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Intransitive Indifference with Unequal Indifference Intervals

PETER C. FISHBURN

Research Analysis Corporation, McLean, Virginia 22101

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An interval order is a binary relation $\prec$ on a set X that is irreflexive and satisfies the condition: if $x \prec y$ and $z \prec w$ then $x \prec w$ or $z \prec y$. An interval order is a special kind of strict partial order (transitive, irreflexive) and a generalization of the semiorder concept. If X is countable then there are real-valued functions u and ρ on X such that ρ is positive and [$x \prec y$ if and only if $u(x) + \rho(x) < u(y)$], if and only if $\prec$ is an interval order.

1. ORDERS

In examining the concept of intransitive indifference, it seems natural to begin with a rather general order notion and to add conditions to that notion to see what effect the additional conditions have on numerical representation possibilities. In this paper we shall do this using $\prec$ (interpreted here as strict preference, reading $x \prec y$ as y is preferred to x) as the primitive binary relation on a set X. Three successively restricted cases for $\prec$ are defined as follows.

D1. $\prec$ on X is a *strict partial order* (Suppes, 1957) if and only if $\prec$ is irreflexive (not $x \prec x$) and transitive ($x \prec y$ and $y \prec z$ imply $x \prec z$).

D2. $\prec$ on X is an *interval order* if and only if $\prec$ is irreflexive and satisfies the following *interval order condition*: if $x \prec y$ and $z \prec w$ then $x \prec w$ or $z \prec y$.

D3. $\prec$ on X is a *semiorder* (Luce, 1956; Scott and Suppes, 1958) if and only if $\prec$ is an interval order and satisfies the following condition: if $x \prec y$ and $y \prec z$ then $x \prec w$ or $w \prec z$ for each w in X.

Because irreflexivity and the interval order property imply that $\prec$ is transitive, an interval order is a strict partial order that satisfies the interval order condition. Following the lead provided by the theory of semiorders (for recent work see Suppes and Zinnes, 1963; Scott, 1964; Krantz, 1967), we shall define indifference ($\sim$) as the absence of strict preference and also adopt the usual equivalence ($\approx$) definition.

D4. $x \sim y$ if and only if (not $x \prec y$, not $y \prec x$). $x \approx y$ if and only if, for all z in X, $x \sim z$ if and only if $y \sim z$.

Since ($x \sim y$, $y \sim z$, $x \prec z$) can hold if $\prec$ is a semiorder, $\sim$ is not necessarily transitive when D1, or D2, or D3 is used. You should have little difficulty in proving the following results.

THEOREM 1. *If $\prec$ on X is a strict partial order then*

1. *$\approx$ is an equivalence (reflexive, symmetric, transitive);*
2. *$(x \prec y, y \approx z)$ implies $x \prec z$; $(x \approx y, y \prec z)$ implies $x \prec z$;*
3. *$x \approx y$ if and only if, for all z in X, ($z \prec x$ if and only if $z \prec y$, $x \prec z$ if and only $y \prec z$).*

Before establishing a representation theorem for interval orders, which uses the results of Theorem 1, some notes on strict partial orders and semiorders will be presented in the next section. These will show the kinds of representation theorems appropriate to D1 and D3, and give an indication of the position of an interval order with respect to the other two orders.

2. Representation Theorems

In the following theorem $X/\approx$ is the set of $\approx$ classes of X.

THEOREM 2. *If $\prec$ on X is a strict partial order and $X/\approx$ is countable then there is a real-valued function u on X such that, for all x and y in X,*

$$x \prec y \text{ implies } u(x) < u(y) \tag{1}$$

$$x \approx y \text{ implies } u(x) = u(y). \tag{2}$$

Proof. Select one element from each $\approx$ class, and let Y denote the selected set. Szpilrajn's extension theorem (1930) says that $\prec$ on Y can be extended to a strict order $<^*$ on Y ($<^*$ is transitive, irreflexive, and if $x \neq y$ then $x <^* y$ or $y <^* x$), with $x <^* y$ if $x \prec y$. It then follows from a theorem by Milgram (1939) (or see Birkhoff, 1948, p. 31; or Suppes and Zinnes, 1963, pp. 26–28) that there is a real-valued function u on Y such that $x <^* y$ if and only if $u(x) < u(y)$. Equation 1 for Y follows from this. Let $u(z) = u(x)$ when x is in Y and $z \approx x$. Equations 1 and 2 for all of X then follow from Theorem 1.

If $x \sim y$ then any one of $u(x) < u(y)$, $u(y) < u(x)$, and $u(x) = u(y)$ might hold for u as in Theorem 2.

Without imposing other conditions on the strict partial order $\prec$, Theorem 2 is about as far as we can go when $X/\approx$ is countable (whether finite or denumerable). In contrast to this, if $\prec$ is a semiorder and $X/\approx$ is finite then a much stronger result is obtained.

THEOREM 3. *If $\prec$ on X is a semiorder and $X/\approx$ is finite then there is a real-valued function u on X such that, for all x and y in X,*

$$x \prec y \text{ if and only if } u(x) + 1 < u(y). \tag{3}$$

Proofs of this are given by Scott and Suppes (1958), Suppes and Zinnes (1963), Scott (1964). Equation 3 is stronger than (1) and (2) in two ways. First, the representation is "if and only if." Thus, by (3), $x \sim y$ if and only if $|u(x) - u(y)| \leqslant 1$. Second, it suggests the idea of a constant *jnd* over the range of u, as represented by the positive constant in (3). [Compare this with (4) below.]

We can think of (3) for semiorders in terms of a closed interval

$$I(x) = [u(x), u(x) + 1]$$

for each x in X. If $I(x)$ and $I(y)$ overlap (intersect) then $x \sim y$. On the other hand, if $I(x) \cap I(y) = \varnothing$, then $x \prec y$ if $I(x)$ is to the left of $I(y)$, and $y \prec x$ if $I(y)$ is to the left on $I(x)$ on the real line.

A main feature of these semiorder indifference intervals is that they all have (or can be made to have) the same length when $X/\approx$ is finite. It therefore is natural to ask about cases where such indifference intervals can be defined but it may be impossible to make them all have the same length, even when $X/\approx$ is finite. These are precisely the cases covered by interval orders.

THEOREM 4. *If $\prec$ on X is an interval order and $X/\approx$ is countable then there are real-valued functions u and ρ on X with $\rho(x) > 0$ for each x in X such that, for all x and y in X,*

$$x \prec y \text{ if and only if } u(x) + \rho(x) < u(y). \tag{4}$$

This is proved in the final section. As with (3) we can define intervals for (4) as follows:

$$J(x) = [u(x), u(x) + \rho(x)] \text{ for each } x \text{ in } X.$$

Then, under the conditions of Theorem 4, $x \sim y$ if and only if $J(x) \cap J(y) \neq \varnothing$; and if $J(x) \cap J(y) = \varnothing$ then $x \prec y$ if $J(x)$ is to the left of $J(y)[J(x) < J(y)]$, and $y \prec x$ if $J(y) < J(x)$.

To see why it may be impossible to make all J have the same length when X is finite, suppose $X = \{x, y, z, w\}$ with

$$x \prec y \prec z \quad \text{and} \quad x \sim w, \quad y \sim w, \quad z \sim w.$$

Then $\prec$ is an interval order but not a semiorder since it violates $[(x \prec y, y \prec z)$ implies $x \prec w$ or $w \prec z]$. To satisfy (4), $J(x)$ must be to the left of $J(y)$, which in turn is to the left of $J(z)$. But $J(w)$ intersects each of these, so that $J(y)$ must be shorter than $J(w)$, and hence $\rho(y) < \rho(w)$.

Thus, the interval order notion applies when we can visualize the elements as represented by indifference intervals on a continuum where it may be necessary to have one interval wholly included within another. For example, we might think of the indifference intervals as controlled by two forces, one of which governs placement of the midpoints, the other governing their ranges or spreads.

3. Interval Graphs

Before proving Theorem 4, I should like to point out the relationship between that theorem and what are called interval graphs.[1] To use our prior notation, $(X, \sim)$ is an interval graph if and only if a real interval $J(x)$ can be assigned to each x in X so that, for all x and y in X,

$$x \sim y \text{ if and only if } J(x) \cap J(y) \neq \varnothing .$$

Conditions that are necessary and sufficient for a graph to be an interval graph are presented by Lekkerkerker and Boland (1962), Gilmore and Hoffman (1964), and Fulkerson and Gross (1965).

The preceding expression is of course a consequence of (4), which goes further in specifying an order on nonintersecting intervals according to $\prec$. Roberts (1968, 1969) also has made some extensions on the theme of interval graphs by bringing in an order $\prec$ and defining $\sim$ as we have above. Much of his work is done in the semiorder context.

Roberts has pointed out to me that the connection between the interval order theorem and interval graphs can be expressed in the following way. If X is countable then the following are equivalent:

A. $\prec$ on X is an interval order.

B. $\prec$ on X is transitive and $(X, \sim)$ is an interval graph.

C. There is a real interval $J(x)$ for each x in X so that, for all x and y in X, $x \prec y$ if and only if $J(x) < J(y)$.

Clearly, C implies B, and it is not hard to see that B implies A. The next section proves that A implies C.

4. Proof of Theorem 4

Throughout, $\Leftrightarrow$ means "if and only if", $\Rightarrow$ means "implies", and if a binary relation of the form $\leqslant'$ is defined then $x <' y \Leftrightarrow (x \leqslant' y, \text{ not } y \leqslant' x)$, and

$$x =' y \Leftrightarrow (x \leqslant' y, y \leqslant' x).$$

[1] I am grateful to a referee for directing me to Fred Roberts, who has alerted me to some of the work done in this area.

In addition, $x \preccurlyeq y \Leftrightarrow (x \prec y$ or $x \sim y)$. We assume that $\prec$ is an interval order.

D5. $x \leqslant^L y \Leftrightarrow (z \prec x \Rightarrow z \prec y)$ for all z in X.

D6. $x \leqslant^R y \Leftrightarrow (y \prec z \Rightarrow x \prec z)$ for all z in X.

The transitivity of $\leqslant^L$ and of $\leqslant^R$ follows immediately from these definitions. Suppose (not $x \leqslant^L y$, not $y \leqslant^L x$). Then, for some z and w in X, $(z \prec x$, not $z \prec y)$ and $(w \prec y$, not $w \prec x)$, which contradict the condition in D2. Hence $\leqslant^L$ is connected: so is $\leqslant^R$. This establishes L1; L2 is immediate from Theorem 1.3.

L1. *$\leqslant^L$ on X is a weak order; $\leqslant^R$ on X is a weak order.*

L2. $x \approx y \Leftrightarrow (x =^L y, x =^R y)$.

Let Y consist of exactly one element from each $\approx$ class of X. For each x in Y let x^* denote an artificial element that corresponds to x, with Y^* the set of artificial elements. We shall now define a binary relation $\leqslant^0$ on $Y \cup Y^*$ as follows, where an element with no asterisk is in Y and an element with an asterisk is in Y^*:

$$x \leqslant^0 y \Leftrightarrow x \leqslant^L y \tag{5}$$

$$x^* \leqslant^0 y^* \Leftrightarrow x \leqslant^R y \tag{6}$$

$$x^* <^0 y \Leftrightarrow x \prec y \tag{7}$$

$$x <^0 y^* \Leftrightarrow x \preccurlyeq y. \tag{8}$$

According to these definitions $x =^0 y^*$ and $x^* =^0 y$ are impossible.

L3. *$\leqslant^0$ on $Y \cup Y^*$ is a weak order.*

To prove connectedness observe that (5) and (6) under L1 take care of a pair x, y or a pair x^*, y^*. With any pair x^*, y, either $x \prec y$ in which case $x^* <^0 y$ (and hence $x^* \leqslant^0 y$), or $y \preccurlyeq x$ in which case $y <^0 x^*$ (and hence $y \leqslant^0 x^*$). Thus $\leqslant^0$ on $Y \cup Y^*$ is connected.

To prove that $\leqslant^0$ is transitive we assume that $a \leqslant^0 b$, $b \leqslant^0 c$, and enumerate the possibilities for (a, b, c) as follows:

1. (x, y, z). $\therefore (x \leqslant^L y, y \leqslant^L z)$. $\therefore x \leqslant^L z$ by L1. $\therefore x \leqslant^0 z$.
2. (x^*, y^*, z^*). $\therefore (x \leqslant^R y, y \leqslant^R z)$. $\therefore x \leqslant^R z$ by L1. $\therefore x^* \leqslant^0 z^*$.
3. (x, y, z^*). $\therefore (x \leqslant^L y, y \preccurlyeq z)$. $(z \prec x) \Rightarrow z \prec y$ by D5, a contradiction. $\therefore x \preccurlyeq z$. $\therefore x <^0 z^*$.
4. (x, y^*, z). $\therefore (x \preccurlyeq y, y \prec z)$. $(z \leqslant^L x) \Rightarrow y \prec x$ by D5, a contradiction. $\therefore x <^L z$. $\therefore x <^0 z$.
5. (x^*, y, z). $\therefore (x \prec y, y \leqslant^L z)$. $\therefore x \prec z$ by D5. $\therefore x^* <^0 z$.
6. (x, y^*, z^*). $\therefore (x \preccurlyeq y, y \leqslant^R z)$. $(z \prec x) \Rightarrow y \prec x$ by D6, a contradiction. $\therefore x \preccurlyeq z$. $\therefore x <^0 z^*$.

7. (x^*, y, z^*). $\therefore (x \prec y, y \preccurlyeq z)$. $(z \leqslant^R x) \Rightarrow z \prec y$ by D6, a contradiction. $\therefore x <^R z$. $\therefore x^* <^0 z^*$.

8. (x^*, y^*, z). $\therefore (x \leqslant^R y, y \prec z)$. $\therefore x \prec z$ by D6. $\therefore x^* <^0 z$.

This concludes the proof of L3.

Assume that $X/\approx$ is countable. Then $Y \cup Y^*$ is countable and it follows from L3 and Theorem 6 in Suppes and Zinnes (1963) that there is a real-valued function f on $Y \cup Y^*$ such that, for all b and c in $Y \cup Y^*$,

$$b \leqslant^0 c \Leftrightarrow f(b) \leqslant f(c). \tag{9}$$

For x in Y let $u(x) = f(x)$, $\rho(x) = f(x^*) - f(x)$. Then, by (9) and (7),

$$x \prec y \Leftrightarrow u(x) + \rho(x) < u(y) \tag{10}$$

for all x and y in Y. Moreover, since $x <^0 x^*$ by (8), $\rho > 0$. Finally, (10) holds for all x and y in X according to L2 and Theorem 1.2 when u and ρ are extended to X in the obvious $\approx$ way.

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