MATH1071
Advanced Calculus & Linear Algebra I

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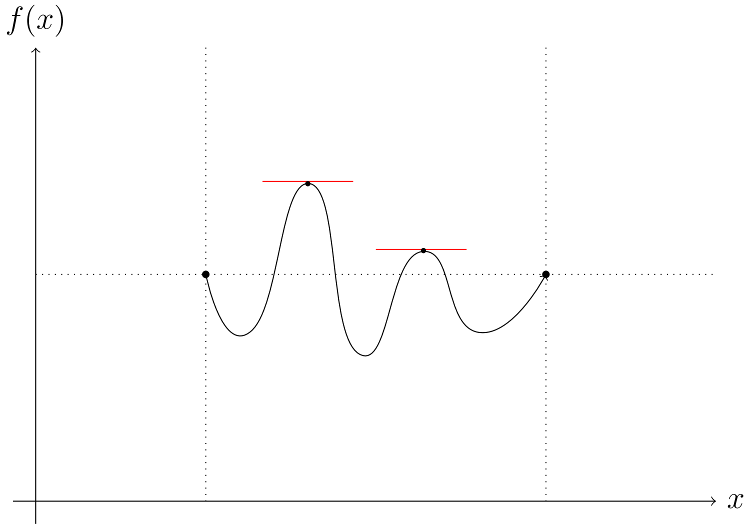
Contents

Lecture 22 - Rolle's Theorem and Mean Value Theorem

Theorem: Rolle's Theorem

Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, and differentiable on (a,b). If $f(a) = f(b)$, then $\exists c \in (a,b)$ such that $f'(c) = 0$



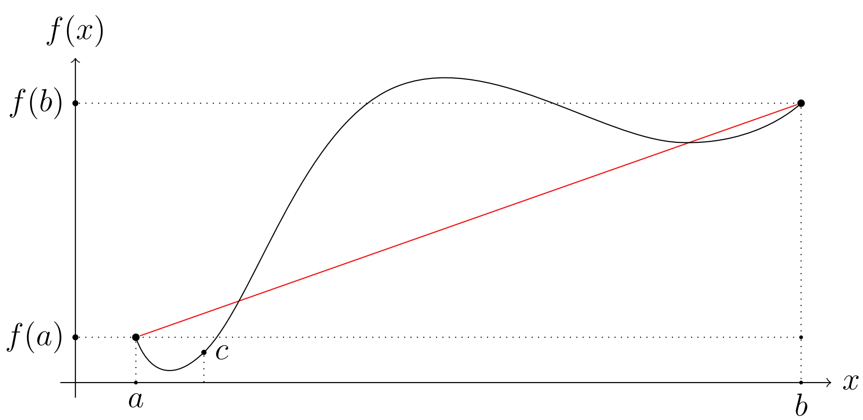
Proof. If $f(x) = f(a)\ \forall x \in (a,b)$, then the theorem is obvious. Assume f is not constant, without loss of generality, assume $\exists x_0 \in (a,b)$ such that $f(x_0) > f(a)$. (If no such x_0 exists then there exists some $x_1 \in (a,b)$ such that $f(x_1) < f(a)$; in this case, a similar argument works).

By the extreme value theorem, f has a global minimum on $[a,b]$, call it c. Since $f(c) \geq f(x_0) > f(a) = f(b)$, we have $c \neq a, c \neq b$. Thus c lies within (a,b) and $f'(c) = 0$. $\square$

Theorem: Mean Value Theorem

Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b). Then $\exists c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Proof. Consider the function

$$\phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

Observe that

$$\begin{aligned}
\phi(a) &= f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) \\
&= 0 \\
\phi(b) &= f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) \\
&= 0
\end{aligned}$$

Applying Rolle's Theorem to $\phi(x)$ we obtain the existence of $c \in (a, b)$ such that

$$\phi(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

Which gives our result

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

□

Example: Applications of MVT

Assuming $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b)

1. If $f'(x) = 0$ on (a, b), then f is continuous on $[a, b]$, or rather $f'(x) \iff f(x) = c$, where c is some constant value. Indeed take $x \in [a, b]$, apply MVT on $[a, x]$ and we conclude that $f(x) - f(a) = f'(c)(x - a)$ for some $c \in [a, x]$. Therefore $f(x) - f(a) = 0$, which gives $f(x) = f(a)$

Theorem: Properties

1. If $f' \geq 0$ on (a, b), then f is monotone decreasing,

   Proof. Take $x, y \in [a, b]$ and assume $x < y$. Need to prove that $f(x) \leq f(y)$. Apply MVT on $[x, y]$. We need to find $f(y) - f(x) = f'(c)(y - x)$, where both $f'(c), (y - x) \geq 0$, for some $c \in (x, y)$. By assumption, $f'(c) \geq 0$. Therefore $f(y) \geq f(x)$. □

2. if $f' \leq 0$ on (a, b), then f is non-zero.

   Proof. Apply (a) to $-f$ □

3. if $f' > 0$, then f is strictly increasing.
4. if $f' < 0$, then f is strictly decreasing.

   Proof. Both (c), (d), analagously proved. □