1071 Final Theorem List

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Contents

1 Squeeze Theorem

Theorem: Squeeze Theorem For Sequences

Suppose $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$ are such that

1. $$a_n \leq b_n \leq c_n \qquad \forall n \in \mathbb{N}$$

2. $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

Then,

$$\lim_{n\to\infty} b_n = L$$

Proof

Observe that,

$$\begin{aligned}
|b_n - L| &= |b_n - a_n + a_n - L| \\
|(b_n - a_n) + (a_n - L)| &\leq |b_n - a_n| + |a_n - L| = b_n - a_n + |a_n - L| \\
&\leq c_n - a_n + |a_n - L| = |c_n - L + L - a_n| + |a_n - L| \\
&\leq |c_n - L| + |L - a_n| + |a_n - L|
\end{aligned}$$

Fix $\varepsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then

$$|a_n - L| = |L - a_n| < \frac{\varepsilon}{3}$$

.
Also, there is $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then

$$|c_n - L| < \frac{\varepsilon}{3}$$

Now, set $N = \max N_1, N_2$. If $n \geq N$ then,

$$|b_n - L| \leq |c_n - L| + |L - a_n| + |a_n - L| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Thus, $\lim\limits_{n\to\infty} b_n = L$.

$\square$

2 A Convergent Sequence is Bounded

Theorem: A Convergent Sequence is Bounded

Convergent sequences are bounded. More precicely, if $(a_n)_{n=1}^{\infty}$ converges, then there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$

Proof

Assume $\lim\limits_{n\to\infty} a_n = L$.
For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - L| < \varepsilon$.
This holds for instance if $\varepsilon = 1$.
Thus, there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then

$$|a_n - L| < 1$$

By the second triangle inequality,

$$|a_n - L| \geq |a_n| - |L|$$

Thus, $|a_n| - |L| < 1$ and $|a_n| < 1 + |L|$
Now, $|a_n| \leq M = \max\{|a_1|, |a_2|, |a_3|, \cdots, |a_{n-1}|, |L| + 1\}$ for all $n \in \mathbb{N}$. $\square$

3 A Bounded, Monotone Sequence Converges

Theorem: A Bounded, Monotone Sequence Converges

A monotone sequence converges if and only if it is bounded.

Proof

$(\implies)$
This is already proven, see 2
$(\impliedby)$
Without loss of generality, assume $(a_n)_{n=1}^{\infty}$ is increasing (If the sequence is monotone decreasing, a similar argument is used).
Define $\alpha = \sup\{a_1, a_2, a_3, \cdots, a_n\} \in \mathbb{R}$.
Since the sequence is bounded, this supremum exists in $\mathbb{R}$.
Let us prove

$$\alpha = \lim_{n\to\infty} a_n$$

Chose $\varepsilon > 0$.
By an earlier theorem, there exists $N \in \mathbb{N}$ such that $a_N \in (\alpha - \varepsilon, \alpha]$.
By monotonicity, if $n \geq N$ then $a_n \geq a_N > \alpha - \varepsilon$. Also, $a_n \leq \alpha$.
Thus, if $n \geq N$ then $\alpha - \varepsilon < a_n \leq \alpha < \alpha + \varepsilon \implies |a_n - a| < \varepsilon$. Thus, $\lim_{n\to\infty} a_n = \alpha$, so $(a_n)_{n=1}^{\infty}$ converges.

$\square$

4 Existence of a Monotone Subsequnce

Theorem: Existence of a Monotone Subsequnce

Every sequence of real numbers has a monotone subsequence.

Proof

We call $k \in \mathbb{N}$ a peak point of $(a_n)_{n=1}^{\infty}$ if $a_k > a_n$ for all $n > k$.
Case 1: There are infinitely many peak points, $n_1 < n_2, n_3 < \cdots$. By definition, $a_{n_1} > a_{n_2} > a_{n_3} > \cdots$. Thus, $(a_{n_k})_{k=1}^{\infty}$ is monotone.
Case 2: There are finitely many peak points. $k_1, k_2, k_3, \cdots, k_l$. Set

$$n_1 = \max\{k_1, \cdots, k_l\} + 1$$

Clearly, n_1 is not peak. Therefore, there is some $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$. Now, n_2 is not peak. Therefore there is $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$. Continue in this way to obtain a monotone subsequence $(a_{n_i})_{i=1}^{\infty}$. $\square$

5 Bolzano-Weiestrass Theorem

Theorem: Bolzano-Weiestrass Theorem

Every bounded sequence has a convergent subsequence.

Proof

Every sequence has a monotone subsequence. Every subsequence of a bounded sequence is bounded, thus, we have a bounded, monotone subsequence which must converge. $\square$

6 Limit Arithmetic

Theorem: Limit Arithmetic

Assume $\lim_{x \to a} f(x) = L \quad \lim_{x \to a} g(x) = m$.
Then,

1. $\lim_{x \to a} (f(x) + g(x)) = L + m$
2. $\lim_{x \to a} (f(x) \cdot g(x)) = L \cdot m$
3. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{m}$

Proof

1. Fix $\varepsilon > 0$. We need to find $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x)+g(x)-L+m| < \varepsilon$.

$$\begin{aligned} |f(x) + g(x) - (L+m)| &= |f(x) + g(x) - (L+m)| \\ &= |(f(x) - L) + (g(x) - m)| \\ &\leq |f(x) - L| + |g(x) - m| \end{aligned}$$

   Since $\lim_{x \to a} f(x) = L, \quad \lim_{x \to a} g(x) = m$, there exists $\delta_1, \delta_2 > 0$ such that if $0 < |x-a| < \delta_1, \delta_2$ then,

$$\begin{aligned} |f(x) - L| &< \frac{\varepsilon}{2} \\ |g(x) - m| &< \frac{\varepsilon}{2} \end{aligned}$$

   Choosing $\delta = \min\{\delta_1, \delta_2\}$, we have that,

$$\begin{aligned} |f(x) - L| + |g(x) - m| &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

2. Fix $\varepsilon > 0$. We need to find $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x)g(x) - Lm| < \varepsilon$.

$$
\begin{aligned}
|f(x)g(x) - Lm| &= |f(x)g(x) - f(x)m + f(x)m - Lm| \\
&= |f(x)(g(x) - m) + m(f(x) - L)| \\
&\leq |f(x)||g(x) - m| + |m||f(x) - L|
\end{aligned}
$$

Since $\lim_{x \to a} f(x) = L$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then, $|f(x) - L| < 1$.
In this case, $|f(x)| - |L| < 1 \implies |f(x)| < 1 + |L|$.
Thus, if $0 < |x - a| < \delta$, then $|f(x)g(x) - Lm| \leq |1 + |L|||g(x) - m| + |m||f(x) - L|$.
There exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then, $|g(x) - m| < \frac{\varepsilon}{2(|m|+1)}$.
Set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$,
If $0 < |x - a| < \delta$ then,

$$
\begin{aligned}
|f(x)g(x) - Lm| &\leq (1 + |L|)|g(x) - m| + |m||f(x) - L| \\
&< \frac{\varepsilon}{2(|L| + 1)} \cdot (|L| + 1) + \frac{\varepsilon}{2(|m| + 1)} \cdot 2(|m| + 1) \qquad \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

3. We need to show first that $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{m}$

   Fix $\varepsilon > 0$.
   Since $\lim_{x \to a} g(x) = m$, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then, $|g(x) - m| < \frac{|m|}{2}$.

$$
\begin{aligned}
|m| &= |m + g(x) - g(x)| \\
&\leq |g(x) - m| + |g(x)| \\
&< \frac{|m|}{2} + |g(x)| \\
\frac{|m|}{2} &< |g(x)| \\
\frac{1}{g(x)} &< \frac{2}{|m|}
\end{aligned}
$$

   Indeed, there exists some $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then $|g(x) - m| < \frac{|m|^2}{2}\varepsilon$.
   Choosing $\delta = \min\{\delta_1, \delta_2\}$, if $0 < |x - a| < \delta$ we have,

$$\begin{aligned}
\left|\frac{1}{g(x)-\frac{1}{m}}\right| &= \left|\frac{m-g(x)}{mg(x)}\right| \\
&= \frac{1}{|mg(x)|}|g(x)-m| \\
&< \frac{1}{|m|}\frac{2}{|m|}|g(x)-L| \\
&< \frac{2}{|m|^2}\frac{|m|^2}{2}\varepsilon \\
&= \varepsilon
\end{aligned}$$

Now we have proven that $\lim_{x\to a}\frac{1}{g(x)} = \frac{1}{m}$, the more general fact follows.

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a} f(x)\frac{1}{g(x)} \\
&= \lim_{x\to a} f(x) \lim_{x\to a}\frac{1}{g(x)} \\
&= L\frac{1}{m} \\
&= \frac{L}{m}
\end{aligned}$$

□

7 Differentiability implies Continuity

Proposition: Differentiability implies Continuity

Suppose f is differentiable at x_0. Then f is continuous at x_0.

Proof

Observe that:

$$\begin{aligned}
\lim_{x\to\ x_0} f(x) &= \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}\cdot(x-x_0)+f(x_0)\right) \\
&= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\cdot\lim_{x\to x_0}(x-x_0)+f(x_0) \\
&= f'(x_0)\cdot(x_0-x_0)+f(x_0) \\
&= 0\cdot f'(x_0)+f(x_0) = f(x_0)
\end{aligned}$$

Therefore, f is continuous at x_0. $\square$

8 Derivative Arithmetic

Theorem: Derivative Arithmetic

Suppose $f, g : (a, b) \to \mathbb{R}$ are both differentiable at $x \in (a, b)$. Then, $f + g, fg, \frac{f}{g}$ are differentiable at x. We need $g(x) \neq 0$ for $\frac{f}{g}$.

1. $(f + g)'(x) = f'(x) + g'(x)$
2. $(fg)'(x) = f'(x)g(x) + g'(x)f(x)$
3. $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

Proof

1.

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)+g'(x)\end{aligned}$$

2.

$$\begin{aligned}(fg)'(x) &= \lim_{h\to 0}\frac{f(x+g)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= f(x)g'(x)+f'(x)g(x)\end{aligned}$$

3.

$$\begin{aligned}\left(\frac{f'(x)}{g'(x)}\right) &= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}\\ &= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}\cdot\frac{g(x)}{g(x)}-\frac{f(x)}{g(x)}\cdot\frac{g(x+h)}{g(x+h)}}{h}\\ &= \lim_{h\to 0}\frac{g(x)f(x+h)-f(x)(g(x+h))}{hg(x)g(x+h)}\\ &= \lim_{h\to 0}\frac{g(x)f(x+h)-f(x)g(x)+f(x)g(x)-f(x)(g(x+h))}{hg(x)g(x+h)}\\ &= \lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))-f(x)(g(x)-g(x+h))}{hg(x)g(x+h)}\\ &= \frac{g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}-f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}}{\lim_{h\to 0}g(x)g(x+h)}\\ &= \frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}\end{aligned}$$

□

9 Rolle's Theorem

Theorem: Rolle's Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof

If $f(x) = f(a)$ for all $x \in (a, b)$, then the theorem is obvious.
Assume f is not constant.
Without loss of generality, assume there exists $x_0 \in (a, b)$ such that $f(x_0) < f(a)$.
(If no such x_0 exists, then there exists $x_1 \in (a, b)$ such that $f(x_1) < f(a)$. In this case, a similar argument works.)
By the Extreme Value theorem, f has a global maximum on $[a, b]$, call this c. Since,

$$f(c) \geq f(x_0) > f(a) = f(b)$$

we know $c \neq a, c \neq b$. Thus, c lies in (a, b), and $f'(c) = 0$ $\square$

10 The Mean Value Theorem

Theorem: The Mean Value Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then, there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof

Consider the function $\phi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a} \cdot (x - a)$.
Observe that

$$\phi(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (a - a) = 0$$

$$\phi(b) = f(b) - f(a) - \frac{f(b) - f(a)}{\cancel{b - a}} \cdot \cancel{(b - a)} = 0$$

Applying Rolle's theorem to $\phi(x)$, we obtain the existence of $c \in (a, b)$ such that,

$$\phi'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

$\square$

11 A Vanishing Derivative Implies a Constant Function

Proposition: Vanishing Derivative Implies a Constant Function

Suppose $f : [a, b] \implies \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$

Proof

Take $x \in (a, b]$. Applying the Mean Value theorem on $[a, x]$, we conclude that $f(x) - f(a) = f'(c)(x - a)$ for some $c \in (a, x)$. Therefore, as $f'(c) = 0$,

$$f(x) - f(a) = 0(x - a)$$
$$f(x) = f(a)$$

Thus, f is constant in $[a, b]$. □

12 A Continuous Function on an interval is Uniformly Continuous

Theorem: A Continuous Function on an interval is Uniformly Continuous

Suppose f is continuous on a closed, bounded interval $[a, b]$. Then, f is uniformly continuous on $[a, b]$.

Proof

By contradiction.
Assume f is not uniformly continuous.
There exists $\varepsilon_0 > 0$ such that for all $\delta > 0$, there exists $x, y \in [a, b]$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$.
Take $\delta = 1$. There exists $x_1, y_1 \in [a, b]$ such that $|x_1 - y_1| < 1$ but $|f(x_1) - f(y_1)| \geq \varepsilon_0$.
Take $\delta = \frac{1}{2}$. There exists $x_2, y_2 \in [a, b]$ such that $|x_2 - y_2| < \frac{1}{2}$ but $|f(x_2) - f(y_2)| \geq \varepsilon_0$.
Continue in this way for $\delta = \frac{1}{n}, \quad \forall n \in \mathbb{N}$.
Then, there exists $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.
We have thus constructed two sequences.

$$(X_n)_{n=1}^{\infty}$$
$$(Y_n)_{n=1}^{\infty}$$

inside $[a, b]$.

Since $[a, b]$ is closed and bounded, $(X_n)_{n=1}^{\infty}$ must have some subsequence $(X_{n_k})_{k=1}^{\infty}$ which converges to some point $x_0 \in [a, b]$.

$$\lim_{k \to \infty} (X_{n_k}) = x_0$$

<u>Claim:</u>
The subsequence $(Y_{n_k})_{k=1}^{\infty}$ converges to x_0.
<u>Proof of the claim:</u>
Given $\varepsilon > 0$, we look for $N \in \mathbb{N}$ such that if $k \geq N$ then, $|Y_{n_k} - x_0| < \varepsilon$.

$$\begin{aligned} |Y_{n_k} - x_0| &= |Y_{n_k} + X_{n_k} - X_{n_k} - x_0| \\ &\leq |Y_{n_k} - X_{n_k}| + |X_{n_k} - x_0| \\ &< \frac{1}{n_k} + |X_{n_k} - x_0| \end{aligned}$$

Since $(X_n)_{n=1}^{\infty}$ converges to x_0, for k large enough, we have $|X_{n_k} - x_0| < \frac{\varepsilon}{2}$.
Then taking k large enough such that $\frac{1}{n_k} < \frac{\varepsilon}{2}$, we have,

$$\begin{aligned} |Y_{n_k} - x_0| &< \frac{1}{n_k} + |X_{n_k} - x_0| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Let us prove that this contradicts the continuity of f.
Since f is continuous and since

$$\lim_{k \to \infty} X_{n_k} = \lim_{n \to \infty} Y_{n_k} = x_0$$

we have that,

$$\lim_{k \to \infty} f(X_{n_k}) = \lim_{n \to \infty} f(Y_{n_k}) = f(x_0)$$

Then, for some $N \in \mathbb{N}$, if $k \geq N$ then $|f(X_{n_k}) - x_0| < \frac{\varepsilon}{4}$ and $|f(Y_{n_k}) - x_0| < \frac{\varepsilon}{4}$.
On the other hand,

$$\begin{aligned} |f(X_{n_k}) - f(Y_{n_k})| &= |f(X_{n_k}) - f(x_0) + f(x_0) - f(Y_{n_k})| \\ &\leq |f(X_{n_k}) - f(x_0)| + |f(x_0) - f(Y_{n_k})| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{2} \end{aligned}$$

Also, we know by construction that

$$|f(X_{n_k}) - f(Y_{n_k})| \geq \varepsilon_0$$

Thus, we have a contradiction. $\square$

13 Integrability Condition

Theorem: Integrability Condition

The function $f : [a, b] \to \mathbb{R}$ is integrable if and only if for every $\varepsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon$$

Proof

$(\implies)$
Assume f is integrable. Fix $\varepsilon > 0$. There exists P_1 such that

$$\int_a^b f(x)\,\mathrm{d}x = \int_{\underline{a}}^b f\,\mathrm{d}x < L(f, P_1) + \frac{\varepsilon}{2}$$

Also, there exists P_2 such that

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^{\overline{b}} f\,\mathrm{d}x > U(f, P_1) - \frac{\varepsilon}{2}$$

Define $P = P_1 \cup P_2$. Then,

$$\begin{aligned} U(f, P) - L(f, P) &\leq U(f, P_2) - L(f, P_1) \\ &< \int_a^b f(x)\,\mathrm{d}x + \frac{\varepsilon}{2} - \int_a^b f(x)\,\mathrm{d}x + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

$(\impliedby)$
Fix $\varepsilon > 0$. There exists P such that,

$$U(f, P) - L(f, P) < \varepsilon$$

Then,

$$\int_a^{\overline{b}} f(x)\,\mathrm{d}x - \int_{\underline{a}}^b f(x)\,\mathrm{d}x \leq U(f, P) - L(f, P) < \varepsilon$$

This means, $\int_a^{\overline{b}} f(x)\,\mathrm{d}x - \int_{\underline{a}}^b f(x)\,\mathrm{d}x = 0$.
Thus, f is integrable. $\square$

14 Finite Discontinuities implies Integrability

Theorem: Finite Discontinuities implies Integrability

If $f : [a, b] \to \mathbb{R}$ is continuous at all but finitely many points, then f is integrable.

Proof

<u>Part 1:</u>
Assume f is continuous on $[a, b]$ then f is uniformly continuous on $[a, b]$.
Fix $\varepsilon > 0$. We will find a partition P such that $U(f, P) - L(f, P) < \varepsilon$. This will imply integrability.
Uniform continuity implies that there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{|b-a|}$.
Choose P so that $|x_i - x_{i-1}| < \delta$ for all $i \in 1, 2, \cdots, n$.
Here $P = \{x_0, x_1, x_2, \cdots, x_{n-1}\}$. Then,

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} \left(\sup_{[x_{i-1}, x_i]} f(x) - \inf_{[x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1})$$

Since f is continuous, by the extreme value theorem, there exists $x_i' \in [x_{i-1}, x_i]$ such that,

$$f(x_i') = \sup_{[x_{i-1}, x_i]} f(x)$$

And there exists some $x_i'' \in [x_{i-1}, x_i]$ such that,

$$f(x_i'') = \inf_{[x_{i-1}, x_i]} f(x)$$

Then,

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} (f(x_i') - f(x_i''))(x_i - x_{i-1}) \\
&< \sum_{i=1}^{n} \frac{\varepsilon}{|b-a|}(x_i - x_{i-1}) \\
&= \frac{\varepsilon}{|b-a|}(x_1 - x_0 + x_2 - x_1 + \cdots + x_n - x_{n-1}) \\
&= \frac{\varepsilon}{|b-a|}(x_n - x_0) \\
&= \frac{\varepsilon}{|b-a|}|b-a| \\
&= \varepsilon
\end{aligned}$$

Thus, f is integrable.
<u>Part 2:</u>
Assume f has exactly 1 discontinuity, assume it is at $C \in (a, b)$. The cases where it is at a or b are treated similarly.

Fix $\varepsilon > 0$. We will find a partition P such that $U(f,P) - L(f,P) < \varepsilon$.
Define $\delta = \min\{\frac{\varepsilon}{8M}, \frac{b-C}{2}, \frac{C-a}{2}\}$.
Where M is such that $|f(x)| \leq M \quad \forall x \in [a,b]$.
By our choice of $\delta, [C-\delta, C+\delta] \subset [a,b]$.
By part 1, f is integrable on $[a, C-\delta]$ and $[C+\delta, b]$.
Therefore, there exist partitions P_1 of $[a, C-\delta]$ and P_2 of $[C+\delta, b]$ such that,

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{4}$$
$$U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{4}$$

Define $P = P_1 \cup P_2$, a partition of $[a,b]$.

$$\begin{aligned}
U(f,P) - L(f,P) &= U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2) + \\
&\sup_{[C-\delta,C+\delta]} f(x) - \inf_{[C+\delta,C-\delta]} f(x)(C+\delta-(C-\delta)) \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2M \cdot 2\delta \\
&\leq \frac{\varepsilon}{2} + \frac{4\varepsilon}{8M} \cdot M \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

<u>Part 3:</u>
If f has more than 1 discontinuity, apply part 2 enough times. $\square$

15 A Monotone Function is Integrable

Theorem: A Monotone Function is Integrable

If $f : [a, b] \to \mathbb{R}$ is monotone, then f is integrable.

Proof

Assume without loss of generality that f is increasing.
Fix $\varepsilon > 0$. We will find a partition $P = \{x_0, x_1, x_2, \cdots, x_n\}$ such that $U(f, P) - L(f, P) < \varepsilon$.
Let P be the partition that splits $[a, b]$ into n equal parts.
Namely, set $x_k = a + k \cdot \frac{b-a}{n}$. Then,

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} \left(\left(\sup_{[x_{i-1}, x_i]} f(x) - \sup_{[x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}) \right) \\
&= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \left(\frac{b-a}{n} \right) \\
&= \frac{b-a}{n} \cdot (f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1})) \\
&= \frac{b-a}{n} \cdot (f(x_n) - f(x_0)) \\
&= \frac{b-a}{n} \cdot (f(b) - f(a))
\end{aligned}$$

Chose $n > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$.
Then, $U(f, P) - L(f, P) < \varepsilon$ and so f is integrable. $\square$

16 The Mean Value Theorem For Integrals

Theorem: The Mean Value Theorem For Integrals

If f is continuous on $[a, b]$, then, there exists some $C \in [a, b]$ such that $\displaystyle\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a)$.

Proof

If f is constant on $[a, b]$, the result is obvious.
Assume f is not constant on $[a, b]$.
Denote $m = \inf_{[a,b]} f(x), M = \sup_{[a,b]} f(x)$.
Since f is continuous, by the extreme value theorem, there exists points $x_m \in [a, b]$ and $x_M \in [a, b]$ such that $m = f(x_m), M = f(x_M)$.
Without loss of generality, assume $x_m < x_M$. Observe that

$$m(a-b) \leq \int_a^b f \, \mathrm{d}x \leq M(b-a)$$

$$m \leq \frac{\int_a^b f \, \mathrm{d}x}{(b-a)} \leq M$$

Apply the intermediate value theorem on $[x_m, x_M]$. Since $f(x_m) = m$ and $f(x_M) = M$, there exists some $C \in [x_m, x_M]$ such that $f(C) = \frac{\int_a^b f \, \mathrm{d}x}{(b-a)}$.
Then,

$$\int_a^b f \, \mathrm{d}x = f(C) \cdot (b-a)$$

□

17 Continuity of an Antiderivative

Theorem: Continuity of an Antiderivative

If f is integrable on $[a, b]$ then the function $F(x) = \int_a^b f(x) \, \mathrm{d}x$ is continuous on $[a, b]$.

Proof

Suppose $C \in [a,b]$. We will show F is continuous at C. Let M be such that $|F(x)| \leq M \quad \forall x \in [a,b]$.
We want to show that $\lim_{h\to 0} F(C+h) = F(C)$ or equivalently, $\lim_{h\to 0}(F(C+h) - F(C)) = 0$.
We begin by showing $\lim_{h\to 0^+}(F(C+h) - F(C)) = 0$.
Indeed, if $h > 0$ then,

$$F(C+h) - F(C) = \int_a^{C+h} f\,\mathrm{d}x - \int_a^C f\,\mathrm{d}x = \int_C^{C+h} f\,\mathrm{d}x$$

Since $-M \leq f \leq M$ on $[a,b]$,

$$\int_C^{C+h} f\,\mathrm{d}x \leq M(\cancel{C} + h - \cancel{C}) = Mh$$

Also, $-Mh \leq \displaystyle\int_C^{C+h} f\,\mathrm{d}x$.
Therefore,

$$|f(C+h) - f(C)| = \left|\int_C^{C+h} f\,\mathrm{d}x\right| \leq Mh$$

This means $\lim_{h\to 0^+}(F(C+h) - F(C)) = 0$ and similarly, $\lim_{h\to 0^-}(F(C+h) - F(C)) = 0$.
$\square$

18 The Fundamental Theorem of Calculus

Theorem: The Fundamental Theorem of Calculus

Assume $f : [a,b] \to \mathbb{R}$ is continuous.
Define $F(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$. Then, F is differentiable on (a,b), $F'(x) = f(x)$.
Indeed, $\displaystyle\int_a^b f\,\mathrm{d}x = F(b) - F(a)$.

Proof

Let us compute $F'(x)$ for $x \in (a,b)$. We want to find $\displaystyle\lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$.
We begin by finding $\displaystyle\lim_{h\to 0^+} \frac{F(x+h) - F(x)}{h}$.
Given $h \in (0, b-x)$, we compute,

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,\mathrm{d}x$$

By the mean value theorem for integrals,

$$\int_x^{x+h} f(t)\,\mathrm{d}t = f(c)(\cancel{x} + h - \cancel{x}) = f(c)(h)$$

Where $c \in [x, x+h]$.
Note that f is uniformly continuous on $[a, b]$. Therefore, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.
If $h < \delta$ then $|c - x| < \delta$. Therefore,

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| = \left|\frac{f(c)h}{h} - f(x)\right| = |f(c) - f(x)| < \varepsilon$$

Thus, $\lim\limits_{h \to 0^+} \dfrac{F(x+h) - F(x)}{h} = f(x)$.
Similarly, $\lim\limits_{h \to 0^-} \dfrac{F(x+h) - F(x)}{h} = f(x)$.
So, $F'(x) = f(x)$.
The Newton-Leibniz formula follows immediately. $\square$

19 Substitution Formula

Theorem: Substitution Formula

If f, g are continuous on $[a, b]$ and g is differentiable on (a, b) with g' continuous, then,

$$\int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u = \int_a^b f(g(x))g'(x)\,\mathrm{d}x$$

Proof

Suppose F is an antiderivative of f.
By the fundamental theorem of calculus, the left hand side of the equation is equal to,

$$F(g(b)) - f(g(a))$$

Next, by the chain rule, we have,

$$(F \circ g)' = (F' \circ g) \cdot g' = (f \circ g)g'$$

This means $F \circ g$ is an anti-derivative of $(f \circ g)g'$.
Therefore, by the fundamental theorem of calculus,

$$\begin{aligned}\int_a^b (f \circ g)g'\,\mathrm{d}x &= (F \circ g)(b) - (F \circ g)(a)\\ &= F(g(b)) - F(g(a))\end{aligned}$$

Thus, $\displaystyle\int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u = \int_a^b f(g(x))g'(x)\,\mathrm{d}x$ holds. $\square$

20 Integration by Parts

Theorem: Integration by Parts

If $u, v : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b),

$$\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$$

$$\int_a^b uv'\,\mathrm{d}x = uv|_a^b - \int_a^b u'v\,\mathrm{d}x$$

Where $uv|_a^b = u(b)v(b) - u(a)v(a)$.

Proof

We know that $uv' = u'v + v'u$.
Integrating both sides, we obtain,

$$\int (uv)'\,\mathrm{d}x = \int u'v\,\mathrm{d}x + \int v'u\,\mathrm{d}x$$

$$uv = \int u'v\,\mathrm{d}x + \int uv'\,\mathrm{d}x$$

$$\int u'v\,\mathrm{d}x = uv - \int v'u\,\mathrm{d}x$$

The formula for the definite integral follows from the fundamental theorem of calculus.

$\square$

21 Integral Test

Theorem: Integral Test

Assume f is a non negative, non increasing continuous function on the interval $[1, \infty)$.
Then, $\displaystyle\int_1^\infty f\,\mathrm{d}x$ and $\displaystyle\sum_{n=1}^\infty f(n)$ converge or diverge together.

Proof

Consider the interval $[1, n+1]$. The set $P = \{1, 2, 3, 4, \cdots, n+1\}$ is a partition of $[1, n+1]$. Since f is non increasing,

$$\inf_{[x_{i-1}, x_i]} f = f(x_i), \quad \sup_{[x_{i-1}, x_i]} f = f(x_{i-1})$$

Consequently,

$$\begin{aligned} U(f, P) &= \sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]} f(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f(x_{i-1})(i + 1 - i) \\ &= \sum_{i=1}^{n} f(x_{i-1}) \\ &= \sum_{i=1}^{n} f(i) \end{aligned}$$

Also, $L(f, P) = \sum_{i=1}^{n} f(i+1)$.

If $\int_1^\infty f \, \mathrm{d}x$ converges, then, $L(f, P) = \sum_{i=1}^{n} f(i+1) = \sum_{i=2}^{n+1} f(i)$ converges as $n \to \infty$.

Thus, $\sum_{i=1}^{\infty} f(i)$ converges.

If $\int_1^\infty f \, \mathrm{d}x$ diverges then $U(f, P) = \sum_{i=1}^{n} f(i)$ diverges as $n \to \infty$

Which means, $\sum_{i=1}^{\infty} f(i)$ diverges. $\square$

22 Limit Comparison Test

Theorem: Limit Comparison Test

Suppose $a_n, b_n > 0$ for all $n \in \mathbb{N}$.
If $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$ (not ∞)
then, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and diverge simultaneously.

Proof

Assume $\sum_{n=1}^{\infty} b_n$ converges. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, there exists some $N \in \mathbb{N}$ such that if $n \geq N\frac{a_n}{b_n} < 2c$ and $a_n < 2cb_n$.

Now, $\sum_{n=1}^{\infty} 2cb_n = 2c\sum_{n=1}^{\infty} b_n$ converges.

By comparison, since $\sum_{n=N}^{\infty} a_n \leq 2c\sum_{n=N}^{\infty} b_n \leq 2c\sum_{n=1}^{\infty} b_n$, the series $\sum_{n=N}^{\infty} a_n$. Hence, $\sum_{n=1}^{\infty} a_n$ converges.

Next, suppose $\sum_{n=1}^{\infty} a_n$ converges. Then, $\lim_{n\to\infty} \frac{b_n}{c_n} = \frac{1}{c} > 0$ and $\sum_{n=1}^{\infty} b_n$ converges by the previous argument. $\square$

23 Ratio Test

Theorem: Ratio Test

Assume $a_n > 0$ for all $n \in \mathbb{N}$.
Then,

1. If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$

   Then, $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$

   Then, $\sum_{n=1}^{\infty} a_n$ diverges.

Proof

We assume that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r \in \mathbb{R}$.

1. Fixing S such that $r < S < 1$.
   Because $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < S$, there exists some $N \in \mathbb{N}$ such that if $n \geq N$ then $\frac{a_{n+1}}{a_n} < S$.
   Then, $\frac{a_{N+1}}{a_N} < S$ and $a_{N+1} < S \cdot a_N$.
   Also, $\frac{a_{N+2}}{a_{N+1}} < S$ then $a_{N+2} < S \cdot a_{N+1} < S^2 a_N$.
   Next, $\frac{a_{N+3}}{a_{N+2}} < S$ then $a_{N+3} < S \cdot a_{N+2} < S^3 a_N$.

Continuing like this, we find $a_{N+k} < S^k \cdot a_N$. Thus,

$$\sum_{k=N}^{\infty} a_n \leq \sum_{k=0}^{\infty} S^k \cdot a_N = a_N \sum_{k=0}^{\infty} S^k = a_N \frac{1}{1-S}$$

Thus, $\sum_{k=N}^{\infty} a_N$ converges. Hence, $\sum_{n=0}^{\infty} a_n$ converges.

2. Assuming $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r > 1$ then there exists some $N \in \mathbb{N}$ such that if $n \geq N$ then $\frac{a_n+1}{a_n} > 1$ then $\frac{a_{N+1}}{a_N} > 1$ so, $a_{N+1} > a_N$. Also, $\frac{a_{N+2}}{a_N+1} > 1$ so $a_{N+2} > a_{N+1} > a_N$ and so on.
   Thus, $a_{N+k} > a_N$ for all $k \in \mathbb{N}$ which means a_n cannot go to 0 as $n \to \infty$.

□

24 The Leibniz Test

Theorem: The Leibniz Test

Assume $(a_n)_{n=1}^{\infty}$ is a sequence such that $a_n \geq 0$, $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$,

Then, the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Proof

Let $S_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$.
Observe that $S_1 \geq S_3 \geq S_5 \geq \cdots$
Indeed,

$$\begin{aligned} S_{2n+3} &= S_{2n+1} - a_{2n+2} + a_{2n+3} \\ &= S_{2n+1} + (a_2n + 3 - a_2n + 2) \\ &\leq S_{2n+1} \end{aligned}$$

Also,

$$S_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2}) \geq S_{2n}$$

And,

$$S_2 \leq S_4 \leq S_6 \leq \cdots$$

Moreover, $S_{2n} \leq S_{2n+1}$ since $S_{2n+1} = S_{2n} + a_{2n+1} \geq S_{2n}$.
We conclude $S_{2n} \leq S_{2n+1} \leq S_1$ and $S - 2n + 1 \geq S_{2n} \geq S_2$.
Thus, the sequences $(S_{2n})_{n=1}^{\infty}$ and $(S_{2n+1})_{n=1}^{\infty}$ are bounded.

Hence, they must converge. Thus, we can say,

$$\lim_{n\to\infty} S_{2n} = \alpha, \quad \lim_{n\to\infty} S_{2n+1} = \beta$$

Now,

$$\begin{aligned}\beta - \alpha &= \lim_{n\to\infty} S_{2n+1} - \lim_{n\to\infty} S_{2n} \\ &= \lim_{n\to\infty} (S_{2n+1} - S_{2n}) \\ &= \lim_{n\to\infty} a_{2n+1} \\ &= 0\end{aligned}$$

Thus, $\beta = \alpha$ and $\lim_{n\to\infty} S_n = \alpha = \beta$ and $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. $\square$

25 An Absolutely Convergent Series implies convergence

Theorem: An Absolutely Convergent Series implies convergence

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

Proof

Use the cauchy criterion.
Fixing $\varepsilon > 0$. We must show that there exists some $N \in \mathbb{N}$ such that if $p > q \geq N$ then

$$\left|\sum_{n=q}^{p} a_n\right| < \varepsilon$$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, there exists some $N \in \mathbb{N}$ such that $\left|\sum_{n=q}^{p} |a_n|\right| < \varepsilon$
By the triangle inequality,

$$\left|\sum_{n=q}^{p} a_n\right| \leq \sum_{n=1}^{\infty} |a_n| = \left|\sum_{n=q}^{p} |a_n|\right| < \varepsilon$$

$\square$

26 Maclaurin Series Examples

Theorem: Maclaurin Series Examples

The formulas and derivations for the functions,

1. $f(x) = e^x$
2. $f(x) = \sin(x)$
3. $f(x) = \cos(x)$
4. $f(x) = \frac{1}{1-x}$

Proof

1. Let $f(x) = e^x$.
   Let us write down the Maclaurin Series.
   Clearly, $f^{(n)}(0) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^x \Big|_{x=0} = e^0 = 1$
   Therefore, the series is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
   This converges absolutely for all $x \in \mathbb{R}$ by the ratio test.

2. Let $f(x) = \sin(x)$.

$$\begin{aligned} f(x) &= \sin(x) \\ f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f'''(x) &= -\cos(x) \\ f''''(x) &= \sin(x) \end{aligned}$$

$$f^{4k}(x) = \sin(x) \qquad f^{4k+1}(x) = \cos(x)$$

   For all $k \in \mathbb{N}$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n &= \frac{\sin(0)}{0!} \cdot x^0 + \frac{\cos(0)}{1!} \cdot x^1 + \frac{-\sin(0)}{2!} \cdot x^2 + \frac{-\cos(0)}{3!} \cdot x^3 + \dots \\ &= x - \frac{1}{3!} \cdot x^3 + \frac{1}{5!} \cdot x^5 \\ &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

<u>Note:</u> the series converges absolutely by the ratio test for all $x \in \mathbb{R}$

$$\limsup_{n\to\infty} \frac{(-1)^{n+1} \cdot \left(\frac{x^{2n+2}}{(2n+2)!}\right)}{(-1)^n \cdot \left(\frac{x^{2n+1}}{(2n+1)!}\right)} = \limsup_{n\to\infty} \frac{-x}{2n+2} = 0 < 1$$

3. Let $f(x) = \cos(x)$.

$$\begin{aligned} f(x) &= \cos(x) \\ f'(x) &= -\sin(x) \\ f''(x) &= -\cos(x) \\ f'''(x) &= \sin(x) \\ f''''(x) &= \cos(x) \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n &= \frac{\cos(0)}{0!} \cdot x^0 + \frac{-\sin(0)}{1!} \cdot x^1 + \frac{-\cos(0)}{2!} \cdot x^2 + \frac{\sin(0)}{3!} \cdot x^3 + \dots \\ &= 1 - \frac{1}{2!} \cdot x^2 + \frac{1}{4!} \cdot x^4 - \frac{1}{6!} \cdot x^6 + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{2n!} \end{aligned}$$

For all $x \in \mathbb{R}$

$$\begin{aligned} f(x) &= \frac{1}{1-x} \\ f'(x) &= \frac{1}{(1-x)^2} \\ f''(x) &= \frac{2}{(1-x)^3} \\ f'''(x) &= \frac{6}{(1-x)^4} \\ f''''(x) &= \frac{24}{(1-x)^5} \end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \frac{1}{(1-0)(0!)} \cdot x^0 + \frac{1}{(1-0)^2(1!)} \cdot x^1 + \frac{2}{(1-0)^3(2!)} \cdot x^2$$
$$+ \frac{6}{(1-0)^4(3!)} \cdot x^3 + \frac{24}{(1-0)^5(4!)} \cdot x^4$$
$$= \sum_{n=0}^{\infty} \frac{n!}{n!} \cdot x^n$$
$$= \sum_{n=0}^{\infty} x^n$$

This is a geometric series. It converges if $x \in (-1, 1)$ and diverges otherwise.

□

27 Inhomogenous Linear System Solutions

Proposition: Inhomogenous Linear System Solutions

If p is a vector such that $Ap = b$, then,

$$\{x \in \mathbb{R}^n | Ax = b\} = \{y + p | y \in \text{NS(A)}\}$$

Proof

Observe that $A(x + y) = Ax + Ay$ and $A(\lambda x) = \lambda Ax$ for all $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.
Assume $y \in \text{NS(A)}$. Let us prove that $A(y + p) = b$.
Indeed, $A(y + p) = Ay + Ap = 0 + b = b$.
Thus,

$$\{x \in \mathbb{R}^n | Ax = b\} \supset \{y + p | y \in \text{NS(A)}\}$$

Now, assume $Ax = b$. Clearly, $x = p + (x - p)$.
The vector $y = x - p$ is in NS(A) because

$$A(x - p) = Ax - Ap = b - b = 0$$

Thus,

$$\{x \in \mathbb{R}^n | Ax = b\} \subset \{y + p | y \in \text{NS(A)}\}$$

□

28 Invertibility of a Matrix and the Null Space

Theorem: Invertibility of a Matrix and the Null Space

The matrix A is invertible if and only if $\text{NS(A)} = \{0\}$.

Proof

($\Longrightarrow$)
Assume A is invertible. We know $0 \in$ NS(A). We want to prove that if $Ax = 0, x = 0$ then,

$$\begin{aligned} x &= (A^{-1}A)x \\ &= A^{-1}(Ax) = A^{-1}0 = 0 \end{aligned}$$

($\Longleftarrow$)
We claim that if NS(A) = 0 then A has a right inverse.
Proof of the claim
If NS(A = 0 then the system $Ax = 0$ has a unique solution.
By an earlier proposition, the system $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.
Take,

$$b^1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad b^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \cdots$$

Solving $Ax = b^i$ for $i = 1, 2, \cdots, n$, we obtain an array of vectors,

$$x^i = \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix}$$

We construct a matrix,

$$C = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

Clearly, $AC = I_n$.
Thus, C is the right inverse of A.
This proves the claim.
Now, let us show that A has a left inverse. Note that C has a left inverse, namely, A.
By the same argument as in ($\Longrightarrow$).
This implies NS(C) = $\{0\}$. By the claim, C has a right inverse.
This inverse is equal to A (proven earlier).
Thus, $AC = CA = I_n$.
Thus, C is the inverse of A. $\square$

29 Linearity of the Determinant

Theorem: Linearity of the Determinant

Suppose $u, v, a_1, \cdots, a_n$ are vectors in $\mathbb{R}^n$. Consider the matrices,

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{r-1} \\ u + \lambda v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} \qquad B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} \qquad C = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

Then, $\det A = \det B + \lambda \det C$

Proof

We will argue with induction.
The result is obvious if $n = 1$. Then, $A = (u + \lambda v), B = u, C = v$ for some $u, v \in \mathbb{R}$.
Then,

$$\det A = u + \lambda v = \det B + \lambda \det C$$

Assume the result holds for $(n-1) \times (n-1)$ matrices. Let us prove the result for A, B, C being $n \times n$ matrices.
<u>Case 1:</u>
Assume $r = 1$.
In this case,

$$\begin{aligned} \det A &= \sum_{j=1}^{n} (-1)^{1+j} (u_j + \lambda v_j) \det \tilde{A_{1j}} \\ &= \sum_{j=1}^{n} (-1)^{1+j} u_j \det \tilde{A}_{1j} + \lambda \sum_{j=1}^{n} (-1)^{1+j} v_j \det \tilde{A_{1j}} \end{aligned}$$

Where $u = (u_1, u_2, \cdots u_n) \qquad v = (v_1, v_2, \cdots, v_n)$.
Now, since $r = 1$, $\tilde{A_{1j}} = \tilde{B_{1j}} = \tilde{C_{1j}}$.
Therefore,

$$\begin{aligned} \det A &= \sum_{j=1}^{n} (-1)^{1+j} u_j \det \tilde{B_{1j}} + \lambda \sum_{j=1}^{n} (-1)^{1+j} v_j \det \tilde{C_{1j}} \\ &= \det B + \lambda \det C \end{aligned}$$

<u>Case 2:</u>
Assume $r > 1$.

In this case, the first rows of A, B and C are the same. In fact, they are a_1.
Now,

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det \tilde{A_{1j}}$$

The matrix $\tilde{A_{ij}}$ is $(n-1) \times (n-1)$. By the induction hypothesis,

$$\det \tilde{A_{1j}} = \det \tilde{B_{1j}} + \lambda \det \tilde{C_{1j}}$$

Therefore,

$$\begin{aligned}
\det A &= \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det \left(\tilde{B_{1j}} + \tilde{C_{1j}} \right) \\
&= \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det \tilde{B_{1j}} + \lambda \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det \tilde{C_{1j}} \\
&= \sum_{j=1}^{n} (-1)^{j+1} B_{1j} \det \tilde{B_{1j}} + \lambda \sum_{j=1}^{n} (-1)^{j+1} C_{1j} \det \tilde{C_{1j}} \\
&= \det B + \lambda \det C
\end{aligned}$$

□

30 Invertibility and the Determinant

Theorem: Invertibility and the Determinant

A matrix A is invertible if and only if $\det A \neq 0$

Proof

Suppose A is some square matrix. ($\Longrightarrow$)
Since $AA^{-1} = I_n$, taking the determinant of this,

$$\det A \det A^{-1} = 1$$

This clearly shows $\det A \neq 0$.
($\Longleftarrow$)
Cosider the matrix

$$G = \frac{1}{\det A} \left((C_{ij})_{i,j=1}^{n} \right)^T$$

Where $C_{ij} = (-1)^{i+j} \det \tilde{A_i} j$.
We claim that

$$GA = AG = I_n$$

Indeed, given $k = 1, \cdots, n$, we find,

$$
\begin{aligned}
(AG)_{KK} &= \sum_{i=1}^{n} A_{ki} G_{ki} \\
&= \sum_{i=1}^{n} A_{ki} (-1)^{i+k} \det \tilde{A_{ki}} \\
&= \frac{1}{\det A} \sum_{i=1}^{n} A_{ki} (-1)^{i+k} \det \tilde{A_{ki}} \\
&= \frac{1}{\det A} \det A = 1
\end{aligned}
$$

Thus, AG has a diagonal of 1s.
If $K \neq L$ then,

$$
\begin{aligned}
(AG)_{KL} &= \sum_{i=1}^{n} A_{Ki} G_{iL} \\
&= \frac{1}{\det A} \sum_{i=1}^{n} A_{Ki} (-1)^{i+L} \det \tilde{A_{Li}}
\end{aligned}
$$

The sum is the determinant of the matrix

$$
\begin{pmatrix}
A_{11} & \cdots & A_{1N} \\
\vdots & & \vdots \\
A_{K1} & \cdots & A_{KN} \\
\vdots & \ddots & \vdots \\
A_{K1} & \cdots & A_{KN} \\
\vdots & \ddots & \vdots \\
A_{N1} & \cdots & A_{NN}
\end{pmatrix}
$$

However, this matrix has two identical rows so its determinant is equal to 0.
This means that AG has zeroes off the diagonal.

$$AG = GA = I_n$$

Thus, A is invertible. $\square$