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Electronic Notes in Theoretical Computer Science

Electronic Notes in Theoretical Computer Science 257 (2009) 35–54

www.elsevier.com/locate/entcs

Bicontinuous Domains and Some Old Problems in Domain Theory

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Abstract

K. Martin and P. Panangaden [12] have made the interesting observation that the causal order in general relativity theory closely relates to domain theoretical notions. From the causal structure of models of space-time which are called strongly hyperbolic they extract the notion of a strongly hyperbolic poset. These are in particular bicontinuous, that is, continuous posets with the property that they are also continuous with respect to the opposite order. Not much is known about bicontinuous posets. In this paper we collect known results, we exhibit substantial examples, we show that these examples are connected with open problems in domain theory, and we indicate directions of research.

Keywords: Domains theory, bicontinuous posets, strongly hyperbolic posets, ordered manifolds.

1 Introduction

K. Martin and P. Panangaden [12] have made the interesting observation that the causal order in general relativity theory closely relates to domain theoretical notions. From the order theoretical structure of models of space-time which are called strongly hyperbolic they extract the notion of a strongly hyperbolic poset. In Section 3 this notion will be presented, although in a different terminology. It is the aim of this paper to exhibit a large range of examples, to illustrate that these examples are closely connected to open problems in domain theory and to indicate an interesting direction for research.

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doi:10.1016/j.entcs.2009.11.025

In section 4 we consider vector space orderings on $\mathbb{R}^n$. These correspond bijectively to (convex, pointed) cones C in $\mathbb{R}^n$. If these cones are closed and have inner points, then $\mathbb{R}^n$ becomes a strongly hyperbolic poset in the above sense. As there is nothing hyperbolic about these examples I prefer to adopt another terminology, although the term *strongly hyperbolic* sounds impressive.

It seems worthwhile to investigate these orders on $\mathbb{R}^n$ from the point of view of domain theory. In fact, some old unsolved problems of domain theory might find their solution in this setting, as I will indicate.

Every continuous poset can be embedded in a continuous dcpo via the round ideal completion. The round ideals of the vector space orderings on $\mathbb{R}^n$ depend essentially on the geometry of the positive cone. The same is true for the opposite notion of a round filter completion. It is desirable to combine these two completions. For this I propose a procedure extending a construction of Martin and Pananagaden. It would be nice to investigate whether this completion is a compactification of $\mathbb{R}^n$ and embeds $\mathbb{R}^n$ in a manifold with boundary. The same procedure might then be applied to strongly hyperbolic models of space-time, but the case of $\mathbb{R}^n$ maybe easier to manage before attacking the more general setting.

2 Preliminaries

If not explained here, we use the terminology of [6]. On a partially ordered set P, a poset, for short, we use the following topologies:

ω the lower topology: the principal filters $\uparrow a$ are a subbasis for the closed sets,

σ the Scott topology: closed sets are the sets C such that $a \leq \bigvee^{\uparrow}_i a_i$ implies $a \in C$, whenever a_i is a directed family in C,

$\lambda = \sigma \vee \omega$ the Lawson topology: the join of the lower and the Scott topology, their opposites:

$\nu = \omega^{op} S$ the upper topology: the principal ideals $\downarrow a$ are a subbasis for the closed sets,

σ^{op} the dual Scott topology: the Scott topology w.r.t. the opposite order,

λ^{op} the dual Lawson topology,

and the following 'symmetric' topologies:

$\sigma^{bi} = \sigma \vee \sigma^{op}$	the bi-Scott topology: the join of the Scott topology with its dual σ^{op}
$\iota = \nu \vee \omega$	the interval topology: the join of the upper and the lower topology.

The sets $\downarrow a$ and $\uparrow b$ generate the closed sets of the interval topology. Thus the closed intervals

$$[a, b] = \uparrow a \cap \downarrow b = \{x \mid a \leq x \leq b\}$$

are closed in the interval topology. [2]

The Scott topology is finer than the upper topology, and the dual Scott topology is finer than the lower topology. The bi-Scott topology is finer than both the Lawson and the dual Lawson topology, and they are both finer than the interval topology.

A poset P is said to be *(conditionally) directed complete*, a *(c)dcpo*, for short, if every (upper bounded) directed subset D has a least upper bound, denoted by $\bigvee^{\uparrow} D$. The order dual is the notion of a *(conditionally) filtered complete* poset, a *(c)fcpo*, for short. A poset that is both (conditionally) directed and filtered complete will be called *(conditionally) bicomplete*, a *(c)bicpo*, for short.

We will use the following notions for a dcpo D:

Definition 2.1 A Scott-continuous function $f \colon D \to D$ is *finitely separated from the identity* if there is a finite set $F \subseteq D$ such that for every $x \in D$ there is a $y \in F$ such that $f(x) \leq y \leq x$. If there is a directed family (f_i) of functions f_i which are finitely separated from the identity with $\mathrm{id} = \sup_i f_i$, then D is called an *FS-domain*.

Definition 2.2 D is *bifinite* if it has a directed family of Scott-continuous retractions ρ_i with finite range such that $\mathrm{id} = \sup_i \rho_i$.

3 Bicontinuous Posets

In this section we introduce the basic notions of bicontnuity, joint bicontinuity and linked bicontinuity.

Recall that in a poset P the *way-below relation* $a \ll b$ is defined as follows: For every directed subset D which has a least upper bound, $b \leq \bigvee^{\uparrow} D$ implies

[2] Martin and Panangaden [12] use the term *interval topology* for what we call bi-Scott topology.

$a \leq d$ for some $d \in D$. One says that P is a *continuous poset* if, for every $b \in P$ the set $\Downarrow b = \{a \in P \mid a \ll b\}$ is directed and $b = \bigvee^{\uparrow} \Downarrow b$.

Dually, one can define the *way-above relation* $a \gg_d b$: For every filtered subset F which has a greatest lower bound, $b \geq \bigwedge_{\downarrow} F$ implies $a \geq f$ for some $f \in F$. One says that P is a *dually continuous poset* if, for every $a \in P$ the set $\Uparrow_d a = \{b \in P \mid a \ll_d b\}$ is filtered and $a = \bigwedge_{\downarrow} \Uparrow_d a$.

Definition 3.1 A poset P that is continuous and dually continuous is called *bicontinuous.* [3]

The Scott topology on a bicontinuous poset has as a basis for the opens the sets of the form $\Uparrow a$, the opposite Scott topology the sets of the form $\Downarrow_d a$. The *bi-Scott* topology is generated by the Scott topology and its opposite; it has the *open intervals*

$$]a, b[= \Uparrow a \cap \Downarrow_d b = \{x \mid a \ll x \ll_d b\}$$

as a basis for the open sets. Because of the interpolation property of the way-below and the way-above relation, every element x also has a neighborhood basis of *closed intervals*

$$[a, b] = \uparrow a \cap \downarrow b = \{x \mid a \leq x \leq b\}, \quad a \ll x \ll_d b$$

As the principal ideals $\downarrow b$ are closed for the Scott topology and as similarly the principal filters are closed for the opposite Scott topology, we have that indeed:

Lemma 3.2 *The closed intervals are closed for the bi-Scott topology.*

Thus, in a bicontinuous poset with the bi-Scott topology every element has a neighborhood basis of closed neighborhoods, i.e., the bi-Scott topology is regular.

Recall the an *ordered space* in the sense of Nachbin [13] is a topological space together with a partial order the graph of which is closed.

Proposition 3.3 *Let C be a Scott-closed set in a continuous poset P and Q a Scott-compact subset of P disjoint from C. Then there is a Scott-open set U containing Q and a lower open set V containing C disjoint from U. In particular, P is an ordered space for the Lawson topology.*

Proof. For every $a \in Q$ there is an $a' \ll a$ with $a' \notin C$. As the Scott-open sets $\Uparrow a'$ cover the Scott-compact set Q, already finitely many of them cover

[3] Note that in [12] the term *bicontinuous* is used for what we call *jointly bicontinuous*.

Q. Thus there is a finite set F such that $Q \subseteq \uparrow F = U$ and $V = P \setminus \uparrow F$ together with U satisfies the first claim. Further, $U \times V$ is an Lawson-open neighborhood of $Q \times C$ not meeting the graph $G_{\leq} = \{(a, b) \mid a \leq b\}$ of the order. □

Moreover, If A is a Scott-closed set and $a \notin A$, then there is a dually Scott-open set U containing A and a Scott-open set V containing a which are disjoint.

One should note that the way-below relation in a bicontinuous poset need not be the opposite of the way-above relation in the sense that $a \ll b$ does not imply $b \gg_d a$ or vice-versa. Indeed, in the unit interval $[0, 1]$ with the usual order is bicontinuous and one has $a \ll b$ iff $a < b$ or $a = b = 0$ whilst $a \gg_d b$ iff $a > b$ or $a = b = 1$, in particular $0 \ll 0$ but $0 \not\gg_d 0$. The situation becomes more complicated on the unit square $[0, 1]^2$ with the coordinatewise order $(a_1, a_2) \leq (b_1, b_2)$ iff $a_1 \leq b_1$ and $a_2 \leq b_2$: Here we have $(a_1, a_2) \ll (b_1, b_2)$ iff $a_1 \ll b_1$ and $a_2 \ll b_2$, explicitly, $a_1 < b_1, a_2 < b_2$ or $a_1 = b_1 = 0, a_2 < b_2$ or $a_1 < b_1, a_2 = b_2 = 0$ or $a_1 = b_1 = a_2 = b_2 = 0$. Thus, $(0, \frac{1}{3}) \ll (0, \frac{2}{3})$ but $(0, \frac{1}{3}) \not\gg_d (0, \frac{2}{3})$. On $\mathbb{R}$ or $\mathbb{R}^2$ the situation is nice in the sense that $a \ll b$ if $b \gg_d a$. This indicates that the 'boundary' is responsible for the mismatch.

Another useful counterexample is the powerset $\mathfrak{P}(X)$ of a infinite set X ordered by inclusion; here $A \ll B$ is equivalent to A being finite and $A \subseteq B$, whilst and $A \ll_d B$ is equivalent to B being cofinite and $A \subseteq B$.

Definition 3.4 A poset is called *jointly bicontinuous* if it is bicontinuous and if the way-below relation coincides with the dual way-below relation. [4]

A bicontinuous poset is locally compact (for the bi-Scott topology) iff each of its points has a closed interval as a neighborhood which is compact or, equivalently, if for every x there are elements $a \ll x \ll_d b$ such that the closed interval $[a, b]$ is compact.

On every continuous poset, the Lawson topology is Hausdorff. On a bicontinuous poset it is coarser than the bi-Scott topology. Thus, on the bi-Scott compact subsets of a bicontinuous poset, both topologies agree. Thus, if a bicontinuous poset is locally compact for the bi-Scott topology, the latter agrees with the Lawson topology.

Definition 3.5 A bicontinuous poset is called *interval-compact* if all of its

[4] Martin and Panangaden [12] use the term *bicontinuous* for what we call *jointly bicontinuous*. They also use a different but equivalent definition: (1) P is a continuous poset, (2) whenever $a \ll b$ and F is a filtered set with $\bigwedge_{\downarrow} F \leq a$, then $f \leq b$ for some $f \in F$, and (3) $\twoheaddownarrow$a is filtered and $\bigwedge_{\downarrow} \twoheaddownarrow = a$ for every a.

closed intervals are compact in the bi-Scott topology. [5]

An interval-compact bicontinuous poset is a conditional bicpo.

It seems of interest to weaken the requirement of compact intervals to the requirement of conditional bicompleteness. Under this weaker hypothesis the construction of the interval domain of Martin and Panangaden [12] works perfectly well. It seems that there bijection between interval domains and globally hyperbolic posets extends to the slightly more general situation by omitting the last condition in their definition of an interval domain.

Definition 3.6 A poset is called *linked bicontinuous* if its Scott topology σ agrees with its upper topology ν and its dual Scott topology σ^{op} agrees with its lower topology.

In a linked bicontinuous poset, the interval topology, the Lawson topology, the dual Lawson topology and the Scott topology all agree.

Even a jointly bicontinuous poset need not be linked bicontinuous as we shall see in the next section; but the Lawson topology and the dual Lawson topology still may agree:

Proposition 3.7 *In an interval-compact bicontinuous poset the Lawson topology agrees with the bi-Scott topology and, similarly, with the dual Lawson topology.*

Proof. As every neighborhood for the bi-Scott topology of a point c contains an open interval $]a,b[$ which is a neighborhood of c, it suffices to show that there is a finite set F such that $c \in \twoheaduparrow a \setminus \uparrow F \subseteq]a,b[$. For this consider $A = (\downarrow b \setminus \twoheaddownarrow b) \cap [a,b]$. As a closed subset of a closed interval, A is compact. For every $y \in A$, there is a $z_y \ll y$ such that $z_y \not\leq c$. As the open sets $uaz_y, y \in A$ are covering A, there is a finite sub-cover. Thus, there is a finite set F such that $A \subseteq \uparrow F$. Thus $c \in \twoheaduparrow a \setminus \uparrow F \subseteq]a,b[$. □

For complete lattices the notions of bicontinuity and linked bicontinuity have been considered in [6, pp. 501–504]. There, a more liberal definition for linked bicontinuity was adopted: A complete lattices was called linked bicontinuous, it its Lawson topology agrees with the dual Lawson topology. For complete lattices this condition implies that the Scott topology agrees

[5] Interval-compact jointly bicontinuous posets have been called strongly hyperbolic posets by Martin and Panangaden [12]. The reason is that they made the interesting observation that they occur in general relativity theory in models of space-time called strongly hyperbolic there. Although this terminology sounds great, we do not want to adopt it. In view of the examples that we will discuss in the next sections the term *strongly hyperbolic* seems inappropriate. There is nothing justifying the term hyperbolic in the definition and or in the examples.

with the upper and the dual Scott topology with the lower topology (see [6, Lemma VII-2.7]). Thus, for complete lattices, both definitions of bicontinuity agree. The preceding proposition shows that in interval-compact bicontinuous poset is linked bicontinuous in the less restricted sense. In the next section we shall see that there are interval-compact bicontinuous posets that are not linked bicontinuous in our more restricted sense.

Let us summarize some of the results from [6, pp. 501–504].

Properties. For a complete lattice L, the following are equivalent:

(i) L is linked bicontinuous.

(ii) L is meet continuous and join continuous and its interval topology is Hausdorff.

(iii) With respect to the interval topology, L is a compact topological lattice with a basis of sublattices.

For a distributive complete lattice, the following are equivalent:

(1) L is bicontinuous.

(2) L is linked bicontinuous.

(3) L is completely distributive.

(4) L is isomorphic to a closed sublattice of some power $[0,1]^I$ of the unit interval with the product order and product topology.

4 Cones and Orders in $\mathbb{R}^n$

In this section we will show that finite dimensional real vector spaces carry lots of partial orders which make them into bicontinuous posets. We endow finite dimensional vector spaces with their usual vector space topology.

A *cone* in $\mathbb{R}^n$ is meant to be subset C with the following properties:

$$(1) \quad C \cap -C = \{0\}, \quad (2) \quad C + C \subseteq C \quad (3) \quad \mathbb{R}_+ \cdot C \subseteq C$$

If we omit property (1), we talk about a *wedge*. For a wedge W, the set $E = W \cap -W$ is a linear subspace called the *edge* of the wedge.

The orders that we want to consider on $\mathbb{R}^n$ first are *vector space orderings* $\leq$ satisfying

$$x \leq y, r \in \mathbb{R}_+ \Rightarrow x + z \leq y + z,\ rx \leq ry$$

Vector space orderings are in a one-to-one correspondence with cones: To every cone C corresponds the vector space ordering $\leq_C$ given by $x \leq_C y$ iff $y - x \in C$ iff $y \in x + C$ and, for every vector space ordering $\leq$, consider its positive cone $C_\leq = \{x \in \mathbb{R}^n \mid x \geq 0\}$.

Example 4.1 The causal order of the Minkowski space of special relativity is the the ordering given by the cone $C = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid t^2 \geq x_1^2 + \ldots + x_n^2\}$.

We fix a cone C in $\mathbb{R}^n$ and the associated order $\leq$ on $\mathbb{R}^n$. We denote by C° the algebraic interior of C which can be characterized as the topological interior of C in the linear subspace $V = C - C$ of $\mathbb{R}^n$. We have the following properties:

Properties.

(i) $x \mapsto -x$ is an order anti-isomorphism.

(ii) $\mathbb{R}^n$ is directed with respect to $\leq$ iff the positive cone C generates $\mathbb{R}^n$ (i.e., $\mathbb{R}^n = C - C$) iff the interior of C in $\mathbb{R}^n$ is nonempty.

(iii)
$$\begin{array}{lcl} C \text{ is closed} & \iff & \text{the graph of } \leq \text{ is closed in } R^n \times \mathbb{R}^n \\ & \iff & \text{every upper bounded directed set has a supremum} \\ & \iff & \text{every lower bounded filtered set has an infimum.} \end{array}$$

(iv) $C^\circ + C \subseteq C^\circ$ and $C^\circ \cup \{0\}$ is a cone.

We are interested in closed cones. It is easy to see that a cone C is closed if and only iff the associated order $\leq$ is closed which means that, whenever we have convergent sequences (x_n) and (y_n) then $x_n \leq y_n$ for all n implies $\lim_n x_n \leq \lim_n y_n$. From now on we suppose that C is a closed generating cone.

Properties.

(i) $\mathbb{R}^n$ is jointly bicontinuous and $x \ll y$ iff $y \in x + C^\circ$ iff $x \in y - C^\circ$ iff $y \gg_d x$, i.e., the way-below relation is the strict order associated with the cone $C^\circ \cup \{0\}$.

(ii) $x \ll y$ and $0 < r \in \mathbb{R} \Rightarrow x + z \ll y + z$ and $rx \ll ry$, i.e., the reflexive hull of $\ll$ is a vector space ordering.

(iii) $\mathbb{R}^n$ is interval-compact.

(iv) The Scott-open sets are the open upper sets, the dually Scott-open sets are the open lower sets. The bi-Scott topology agrees with the usual topology.

Proposition 4.2 *Addition on $\mathbb{R}^n$ is Scott-continuous and dually Scott-continuous.*

Example 4.3 $\mathbb{R}^n$ is not linked bicontinuous (with respect to closed vector space orderings as above). Indeed, the Scott topology is strictly finer than the upper topology.

Consider for example $\mathbb{R}^2$ with the usual coordinatewise order the positive cone of which is $C = (\mathbb{R}_+)^2$. The Scott topology consists of all open upper sets. Thus the half space $U = \{(x, y) \mid y > 0\}$ is Scott-open and the complement $A = \{(x, y) \mid y \leq 0\}$ is Scott-closed. But A is not closed for the upper topology. Indeed, a subbasis for the closed sets in the upper topology is a finite union of principal ideals $\downarrow(x_i, y_i), i = 1, \ldots, n$. Finitely many elements have an upper bound. Thus every subbasic closed set, hence every closed set for the upper topology has an upper bound. As the Scott-closed half space A does not have any upper bound, it cannot be closed for the upper topology. A similar argument holds for any vector space order on $\mathbb{R}^n$ with a generating positive cone.

In view of the previous observation, one might expect that the Lawson topology is strictly finer than the bi-Scott topology which agrees with the usual topology on $\mathbb{R}^n$. This is not the case by Proposition 3.7, as $\mathbb{R}^n$ is interval-compact.

These observations illustrate that the the various topologies behave differently for non-complete posets than in the complete case. Indeed if we complete $\mathbb{R}$ by adding $\pm\infty$, thus obtaining the extended reals $\widehat{\mathbb{R}}$, then $\widehat{\mathbb{R}}^n$ is a completely distributive complete lattice which is linked bicontinuous.

Vector space orderings on $\mathbb{R}^n$ yield lots of examples of jointly bicontinuous posets. A slight generalization leads to still more examples.

Example 4.4 Translation invariant orders on $\mathbb{R}^n$. Let P be a subset of $\mathbb{R}^n$ with the following properties: (1) $\quad P \cap -P = \{0\}$, $\quad$ (2) $\quad P + P \subseteq P$.

These sets P are in a bijective correspondence with the translation invariant ($x \leq y \Rightarrow x + z \leq y + z$) orders on $\mathbb{R}^n$ given as above: For given P define $x \leq_P y$ iff $y - x \in P$. Again such an order is closed if and only if C is closed.

We consider special sets P of this type: Let $f \colon \mathbb{R}^{n-1} \to \mathbb{R}$ be a continuous function with the following properties:

$$(1) \quad 0 = f(0) < f(x) + f(-x) \text{ for } x \neq 0, \quad (2) \quad f(x + y) \leq f(x) + f(y),$$

Let $P \subseteq \mathbb{R}^n$ be the epigraph of the function F, that is,

$$P = \{(x, r) \mid f(x) \leq r\}$$

Property (1) corresponds to $P \cap -P = \{0\}$ and property (2) is equivalent to $P + P \subseteq P$; moreover P is closed. Note that P is a cone iff f is positively homogeneous, in addition. The interior of P is the set

$$P^\circ = \{(x, r) \mid f(x) > r\}$$

Again, $P^\circ + P \subseteq P^\circ$. Equipped with the order relation $\leq$ associated with P, $\mathbb{R}^n$ becomes a jointly bicontinuous poset with $x \ll y$ iff $y \in x + P^\circ$ iff $x \in y - P^\circ$ iff $y \gg_d x$.

In this slightly more general situation, closed intervals need not be compact in the bi-Scott topology. Take, for example, the epigraph of the cissoid of Diocles given by the equation $(y-1)x^2 = y^3$). In this example there are still small compact intervals in the sense that every point has a neighborhood basis of compact intervals. This is sufficient to prove that the the Lawson topology agrees with the bi-Scott topology (see Proposition 3.7. But in general, the bi-Scott topology might be strictly courser than the usual topology on $\mathbb{R}^n$. As both are Hausdorff, they agree on the bounded subsets of $\mathbb{R}^n$.

One may replace the finite dimensional vector space $\mathbb{R}^n$ by a Lie group G with an invariant $(x \leq y \Rightarrow xz \leq yz, zx \leq zy)$ closed order. These orders correspond to closed subsets P with the properties

$$(1)\ P \cap P^{-1} = \{e\}, \quad (2)\ P \cdot P \subseteq P, \quad (3)\ z^{-1}Pz \subseteq P$$

In order to be close to vector space orderings we suppose in addition that P is topologically generated by the intersection of P with any neighbourhood of the identity e. Lie groups equipped with such orderings have the same properties as we have seen before for finite dimensional vector spaces. In vector spaces we have lots of closed cones. For Lie groups, it is a difficult question to decide whether there are invariant orderings at all. Clearly compact Lie groups cannot have invariant closed orderings. E. B. Vinberg [16] has shown that a simple Lie group has an invariant closed order if and only if its center is infinite.

One may replace $\mathbb{R}^n$ by a smooth manifold M. We have already cited the paper of Martin and Panangaden [12], where they consider space-time manifolds with the causal order. Quite more generally, Lawson [10] has introduced orders on manifolds induced by cone fields. If we assign to every point x of a smooth manifold M a cone $C(x)$ in the tangent space of M at x, we have a *cone field*. We write $x \prec y$ if there is a continuous piecewise smooth curve from x to y whose derivative (tangent vector) belongs to the tangent cone $C(z)$ in each of its smooth points z. The relation $\prec$ is reflexive and transitive, that is, a preorder. We consider the closure $\leq$ of the relation $\prec$ in $M \times M$ which is again a preorder. If $\leq$ is a partial order we call it the conal order and we say that M is globally ordered. In the case of a homogeneous spaces $M = G/H$, where G is a Lie group and H a closed subgroup, the tangent cone is shifted around by the group action. Here one encounters again orders which are linked bicontinuous. In this case, interval compactness is an important issue as this property is needed to define the Volterra algebra. Neeb [14] has characterized those globally orderable homogeneous spaces for which the order intervals are compact.

It seems worthwhile to look at the quite vast literature on causal orders on manifolds in view of their domain theoretical properties.

5 Cone Orders on Subsets of $\mathbb{R}^n$

We return to the situation of an order $\leq$ on $\mathbb{R}^n$ given by a closed generating cone C. We may restrict this order to subsets A of $\mathbb{R}^n$. We only will consider closed convex subsets.

As a first case, we restrict the order $\leq_C$ defined by the closed generating cone C to the cone C itself. Then C remains bicontinuous. The way-above relation is the one induced from the way-above relation on $\mathbb{R}^n$, that is, for $a, b \in C$ we have $b \gg_d a$ iff $b - a \in C^\circ$. But the way-below relation changes on the boundary of C. Thus $(C, \leq_C)$ is not jointly bicontinuous. For the characterization of the way below relation on C need some notation: For $b \in C$, let

$$C_b = \{x \in C \mid x \leq_C rb \text{ for some } r \in \mathbb{R}_+\} = C \cap (\mathbb{R}_+ \cdot b - C)$$

As $(\mathbb{R}_+ \cdot b - C)$ is a wedge in $\mathbb{R}^n$, C_b is a subcone and a lower subset of C, that is, C_b is a *face* of C. We say that C_b is the face of C generated by b. Every face of C of this form. By [17, Corollary 2.6.3] every face of a closed convex set in $\mathbb{R}^n$ is closed.

Proposition 5.1 *For $a, b \in C$, we have $a \ll_C b$ iff $b \in a + (C_b)^\circ$, where $(C_b)^\circ$ is the algebraic interior of C_b.*

Remark 5.2 There is a linear functional φ on $\mathbb{R}^n$ such that $\varphi(x) > 0$ for every nonzero element x of C. We denote by H the hyperplane $\varphi^{-1}(0)$ which meets C only in $\{0\}$. The set

$$K = \{c \in C \mid \varphi(x) = 1\}$$

is a compact convex subset of C with the property that for every $x \neq 0$ in C there is a unique positive real number r such that $rx \in K$. Every such subset K is called a *base* of C. The convex hull of K and $\{0\}$, i.e., the set

$$S = \{c \in C \mid \varphi(x) \leq 1\}$$

is called a *kegelspitz* of C.

Proposition 5.3 *The kegelspitz S ordered by $\leq_C$ is a continuous dcpo. Its Lawson topology is the usual topology.*

The kegelspitz S is no longer bicontinuous. Indeed the maximal elements of a dually continuous poset have to be dually compact. The maximal elements of S are those in K, but none of these elements is compact. The way-above relation in S is the restriction of the way-below relation on $\mathbb{R}^n$ to S. Thus, S is almost dually continuous in the sense that every non-maximal element is the inf of a filtered family of element way-above.

As a kegelspitz is Lawson-compact, one can ask whether it is an FS-domain or a retract of a bifinite one. If the cone C is a simplicial cone, i.e., if the base K of the cone is a simplex which is equivalent to saying that the associated order $\leq C$ is a lattice order, then $(S, \leq_C)$ is a conditionally complete continuous poset. At the opposite, if the base K of the cone is a sphere, then S is not an FS-domain.

Now let us consider the cone order $\leq_C$ on the opposite cone $-C$. We may add $\perp$ as a smallest element to $-C$. Then $-C_\perp$ becomes bicomplete, but the property of being dually continuous is lost. (Going down to $\perp$ along an extreme ray of $-C$ shows that $\perp \not\ll 0$.) We claim (compare Jung [8]):

Proposition 5.4 *$-C_\perp$ ordered by $\leq_C$ is an FS-domain.*

This leads us to the old problem raised by Jung [8] whether the class of FS-domains is strictly bigger than the class of retracts of bifinite domains. We conjecture that the answer is yes. But until now this question is open. There is hope that a counterexample can be found through cone orders.

Problem 5.5 *(Jung, Lawson) Is $-C_\perp$ ordered by $\leq_C$ a retract of a bifinite domain?*

This problem has been asked by Lawson and Jung for the case where C is an ice-cream cone, that is a three-dimensional cone with a disc K as base. Possibly polyhedral cones are easier to handle than 'round' cones.

Now let D be a closed cone in $\mathbb{R}^n$ properly containing the cone C. Then D is dually continuous, but not a continuous poset with respect to $\leq_C$. Indeed, D has an extreme ray R which does not belong to C. The points a on the extreme ray are minimal in D with respect to $\leq_C$ but not compact.

At the other hand, if D is a closed cone contained in C, then D is a continuous poset with respect to $\leq_C$. The way-below relation is the one induced by the way-below relation on C. Any kegelspitz S of D is a Lawson compact continuous dcpo. Again one may ask for its domain theoretical properties.

6 Vector Space Orderings and the Formal Ball Model

Let me point out an interesting class of jointly bicontinuous posets, the *extended formal ball model* due to Tsuiki and Hattori [15]: For a metric space X with metric d, the extended formal ball model is the set $B(X) = X \times \mathbb{R}$ with the order $(x, r) \le (y, s)$ iff $d(x, y) \le r - s$. For the formal ball model – as studied by Edalat and Heckmann [2] – the above poset is restricted to $B_+(X) =_{def} X \times \mathbb{R}_+$.

In the extended formal ball model $B(X)$ as in the traditional formal ball model $B_+(X)$, the way-below relation is given by $(x, r) \ll (y, s)$ iff $d(x, y) < r - s$ and $B(X)$ is a continuous poset.

From the definition one sees immediately that $B(X)$ has an involutory order anti-isomorphism $(x, r) \mapsto (x, -r)$. Thus every order theoretical property of $B(X)$ will also be true for the opposite order and the way-below relation is the opposite of the way-above relation. We have:

Proposition 6.1 *For any metric space X, the extended formal ball model $B(X)$ is a jointly bicontinuous poset.*

For the various topologies on $B(X)$, one has:

Lemma 6.2 [15, Lemma 6, Proposition 8]) *The bi-Scott topology on $B(X) = X \times \mathbb{R}$ agrees with the product of the metric topology on X and the usual Hausdorff topology on $\mathbb{R}$. If the metric space X is totally bounded, in particular if X is a compact metric space, then the upper topology equals the Scott topology and the lower topology the dual Scott topology on $B(X)$, that is, $B(X)$ is linked bicontinuous.*

As the name says, the formal balls (x, r) represent formally the concrete closed balls $B(x, r)$ with center point x and radius r. Let $B^*(X)$ denote the set of all concrete closed balls $B(x, r), x \in X, r \ge 0$ ordered by reverse inclusion. There is a natural surjection $\eta = ((x, r) \mapsto B(x, r))$ from the set $B_+(X)$ of formal balls to the set $B^*(X)$ of concrete balls. This map preserves the order, but need not be injective as a metric space may contain holes and as the metric need not assume arbitrary real values. It is difficult to develop an intuition for balls with negative radii. In the following we give a concrete representation of the extended formal ball model for normed vector spaces X. In this case we consider the vector space $X \times \mathbb{R}$ and therein the subset $C =_{def} \{(x, r) \mid ||x|| \le r\}$ which is easily verified to be a cone. The set $B(0, 1) \times \{1\}$ is a basis of the cone C, where $B(0, 1)$ is the unit ball in X. As for extended formal balls we have $(x, r) \le (y, s)$ iff $||x - y|| \le r - s$, we have:

Proposition 6.3 *If X is a normed vector space, the extended formal ball* *model $B(X)$ is order isomorphic to the vector space $X \times \mathbb{R}$ ordered by $-C$ as positive cone.*

The poset $B(X)$ never is a dcpo as $(x, -n), n \in \mathbb{N}$, is an increasing sequence without any upper bound.

Proposition 6.4 *For a metric space X, the extended formal ball model $B(X)$ is conditionally directed complete if and only if X is a complete metric space.*

Indeed, if (x_i, r_i) is a directed set in $B(X)$ bounded by (y, s), then the r_i form a filtered set of real numbers bounded below by s. Thus the r_i converge to some $r = \inf_i r_i \geq s$. As a consequence the x_i form a Cauchy net.If X is complete, the Cauchy net x_i converges to some x, and $(x, r) = \sup_i(x_i, r_i)$ in $B(X)$.

Let us rise the question, under which condition intervals are compact for the bi-Scott topology.

7 The Probabilistic Powerdomain over Finite Posets

A special case of cone orders on subcones or a kegelspitz thereof leads us to another old open problem in domain theory.

Let us concentrate on the probabilistic powerdomain over a finite poset P. We denote by $\mathcal{O}P$ the collection of all upper sets $U \subseteq P$. Ordered by inclusion, $\mathcal{O}P$ is a finite distributive lattice, and every finite distributive lattice L occurs in this way, namely L is (isomorphic to) the collection of upper sets of the set P of $\vee$-irreducible elements of L.

In the vector space $\mathbb{R}^P$ we consider the polyhedral cone C defined by the set of inequalities

$$\sum_{p \in U} x_p \geq 0, \ U \in \mathcal{O}P$$

Clearly C contains the cone $D = \mathbb{R}^P_+$,

Example 7.1 Consider a poset with four elements 1, 2, 3, 4, where 1 and 2 are incomparable, 3 and 4 are incomparable, and where 1 and 2 are both dominated by 3 and 4. The cone C in $\mathbb{R}^4$ is given by the five linear inequalities

$$\begin{aligned} x_3 &\geq 0 \\ x_4 &\geq 0 \\ x_1 + x_3 + x_4 &\geq 0 \\ x_2 + x_3 + x_4 &\geq 0 \\ x_1 + x_2 + x_3 + x_4 &\geq 0 \end{aligned}$$

We may view $\mathbb{R}^P$ as the set of all measures on the finite set P. The order $\leq_C$ has been called the stochastic order by Edwards [1]. We may restrict the stochastic order to the simplicial cone $\mathbb{R}^P_+$, the set of all positive measures on P, to $\mathcal{V}_{\leq 1}P$ and $\mathcal{V}_1 P$, the subprobability and the probability measures, that is, the set of all $x \in \mathbb{R}^P_+$ such that $\sum_{p\in P} x_p \leq 1$ and $\sum_{p\in P} x_p = 1$, respectively.

Problem 7.2 *Is the subprobabilistic and the probabilistic powerdomain on a finite poset with the stochastic order an FS-domain? or even a retract of a bifinite domain?*

This question is of interest as one would like to have a cartesian closed category of continuous dcpos which is stable under the probabilistic powerdomain construction.

An answer to this problem is only known for very special cases only: If the poset P is a tree, then $\mathcal{V}_{\leq 1}P$ is a retract of a bifinite domain. If P is a finite root system, then $\mathcal{V}_{\leq 1}P$ is an FS-domain (see Jung and Tix [9]).

8 Completions

Every continuous poset P can be embedded in a continuous dcpo by passing to the *round ideal completion* $\mathcal{J}P$. Recall that a *round ideal* is a directed lower set J with the property that for every $x \in J$ there is a $y \in J$ with $x \ll y$. The set $\mathcal{J}P$ of round ideals is ordered by inclusion. The way-below relation between round ideals is given by $I \ll J$ iff there is an $a \in J$ such that $I \subseteq \downarrow a$. The embedding of P into $\mathcal{J}P$ is given by $x \mapsto \twoheaddownarrow x$. In a conditionally directed complete continuous poset, every upper bounded round ideal is of the form $\twoheaddownarrow x$ and the round ideal completion just adds unbounded round ideals on top. In a similar way, one defines *round filters* and the *round filter completion* $\mathcal{F}P$ of a dually continuous poset.

The round ideal completion of a jointly bicontinuous poset need not be bicontinuous. For example, $P = \{(x, y) \in [1, 2]^2 \mid 0 < x, 0 < y, x+y < 1\}$ with the coordinatewise order is jointly bicontinuous. The round ideal completion adds the points (x, y) with $x + y = 1$. These new points are all maximal but not compact for the opposite order. Thus, the round ideal completion is not dually continuous.

Example 8.1 Consider $\mathbb{R}^2$ with the coordinatewise order (induced by the cone cone $C = \mathbb{R}^2_+$). The round ideals are I_{1r} given by the inequality $x_1 < r$ and I_{2r} given by the inequality $x_2 < r$, plus the whole of $\mathbb{R}^2$. The round ideal completion can be identified with $(\mathbb{R} \cup \{+\infty\})^2$ which is again dually continuous. The round filter completion of it becomes $(\mathbb{R} \cup \{-\infty\})^2$. One may consider $(\mathbb{R} \cup \{+\infty - \infty\})^2$ to be jointly a round ideal and round filter completion, but notice that the two points $(-\infty, +\infty)$ and $(+\infty, -\infty)$ are neither in the round ideal nor in the round filter completion.

In the following proposition we give a description of the round ideals of $\mathbb{R}^n$ ordered by a closed generating cone C. They are determined by the geometry of the cone.

Proposition 8.2 *Consider any face F of C. Then $I_F = F - C^\circ$ is a round ideal and all of its translates $x + I_F$ are round ideals, too, and these are all the round ideals of $(\mathbb{R}^n, \leq_C)$.*

Proof. It is easy to check that I_F is a round ideal for every face F of the cone C. Conversely, let J be a round ideal. If J has an upper bound in $\mathbb{R}^n$, then $J = \mathop{\downarrow\!\!\!\downarrow} a$ where $a = \bigvee^{\uparrow} J$. In this case $J = a + \mathop{\downarrow\!\!\!\downarrow} 0 = a + \{0\} - C^\circ = J = a + I_F$ with $F = \{0\}$. Thus, we may assume J to have no upper bound. If $J = \mathbb{R}^n$, then $J = I_F$ with $F = C$. Thus we may suppose also that J is a proper subset of $\mathbb{R}^n$. By replacing J by $a + J$ for an appropriate a, we may suppose that J contains an interior point e of C. As J is a proper subset, there is a positive real number such that $re \not{i}nC$. As J is open, there is a smallest r with this property. Replace J by $-re + J$. Then $0 \notin J$ but $\mathop{\downarrow\!\!\!\downarrow} 0 = -C^\circ \subseteq J$. As J is convex and directed, its closure $\overline{J}$ is convex and directed, too. As $x - C \subseteq \overline{J}$ for all $x \in \overline{J}$, $\overline{J}$ is a lower set and it contains 0. We claim that $F = C \cap \overline{J}$ is a face of C. It is indeed a closed convex lower subset of C. Further, F is directed. If F were bounded, the least upper bound of F would belong to F, too. But as $\overline{J}$ is unbounded, F cannot have a least upper bound. Thus F is a face of C and $I_F = F - C^\circ \subseteq J$. Conversely, let $x \in J$. Choose $y \in J$ with $x \ll y$. As $\overline{J}$ is directed, there is a common upper bound a of y and 0 in $\overline{J}$. Then $a \in F$ and $x \ll a$, i.e. $x \in F - C^\circ = I_F$. $\square$

In the preceding proposition we have given an explicit description of the individual round ideals.

Problem 8.3 *Give an explicit description of the round ideal completion of $(\mathbb{R}^n, \leq_C)$ as a continuous dcpo and as a topological space.*

It would be nice to combine the round ideal and the round filter completion with the aim of obtaining a poset which is compact in the bi-Scott topology, as far as this is possible. One attempt to achieve this goal could consist in an appropriate extension of the interval domain construction as performed by Martin and Panangaden [12]:

For any poset P one may consider the set $\mathcal{I}P$ of all closed intervals $[x, y]$. The set P is embedded into $\mathcal{I}P$ by the map $x \mapsto [x, x]$. There are two natural orders on $\mathcal{I}P$: firstly the inclusion order $[x, y] \subseteq [x', y']$ and the Egli-Milner order $[x, y] \sqsubseteq [x', y']$ if $x \leq x'$ and $y \leq y'$. With respect to inclusion, the elements of P are identified with the minimal elements of $\mathcal{I}P$. With respect to the Egl-Milner order, P is order embedded into $\mathcal{I}P$.

Alternatively, $\mathcal{I}P$ may be identified with the graph of the order on P, that is, the set $\{(x, y) \mid x \leq y\} \subseteq P \times P$, and P is identified with the diagonal in $P \times P$ via $x \mapsto (x, x)$. The Egli-Milner order corresponds to the coordinatewise order $(x, y) \sqsubseteq (x', y')$ iff $x \leq x'$ and $y \leq y'$ whilst the inclusion order corresponds to the order $(x, y) \subseteq (x', y')$ iff $x \geq x'$ and $y \leq y'$.

Suppose now that P is a bi-Scott-dense subset of a jointly bicontinuous poset M with the induced order and the way-below of M restricted to P. The intersection $B = J \cap F$ of a round ideal J with a round filter F will be called a *round subset* of P, provided that this intersection $F \cap J$ is nonempty. Equivalently, a subset B of P is round if B is directed, filtered, order convex and if for every $x \in I$ there is are elements $a, b \in I$ with $a \ll x \ll b$. Denote by $\mathcal{B}P$ the set of all round subsets of P. Again we have two natural orders on $\mathcal{B}P$, the inclusion $B \subseteq B'$ and the Egli-Milner order $B \sqsubseteq B'$ iff $\uparrow B \supseteq \uparrow B'$ and $\downarrow B \subseteq \downarrow B'$.

Note that F and J are completely determined by $B = F \cap J$, namely $F = \uparrow B$ and $J = \downarrow B$. Thus, $\mathcal{B}P$ may be identified with the set $\{(F, J) \mid J \cap F \neq \emptyset\} \subseteq \mathcal{F}P \times \mathcal{J}P$. The inclusion order on $\mathcal{B}P$ corresponds to the order $(F, J) \leq (F', J')$ iff $F \subseteq F'$ and $J \subseteq J'$ whilst the Egli-Milner order corresponds to the order $(F, J) \sqsubseteq (F', J')$ iff $F \supseteq F'$ and $J \subseteq J'$.

Among the round subsets we have the open intervals $]a, b[$. Martin and Panangaden [12] have shown the following: With respect to the relation $]a, b[\ll]a', b'[$ iff $a \ll a'$ and $b' \ll b$ the open intervals form a dual abstract basis $\mathcal{B}'P$. If M is a strongly hyperbolic poset, the round filter completion $\mathcal{F}\mathcal{B}'P$ is order isomorphic to the dually continuous poset of closed intervals, and M is homeomorphic to the space of its minimal elements with the dual Scott topology.

We extend this construction to the collection $\mathcal{B}P$ of all round subsets of P: We define a relation $B_1 \gg B_2$ if there are elements a, b such that $B_1 \supseteq [a, b] \supseteq B_2$. In this way, $\mathcal{B}I$ becomes a dual abstract basis. Indeed, if $B \gg B_j$ for $j = 1, \ldots, n$ then there are elements a_j, b_j such that $B \supseteq [a_j, b_j] \supseteq B_j$ for $j = 1, \ldots, n$. As B is filtered and directed, there are elements a, b in B such that $a \leq a_j$ and $b_j \leq b$ for $j = 1, \ldots, n$, hence $B \supseteq [a, b] \supseteq [a_j, b_j] \supseteq B_j$ for $j = 1 \ldots, n$, that is, $B \gg B_j$ for $j = 1, \ldots, n$. The round filter completion $\mathcal{F}\mathcal{B}P$ is a continuous fcpo with respect to inclusion. The fcpo $\mathcal{I}M$ of closed intervals of M is embedded in $\mathcal{F}\mathcal{B}P$ by assigning to every closed interval $[a, b], a, b \in M$ the round filter of all round sets $B \subseteq P$ containing some

$a' \ll a$ and some $b' \gg b$. Identifying the elements x of M with the minimal intervals $[x, x]$ we obtain a natural embedding of M onto a dense subset of the set $\mathrm{Max}(\mathcal{FB}P)$ of maximal elements of $\mathcal{FB}P$. The bi-Scott topology on M corresponds to the topology on $\mathrm{Max}(\mathcal{FB}P)$ induced by the Scott topology on $\mathcal{FB}P$. We summarize:

Proposition 8.4 *Let P be a bi-Scott-dense subset of a jointly continuous poset M. The set $\mathcal{B}P$ of all round subsets of P with the relation $\geq$ as above is a dually abstract basis. Its round filter completion $\mathcal{FB}P$ is a continuous poset which contains an isomorphic copy of the domain of closed intervals $[a, b], a, b \in M$ and M wit its bi-Scott topology is topologically embedded as a dense subset in the space $\mathrm{Max}(\mathcal{FB}P)$ of maximal elements with the induced Scott topology.*

Question: Under which hypothesis is $\mathrm{Max}(\mathcal{FB}P)$ a compact Hausdorff space?

9 Bicontinuous Function Spaces

For two topological spaces X and Y let $[X \to Y]$ denote the set of continuous functions $f\colon X \to Y$ endowed with the partial order $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$, where $\leq$ is the specialization order on Y. We ask the question, when $[X \to Y]$ is a bicontinuous poset.

Our results are modelled along the lines of the following criterion for the continuity of $[X \to Y]$ in [6, Theorem 4.7] :

Theorem 9.1 *Let X be a nonempty topological space and L a complete lattice with the Scott topology and at least two elements. Then $[X \to L]$ is a continuous lattice if and only if the lattice $\mathcal{O}(X)$ of open subsets of X and the lattice L are continuous.*

For complete distributivity, independently Erné [4] and Heckmann, Huth, Mislove [7] (the latter under the restriction of soberness) have proved:

Theorem 9.2 *Let X be a nonempty topological space and L a complete lattice with the Scott topology and at least two elements. Then $[X \to L]$ is completely distributive if and only if the lattice $\mathcal{O}(X)$ of open subsets of X and the lattice L are completely distributive.*

The topological spaces X for which the lattice $\mathcal{O}(X)$ of open subsets is completely distributive are called *c-spaces*; they have been characterized independently by Erné [3] and Ershov [5] as those spaces in which every point has a neighborhood basis of principal filters $\uparrow a$. All continuous posets are c-spaces when endowed with the Scott topology.

The following has been proved by Heckmann, Huth and Mislove [7, Theorem 11]. In fact, we slightly generalize their result by dropping the condition of soberness:

Theorem 9.3 *Let X be a nonempty topological space and L a complete lattice with the Scott topology and at least two elements. Then $[X \to L]$ is bicontinuous if and only if the lattice $\mathcal{O}(X)$ of open subsets of X and the lattice L are bicontinuous.*

Proof. We just show how to reduce our result to the case of a sober space X which was handled by [7, Theorem 11]. We first note that a distributive complete lattice is bicontinuous iff it is completely distributive by the properties at the end of Section 3. Thus $\mathcal{O}(X)$ is bicontinuous iff X is a c-space. The sobrification X^s of a c-space X is a continuous dcpo with its Scott topology (see [11]). As every continuous function $f\colon X \to L$ has a unique continuous extension to the sobrification X^s, the two function spaces $[X \to L]$ and $[X^s \to L]$ agree. □

Corollary 9.4 ([7, corollary 19]) *The category of bicontinuous lattices and Scott-continuous maps is cartesian closed.*

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