, and W. Struckmann, editors. *Workshop Domains II*. Technische Universität Braunschweig, May 3–5, 1996. 238 pp.

[1997, Birmingham] A. Edalat, A. Jung, K. Keimel, and M. Kwiatkowska, editors. *Comprox III. Third Workshop on Computation and Approximation. Birmingham, England, 11–13 September 1997*, volume 13 of *Electronic Notes in Theoretical Computer Science*, 1998. ⟨http://www.elsevier.nl/locate/entcs/⟩.

[1997, München] U. Berger, K.-H. Niggl, and B. Reus, editors. *Workshop Domains III*, Ludwig Maximilians Universität München, May 29–31 1997. Institut für Informatik. 152 pp.

[1997, Pittsburgh] S. Brookes and M. Mislove, editors. *Mathematical Foundations of Programming Semantics Thirteenth Annual Conference. Carnegie Mellon University, Pittsburgh, PA, USA, March 23–26, 1997*, volume 6 of *Electronic Notes in Theoretical Computer Science*. Elsevier, 1997. ⟨http://www.elsevier.nl/locate/entcs/⟩.

[1999, Shanghai] K. Keimel, G.-Q. Zhang, Y.-M. Liu, and Y.-X. Chen, editors. *Domains and Processes, Proceedings of the First International Conference on Domain Theory, Shanghai, China, October 1999*, volume 1 of *Semantic Structures in Computation*. Kluwer, 2001. xiv+273 pp.

Articles

[Abadi *et al.*, 1989] M. Abadi, B. Pierce, and G.D. Plotkin. Faithful ideal models for recursive polymorphic types. In *Proceedings of the Fourth Annual IEEE Symposium on Logic in Computer Science*, Washington, 1989, pages 216–225. IEEE Computer Society Press, 1989.

[Abramsky, 1990] S. Abramsky. A generalized Kahn principle for abstract asynchronous networks. In [1989, New Orleans], pages 1–21.

[Abramsky, 1991a] S. Abramsky. A domain equation for bisimulation. *Information and Computation*, 92:161–218, 1991.

[Abramsky, 1991b] S. Abramsky. Domain theory in logical form. *Annals of Pure and Applied Logic*, 51:1–77, 1991.

[Abramsky and Jung, 1994] S. Abramsky and A. Jung. Domain theory. In [Abramsky *et al.*, B1992 ff.], volume 3, pages 1–168.

[Abramsky and Ong, 1993] S. Abramsky and C.-H.L. Ong. Full abstraction in the lazy lambda calculus. *Information and Computation*, 105:159–267, 1993.

[Adámek, 1982] J. Adámek. Construction of free continous algebras. *Algebra Universalis*, 14:140–166, 1982. [MR 83a:08016].

[Adámek, 1997] J. Adámek. A categorical generalization of Scott domains. *Mathematical Structures in Computer Science*, 7:419–444, 1997.

[Adámek and Koubek, 1979] J. Adámek and V. Koubek. Least fixed point of a functor. *Journal of Computer and System Sciences*, 19:163–178, 1979. [MR 82d:18004].

[Adámek *et al.*, 1981] J. Adámek, E. Nelson, and J. Reiterman. Tree constructions of free continuous algebras. *Journal of Computer and System Sciences*, 24:114–146, 1981. [MR 84e:06008].

[Adams, 1975] M.E. Adams. The poset of prime ideals of a distributive lattice. *Algebra Universalis*, 5:141–142, 1975. [MR 52:2986].

[Adaricheva *et al.*, 2001] K.V. Adaricheva, V.A. Gorbunov, and M.V. Semonova. On continuous noncomplete lattices. *Algebra Universalis*, 46:215–230, 2001.

[Albert, 1984] M.H. Albert. Iteratively algebraic posets have the a.c.c. *Semigroup Forum*, 30:371–373, 1984.

[Alexandrov, 1937] P.S. Alexandrov. Diskrete Räume. *Matematicheskij Sbornik*, 2:501–519, 1937.

[Alvarez-Manilla *et al.*, 1998] M. Alvarez-Manilla, A. Edalat, and N. Saheb-Djahromi. An extension result for continuous valuations. In [1994, Birmingham].

[Alvarez-Manilla *et al.*, 2000] M. Alvarez-Manilla, A. Edalat, and N. Saheb-Djahromi. An extension result for continuous valuations. *Journal of the London Mathematical Society*, 61:629–640, 2000.

[Amadio, 1991a] R. Amadio. Bifinite domains: stable case. In D.H. Pitt, P.-L. Curien, S. Abramsky, A.M. Pitts, A. Poigne, and D.E. Rydeheard, editors, *Category Theory and Computer Science. Paris, France, September 3–6, 1991. Proceedings*, volume 530 of *Lecture Notes in Computer Science*, pages 16–33. Springer-Verlag, 1991.

[Amadio, 1991b] R. Amadio. Domains in a realizability framework. In S. Abramsky and T.S.E. Maibaum, editors, *TAPSOFT '91. Proceedings of the International Joint Conference on Theory and Practice of Software Development, Brighton, UK, April 8–12, 1991. Vol. 1: Colloquium on Trees in Algebra and Programming (CAAP '91),* volume 493 of *Lecture Notes in Computer Science*, pages 241–263. Springer-Verlag, 1991.

[Amadio, 1995] R. Amadio. A quick construction of a retraction of all retractions for stable bifinites. *Information and Computation*, 116(2):272–274, 1995.

[Amadio *et al.*, 1986] R. Amadio, K. Bruce, and G. Longo. The finitary projection model and the solution of higher order domain equations. In *Proceedings of the IEEE Symposium on Logic in Computer Science*, pages 122–130, 1986. IEEE Computer Society Press, 1986.

[America and Rutten, 1989] P. America and J.J.M.M. Rutten. Solving reflexive domain equations in a category of complete metric spaces. *Journal of Computer and System Sciences*, 39(3):343–375, 1989.

[Anderson, 1959] L.W. Anderson. On the breadth and co-dimension of a topological lattice. *Pacific Journal of Mathematics*, 9:327–333, 1959. [MR 21:4206].

[Anderson and Ward, 1961] L.W. Anderson and L.E. Ward, Jr. A structure theorem for topological semilattices. *Proceedings of the Glasgow Mathematical Association*, 5:1–3, 1961. [MR 26:5546].

[Atsumi, 1966] K. Atsumi. On complete lattices having the Hausdorff interval topology. *Proceedings of the American Mathematical Society*, 17:197–199, 1966. [MR 32:4674].

[Attardi, 1974] G. Attardi. Caratterizzazione dei reticoli continui per la teoria della computazione di Dana Scott. *Calcolo*, 11:33–46, 1974. [MR 51:84].

[Aumann, 1955] G. Aumann. Bemerkung über Galois-Verbindungen. *Bayerische Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. Sitzungsberichte*, pages 281–284, 1955. [MR 17-1180].

[Austin, 1963] C.W. Austin. Duality theorems for some commutative semigroups. *Transactions of the American Mathematical Society*, 109:245–256, 1963. [MR 27:3737].

[Baartz, 1967] A.P. Baartz. The measure algebra of a locally compact semigroup. *Pacific Journal of Mathematics*, 21:199–214, 1967. [MR 35:4678].

[Baker and Stralka, 1970] K.A. Baker and A.R. Stralka. Compact distributive lattices of finite breadth. *Pacific Journal of Mathematics*, 34:311–320, 1970. [MR 44:129].

[Banaschewski, 1969] B. Banaschewski. Frames and compactifications. In H. Poppe, J. Flachsmeyer and F. Terpe, editors, *Contributions to Extension Theory of Topological Structures, Berlin 1967*, pages 29–33. Deutscher Verlag der Wissenschaften, 1969.

[Banaschewski, 1970] B. Banaschewski. Injectivity and essential extensions in equational classes of algebras. In C.H. Wenzel, editor, *Proceedings, Conference on Universal Algebra, Queen's University, Kingston, Ontario, 1969*, pages 131–147. Queen's University, 1970. [MR 41:3354].

[Banaschewski, 1977] B. Banaschewski. Essential extensions of T_0-spaces. *General Topology and Its Applications*, 7:233–246, 1977. [MR 56:16557].

[Banaschewski, 1978] B. Banaschewski. Hulls, kernels, and continuous lattices. *Houston Journal of Mathematics*, 4:517–525, 1978. [MR 80j:06009].

[Banaschewski, 1980] B. Banaschewski. The duality of distributive continuous lattices. *Canadian Journal of Mathematics*, 32:385–394, 1980. [MR 81m:54072].

[Banaschewski, 1981a] B. Banaschewski. Coherent frames. In [1979, Bremen], pages 1–11.

[Banaschewski, 1981b] B. Banaschewski. The duality of distributive σ-continuous lattices. In [1979, Bremen], pages 12–19.

[Banaschewski, 1985] B. Banaschewski. On the topologies of injective spaces. In [1982, Bremen]. pages 1–8.

[Banaschewski and Brümmer, 1988] B. Banaschewski and G.C.L. Brümmer. Stably continuous frames. *Mathematical Proceedings of the Cambridge Philosophical Society*, 104:7–19, 1988.

[Banaschewski and Nelson, 1982] B. Banaschewski and E. Nelson. Completions of partially ordered sets. *SIAM Journal on Computing*, 11:521–528, 1982. [MR 84k:06001].

[Bandelt, 1980a] H.-J. Bandelt. Regularity and complete distributivity. *Semigroup Forum*, 19:123–126, 1980. [MR 81d:20058].

[Bandelt, 1980b] H.-J. Bandelt. The tensor product of continuous lattices. *Mathematische Zeitschrift*, 172:89–96, 1980. [MR 81j:06014].

[Bandelt, 1981] H.-J. Bandelt. Complemented continuous lattices. *Archiv der Mathematik*, 36:474–475, 1981. [MR 82m:06006].

[Bandelt, 1982a] H.-J. Bandelt. M-distributive lattices. *Archiv der Mathematik*, 39:436–442, 1982.

[Bandelt, 1982b] H.-J. Bandelt. Tight residuated mappings and d-extensions. In E. Fried, B. Csákány and E.T. Schmidt, editors, *Universal Algebra*, pages 61–72. North-Holland, 1982. [MR 83f:06007].

[Bandelt, 1983] H.-J. Bandelt. Coproducts of bounded (α, β)-distributive lattices. *Algebra Universalis*, 17:92–100, 1983. [MR 84i:06011].

[Bandelt and Erné, 1983] H.-J. Bandelt and M. Erné. The category of Z-continuous posets. *Journal of Pure and Applied Algebra*, 30:219–226, 1983.

[Bandelt and Erné, 1984] H.-J. Bandelt and M. Erné. Representations and embeddings of M-distributive lattices. *Houston Journal of Mathematics*, 10:315–324, 1984.

[Baranga, 1996] A. Baranga. Z-continuous posets. *Discrete Mathematics*, 152:33–45, 1996.

[Barendregt *et al.*, 1983] H. Barendregt, M. Coppo, and M. Dezani. A filter lambda model and the completeness of type assignment. *Journal of Symbolic Logic*, 48:931–940, 1983.

[Barr, 1992] M. Barr. Algebraically compact functors. *Journal of Pure and Applied Algebra*, 82:211–231, 1992.

[Beer, 1982] G. Beer. Upper semicontinuous functions and the Stone approximation theorem. *Journal of Approximation Theory*, 34:1–11, 1982. [MR 83h:26005].

[Berardi, 1991] S. Berardi. Retractions on dI-domains as a model for type:type. *Information and Computation*, 94:204–231, 1991.

[Berger, 1993] U. Berger. Total sets and objects in domain theory. *Annals of Pure and Applied Logic*, 60:91–117, 1993.

[Berline, 1992] C. Berline. Rétractions et interprétation interne du polymorphisme: le problème de la rétraction universelle. *Informatique Théorique et Applications*, 26(1):59–91, 1992.

[Berline, 2000] C. Berline. From computation to foundations via functions and application: the λ-calculus and its webbed models. *Theoretical Computer Science*, 249:81–161, 2000.

[Berry, 1978] G. Berry. Stable models of typed λ-calculi. In *Proceedings of the Fifth International Colloquium on Automata, Languages and Programming,* volume 62 of *Lecture Notes in Computer Science*, pages 72–89. Springer-Verlag, 1978.

[Birkedal, 2000] L. Birkedal. Developing theories of types and computability via realizability. *Electronic Notes in Theoretical Computer Science*, 34:280pp., 2000.

[Birkhoff and Frink, 1948] G. Birkhoff and O. Frink. Representations of lattices by sets. *Transactions of the American Mathematical Society*, 64:299–316, 1948.

[Blanck, 1997] J. Blanck. Domain representability of metric spaces. *Annals of Pure and Applied Logic*, 83:225–247, 1997.

[Blanck, 1998] J. Blanck. Domain representations of topological spaces. In [1997, Birmingham].

[Blanck, 1999] J. Blanck. Effective domain representations of $\mathcal{H}(X)$ the space of compact subsets. *Theoretical Computer Science*, 219:19–48, 1999.

[Bonsangue *et al.*, 1995] M.M. Bonsangue, B. Jacobs, and J.N. Kok. Duality beyond sober spaces: topological spaces and observation frames. *Theoretical Computer Science*, 151(1):79–124, 1995.

[Bosbach, 1982] B. Bosbach. A representation theorem for completely join-distributive algebraic lattices. *Periodica Mathematica Hungarica*, 13:113–118, 1982.

[Brown, 1965] D.R. Brown. Topological semilattices on the 2-cell. *Pacific Journal of Mathematics*, 15:35–46, 1965. [MR 31:725].

[Brown and Stralka, 1973] D.R. Brown and A.R. Stralka. Problems on compact semilattices. *Semigroup Forum*, 6:265–270, 1973. [MR 51:8326].

[Brown and Stralka, 1977] D.R. Brown and A.R. Stralka. Compact totally instable zero-dimensional semilattices. *General Topology and Its Applications*, 7:151–159, 1977. [MR 55:3144].

[Bruns, 1961] G. Bruns. Distributivität und subdirekte Zerlegbarkeit vollständiger Verbände. *Archiv der Mathematik*, 12:61–66, 1961. [MR 23:A1561].

[Bruns, 1962a] G. Bruns. Darstellungen und Erweiterungen geordneter Mengen. I. *Journal für die Reine und Angewandte Mathematik*, 209:167–200, 1962. [MR 26:1270a].

[Bruns, 1962b] G. Bruns. Darstellungen und Erweiterungen geordneter Mengen. II. *Journal für die Reine und Angewandte Mathematik*, 210:1–23, 1962. [MR 26:1270b].

[Bruns, 1967] G. Bruns. A lemma on directed sets and chains. *Archiv der Mathematik*, 18:561–563, 1967. [MR 36:3683].

[Bucciarelli, 1997] A. Bucciarelli. Logical reconstruction of bi-domains. In P. de Groote and J.R. Hindley, editors, *Typed Lambda Calculi and Applications. Third International Conference on Typed Lambda Calculi and Applications, TLCA '97, Nancy, France, April 2–4, 1997, Proceedings*, volume 1210 of *Lecture Notes in Computer Science*, pages 99–111. Springer-Verlag, 1997.

[Bucciarelli and Ehrhard, 1991a] A. Bucciarelli and T. Ehrhard. Extensional embedding of a strongly stable model of PCF. In J. Leach Albert, B. Monien, and M. Rodriguez Artalejo, editors, *Automata, Languages and Programming*, volume 510 of *Lecture Notes in Computer Science*, pages 35–46. Springer-Verlag, 1991.

[Bucciarelli and Ehrhard, 1991b] A. Bucciarelli and T. Ehrhard. Sequentiality and strong stability. In *Sixth Annual IEEE Symposium on Logic in Computer Science*, pages 138–145. IEEE Computer Society Press, 1991.

[Bucciarelli and Ehrhard, 1993] A. Bucciarelli and T. Ehrhard. A theory of sequentiality. *Theoretical Computer Science*, 113:273–291, 1993.

[Bucciarelli and Ehrhard, 1994] A. Bucciarelli and T. Ehrhard. Sequentiality in an extensional framework. *Information and Computation*, 110:265–296, 1994.

[Büchi, 1952] J.R. Büchi. Representation of complete lattices by sets. *Portugaliae Mathmatica*, 11:151–167, 1952. [MR 14:940].

[Bulman-Fleming *et al.*, 1979] S. Bulman-Fleming, I. Fleischer, and K. Keimel. The semilattices with distinguished endomorphisms which are equationally compact. *Proceedings of the American Mathematical Society*, 73:7–10, 1979. [MR 80d:08008].

[Buneman *et al.*, 1991] P. Buneman, A. Jung, and A. Ohori. Using powerdomains to generalize relational databases. *Theoretical Computer Science*, 91:23–55, 1991.

[Carruth, 1968] J.H. Carruth. A note on partially ordered compacta. *Pacific Journal of Mathematics*, 24:229–231, 1968. [MR 36:5902].

[Chen, 1997a] Y.-X. Chen. Cartesian closedness of category stable functions and locally algebraic lattices. *Acta Mathematica Sinica*, 40:597–602, 1997.

[Chen, 1997b] Y.-X. Chen. Stone duality and representation of stable domains. *Journal of Computer Mathematics with Applications*, 34:27–41, 1997.

[Chen, 1998a] Y.-X. Chen. Characterizations of stable functions on L-domains. *Chinese Quarterly Journal of Mathematics*, 13:1–7, 1998.

[Chen, 1998b] Y.-X. Chen. Semilattice and representation of domain. *Acta Mathematica Sinica*, 41:737–742, 1998.

[Chen, 2001] Y.-X. Chen. Totality and maximality of a class of functions on domains. *Chinese Journal of Computers*, 24:680–684, 2001.

[Choe, 1969] T.H. Choe. Intrinsic topologies in a topological lattice. *Pacific Journal of Mathematics*, 28:49–52, 1969. [MR 39:1365].

[Choe, 1971] T.H. Choe. Locally compact lattices with small lattices. *Michigan Mathematical Journal*, 18:81–85, 1971.

[Choe and Park, 1979] T.H. Choe and Y.S. Park. Wallman's type order compactification. *Pacific Journal of Mathematics*, 82:339–347, 1979. [MR 80m:54034].

[Church, 1940] A. Church. A formulation of the simple theory of types. *Journal of Symbolic Logic*, 5:56–68, 1940.

[Clark and Eberhart, 1968] C.E. Clark and C. Eberhart. A characterization of compact connected planar lattices. *Pacific Journal of Mathematics*, 24:233–240, 1968. [MR 38:5178].

[Clinkenbeard, 1981] D. Clinkenbeard. The lattice of closed congruences on a topological lattice. *Transactions of the American Mathematical Society*, 263:457–467, 1981. [MR 82d:06005].

[Coppo *et al.*, 1987] M. Coppo, M. Dezani-Ciancaglini, and M. Zacchi. Type theories, normal forms and D_∞ λ-models. *Information and Computation*, 72:85–116, 1987.

[Coquand *et al.*, 1987] T. Coquand, C. Gunter, and G. Winskel. dI-domains as a model of polymorphism. In [1987, New Orleans], pages 344–363.

[Coquand *et al.*, 1988] T. Coquand, C. Gunter, and G. Winskel. Domain theoretic models of polymorphism. *Information and Computation*, 81:123–167, 1988.

[Cousot and Cousot, 1979] P. Cousot and R. Cousot. Constructive versions of Tarski's fixed point theorems. *Pacific Journal of Mathematics*, 82:43–57, 1979. [MR 82d:06004].

[Crawley, 1976] W. Crawley. A note on epimorphisms of compact Lawson semilattices. *Semigroup Forum*, 13:92–94, 1976. [MR 54:5380].

[Cunningham and Roy, 1974] F. Cunningham and N.M. Roy. Extreme functionals on an upper semicontinuous function space. *Proceedings of the American Mathematical Society*, 42:461–465, 1974. [MR 48:6921].

[Curien *et al.*, 2000] P.-L. Curien, G. Plotkin, and G. Winskel. Bistructures, bidomains and linear logic. In G.D. Plotkin, C.P. Stirling, and M. Tofte, editors, *Proof, Language, and Interaction: Essays in Honour of Robin Milner*. MIT Press, 2000.

[Curtis, 1974] D.W. Curtis. The hyperspace of subcontinua of a Peano continuum. In R. Alò, editor, *General Topology and Its Applications*, volume 378 of *Lecture Notes in Mathematics*, pages 108–118. Springer-Verlag, 1974.

[Curtis and Schori, 1974] D.W. Curtis and R.M. Schori. 2^X and $C(X)$ are homeomorphic to the Hilbert cube. *Bulletin of the American Mathematical Society*, 80:927–931, 1974.

[Davies, 1968] E.B. Davies. The existence of characters on topological lattices. *The Journal of the London Mathematical Society*, 43:217–220, 1968. [MR 37:112].

[Davis, 1955] A.C. Davis. A characterization of complete lattices. *Pacific Journal of Mathematics*, 5:311–319, 1955. [MR 17:574].

[Day, 1975] A. Day. Filter monads, continuous lattices and closure systems. *Canadian Journal of Mathematics*, 27:50–59, 1975. [MR 51:3258].

[Day and Kelly, 1970] B.J. Day and G.M. Kelly. On topological quotient maps preserved by pullbacks or products. *Proceedings of the Cambridge Philosophical Society*, 67:553–558, 1970. [MR 40:8024].

[de Bakker, 1976] J.W. de Bakker. Least fixed points revisited. *Theoretical Computer Science*, 2:155–181, 1976. [MR 53:12055].

[de Bakker and Warmerdam, 1991] J.W. de Bakker and J.H.A. Warmerdam. Four domains for concurrency. *Theoretical Computer Science*, 90(1):127–149, 1991.

[Derderian, 1967] J.C. Derderian. Residuated mappings. *Pacific Journal of Mathematics*, 20:35–43, 1967. [MR 35:1511].

[Dilworth and Crawley, 1960] R.P. Dilworth and P. Crawley. Decomposition theory for lattices without chain conditions. *Transactions of the American Mathematical Society*, 96:1–22, 1960. [MR 22:9461].

[Dixmier, 1968] J. Dixmier. Sur les espaces localement quasi-compacts. *Canadian Journal of Mathematics*, 20:1093–1100, 1968. [MR 38:5171].

[Dobbs, 1982] D.E. Dobbs. Posets admitting a unique order-compatible topology. *Discrete Mathematics*, 41:235–240, 1982. [MR 84a:06007].

[Dolecki *et al.*, 1993] S. Dolecki, G.H. Greco, and A. Lechicki. When do the upper Kuratowski topology (homeomorphically, Scott topology) and the co-compact topology coincide? *Transactions of the American Mathematical Society*, 347:2869–2884, 1993.

[Dowker and Papert, 1966] C.H. Dowker and D. Papert. Quotient frames and subspaces. *Proceedings of the London Mathematical Society*, 16:275–296, 1966. [MR 34:2510].

[Drake and Thron, 1965] D. Drake and W.J. Thron. On the representation of an abstract lattice as the family of closed subsets of a topological space. *Transactions of the American Mathematical Society*, 120:57–71, 1965. [MR 32:6390].

[Droste, 1989] M. Droste. Event structures and domains. *Theoretical Computer Science*, 68:37–48, 1989.

[Droste, 1990] M. Droste. Non-deterministic information systems and their domains. *Theoretical Computer Science*, 75:289–309, 1990.

[Droste, 1991] M. Droste. Universal homogeneous event structures and domains. *Information and Computation*, 94:48–61, 1991.

[Droste, 1992] M. Droste. Finite axiomatizations for universal domains. *Journal of Logic and Computation*, 2:119–131, 1992.

[Droste, 1993] M. Droste. On stable domains. *Theoretical Computer Science*, 111:89–101, 1993.

[Droste and Göbel, 1990] M. Droste and R. Göbel. Universal domains in the theory of denotational semantics of programming languages. In *Fifth Annual IEEE Symposium on Logic in Computer Science*, pages 19–34. IEEE Computer Society Press, 1990.

[Droste and Göbel, 1991] M. Droste and R. Göbel. Universal information systems. *International Journal of Foundations of Computer Science*, 1:413–424, 1991.

[Droste and Göbel, 1993] M. Droste and R. Göbel. Universal domains and the amalgamation property. *Mathematical Structures in Computer Science*, 3:137–159, 1993.

[Dwinger, 1979] Ph. Dwinger. Classes of completely distributive complete 1attices. *Indagationes Mathematicae*, 41:411–42, 1979. [MR 81m:06029].

[Dwinger, 1981] Ph. Dwinger. Structure of completely distributive complete lattices. *Indagationes Mathematicae*, 43:361–373, 1981.

[Dwinger, 1982] Ph. Dwinger. Characterization of the complete homomorphic images of a completely distributive complete lattice, I. *Indagationes Mathematicae*, 44:403–414, 1982.

[Dwinger, 1983] Ph. Dwinger. Characterization of the complete homomorphic images of a completely distributive complete lattice, II. *Indagationes Mathematicae*, 45:43–49, 1983.

[Dyer and Shields, 1959] E. Dyer and A. Shields. Connectivity of topological lattices. *Pacific Journal of Mathematics*, 9:443–448, 1959. [MR 21:4205].

[Edalat, 1992] A. Edalat. Continuous I-categories. In A. Nerode and M. Taitslin, editors, *Logical Foundations of Computer Science*, volume 620 of *Lecture Notes in Computer Science*, pages 127–138. Springer-Verlag, 1992.

[Edalat, 1993] A. Edalat. Dynamical systems, measures and fractals via domain theory: extended abstract. In G. Burn, S. Gay, and M. Ryan, editors, *Theory and Formal Methods 1993*, Workshops in Computing, pages 82–99. Springer-Verlag, 1993.

[Edalat, 1995a] A. Edalat. Domain of computation of a random field in statistical physics. In C. Hankin *et al.*, editors, *Theory and Formal Methods 1994: Proceedings of the Second Imperial College Workshop*. IC Press, 1995.

[Edalat, 1995b] A. Edalat. Domain theory and integration. *Theoretical Computer Science*, 151:163–193, 1995.

[Edalat, 1995c] A. Edalat. Domain theory in learning processes. In [1995, New Orleans] 18 pp.

[Edalat, 1995d] A. Edalat. Domain theory in stochastic processes. In *Proceedings of the Tenth Annual IEEE Symposium on Logic in Computer Science*, pages 244–254. IEEE Computer Society Press, 1995.

[Edalat, 1995e] A. Edalat. Dynamical systems, measures and fractals via domain theory. *Information and Computation*, 120(1):32–48, 1995.

[Edalat, 1996] A. Edalat. Power domains and iterated function systems. *Information and Computation*, 124:182–197, 1996.

[Edalat, 1997a] A. Edalat. Domains for computation in mathemetics, physics and exact real arithmetic. *Bulletin of Symbolic Logic*, 3(4):401–452, 1997.

[Edalat, 1997b] A. Edalat. When Scott is weak on the top. *Mathematical Structures in Computer Science*, 7:401–417, 1997.

[Edalat and Escardó, 1996] A. Edalat and M.H. Escardó. Integration in Real PCF (extended abstract). In *Proceedings of the Eleventh Annual IEEE Symposium on Logic in Computer Science*, pages 382–393. IEEE Computer Society Press, 1996.

[Edalat and Heckmann, 1998] A. Edalat and R. Heckmann. A computational model for metric spaces. *Theoretical Computer Science*, 193:53–73, 1998.

[Edalat and Negri, 1998] A. Edalat and S. Negri. The generalized Riemann integral on locally compact spaces. *Topology and Its Applications*, 89:121–150, 1998.

[Edalat and Potts, 1997] A. Edalat and P.J. Potts. A new representation for exact real numbers. In [1997, Pittsburgh].

[Edalat and Smyth, 1991] A. Edalat and M.B. Smyth. Categories of information systems. In D.H. Pitt, P.L. Curien, S. Abramsky, A.M. Pitts, A. Poigné, and D.E. Rydeheard, editors, *Category Theory and Computer Science*, pages 37–52. Springer-Verlag, 1991.

[Edalat and Smyth, 1993a] A. Edalat and M.B. Smyth. I-categories as a framework for solving domain equations. *Theoretical Computer Science*, 115(1):77–106, 1993.

[Edalat and Smyth, 1993b] A. Edalat and M.B. Smyth. Information categories. *Applied Categorical Structures*, 1:197–232, 1993.

[Edalat and Sünderhauf, 1999] A. Edalat and P. Sünderhauf. Computable Banach spaces via domain theory. *Theoretical Computer Science*, 210:169–184, 1999.

[Edmondson, 1956] D.E. Edmondson. A non-modular compact connected topological lattice. *Proceedings of the American Mathematical Society*, 7:1157–1158, 1956. [MR 18:461].

[Edmondson, 1969a] D.E. Edmondson. A modular topological lattice. *Pacific Journal of Mathematics*, 29:271–277, 1969. [MR 39:4062].

[Edmondson, 1969b] D.E. Edmondson. Modularity in topological lattices. *Proceedings of the American Mathematical Society*, 21:81–82, 1969. [MR 39:2132].

[Ehrhard, 1993] T. Ehrhard. Hypercoherences: a strongly stable model of linear logic. *Mathematical Structures in Computer Science*, 3:365–386, 1993.

[Eilenberg and Kelly, 1966] S. Eilenberg and G.M. Kelly. Closed categories. In S. Mac Lane, S. Eilenberg, D.K. Harrison and H. Röhrl, editors, *Proceedings of the Conference on Categorical Algebra at La Jolla, La Jolla, Calif., June 7–12, 1965*, pages 421–562. Springer-Verlag, 1966. [MR 37:1432].

[Erker, 1998] T. Erker. Right Kan spaces and essentially complete T_0-spaces. In [1997, Birmingham].

[Erker *et al.*, 1998] T. Erker, M.H. Escardó, and K. Keimel. The way-below relation of function spaces over semantic domains. *Topology and Its Applications*, 89:61–74, 1998.

[Erné, 1981a] M. Erné. A completion-invariant extension of the concept of continuous lattices. In [1979, Bremen], pages 45–60.

[Erné, 1981b] M. Erné. Scott convergence and Scott topology in partially ordered sets, II. In [1979, Bremen], pages 61–96.

[Erné, 1985] M. Erné. Posets isomorphic to their extensions. *Order*, 2:199–210, 1985.

[Erné, 1994] M. Erné. Algebraic ordered sets and their generalizations. In I. Rosenberg and G. Sabidussi, editors, *Algebras and Orders, Proceedings Montreal 1992*, pages 113–192. Kluwer, 1994.

[Erné, 1999] M. Erné. Z-continuous posets and their topological manifestation. *Applied Categorical Structures*, 7:31–70, 1999.

[Erné and Gatzke, 1985] M. Erné and H. Gatzke. Convergence and continuity in partially ordered sets and semilattices. In [1982, Bremen], pages 9–40.

[Eršov, 1972a] Yu.L. Eršov. Computable functionals of finite types. *Algebra i Logika*, 11:367–437, 492, 1972. (In Russian.) *Algebra and Logic*, 11:203–242, 1974. (English translation.) [MR 50:12688].

[Eršov, 1972b] Yu.L. Eršov. Continuous lattices and A-spaces. *Doklady Akademii Nauk SSSR*, 207:523–526, 1972. (In Russian.) *Soviet Mathematics Doklady*, 13:1551–1555, 1973. (English translation.) [MR 51:2892].

[Eršov, 1972c] Yu.L. Eršov. Everywhere-defined continuous functionals. *Algebra and Logic*, 11: 363–368, 1972. Translation from the Russian in *Algebra i Logika* 11:656–665, 1972.

[Eršov, 1973] Yu.L. Eršov. The theory of A-spaces. *Algebra i Logika*, 12:369–416, 492, 1973. (In Russian). *Algebra and Logic*, 12:209–232, 1975. (English translation.) [MR 54:7236].

[Eršov, 1975] Yu.L. Eršov. Theorie der Nummerierungen II. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 21:473–584, 1975. Translation of *Teoriya numeratsii II*, Novosibirsk 1973 (in Russian).

[Eršov, 1976] Yu.L. Eršov. Model C of partial continuous functions. In R.O. Gandy and J.M.E. Hyland, editors, *Logic Colloquium '76, Oxford, 1976*, volume 87 of *Studies in Logic and Foundations of Mathematics*, pages 455–467. North-Holland, 1976. [MR 58:21541].

[Eršov, 1993] Yu.L. Eršov. Theory of domains and nearby. In D. Bjorner *et al.*, editors, *Formal Methods in Programming and Their Applications*, volume 735 of *Lecture Notes in Computer Science*, pages 1–7. Springer-Verlag, 1993.

[Eršov, 1997] Yu.L. Eršov. The bounded complete hull of an α-space. *Theoretical Computer Science*, 175:3–13, 1997.

[Eršov, 1999a] Yu.L. Eršov. δ-Spaces. *Algebra and Logic*, 38:367–373, 1999.

[Eršov, 1999b] Yu.L. Eršov. The injective hull and the bc-hull of a topological space. *Novi Sad Journal of Mathematics*, 29:1–6, 1999.

[Eršov, 1999c] Yu.L. Eršov. On d-spaces. *Theoretical Computer Science*, 224:59–72, 1999.

[Eršov, 1999d] Yu.L. Eršov. On essential extensions of t_0-spaces. *Doklady Mathematics*, 60:184–187, 1999.

[Escardó, 1996a] M.H. Escardó. PCF extended with real numbers. *Theoretical Computer Science*, 162(1):79–115, 1996.

[Escardó, 1996b] M.H. Escardó. Real PCF extended with $\exists$ is universal. In A. Edalat, S. Jourdan, and G. McCusker, editors, *Advances in Theory and Formal Methods of Computing: Proceedings of the Third Imperial College Workshop, April 1996*, pages 13–24, Christ Church, Oxford, 1996. IC Press, 1996.

[Escardó, 1998a] M.H. Escardó. Effective and sequential definition by cases on the reals via infinite signed-digit numerals. In [1997, Birmingham].

[Escardó, 1998b] M.H. Escardó. Properly injective spaces and function spaces. *Topology and Its Applications*, 89:75–120, 1998.

[Escardó and Streicher, 1999] M.H. Escardó and T. Streicher. Induction and recursion on the partial real line with applications to Real PCF. *Theoretical Computer Science*, 210:69–83, 1999.

[Evans, 1980] E. Evans. Complete co-atomic lattices with enough primes. *Semigroup Forum*, 21:113–126, 1980. [MR 83b:06005].

[Everett, 1944] C.J. Everett. Closure operators and Galois theory in lattices. *Transactions of the American Mathematical Society*, 55:514–525, 1944. [MR 6-36].

[Fell, 1962] J.M.G. Fell. A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space. *Proceedings of the American Mathematical Society*, 13:472–476, 1962. [MR 25:2573].

[Fiech, 1996] A. Fiech. Colimits in the category DCPO. *Mathematical Structures in Computer Science*, 5:455–468, 1996.

[Fiech and Huth, 1994] A. Fiech and M. Huth. Algebraic domains of natural tranformations. *Theoretical Computer Science*, 136:57–78, 1994.

[Fiore, 1997] M.P. Fiore. An enrichment theorem for an axiomatisation of categories of domains and continuous functions. *Mathematical Structures in Computer Science*, 7:591–618, 1997.

[Fiore and Plotkin, 1994] M.P. Fiore and G.D. Plotkin. An axiomatisation of computationally adequate domain theoretic models of FPC. In *Proceedings of the Ninth Annual IEEE Symposium on Logic in Computer Science*, pages 92–102. IEEE Computer Society Press, 1994.

[Fiore and Plotkin, 1997] M.P. Fiore and G.D. Plotkin. An extension of models of axiomatic domain theory to models of synthetic domain theory. In *Proceedings of the CSL'96 Conference*, volume 1258 of *Lecture Notes in Computer Science*, pages 129–149. Springer-Verlag, 1997.

[Fiore and Rosolini, 1997a] M.P. Fiore and G. Rosolini. The category of cpos from a synthetic viewpoint. In [1997, Pittsburgh].

[Fiore and Rosolini, 1997b] M.P. Fiore and G. Rosolini. Two models of synthetic domain theory. *Journal of Pure and Applied Algebra*, 116:151–162, 1997.

[Fiore *et al.*, 1996] M.P. Fiore, E. Moggi, and D. Sangiorgi. A fully-abstract model for the pi-calculus. In *Proceedings of the Eleventh Annual IEEE Symposium on Logic in Computer Science*, pages 43–54. IEEE Computer Society Press, 1996.

[Fiore *et al.*, 1997] M.P. Fiore, G.D. Plotkin, and A.J. Power. Complete cuboidal sets in axiomatic domain theory. In *Proceedings of the Twelfth Annual IEEE Symposium on Logic in Computer Science*, pages 268–279. IEEE Computer Society Press, 1997.

[Flachsmeyer, 1964] J. Flachsmeyer. Verschiedene Topologisierungen im Raum der abgeschlossenen Mengen. *Mathematische Nachrichten*, 26:321–337, 1964. [MR 30:4233].

[Floyd, 1955] E.E. Floyd. Boolean algebras with pathological order topologies. *Pacific Journal of Mathematics*, 5:687–689, 1955. [MR 17-450].

[Freyd, 1990] P.J. Freyd. CPO-categories. In *Fifth Annual IEEE Symposium on Logic in Computer Science*, pages 498–507. IEEE Computer Society Press, 1990.

[Freyd, 1991] P.J. Freyd. Algebraically complete categories. In A. Carboni, M.C. Pedicchio, and G. Rosolini, editors, *Como Category Theory Conference*, volume 1488 of *Lecture Notes in Mathematics*, pages 95–104. Springer-Verlag, 1991.

[Freyd, 1992] P.J. Freyd. Remarks on algebraically compact categories. In M.P. Fourman, P.T. Johnstone, and A.M. Pitts, editors, *Applications of Categories in Computer Science*, volume 177 of *London Mathematical Society Lecture Notes*, pages 95–106. Cambridge University Press, 1992.

[Friedberg, 1972] M. Friedberg. Metrizable approximations of semigronps. *Colloquium Mathematicum*, 25:63–69 and p. 164, 1972. [MR 46:3678].

[Frink, 1942] O. Frink. Topology in lattices. *Transactions of the American Mathematical Society*, 51:569–582, 1942. [MR 3-313].

[Frink, 1954] O. Frink. Ideals in partially ordered sets. *American Mathematical Monthly*, 61:223–234, 1954. [MR 15-848].

[Geissinger and Graves, 1972] L. Geissinger and W. Graves. The category of complete algebraic lattices. *Journal of Combinatorial Theory (A)*, 13:332–338, 1972. [MR 46:5193].

[Gierz, 1982a] G. Gierz. Bündel und stetige Verbände. In [1981, Bremen], pages 51–58.

[Gierz, 1982b] G. Gierz. Colimits of continuous lattices. *Journal of Pure and Applied Algebra*, 23:137–144, 1982. [MR 83b:06019].

[Gierz and Keimel, 1976] G. Gierz and K. Keimel. Topologische Darstellung von Verbänden. *Mathematische Zeitschrift*, 150:83–99, 1976. [MR 55:2694].

[Gierz and Keimel, 1977] G. Gierz and K. Keimel. A lemma on primes appearing in algebra and analysis. *Houston Journal of Mathematics*, 3:207–224, 1977. [MR 57:193].

[Gierz and Keimel, 1981] G. Gierz and K. Keimel. Continuous ideal completions and compactifications. In [1979, Bremen], pages 97–124.

[Gierz and Lawson, 1981] G. Gierz and J.D. Lawson. Generalized continuous and hypercontinuous lattices. *The Rocky Mountain Journal of Mathematics*, 11:271–296, 1981. [MR 82h:54069].

[Gierz and Stralka, 1985] G. Gierz and A.R. Stralka. Natural topologies, essential extensions, reductive lattices and congruence extension. In [1982, Bremen], pages 41–55.

[Gierz and Stralka, 1995] G. Gierz and A.R. Stralka. Homogeneous sublattices of Euclidean space. *Houston Journal of Mathematics*, 21:297–317, 1995.

[Gierz *et al.*, 1983a] G. Gierz, J.D. Lawson, and A.R. Stralka. Metrizability conditions for completely distributive lattices. *Canadian Mathematical Bulletin*, 26:446–453, 1983. [MR 85c:06013].

[Gierz *et al.*, 1983b] G. Gierz, J.D. Lawson, and A.R. Stralka. Quasicontinuous posets. *Houston Journal of Mathematics*, 9:191–208, 1983. [MR 85b:06009].

[Girard, 1986] J.-Y. Girard. The system F of variable types: fifteen years later. *Theoretical Computer Science*, 45:159–192, 1986.

[Girard, 1987] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[Giuli, 1971] E. Giuli. Una caratterizzazione degli spazi ad aperti localmente compatti. *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Mathematiche e Naturali. Serie VIII.*, 50:24–28, 1971. [MR 45:7667].

[Goguen *et al.*, 1977] J.A. Goguen, J.W. Thatcher, E.G. Wagner, and J.B. Wright. Initial algebra semantics and continuous algebras. *Journal of the Association for Computing Machinery*, 24:68–95, 1977.

[Gouy and Jiang, 1995] X. Gouy and Y. Jiang. Universal retractions on DI-domains. *Information and Computation*, 119(2):252–257, 1995.

[Graham, 1988] S. Graham. Closure properties of a probabilistic powerdomain construction. In [1987, New Orleans], pages 213–233.

[Gross, 1967] J.I. Gross. A third definition of local compactness. *American Mathematical Monthly*, 74:1120–1122, 1967.

[Gruchalski, 1996] A. Gruchalski. Computability on dI-domains. *Information and Computation*, 124:7–19, 1996.

[Guessarian, 1979] I. Guessarian. On continuous completions. In K. Weihrauch, editor, *Theoretical Computer Science: Fourth GI Conference, Aachen, March 26–28, 1979*, volume 67 of *Lecture Notes in Computer Science*, pages 142–152. Springer-Verlag, 1979. [MR 83e:06010].

[Gunter, 1985] C. Gunter. Comparing categories of domains. In [1989, Manhattan], pages 101–121.

[Gunter, 1986] C. Gunter. The largest first-order axiomatizable cartesian closed category of domains. In *Symposium on Logic in Computer Science*, pages 142–148. IEEE Computer Society Press, 1986.

[Gunter, 1987] C. Gunter. Universal profinite domains. *Information and Computation*, 72:1–30, 1987.

[Gunter, 1992] C. Gunter. The mixed power domain. *Theoretical Computer Science*, 103:311–334, 1992.

[Gunter and Jung, 1988] C. Gunter and A. Jung. Coherence and consistency in domains. In *Third Annual Symposium on Logic in Computer Science*, pages 309–317. IEEE Computer Society Press, 1988.

[Gunter and Jung, 1989] C. Gunter and A. Jung. Coherence and consistency in domains. *Journal of Pure and Applied Algebra*, 63:49–66, 1989.

[Gunter and Scott, 1990] C. Gunter and D.S. Scott. Semantic Domains. In [van Leeuwen, B1990], pages 633–674.

[Heckmann, 1990] R. Heckmann. Set domains. In N. Jones, editor, *ESOP '90*, volume 432 of *Lecture Notes in Computer Science*, pages 177–196. Springer-Verlag, 1990.

[Heckmann, 1991a] R. Heckmann. Lower and upper power domain constructions commute on all cpos. *Information Processing Letters*, 40(1):7–11, 1991.

[Heckmann, 1991b] R. Heckmann. Power domain constructions. *Science of Computer Programming*, 17:77–117, 1991.

[Heckmann, 1992a] R. Heckmann. Power domains supporting recursion and failure. In J.-C. Raoult, editor, *CAAP '92*, volume 581 of *Lecture Notes in Computer Science*, pages 165–181. Springer-Verlag, 1992.

[Heckmann, 1992b] R. Heckmann. An upper power domain construction in terms of strongly compact sets. In [1991, Pittsburgh], pages 272–293.

[Heckmann, 1993a] R. Heckmann. Observable modules and power domain constructions. In [1991, Dagstuhl], pages 159–187.

[Heckmann, 1993b] R. Heckmann. Power domains and second order predicates. *Theoretical Computer Science*, 111:59–88, 1993.

[Heckmann, 1994a] R. Heckmann. Probabilistic domains. In S. Tison, editor, *CAAP '94*, volume 787 of *Lecture Notes in Computer Science*, pages 142–156. Springer-Verlag, 1994.

[Heckmann, 1994b] R. Heckmann. Probabilistic power domains, information systems, and locales. In [1993, New Orleans], pages 410–437. Springer-Verlag, 1994.

[Heckmann, 1994c] R. Heckmann. Stable power domains. *Theoretical Computer Science*, 136:21–56, 1994.

[Heckmann, 1995] R. Heckmann. Lower bag domains. *Fundamenta Informaticae*, 24(3):259–281, 1995.

[Heckmann, 1996] R. Heckmann. Spaces of valuations. In S. Andima, R.C. Flagg, G. Itzkowitz, Y. Kong, R. Kopperman, and P. Misra, editors, *Papers on General Topology and Its Applications*: *Eleventh Summer Conference at the University of Southern Maine*, volume 806 of *Annals of the New York Academy of Science*, pages 174–200 1996.

[Heckmann, 1997] R. Heckmann. Abstract valuations: a novel representation of Plotkin power domain and Vietoris hyperspace. In [1997, Pittsburgh].

[Heckmann, 2001] R. Heckmann. Characterizing FS-domains by means of power domains. *Theoretical Computer Science*, 264:195–203, 2001.

[Heckmann and Huth, 1998a] R. Heckmann and M. Huth. A duality theory for quantitative semantics. In M. Nielsen and W. Thomas, editors, *Computer Science Logic.*

Eleventh International Workshop, volume 1414 of *Lecture Notes in Computer Science*, pages 255–274. EACSL, Springer-Verlag, 1998.

[Heckmann and Huth, 1998b] R. Heckmann and M. Huth. Quantitative semantics, topology, and possibility measures. *Topology and Its Applications*, 89:151–178, 1998.

[Hennessy and Plotkin, 1979] M.C.B. Hennessy and G.D. Plotkin. Full abstraction for a simple parallel programming language. In J. Beçvar, editor, *Mathematical Foundations of Computer Science*, volume 74 of *Lecture Notes in Computer Science*, Springer-Verlag, 1979.

[Higgs, 1971] D.A. Higgs. Lattices isomorphic to their ideal lattices. *Algebra Universalis*, 1:71–72, 1971. [MR 45:123].

[Hochster, 1969] M. Hochster. Prime ideal structure in commutative rings. *Transactions of the American Mathematical Society*, 142:43–60, 1969. [MR 40:4257].

[Hoffmann, 1975] R.-E. Hoffmann. Charakterisierung nüchterner Räume. *Manuscripta Mathematica*, 15:185–191, 1975. [MR 51:11405].

[Hoffmann, 1977] R.-E. Hoffmann. Irreducible filters and sober spaces. *Manuscripta Mathematica*, 22:365–380, 1977. [MR 57:41071].

[Hoffmann, 1979a] R.-E. Hoffmann. Continuous posets and adjoint sequences. *Semigroup Forum*, 18:173–188, 1979. [MR 80h:18002].

[Hoffmann, 1979b] R.-E. Hoffmann. Essentially complete T_0-spaces. *Manuscripta Mathematica*, 27:401–432, 1979. [MR 81c:54060].

[Hoffmann, 1979c] R.-E. Hoffmann. On the sobrification remainder ${}^sX - X$. *Pacific Journal of Mathematics*, 83:145–156, 1979. [MR 81b:54024].

[Hoffmann, 1979d] R.-E. Hoffmann. Sobrification of partially ordered sets. *Semigroup Forum*, 17:123–138, 1979. [MR 81c:54013].

[Hoffmann, 1979e] R.-E. Hoffmann. Topological spaces admitting a “dual”. In H. Herrlich and G. Preuss, editors, *Categorical Topology, Proceedings of the International Conference, Berlin, August 27th to September 2nd, 1978*, volume 719 of *Lecture Notes in Mathematics*, pages 157–166. Springer-Verlag, 1979. [MR 80j:54001].

[Hoffmann, 1981a] R.-E. Hoffmann. Projective sober spaces. In [1979, Bremen], pages 125–158. [MR 82b:54017]. (Corrections in Zentralblatt für Mathematik, vol. 476, 1982, review 06004.).

[Hoffmann, 1981b] R.-E. Hoffmann. Continuous posets, prime spectra of completely distributive complete lattices, and Hausdorff compactifications. In [1979, Bremen], pages 159–208. (Corrections in *Zentralblatt für Mathematik*, vol. 476, 1982, review 06005.)

[Hoffmann, 1982] R.-E. Hoffmann. Essentially complete T_0-spaces, II. A lattice-theoretic approach. *Mathematische Zeitschrift*, 179:73–90, 1982. [MR 83i:54038].

[Hoffmann, 1985a] R.-E. Hoffmann. The Fell compactification revisited. In [1982, Bremen], pages 57–116.

[Hoffmann, 1985b] R.-E. Hoffmann. The trace of the weak topology and of the Γ-topology of L^{op} coincide on the pseudo-meet-prime elements of a continuous lattice L. In [1982, Bremen], pages 117–119.

[Hoffmann, 1985c] R.-E. Hoffmann. The injective hull and the CL-compactification of a continuous poset. *Canadian Journal of Mathematics*, 37:810–835, 1985.

[Hofmann, 1970] K.H. Hofmann. A general invariant metrization theorem for compact spaces. *Fundamenta Mathematicae*, 68:281–296, 1970. [MR 42:2428].

[Hofmann, 1978] K.H. Hofmann. Continuous lattices, topology and topological algebra. In W. Kuperberg, G.M. Reed, and Ph. Zenor, editors, *Topology Proceedings, 2, Auburn, AL, 1977*, pages 179–212. Auburn University, 1978. [MR 80k:06011].

[Hofmann, 1980] K.H. Hofmann. A note on Baire spaces and continuous lattices. *Bulletin of the Australian Mathematical Society*, 21:265–279, 1980. [MR 81k:54049].

[Hofmann, 1984a] K.H. Hofmann. Order aspects of the essential hull of a topological T_0-space. In M. Pouzet and D. Richard, editors, *Order, Description and Roles*, volume 23 of *Annals of Discrete Mathematics*, pages 193–205. North-Holland, 1984.

[Hofmann, 1984b] K.H. Hofmann. Stably continuous frames, and their topological manifestations. In H.L. Bentley, H. Herrlich, M. Rajagopalan, and H. Wolff, editors, *Categorical Topology*, volume 5 of *Sigma Series in Pure Mathematics*, pages 282–307. Heldermann Verlag, 1984.

[Hofmann, 1985] K.H. Hofmann. Complete distributivity and the essential hull of a T_0-space. In [1982, Bremen], pages 121–127.

[Hofmann and Lawson, 1976] K.H. Hofmann and J.D. Lawson. Irreducibility and generation in continuous lattices. *Semigroup Forum*, 13:307–353, 1976. [MR 55:7868].

[Hofmann and Lawson, 1978] K.H. Hofmann and J.D. Lawson. The spectral theory of distributive continuous lattices. *Transactions of the American Mathematical Society*, 246:285–310, 1978. [MR 80c:54045].

[Hofmann and Lawson, 1984] K.H. Hofmann and J.D. Lawson. On the order theoretical foundation of a theory of quasicompactly generated spaces without separation axiom. *Journal of the Australian Mathematical Society (Series A)*, 36:194–212, 1984.

[Hofmann and Mislove, 1975] K.H. Hofmann and M.W. Mislove. Epics of compact Lawson semilattices are surjective. *Archiv der Mathematik*, 26:337–345, 1975. [MR 51:12636].

[Hofmann and Mislove, 1976] K.H. Hofmann and M.W. Mislove. Amalgamation in categories with concrete duals. *Algebra Universalis*, 6:327–347, 1976. [MR 56:5676].

[Hofmann and Mislove, 1977] K.H. Hofmann and M.W. Mislove. The lattice of kernel operators and topological algebra. *Mathematische Zeitschrift*, 154:175–188, 1977. MR 56:2884].

[Hofmann and Mislove, 1981] K.H. Hofmann and M.W. Mislove. Local compactness and continuous lattices. In [1979, Bremen], pages 209–248.

[Hofmann and Mislove, 1985] K.H. Hofmann and M.W. Mislove. Free objects in the category of completely distributive lattices. In [1982, Bremen], pages 129–150.

[Hofmann and Stralka, 1973] K.H. Hofmann and A.R. Stralka. Push-outs and strict projective limits of semilattices. *Semigroup Forum*, 5:243–261, 1973. [MR 47:5167].

[Hofmann and Stralka, 1976] K.H. Hofmann and A.R. Stralka. The algebraic theory of compact Lawson semilattices: applications of Galois connections to compact semilattices. *Dissertationes Mathematicae*, 137:1–54, 1976. [MR 55:213].

[Hofmann *et al.*, 1973] K.H. Hofmann, M.W. Mislove, and A.R. Stralka. Dimension raising maps in topological algebra. *Mathematische Zeitschrift*, 135:1–36, 1973. [MR 49:3019].

[Hofmann *et al.*, 1975] K.H. Hofmann, M.W. Mislove, and A.R. Stralka. On the dimensional capacity of semilattices. *Houston Journal of Mathematics*, 1:43–55, 1975. [MR 54:452].

[Hosono and Sato, 1977] Ch. Hosono and M. Sato. The retracts in P_ω do not form a continuous lattice – a solution to Scott's problem. *Theoretical Computer Science*, 4:137–142, 1977. [MR 58:8442].

[Hrbacek, 1987] K. Hrbacek. Convex powerdomains I. *Information and Computation*, 74:198–225, 1987.

[Hrbacek, 1988] K. Hrbacek. A powerdomain construction. In [1987, New Orleans], pages 200–212.

[Hrbacek, 1989] K. Hrbacek. Convex powerdomains II. *Information and Computation*, 81:290–317, 1989.

[Hrbacek, 1991] K. Hrbacek. Continuous completions. *Algebra Universalis*, 28:230–244, 1991.

[Huth, 1992] M. Huth. Cartesian closed categories of domains and the space Proj(D). In [1991, Pittsburgh], pages 259–271.

[Huth, 1994] M. Huth. Linear domains and linear maps. In [1993, New Orleans], pages 438–453.

[Huth, 1995a] M. Huth. A maximal monoidal closed category of distributive algebraic domains. *Information and Computation*, 116(1):10–25, 1995.

[Huth, 1995b] M. Huth. Zero dimensional and connected domains. *Semigroup Forum*, 51:63–71, 1995.

[Huth, 1997] M. Huth. A powerdomain of possibility measures. In [1997, Pittsburgh] 12 pp.

[Huth *et al.*, 1994] M. Huth, A. Jung, and K. Keimel. Linear types, approximation, and topology. In *Proceedings of the Ninth Annual IEEE Symposium on Logic in Computer Science*, pages 110–114. IEEE Computer Society Press, 1994.

[Huth *et al.*, 2000] M. Huth, A. Jung, and K. Keimel. Linear types, approximation, and topology. *Mathematical Structures in Computer Science*, 10:719–746, 2000.

[Huwig and Poigné, 1990] H. Huwig and A. Poigné. A note on inconsistencies caused by fixpoints in a cartesian closed category. *Theoretical Computer Science*, 73:101–112, 1990.

[Hyland, 1979] J.M.E. Hyland. Continuity in spatial toposes. In [1978, Durham], pages 442–465. [MR 83g:18009].

[Hyland, 1981] J.M.E. Hyland. Function spaces in the category of locales. In [1979, Bremen], pages 264–281.

[Hyland, 1991] J.M.E. Hyland. First steps in synthetic domain theory. In A. Carboni, C. Pedicchio, and G. Rosolini, editors, *Conference on Category Theory 1990*, volume 1488 of *Lecture Notes in Mathematics*, pages 131–156. Springer-Verlag, 1991.

[Isbell, 1972] J.R. Isbell. Atomless parts of spaces. *Mathematica Scandinavica*, 31:5–32, 1972. [MR 50:11184].

[Isbell, 1975a] J.R. Isbell. Function spaces and adjoints. *Mathematica Scandinavica*, 36:317–339, 1975. [MR 53:9134].

[Isbell, 1975b] J.R. Isbell. Meet-continuous lattices. *Symposia Mathematica*, 16:41–54, 1975.

[Isbell, 1982a] J.R. Isbell. Completion of a construction of Johnstone. *Proceedings of the American Mathematical Society*, 85:333–334, 1982. [MR 83i:06011].

[Isbell, 1982b] J.R. Isbell. Direct limits of meet-continuous lattices. *Journal of Pure and Applied Algebra*, 23:33–35, 1982. [MR 83a:18015].

[Isbell, 1985] J.R. Isbell. Discontinuity of meets and joins. In [1982, Bremen], pages 151–154.

[Isbell, 1986] J.R. Isbell. General function spaces, products and continuous lattices. *Mathematical Proceedings of the Cambridge Philosophical Society*, 100:193–205, 1986.

[Iwamura, 1944] T. Iwamura. A lemma on directed sets. *Zenkoku Shijo Sugaku Danwakai*, 262:107–111, 1944. (In Japanese.)

[Jagadeesan, 1990] R. Jagadeesan. L-domains and lossless powerdomains. In M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Mathematical Foundations of Programming Semantics*, volume 442 of *Lecture Notes in Computer Science*, pages 364–372. Springer-Verlag, 1990.

[Jansen, 1975] S.L. Jansen. Représentation d'un ensemble ordonné dans un sous-treillis d'un treillis complet et complètement distributif. *Comptes Rendus de l'Académie des Sciences, Paris, Série A-B*, 281:A127–A128, 1975. [MR 52:10514].

[Jarzembski, 1982] G. Jarzembski. Free ω-complete algebras. *Algebra Universalis*, 14:231–234, 1982. [MR 83a:08017).

[Jiang, 1992] Y. Jiang. La sémantique continue du lambda-calcul est incompatible avec l'existence d'une rétraction universelle. *Comptes Rendus de l'Académie des Sciences, Série I*, 314:779–782, 1992.

[Jim and Meyer, 1994] T. Jim and A. Meyer. Full abstraction and the context lemma. In T. Ito and A.R. Meyer, editors, *Theoretical Aspects of Computer Software. International Conference TACS '91, Sendai, Japan, September 24–27, 1991. Proceedings*, volume 526 of *Lecture Notes in Computer Science*, pages 663–696. Springer-Verlag, 1994.

[Johnstone, 1981] P.T. Johnstone. Scott is not always sober. In [1979, Bremen], pages 282–283.

[Johnstone and Joyal, 1982] P.T. Johnstone and A. Joyal. Continuous categories and exponentiable toposes. *Journal of Pure and Applied Algebra*, 25:255–296, 1982. [MR 83k:18005].

[Jones and Plotkin, 1989] C. Jones and G. Plotkin. A probabilistic powerdomain of evaluations. In *Proceedings of the Fourth Annual Symposium on Logic in Computer Science*, pages 186–195. IEEE Computer Society Press, 1989.

[Jónsson, 1967] B. Jónsson. Algebras whose congruence lattices are distributive. *Symposia Mathematica*, 21:110–121, 1967. [MR 38:5689].

[Jung, 1988] A. Jung. New results on hierarchies of domains. In [1987, New Orleans], pages 303–310.

[Jung, 1990a] A. Jung. Cartesian closed categories of algebraic CPO's. *Theoretical Computer Science*, 70:233–250, 1990.

[Jung, 1990b] A. Jung. The classification of continuous domains. In *Proceedings of the Fifth Annual IEEE Symposium on Logic in Computer Science*, pages 35–40. IEEE Computer Society Press, 1990.

[Jung, 1991] A. Jung. The dependent product construction in various categories of domains. *Theoretical Computer Science*, 79:359–364, 1991.

[Jung and Puhlmann, 1995] A. Jung and H. Puhlmann. Types, logic, and semantics for nested databases. In [1995, New Orleans].

[Jung and Stoughton, 1993] A. Jung and A. Stoughton. Studying the fully abstract model of PCF within its continuous function model. In M. Bezem and J.F. Groote, editors, *Typed Lambda Calculi and Applications. International Conference on Types Lambda Calculi and Applications, TLCA '93, March 16–18, 1993, Utrecht, The Netherlands*, volume 664 of *Lecture Notes in Computer Science*, pages 230–244. Springer-Verlag, 1993.

[Jung and Sünderhauf, 1996] A. Jung and P. Sünderhauf. On the duality of compact vs. open. In S. Andima, R.C. Flagg, G. Itzkowitz, Y. Kong, R. Kopperman, and P. Misra, editors, *Papers on General Topology and Applications: Eleventh Summer Conference at the University of Southern Maine*, volume 806 of *Annals of the New York Academy of Sciences*, pages 214–230, 1996.

[Jung and Tix, 1998] A. Jung and R. Tix. The troublesome probabilistic powerdomain. In [1997, Birmingham].

[Jung *et al.*, 1992] A. Jung, L. Libkin, and H. Puhlmann. Decomposition of domains. In [1991, Pittsburgh], pages 235–258.

[Jung *et al.*, 1996] A. Jung, M.P. Fiore, E. Moggi, P. O'Hearn, J. Riecke, G. Rosolini, and I. Stark. Domains and denotational semantics: history, accomplishments and open problems. *Bulletin of EATCS*, 59:227–256, 1996.

[Jung *et al.*, 1997] A. Jung, M. Kegelmann, and M.A. Moshier. Multi lingual sequent calculus and coherent spaces. In [1997, Pittsburgh], 18 pp.

[Kaganovsky, 1998] A. Kaganovsky. Computing with exact real numbers in a radix-r system. In [1997, Birmingham].

[Kahn and Plotkin, 1993] G. Kahn and G.D. Plotkin. Concrete domains. *Theoretical Computer Science*, 121:187–277, 1993. Translation of a technical report from 1978.

[Kamara, 1978] M. Kamara. Treillis continus et treillis complètement distributifs. *Semigroup Forum*, 16:387–388, 1978. [MR 58:21881].

[Kamimura and Tang, 1984] T. Kamimura and A. Tang. Total objects of domains. *Theoretical Computer Science*, 34:275–288, 1984.

[Kamimura and Tang, 1986] T. Kamimura and A. Tang. Retracts of SFP objects. In [1985, Manhattan], pages 135–148.

[Kamimura and Tang, 1988] T. Kamimura and A. Tang. Continuous auxiliary relations. In [1987, New Orleans], pages 364–371.

[Kanda, 1979] A. Kanda. Fully effective solutions of recursive domain equations. In J. Beçvar, editor, *Mathematical Foundations of Computer Science*, volume 74 of *Lecture Notes in Computer Science*. Springer-Verlag, 1979.

[Kanda, 1982] A. Kanda. A categorization of effective retract calculus. In [1981, Bremen], pages 180–203.

[Keimel, 1998] K. Keimel. Bi-continuous valuations. In [1997, Birmingham].

[Keimel and Gierz, 1982] K. Keimel and G. Gierz. Halbstetige Funktionen und stetige Verbände. In [1981, Bremen], pages 59–67.

[Keimel and Paseka, 1994] K. Keimel and J. Paseka. A direct proof of the Hofmann–Mislove theorem. *Proceedings of the American Mathematical Society*, 120:301–303, 1994.

[Keimel and Wieczorek, 1988] K. Keimel and A. Wieczorek. Kakutani property of the polytopes implies the Kakutani property of the whole space. *Journal of Mathematical Analysis and Its Applications*, 130:97–109, 1988.

[Kerth, 1994] R. Kerth. Définissabilité dans les domaines réflexifs. *Comptes Rendus de l'Académie des Sciences, Série I*, 318:685–688, 1994.

[Koch, 1959] R.J. Koch. Arcs in partially ordered spaces. *Pacific Journal of Mathematics*, 9:723–728, 1959. [MR 21:7269].

[Koch, 1965] R.J. Koch. Connected chains in quasi-ordered spaces. *Fundamenta Mathematica*, 56:245–249, 1965. [MR 30:5263).

[Kopperman, 1995] R. Kopperman. Asymmetry and duality in topology. *Topology and Its Applications*, 66:1–39, 1995.

[Kou, 1999] H. Kou. u_k-admitting **dcpo**s need not be sober. In [1999, Shanghai], pages 41–50.

[Kou *et al.*, 2001] H. Kou, Y.-M. Liu, and M.-K. Luo. On meet-continuous dcpos. In J. Lawson, G.-Q. Zhang, Y.-M. Liu, and M.-K. Luo, editors, *Domains and Processes II*, Semantic Structures in Computation. Kluwer, 2001. to appear.

[Kristiansen and Normann, 1994] L. Kristiansen and D. Normann. Interpreting higher computations as types with totality. *Archive for Mathematical Logic*, 33:243–259, 1994.

[Kristiansen and Normann, 1995] L. Kristiansen and D. Normann. Semantics for some constructors of type theory. In M. Behara, R. Fritsch, and R.G. Lintz, editors, *Symposia Gaussiana, Conference A*, pages 201–224. Walter de Gruyter, 1995.

[Kristiansen and Normann, 1997] L. Kristiansen and D. Normann. Total objects in inductively defined types. *Archives for Mathematical Logic*, 36:405–436, 1997.

[Lambek, 1968] J. Lambek. A fixpoint theorem for complete categories. *Mathematische Zeitschrift*, 103:151–161, 1968. [MR 37:270].

[Lambrinos, 1974] P.Th. Lambrinos. Locally bounded spaces. *Proceedings of the Edinburgh Mathematical Society*, 19:321–325, 1974. [MR 52:15373].

[Lambrinos, 1985] P.Th. Lambrinos. On the exponential law for function spaces equipped with the compact open topology. In [1982, Bremen], pages 181–190.

[Lambrinos and Papadopoulos, 1985] P.Th. Lambrinos and B. Papadopoulos. The (strong) Isbell topology and (weakly) continuous lattices. In [1982, Bremen], pages 191–211.

[Larsen and Winskel, 1984] K.G. Larsen and G. Winskel. Using information systems to solve recursive domain equations effectively. In [1984, Sophia Antipolis], pages 109–130.

[Lassez *et al.*, 1982] J.-L. Lassez, V.L. Nguyen, and E.A. Sonenberg. Fixed point theorems and semantics: a folk tale. *Information Processing Letters*, 14:112–116, 1982.

[Lau, 1972] A.Y.W. Lau. Small semilattices. *Semigroup Forum*, 4:150–155, 1972. [MR 45:1808].

[Lau, 1976] A.Y.W. Lau. Existence of n-cells in Peano semilattices. In S.P. Franklin and B.V. Smith Thomas, editors, *Topology (Proceedings, Memphis State University), Memphis, TN, 1975*, volume 24 of *Lecture Notes in Pure and Applied Mathematics*, pages 197–200. Marcel Dekker, 1976. [MR 55:3148].

[Laursen and Sinclair, 1975] K.B. Laursen and A.M. Sinclair. Lifting matrix units in C^*-algebras, II. *Mathematica Scandinavica*, 37:167–172, 1975. [MR 53:1281].

[Lawson, 1969] J.D. Lawson. Topological semilattices with small semilattices. *The Journal of the London Mathematical Society*, 1:719–724, 1969. [MR 40:6516].

[Lawson, 1970] J.D. Lawson. Lattices with no interval homomorphisms. *Pacific Journal of Mathematics*, 32:459–465, 1970. [MR 51:1019].

[Lawson, 1972] J.D. Lawson. Dimensionally stable semilattices. *Semigroup Forum*, 5:181–185, 1972. [MR 47:4874].

[Lawson, 1973] J.D. Lawson. Intrinsic topologies in topological lattices and semilattices. *Pacific Journal of Mathematics*, 44:593–602, 1973. [MR 47:6580].

[Lawson, 1976a] J.D. Lawson. Additional notes on continuity in semitopological semigroups. *Semigroup Forum*, 12:265–280, 1976. [MR 53:9175].

[Lawson, 1976b] J.D. Lawson. Embeddings of compact convex sets and locally compact cones. *Pacific Journal of Mathematics*, 66:443–453, 1976. [MR 55:13213].

[Lawson, 1977] J.D. Lawson. Compact semilattices which must have a basis of subsemilattices. *The Journal of the London Mathematical Society*, 16:367–371, 1977. [MR 57:518].

[Lawson, 1979] J.D. Lawson. The duality of continuous posets. *Houston Journal of Mathematics*, 5:357–394, 1979. [MR 81i:06003].

[Lawson, 1980] J.D. Lawson. Algebraic conditions leading to continuous lattices. *Proceedings of the American Mathematical Society*, 78:477–481, 1980. [MR 81g:06002].

[Lawson, 1982] J.D. Lawson. Valuations on continuous lattices. In [1981, Bremen], pages 204–225.

[Lawson, 1985] J.D. Lawson. Obtaining the T_0-essential hull. In [1982, Bremen], pages 213–217.

[Lawson, 1987] J.D. Lawson. The versatile continuous order. In [1987, New Orleans].

[Lawson, 1997] J.D. Lawson. Spaces of maximal points. *Mathematical Structures in Computer Science*, 7:543–555, 1997.

[Lawson, 1998a] J.D. Lawson. Computation on metric spaces via domain theory. *Topology and Its Applications*, 85:247–263, 1998.

[Lawson, 1998b] J.D. Lawson. The upper interval topology, property M and compactness. In [1997, Birmingham].

[Lawson, 1999] J.D. Lawson. Encounters between topology and domain theory. In [1999, Shanghai], pages 1–32.

[Lawson and Mislove, 1990] J.D. Lawson and M. Mislove. Problems in domain theory and topology. In J.V. Mill and G.M. Reed, editors, *Open Problems in Topology*, pages 351–372. North-Holland, 1990.

[Lawson and Williams, 1970] J.D. Lawson and W. Williams. Topological semilattices and their underlying spaces. *Semigroup Forum*, 1:209–223, 1970. [MR 42:3221].

[Lawson *et al.*, 1977] J.D. Lawson, J.R. Liukkonen, and M.W. Mislove. Measure algebras of semilattices with finite breadth. *Pacific Journal of Mathematics*, 69:125–139, 1977. [MR 58:12191].

[Lea, 1972] J.W. Lea, Jr. An embedding theorem for compact semilattices. *Proceedings of the American Mathematical Society*, 34:325–331, 1972. [MR 55:10337].

[Lea, 1974] J.W. Lea, Jr. Sublattices generated by chains in modular topological lattices. *Duke Mathematical Journal*, 41:241–246, 1974. [MR 55:2689].

[Lea, 1976a] J.W. Lea, Jr. Breadth two topological lattices with connected sets of irreducibles. *Transactions of the American Mathematical Society*, 219:337–345, 1976. [MR 53:5401].

[Lea, 1976b] J.W. Lea, Jr. Continuous lattices and compact Lawson semilattices. *Semigroup Forum*, 13:387–388, 1976. [MR 55:7864].

[Lea, 1980] J.W. Lea, Jr. Continuous modular lattices of breadth two. *Semigroup Forum*, 19:387–388, 1980. [MR 81d:06009].

[Li, 1994] Y.-M. Li. The structures of lower convergence of continuous posets. *Journal of Engineering Mathematics, Xi'an*, 11:1–7, 1994. (In Chinese.)

[Liber, 1978] S.A. Liber. Free compact lattices. *Mathematical Notes of the Academy of Sciences of the USSR*, 24:832–835, 1978. (English translation.) [MR 80c:06014].

[Liu and Liang, 1996] Y.-M. Liu and J.-H. Liang. Solutions to two problems of J.D. Lawson and M. Mislove. *Topology and Its Applications*, 69:153–164, 1996.

[Liukkonen and Mislove, 1983] J.R. Liukkonen and M.W. Mislove. Measure algebras of locally compact semilattices. In K.H. Hofmann, H. Jürgensen, and H.J. Weinert, editors, *Recent Developments in the Algebraic, Analytical, and Topological Theory of Semigroups, Oberwolfach, 1981*, volume 998 of *Lecture Notes in Mathematics*, pages 202–214. Springer-Verlag, 1983.

[Longley and Simpson, 1997] J.R. Longley and A.K. Simpson. A uniform appraoch to domain theory in realizability models. *Mathematical Structures in Computer Science*, 7(5):453–468, 1997.

[Lystad and Stralka, 1979] G.S. Lystad and A.R. Stralka. Semilattices having bialgebraic congruence lattices. *Pacific Journal of Mathematics*, 85:131–143, 1979. [MR 81i:06004].

[Lystad and Stralka, 1981] G.S. Lystad and A.R. Stralka. Lawson semilattices with bialgebraic congruence lattices. In L.F. McAuley and M.M. Rao, editors, *General Topology and Modern Analysis, University of California, Riverside, 1980*, pages 247–254. Academic Press, 1981. [MR 82h:06006].

[MacNeille, 1937] H. M. MacNeille. Partially ordered sets. *Transactions of the American Mathematical Society*, 42:416–460, 1937.

[MacQueen *et al.*, 1986] D. MacQueen, G.D. Plotkin, and R. Sethi. An ideal model for recursive polymorphic types. *Information and Control*, 71:95–130, 1986.

[Main, 1985] M. Main. Free constructions of powerdomains. In [1985, Manhattan], pages 162–183.

[Markowsky, 1976] G. Markowsky. Chain-complete posets and directed sets with applications. *Algebra Universalis*, 6:53–68, 1976. [MR 53:2764].

[Markowsky, 1977] G. Markowsky. Categories of chain-complete posets. *Theoretical Computer Science*, 4:125–135, 1977. [MR 56:11859].

[Markowsky, 1979] G. Markowsky. Free completely distributive lattices. *Proceedings of the American Mathematical Society*, 74:227–228, 1979. [MR 80c:06017].

[Markowsky, 1981a] G. Markowsky. A motivation and generalization of Scott's notion of a continuous lattice. In [1979, Bremen], pages 298–307.

[Markowsky, 1981b] G. Markowsky. Propaedeutic to chain-complete posets with basis. In [1979, Bremen], pages 308–314.

[Markowsky and Rosen, 1976] G. Markowsky and B.K. Rosen. Bases for chain-complete posets. *IBM Journal of Research and Development*, 20:138–147, 1976. [MR 52:13523 (EA 54)].

[Martin, 1998] K. Martin. Domain theoretic models of topological spaces. In [1997, Birmingham].

[McWaters, 1969] M.M. McWaters. A note on topological semilattices. *The Journal of the London Mathematical Society*, 1:64–66, 1969. [MR 39:4314].

[Meseguer, 1980] J. Meseguer. Varieties of chain-complete algebras. *Journal of Pure and Applied Algebra*, 19:347–383, 1980. [MR 82g:18005].

[Meseguer, 1983] J. Meseguer. Order completion monads. *Algebra Universalis*, 16:63–82, 1983. [MR 84i:18005].

[Meyer and de Vink, 1988] J.J.C. Meyer and E. P. de Vink. Applications of compactness in the Smyth powerdomain of streams. *Theoretical Computer Science*, 57(2/3):251–282, 1988.

[Michael, 1951] E. Michael. Topologies on spaces of subsets. *Transactions of the American Mathematical Society*, 71:152–182, 1951. [MR 13-54].

[Michael, 1968] E. Michael. Local compactness and cartesian products of quotient maps and k-spaces. *Annales de l'Institut Fourier*, 18:281–286, 1968. [MR 39:6256].

[Milner, 1972] R. Milner. Implementation and application of Scott's logic of continuous functions. In *Conference on Proving Assertions about Programs*, pages 1–6. SIGPLAN 1, 1972.

[Mislove, 1982] M.W. Mislove. An introduction to the theory of continuous lattices. In [1981, Banff], pages 379–406. [MR 83k:06012].

[Mislove, 1998] M.W. Mislove. Topology, domain theory and theoretical computer science. *Topology and Its Applications*, 89:3–59, 1998.

[Mislove *et al.*, 1991] M.W. Mislove, L.S. Moss, and F.J. Oles. Non-well-founded sets modeled as ideal fixed points. *Information and Computation*, 93:16–54, 1991.

[Mitchell and Plotkin, 1988] J. Mitchell and G.D. Plotkin. Abstract types have existential type. *ACM Transactions on Programming Languages and Systems*, 10(3):470–502, 1988.

[Möbus, 1991] A. Möbus. Equational theory of continuous lattices. *Semigroup Forum*, 42:77–82, 1991.

[Moore and Smith, 1922] E.H. Moore and H.L. Smith. A general theory of limits. *American Journal of Mathematics*, 44:102–121, 1922.

[Mulvey, 1986] C.J. Mulvey. &. *Supplemento ai Rendiconti del Circolo Matematico di Palermo*, pages 99–104, 1986. Serie II no. 12.

[Nachbin, 1949] L. Nachbin. On a characterization of the lattice of all ideals of a Boolean ring. *Fundamenta Mathematicae*, 36:137–142, 1949. [MR 11-712].

[Nait-Abdallah, 1982] A. Nait-Abdallah. A universal domain with types: A_∞. In [1981, Bremen], pages 226–259.

[Nelson, 1981] E. Nelson. Z-continuous algebras. In [1979, Bremen], pages 315–334.

[Nerode, 1959] A. Nerode. Some Stone spaces and recursion theory. *Duke Mathematical Journal*, 26:397–406, 1959.

[Niederle, 1992] J. Niederle. On Z-continuous tolerances of Z-distributive lattices. *Czechoslovak Mathematical Journal*, 42 (117):95–100, 1992.

[Nielsen *et al.*, 1981] M. Nielsen, G.D. Plotkin, and G. Winskel. Petri nets, event structures and domains, Part I. *Theoretical Computer Science*, 13(1):85–108, 1981.

[Norberg, 1989] T. Norberg. Existence theorems for measures and continuous posets, with applications to random set theory. *Mathematica Scandinavica*, 64:15–55, 1989.

[Normann, 1980] D. Normann. The recursion theory of the continuous functionals. In F. Drake and S. Wainer, editors, *Recursion Theory: Its Generalisations and Applications*, pages 171–181. Cambridge University Press, 1980.

[Normann, 1990] D. Normann. Formalizing the notion of total information. In P. Petkov, editor, *Mathematical Logic*, pages 67–94. Plenum Press, 1990.

[Normann, 1996] D. Normann. A hierarchy of domains with totality, but without density. In Cooper, Slaman, and S. Wainer, editors, *Computability, Enumerability, Unsolvability*, pages 233–257. Cambridge University Press, 1996.

[Normann, 1997] D. Normann. Closing the gap between the continuous functionals and recursion. *Archives for Mathematical logic*, 36:269–287, 1997.

[Novak, 1982a] D. Novak. Generalization of continuous posets. *Transactions of the American Mathematical Society*, 272:645–667, 1982. [MR 83i:06007].

[Novak, 1982b] D. Novak. On a duality between the concepts "finite" and "directed". *Houston Journal of Mathematics*, 8:545–563, 1982. [MR 84i:06007].

[Numakura, 1957] K. Numakura. Theorems on compact totally disconnected semigroups and lattices. *Proceedings of the American Mathematical Society*, 8:623–626, 1957. [MR 19-290].

[Ore, 1944] O. Ore. Galois connexions. *Transactions of the American Mathematical Society*, 55:493–513, 1944. [MR 6-36].

[Palmgren, 1993] E. Palmgren. An information system interpretation of Martin-Löf's partial type theory with universes. *Information and Computation*, 106:26–60, 1993.

[Palmgren and Stoltenberg-Hansen, 1990] E. Palmgren and V. Stoltenberg-Hansen. Domain interpretations of Martin-Löf's partial type theory. *Annals of Pure and Applied Logic*, 48:135–196, 1990.

[Palmgren and Stoltenberg-Hansen, 1995] E. Palmgren and V. Stoltenberg-Hansen. Logically presented domains. *In Proceedings of the Tenth Annual IEEE Symposium on Logic in Computer Science*, pages 455–463. IEEE Computer Society Press, 1995.

[Papert, 1959] S. Papert. Which distributive lattices are lattices of closed sets? *Proceedings of the Cambridge Philosophical Society*, 55:172–176, 1959. [MR 21:3354].

[Park, 1969] D.M. Park. Fixpoint induction and proofs of program properties. In B. Meltzer and D. Michie, editors, *Machine Intelligence*, volume 5, pages 59–78. Edinburgh University Press, 1969.

[Pedicchio, 1984] M.C. Pedicchio. On the category of topological topologies. *Cahiers de Topologie et Géométrie Différentielle*, 25:3–13, 1984.

[Phoa, 1990] W. Phoa. Effective domains and intrinsic structure. In *Proceedings of the Fifth Annual IEEE Symposium on Logic in Computer Science*, pages 366–377. IEEE Computer Society Press, 1990.

[Phoa, 1994] W.K. Phoa. From term models to domains. *Information and Computation*, 109:211–255, 1994.

[Pickert, 1952] G. Pickert. Bemerkungen über Galois-Verbindungen. *Archiv der Mathematik*, 3:285–289, 1952. [MR 14-529].

[Pitts, 1993] A.M. Pitts. Relational properties of recursively defined domains. In *Eighth Annual IEEE Symposium on Logic in Computer Science*, pages 86–97. IEEE Computer Society Press, 1993.

[Pitts, 1994] A.M. Pitts. A co-induction principle for recursively defined domains. *Theoretical Computer Science*, 124:195–219, 1994.

[Pitts, 1996] A.M. Pitts. Relational properties of domains. *Information and Computation*, 127:66–90, 1996.

[Plotkin, 1976] G.D. Plotkin. A powerdomain construction: semantics and correctness of programs. *SIAM Journal on Computing*, 5:452–487, 1976. [MR 56:4224].

[Plotkin, 1977] G.D. Plotkin. LCF considered as a programming language. *Theoretical Computer Science*, 5:223–255, 1977.

[Plotkin, 1978] G.D. Plotkin. T^ω as a universal domain. *Journal of Computer and System Sciences*, 17:209–236, 1978. [MR 80d:68105].

[Plotkin, 1980] G.D. Plotkin. Dijkstra's predicate transformers and Smyth's power domains. In D. Bjorner, editor, *Abstract Software Specifications*, volume 86 of *Lecture Notes in Computer Science*, pages 527–553. Springer-Verlag, 1980.

[Plotkin, 1982] G.D. Plotkin. A powerdomain for countable non-determinism. In M. Nielsen and E.M. Schmidt, editors, *International Conference on Automata, Languages and Programming*, volume 140 of *Lecture Notes in Computer Science*, pages 418–428. Springer-Verlag, 1982.

[Plotkin, 1993] G.D. Plotkin. Set-theoretical and other elementary models of the λ-calculus. *Theoretical Computer Science*, 121:351–410, 1993. In a *Collection of Contributions in Honour of Corrado Böhm on the Occasion of His Seventieth Birthday*.

[Plotkin, 1999] G.D. Plotkin. Full abstraction, totality and PCF. *Mathematical Structures in Computer Science*, 9:1–20, 1999.

[Plotkin and Abadi, 1993] G.D. Plotkin and M. Abadi. A logic for parametric polymorphism. In M. Bezem and J.F. Groote, editors, *Typed Lambda Calculi and Applications. International Conference on Typed Lambda Calculi and Applications, TLCA '93, March 16–18, 1993, Utrecht, The Netherlands*, volume 664 of *Lecture Notes in Computer Science*, pages 361–375. Springer-Verlag, 1993.

[Plotkin and Winskel, 1994] G.D. Plotkin and G. Winskel. Bistructures, bidomains and linear logic. In *In Proceedings of ICALP 1994*, volume 820 of *Lecture Notes in Computer Science*, pages 352–363. Springer-Verlag, 1994.

[Priestley, 1970] H.A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bulletin of the London Mathematical Society*, 2:186–190, 1970. [MR 42:153].

[Priestley, 1972] H.A. Priestley. Ordered topological spaces and the representation of distributive lattices. *Proceedings of the London Mathematical Society*, 24:507–530, 1972. [MR 46:109].

[Priestley, 1975] H.A. Priestley. The construction of spaces dual to pseudo-complemented distributive lattices. *The Quarterly Journal of Mathematics (Second Series)*, 26:215–228, 1975. [MR 52:13548].

[Priestley, 1984a] H.A. Priestley. Catalytic distributive lattices and compact zero-dimensional topological lattices. *Algebra Universalis*, 1984.

[Priestley, 1984b] H.A. Priestley. Ordered sets and duality for distributive lattices. In M. Pouzet and D. Richard, editors, *Orders: Description and Roles, Annals of Discrete Mathematics*, 23:39–60, 1984. North-Holland Publishing Company.

[Priestley, 1985] H.A. Priestley. Algebraic lattices as dual spaces of distributive lattices. In [1982, Bremen], pages 237–249.

[Puhlmann, 1993] H. Puhlmann. The snack powerdomain for database semantics. In A.M. Borzyszkowski and S. Sokołowski, editors, *Mathematical Foundations of Computer Science*, volume 711 of *Lecture Notes in Computer Science*, pages 650–659 Springer-Verlag, 1993.

[Puhlmann, 1998] H. Puhlmann. Re-grouping information in a domain theoretic data model. *Mathematical Structures in Computer Science*, 8:67–92, 1998.

[Raney, 1952] G.N. Raney. Completely distributive complete lattices. *Proceedings of the American Mathematical Society*, 3:677–680, 1952. [MR 14-612].

[Raney, 1953] G.N. Raney. A subdirect-union representation for completely distributive complete lattices. *Proceedings of the American Mathematical Society*, 4:518–522, 1953. [MR 15-389].

[Raney, 1960] G.N. Raney. Tight Galois connections and complete distributivity. *Transactions of the American Mathematical Society*, 97:418–426, 1960. [MR 22:10928].

[Rauch, 1982] M. Rauch. Stetige Verbände in der axiomatischen Potentialtheorie. In [1981, Bremen], pages 260–308.

[Rennie, 1951] B.C. Rennie. Lattices. *Proceedings of the London Mathematical Society*, 52:386–400, 1951. [MR 13-7].

[Reus and Streicher, 1997] B. Reus and T. Streicher. General synthetic domain theory – a logical approach (extended abstract). In E. Moggi and G. Rosolini, editors, *Seventh Conference on Category Theory in Computer Science*, volume 1290 of *Lecture Notes in Computer Science*, pages 293–313. Springer-Verlag, 1997.

[Reynolds, 1975] J.C. Reynolds. On the interpretation of Scott domains. *Symposia Mathematica*, 15:123–135, 1975. [MR 54:1704].

[Richter, 1997] G. Richter. An elementary approach to exponential spaces. In *Mathematik-Arbeitspapiere*, volume 48, pages 391–396. Universität Bremen, 1997.

[Roberts, 1977] J.W. Roberts. A compact convex set with no extreme points. *Studia Mathematica*, 60:255–266, 1977. [MR 57:10595].

[Rounds and Zhang, 1995] W. Rounds and G.-Q. Zhang. Domain theory meets default logic. *Journal of Logic and Computation*, 5:1–25, 1995.

[Rudin, 1981] M.E. Rudin. Directed sets which converge. In L.F. McAuley and M.M. Rao, editors, *General Topology and Modern Analysis, University of California, Riverside, 1980*, pages 305–307. Academic Press, 1981. [MR 82f:54006].

[Rutten, 1996] J.J.M.M. Rutten. Elements of generalized ultrametric domain theory. *Theoretical Computer Science*, 170:349–381, 1996.

[Rutten, 1998] J.J.M.M. Rutten. Weighted colimits and formal balls in generalized metric spaces. *Topology and Its Applications*, 89:179–202, 1998.

[Rutten, 2000] J.J.M.M. Rutten. Universal coalgebra: a theory of systems. *Theoretical Computer Science*, 249:3–80, 2000.

[Rutten and Turi, 1993] J.J.M.M. Rutten and D. Turi. On the foundations of final semantics: non-standard sets, metric spaces, partial orders. In J.W. de Bakker, W.-P. de Roever, and G. Rozenberg, editors, *Semantics: Foundations and Applications REX Workshop, Beekbergen, The Netherlands, June 1–4, 1992*, volume 666 of *Lecture Notes in Computer Science*, pages 477–530. Springer-Verlag, 1993.

[Saheb-Djahromi, 1980] N. Saheb-Djahromi. CPO's of measures for non-determinism. *Theoretical Computer Science*, 12(1):19–37, 1980.

[Sambin *et al.*, 1996] G. Sambin, S. Valentini, and P. Virgili. Constructive domain theory as a branch of intuitionistic pointfree topology. *Theoretical Computer Science*, 169:319–342, 1996.

[Schalk, 1993] A. Schalk. Domains arising as algebras for powerspace constructions. *Journal of Pure and Applied Algebra*, 89:305–328, 1993.

[Schnare, 1965] P.S. Schnare. Two definitions of local compactness. *American Mathematical Monthly*, 72:764–765, 1965.

[Schwarz, 1981] F. Schwarz. “Continuity” properties in lattices of topological structures. In [1979, Bremen], pages 335–347.

[Schwarz, 1982] F. Schwarz. Exponential objects in categories of (pre) topological spaces and their natural function spaces. *La Société Royale du Canada. L'Académie des Sciences. Comptes Rendus Mathématiques*, 4:321–326, 1982. [MR 84g:18016].

[Schwarz and Weck, 1985] F. Schwarz and S. Weck. Scott topology, Isbell topology and continuous convergence. In [1982, Bremen], pages 251–272.

[Sciore and Tang, 1978] E. Sciore and A. Tang. Admissible coherent c.p.o.'s. In G. Ausiello and C. Böhm, editors, *Proceedings of the Fifth International Colloquium on Automata, Languages and Programming, Udine, 1978*, volume 62 of *Lecture Notes in Computer Science*, pages 440–456. Springer-Verlag, 1978.

[Scott, 1970] D.S. Scott. Outline of a mathematical theory of computation. In *Proceedings of the Fourth Annual Princeton Conference on Information Sciences and Systems*, pages 169–176. Princeton University Press, 1970.

[Scott, 1971] D.S. Scott. The lattice of flow diagrams. In E. Engeler, editor, *Symposium on Semantics of Algorithmic Languages*, volume 188 of *Lecture Notes in Mathematics*, pages 311–366. Springer-Verlag, 1971.

[Scott, 1972a] D.S. Scott. Continuous lattices. In F.W. Lawvere, editor, *Toposes, Algebraic Geometry and Logic, Dalhousie University, Halifax, Nova Scotia, January 16–19, 1971*, volume 274 of *Lecture Notes in Mathematics*, pages 97–136. Springer-Verlag, 1972. [MR 53:7879].

[Scott, 1972b] D.S. Scott. Lattice theory, data types, and semantics. In R. Rustin, editor, *Formal Semantics of Programming Languages*, volume 2 of *Courant Computer Science Symposia, New York, 1970*, pages 65–106. Prentice-Hall, 1972. [MR 56:7304].

[Scott, 1972c] D.S. Scott. Mathematical concepts in programming language semantics. In *AFIPS Conference Proceedings*, volume 40, pages 225–234. AFIPS Press, 1972.

[Scott, 1973] D.S. Scott. Models for various type-free calculi. In P. Suppes, editor, *Logic, Methodology and Philosophy of Science IV, Bucureşti, 1971*, pages 157–187. North-Holland, 1973. [MR 57:15987].

[Scott, 1975a] D.S. Scott. Data types as lattices. In G. Muller *et al.*, editors, *Proceedings of the International Summer Institute and Logic Colloquium, Kiel*, volume 499 of *Lecture Notes in Mathematics*, pages 579–651. Springer-Verlag, 1975.

[Scott, 1975b] D.S. Scott. Some philosophical issues concerning theories of combinators. In [1975, Rome], pages 346–366. [MR 57:15986].

[Scott, 1976] D.S. Scott. Data types as lattices. *SIAM Journal on Computing*, 5:522–587, 1976. [MR 55:10262].

[Scott, 1980a] D.S. Scott. Lambda calculus: some models, some philosophy. In J. Barwise, H.J. Keisler, and K. Kunen, editors, *The Kleene Symposium, 1978*, pages 223–265. North-Holland, 1980. [MR 82d:03024].

[Scott, 1980b] D.S. Scott. Relating theories of the λ-calculus. In J.P. Seldin and J.R. Hindley, editors, *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 403–450. Academic Press, 1980. [MR 82d:03025].

[Scott, 1982a] D.S. Scott. Domains for denotational semantics. In M. Nielsen and E.M. Schmidt, editors, *Automata, Languages and Programming, Ninth Colloquium*

Aarhus, Denmark, July 12–16, 1982 (ICALP 82), volume 140 of *Lecture Notes in Computer Science*, pages 577–613. Springer-Verlag, 1982. [MR 83m:68029].

[Scott, 1982b] D.S. Scott. Lectures on a mathematical theory of computation. In M. Broy and G. Schmidt, editors, *Theoretical Foundations of Programming Methodology*, pages 145–292. D. Reidel 1982.

[Scott, 1982c] D.S. Scott. Some ordered sets in computer science. In [1981, Banff], pages 677–718. D. Reidel 1982. [MR 83j:06007].

[Scott, 1993] D.S. Scott. A type theoretic alternative to ISWIM, CUCH, OWHY. *Theoretical Computer Science*, 121:411–440, 1993. Manuscript, University of Oxford, 1969(!).

[Scott and Strachey, 1971] D.S. Scott and C. Strachey. Toward a mathematical semantics for computer languages. In *Proceedings of the Symposium on Computers and Automata*, volume 21 of *Microwave Reserach Institute Symposia Series*, pages 19–46. Polytechnic Institute of Brooklyn Press, 1971. Also: Technical Monograph PRG-6, Programming Research Group, University of Oxford, 1971.

[Shirley and Stralka, 1971] E.D. Shirley and A.R. Stralka. Homomorphisms on connected topological lattices. *Duke Mathematical Journal*, 38:483–490, 1971. [MR 43:5497].

[Shmuely, 1974] Z. Shmuely. The structure of Galois connections. *Pacific Journal of Mathematics*, 54:209–225, 1974. [MR 51:12630].

[Sigstam, 1995] I. Sigstam. Formal spaces and their effective presentations. *Archive for Mathematical Logic*, 34:211–246, 1995.

[Sigstam and Stoltenberg-Hansen, 1997] I. Sigstam and V. Stoltenberg-Hansen. Representations of locally compact spaces by domains and formal spaces. *Theoretical Computer Science*, 179:319–331, 1997.

[Smyth, 1977] M.B. Smyth. Effectively given domains. *Theoretical Computer Science*, 5:257–274, 1977. [MR 58:10366].

[Smyth, 1978] M.B. Smyth. Power domains. *Journal of Computer and System Sciences*, 16:23–36, 1978. [MR 57:8128].

[Smyth, 1983a] M.B. Smyth. The largest cartesian closed category of domains. *Theoretical Computer Science*, 27:109–119, 1983.

[Smyth, 1983b] M.B. Smyth. Power domains and predicate transformers: a topological view. In J. Díaz, editor, *ICALP 83, Proceedings of the Tenth International Colloquium on Automata, Languages, and Programming, Barcelona, July 18–22, 1983*, volume 154 of *Lecture Notes in Computer Science*, pages 662–675. Springer-Verlag, 1983.

[Smyth, 1986] M.B. Smyth. Finite approximation of spaces. In D. Pitt, S. Abramsky, A. Poigné, and D. Rydeheard, editors, *Category Theory and Computer Programming*, volume 240 of *Lecture Notes in Computer Science*, pages 225–241. Springer-Verlag, 1986.

[Smyth, 1988] M.B. Smyth. Quasi unifomities: reconciling domains with metric spaces. In [1987, New Orleans]. pages 236–253.

[Smyth, 1991] M.B. Smyth. Totally bounded spaces and compact ordered spaces as domains of computation. In G.M. Reed, A.W. Roscoe, and R.F. Wachter, editors, *Topology and Category Theory in Computer Science*, pages 207–229. Clarendon Press, 1991.

[Smyth, 1992a] M.B. Smyth. I-categories and duality. In M.P. Fourman, P.T. Johnstone, and A.M. Pitts, editors, *Applications of Categories in Computer Science*, volume 177 of *London Mathematical Society Lecture Notes*, pages 270–287. Cambridge University Press, 1992.

[Smyth, 1992b] M.B. Smyth. Stable compactification I. *Journal of the London Mathematical Society*, 45:321–340, 1992.

[Smyth, 1992c] M.B. Smyth. Topology. In [Abramsky *et al.*, B1992 ff.], volume 1, pages 641–761.

[Smyth and Plotkin, 1978] M.B. Smyth and G.D. Plotkin. The category-theoretic solution of recursive domain equations. In *Eighteenth Annual Symposium on Foundations of Computer Science, Providence, RI, October 31 to November 2, 1977*, pages 13–29. IEEE Computer Society Press, 1978. [MR 58:8459].

[Smyth and Plotkin, 1982] M.B. Smyth and G.D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM Journal on Computing*, 11:761–783, 1982. [MR 84d:68052].

[Speed, 1972] T.P. Speed. Profinite posets. *Bulletin of the Australian Mathematical Society*, 6:177–183, 1972. [MR 45:4371].

[Spreen, 1984] D. Spreen. On r.e. inseparability of cpo index sets. In E. Bürger, editor, *Logic and Machines: Decision Problems and Complexity*, volume 171 of *Lecture Notes in Computer Science*, pages 103–117. Springer-Verlag, 1984.

[Spreen, 1988] D. Spreen. Computable one-to-one enumerations of effective domains. In [1987, New Orleans], pages 372–384.

[Spreen, 1989] D. Spreen. A characterization of effective topologies. In K. Ambos-Spies *et al.*, editors, *Recursion Theory Week, Proceedings of Oberwolfach 1989*, volume 1432 of *Lecture Notes in Mathematics*, pages 363–388. Springer-Verlag, 1989.

[Spreen, 1990] D. Spreen. Computable one-to-one enumerations of effective domains. *Information and Computation*, 84:26–46, 1990.

[Spreen, 1998] D. Spreen. On effective topological spaces. *The Journal of Symbolic Logic*, 63:185–221, 1998.

[Stepp, 1971a] J.W. Stepp. Semilattices which are embeddable in a product of min intervals. *Proceedings of the American Mathematical Society*, 28:81–86, 1971. [MR 43:1895].

[Stepp, 1971b] J.W. Stepp. Semilattices which are embeddable in a product of min intervals. *Semigroup Forum*, 2:80–82, 1971. [MR 44:5260].

[Stepp, 1972] J.W. Stepp. The lattice of ideals of a compact semilattice. *Semigroup Forum*, 5:176–180, 1972. [MR 49:4882].

[Stepp, 1975a] J.W. Stepp. Algebraic maximal semilattices. *Pacific Journal of Mathematics*, 58:243–248, 1975. [MR 51:12640].

[Stepp, 1975b] J.W. Stepp. The free compact lattice generated by a topological semilattice. *Journal für die Reine und Angewandte Mathematik*, 273:77–86, 1975. [MR 51:7974].

[Stoltenberg-Hansen and Tucker, 1991] V. Stoltenberg-Hansen and J.V. Tucker. Algebraic and fixed point equations over inverse limits of algebras. *Theoretical Computer Science*, 87:1–24, 1991.

[Stoltenberg-Hansen and Tucker, 1995] V. Stoltenberg-Hansen and J.V. Tucker. Effective algebras. In S. Abramsky, D.M. Gabbay, and T.S.E. Maibaum, editors, *Semantic*

Modelling, volume 4 of *Handbook of Logic in Computer Science*, pages 357–526. Oxford University Press, 1995.

[Stone, 1936] M.H. Stone. The theory of representations for Boolean algebras. *Transactions of the American Mathematical Society*, 40:37–111, 1936.

[Stone, 1937] M.H. Stone. Topological representation of distributive lattices and Brouwerian logics. *Časopis pro Pěstování Matematiky a Fysiky*, 67:1–25, 1937.

[Stralka, 1970] A.R. Stralka. Locally convex topological lattices. *Transactions of the American Mathematical Society*, 151:629–640, 1970. [MR 41:9216].

[Stralka, 1972] A.R. Stralka. The lattice of ideals of a compact semilattice. *Proceedings of the American Mathematical Society*, 33:175–180, 1972. [MR 46:7439].

[Stralka, 1977] A.R. Stralka. Congruence extension and amalgamation in CL. *Semigroup Forum*, 13:355–375, 1977. [MR 58:10640].

[Stralka, 1979] A.R. Stralka. Quotients of products of compact chains. *Bulletin of the London Mathematical Society*, 11:1–4, 1979. [MR 80h:22004].

[Stralka, 1980] A.R. Stralka. A partially ordered space which is not a Priestley space. *Semigroup Forum*, 20:293–297, 1980. [MR 82f: 54051].

[Stralka, 1981] A.R. Stralka. Fundamental congruences on Lawson semilattices. In [1979, Bremen], pages 348–359.

[Strauss, 1968] D.P. Strauss. Topological lattices. *Proceedings of the London Mathematical Society*, 18:217–230, 1968. [MR 37:3532].

[Streicher and Reus, 1998] Th. Streicher and B. Reus. Classical logic, continuation semantics and abstract machines. *Journal of Functional Programming*, 8:543–572, 1998.

[Sünderhauf, 1995a] P. Sünderhauf. Constructing a quasi-uniform function space. *Topology and Its Applications*, 67:1–27, 1995.

[Sünderhauf, 1995b] P. Sünderhauf. A faithful computational model of the real numbers. *Theoretical Computer Science*, 151:277–294, 1995.

[Tang, 1979] A. Tang. Chain properties in $P\omega$. *Theoretical Computer Science*, 9:153–172, 1979. [MR 80:68009].

[Tang, 1981] A. Tang. Wadge reducibility and Hausdorff difference hierarchy in $P\omega$. In [1979, Bremen], pages 360–371.

[Tarski, 1955] A. Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5:285–309, 1955. [MR 17:574].

[Taylor, 1986] P. Taylor. Internal completeness of categories of domains. In D. Pitt, editor, *Category Theory and Computer Programming (Guildford, September 1985)*, volume 240 of *Lecture Notes in Computer Science*, pages 449–465. Springer-Verlag, 1986.

[Taylor, 1989] P. Taylor. Quantitative domains, groupoids and linear logic. In D. Pitt, editor, *Category Theory and Computer Science*, volume 389 of *Lecture Notes in Computer Science*, pages 155–181, Manchester, 1989. Springer-Verlag, 1989.

[Taylor, 1990] P. Taylor. An algebraic approach to stable domains. *Journal of Pure and Applied Algebra*, 64:171–203, 1990.

[Taylor, 1991] P. Taylor. The fixed point property in synthetic domain theory. In *Sixth Annual Symposium on Logic in Computer Science*, pages 152–160. IEEE Computer Society Press, 1991.

[Thron, 1962] W.J. Thron. Lattice-equivalence of topological spaces. *Duke Mathematical Journal*, 29:671–679, 1962. [MR 26:4307].

[Tiller, 1981] J. Tiller. Augmented compact spaces and continuous lattices. *Houston Journal of Mathematics*, 7:441–453, 1981. [MR 82m:06008].

[Tix, 2001] R. Tix. Some results on Hahn–Banach-type theorems for continuous *D-cones. Theoretical Computer Science*, 264:205–218, 2001.

[Tsuiki, 1998] H. Tsuiki. A computationally adequate model for overloading via domain-valued functors. *Mathematical Structures in Computer Science*, 8:321–349, 1998.

[van de Vel, 1985] M. van de Vel. Lattices and semilattices: a convex point of view. In [1982, Bremen], pages 279–302.

[Venugopalan, 1986] G. Venugopalan. Z-continuous posets. *Houston J. Math.*, 12:275–294, 1986.

[Vickers, 1987] S. Vickers. A fixpoint construction of the p-adic domain. In D. H. Pitt, A. Poigné, and D.E. Rydeheard, editors, *Category Theory and Computer Science*, volume 283 of *Lecture Notes in Computer Science*, pages 270–289. Springer-Verlag, 1987.

[Vickers, 1993] S.J. Vickers. Information systems for continuous posets. *Theoretical Computer Science*, 114:201–229, 1993.

[Vietoris, 1922] L. Vietoris. Bereiche zweiter Ordnung. *Monatshefte für Mathematik und Physik*, 32:258–280, 1922.

[Vietoris, 1923] L. Vietoris. Kontinua zweiter Ordnung. *Monatshefte für Mathematik und Physik*, 33:49–62, 1923.

[Wadsworth, 1976] C. Wadsworth. The relation between computational and denotational properties for Scott's D_∞-models of the λ-calculus. *SIAM Journal on Computing*, 5:488–521, 1976. [MR 58:21493].

[Wallace, 1945] A.D. Wallace. A fixed-point theorem. *Bulletin of the American Mathematical Society*, 51:413–416, 1945. [MR 6-278].

[Wallace, 1957] A.D. Wallace. The center of a compact lattice is totally disconnected. *Pacific Journal of Mathematics*, 7:1237–1238, 1957. [MR 20:823].

[Wand, 1979] M.Wand. Fixed point constructions in order-enriched categories. *Theoretical Computer Science*, 8:13–30, 1979. [MR 80e:18002].

[Ward, 1955] A.J. Ward. On relations between certain intrinsic topologies in partially ordered sets. *Proceedings of the Cambridge Philosophical Society*, 51:254–261, 1955. [MR 17-67].

[Ward, 1965a] A.J. Ward. Concerning Koch's theorem on the existence of arcs. *Pacific Journal of Mathematics*, 15:347–355, 1965. [MR 31:6206].

[Ward, 1965b] A.J. Ward. On a conjecture of R.J. Koch. *Pacific Journal of Mathematics*, 15:1429–1433, 1965. [MR 32:6397].

[Ward, 1969] A.S. Ward. Problem. In D.R. Kurepa, editor, *Topology and Its Applications, Belgrad, 1969*, pages 351–352. Savez Društava Matematičara Fizičara i Astronoma, 1969.

[Weaver, 2001] N. Weaver. A prime C^*-algebra that is not primitive. arXiv:math. OA/0106252 v 2 6 Jul 2001.

[Weck, 1981] S. Weck. Scott convergence and Scott topology in partially ordered sets I. In [1979, Bremen], pages 372–383.

[Weihrauch and Schreiber, 1979] K. Weihrauch and U. Schreiber. Metric spaces defined by weighted algebraic cpo's. In L. Budach, editor, *Fundamentals of Computation Theory, Proceedings, Berlin/Wendisch-Rietz, 1979, Berlin, 1979*, pages 516–522. Akademie-Verlag, 1979. [MR 81e:68055].

[Weihrauch and Schreiber, 1981] K. Weihrauch and U. Schreiber. Embedding metric spaces into cpo's. *Theoretical Computer Science*, 16:5–24, 1981. [MR 83i: 06023].

[Winskel, 1983] G. Winskel. A note on powerdomains and modality. In M. Karpinski, editor, *Foundations of Computation Theory 1983, Proceedings of the 1983 International FCT-Conference, Borgholm, Sweden, August 21–27, 1983*, volume 158 of *Lecture Notes in Computer Science*, pages 505–514. Springer-Verlag, 1983.

[Winskel and Larsen, 1984] G. Winskel and K.G. Larsen. Using information systems to solve recursive domain eqations effectively. In [1984, Sophia Antipolis], pages 109–130.

[Wojdysławski, 1939] M. Wojdysławski. Retractes absolus et hyperspaces des continus. *Fundamenta Mathematicae*, 32:184–192, 1939.

[Wolk, 1958] E.S. Wolk. Order-compatible topologies on a partially ordered set. *Proceedings of the American Mathematical Society*, 9:524–529, 1958. [MR 20:3079].

[Wright *et al.*, 1978] J.B. Wright, E.G.Wagner, and J.W. Thatcher. A uniform approach to inductive posets and inductive closure. *Theoretical Computer Science*, 7:57–77, 1978. [MR 58:404].

[Wright *et al.*, 1986] J.B. Wright, E.G.Wagner, and J.W. Thatcher. A uniform approach to inductive posets and inductive closure. *Theoretical Computer Science*, 7:57–77, 1986.

[Wulfsohn, 1972] A. Wulfsohn. A compactification due to Fell. *Canadian Mathematical Bulletin*, 15:145–146, 1972. [MR 48:3004].

[Wyler, 1977] O. Wyler. Injective spaces and essential extensions in TOP. *General Topology and Its Applications*, 7:247–249, 1977. [MR 56:16559].

[Wyler, 1981a] O. Wyler. Dedekind complete posets and Scott topologies. In [1979, Bremen], pages 384–389.

[Wyler, 1981b] O. Wyler. Algebraic theories of continuous lattices. In [1979, Bremen], pages 390–413.

[Wyler, 1985] O. Wyler. Algebraic theories for continuous semilattices. *Archive for Rational Mechanics and Analysis*, 90:95–113, 1985.

[Xu, 1995a] X.-Q. Xu. Construction of homomorphisms of M-continuous lattices. *Transactions of the American Mathematical Society*, 347:3167–3175, 1995.

[Xu, 1995b] X.-Q. Xu. Embeddings of M-continuous lattices in hilbert cubes. *Acta Mathematica Sinica*, 38:827–830, 1995. (In Chinese.)

[Xu, 1995c] X.-Q. Xu. Embeddings of Z-domains in cubes. In *Fuzzy Logic and Its Applications, Theory Decisions Library Series D*, pages 333–342. Kluwer, 1995.

[Xu, 2000] X.-Q. Xu. Strictly copmplete regularity of the Lawson topology on a continuous poset. *Topology and Its Applications*, 103:37–42, 2000.

[Zhang, 1989] G.-Q. Zhang. DI-domains as information systems. In *Proceedings of the Sixteenth International Colloquium on Automata, Languages, and Programming*, volume 372 of *Lecture Notes in Computer Science*, pages 773–788. Springer-Verlag, 1989.

[Zhang, 1992a] G.-Q. Zhang. DI-domains as prime information systems. *Information and Computation*, 100:151–177, 1992.

[Zhang, 1992b] G.-Q. Zhang. Disjunctive systems and L-domains. In *Proceedings of the Nineteenth International Colloquium on Automata, Languages, and Programming*, volume 623 of *Lecture Notes in Computer Science*, pages 284–295. Springer-Verlag, 1992.

[Zhang, 1992c] G.-Q. Zhang. Stable neighborhoods. *Theoretical Computer Science*, 93:143–157, 1992.

[Zhang, 1993] G.-Q. Zhang. Some monoidal closed categories of stable domains and event structures. *Mathematical Structures in Computer Science*, 3:259–276, 1993.

[Zhang, 1994a] G.-Q. Zhang. A representation of SFP. *Information and Computation*, 110:233–263, 1994.

[Zhang, 1994b] G.-Q. Zhang. Universal quasi-prime algebraic domains. In [1993, New Orleans], pages 454–473.

[Zhang, 1995] G.-Q. Zhang. Maximal stable functions. *Theoretical Computer Science*, 146:331–339, 1995.

[Zhang, 1996a] G.-Q. Zhang. Quasi-prime algebraic domains. *Theoretical Computer Science*, 155:221–264, 1996.

[Zhang, 1996b] G.-Q. Zhang. The largest cartesian closed category of stable domains. *Theoretical Computer Science*, 166:203–219, 1996.

[Zhang and Chen, 2001] G.-Q. Zhang and Y.-X. Chen. Domains via graph. *Journal of Computer Science and Technology*, 16:505–521, 2001.

[Zhang and Rounds, 1997a] G.-Q. Zhang and W. Rounds. Defaults in domain theory. *Theoretical Computer Science*, 177:155–182, 1997.

[Zhang and Rounds, 1997b] G.-Q. Zhang and W. Rounds. Nonmonotonic consequences of default domain theory. *Annals of Mathematics and Artificial Intelligence*, 20:227–265, 1997.

[Zhang and Rounds, 1997c] G.-Q. Zhang and W. Rounds. Resolution in the Smyth power-domain. In [1997, Pittsburgh] 14 pp.

[Zhang, 1993] H. Zhang. A note on continuous partially ordered sets. *Semigroup Forum*, 47:101–104, 1993.

Dissertations and Master's Theses

[Abramsky, D1987] S. Abramsky. *Domain Theory and the Logic of Observable Properties*. PhD thesis, Queen Mary College, University of London, 1987.

[Alvarez-Manilla, D1996] M. Alvarez-Manilla. A generalization of the Riemann–Stieltjes integral using domain theory. Master's thesis, Department of Computing, Imperial College, University of London, 1996.

[Alvarez-Manilla, D2000] M. Alvarez-Manilla. *Measure Theoretic Results for Continuous Valuations on Partially Ordered Structures*. PhD thesis, Imperial College, University of London, 2000.

[Anderson, D1956] L.W. Anderson. *Topological Lattices*. PhD thesis, Tulane University, New Orleans, 1956.

[Berger, D1990] U. Berger. *Totale Objekte und Mengen in der Bereichstheorie*. PhD thesis, Universität München, 1990.

[Berry, D1979] G. Berry. *Modèles complètement adéquats et stables des lambda-calculs typés.* Thèse de doctorat d'etat, Université Paris VII, 1979.

[Blanck, D1997] J. Blanck. *Computability on Topological Spaces by Effective Domain Representations*. Uppsala dissertations in mathematics 7, Department of Mathematics, Uppsala University, 1997.

[Bonsangue, D1996] M.M. Bonsangue. *Topological Duality in Semantics*. PhD thesis, Vrije Universiteit, Amsterdam, 1996. A revised version appeared as Volume 8 of

Electronic Notes in Theoretical Computer Science, Elsevier. ⟨http://www.elsevier.nl/locate/entcs/⟩.

[Bowen, D1981] W.G. Bowen. *Lattice Theory and Topology*. D. Phil. thesis, Oxford University, 1981.

[Bracho, D1983] F. Bracho. *Continuously Generated Fixed Points*. D. Phil. thesis, Oxford University, 1983. 223 pp.

[Bucciarelli, D1993] A. Bucciarelli. *Sequential Models of PCF: Some Contributions to the Domain-Theoretic Approach to Full Abstraction*. PhD thesis, Universitá di Pisa, 1993.

[Clinkenbeard, D1976] D. Clinkenbeard. *Lattices of Congruences on Compact Topological Lattices*. Doctoral Dissertation, University of California at Riverside, 1976. 54 pp.

[Dupré, D1996] C. Dupré. *Domain Models of Typed Lambda-Calculi – towards a Theory of Concurrency in Typed Functional Setting*. PhD thesis, University of Siegen, Germany, 1996.

[Erker, D1997] T. Erker. Stetigkeit und Ordnung in Funktionenräumen. Master's thesis, Technische Universität Darmstadt, 1997. 61pp.

[Erné, D1979] M. Erné. *Verallgemeinerungen der Verbandstheorie, II. m-Ideale in halbgeordneten Mengen und Hüllenräumen*. Habilitationsschrift, Universität Hannover, 1979. 132 pp.

[Escardó, D1996] M.H. Escardó. *PCF Extended with Real Numbers: a Domain-Theoretic Approach to Higher-Order Exact Real Number Computation*. PhD thesis, Imperial College, University of London, 1996. Also Technical Report ECS-LFCS-97-374, Department of Computer Science, University of Edinburgh, 1997. ⟨http://www.dcs.ed.ac.uk/lfcsreps/EXPORT/97/ECS-LFCS-97-374/index.html⟩.

[Gerritse, D1990] G.J.J. Gerritse. *Supremum self-decomposability*. PhD thesis, Universiteit Nijmegen, 1990.

[Gerritse, D1995] B.E.W. Gerritse. *Large Deviations*. PhD thesis, Universiteit Nijmegen, 1995.

[Gerstmann, D1983] H. Gerstmann. *n-Distributivgesetze*. PhD thesis, Universität Hannover, 1983.

[Greb, D1996] R. Greb. Die Kategorie der stark finiten Sequenzenstrukturen. Master's thesis, University of Siegen, Germany, 1996.

[Gruchalski, D1995] A. Gruchalski. *Constructive Domain Models of Typed Lambda-Calculi*. PhD thesis, University of Siegen, Germany, 1995.

[Gunter, D1985] C. Gunter. *Profinite Solutions for Recursive Domain Equations*. PhD thesis, University of Wisconsin, 1985.

[Haack, D1993] C. Haack. Die Hyperlimeskonstruktion. Master's thesis, Technische Hochschule Darmstadt, 1993. 86 pp.

[Heckmann, D1990] R. Heckmann. *Power Domain Constructions*. PhD thesis, Universität des Saarlandes, 1990.

[Henhapl, D1995] B. Henhapl. Ein voll abstraktes, extensionales Modell für endliches PCF. Master's thesis, Technische Hochschule Darmstadt, 1995. 57pp.

[Holwerda, D1993] H. Holwerda. *Topology and Order. Some Investigations Motivated by Probability Theory*. PhD thesis, Universiteit Nijmegen, 1993.

[Hoofman, D1992] R. Hoofman. *Non-stable Models of Linear Logic*. PhD thesis, University of Utrecht, 1992.

[Huth, D1990] M. Huth. *Projection-Stable and Zero Dimensional Domains*. PhD thesis, Tulane University, New Orleans, 1990.

[Jamison, D1974] R.E. Jamison. *A General Theory of Convexity*. Doctoral Dissertation, University of Washington, 1974. 127 pp.

[Jones, D1980] L.W. Jones, Jr. *Freeness and Continuity in Semilattices*. Doctoral Dissertation, Tulane University, New Orleans, 1980. vi+146 pp.

[Jones, D1989] C. Jones. *Probabilistic Non-determinism*. PhD thesis, University of Edinburgh, 1989. LFCS report ECS-LFCS-90-105 (also published as Department of Computer Science Report CST-63-90). 201 pp.

[Jung, D1983] A. Jung. Stetige Verbände und ein Approximationssatz für oberhalbstetige Funktionen. Master's thesis, Technische Hochschule Darmstadt, 1983. ii+40 pp.

[Jung, D1988] A. Jung. *Cartesian Closed Categories of Domains*. PhD thesis, Technische Hochschule Darmstadt, 1988.

[Jung, D1994] A. Jung. *Mathematical Structures in the Theory of Programming Languages*. Habilitationsschrift, Technische Hochschule Darmstadt, February 1994. 162 pp.

[Kegelmann, D1995] M. Kegelmann. Faktorisierungssysteme auf Bereichen. Master's thesis, Technische Hochschule Darmstadt, September 1995. 63pp.

[Kegelmann, D1999] M. Kegelmann. *Continuous Domains in Logical Form*. PhD thesis, University of Birmingham, June 1999. 178 pp.

[Kerth, D1995] R. Kerth. *Isomorphisme et équivalence équationnelle entre modèles du lambda-calcul*. PhD thesis, Université Paris 7-Denis Diderot, Paris, France, 1995.

[Kirch, D1993] O. Kirch. Bereiche und Bewertungen. Master's thesis, Technische Hochschule Darmstadt, 1993. 77pp.

[Knobel, D1990] A. Knobel. *Constructive λ-Models*. PhD thesis, University of Edinburgh, 1990. LFCS report ECS-LFCS-90-120 (also published as Department of Computer Science Report CST-68-90).

[Kristiansen, D1993] L. Kristiansen. *Totality in Qualitative Domains*. PhD thesis, University of Oslo, 1993.

[Lawson, D1967] J.D. Lawson. *Vietoris Mappings and Embeddings of Topological Semilattices*. Doctoral Dissertation, University of Tennessee, 1967. 99 pp.

[Longley, D1994] J.R. Longley. *Realizability Toposes and Language Semantics*. PhD thesis, University of Edinburgh, 1994.

[Martin, D2000] K. Martin. *A Foundation for Computation*. PhD thesis, Tulane University, New Orleans, 2000.

[McCarty, D1984] D.C. McCarty. *Realizability and Recursive Mathematics*. D. Phil. thesis, Oxford University, 1984.

[Meyer, D1990] K. Meyer. Lokale Bereiche. Master's thesis, Technische Hochschule Darmstadt, 1990. 79pp.

[Milne, D1973] R.E. Milne. *The Formal Semantics of Computer Languages and Their Implementations*. D. Phil. thesis, University of Cambridge 1973.

[Moggi, D1988] E. Moggi. *The Partial Lambda-Calculus*. PhD thesis, University of Edinburgh, 1988. LFCS report ECS-LFCS-88-63 (also published as Department of Computer Science Report CST-53-88).

[Mulmuley, D1985] K. Mulmuley. *Full Abstraction and Semantic Equivalence*. PhD thesis, Carnegie Mellon University, 1985.

[Nait-Abdallah, D1980] A. Nait-Abdallah. *Faisceaux et sémantique des programmes*. Thèse d'etat, University of Paris, 1980. Technical Report, no. cs-82-08, University of Waterloo, 1982, 236 pp.

[Nicolaou, D1993] N. Nicolaou. Fractal image processing on the digitised screen. Master's thesis, Imperial College, University of London, September 1993.

[Nielsen, D1984] F. Nielsen. *Abstract Interpretation Using Domain Theory*. PhD thesis, University of Edinburgh, 1984. Department of Computer Science Report CST-31-84.

[Niño Salcedo, D1981] J. Niño Salcedo. *On Continuous Posets and Their Applications*. Doctoral Dissertation, Tulane University, New Orleans, 1981. iii+105 pp.

[Nüßler, D1992] K.-J. Nüßler. *Ordnungstheoretische Modelle für nicht-deterministische Programmiersprachen*. PhD thesis, Universität GH Essen, 1992.

[Ong, D1988] C.-H.L. Ong. *The Lazy λ-Calculus: an Investigation into the Foundations of Functional Programming*. PhD thesis, Imperial College, University of London, 1988.

[Parry, D1996] J. Parry. A new algorithm for computing fractal dimension. Master's thesis, Department of Computing, Imperial College, University of London, 1996.

[Phoa, D1991] W. Phoa. *Domain Theory in Realizability Toposes*. D. Phil. thesis, University of Cambridge, 1991.

[Platek, D1964] R. Platek. *New Foundations for Recursion Theory*. PhD thesis, Stanford University, 1964.

[Potts, D1998] P.J. Potts. *Exact Real Arithmetic Using Möbius Transformations*. PhD thesis, Imperial College, University of London, 1998.

[Puhlmann, D1990] H. Puhlmann. Verallgemeinerung relationaler Schemata in Datenbanken mit Informationsordnung. Master's thesis, Technische Hochschule Darmstadt, 1990. 78 pp.

[Puhlmann, D1991] H. Puhlmann. A semantics for generalized database relations. Master's thesis, Imperial College, University of London, 1991.

[Puhlmann, D1995] H. Puhlmann. *Die Snack-Potenzkonstruktion: Grundlage einer Semantik genesteter Datenbanken*. PhD thesis, Technische Hochschule Darmstadt, July 1995. 88 pp.

[Reus, D1995] B. Reus. *Program Verification in Sythetic Domain Theory*. PhD thesis, Ludwig-Maximilians Universität München, 1995. Shaker Verlag Aachen, 1996.

[Rosolini, D1986] G. Rosolini. *Continuity and Effectiveness in Topoi*. D. Phil. thesis, Oxford University, 1986.

[Rothe, D1991] M. Rothe. Retraktionen auf stetigen Bereichen. Master's thesis, Technische Hochschule Darmstadt, 1991. 34pp.

[Schalk, D1990] A. Schalk. Bereiche und topologischer Zusammenhang. Master's thesis, Technische Hochschule Darmstadt, 1990. 70 pp.

[Schalk, D1993] A. Schalk. *Algebras for Generalized Power Constructions*. Doctoral thesis, Technische Hochschule Darmstadt, 1993. 174 pp.

[Schneider, D1998] K.U. Schneider. Dualitäten für Z-distributive und Z-algebraische Verbände. Master's thesis, Universität Hannover, 1998.

[Schulz, D1997] H. Schulz. Berechenbarkeit auf reellen Zahlen – ein Vergleich. Master's thesis, University of Siegen, Germany, 1997.

[Schwarz, D1983] F. Schwarz. *Funktionenräume und exponentiale Objekte in punktetrennend initialen Kategorien*. Doctoral Dissertation, Universität Bremen, 1983. vi+193 pp.

[Simpson, D1994] A.K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh, 1994. LFCS report ECS-LFCS-94-308 (also published as Department of Computer Science Report CST-114-94).

[Spreen, D1985] D. Spreen. *Rekursionstheorie auf Teilmengen partieller Funktionen*. Habilitationsschrift, University of Aachen, 1985.

[Stoughton, D1986] A. Stoughton. *Abstract Models of Programming Languages*. PhD thesis, University of Edinburgh, 1986. Department of Computer Science Report CST-40-86. (A revision, with additions, of 123 pp., was published under the same title in the series *Research Notes in Theoretical Computer Science*, Pitman/Wiley, 1988.)

[Streicher, D1982] T. Streicher. Scott domains, computability and the definability problem. Master's thesis, Johannes Kepler Universität Linz, 1982.

[Strüder, D1995] C. Strüder. Stetige Operationen auf kontinuierlichen Bereichen. Master's thesis, Technische Hochschule Darmstadt, April 1995. 38 pp.

[Sünderhauf, D1991] P. Sünderhauf. Vervollständigung quasi-uniformer Räume. Master's thesis, Technische Hochschule Darmstadt, 1991. 64pp.

[Sünderhauf, D1994] P. Sünderhauf. *Discrete Approximation of Spaces*. PhD thesis, Technische Universität Darmstadt, 1994. 91 pp.

[Tang, D1974] A. Tang. *Recursion Theory and Descriptice Set Theory in Effectively Given T_0-Spaces*. PhD thesis, Princeton University, 1974.

[Taylor, D1983] P. Taylor. Applications of continuous lattices to λ-calculus and denotational semantics. Master's thesis, University of Cambridge, Department of Pure Mathematics and Mathematical Statistics, May 1983. 45 pp.

[Taylor, D1986] P. Taylor. *Recursive Domains, Indexed Category Theory and Polymorphism*. D. Phil. thesis, University of Cambridge 1986.

[Tiller, D1980] J. Tiller. *Continuous Lattices and Convexity Theory*. Doctoral Dissertation, McMaster University, Hamilton, Ontario, 1980. vii+106 pp.

[Tix, D1995] R. Tix. Stetige Bewertungen auf topologischen Räumen. Master's thesis, Technische Universität Darmstadt, June 1995. ⟨http://www.mathematik.tu-darmstadt.de/ags/ag14/papers/tix/⟩, 51 pp.

[Tix, D1999] R. Tix. *Continuous D-Cones: Convexity and Powerdomain Constructions*. PhD thesis, Technische Universität Darmstadt, 1999.

[Turi, D1996] D. Turi. *Functorial Operational Semantics and Its Denotational Dual*. PhD thesis, Vrije Universiteit, Amsterdam, 1996.

[van Breugel, D1994] F. van Breugel. *Topological Models in Comparative Semantics*. PhD thesis, Vrije Universiteit, Amsterdam, 1994.

[Vuillemin, D1974] J. Vuillemin. *Syntaxe, sémantique et axiomatique d'un langage de programmation simple*. Thèse d'état, University of Paris, 1974.

[Waagbø, D1997] G. Waagbø. *Domains-with-Totality Semantics for Intuitionistic Type Theory*. PhD thesis, University of Oslo, 1997.

[Wagner, D1994] K.R.Wagner. *Solving Domain Equations with Enriched Categories*. PhD thesis, Carnegie Mellon University, 1994.

[Weidner, D1995] G. Weidner. Bewertungen auf stetigen Bereichen. Master's thesis, Technische Hochschule Darmstadt, 1995. 45pp.

[Wilson, D1978] R.L. Wilson. *Intrinsic Topologies on Partially Ordered Sets and Results on Compact Semigroups*. Doctoral Dissertation, University of Tennessee, 1978. 167 pp.

[Winskel, D1981] G. Winskel. *Events in Computation*. PhD thesis, University of Edinburgh, 1981. Department of Computer Science Report CST-10-80.

[Zhang, D1989] G.-Q. Zhang. *Logics of Domains*. D. Phil. thesis, University of Cambridge, 1989. Computer Laboratory Technical Report No. 185.

[Zhang, D1993] H. Zhang. *Dualities of Domains*. PhD thesis, Tulane University, New Orleans, 1993.

Memos Circulated in the Seminar on Continuity in Semilattices (SCS)

[SCS 1] 19. January 1976, J.D. Lawson. More notes on spreads.

[SCS 2] 19. January 1976, K.H. Hofmann. Notes on Memo [SCS 1].

[SCS 3] 29. January 1976, K. Keimel. Equationally compact SENDOs are retracts of compact ones.

[SCS 4] 30. March 1976, D.S. Scott. Note on continuous lattices.

[SCS 5] 19. April 1976, K.H. Hofmann. Notes on chains in *CL*-objects.

[SCS 6] 28. May 1976, J.H. Carruth *et al.* More notes on chains in *CL*-objects.

[SCS 7] 15. June 1976, J.H. Carruth *et al.* Still more notes on chains in *CL*-objects.

[SCS 8] 28. June 1976, K.H. Hofmann and M.W. Mislove. On the theorem of Lawson's that all compact locally connected finite dimensional semilattices are *CL*.

[SCS 9] 7. July 1976, K.H. Hofmann and M.W. Mislove. Commentary on Scott's function spaces.

[SCS 10] 12. July 1976, J.D. Lawson. Points with small semilattices.

[SCS 11] 20. July 1976, K.H. Hofmann and M.W. Mislove. Errata and corrigenda to Memo [SCS 9].

[SCS 12] 1. August 1976, G. Gierz, K.H. Hofmann, K. Keimel, and M.W. Mislove. Relations with the interpolation property and continuous lattices.

[SCS 13] 10. August 1976, K. Keimel. Complements to relations with the interpolation property and continuous lattices.

[SCS 14] 18. August 1976, M.W. Mislove. On Memo [SCS 10].

[SCS 15] 23. August 1976, D.S. Scott. Continuous lattices and universal algebra.

[SCS 16] 1. September 1976, K.H. Hofmann and J.R. Liukkonen. The random unit interval (another example of a *CL*-object).

[SCS 17] 20. September 1976, K.H. Hofmann. The space of lower semicontinuoue functions into a *CL*-object, applications (Part I): copowers in *CL*.

[SCS 18] 21. September 1976, A. Day. Continuous lattices and universal algebra.

[SCS 19] 30. September 1976, K. Keimel and M.W. Mislove. Several remarks: 1. The closed subsemilattices of a continuous lattice form a continuous lattice; 2. When do the prime elements of a distributive lattice form a closed subset; 3. On lower semicontinuous function spaces; 4. On the continuity of the congruence lattice of a continuous lattice.

[scs 20] 23. October 1976, K.H. Hofmann. More on the coproduct. Errata and addenda.
[scs 21] 10. November 1976, J.H. Carruth, C.E. Clark, E. Evans, J.W. Lea, Jr, and R.L. Wilson. $\leq (n)$.
[scs 22] 10. November 1976, G. Gierz. Representation of colimits in *CL*. Parts I and II.
[scs 23] 19. November 1976, J.D. Lawson. Non-continuous lattices.
[scs 24] 19. November 1976, K.H. Hofmann and K. Keimel. Editorial.
[scs 25] 23. November 1976, K.H. Hofmann. Observations.
[scs 26] 30. November 1976, D.S. Scott. A reply to an editorial.
[scs 27] 8. December 1976, M.W. Mislove. Closure operators and kernel operators in *CL*.
[scs 28] 15. December 1976, K. Keimel and M.W. Mislove. The lattice of open subsets of a topological space.
[scs 29] 28. December 1976, K.H. Hofmann and O. Wyler. On the closedness of the set of primes in a continuous lattice.
[scs 30] 4. January 1977, J.D. Lawson. Continuous semilattices and duality.
[scs 31] 13. January 1977, K.H. Hofmann. The lattice of ideals of a C^*-algebra.
[scs 32] 8. February 1977, K.H. Hofmann and J.D. Lawson. The spectral theory of distributive continuous lattices.
[scs 33] 14. March 1977, K.H. Hofmann and J.D. Lawson. Complement to Memo [scs 32].
[scs 34] 8. April 1977, G. Gierz and K.H. Hofmann. On complete lattices for which $\mathcal{O}(L)$ is continuous – a lattice theoretical characterization of *CS*.
[scs 35] 18. April 1977, O. Wyler. Dedekind complete posets and Scott topologies.
[scs 36] 16. May 1977, D.S. Scott. Quotients of distributive continuous lattices: a result of S.A. Jalali.
[scs 37] 20. May 1977, O. Wyler. Comments on the spectral theory of continuous lattices.
[scs 38] 1. July 1977, M. Kamara. Treillis continus et treillis complètement distributifs.
[scs 39] 15. July 1977, A.R. Stralka. Quotients of cubes.
[scs 40] 28. July 1977, M.W. Mislove. A new approach to some results of Lawson, Gierz and Hofmann.
[scs 41] 25. September 1977, K.H. Hofmann and D.S. Scott. An exercise on the spectrum of function spaces.
[scs 42] 2. November 1977, G. Gierz and J.D. Lawson. Generalized continuous lattices.
[scs 43] 18. January 1978, K.H. Hofmann. Locally quasicompact sober spaces are Baire spaces.
[scs 44] 9. February 1978, H. Bauer and K. Keimel. Remark on the Memo [scs 43].
[scs 45] 15. April 1978, H. Bauer. Antichains and equational compactness.
[scs 46] 19. May 1978, G. Gierz, J.D. Lawson, and M.W. Mislove. A result about $\mathcal{O}(X)$.
[scs 47] 28. May 1978, K.H. Hofmann. Equivalence des espaces de Batbédat et des treillis algébriques.
[scs 48] 29. November 1978, K.H. Hofmann and J. Niño. Projective limits in *CL* and Scott's construction.
[scs 49] 30. November 1978, K.H. Hofmann and F. Watkins. A review of a theorem of Dixmier's.
[scs 50] 30. May 1979, K.H. Hofmann and L.W. Jones, Jr. Scott continuous closure operators and modal operators. More self functors to which the Scott construction applies.

[scs 51] 30. May 1979, K.H. Hofmann and F. Watkins. A new lemma on primes and a topological characterization of the category *DCL* of continuous Heyting algebras and *CL*-morphisms.

[scs 52] 11. June 1979, K.H. Hofmann and K. Keimel. Bemerkungen zum "Neuen Lemma".

[scs 53] 27. November 1979, K.H. Hofmann. Completely distributive algebraic lattices.

[scs 54] 29. February 1980, K.H. Hofmann. *CL*-projective limits of distributive continuous lattices are distributive.

[scs 55] 19. March 1980, J.R. Isbell. MC direct limits.

[scs 56] 26. October 1980, K.H. Hofmann and K. Keimel. On a question of O. Wyler.

[scs 57] 19. November 1980, K.H. Hofmann and J.D. Lawson. On the duality of semilattices.

[scs 58] 10. September 1981, R.-E. Hofmann. The *CL*-compactification of a continuous poset. Abstract.

[scs 59] 22. November 1981, J.R. Isbell. Sober quotients.

[scs 60] 23. November 1981, K.H. Hofmann. Miniworkshop on continuous lattices at Tulane University, Nov. 19–21, 1981.

[scs 61] 24. November 1981, K.H. Hofmann. The category *CD* of completely distributive lattices and their free objects.

[scs 62] 1. December 1981, R.-E. Hoffmann. Continuous posets: injective hull and MacNeille completion.

[scs 63] 11. March 1982, R.-E. Hoffmann. The Fell compactification. Abstract.

[scs 64] 28. May 1982, K.H. Hofmann and M.W. Mislove. A continuous poset whose compactification is not a continuous poset. The square is the injective hull of a discontinuous *CL*-compact poset.

[scs 65] 8. June 1982, K.H. Hofmann. Bernhardina – the essential hull revisited.

[scs 66] 2. July 1982, K.H. Hofmann and M.W. Mislove. Revision of [scs 64].

[scs 67] 5. July 1982, J.D. Lawson. A strict extension of previous results on essential extensions.

[scs 68] 16. July 1982, K.H. Hofmann. A remark on the complete distributivity of algebraic lattices.

[scs 69] 23. July 1982, M. Erné. Order generation and distributive laws in complete lattices.

[scs 70] 25. July 1982, M. Erné. Freedom for completely distributive lattices (over continuous posets).

[scs 71] 28. July 1982, R.-E. Hoffmann. Two remarkable continuous posets and an appendix to "The *CL*-compactification and the injective hull of a continuous poset".

[scs 72] 1. August 1982, M. Erné. Algebraic posets and compactly generated posets.

[scs 73] 14. September 1982, M. Erné and H. Gatzke. Meet-continuous lattices in which meet is not continuous.

[scs 74] 12. November 1982, H. Dobbertin. Distributive semilattices.

[scs 75] 18. November 1982, H. Dobbertin. Distributive semilattices, Heyting algebras, and *V*-homomorphisms.

[scs 76] 9. January 1983, R.-E. Hoffmann. The trace of the weak topology and of the Γ-topology of L^{op} coincide on the pseudo-meet-prime elements of a continuous lattice L.

[SCS 77] 12. January 1983, K.H. Hofmann. On the pseudo-spectrum of a continuous distributive lattice.

[SCS 78] 14. February 1983, R.-E. Hoffmann. "Duality" for distributive compact multiplicative continuous lattices.

[SCS 79] 14. July 1983, O. Wyler. The lower topology for continuous lattices is a monadic functor.

[SCS 80] 1. September 1983, O. Wyler. Compact ordered spaces and prime Wallman compactifications: summary of results.

[SCS 81] 23. November 1983, G. Gierz, J.D. Lawson, and A.R. Stralka. Intrinsic topologies on semilattices of finite breadth.

[SCS 82] 1. December 1983, G. Gierz and A.R. Stralka. Compactifying distributive lattices.

[SCS 83] 5. December 1983, K. Keimel. Continuous lattices, general convexity spaces, and a fixed-point theorem.

[SCS 84] 6. December 1983, G. Gierz and A.R. Stralka. The Zariski topology on semilattices and essential extensions.

[SCS 85] 19. January 1984, K. Keimel. The space of compact convex subsets of a locally convex topological vector space.

[SCS 86] February 1984, K. Keimel. Korovkin theorems for multivalued functions.

[SCS 87] 23. February 1984, J. Tiller. The way-below relation is not what I think.

[SCS 88] 1. April 1984, K. Keimel. A proof of a theorem of BB.

[SCS 89] 1. May 1984, M. Erné. Continuity concepts for partially ordered sets.

[SCS 90] 2. May 1984, H. Dobbertin. About polytopes of valuations on finite distributive lattices.

[SCS 91] 7. May 1984, M. Erné. Compactly generated and continuous closure systems.

[SCS 92] 14. May 1984, M. Erné. Products of continuous partially ordered sets.

[SCS 93] 15. May 1984, H. Dobbertin. Refinement monoids.

[SCS 94] 3. July 1984, O. Wyler. Algebraic theories for proper filter monads.

[SCS 95] 10. March 1985, M. Erné. Fixed point constructions for standard completions.

[SCS 96] 18. March 1985, M. Erné. Generators and weights of completely distributive lattices.

[SCS 97] 5. October 1985, M. Erné. $\mathcal{Z}$-continuous posets, $\mathcal{Z}$-ary closure spaces and generalized soberness.

[SCS 98] 4. June 1986, M. Erné. $\mathcal{Z}$-continuity, $\mathcal{Z}$-hypercompactness and complete distributivity.

List of Symbols

$\bot$	Bottom or zero of a poset *5* O-1.8
$\nabla_L(a)$	$\uparrow a \cap \operatorname{Spec} L$ *408* V-4.1
$\prec^{\text{sup}}$	The sup closure of $\prec$ on L *72* I-1.29
$\top$	Top or identity of a poset *5* O-1.8
$\nearrow(T)$	Set of suprema of directed subsets of T *448* VI-2.8
$\nearrow_\omega(T)$	Set of suprema of directed countable subsets of T *448* VI-2.8
$\downarrow X$	Lower set of X *2* O-1.3
$\downarrow x$	Lower set of the point x *2* O-1.3
$\uparrow X$	Upper set of X *2* O-1.3
$\uparrow x$	Upper set of the point x *2* O-1.3
$\twoheaduparrow M$	Way-above set of M *72* I-1.30(iii)
$\twoheaduparrow x$	Way-above set of the point x *51 remarks following* I-1.2
$\twoheaddownarrow x$	Way-below set of the point x *51 remarks following* I-1.2
$(S \to T)$	Order preserving maps from S to T *91* I-2.21(i)
$[A]$	The order convex hull of A *441* VI-1.5
$[S \to T]$	Maps in $(S \to T)$ preserving directed sups *91* I-2.21(iii)
$[S \to T]$	The **dcpo** of Scott-continuous maps $f\colon S \to T$ *162* II-2.7
$[X, Y]$	$\mathit{TOP}(X, Y)$ with the Isbell topology *188* II-4.1
$\lvert L\rvert\colon I \to C$	Constant functor with value L *305* IV-4.1
$\bigvee^{\uparrow} X$	Supremum of the directed set X *1* O-1.1
$\bigwedge X$	Infimum of the set X *1* O-1.1
$\bigvee X$	Supremum of the set X *1* O-1.1
2	The Sierpinski space *136* II-1.5(3)
2^X	Power set of X *13* O-2.7(1)
A^-	Topological closure of A *42 remarks preceding* O-5.1
$\operatorname{App}(L)$	Approximating auxiliary relation on L *59* I-1.13
$\operatorname{Aux}(L)$	Auxiliary relation on L *57* I-1.11
$\mathcal{C}(L)$	All closure systems on L *29 remarks preceding* O-3.13
$\mathrm{C}(X, \mathbb{R}^*)$	Continuous extended real-valued functions on X *17* O-2.10
cBa	Complete Boolean algebra *12* O-2.6
(CD)	Complete distributive law *85* I-2.8

List of Categories

$DCPO_G$	Category of **dcpo**s and directed sup preserving upper adjoints *270* IV-1.9
DCPOFILT	Category of **dcpo**s and maps under which preimages of open filters are open filters *281* IV-2.2
DL	Continuous distributive lattices and *CL*-maps preserving spectra *422 remarks preceding* V-5.20
DLat	Distributive lattices with 0 and 1 and all 0 and 1 preserving lattice maps *423* V-5.22
DOM	Domains and Scott-continuous maps *158* II-2.2
DOM_D	Domains and lower adjoints d preserving the way-below relation *270* IV-1.9
DOM_G	Domains and directed sup preserving upper adjoints *270* IV-1.9
DOMFILT	Domains and maps under which preimages of open filters are open filters *281* IV-2.2
FRM	Frames and maps preserving arbitrary sups and finite infs *170 remarks preceding* II-2.24
FRM_0	Full subcategory of *FRM* of lattices where the primes order generate *411 remarks preceding* V-4.7
FS	*FS*-domains and Scott-continuous functions *168* II-2.19
GRAPH	Sup semilattices and monotone relations *175* II-2.36
H	Distributive continuous and join continuous lattices L such that Spec L is closed with respect to arbitrary sups, and maps preserving arbitrary sups, finite infs, the way-below relation and whose adjoint preserves finite sups *519 remarks preceding* VII-4.5
INF	Complete lattices and inf preserving maps *266* IV-1.1
$INF^{\uparrow}$	Complete lattices and inf and directed sup preserving maps *270* IV-1.9
LAT	Lattices with 0 and lattice homomorphisms preserving 0 *274* IV-1.17
LCSOB	Locally compact sober spaces and proper maps *422 remarks preceding* V-5.20
LDOM	L-domains and Scott-continuous functions *173* II-2.32
POID	Posets and maps for which preimages of ideals are ideals *272* IV-1.13
POSET	Posets and monotone maps *5* O-1.9
$POSET_D$	Posets and lower adjoints *266* IV-1.1
$POSET_G$	Posets and upper adjoints *266* IV-1.1
SCFRM	Stably continuous frames and functions preserving arbitrary sups, finite infs and the relation $\ll$ *488* VI-7
$SCFRM_1$	Stably continuous frames with compact 1 and functions preserving arbitrary sups, finite infs and the relation $\ll$ *488* VI-7
SCTOP	Stably compact spaces and proper maps *482* VI-6.22
SEM	Sup semilattices with 0 and sup semilattice homomorphisms preserving 0 *272* IV-1.13
SEMI	Sup semilattices with 0 and monotone maps *175* II-2.36
SET	Sets and functions *161 remarks preceding* II-2.5

SLCTOP	Stably locally compact spaces and proper maps *488* VI-7
SOB	Sober spaces and continuous maps *43* O-5.6
SUP	Complete lattices and sup preserving maps *158* II-2.2, *266* IV-1.1
SUP$^\wedge$	Complete lattices and arbitrary sups and finite infs preserving maps *411 remarks following* V-4.6
SUP^0	Complete lattices and *SUP*-maps d where for each Scott open set U the set $\uparrow d(U)$ is Scott open *270* IV-1.9
TCPOSP	Totally order disconnected compact pospaces and monotone continuous maps *491* VI-7.10
TOP	T_0 spaces and continuous maps *42* O-5.1
UPS	Complete lattices and Scott-continuous maps *158* II-2.2

Index

A

B

C

D

E

F

G

H

I

J

K

L

M

T

U