

A detailed reference of MCMC algorithms

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1 Introduction

All the code for this project (including code to generate the report and its graphs) can be found at <https://github.com/bollu/sampleraytracer>.

1.1 Fundamental Problem of MCMC sampling

Given a weak, simple sampler of the form `rand` : $S \rightarrow S \times \text{int}$, build a sampler `sampler`(P, X) : $T \rightarrow T \times X$ which returns value distributed according to some *un-normalized* distribution of choice $P : X \rightarrow \mathbb{R}$.

The un-normalized constraint is important: it is what allows us to exploit MCMC to a wide variety of scenarios.

1.2 Sampling use case 1. Simulation And Sampling

Often, we do really want samples from a particular distribution. For example, we might often want to apply Bayes' rule and sample from the posterior distribution. Recall the formula:

$$P(Y|X) = P(X|Y).P(Y)/P(X)$$

1.3 Sampling use case 2. Gradient Free Optimisation

We want to maximize a function $f : X \rightarrow \mathbb{R}$. However, we lack gradients for f , hence we cannot use techniques such as gradient descent, or other techniques from convex optimisation.

In such a case, we can consider f as some sort of unnormalized probability distribution, and use MCMC to sample from f .

1.4 Sampling use case 3. Numerical Integration

2 Where it all begins: The Metropolis Hastings sampler

2.1 The big idea

We wish to sample from a distribution \mathfrak{P} , but we do not know how to do so. The idea is that we build a markov chain $M[\mathfrak{P}, \text{Prop}]$ where $\mathfrak{P}, \text{Prop}$ are supplied by the user. We will show that the *stationary distribution* of $M[\mathfrak{P}, \text{Prop}]$ is going to be \mathfrak{P} . This will ensure that if we interpret *states of* M as samples, these samples will be distributed according to \mathfrak{P} .

2.2 Detail balance: A tool for proofs

We will first require a condition that will enable to rapidly establish that some distribution $\mathfrak{P} : X \rightarrow \mathbb{R}$ is the stationary distribution of a markov chain M . If the transition kernel $K : X \times X \rightarrow \mathbb{R}$ of $M \equiv (X, K)$ is such that:

$$\forall x, x' \in X, \mathfrak{P}(x)K(x, x') = \mathfrak{P}(x')K(x', x)$$

Then \mathfrak{P} is said to be **detail balanced** with respect to M .

Theorem 1 *If \mathfrak{P} is detail balanced to $M \equiv (X, K : X \rightarrow (X \rightarrow [0, 1]))$, then \mathfrak{P} is the stationary distribution of M .*

Proof 1.1 *Let \mathfrak{P} be detail balanced to M . This means that:*

$$\forall x, x' \in X, \mathfrak{P}(x)K(x)(x') = \mathfrak{P}(x')K(x')(x)$$

Let us say that we are in state \mathfrak{P} . We wish to find the probability distribution after one step of transition. The probability of being in some state x'_0 is going to be:

$$Pnext(x'_0) \equiv \sum_x \mathfrak{P}(x)K(x)(x'_0)$$

since we have $\mathfrak{P}(x)$ probability to be at a given x , and $K(x, x'_0)$ probability to go from x to x'_0 . If we add over all possible $x \in X$, we get the probability of all states to enter in x'_0 . Manipulating $Pnext$, we get:

$$\begin{aligned} Pnext(x'_0) &\equiv \sum_x \mathfrak{P}(x)K(x)(x'_0) \\ &= \sum_x \mathfrak{P}(x'_0)K(x'_0)(x) \quad (\text{by detail balance}) \\ &= \mathfrak{P}(x'_0) \sum_x K(x'_0)(x) \quad (P(x_0) \text{ is constant}) \\ &= \mathfrak{P}(x'_0) \cdot 1 \quad (K(x'_0) \text{ is a distribution which is being summed over}) \\ &= \mathfrak{P}(x'_0) \quad (\text{eliminate multiplication with 1}) \end{aligned}$$

Hence, $Pnext(x'_0) = \mathfrak{P}(x'_0)$ if \mathfrak{P} is the current state, and \mathfrak{P} is in detail balance with the kernel K . This means that \mathfrak{P} is the stationary distribution of M . \triangle

2.3 Metropolis Hastings

There are three key players in the metropolis hasting sampler:

- 1 $\mathfrak{P} : X \rightarrow \mathbb{R}$: the probability distribution we wish to sample from.

- 2 $\text{Prop} : X \rightarrow (X \rightarrow \mathbb{R})$. For each $x_0 \in X$, provide a distribution $\text{Prop}(x) : X \rightarrow \mathbb{R}$ that is used to sample points around x_0 . Prop for *proposal*.
- 3 $M[\mathfrak{P}, \text{Prop}] \equiv (X, K[\mathfrak{P}, \text{Prop}] : X \rightarrow \mathbb{R})$: The Metropolis Markov chain we will sample from, whose stationary distribution is $\mathfrak{P} - M$ for *Markov*.

We want the stationary distribution of $M[\mathfrak{P}, \text{Prop}]$ to be \mathfrak{P} . We also wish for $K[\mathfrak{P}, \text{Prop}](x_0) \sim \text{Prop}(x_0)$: That is, at a point x_0 , we want to choose new points in a way that is 'controlled' by the proposal distribution $\text{Prop}(x_0)$, since this will allow us to 'guide' the markov chain towards regions where \mathfrak{P} is high. If we had a gradient, then we could use \mathfrak{P}' to 'move' from the current point x_0 to a new point. Since we lack a gradient, we will provide a custom $\text{Prop}(x_0)$ for each x_0 that will tell us how to pick a new x' , in a way that will improve \mathfrak{P} . So, we tentatively define

$$K[\mathfrak{P}, \text{Prop}](x)(x') \stackrel{?}{=} \text{Prop}(x)(x').$$

Recall that for \mathfrak{P} to be a stationary distribution of K , it is sufficient for \mathfrak{P} to be in detail balance for K . So, we write:

$$\begin{aligned} \mathfrak{P}(x)K(x, x') &\stackrel{?}{=} \mathfrak{P}(x')K(x', x) \quad (\text{going forward equally likely as coming back}) \\ \mathfrak{P}(x)\text{Prop}(x)(x') &\stackrel{?}{=} \mathfrak{P}(x')\text{Prop}(x')(x) \quad (\text{this is a hard condition to satisfy}) \end{aligned}$$

This is far too complicated a condition to impose on Prop and \mathfrak{P} , and there is no reason for this condition to be satisfied in general. Hence, we add a custom "fudge factor" $\alpha \in X \rightarrow (X \rightarrow \mathbb{R})$ that tells us how often to transition from x to x' . We redefine the kernel as:

$$K[\mathfrak{P}, \text{Prop}](x)(x') \equiv \text{Prop}(x)(x')\alpha(x)(x')$$

Redoing detail balance with this new K , we get:

$$\begin{aligned} \mathfrak{P}(x)K(x, x') &\stackrel{?}{=} \mathfrak{P}(x')K(x', x) \quad (\text{going forward equally likely as coming back}) \\ \mathfrak{P}(x)\text{Prop}(x)(x')\alpha(x)(x') &= \mathfrak{P}(x')\text{Prop}(x')(x)\alpha(x')(x) \quad (\text{Fudge a hard condition with } \alpha) \\ \frac{\alpha(x)(x')}{\alpha(x')(x)} &= \frac{\mathfrak{P}(x')\text{Prop}(x)(x')}{\mathfrak{P}(x)\text{Prop}(x')(x)} \quad (\text{Find conditions for } \alpha) \end{aligned}$$

What we have above is a *constraint* for α . We now need to *pick* an α that satisfies this. A reasonable choice is:

$$\alpha(x)(x') \equiv \min \left(1, \frac{\mathfrak{P}(x')\text{Prop}(x)(x')}{\mathfrak{P}(x)\text{Prop}(x')(x)} \right)$$

since K is a transition kernel, we cannot have its entries be greater than 1. Hence, we choose to clamp it with a $\min(1, \cdot)$. This finally gives us the kernel as:

$$\begin{aligned} K[\mathfrak{P}, \text{Prop}](x)(x') &\equiv \text{Prop}(x)(x')\alpha(x)(x') \\ K[\mathfrak{P}, \text{Prop}](x)(x') &= \text{Prop}(x)(x') \min\left(1, \frac{\mathfrak{P}(x')\text{Prop}(x')(x)}{\mathfrak{P}(x)\text{Prop}(x)(x')}\right) \\ K[\mathfrak{P}, \text{Prop}](x)(x') &= \min\left(\text{Prop}(x)(x'), \frac{\mathfrak{P}(x')\text{Prop}(x')(x)}{\mathfrak{P}(x)}\right) \end{aligned}$$

We can make sure that detail balance is satisfied:

$$\begin{aligned} \mathfrak{P}(x)K[\mathfrak{P}, \text{Prop}](x)(x') &= \mathfrak{P}(x) \min\left(\text{Prop}(x)(x'), \frac{\mathfrak{P}(x')\text{Prop}(x')(x)}{\mathfrak{P}(x)}\right) \\ &= \min\left(\mathfrak{P}(x)\text{Prop}(x)(x'), \mathfrak{P}(x) \frac{\mathfrak{P}(x')\text{Prop}(x')(x)}{\mathfrak{P}(x)}\right) \\ &= \min(\mathfrak{P}(x)\text{Prop}(x)(x'), \mathfrak{P}(x')\text{Prop}(x')(x)) \end{aligned}$$

Note that the above right-hand-side is symmetric in x and x' , and hence we can state that:

$$\mathfrak{P}(x)K[\mathfrak{P}, \text{Prop}](x)(x') = \min(\mathfrak{P}(x)\text{Prop}(x)(x'), \mathfrak{P}(x')\text{Prop}(x')(x)) = \mathfrak{P}(x')K[\mathfrak{P}, \text{Prop}](x')(x)$$

Hence, we can wrap up, stating that our design of K does indeed give us a markov chain whose stationary distribution is \mathfrak{P} , since \mathfrak{P} is detail balanced with $K[\mathfrak{P}, \text{Prop}]$. As an upshot, we also gained a level of control with Prop , where we are able to provide "good" samples for a given point.

2.4 Simplification when proposal is symmetric

If our function Prop is symmetric: $\forall x, x' \text{Prop}(x)(x') = \text{Prop}(x')(x)$, then a lot of the above derivation becomes much simpler. We will perform those simplifications here for pedagogy.

When we have $\text{Prop}(x)(x') = \text{Prop}(x')(x)$, we can simply α :

$$\begin{aligned} \alpha(x)(x') &\equiv \min\left(1, \frac{\mathfrak{P}(x')\text{Prop}(x')(x)}{\mathfrak{P}(x)\text{Prop}(x)(x')}\right) \\ \alpha(x)(x') &= \min\left(1, \frac{\mathfrak{P}(x')}{\mathfrak{P}(x)}\right) \quad [\text{cancelling: } \text{Prop}(x)(x') = \text{Prop}(x')(x)] \end{aligned}$$

This also makes the kernel look a lot more pleasing:

$$K[\mathfrak{P}, \text{Prop}](x)(x') \equiv \text{Prop}(x)(x') \alpha(x)(x')$$

$$K[\mathfrak{P}, \text{Prop}](x)(x') = \text{Prop}(x)(x') \min \left(1, \frac{\mathfrak{P}(x')}{\mathfrak{P}(x)} \right)$$

```

# prob is the distribution to sample from;
# symproposol is the *symmetric* proposal function
# symproposol: X -> X; produces a new 'X' from a
#             given 'X' with some distribution.
# prob: X -> |R: gives probability of point 'xi'.
# N: number of markov chain walks before returning a new sample.
def metropolis_hastings(prob, symproposol, x0, N):
    x = x0
    while True:
        for i in range(N):
            # xnext chosen with Prop(x)(x') prob.
            xnext = symproposol(x); px = prob(x); pxnext = prob(xnext);
            # x' chosen with Prop(x)(x') * alpha prob.
            r = uniform01(); alpha = min(1, pxnext / px); if r < alpha: x = xnext
        yield x

```

2.5 Implementing MH: Devil in the Details

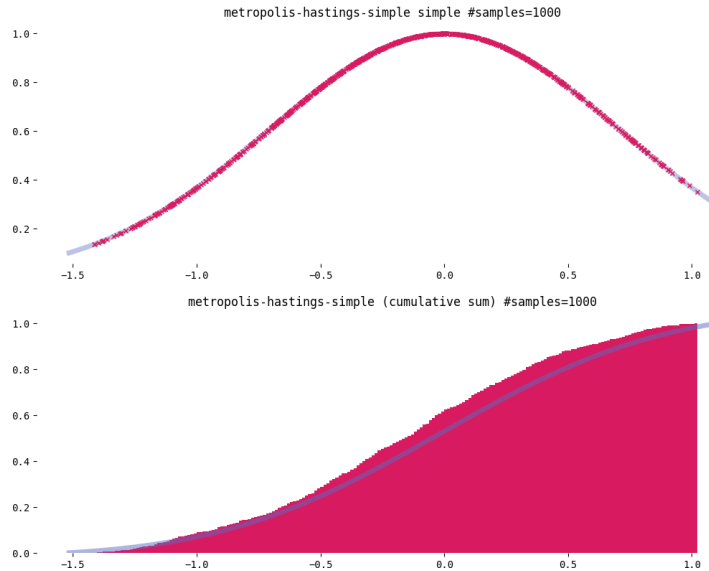
2.5.1 Naive implementation

Let us try to transcribe the equations we have derived for symmetric proposal into code, and see what happens. Let us choose $\mathfrak{P}(x) \equiv \text{normal}(0, 1) = e^{-x^2/2}$ and the proposal function to be $P(x_0)(x) = \text{normal}(x_0, 1e-2)$ is, a Gaussian centered around x_0 with standard deviation $1e-2$. This gives us the code:

```

# mcmc1d.py
def mhsimple(x0, prob, prop):
    yield x0; x = x0;
    while true:
        xnext = prop(x); p = prob(x); pnext = prob(xnext)
        r = np.random.uniform() + 1e-5;
        if r < pnext/p: x = xnext
        yield xnext
    ...
def exp(x): return np.exp(-x*x)
def expprop(x): return np.random.normal(loc=x, scale=1e-1)
    ...
nsamples = 1000
xs = list(itertools.islice(mhsimple(0, exp, expprop), nsamples))

```

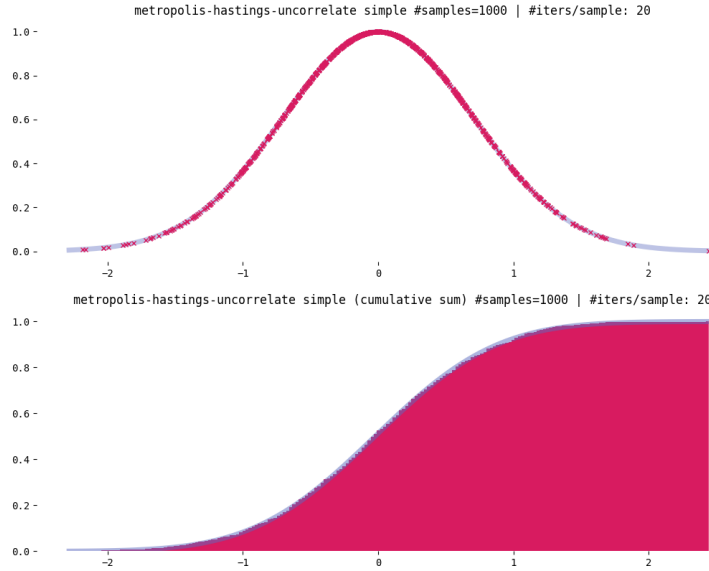


We can see from the plots of the raw samples and the cumulative distribution that we wind up overshooting. This is because the markov chain, by definition, has correlations between samples; However, we are supposed to be drawing *independent* samples from \mathfrak{P} for plotting/downstream use.

2.5.2 Sampling per `iters_per_sample` steps

The solution is to take samples that are *spaced out* — that is, we do not consider each step from the markov chain as a sample. Rather, we consider the state we are in after some `iters_per_sample` steps to be a sample. In code, this is now:

```
# mcmc1d.py
def mh_uncorr(x0, prob, prop, iters_per_sample):
    yield x0; x = x0;
    while True:
        for i in range(iters_per_sample):
            xnext = prop(x); p = prob(x); pnext = prob(xnext)
            r = np.random.uniform() + 1e-5;
            if r < min(1, pnext/p): x = xnext
        yield xnext
...
def exp(x): return np.exp(-x*x)
def expprop(x): return np.random.normal(loc=x, scale=1e-1)
...
NSAMPLES = 1000; ITERS_PER_SAMPLE = 20
xs = list(itertools.islice(mhsimple(0, exp, expprop, ITERS_PER_SAMPLE), NSAMPLES))
```

3 Hamiltonian Monte Carlo

If our target probability density $\mathfrak{P} : X \rightarrow [0, 1]$ is *differentiable*, then we can use the derivative of $\mathfrak{P}(x)$ to provide better proposals. The idea is as follows:

- Interpret the probability landscape as a potential, with points of high probability being "valleys" and points of low probability being "peaks" (ie, invert the probability density with a transform such as $U[\mathfrak{P}](x) = e^{-\mathfrak{P}(x)}$. This way, a ball rolling on this terrain will try to move towards the valleys — which are the locations of high probability \mathfrak{P} .
- For a proposal at a position $x_0 \in X$, keep a ball at x_0 , *randomly choose its velocity*, simulate the ball according to classical mechanics (Newton's laws of motion) for a fixed duration $D \in \mathbb{R}$ and propose the final position as the final position of the ball. This is reversible and detail balanced because *classical mechanics is reversible and detail balanced*.

We need the derivative of $\mathfrak{P}(x)$ to be able to simulate the ball according to Newton's laws. If we can do this, though, we are able to cover large amounts of terrain.

3.1 Newton’s laws of motion: Hamilton’s equations

$$\frac{\partial \mathbf{q}}{\partial t} = \frac{\partial H}{\partial \mathbf{p}} \quad \frac{\partial \mathbf{p}}{\partial t} = -\frac{\partial H}{\partial \mathbf{q}}$$

In our case, the choice of the hamiltonian will consider the negative-log-probability-density to be the *energy*. So, higher probability has lower energy. We know that particles like to go towards lower energy states.

$$H(\mathbf{q}, \mathbf{p}) \equiv U(\mathbf{q}) + K(\mathbf{p}) \quad K(\mathbf{p}) \equiv \mathbf{p}^T M^{-1} \mathbf{p} / 2$$

where M is a symmetric positive-definite matrix known as a “mass matrix”.

3.2 Modifying Hamilton’s equations for generating proposals

We wish to maximize $\mathfrak{P}(x)$. To do this, we will consider a new energy function $U[\mathfrak{P}](x) \equiv e^{-\mathfrak{P}(x)}$. When $U[\mathfrak{P}] = e^{-U[\mathfrak{P}](x)}$ is minimized, then $\mathfrak{P}(x)$ will be maximized.

We use the proposal function as:

- We are currently at the location \mathbf{q}_0 .
- Pick a uniform random momentum \mathbf{p} .
- Simulate the system according to the differential equations mentioned above, to produce a new $(\mathbf{q}', \mathbf{p}')$.
- Use this $(\mathbf{q}', \mathbf{p}')$ as the proposed point from the proposal.
- Use metropolis-hastings accept-reject.

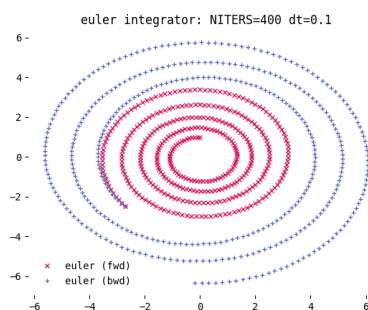
3.3 Hamiltonian for our simulation

If we have a target probability distribution $Prob(\mathbf{q}) : \mathbb{R}^n \rightarrow \mathbb{R}$ we choose the Hamiltonian to be $H(\mathbf{p}, \mathbf{q}) \equiv Prob(\mathbf{q}) + \mathbf{p}^T \mathbf{p} / 2$.

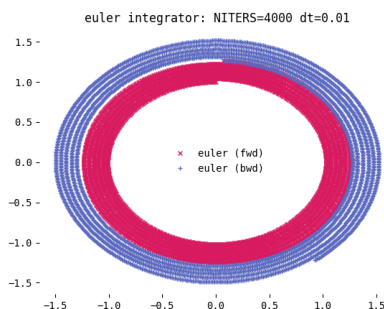
3.4 Simulation: Euler integration

Unfortunately, it turns out that running this simulation is in fact numerically unstable on using a naive simulation schemes. For example, let us say that we wish to simulate the orbit of a planet. Recall that we want the proposal to be symmetric: so, if we simulate the trajectory of the planet for N timesteps, each timestep of time δt , and then *reverse* the momentum of the planet, run the next phase of the simulation for N timesteps with timestep Δt , we should begin where we started. However, if we try to use the euler integration equations, here is what we see:

```
def euler(dhdp, dhdq, q, p, dt):
    pnew = p + -dhdq(q, p) * dt
    qnew = q + dhdp(q, p) * dt
    return (qnew, pnew)
```



We can clearly see the forward trajectory (in pink) spiralling out, and the backward trajectory (in blue), spiralling out even more. We can attempt to fix this by making Δt smaller:



to no avail. Indeed, this is a **fundamental limitation of euler integration**. Hence, we will need to explore more refined integration schemes.

3.5 Simulation: Symplectic integrator

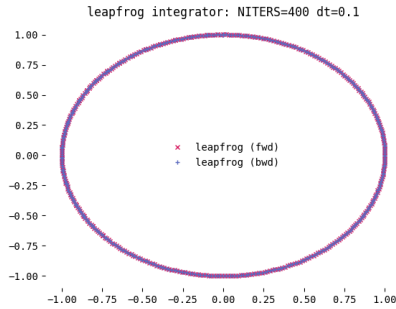
The type of integrators that will allow us to get 'reasonable orbits' that do not decay with time are known as *symplectic integrators*. (An aside: the word *symplectic* comes from Weyl, who substituted the latin root in the word *complex* by the corresponding greek root. It is a branch of differential geometry which formalizes the constructs needed to carry out hamiltonian mechanics on spaces that are more complicated than Euclidian \mathbb{R}^n).

```
## dq/dt = dH/dp|_{p0, q0}, dp/dt = -dH/dq|_{p0, q0}
def leapfrog(dhdp, dhdq, q0, p0, dt):
    p0 += -dhdq(q0, p0) * 0.5 * dt # kick: half step momentum
    q0 += dhdp(q0, p0) * dt # drift: full step position
```

```

p0 += -dhdq(q0, p0) * 0.5 * dt # kick: half step momentum
return (q0, p0)

```



It is clear from the plots that the leapfrog integrator is stable: orbits stay as orbits. Indeed, we can prove the symplecticity/energy-preserving property of this integrator.

3.6 Proof that leapfrog is symplectic

We will show that in phase space, the leapfrog integrator preserves infinitesimal volumes of the form $dV \equiv d\mathbf{p} \times d\mathbf{q}$.

3.7 Simulation: Final

We no longer need to choose how many steps to walk with a no-U-turn-sampler. It prevents us from re-walking energy orbits, by detecting when we have completed traversing an orbit and are going to take a "U-turn". The details are quite complex, so we may not cover this here.