1071 Final Theorem List

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Contents

1 Squeeze Theorem

Theorem: Squeeze Theorem For Sequences

Suppose $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$ are such that

1.
$$a_n \leq b_n \leq c_n \qquad \forall n \in \mathbb{N}$$

2.
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

Then,
$$\lim_{n\to\infty} b_n = L$$

Proof

Observe that,

$$|b_n - L| = |b_n - a_n + a_n - L|$$
$$|(b_n - a_n) + (a_n - L)| \leq |b_n - a_n| + |a_n - L| = b_n - a_n + |a_n - L|$$
$$\leq c_n - a_n + |a_n - L| = |c_n - L + L - a_n| + |a_n - L|$$
$$\leq |c_n - L| + |L - a_n| + |a_n - L|$$

Fix $\varepsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then

$$|a_n - L| = |L - a_n| < \frac{\varepsilon}{3}$$

.
Also, there is $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then

$$|c_n - L| < \frac{\varepsilon}{3}$$

Now, set $N = \max N_1, N_2$. If $n \geq N$ then,

$$|b_n - L| \leq |c_n - L| + |L - a_n| + |a_n - L| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Thus, $\lim_{n\to\infty} b_n = L$.

$\square$

2   A Convergent Sequence is Bounded

> **Theorem: A Convergent Sequence is Bounded**
>
> Convergent sequences are bounded. More precicely, if $(a_n)_{n=1}^{\infty}$ converges, then there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$

> **Proof**
>
> Assume $\lim\limits_{n \to \infty} a_n = L$.
> For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - L| < \varepsilon$.
> This holds for instance if $\varepsilon = 1$.
> Thus, there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then
>
> $$|a_n - L| < 1$$
>
> By the second triangle inequality,
>
> $$|a_n - L| \geq |a_n| - |L|$$
>
> Thus, $|a_n| - |L| < 1$ and $|a_n| < 1 + |L|$
> Now, $|a_n| \leq M = \max\{|a_1|, |a_2|, |a_3|, \cdots, |a_{n-1}|, |L| + 1\}$ for all $n \in \mathbb{N}$. $\qquad\square$

3 A Bounded, Monotone Sequence Converges

Theorem: A Bounded, Monotone Sequence Converges

A monotone sequence converges if and only if it is bounded.

Proof

($\implies$)
This is already proven, see 2
($\impliedby$)
Without loss of generality, assume $(a_n)_{n=1}^\infty$ is increasing (If the sequence is monotone decreasing, a similar argument is used).
Define $\alpha = \sup\{a_1, a_2, a_3, \cdots, a_n\} \in \mathbb{R}$.
Since the sequence is bounded, this supremum exists in $\mathbb{R}$.
Let us prove

$$\alpha = \lim_{n \to \infty} a_n$$

Chose $\varepsilon > 0$.
By an earlier theorem, there exists $N \in \mathbb{N}$ such that $a_N \in (\alpha - \varepsilon, \alpha]$.
By monotonicity, if $n \geq N$ then $a_n \geq a_N > \alpha - \varepsilon$. Also, $a_n \leq \alpha$.
Thus, if $n \geq N$ then $\alpha - \varepsilon < a_n \leq \alpha < \alpha + \varepsilon \implies |a_n - \alpha| < \varepsilon$. Thus, $\lim_{n \to \infty} a_n = \alpha$, so $(a_n)_{n=1}^\infty$ converges.

$\square$

Explanation

This theorem continues from the previous theorem by proving the other direction, if a sequence is bounded, it must converge. That means, if there is some point that the sequence cannot go above, then the sequence must eventually tend to a point. This is ensured by assuming the sequence is monotone as then the sequence will only get larger or smaller to the bound.
The part that uses an 'earlier theorem' simply says that for some large enough N, we have the element is delta away from the limit. Monotonicity implies that each succesive term is larger than the last (or smaller).

4 Existence of a Monotone Subsequnce

Theorem: Existence of a Monotone Subsequnce

Every sequence of real numbers has a monotone subsequence.

Proof

We call $k \in \mathbb{N}$ a peak point of $(a_n)_{n=1}^\infty$ if $a_k > a_n$ for all $n > k$.
<u>Case 1:</u> There are infinitely many peak points, $n_1 < n_2, n_3 < \cdots$. By definition, $a_{n_1} > a_{n_2} > a_{n_3} > \cdots$. Thus, $(a_{n_k})_{k=1}^\infty$ is monotone.
<u>Case 2:</u> There are finitely many peak points. $k_1, k_2, k_3, \cdots, k_l$. Set

$$n_1 = \max\{k_1, \cdots, k_l\} + 1$$

Clearly, n_1 is not peak. Therefore, there is some $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$. Now, n_2 is not peak. Therefore there is $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$. Continue in this way to obtain a monotone subsequence $(a_{n_i})_{i=1}^\infty$. $\square$

Explanation

This theorem states that each sequence has a monotone subsequence. It is proven using the idea of peak points. If we have infinite peak points, then the subsequence will only be able to decrease by the definition of a peak point.
The second case examines if there are finite peak points. Here, the argument that is used is that by definition of a peak point, we can say that there are elements bigger than those bigger than the peak points. This creates some monotone increasing sequence.

5 Bolzano-Weierstrass Theorem

Theorem: Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

Proof

Every sequence has a monotone subsequence. Every subsequence of a bounded sequence is bounded, thus, we have a bounded, monotone subsequence which must converge. $\square$

6 Limit Arithmetic

Theorem: Limit Arithmetic

Assume $\lim\limits_{x \to a} f(x) = L \quad \lim\limits_{x \to a} g(x) = m$.
Then,

1. $\lim\limits_{x \to a}(f(x) + g(x)) = L + m$

2. $\lim\limits_{x \to a}(f(x) \cdot g(x)) = L \cdot m$

3. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{m}$

Proof

1. Fix $\varepsilon > 0$. We need to find $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x)+g(x)-L+m| < \varepsilon$.

$$
\begin{aligned}
|f(x) + g(x) - (L+m)| &= |f(x) + g(x) - (L+m)| \\
&= |(f(x) - L) + (g(x) - m)| \\
&\leq |f(x) - L| + |g(x) - m|
\end{aligned}
$$

Since $\lim\limits_{x \to a} f(x) = L, \quad \lim\limits_{x \to a} g(x) = m$, there exists $\delta_1, \delta_2 > 0$ such that if $0 < |x-a| < \delta_1, \delta_2$ then,

$$
\begin{aligned}
|f(x) - L| &< \frac{\varepsilon}{2} \\
|g(x) - m| &< \frac{\varepsilon}{2}
\end{aligned}
$$

Choosing $\delta = \min\{\delta_1, \delta_2\}$, we have that,

$$
\begin{aligned}
|f(x) - L| + |g(x) - m| &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

2. Fix $\varepsilon > 0$. We need to find $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x)g(x) - Lm| < \varepsilon$.

$$|f(x)g(x) - Lm| = |f(x)g(x) - f(x)m + f(x)m - Lm|$$
$$= |f(x)(g(x) - m) + m(f(x) - L)|$$
$$\leq |f(x)||g(x) - m| + |m||f(x) - L|$$

Since $\lim_{x \to a} f(x) = L$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then, $|f(x) - L| < 1$.
In this case, $|f(x)| - |L| < 1 \implies |f(x)| < 1 + |L|$.
Thus, if $0 < |x - a| < \delta$, then $|f(x)g(x) - Lm| \leq |1 + |L|||g(x) - m| + |m||f(x) - L|$.
There exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then, $|g(x) - m| < \frac{\varepsilon}{2(|m|+1)}$.
Set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$,
If $0 < |x - a| < \delta$ then,

$$|f(x)g(x) - Lm| \leq (1 + |L|)|g(x) - m| + |m||f(x) - L|$$
$$< \frac{\varepsilon}{2(|L| + 1)} \cdot (|L| + 1) + \frac{\varepsilon}{2(|m| + 1)} \cdot 2(|m| + 1) \qquad \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$= \varepsilon$$

3. We need to show first that $\lim_{x \to a} \dfrac{1}{g(x)} = \dfrac{1}{m}$

Fix $\varepsilon > 0$.
Since $\lim_{x \to a} g(x) = m$, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then, $|g(x) - m| < \frac{|m|}{2}$.

$$|m| = |m + g(x) - g(x)|$$
$$\leq |g(x) - m| + |g(x)|$$
$$< \frac{|m|}{2} + |g(x)|$$
$$\frac{|m|}{2} < |g(x)|$$
$$\frac{1}{g(x)} < \frac{2}{|m|}$$

Indeed, there exists some $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then $|g(x) - m| < \frac{|m|^2}{2}\varepsilon$.
Choosing $\delta = \min\{\delta_1, \delta_2\}$, if $0 < |x - a| < \delta$ we have,

$$\left| \frac{1}{g(x)} - \frac{1}{m} \right| = \left| \frac{m - g(x)}{mg(x)} \right|$$
$$= \frac{1}{|mg(x)|}|g(x) - m|$$
$$< \frac{1}{|m|}\frac{2}{|m|}|g(x) - L|$$

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$$< \frac{2}{|m|^2} \frac{|m|^2}{2} \varepsilon$$
$$= \varepsilon$$

Now we have proven that $\lim\limits_{x \to a} \dfrac{1}{g(x)} = \dfrac{1}{m}$, the more general fact follows.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) \frac{1}{g(x)}$$
$$= \lim_{x \to a} f(x) \lim_{x \to a} \frac{1}{g(x)}$$
$$= L \frac{1}{m}$$
$$= \frac{L}{m}$$

$\square$

Explanation

This theorem basically describes the basic arithmetic that works with limits. The proof of parts 1 and 2 are trivial, they come directly from the definition of the limit. Part 3 works by proving a specific case of $\frac{1}{g(x)}$ then applying the multiplication proven in 2.

7 Differentiability implies Continuity

Proposition: Differentiability implies Continuity

Suppose f is differentiable at x_0. Then f is continuous at x_0.

Observe that:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) + f(x_0) \right)$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) + f(x_0)$$

$$= f'(x_0) \cdot (x_0 - x_0) + f(x_0)$$

$$= 0 \cdot f'(x_0) + f(x_0) = f(x_0)$$

Therefore, f is continuous at x_0. □

Explanation

This theorem says that a function being differentiable at a point will ensure its continuous. This is because of the fact that for a function to be differentiable, it needs to approach some limit from the left and right. Those being the same will imply a derivative and indeed will imply continuity.

The proof adds a clever 0 to give the definition of the derivative.

8 Derivative Arithmetic

Theorem: Derivative Arithmetic

Suppose $f, g : (a, b) \to \mathbb{R}$ are both differentiable at $x \in (a, b)$. Then, $f + g, fg, \frac{f}{g}$ are differentiable at x. We need $g(x) \neq 0$ for $\frac{f}{g}$.

1. $(f + g)'(x) = f'(x) + g'(x)$

2. $(fg)'(x) = f'(x)g(x) + g'(x)f(x)$

3. $\left(\frac{f}{g} \right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$

Proof

1.

$$
\begin{aligned}
(f+g)'(x) &= \lim_{h \to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x)
\end{aligned}
$$

2.

$$
\begin{aligned}
(fg)'(x) &= \lim_{h \to 0} \frac{f(x+g)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \\
&= \lim_{h \to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h \to 0} f(x+h) \lim_{h \to 0}\frac{g(x+h)-g(x)}{h} + g(x)\lim_{h \to 0}\frac{f(x+h)-f(x)}{h} \\
&= f(x)g'(x)+f'(x)g(x)
\end{aligned}
$$

3.

$$
\begin{aligned}
\left(\frac{f'(x)}{g'(x)}\right) &= \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h} \\
&= \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} \cdot \frac{g(x)}{g(x)} - \frac{f(x)}{g(x)} \cdot \frac{g(x+h)}{g(x+h)}}{h} \\
&= \lim_{h \to 0} \frac{g(x)f(x+h)-f(x)(g(x+h))}{hg(x)g(x+h)} \\
&= \lim_{h \to 0} \frac{g(x)f(x+h)-f(x)g(x)+f(x)g(x)-f(x)(g(x+h))}{hg(x)g(x+h)} \\
&= \lim_{h \to 0} \frac{g(x)(f(x+h)-f(x))-f(x)(g(x)-g(x+h))}{hg(x)g(x+h)} \\
&= \frac{g(x)\lim_{h \to 0}\frac{f(x+h)-f(x)}{h}-f(x)\lim_{h \to 0}\frac{g(x+h)-g(x)}{h}}{\lim_{h \to 0} g(x)g(x+h)} \\
&= \frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}
\end{aligned}
$$

$\square$

9   Rolle's Theorem

Theorem: Rolle's Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof

If $f(x) = f(a)$ for all $x \in (a, b)$, then the theorem is obvious.
Assume f is not constant.
Without loss of generality, assume there exists $x_0 \in (a, b)$ such that $f(x_0) > f(a)$.
(If no such x_0 exists, then there exists $x_1 \in (a, b)$ such that $f(x_1) < f(a)$. In this case, a similar argument works.)
By the Extreme Value theorem, f has a global maximum on $[a, b]$, call this c. Since,

$$f(c) \geq f(x_0) > f(a) = f(b)$$

we know $c \neq a, c \neq b$. Thus, c lies in (a, b), and $f'(c) = 0$ □

Explanation

Rolle's Theorem takes some function that is differentiable and continuous. It places a condition on it such that the end points are equal. If this is so, there is a point of derivative 0 between them. This is because if the end points are equal, either the function is constant or it changes direction at some point. This change in direction will have a derivative of 0.
The proof outlines the initial trivial case of a constant function. It then assumes that we have some value that is greater than the end points (this does not lose generality as a similar argument follows for value less than the end point). The extreme value theorem predicts some global maximum in the interval. This is larger than the assumed point and is thus larger than the end points meaning there is a derivative 0 within the interval.

10 The Mean Value Theorem

Theorem: The Mean Value Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then, there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof

Consider the function $\phi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a} \cdot (x - a)$.
Observe that

$$\phi(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (a - a) = 0$$

$$\phi(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (b - a) = 0$$

Applying Rolle's theorem to $\phi(x)$, we obtain the existence of $c \in (a, b)$ such that,

$$\phi'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

$\square$

Explanation

This theorem is similar to Rolle's Theorem. It says that there is some derivative of a function at a point that will be equal to the average gradient between the end points.
The proof is completed by considering a function that when Rolle's Theorem is applied to will give exactly what is wanted.

11 A Vanishing Derivative Implies a Constant Function

Proposition: Vanishing Derivative Implies a Constant Function

Suppose $f : [a, b] \implies \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$

Take $x \in (a, b]$. Applying the Mean Value theorem on $[a, x]$, we conclude that $f(x) - f(a) = f'(c)(x - a)$ for some $c \in (a, x)$. Therefore, as $f'(c) = 0$,

$$f(x) - f(a) = 0(x - a)$$
$$f(x) = f(a)$$

Thus, f is constant in $[a, b]$. □

Explanation

This is a specific case of the mean value theorem. We use the fact that the mean value theorem gives and set the derivative to be 0. This gives us a constant function.

12 A Continuous Function on an interval is Uniformly Continuous

Theorem: A Continuous Function on an interval is Uniformly Continuous

Suppose f is continuous on a closed, bounded interval $[a, b]$. Then, f is uniformly continuous on $[a, b]$.

Proof

By contradiction.
Assume f is not uniformly continuous.
There exists $\varepsilon_0 > 0$ such that for all $\delta > 0$, there exists $x, y \in [a, b]$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$.
Take $\delta = 1$. There exists $x_1, y_1 \in [a, b]$ such that $|x_1 - y_1| < 1$ but $|f(x_1) - f(y_1)| \geq \varepsilon_0$.
Take $\delta = \frac{1}{2}$. There exists $x_2, y_2 \in [a, b]$ such that $|x_2 - y_2| < \frac{1}{2}$ but $|f(x_2) - f(y_2)| \geq \varepsilon_0$.
Continue in this way for $\delta = \frac{1}{n}, \quad \forall n \in \mathbb{N}$.
Then, there exists $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.
We have thus constructed two sequences.

$$(X_n)_{n=1}^{\infty}$$
$$(Y_n)_{n=1}^{\infty}$$

inside $[a, b]$.
Since $[a, b]$ is closed and bounded, $(X_n)_{n=1}^{\infty}$ must have some subsequence $(X_{n_k})_{k=1}^{\infty}$ which

converges to some point $x_0 \in [a, b]$.

$$\lim_{k \to \infty} (X_{n_k}) = x_0$$

Claim:
The subsequence $(Y_{n_k})_{k=1}^{\infty}$ converges to x_0.
Proof of the claim:
Given $\varepsilon > 0$, we look for $N \in \mathbb{N}$ such that if $k \geq N$ then, $|Y_{n_k} - x_0| < \varepsilon$.

$$\begin{aligned}
|Y_{n_k} - x_0| &= |Y_{n_k} + X_{n_k} - X_{n_k} - x_0| \\
&\leq |Y_{n_k} - X_{n_k}| + |X_{n_k} - x_0| \\
&< \frac{1}{n_k} + |X_{n_k} - x_0|
\end{aligned}$$

Since $(X_n)_{n=1}^{\infty}$ converges to x_0, for k large enough, we have $|X_{n_k} - x_0| < \frac{\varepsilon}{2}$.
Then taking k large enough such that $\frac{1}{n_k} < \frac{\varepsilon}{2}$, we have,

$$\begin{aligned}
|Y_{n_k} - x_0| &< \frac{1}{n_k} + |X_{n_k} - x_0| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Let us prove that this contradicts the continuity of f.
Since f is continuous and since

$$\lim_{k \to \infty} X_{n_k} = \lim_{n \to \infty} Y_{n_k} = x_0$$

we have that,

$$\lim_{k \to \infty} f(X_{n_k}) = \lim_{n \to \infty} f(Y_{n_k}) = f(x_0)$$

Then, for some $N \in \mathbb{N}$, if $k \geq N$ then $|f(X_{n_k}) - x_0| < \frac{\varepsilon}{4}$ and $|f(Y_{n_k}) - x_0| < \frac{\varepsilon}{4}$.
On the other hand,

$$\begin{aligned}
|f(X_{n_k}) - f(Y_{n_k})| &= |f(X_{n_k}) - f(x_0) + f(x_0) - f(Y_{n_k})| \\
&\leq |f(X_{n_k}) - f(x_0)| + |f(x_0) - f(Y_{n_k})| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&= \frac{\varepsilon}{2}
\end{aligned}$$

Also, we know by construction that

$$|f(X_{n_k}) - f(Y_{n_k})| \geq \varepsilon_0$$

Thus, we have a contradiction. $\qquad \square$

13 Integrability Condition

Theorem: Integrability Condition

The function $f : [a,b] \to \mathbb{R}$ is integrable if and only if for every $\varepsilon > 0$, there exists a partition P such that

$$U(f,P) - L(f,P) < \varepsilon$$

Proof

$(\implies)$
Assume f is integrable. Fix $\varepsilon > 0$. There exists P_1 such that

$$\int_a^b f(x)\,\mathrm{d}x = \int_{\underline{a}}^b f\,\mathrm{d}x < L(f,P_1) + \frac{\varepsilon}{2}$$

Also, there exists P_2 such that

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^{\overline{b}} f\,\mathrm{d}x > U(f,P_1) - \frac{\varepsilon}{2}$$

Define $P = P_1 \cup P_2$. Then,

$$U(f,P) - L(f,P) \leq U(f,P_2) - L(f,P_1)$$
$$< \int_a^b f(x)\,\mathrm{d}x + \frac{\varepsilon}{2} - \int_a^b f(x)\,\mathrm{d}x + \frac{\varepsilon}{2}$$
$$= \varepsilon$$

$(\Longleftarrow)$

Fix $\varepsilon > 0$. There exists P such that,

$$U(f, P) - L(f, P) < \varepsilon$$

Then,

$$\int_a^{\overline{b}} f(x)\,\mathrm{d}x - \int_{\underline{a}}^b f(x)\,\mathrm{d}x \leq U(f, P) - L(f, P) < \varepsilon$$

This means, $\displaystyle\int_a^{\overline{b}} f(x)\,\mathrm{d}x - \int_{\underline{a}}^b f(x)\,\mathrm{d}x = 0$.

Thus, f is integrable. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Explanation

This is almost considered the definition of Riemann Integration. It states that a function is integrable if and only if we can create some partitions such that the distance between the lower and upper sums is small.

In the forward direction, we know f is integrable. We start with some integral and a partition to describe the lower sum. We do the same for the upper sum. Defining a partition as the sum of these two allows us to create an inequality using the assumed partitions and this new union.

The reverse direction gives us the distance between the partitions and we must show that we have an integrable function. We create an inequality using the upper and lower integrals of f. We then can see that the upper and lower integral are the same which means the function is integrable.

14 Finite Discontinuities implies Integrability

Theorem: Finite Discontinuities implies Integrability

If $f : [a, b] \to \mathbb{R}$ is continuous at all but finitely many points, then f is integrable.

Proof

Part 1:

Assume f is continuous on $[a, b]$ then f is uniformly continuous on $[a, b]$.

Fix $\varepsilon > 0$. We will find a partition P such that $U(f, P) - L(f, P) < \varepsilon$. This will imply integrability.

Uniform continuity implies that there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{|b-a|}$.

Choose P so that $|x_i - x_{i-1}| < \delta$ for all $i \in 1, 2, \cdots, n$.

Here $P = \{x_0, x_1, x_2, \cdots, x_{n-1}\}$. Then,

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} \left(\sup_{[x_{i-1},x_i]} f(x) - \inf_{[x_{i-1},x_i]} f(x) \right)(x_i - x_{i-1})$$

Since f is continuous, by the extreme value theorem, there exists $x_i' \in [x_{i-1}, x_i]$ such that,

$$f(x_i') = \sup_{[x_{i-1},x_i]} f(x)$$

And there exists some $x_i'' \in [x_{i-1}, x_i]$ such that,

$$f(x_i'') = \inf_{[x_{i-1},x_i]} f(x)$$

Then,

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} \left(f(x_i') - f(x_i'') \right)(x_i - x_{i-1})$$

$$< \sum_{i=1}^{n} \frac{\varepsilon}{|b-a|}(x_i - x_{i-1})$$

$$= \frac{\varepsilon}{|b-a|}(x_1 - x_0 + x_2 - x_1 + \cdots + x_n - x_{n-1})$$

$$= \frac{\varepsilon}{|b-a|}(x_n - x_0)$$

$$= \frac{\varepsilon}{|b-a|}|b-a|$$

$$= \varepsilon$$

Thus, f is integrable.

Part 2:

Assume f has exactly 1 discontinuity, assume it is at $C \in (a,b)$. The cases where it is at a or b are treated similarly.

Fix $\varepsilon > 0$. We will find a partition P such that $U(f,P) - L(f,P) < \varepsilon$.

Define $\delta = \min\{\frac{\varepsilon}{8M}, \frac{b-C}{2}, \frac{C-a}{2}\}$.

Where M is such that $|f(x)| \leq M \quad \forall x \in [a,b]$.

By our choice of δ, $[C-\delta, C+\delta] \subset [a,b]$.

By part 1, f is integrable on $[a, C-\delta]$ and $[C+\delta, b]$.

Therefore, there exist partitions P_1 of $[a, C-\delta]$ and P_2 of $[C+\delta, b]$ such that,

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{4}$$

$$U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{4}$$

Define $P = P_1 \cup P_2$, a partition of $[a, b]$.

$$U(f, P) - L(f, P) = U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2)+$$

$$\left(\sup_{[C-\delta, C+\delta]} f(x) - \inf_{[C+\delta, C-\delta]} \right) f(x)(C + \delta - (C - \delta))$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2M \cdot 2\delta$$

$$\leq \frac{\varepsilon}{2} + \frac{4\varepsilon}{8M} \cdot M$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

Part 3:
If f has more than 1 discontinuity, apply part 2 enough times. $\square$

Explanation

This theorem describes how a continuous function at all but finitely many points is still integrable. The idea behind this is that if we can take an area around each discontinuity od width epsilon, make the width very small and then as long as it isnt infinitely tall, the area will be negligible, thus making it still integrable.

Part 1 of the proof looks at a function with no discontinuities. This is relatively simple. We know the function is uniformly continuous so we use that definition and we need to take a partition such that the upper and lower sum is less than epsilon. We expand this into the summation notation and use extreme value theorem to predict two extremes, the sup and inf. This is then rearranged to show the partition is less than epsilon.

The second part of the proof looks at 1 discontinuity. We assume that the discontinuity is at a point C and then set delta to be the minimum of some values. We know that the function is bounded. By the choice of delta, no matter what, there is some distance around C within the interval. By part 1, we have that its integegrable on either side of C. We then make those the partitions. Defining a third partition as the union of the two halfs, we get the entire interval minus a bit around C. We can define the third partition by adding the other two and adding the middle section around C. This is then rearranged to give less than epsilon which shows integrability.

Part 3 simply applies part 2 multiple times.

15   A Monotone Function is Integrable

Theorem: A Monotone Function is Integrable

If $f : [a, b] \to \mathbb{R}$ is monotone, then f is integrable.

Assume without loss of generality that f is increasing.
Fix $\varepsilon > 0$. We will find a partition $P = \{x_0, x_1, x_2, \cdots, x_n\}$ such that $U(f, P) - L(f, P) < \varepsilon$.
Let P be the partition that splits $[a, b]$ into n equal parts.
Namely, set $x_k = a + k \cdot \frac{b-a}{n}$. Then,

$$
\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} \left(\left(\sup_{[x_{i-1}, x_i]} f(x) - \inf_{[x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}) \right) \\
&= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \left(\frac{b-a}{n} \right) \\
&= \frac{b-a}{n} \cdot (f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1})) \\
&= \frac{b-a}{n} \cdot (f(x_n) - f(x_0)) \\
&= \frac{b-a}{n} \cdot (f(b) - f(a))
\end{aligned}
$$

Chose $n > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$.
Then, $U(f, P) - L(f, P) < \varepsilon$ and so f is integrable. $\qquad \square$

This theorem states that if a function is monotone, it is integrable. The proof is fairly simple, it requires us to find a partition that splits the interval evenly. We then expand the upper and lower sum which allows us to find a simple expression for the upper - lower sum. We then choose n to be larger than something which allows the expression to be less than epsilon and we are done.

16   The Mean Value Theorem For Integrals

Theorem: The Mean Value Theorem For Integrals

If f is continuous on $[a, b]$, then, there exists some $C \in [a, b]$ such that $\int_a^b f(x)\, \mathrm{d}x = f(c)(b-a)$.

21

> **Proof**
>
> If f is constant on $[a, b]$, the result is obvious.
> Assume f is not constant on $[a, b]$.
> Denote $m = \inf_{[a,b]} f(x), M = \sup_{[a,b]} f(x)$.
> Since f is continuous, by the extreme value theorem, there exists points $x_m \in [a, b]$ and $x_M \in [a, b]$ such that $m = f(x_m), M = f(x_M)$.
> Without loss of generality, assume $x_m < x_M$. Observe that
>
> $$m(a - b) \leq \int_a^b f \, \mathrm{d}x \leq M(b - a)$$
>
> $$m \leq \frac{\int_a^b f \, \mathrm{d}x}{(b - a)} \leq M$$
>
> Apply the intermediate value theorem on $[x_m, x_M]$. Since $f(x_m) = m$ and $f(x_M) = M$, there exists some $C \in [x_m, x_M]$ such that $f(C) = \frac{\int_a^b f \, \mathrm{d}x}{(b-a)}$.
> Then,
>
> $$\int_a^b f \, \mathrm{d}x = f(C) \cdot (b - a)$$
>
> $\square$

> **Explanation**
>
> his theorem states the existance of a value such that a rectangle made of height c and width b - a will be the value of the definite integral between a and b.
> We start by assuming the function is not constant as a constant function's area is a rectangle anyway.
> We use the extreme value theorem to give us the global max and min at the supremum and infimum. We then use this to bound the integral from either side. Rearranging the inequality, we can then apply the intermediate value theorem which states that a continuous function will attain every point between an interval. Thus we know there is some value that gives us the inequality and so we are done.

17 Continuity of an Antiderivative

Theorem: Continuity of an Antiderivative

If f is integrable on $[a, b]$ then the function $F(x) = \int_a^b f(x)\,\mathrm{d}x$ is continuous on $[a, b]$.

Proof

Suppose $C \in [a, b]$. We will show F is continuous at C. Let M be such that $|F(x)| \le M \quad \forall x \in [a, b]$.

We want to show that $\lim_{h \to 0} F(C + h) = F(C)$ or equivalently, $\lim_{h \to 0}(F(C + h) - F(C)) = 0$.

We begin by showing $\lim_{h \to 0^+}(F(C + h) - F(C)) = 0$.

Indeed, if $h > 0$ then,

$$F(C + h) - F(C) = \int_a^{C+h} f\,\mathrm{d}x - \int_a^C f\,\mathrm{d}x = \int_C^{C+h} f\,\mathrm{d}x$$

Since $-M \le f \le M$ on $[a, b]$,

$$\int_C^{C+h} f\,\mathrm{d}x \le M(\cancel{C} + h - \cancel{C}) = Mh$$

Also, $-Mh \le \int_C^{C+h} f\,\mathrm{d}x$.

Therefore,

$$|f(C + h) - f(C)| = \left| \int_C^{C+h} f\,\mathrm{d}x \right| \le Mh$$

This means $\lim_{h \to 0^+}(F(C + h) - F(C)) = 0$ and similarly, $\lim_{h \to 0^-}(F(C + h) - F(C)) = 0$.

$\square$

Explanation

This theorem states that if a function is continous, the integral of the function will be continuous.

The proof shows that the limit is equal to the value for any arbitrary point. We can than take the bound of the function and evaluate the definite integral to give an expression which is less that the integral. This allows us to create an inequality with absolute values which is exactly what we want when we take h to 0 as it will just give the limit we need.

18   The Fundamental Theorem of Calculus

Theorem: **The Fundamental Theorem of Calculus**

Assume $f : [a, b] \to \mathbb{R}$ is continuous.

Define $F(x) = \displaystyle\int_a^x f(t)\,dt$. Then, F is differentiable on (a, b), $F'(x) = f(x)$.

Indeed, $\displaystyle\int_a^b f\,dx = F(b) - F(a)$.

Proof

Let us compute $F'(x)$ for $x \in (a, b)$. We want to find $\displaystyle\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$.

We begin by finding $\displaystyle\lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h}$.

Given $h \in (0, b - x)$, we compute,

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$$

By the mean value theorem for integrals,

$$\int_x^{x+h} f(t)\,dt = f(c)(\cancel{x} + h - \cancel{x}) = f(c)(h)$$

Where $c \in [x, x+h]$.

Note that f is uniformly continuous on $[a, b]$. Therefore, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

If $h < \delta$ then $|c - x| < \delta$. Therefore,

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = \left| \frac{f(c)h}{h} - f(x) \right| = |f(c) - f(x)| < \varepsilon$$

Thus, $\displaystyle\lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h} = f(x)$.

Similarly, $\displaystyle\lim_{h \to 0^-} \frac{F(x+h) - F(x)}{h} = f(x)$.

So, $F'(x) = f(x)$.

The Newton-Leibniz formula follows immediately.   □

19 Substitution Formula

> **Theorem: Substitution Formula**
>
> If f, g are continuous on $[a, b]$ and g is differentiable on (a, b) with g' continuous, then,
>
> $$\int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u = \int_{a}^{b} f(g(x))g'(x)\,\mathrm{d}x$$

> **Proof**
>
> Suppose F is an antiderivative of f.
> By the fundamental theorem of calculus, the left hand side of the equation is equal to,
>
> $$F(g(b)) - f(g(a))$$
>
> Next, by the chain rule, we have,
>
> $$(F \circ g)' = (F' \circ g) \cdot g' = (f \circ g)g'$$
>
> This means $F \circ g$ is an anti-derivative of $(f \circ g)g'$.
> Therefore, by the fundamental theorem of calculus,
>
> $$\int_{a}^{b} (f \circ g)g'\,\mathrm{d}x = (F \circ g)(b) - (F \circ g)(a)$$
> $$= F(g(b)) - F(g(a))$$
>
> Thus, $\int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u = \int_{a}^{b} f(g(x))g'(x)\,\mathrm{d}x$ holds. $\qquad \square$

20   Integration by Parts

Theorem: Integration by Parts

If $u, v : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b),

$$\int uv' \, \mathrm{d}x = uv - \int u'v \, \mathrm{d}x$$

$$\int_a^b uv' \, \mathrm{d}x = uv|_a^b - \int_a^b u'v \, \mathrm{d}x$$

Where $uv|_a^b = u(b)v(b) - u(a)v(a)$.

Proof

We know that $uv' = u'v + v'u$.
Integrating both sides, we obtain,

$$\int (uv)' \, \mathrm{d}x = \int u'v \, \mathrm{d}x + \int v'u \, \mathrm{d}x$$

$$uv = \int u'v \, \mathrm{d}x + \int uv' \, \mathrm{d}x$$

$$\int u'v \, \mathrm{d}x = uv - \int v'u \, \mathrm{d}x$$

The formula for the definite integral follows from the fundamental theorem of calculus.

$\square$

This theorem is really a justification of a formula used to integrate the product rule. It takes the product of two functions. Integrating both sides of the product rule gives us exactly the integration by parts formulua.

21   Integral Test

Theorem: Integral Test

Assume f is a non negative, non increasing continuous function on the interval $[1, \infty)$. Then, $\displaystyle\int_1^\infty f \, \mathrm{d}x$ and $\displaystyle\sum_{n=1}^\infty f(n)$ converge or diverge together.

Proof

Consider the interval $[1, n+1]$. The set $P = \{1, 2, 3, 4, \cdots, n+1\}$ is a partition of $[1, n+1]$. Since f is non increasing,

$$\inf_{[x_{i-1}, x_i]} f = f(x_i), \quad \sup_{[x_{i-1}, x_i]} f = f(x_{i-1})$$

Consequently,

$$U(f, P) = \sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]} f(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} f(x_{i-1})(i + 1 - i)$$

$$= \sum_{i=1}^{n} f(x_{i-1})$$

$$= \sum_{i=1}^{n} f(i)$$

Also, $L(f, P) = \sum_{i=1}^{n} f(i+1)$.

If $\int_{1}^{\infty} f \, \mathrm{d}x$ converges, then, $L(f, P) = \sum_{i=1}^{n} f(i+1) = \sum_{i=2}^{n+1} f(i)$ converges as $n \to \infty$.

Thus, $\sum_{i=1}^{\infty} f(i)$ converges.

If $\int_{1}^{\infty} f \, \mathrm{d}x$ diverges then $U(f, P) = \sum_{i=1}^{n} f(i)$ diverges as $n \to \infty$

Which means, $\sum_{i=1}^{\infty} f(i)$ diverges. $\qquad \square$

22 Limit Comparison Test

Theorem: Limit Comparison Test

Suppose $a_n, b_n > 0$ for all $n \in \mathbb{N}$.

If $\lim\limits_{n \to \infty} \dfrac{a_n}{b_n} = c > 0$ (not ∞)

then, $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ converge and diverge simultaneously.

Proof

Assume $\sum\limits_{n=1}^{\infty} b_n$ converges. Since $\lim\limits_{n \to \infty} \dfrac{a_n}{b_n} = c$, there exists some $N \in \mathbb{N}$ such that if $n \geq N$, $\dfrac{a_n}{b_n} < 2c$ and $a_n < 2cb_n$.

Now, $\sum\limits_{n=1}^{\infty} 2cb_n = 2c \sum\limits_{n=1}^{\infty} b_n$ converges.

By comparison, since $\sum\limits_{n=N}^{\infty} a_n \leq 2c \sum\limits_{n=N}^{\infty} b_n \leq 2c \sum\limits_{n=1}^{\infty} b_n$, the series $\sum\limits_{n=N}^{\infty} a_n$. Hence, $\sum\limits_{n=1}^{\infty} a_n$ converges.

Next, suppose $\sum\limits_{n=1}^{\infty} a_n$ converges. Then, $\lim\limits_{n \to \infty} \dfrac{b_n}{a_n} = \dfrac{1}{c} > 0$ and $\sum\limits_{n=1}^{\infty} b_n$ converges by the previous argument. $\qquad \square$

23 Ratio Test

> **Theorem: Ratio Test**
>
> Assume $a_n > 0$ for all $n \in \mathbb{N}$.
> Then,
>
> 1. If $\displaystyle\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$
>
>   Then, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.
>
> 2. If $\displaystyle\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$
>
>   Then, $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

> **Proof**
>
> We assume that $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r \in \mathbb{R}$.
>
> 1. Fixing S such that $r < S < 1$.
>   Because $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < S$, there exists some $N \in \mathbb{N}$ such that if $n \geq N$ then $\frac{a_{n+1}}{a_n} < S$.
>   Then, $\frac{a_{N+1}}{a_N} < S$ and $a_{N+1} < S \cdot a_N$.
>   Also, $\frac{a_{N+2}}{a_{N+1}} < S$ then $a_{N+2} < S \cdot a_{N+1} < S^2 a_N$.

Next, $\frac{a_{N+3}}{a_{N+2}} < S$ then $a_{N+3} < S \cdot a_{N+2} < S^3 a_N$.

Continuing like this, we find $a_{N+k} < S^k \cdot a_N$. Thus,

$$\sum_{k=N}^{\infty} a_n \leq \sum_{k=0}^{\infty} S^k \cdot a_N = a_N \sum_{k=0}^{\infty} S^k = a_N \frac{1}{1-S}$$

Thus, $\displaystyle\sum_{k=N}^{\infty} a_N$ converges. Hence, $\displaystyle\sum_{n=0}^{\infty} a_n$ converges.

2. Assuming $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r > 1$ then there exists some $N \in \mathbb{N}$ such that if $n \geq N$ then $\frac{a_{n+1}}{a_n} > 1$ then $\frac{a_{N+1}}{a_N} > 1$ so, $a_{N+1} > a_N$. Also, $\frac{a_{N+2}}{a_{N+1}} > 1$ so $a_{N+2} > a_{N+1} > a_N$ and so on.

Thus, $a_{N+k} > a_N$ for all $k \in \mathbb{N}$ which means a_n cannot go to 0 as $n \to \infty$.

$\square$

Explanation

This theorem is very strong at proving series convergence or divergence.

We say that taking the limit of the ratio of two consecutive terms. If the limit is less than 1, the series will converge. If it is greater than 1, the series will diverge. For the first statement, we assume some value larger than the limit. We then say that by definition, the ratio will be less than that value. Rearranging this, we can see than the larger term of the sequence is less than the value multiplied by the smaller term. We then apply this idea multiple times, creating an inequality for the larger term of the sequence. We then take the infinite sum of this inequality and see that the series past a certain point converges which implies the series from 1 to infinity converges.

For the second statement, we know the ratio converges to something less than 1. Applying the definition of the limit, we find that the $a_{n+1} > a_n$. This follows for all n and so we have some sequence that cannot converge to 0 which implies divergence.

---

24 The Leibniz Test

Theorem: The Leibniz Test

Assume $(a_n)_{n=1}^{\infty}$ is a sequence such that $a_n \geq 0$, $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$ and $\displaystyle\lim_{n\to\infty} a_n = 0$,

Then, the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Let $S_n = \displaystyle\sum_{k=1}^{n} (-1)^{k+1} a_k$.

Observe that $S_1 \geq S_3 \geq S_5 \geq \cdots$

Indeed,

$$
\begin{aligned}
S_{2n+3} &= S_{2n+1} - a_{2n+2} + a_{2n+3} \\
&= S_{2n+1} + (a_2 n + 3 - a_2 n + 2) \\
&\leq S_{2n+1}
\end{aligned}
$$

Also,

$$
S_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2}) \geq S_{2n}
$$

And,

$$
S_2 \leq S_4 \leq S_6 \leq \cdots
$$

Moreover, $S_{2n} \leq S_{2n+1}$ since $S_{2n+1} = S_{2n} + a_{2n+1} \geq S_{2n}$.

We conclude $S_{2n} \leq S_{2n+1} \leq S_1$ and $S - 2n + 1 \geq S_{2n} \geq S_2$.

Thus, the sequences $(S_{2n})_{n=1}^{\infty}$ and $(S_{2n+1})_{n=1}^{\infty}$ are bounded.

Hence, they must converge. Thus, we can say,

$$
\lim_{n\to\infty} S_{2n} = \alpha, \quad \lim_{n\to\infty} S_{2n+1} = \beta
$$

Now,

$$
\begin{aligned}
\beta - \alpha &= \lim_{n\to\infty} S_{2n+1} - \lim_{n\to\infty} S_{2n} \\
&= \lim_{n\to\infty} (S_{2n+1} - S_{2n}) \\
&= \lim_{n\to\infty} a_{2n+1} \\
&= 0
\end{aligned}
$$

Thus, $\beta = \alpha$ and $\displaystyle\lim_{n\to\infty} S_n = \alpha = \beta$ and $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges. $\qquad\square$

25 An Absolutely Convergent Series implies convergence

> **Theorem: An Absolutely Convergent Series implies convergence**
>
> If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

> **Proof**
>
> Use the cauchy criterion.
> Fixing $\varepsilon > 0$. We must show that there exists some $N \in \mathbb{N}$ such that if $p > q \geq N$ then
>
> $$\left| \sum_{n=q}^{p} a_n \right| < \varepsilon$$
>
> Since $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, there exists some $N \in \mathbb{N}$ such that $\left| \displaystyle\sum_{n=q}^{p} |a_n| \right| < \varepsilon$
>
> By the triangle inequality,
>
> $$\left| \sum_{n=q}^{p} a_n \right| \leq \sum_{n=1}^{\infty} |a_n| = \left| \sum_{n=q}^{p} |a_n| \right| < \varepsilon$$
>
> $\square$

26   Maclaurin Series Examples

Theorem: Maclaurin Series Examples

The formulas and derivations for the functions,

1. $f(x) = e^x$

2. $f(x) = \sin(x)$

3. $f(x) = \cos(x)$

4. $f(x) = \frac{1}{1-x}$

Proof

1. Let $f(x) = e^x$.
   Let us write down the Maclaurin Series.
   Clearly, $f^{(n)}(0) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^x \Big|_{x=0} = e^0 = 1$
   Therefore, the series is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
   This converges absolutely for all $x \in \mathbb{R}$ by the ratio test.

2. Let $f(x) = \sin(x)$.

$$
\begin{aligned}
f(x) &= \sin(x) \\
f'(x) &= \cos(x) \\
f''(x) &= -\sin(x) \\
f'''(x) &= -\cos(x) \\
f''''(x) &= \sin(x)
\end{aligned}
$$

$$
f^{4k}(x) = \sin(x) \qquad f^{4k+1}(x) = \cos(x)
$$

For all $k \in \mathbb{N}$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \frac{\sin(0)}{0!} \cdot x^0 + \frac{\cos(0)}{1!} \cdot x^1 + \frac{-\sin(0)}{2!} \cdot x^2 + \frac{-\cos(0)}{3!} \cdot x^3 + \dots$$

$$= x - \frac{1}{3!} \cdot x^3 + \frac{1}{5!} \cdot x^5$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$$

<u>Note:</u> the series converges absolutely by the ratio test for all $x \in \mathbb{R}$

$$\limsup_{n \to \infty} \frac{(-1)^{n+1} \cdot \left(\frac{x^{2n+2}}{(2n+2)!} \right)}{(-1)^n \cdot \left(\frac{x^{2n+1}}{(2n+1)!} \right)} = \limsup_{n \to \infty} \frac{-x}{2n+2} = 0 < 1$$

3. Let $f(x) = \cos(x)$.

$$f(x) = \cos(x)$$
$$f'(x) = -\sin(x)$$
$$f''(x) = -\cos(x)$$
$$f'''(x) = \sin(x)$$
$$f''''(x) = \cos(x)$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \frac{\cos(0)}{0!} \cdot x^0 + \frac{-\sin(0)}{1!} \cdot x^1 + \frac{-\cos(0)}{2!} \cdot x^2 + \frac{\sin(0)}{3!} \cdot x^3 + \dots$$

$$= 1 - \frac{1}{2!} \cdot x^2 + \frac{1}{4!} \cdot x^4 - \frac{1}{6!} \cdot x^6 + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{2n!}$$

For all $x \in \mathbb{R}$

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f'''(x) = \frac{6}{(1-x)^4}$$

$$f''''(x) = \frac{24}{(1-x)^5}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \frac{1}{(1-0)(0!)} \cdot x^0 + \frac{1}{(1-0)^2(1!)} \cdot x^1 + \frac{2}{(1-0)^3(2!)} \cdot x^2$$

$$+ \frac{6}{(1-0)^4(3!)} \cdot x^3 + \frac{24}{(1-0)^5(4!)} \cdot x^4$$

$$= \sum_{n=0}^{\infty} \frac{n!}{n!} \cdot x^n$$

$$= \sum_{n=0}^{\infty} x^n$$

This is a geometric series. It converges if $x \in (-1, 1)$ and diverges otherwise.

$\square$

Explanation

This isnt really a theorem, it is a collection of examples of common Maclaurin series.
A Maclaurin series is an approximation of a function at values close to 0.
The derivation of each example has an infinite sum of the ratio of the nth derivative and n factorial multiplied by x to the power of n. So, for each function we need to find the nth derivative and we are done.

27   Inhomogenous Linear System Solutions

Proposition: Inhomogenous Linear System Solutions

If p is a vector such that $Ap = b$, then,

$$\{x \in \mathbb{R}^n | Ax = b\} = \{y + p | y \in \text{NS(A)}\}$$

Proof

Observe that $A(x + y) = Ax + Ay$ and $A(\lambda x) = \lambda Ax$ for all $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.
Assume $y \in \text{NS(A)}$. Let us prove that $A(y + p) = b$.
Indeed, $A(y + p) = Ay + Ap = 0 + b = b$.
Thus,

$$\{x \in \mathbb{R}^n | Ax = b\} \supset \{y + p | y \in \text{NS(A)}\}$$

Now, assume $Ax = b$. Clearly, $x = p + (x - p)$.
The vector $y = x - p$ is in NS(A) because

$$A(x - p) = Ax - Ap = b - b = 0$$

Thus,

$$\{x \in \mathbb{R}^n | Ax = b\} \subset \{y + p | y \in \text{NS(A)}\}$$

□

Explanation

This proposition states that if we have some vector multiplied by a matrix, then the vector is given by some x - the null space of the matrix.
All of the solutions to Ax = b (including the one you found, p) are the same as 1 solution to Ax = 0 and adding p.
We start the proof with some properties assumed on the matrix and defining variables. Then, we need to prove $A(y + p) = 0$ as this will ensure $x = y + p$. This is quite obvious. Indeed, we also need to show that $x = p + y$ which is also quite obvious. Then we are done.

---

28 Invertibility of a Matrix and the Null Space

Theorem: Invertibility of a Matrix and the Null Space

The matrix A is invertible if and only if NS(A) = {0}.

Proof

($\implies$)
Assume A is invertible. We know $0 \in \text{NS(A)}$. We want to prove that if $Ax = 0, x = 0$ then,

$$\begin{aligned} x &= (A^{-1}A)x \\ &= A^{-1}(Ax) = A^{-1}0 = 0 \end{aligned}$$

($\impliedby$)
We claim that if NS(A) = 0 then A has a right inverse.
Proof of the claim
If NS(A = 0) then the system $Ax = 0$ has a unique solution.
By an earlier proposition, the system $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.
Take,

$$b^1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad b^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \dots$$

Solving $Ax = b^i$ for $i = 1, 2, \cdots, n$, we obtain an array of vectors,

$$x^i = \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix}$$

We construct a matrix,

$$C = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

Clearly, $AC = I_n$.
Thus, C is the right inverse of A.
This proves the claim.
Now, let us show that A has a left inverse. Note that C has a left inverse, namely, A.
By the same argument as in ($\implies$).
This implies NS(C) = $\{0\}$. By the claim, C has a right inverse.
This inverse is equal to A (proven earlier).
Thus, $AC = CA = I_n$.
Thus, C is the inverse of A. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Explanation

This theorem looks at the nullspace of a matrix and concludes that if is zero then the matrix is invertible. Indeed, if the matrix is invertible the null space is zero.
The forward direction follows directly from the definition of invertibility.
The reverse direction is a bit more involved.
We must prove that if the nullspace is zero then the matrix is invertible.
We start by showing it has a right inverse. We construct a matrix using the unique solutions to the system $Ax = b$. This is clearly the inverse of A on the right as we have defined the matrix of b to be the identity matrix.
Now, we show that A has a left inverse. Obviously, A is the left inverse of C. Thus, the nullspace of C is zero and so we have shown that C has a right inverse. This is just A. This proves that A is invertible and we are done.

29   Linearity of the Determinant

Theorem: Linearity of the Determinant

Suppose $u, v, a_1, \cdots, a_n$ are vectors in $\mathbb{R}^n$. Consider the matrices,

$$
A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{r-1} \\ u + \lambda v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} \qquad B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} \qquad C = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}
$$

Then, $\det A = \det B + \lambda \det C$

Proof

We will argue with induction.
The result is obvious if $n = 1$. Then, $A = (u + \lambda v), B = u, C = v$ for some $u, v \in \mathbb{R}$.
Then,

$$\det A = u + \lambda v = \det B + \lambda \det C$$

Assume the result holds for $(n-1) \times (n-1)$ matrices. Let us prove the result for A, B, C being $n \times n$ matrices.
<u>Case 1:</u>
Assume $r = 1$.
In this case,

$$
\begin{aligned}
\det A &= \sum_{j=1}^{n} (-1)^{1+j} (u_j + \lambda v_j) \det \tilde{A}_{1j} \\
&= \sum_{j=1}^{n} (-1)^{1+j} u_j \det \tilde{A}_{1j} + \lambda \sum_{j=1}^{n} (-1)^{1+j} v_j \det \tilde{A}_{1j}
\end{aligned}
$$

Where $u = (u_1, u_2, \cdots u_n) \qquad v = (v_1, v_2, \cdots, v_n)$.
Now, since $r = 1$, $\tilde{A}_{1j} = \tilde{B}_{1j} = \tilde{C}_{1j}$.
Therefore,

$$
\begin{aligned}
\det A &= \sum_{j=1}^{n} (-1)^{1+j} u_j \det \tilde{B}_{1j} + \lambda \sum_{j=1}^{n} (-1)^{1+j} v_j \det \tilde{C}_{1j} \\
&= \det B + \lambda \det C
\end{aligned}
$$

Case 2:

Assume $r > 1$.

In this case, the first rows of A, B and C are the same. In fact, they are a_1.

Now,

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det \tilde{A}_{1j}$$

The matrix $\tilde{A}_{ij}$ is $(n-1) \times (n-1)$. By the induction hypothesis,

$$\det \tilde{A}_{1j} = \det \tilde{B}_{1j} + \lambda \det \tilde{C}_{1j}$$

Therefore,

$$
\begin{aligned}
\det A &= \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det \left(\tilde{B}_{1j} + \tilde{C}_{1j} \right) \\
&= \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det \tilde{B}_{1j} + \lambda \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det \tilde{C}_{1j} \\
&= \sum_{j=1}^{n} (-1)^{j+1} B_{1j} \det \tilde{B}_{1j} + \lambda \sum_{j=1}^{n} (-1)^{j+1} C_{1j} \det \tilde{C}_{1j} \\
&= \det B + \lambda \det C
\end{aligned}
$$

$\square$

Explanation

This theorem is very useful in calculating determinants. It says that if we have a matrix that is the same as another matrix with a row containing some augmentation of the other row, we can take the determinant equal to the det of the similar matrix and the scalar multiple of a similar matrix again.

We use induction for this proof. The base case of a 1x1 matrix is obvious since the determinant is just the element in the matrix which holds under addition and multiplication. We assume it holds for $(n-1)x(n-1)$ matrices and now prove it holds for nxn matrices.

We write the determinant of A as the expanded series formula. We can then split this into 2 series. Whih is obviously the result we want. This holds when the row in question is the first row.

We now look at when the row in question is anywhere. The definition of the determinant includes a matrix which is $(n-1)x(n-1)$ by the assumption, the theorem holds here. We can rewrite this then as the determinant of the smaller B and C. Expanding this again we then get the result we need.

30   Invertibility and the Determinant

Theorem: Invertibility and the Determinant

A matrix A is invertible if and only if $\det A \neq 0$

Proof

Suppose A is some square matrix. ($\Longrightarrow$)
Since $AA^{-1} = I_n$, taking the determinant of this,

$$\det A \det A^{-1} = 1$$

This clearly shows $\det A \neq 0$.
($\Longleftarrow$)
Cosider the matrix

$$G = \frac{1}{\det A} \left((C_{ij})_{i,j=1}^n \right)^T$$

Where $C_{ij} = (-1)^{i+j} \det \tilde{A}_i j$.
We claim that

$$GA = AG = I_n$$

Indeed, given $k = 1, \cdots, n$, we find,

$$
\begin{aligned}
(AG)_{KK} &= \sum_{i=1}^n A_{ki} G_{ki} \\
&= \sum_{i=1}^n A_{ki} \frac{1}{\det A} (-1)^{i+k} \det \tilde{A}_{ki} \\
&= \frac{1}{\det A} \sum_{i=1}^n A_{ki} (-1)^{i+k} \det \tilde{A}_{ki} \\
&= \frac{1}{\det A} \det A = 1
\end{aligned}
$$

Thus, AG has a diagonal of 1s.
If $K \neq L$ then,

$$
\begin{aligned}
(AG)_{KL} &= \sum_{i=1}^n A_{Ki} G_{iL} \\
&= \frac{1}{\det A} \sum_{i=1}^n A_{Ki} (-1)^{i+L} \det \tilde{A}_{Li}
\end{aligned}
$$

The sum is the determinant of the matrix

$$
\begin{pmatrix}
A_{11} & \cdots & A_{1N} \\
\vdots & & \vdots \\
A_{K1} & \cdots & A_{KN} \\
\vdots & \ddots & \vdots \\
A_{K1} & \cdots & A_{KN} \\
\vdots & \ddots & \vdots \\
A_{N1} & \cdots & A_{NN}
\end{pmatrix}
$$

However, this matrix has two identical rows so its determinant is equal to 0.
This means that AG has zeroes off the diagonal.

$$AG = GA = I_n$$

Thus, A is invertible. □

> **Explanation**
>
> This theorem says that a matrix is invertible iff the determinant is not 0. That is, we can find an inverse to any square matrix that has a determinant not 0.
> The forward direction is very simple.
> The reverse direction considers a new matrix that is defined as the reciprocal of the determinant multiplied by the transpose of another matrix.
> We want to show that this matrix is the inverse of A.
> This is done by expanding the multiplication of $(AG)_{KK}$ into series form. We can take the reciprocal of det A out as it is just a constant. This then gives us the formula for the determinant of A. This means that any point of KK in AG is 1. Thus, the diagonal is 1.
> We now look at everyone off the diagonal. Expanding this into series form, we can again take out the reciprocal of det A. We can change the value of L to K as it is just a determinant at a different row. Indeed, when taking the determinant at row L, we will take it out. Thus, we can interchange the variables. This gives us a matrix with 2 identical rows which obivously has determinant of 0. So, we have shown it has zeros off the diagonal. This is just the identity matrix and we are done.

Authors note

These following proofs are not on the final list but are useful or used throughout as assumed knowledge.
Thanks.

31 Cauchy Criterion

The series

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $q > p \geq N$ then

$$\left| \sum_{k=p+1}^{q} a_k \right| < \varepsilon$$

Proof

Denote the partial sum as follows,

$$s_k = \sum_{n=1}^{k} a_n$$

$\sum_{n=1}^{\infty} a_n$ converges if and only if $(s_n)_{n=1}^{\infty}$ converges if and only if $(s_n)_{n=1}^{\infty}$ is Cauchy. This means for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $p, q \geq N$

$$|s_q - s_p| < \varepsilon$$

Now,

$$|s_q - s_p| = \left| \sum_{n=1}^{q} a_n - \sum_{n=1}^{p} a_n \right| = \left| \sum_{n=p+1}^{q} a_n \right| < \varepsilon$$

$\square$