JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **44**, 384–404 (1973)

Conjugate Convex Functions in Optimal Stochastic Control*

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Submitted by J. L. Lions

INTRODUCTION

This paper is concerned with the applications of general methods of convex analysis to problems of optimal stochastic control. In particular we will define what dual problems are in optimal stochastic control, and what the coextremality conditions for dual optimums are. The problem that we solve here being more general than a purely deterministic one, the results which are given include the results of deterministic control.

The methods and the exposition of the results are very similar to the corresponding methods used by Rockafellar in [13], to which we will refer constantly.

One of the apparent shortcomings of the method is that, using strictly variational methods, it must suppose that the information σ-fields are fixed. In some cases, where the information σ-fields are generated by the state variable, it is possible to apply the duality methods to a modified problem. But they will not give us the strong results it is possible to obtain by studying more specialized problems, as existence of optimal Markov controls. We develop other methods in [2] for this type of problem. The obvious reason is that, by developing a formalism applicable to purely deterministic cases, as to stochastic cases, it does not use the stochastic features of the problem in some purely stochastic cases.

Because of their very technical nature, existence results will not be given here, but are developed extensively in [1] and [2].

The results of probability theory which we use can be found in [7] and [8], which we will take as references.

* This paper has been written after the second part of a thesis to be submitted at the Faculté des Sciences de Paris. It has been partially supported by I.R.I.A. (Institut de Recherches en Informatique et Automatique) 78 Rocquencourt, France.

1. Notations

$(\Omega, \mathscr{F}, P)$ is a complete probability space.

$\{\mathscr{F}_t\}_{t \in R^+}$ is an increasing sequence of complete sub σ-fields of $\mathscr{F}$, which has the following properties:

(a) It is right-continuous [7, IV, 30].

(b) It has no time of discontinuity [7, VII D.39].

This last assumption is not strictly necessary, but we make it to simplify the results.

$\mathscr{T}$ is the σ-field of $\Omega \times [0, +\infty[$ of the well measurable sets [7, VIII D.14]. $\mathscr{T}^*$ is its completion for the measure $dP \otimes dt$.[1]

V is a n-dimensional vector space ($n \geqslant 1$).

w is a m-dimensional Brownian motion on $(\Omega, \mathscr{F}, P)$, nonanticipating relative to $\{\mathscr{F}_t\}_{t \in R^+}$. w may be defined equivalently as a square-integrable a.s. continuous martingale on $(\Omega, \mathscr{F}, P)$ with values in R^m, such that, by writing $w = (w_1, \ldots, w_m)$, we have, with the notations of [8],

$$d\langle w_i, w_j\rangle = \delta_{ij}\, dt. \tag{1.1}$$

This definition is correct by the result of Levy [8, p. 110]. Moreover, we extend the definitions for $m = 0$, by taking, conventionally, w as the one-dimensional null process.

w having continuous paths, formula (1.1) and the results of [8] show that it is possible to define unambiguously the stochastic integral of a $\mathscr{T}^*$ class of $\mathscr{T}$-measurable processes H such that for any t, we have:

$$E \int_0^t | H_s |^2\, ds < +\infty. \tag{1.2}$$

We can use for that purpose the "classical" definitions of the stochastic integral of [8, Remark, p. 80] or the definition of [8, Theorem 7, p. 86].

More generally, we could have supposed that w is only a square-integrable martingale, such that $d\langle w_i, w_j\rangle$ is a.s. absolutely continuous with respect to the Lebesgue measure on $[0, +\infty[$, by taking the "classical" definition of the stochastic integral for well measurable processes. The reader can do this extension easily.

For any stopping time σ, L_2^σ is the space of square-integrable $\mathscr{F}_\sigma$-measurable random variables, with values in V.

S is a a.s. nonnull stopping time, a.s. bounded by a finite constant T.

[1] For our purpose $\mathscr{T}$ could have been only the σ-field of nonanticipating sets.

L_{21} is the space of the $dP \otimes dt$ classes u of $\mathscr{T}^*$-measurable functions with values in V, such that

$$E\left(\int_0^S |u_t|\, dt\right)^2 < +\infty. \tag{1.3}$$

We define a norm on L_{21} by

$$\|u\|_{21} = \left\{E\left(\int_0^S |u_t|\, dt\right)^2\right\}^{1/2}. \tag{1.4}$$

$L_{2\infty}$ is the space of the $dP \otimes dt$ classes y of $\mathscr{T}^*$-measurable functions with values in V, such that

$$E(\operatorname*{sup\,ess}_{0\leqslant t\leqslant S} |x_t|^2) < +\infty. \tag{1.5}$$

We define then a norm on $L_{2\infty}$ by

$$\|x\|_{2\infty} = \{E(\operatorname*{sup\,ess}_{0\leqslant t\leqslant S} |x_t|^2)\}^{1/2}. \tag{1.6}$$

L_{22} is the space of the $dP \otimes dt$ classes H of $\mathscr{T}^*$-measurable functions with values in V^m such that

$$E\int_0^S |H_t|^2\, dt < +\infty. \tag{1.7}$$

We define a norm on L_{22} by

$$\|H\|_{22} = \left(E\int_0^S |H_t|^2\, dt\right)^{1/2}. \tag{1.8}$$

By convention, we assume that the elements of L_{21}, $L_{2\infty}$, L_{22} are equal to 0 for $t > S$.

Duality brackets are then defined:

(a) between $L_2{}^\sigma$ and $L_2{}^\sigma$ by the standard scalar product,

(b) between L_{21} and $L_{2\infty}$ by

$$E\int_0^S \langle u_t, y_t\rangle\, dt, \tag{1.9}$$

(c) between L_{22} and L_{22} by

$$E\int_0^S \langle H_t, H_t'\rangle\, dt. \tag{1.10}$$

We consider on the previous spaces only locally convex topologies compatible with the duality previously defined. In particular, L_2^{σ} and L_{22} being Hilbert spaces, the norm topology is compatible with the duality.

$\underline{L}$ is the space of square-integrable martingales with values in V stopped at S, null at 0. $\underline{L}$ can be identified to a closed subspace of L_2^S, on which we put the induced topology.

W is the subspace of $\underline{L}$ generated by the stochastic integrals relative to w of elements of L_{22}. W is a stable space, in the sense of [8, p. 80, no. 4]. Let $W^{\perp}$ be the orthogonal of W in $\underline{L}$ in the sense of [8, p. 81, Theorem 5].

We suppose then that $W^{\perp}$ is decomposed in the sum of two orthogonal subspaces of martingales W_1 and W_2:

$$W^{\perp} = W_1 \oplus W_2 . \tag{1.11}$$

Practically, W_1 will be either $W^{\perp}$ or $\{0\}$. This decomposition will be justified afterwards. In particular, if $\{\mathscr{F}_t\}_{t\in R^+}$ is the family of σ-fields generated by w, a result of Ito [8, p. 135] shows that $W^{\perp} = \{0\}$.

PROPOSITION I–1. *Let $(x_0, \dot{x}, H, M)$ and $(p_0, \dot{p}, H', M')$ be two elements of $L_2^0 \times L_{21} \times L_{22} \times W^{\perp}$. Then, if we define the right continuous processes x and p by*

$$\begin{aligned} x_t &= x_0 + \int_0^t \dot{x}_s\, ds + \int_0^t H_s \cdot dw_s + M_t , \\ p_t &= p_0 + \int_0^t \dot{p}_s\, ds + \int_0^t H_s' \cdot dw_s + M_t', \end{aligned} \tag{1.12}$$

then the process N_t defined by

$$\begin{aligned} N_t = \langle p_t , x_t\rangle - \langle p_0 , x_0\rangle &- \int_0^t \langle \dot{p}_s , x_s\rangle\, ds - \int_0^t \langle p_s , \dot{x}_s\rangle\, ds \\ &- \int_0^t \langle H_s , H_s'\rangle\, ds - \langle M_t , M_t'\rangle \end{aligned} \tag{1.13}$$

is a martingale, null at the origin.

Proof. The formula of change of variables in [8, p. 111] on local semimartingales proves that

$$\begin{aligned} \langle p_t , x_t\rangle = \langle p_0 , x_0\rangle &+ \int_0^t \langle \dot{p}_s , x_s\rangle\, ds + \int_0^t \langle p_s , \dot{x}_s\rangle\, ds \\ &+ \int_0^t \langle H_s , H_s'\rangle\, ds + \int_0^t d[M_s , M_s'] + \int_0^t \langle x_s , H_s'\rangle \cdot dw_s \\ &+ \int_0^t \langle p_s , H_s\rangle \cdot dw_s + \int_0^t \langle x_{s-} , dM_s'\rangle + \int_0^t \langle p_{s-} , dM_s\rangle. \end{aligned} \tag{1.14}$$

Moreover, we known that

$$\langle M_t, M_t' \rangle - \int_0^t d[M_s, M_s']$$

is a martingale. (1.14) proves, then, that N is a local martingale, stopped at S. What is left to be proved is that for any t in $[0, T]$, $\{N_{t \wedge S'}\}_{S' \text{ stopping time}}$ is a uniformly integrable family of random variables.

Remark 2 of VI in [7] proves that

$$\begin{aligned}
&E(\sup_{0 \leqslant t \leqslant T} |M_t|^2) \leqslant 4E|M_s|^2, \\
&E(\sup_{0 \leqslant t \leqslant T} |M_t'|^2) \leqslant 4E|M_s'|^2, \\
&E\left(\sup_{0 \leqslant t \leqslant T} \left| \int_0^t H_s \cdot dw_s \right|^2\right) \leqslant 4\|H\|_{22}^2, \\
&E\left(\sup_{0 \leqslant t \leqslant T} \left| \int_0^t H_s' \cdot dw_s \right|^2\right) \leqslant 4\|H\|_{22}^2 .
\end{aligned} \tag{1.15}$$

We deduce immediately that

$$\begin{aligned}
&E(\sup_{0 \leqslant t \leqslant T} |x_t|^2) < +\infty \\
&E(\sup_{0 \leqslant t \leqslant T} |p_t|^2) < +\infty.
\end{aligned} \tag{1.16}$$

We have then

$$|\langle p_t, x_t \rangle| \leqslant \sup_{0 \leqslant u \leqslant T} |p_u| \sup_{0 \leqslant u \leqslant T} |x_u| . \tag{1.17}$$

In the same way, we have

$$\begin{aligned}
&\left| \int_0^t \langle \dot{p}_s, x_s \rangle \, ds \right| \leqslant \left(\int_0^t |\dot{p}_s| \, ds \right) \sup_{0 \leqslant u \leqslant T} |x_u| , \\
&\left| \int_0^t \langle p_s, \dot{x}_s \rangle \, ds \right| \leqslant \left(\int_0^t |\dot{x}_s| \, ds \right) \sup_{0 \leqslant u \leqslant T} |p_u| , \\
&\left| \int_0^t \langle H_s, H_s' \rangle \, ds \right| \leqslant \left(\int_0^t |H_s|^2 \, ds \right)^{1/2} \left(\int_0^t |H_s'|^2 \, ds \right)^{1/2}, \\
&|\langle M_t, M_t' \rangle| \leqslant \sup_{0 \leqslant u \leqslant T} |M_u| \sup_{0 \leqslant u \leqslant T} |M_u'| .
\end{aligned} \tag{1.18}$$

The assumptions that we have made prove then that each of the random variables on the right side of (1.18) is integrable. (1.17) and (1.18) prove that

$$E(\sup_{0 \leqslant u \leqslant T} |N_u|) < +\infty. \tag{1.19}$$

Thus, the result is proved. ∎

COROLLARY. *Under the assumptions of Proposition* I-1, *one has*

$$E(\langle p_S, x_S\rangle) = E(\langle p_0, x_0\rangle) + E\int_0^S (\langle \dot{p}_t, x_t\rangle + \langle p_t, \dot{x}_t\rangle)\,dt$$
$$+ E\int_0^S \langle H_t, H_t'\rangle\,dt + E(\langle M_S, M_S'\rangle).$$

Proof. N_t being a martingale, N_T has a null mean, and $N_T = N_S$. The result follows. ∎

2. The Problem of Control

A. *Preliminaries*

Let L be a normal convex integrand in the sense of [10], defined on $(\Omega \times [0, +\infty[) \times V \times V \times V^m$, $\Omega \times [0, +\infty[$ being considered as the measured space:

$$(\Omega \times [0, +\infty[, \mathscr{T}^*, dP \otimes dt).$$

Let L^* be the dual integrand of L. L^* is then normal, by [10].

Let M be defined by

$$M(\omega, t, p, s, H') = L^*(\omega, t, s, p, H'). \tag{2.1}$$

Let l_0 and l_S be two convex lower semi-continuous functionals defined respectively on L_2^0 and L_2^S with values in $R \cup \{+\infty\}$, and nonidentically $+\infty$. Let l_0^* and l_S^* be their duals. We define l and m on $L_2^0 \times L_2^S$ by

$$\begin{aligned} l(c_0, c_S) &= l_0(c_0) + l_S(c_S), \\ m(c_0, c_S) &= l_0^*(c_0) + l_S^*(-c_S). \end{aligned} \tag{2.2}$$

We make the following assumptions on L and M.

ASSUMPTION II-1. *One can find* (p_0, s_0, H_0') *in* $L_{2\infty} \times L_{21} \times L_{22}$ *such that*

$$E\int_0^S M(\omega, t, p_0(\omega, t), s_0(\omega, t), H_0'(\omega, t))\,dt < +\infty, \tag{2.3}$$

ASSUMPTION II-2. *One can find* (x_0, y_0, H_0) *in* $L_{2\infty} \times L_{21} \times L_{22}$ *such that*

$$E\int_0^S L(\omega, t, x_0(\omega, t), y_0(\omega, t), H_0(\omega, t))\,dt < +\infty. \tag{2.4}$$

Let us notice here that all the spaces of measurable functions which have been introduced are decomposable in the sense of [10], and this will entitle us to use the results of Rockafellar in [10] and [11].

B. *The Problem of Control*

We define R, R_1 and R_2 by

$$\begin{aligned} R &= L_2{}^0 \times L_{21} \times L_{22} \times W^{\perp}, \\ R_1 &= L_2{}^0 \times L_{21} \times L_{22} \times W_1, \\ R_2 &= L_2{}^0 \times L_{21} \times L_{22} \times W_2. \end{aligned} \tag{2.5}$$

To any $(x_0, \dot{x}, H, M)$ in R we associate the process x_t by

$$x_t = x_0 + \int_0^t \dot{x}_s \, ds + \int_0^t H_s \cdot dw_s + M_t. \tag{2.6}$$

The proof of Proposition I-1 shows that

$$E(\sup_{0 \leqslant t \leqslant T} | x_t |^2) < +\infty.$$

x defines, then, an element of $L_{2\infty}$.

Moreover, the general properties of stochastic processes say that decomposition (2.6) is unique. We can identify, then, R to a space of right-continuous stochastic processes.

In the same way, R_1 and R_2 will be identified to the stochastic processes that their elements define.

DEFINITION II-1. $\Phi_{l,L}$ is the functional defined on R by

$$x = (x_0, \dot{x}, H, M) \xrightarrow[\Phi_{l,L}]{} \begin{cases} l(x_0, x_S) + E \int_0^S L(\omega, t, x(\omega, t), \dot{x}(\omega, t), H(\omega, t)) \, dt \\ \qquad\qquad \text{if} \quad x \in R_1, \\ +\infty \qquad \text{if} \quad x \notin R_1. \end{cases} \tag{2.7}$$

$\Phi_{m,M}$ is the functional defined on R by

$$p = (p_0, \dot{p}, H', M') \xrightarrow[\Phi_{m,M}]{} \begin{cases} m(p_0, p_S) \\ \quad + E \int_0^S M(\omega, t, p(\omega, t), \dot{p}(\omega, t), H'(\omega, t)) \, dt \\ \qquad\qquad \text{if} \quad p \in R_2, \\ +\infty \qquad \text{if} \quad p \notin R_2. \end{cases} \tag{2.8}$$

Assumptions II-1 and II-2 prove that $\Phi_{l,L}$ and $\Phi_{m,M}$ are defined unambiguously. Moreover, they are obviously convex on R.

DEFINITION II-2. The problem of control consists in the minimization of $\Phi_{l,L}$ on R.

The dual problem of control consists in the minimization of $\Phi_{m,M}$ on R. ∎

The distinction between a problem of control and its dual is arbitrary. The reader will see easily that the dual problem of the dual problem of control is the initial problem of control.

EXAMPLE II-1. We assume that $\{\mathscr{F}_t\}_{t\in R^+}$ is the family of σ-algebras generated by w. The continuity of the martingales relative to $\{\mathscr{F}_t\}_{t\in R^+}$ [8, p. 135] proves that the assumptions made in Section 1 are satisfied. Moreover, the same results show that $W^\perp = \{0\}$.

Let (A, B, C, D) be a family of $\mathscr{T}^*$ measurable of matrices bounded on $\Omega \times [0, T]$. Let U_d be a compact convex set of a finite-dimensional space U.

For u $\mathscr{T}^*$ measurable with values in U_d, let Z be the solution of

$$\begin{cases} dZ = (AZ + Cu)\,dt + (BZ + Du)\cdot dw, \\ Z_0 = 0. \end{cases} \tag{2.9}$$

(For general existence and uniqueness results see [1]).

Let K be a positive normal finite convex integrand on

$$(\Omega \times [0, +\infty[) \times V \times U,$$

$\Omega \times [0, +\infty[$ being considered as the measured space

$$(\Omega \times [0, +\infty[, \mathscr{T}^*, dP \otimes dt).$$

We want to minimize

$$E\int_0^T K(\omega, t, Z(\omega, t), u(\omega, t))\,dt. \tag{2.10}$$

Let (L, l) be defined by

$$\begin{aligned} L(\omega, t, x, y, H) &= \begin{cases} K(\omega, t, x, u) & \text{if} \quad \begin{cases} y = Ax + Bu, u \in U_d, \\ H = Cx + Du, \end{cases} \\ +\infty \text{ elsewhere} \end{cases} \\ l(x_0) &= \begin{cases} 0 & \text{if} \quad x_0 = 0 \text{ a.s.}, \\ +\infty & \text{elsewhere.} \end{cases} \end{aligned} \tag{2.11}$$

It is then equivalent to minimize $\Phi_{l,L}$ on R; this comes from the fact that it is proved in [1] that if Z is a solution of (2.8), then

$$E(\sup_{0\leqslant t\leqslant T} | Z_T |^2) < +\infty.$$

Consequently, $(AZ + Cu)$ and $(BZ + Du)$ are in L_{21} and L_{22}, respectively.

EXAMPLE II-2. In Example II-1, let us assume now that w is replaced by η, where η is a square-integrable martingale such that

$$d\langle \eta_i, \eta_j \rangle = r_{ij}(t)\, dt. \tag{2.12}$$

By the adjunction procedure (generalized to the multidimensional case) given in [4, p. 449], we can find H^0 defined on $[0, +\infty[$, with values in V^m, and a Brownian motion w such that for any $T > 0$,

$$\int_0^T | H_s^0 |^2\, ds < +\infty. \tag{2.13}$$

$$-\eta_t = \int_0^t H_s^0 \cdot dw_s\,. \tag{2.14}$$

We then change the definition of L into

$$\begin{cases} L(\omega, t, x, y, H) = K(\omega, t, x, u) & \text{when} \quad \begin{cases} y = Ax + Cu, \\ H = (Bx + Du) \cdot H^0, \end{cases} \\ +\infty & \text{elsewhere.} \end{cases} \tag{2.15}$$

$\{\mathscr{F}_t\}_{t \in R^+}$ will be the family of σ-fields generated by w.

EXAMPLE II-3. Let f, σ, and K be functions defined on

$$\Omega \times [0, T] \times V \times U$$

with values respectively in V, V^m, and R. U is assumed to be a compact metrizable space. We assume, moreover, that for a.s. $\omega(f(\omega, \cdot), \sigma(\omega, \cdot), K(\omega, \cdot))$ are continuous on $[0, T] \times V \times U$; for every (x, u) in $V \times U$, $(f(\cdot, x, u), \sigma(\cdot, x, u), K(\cdot, x, u))$ are $\mathscr{T}^*$ measurable processes:

One can find $k \geqslant 0$ such that for every (x, u) in $V \times U$, one has a.s.

$$| f(\omega, t, x, u)| + | \sigma(\omega, t, x, u)| + | K(\omega, t, x, u)|^{1/2} \leqslant k(1 + | x |^2)^{1/2}; \tag{2.16}$$

One can find $k' \geqslant 0$ such that for any (x, x', u) in $V \times V \times U$, one has a.s.

$$| f(\omega, t, x, u) - f(\omega, t, x, u)| + | \sigma(\omega, t, x, u) - \sigma(\omega, t, x, u)| \leqslant K' \,| x' - x |; \tag{2.17}$$

K is positive.

For u $\mathscr{T}^*$-measurable, let x be the only solution of

$$\begin{aligned} dx &= f(\omega, t, x, u_t)\, dt + \sigma(\omega, t, x, u_t)\, dw, \\ x(0) &= x_0\,. \end{aligned} \tag{2.18}$$

By using the methods of Gikhman–Shorokhod in [5], existence and uniqueness of the solution of (2.18) follow immediately. Moreover we will have

$$E(\sup_{0\leqslant t\leqslant T} |x_t|^2) < +\infty. \tag{2.19}$$

Then (2.16) will prove that $f(\omega, t, x(\omega, t), u(\omega, t))$, $\sigma(\omega, t, x(\omega, t), u(\omega, t))$ are in L_{21} and L_{22}, respectively.

The goal of the problem of control is to find u minimizing

$$E\int_0^T K(\omega, t, x(\omega, t), u(\omega, t))\, dt. \tag{2.20}$$

We change the problem in the following way (the method is very similar to the treatment of Example 3 in [13]): Let L be defined by

$$\begin{aligned} L(\omega, t, x, y, H) &= \inf K(\omega, t, x, u),\\ f(\omega, t, x, u) &= y,\\ \sigma(\omega, t, x, u) &= H, \end{aligned} \tag{2.21}$$

giving to this expression the value $+\infty$ if there is no u such that

$$\begin{aligned} f(\omega, t, x, u) &= y,\\ \sigma(\omega, t, x, u) &= H. \end{aligned} \tag{2.22}$$

Then for any (ω, t), $L(\omega, t, \cdot)$ is lower semicontinuous. Moreover, if (x, y, H) are $\mathscr{T}^*$-measurable functions, then $L(\omega, t, x(\omega, t), y(\omega, t), H(\omega, t))$ is $\mathscr{T}^*$-measurable. This follows from the fact that

$$\begin{aligned}(\omega, t) \underset{\Gamma}{\rightarrow} \{u : f(\omega, t, x(\omega, t), u) &= y(\omega, t),\\ \sigma(\omega, t, x(\omega, t), u) &= H(\omega, t)\},\end{aligned}$$

is a $\mathscr{T}^*$-measurable correspondence, by Theorem 2 of [11], because its graph is $\mathscr{T}^* \otimes B(U)$ measurable. If u_n is a countable family of $\mathscr{T}^*$-measurable selections of Γ, such that $\{u_n(\omega, t)\}$ is dense in $\Gamma(\omega, t)$ for any (ω, t), then

$$L(\omega, t, x(\omega, t), y(\omega, t), H(\omega, t), = \inf_n K(\omega, t, x(\omega, t), u_n(\omega, t)) \tag{2.23}$$

because of the continuity of K.

Besides,

$$(\omega, t) \underset{\Delta}{\rightarrow} \{x, f(\omega, t, x, u), \sigma(\omega, t, x, u); u \in U_d\}$$

is a $\mathscr{T}^*$-measurable correspondence, because, if $\{x_n, u_n\}_{n\in N}$ is a dense countable family in $V \times U$,

$$\{x_n, f(\omega, t, x_n, u_n), \sigma(\omega, t, x_n, u_n)\}_{n\in N}$$

is dense in $\Delta(\omega, t)$ for every (ω, t).

By writing: $W_1 = \{0\}$, the given control problem is equivalent to minimizing the integral:

$$E \int_0^T L(\omega, t, x(\omega, t), \dot{x}(\omega, t), H(\omega, t))\, dt \tag{2.24}$$

for the x of R_1 satisfying $x(0) = x_0$.

Indeed, the integral is well-defined, by the measurability of $L(\omega, t, x(\omega, t), \dot{x}(\omega, t), H(\omega, t))$ and its positivity. Moreover, by using the criteria of Theorem 2 of [11], it can be seen that the integral is finite if and only if one can find u $\mathscr{T}^*$-measurable with values in U such that:

$$dP \otimes dt \text{ a.s.:} \qquad \begin{aligned} f(\omega, t, x(\omega, t), u(\omega, t)) &= \dot{x}(\omega, t) \\ \sigma(\omega, t, x(\omega, t), u(\omega, t)) &= H(\omega, t). \end{aligned} \tag{2.25}$$

If for $dP \otimes dt$ almost every (ω, t), $L(\omega, t, \cdot)$ is convex, then L is a normal convex integrand, by the measurability of Δ, because we can apply the criteria (a) and (b) of [11] part I.

In this case, the approach that we have taken can be used: one checks in particular that Assumptions II-1 and II-2 are satisfied; this follows from the inequalities,

$$0 \leqslant L(\omega, t, x\colon y, H) \leqslant k(1 + |x|^2). \tag{2.26}$$

The first part of the inequality gives then

$$M(\omega, t, 0, 0, 0) \leqslant 0. \tag{2.27}$$

(2.26) and (2.27) prove that II-1 and II-2 are satisfied.

3. Perturbation Methods

We define a duality between R (which we have identified to a space of right continuous stochastic processes) and $R' = L_{2\infty} \times L_2^S$ by

$$\{x = (x_0, \dot{x}, H, M), (y, b)\} \to E \int_0^S \langle \dot{x}_t, y_t \rangle\, dt + E\langle x_S, b\rangle. \tag{3.1}$$

This duality defines on R and R' a locally convex topology, which is Hausdorff. Indeed, if for any (y, b) in R', $\langle x, (y, b)\rangle = 0$ then obviously $\dot{x} = 0$, $x_S = 0$. Then x, being a martingale stopped at S, is null and $x_0 = 0$, $H = 0$ and $M = 0$; if for any x in R, $\langle x, (y, b)\rangle = 0$, then $y = 0$. Moreover, the process $E^{\mathscr{F}_t} b$ is in R. Then, obviously, $b = 0$.

DEFINITION III-1. For (y, b) in R', we define the functional $\Phi^{y,b}_{l,L}$ on R by:

$$x = (x_0, \dot{x}, H, M) \xrightarrow[\Phi^{y,b}_{l,L}]{} \begin{cases} l(x_0, x_S - b) \\ \quad + E\int_0^S L(\omega, t, (x+y)(\omega, t), \dot{x}(\omega, t), H(\omega, t))\, dt \\ \qquad \text{if} \quad x \text{ is in } R_1, \\ +\infty \qquad \text{elsewhere.} \end{cases} \tag{3.2}$$

We define, in the same way, the functional $\Phi^{y,b}_{m,M}$ on R by

$$p = (p_0, \dot{p}, H', M) \xrightarrow[\Phi^{y,b}_{m,M}]{} \begin{cases} m(p_0, p_S - b) \\ + E\int_0^S M(\omega, t, (p+y)(\omega, t), \dot{p}(\omega, t), H(\omega, t))\, dt \\ \qquad \text{if} \quad p \text{ is in } R_2, \\ +\infty \qquad \text{elsewhere.} \end{cases} \tag{3.2'}$$

$\Phi^{y,b}_{l,L}$ and $\Phi^{y,b}_{m,M}$ are then convex functionals on R. We define $\varphi_{l,L}$ and $\varphi_{m,M}$ on R' by

$$\begin{aligned} \varphi_{l,L}(y, b) &= \inf_{x \in R} \Phi^{y,b}_{l,L}(x), \\ \varphi_{m,M}(y, b) &= \inf_{p \in R} \Phi^{y,b}_{m,M}(p). \end{aligned} \tag{3.3}$$

THEOREM III-1. *$\varphi_{l,L}$ and $\varphi_{m,M}$ are convex functionals on R', and their duals are defined by*

$$\begin{aligned} \varphi^*_{l,L} &= \Phi_{m,M}, \\ \varphi^*_{m,M} &= \Phi_{l,L}. \end{aligned} \tag{3.4}$$

In the same way one has

$$\Phi^*_{l,L}(y, b) = \liminf_{(y',b') \to (y,b)} \varphi_{m,M}(y', b'), \tag{3.5}$$

except in the case where $\Phi_{l,L}$ is identically $+\infty$, and where $\varphi_{m,M}$ is equal to $+\infty$ on a neighborhood of (y, b) (the topology is any topology compatible with the duality (R, R')). One has the corresponding result for $\Phi^*_{m,M}$.

Proof. This result is closely related to Theorem 3 of [13]. We will prove only its first part, the second following from general convex analysis results as used in [13].

For p in R, we must calculate

$$\begin{aligned}\varphi_{l,L}^*(p) = \sup_{\substack{x\in R_1\\(y,b)\in R'}} & E\int_0^S \langle \dot{p}_t, y_t\rangle\,dt + E\langle p_S, b\rangle \\ & - E\int_0^S L(\omega, t, (x+y)(\omega, t), \dot{x}(\omega, t), H(\omega, t))\,dt \\ & - l_0(x_0) - l_S(x_S - b).\end{aligned}\tag{3.6}$$

But, x defining an element of $L_{2\infty}$, we have

$$\begin{aligned}\varphi_{l,L}^*(p) = \sup_{\substack{x\in R_1\\(z,b')\in L_{2\infty}\times L_2{}^S}} & E\int_0^S \langle \dot{p}_t, z_t\rangle\,dt - E\int_0^S \langle \dot{p}_t, x_t\rangle\,dt - E\langle p_S, b'\rangle \\ & + E\langle p_S, x_S\rangle - E\int_0^S L(\omega, t, z(\omega, t), \dot{x}(\omega, t), H(\omega, t))\,dt \\ & - l_0(x_0) - l_S(b').\end{aligned}\tag{3.7}$$

But x in R_1 and p in R may be written

$$\begin{aligned} x_t &= x_0 + \int_0^t \dot{x}_s\,ds + \int_0^t H_s\cdot dw_s + M_{1t} \qquad && M_1\in W_1, \\ p_t &= p_0 + \int_0^t \dot{p}_s\,ds + \int_0^t H_s'\cdot dw + M_{1t}' + M_{2t}' \qquad && M_1'\in W_1,\ M_2'\in W_2. \end{aligned}\tag{3.8}$$

By Proposition I-1, and the definitions of R, R_1, and R_2, we have

$$\begin{aligned} E\langle p_S, x_S\rangle = E\langle p_0, x_0\rangle &+ E\int_0^S \langle \dot{p}_t, x_t\rangle\,dt + E\int_0^S \langle p_t, \dot{x}_t\rangle\,dt \\ &+ E\int_0^S \langle H_t, H_t'\rangle\,dt + E\langle M_{1S}, M_{1S}'\rangle. \end{aligned}\tag{3.9}$$

We deduce

$$\begin{aligned}\varphi_{l,L}^*(p) = \sup_{M_1\in W_1} E\langle M_{1S}', M_{1S}\rangle & + \sup_{(\dot{x},H,z)\in L_{21}\times L_{22}\times L_{2\infty}} E\int_0^S \langle \dot{p}_t, z_t\rangle\,dt \\ & + E\int_0^S \langle p_t, \dot{x}_t\rangle\,dt + E\int_0^S \langle H_t', H_t\rangle\,dt \\ & - E\int_0^S L(\omega, t, z(\omega, t), \dot{x}(\omega, t), H(\omega, t))\,dt \\ & + \sup_{(x_0,b')\in L_2{}^0\times L_2{}^S} E\langle p_0, x_0\rangle - E\langle p_S, b'\rangle - l_0(x_0) - l_S(b').\end{aligned}\tag{3.10}$$

But we have

$$\sup_{M_1 \in W_1} E\langle M'_{1S}, M_{1S}\rangle = \begin{cases} 0 & \text{if} \quad M_1' = 0, \\ +\infty & \text{if} \quad M_1' \neq 0. \end{cases} \tag{3.11}$$

Moreover, the results of [10], which can be applied because all the considered spaces are decomposable and because of Assumptions II-1 and II-2, prove that

$$\begin{aligned} \sup_{(\dot{x},H,z)\in L_{21}\times L_{22}\times L_{2\infty}} & E\int_0^S \langle \dot{p}_t, z_t\rangle\, dt + E\int_0^S \langle p_t, \dot{x}_t\rangle\, dt \\ & + E\int_0^S \langle H_t', H_t\rangle\, dt - E\int_0^S L(\omega, t, z(\omega, t), \dot{x}(\omega, t), H(\omega, t))\, dt \\ & = E\int_0^S L^*(\omega, t, \dot{p}(\omega, t), p(\omega, t), H'(\omega, t))\, dt, \end{aligned} \tag{3.12}$$

and this last quantity is never $-\infty$. Finally, we have

$$\sup_{(x_0,b')\in L_2^0\times L_2^S} E\langle p_0, x_0\rangle - E\langle p_S, b'\rangle - l_0(x_0) - l_S(b') = l_0^*(p_0) + l_S^*(-p_S) \tag{3.13}$$

and this last quantity is never $-\infty$.

By adding (3.11), (3.12), and (3.13), we find

$$\varphi_{l.L}^*(p) = \Phi_{m,M}(p). \tag{3.14}$$

The second part of (3.4) will be proved in the same way. ∎

Remark. This result proves that $\Phi_{l,L}$ and $\Phi_{m,M}$ are lower-semicontinuous on R for any topology compatible with the duality (R, R').

4. Duality of Infima

When x is in R_1 and p in R_2, we have

$$l(x_0, x_S) + m(p_0, p_S) \geqslant E\langle x_0, p_0\rangle - E\langle x_S, p_S\rangle. \tag{4.1}$$

Moreover, we will have

$$\begin{aligned} & E\int_0^S L(\omega, t, x(\omega, t), \dot{x}(\omega, t), H(\omega, t))\, dt \\ & \quad + E\int_0^S M(\omega, t, p(\omega, t), \dot{p}(\omega, t), H'(\omega, t))\, dt \\ & \geqslant E\int_0^S \langle x(\omega, t), \dot{p}(\omega, t)\rangle\, dt + E\int_0^S \langle \dot{x}(\omega, t), p(\omega, t)\rangle\, dt \\ & \quad + E\int_0^S \langle H(\omega, t), H'(\omega, t)\rangle\, dt. \end{aligned} \tag{4.2}$$

But this last quantity is by Proposition I-1 precisely

$$E\langle p_S, x_S\rangle - E\langle p_0, x_0\rangle. \tag{4.3}$$

Moreover, knowing that $\Phi_{l,L}$ and $\Phi_{m,M}$ are respectively equal to $+\infty$ out of R_1 and of R_2, we have the following.

PROPOSITION IV-1. *For any couple (x, p) in $R \times R$, we have*

$$\Phi_{l,L}(x) + \Phi_{m,M}(p) \geqslant 0. \tag{4.4}$$

In particular,

$$\inf\{\Phi_{l,L}(x), x \in R_1\} \geqslant -\inf\{\Phi_{m,M}(p): p \in R_2\}. \tag{4.5}$$

THEOREM IV-1. *If $\Phi_{l,L}$ or $\Phi_{m,M}$ are not identically $+\infty$, the following assertions are equivalent:*

(a) $$\inf_{x\in R} \Phi_{l,L}(x) = -\inf_{p\in R} \Phi_{m,M}(p), \tag{4.6}$$

(b) $$\inf_{x\in R} \Phi_{l,L}(x) = \liminf_{(y,b)\to(0,0)} \inf_{x\in R} \Phi_{l,L}^{y,b}(x), \tag{4.7}$$

(c) $$\inf_{p\in R} \Phi_{m,M}(p) = \liminf_{(y,b)\to(0,0)} \inf_{p\in R} \Phi_{m,M}^{y,b}(p). \tag{4.8}$$

Proof. Each of the stated relations is equivalent to the lower semicontinuity at $(0, 0)$ of $\varphi_{l,L}$ and $\varphi_{m,M}$. ∎

Remark. All the proofs concerning purely convex analysis results are formally the same as the ones in the purely deterministic case. We refer to [13] for more detailed proofs of these points.

DEFINITION IV-1. x in R_1 and p in R_2 will be said to be coextremal if

(a) $dP \otimes dt$ a.s.

$$(\dot{p}(\omega, t), p(\omega, t), H'(\omega, t)) \in \partial L(\omega, t, x(\omega, t), \dot{x}(\omega, t), H(\omega, t)), \tag{4.9}$$

(b) $p_0 \in \partial l_0(x_0), \qquad -p_S \in \partial l_S(x_S)$. ∎ (4.10)

The conditions of coextremality may be written

(a′) $dP \otimes dt$ a.s.

$$(\dot{x}(\omega, t), x(\omega, t), H(\omega, t)) \in \partial M(\omega, t, p(\omega, t), \dot{p}(\omega, t), H'(\omega, t)), \tag{4.9′}$$

(b′) $x_0 \in \partial m_0(p_0), \qquad -x_S \in \partial m_S(p_S)$. (4.10′)

The definition of coextremality is then symmetric with respect to the two control problems.

THEOREM IV-2. *The following assertions are equivalent:*

(a) *x and p are coextremal;*

(b) *x minimizes $\Phi_{l,L}$ on R, p minimizes $\Phi_{m,M}$ on R, and the equivalent conditions of Theorem* IV-1 *are satisfied;*

(c) $\Phi_{l,L}(x) = -\Phi_{m,M}(p)$.

Proof. The conditions of coextremality are equivalent to

$$\begin{aligned} &L(\omega, t, x(\omega, t), \dot{x}(\omega, t), H(\omega, t)) + M(\omega, t, p(\omega, t), \dot{p}(\omega, t), H'(\omega, t)) \\ &\quad - \langle x(\omega, t), \dot{p}(\omega, t)\rangle - \langle \dot{x}(\omega, t), p(\omega, t)\rangle - \langle H(\omega, t), H'(\omega, t)\rangle = 0, \\ &\quad dP \otimes dt \text{ a.s.} \end{aligned} \tag{4.11}$$

$$l(x_0, x_S) + m(p_0, p_S) - E\langle x_0, p_0\rangle + E\langle x_S, p_S\rangle = 0. \tag{4.12}$$

By taking (4.11) and (4.12), and using the same argument as for Proposition IV-1, we have

$$\Phi_{l,L}(x) + \Phi_{m,M}(p) = 0. \tag{4.13}$$

$\Phi_{l,L}$ and $\Phi_{m,M}$ taking nowhere the value $-\infty$, (4.13) proves that $\Phi_{l,L}(x)$ and $\Phi_{m,M}(p)$ are both finite, and Proposition IV-1 proves that x minimizes $\Phi_{l,L}$ on R, and p minimizes $\Phi_{m,M}$ on R. Moreover, condition (a) of Theorem IV-1 is satisfied.

Conversely if condition (b) is satisfied, by Proposition IV-1 and Theorem IV-1, we will have

$$\Phi_{l,L}(x) = -\Phi_{m,M}(p).$$

The same proof as the one used to prove Proposition IV-1 shows that (4.11) and (4.12) are satisfied. But (4.11) and (4.12) are precisely the coextremality conditions. ▮

EXAMPLE IV-1. We take again example II-1. For k in L_{21}, let p be the unique solution of

$$dp = (k - A^*p - B^*H)\,dt + H \cdot dw + dM, \qquad p_S = 0, \tag{4.14}$$

with (p_0, H, M) in $L_2{}^0 \times L_{22} \times W^{\perp}$; existence and uniqueness are generally nontrivial in the multidimensional case and are proved in [1].

The dual problem consists, then, in the minimization of

$$E \int_0^S L^*(\omega, t, k(\omega, t), (C^*p + D^*H)(\omega, t))\,dt. \tag{4.15}$$

The coextremality conditions can be written

$$(k(\omega, t), (C^*p + D^*H)(\omega, t)) \in \partial L(\omega, t, x(\omega, t), u(\omega, t))\ dP \otimes dt \text{ a.s.} \tag{4.16}$$

For an extensive analysis, especially of the linear-quadratic case with random coefficients, we refer to [1].

5. A Pontryagin-Type Principle for Ito Equations

We consider again Example II-3. In the same way as in [13], we are going to deduce a generalized Pontryagin principle for Ito equations. We assume that the convexity assumptions given in Example II-3 are satisfied.

We have here

$$W_1 = \{0\}, \qquad W_2 = W^{\perp}. \tag{5.1}$$

Let us then write that x in R_1 and p in R_2 are coextremal. We will have $dP \otimes dt$ a.s.

$$\begin{aligned}\langle x(\omega, t), \dot{x}(\omega, t)\rangle + \langle \dot{x}(\omega, t), p(\omega, t)\rangle + \langle H(\omega, t), H'(\omega, t)\rangle \\ - L(\omega, t, x(\omega, t), \dot{x}(\omega, t), H(\omega, t)) \\ = L^*(\omega, t, \dot{p}(\omega, t), p(\omega, t), H'(\omega, t)), \qquad p_T = 0.\end{aligned} \tag{5.2}$$

But

$$\begin{aligned}&L(\omega, t, x(\omega, t), \dot{x}(\omega, t), H(\omega, t)) \\ &\quad = \inf_{u \in U} K(\omega, t, x(\omega, t), u) \begin{cases} f(\omega, t, x(\omega, t), u) = \dot{x}(\omega, t), \\ \sigma(\omega, t, x(\omega, t), u) = H(\omega, t). \end{cases}\end{aligned} \tag{5.3}$$

Then

$$\begin{aligned}&L^*(\omega, t, \dot{p}(\omega, t), p(\omega, t), H'(\omega, t)) \\ &\quad = \sup_{x \in V} \sup_{(v,H) \in V \times V^m} \{\langle x, \dot{p}(\omega, t)\rangle + \langle v, p(\omega, t)\rangle \\ &\qquad + \langle H, H'(\omega, t)\rangle - L(\omega, t, x, v, H)\}.\end{aligned} \tag{5.4}$$

Equivalently,

$$\begin{aligned}&L^*(\omega, t, \dot{p}(\omega, t), p(\omega, t), H'(\omega, t)) \\ &\quad = \sup_{x \in V} \max_{u \in U} \langle x, \dot{p}(\omega, t)\rangle + \langle f(\omega, t, x, u), p(\omega, t)\rangle \\ &\qquad + \langle \sigma(\omega, t, x, u), H'(\omega, t)\rangle - K(\omega, t, x, u).\end{aligned} \tag{5.5}$$

By comparing with (5.2), we see there must exist $u(\omega, t)$ with values in U such that

u is $\mathscr{T}^*$-measurable.

$dP \otimes dt$ a.s.

$$\begin{aligned}\dot{x}(\omega, t) &= f(\omega, t, x(\omega, t), u(\omega, t)), \\ H(\omega, t) &= \sigma(\omega, t, x(\omega, t), u(\omega, t)),\end{aligned} \tag{5.6}$$

$dP \otimes dt$ a.s. the "sup max" in (5.5) is attained at $u(\omega, t)$.

The possibility of a $\mathscr{T}^*$-measurable choice of u follows from the $\mathscr{T}^*$-measurability of the set-valued function Δ' defined by

$$(\omega, t) \underset{\Delta'}{\to} \{u \in U: \langle f(\omega, t, x(\omega, t), u), p(\omega, t)\rangle + \langle \sigma(\omega, t, x(\omega, t), u'), H'(\omega, t)\rangle - K(\omega, t, x(\omega, t), u) = \varphi(\omega, t)\}. \tag{5.7}$$

$\varphi(\omega, t)$ being the maximum in u of the lefthand member of the equality defining $\Delta'(\omega, t)$. φ is then $\mathscr{T}^*$-measurable, because we can take the "maximum" on a countable dense subset of U. Δ' is $\mathscr{T}^*$-measurable (with nonempty values), because its graph is measurable. We then apply Theorem 2 of [11].

We will have, from (5.5)

$$\langle f(\omega, t, x(\omega, t), u(\omega, t)), p(\omega, t)\rangle + \langle \sigma(\omega, t, x(\omega, t), u(\omega, t)), H'(\omega, t)\rangle - K(\omega, t, x(\omega, t), u(\omega, t)) = \varphi(\omega, t), \tag{5.9}$$

$$\begin{aligned} &\langle x(\omega, t), \dot{p}(\omega, t)\rangle + \langle f(\omega, t, x(\omega, t), u(\omega, t)), p(\omega, t)\rangle \\ &\quad + \langle \sigma(\omega, t, x(\omega, t), u(\omega, t)), H'(\omega, t)\rangle - K(\omega, t, x(\omega, t), u(\omega, t)) \\ &= \max_{x \in V} \langle x, \dot{p}(\omega, t)\rangle + \langle f(\omega, t, x, u(\omega, t)), p(\omega, t)\rangle \\ &\quad + \langle \sigma(\omega, t, x, u(\omega, t), H'(\omega, t)\rangle - K(\omega, t, x, u(\omega, t)). \end{aligned} \tag{5.10}$$

If f, σ, K are differentiable in x, (5.10) implies

$$\begin{aligned} \dot{p}(\omega, t) = &- \left\langle \frac{\partial}{\partial x} f(\omega, t, x(\omega, t), u(\omega, t)), p \right\rangle \\ &- \left\langle \frac{\partial}{\partial x} \sigma(\omega, t, x(\omega, t), u(\omega, t)), H'(\omega, t) \right\rangle \\ &+ \frac{\partial}{\partial x} K(\omega, t, x(\omega, t), u(\omega, t)). \end{aligned} \tag{5.11}$$

From (5.2), (5.9), and (5.11), we deduce

THEOREM V-1. *Under the assumptions of Example* II-3, *let us suppose that* (f, σ, K) *are differentiable in* x. *Let* $\mathscr{H}$ *be the random functional be defined by*

$$\mathscr{H}(\omega, t, x, u, p, H') = \langle f(\omega, t, x, u), p\rangle + \langle \sigma(\omega, t, x, u), H'\rangle - K(\omega, t, x, u). \tag{5.12}$$

Then for x *in* R_1 *and* p *in* R_2 *to be coextremal, it is necessary that one can find* u $\mathscr{T}^*$*-measurable with values in* U, H' *in* L_{22} *and* M *in* $W^{\perp}$ *such that*

$$\begin{aligned} dx &= f(\omega, t, x(\omega, t), u(\omega, t))\, dt + \sigma(\omega, t, x(\omega, t), u(\omega, t)) \cdot dw, \\ x(0) &= x_0, \end{aligned} \tag{5.13}$$

$$\begin{aligned} dp &= -\frac{\partial \mathscr{H}}{\partial x}\, dt + H' \cdot dw + dM, \\ p_T &= 0, \end{aligned} \tag{5.14}$$

$dP \otimes dt$ a.s.

$$\begin{aligned}&\mathscr{H}(\omega, t, x(\omega, t), u(\omega, t), p(\omega, t), H'(\omega, t))\\&\quad = \max_{u \in U} \mathscr{H}(\omega, t, x(\omega, t), u, p(\omega, t), H'(\omega, t)). \quad \blacksquare\end{aligned} \tag{5.15}$$

This result allows us to make very clearly the connection between deterministic optimization and stochastic optimization. Moreover, all the various necessary and sufficient conditions for optimality of a given control derive from these conditions. For various applications of this principle, especially to the linear quadratic case with random coefficients, we refer to [1]. It is a remarkable feature of the dual problem that the duat state variable p can have jumps, corresponding to the jumps of M.

Relation between the Stochastic Pontryagin Principle and the Dynamic Programming Equation

We assume here that (f, σ, K) do not depend on ω, and that $\{\mathscr{F}_t\}_{t \in R^+}$ is the family of σ-algebras generated by w. Moreover, to simplify the calculations, we suppose that $n = 1$.

If we assume (without having any justification other than a purely intuitive approach) that $p(\omega, t) = p(t, x(\omega, t))$, p being a sufficiently smooth function of (t, x), then by the Ito formula, Theorem V-1 can be written, knowing that here $W^\perp = \{0\}$.

$$\begin{gathered}\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x} f + \frac{1}{2} \frac{\partial^2 p}{\partial x^2} | \sigma |^2 = - \frac{\partial \mathscr{H}}{\partial x} \cdot dP \otimes dt \text{ a.s.},\\ \frac{\partial p}{\partial x} \sigma = H'.\end{gathered} \tag{5.16}$$

Without justification, we can write that, at least formally, p is a solution of the partial differential equation

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x} f + \frac{1}{2} \frac{\partial^2 p}{\partial x^2} | \sigma |^2 = - \frac{\partial f}{\partial x} p - \frac{\partial \sigma}{\partial x} \sigma \frac{\partial p}{\partial x} + \frac{\partial K}{\partial x}. \tag{5.17}$$

Moreover, we maximize in u the expression

$$f(t, x, u)\, p + \sigma(t, x, u)\, H' - K(t, x, u) \tag{5.18}$$

to get an optimal control u_0.

If we write (formally) the derivative of (5.18), we get

$$\frac{\partial f}{\partial u} p + \frac{\partial \sigma}{\partial u} H' - \frac{\partial K}{\partial u} = 0. \tag{5.19}$$

But H' is equal to $(\partial p/\partial x)\,\sigma$. (5.19) is then a "formal" condition for $\mathscr{H}_0(t, x, u)$ defined by

$$\mathscr{H}_0\left(t, x, p, \frac{\partial p}{\partial x}\right) = f(t, x, u)\, p + \frac{1}{2} \mid \sigma \mid^2 (t, x, u) \frac{\partial p}{\partial x} - K(t, x, u), \qquad (5.20)$$

to be extremum at u_0.

If we consider now the dynamic programming equation (see, for instance [6, p. 105])

$$\frac{\partial V}{\partial t} = - \max_{u \in U} \left(f(t, x, u) \frac{\partial V}{\partial x} + \frac{1}{2} \mid \sigma \mid^2 (t, x, u) \frac{\partial^2 V}{\partial x^2} - K(t, x, u) \right), \quad (5.21)$$

we see that by comparing the right sides of (5.12) and (5.20), by writing formally

$$\frac{\partial V}{\partial x} = p, \qquad (5.22)$$

then (5.17) is the derivative in x of (5.21). ∎

These calculations are purely formal. In particular, we must remember that in the general duality approach, we optimize with respect to the controls, nonanticipating with respect to w, whereas in the partial differential equation approach, we optimize with pure Markov controls. More generally, it is in most cases not true that x generates the same σ-fields as w; this prevents us in general from identifying the two problems. For a general case in which the actual solution is Markov, we refer to [2]. For existence results in the general case, we refer to [1].

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