Copula Simulation in Portfolio Allocation Decisions

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Outline

- Introduction
- Theory
- Simulation Study:
  Data and Methodology
- Results
- Concluding Remarks
- References

Introduction

The **objective of the research** is twofold:

- simulate the risk (CVaR) of various two-asset portfolios
- compare the estimation capabilities of different models.

Significance of the topic: Basel III -

the proposed fundamental review of the trading book

Theory

Modeling **portfolio risk** requires

- selection of an "appropriate" ***measure of risk***
- modeling ***statistical dependence*** (co-movement) between random variables, i.e. return or loss components of the portfolio.

Traditional portfolio theory (Markowitz, 1952) assumes that **risk factors** are **normally** (elliptically) **distributed**.

In case of **real-world portfolios** two difficult problems arise (Dowd, 2005)

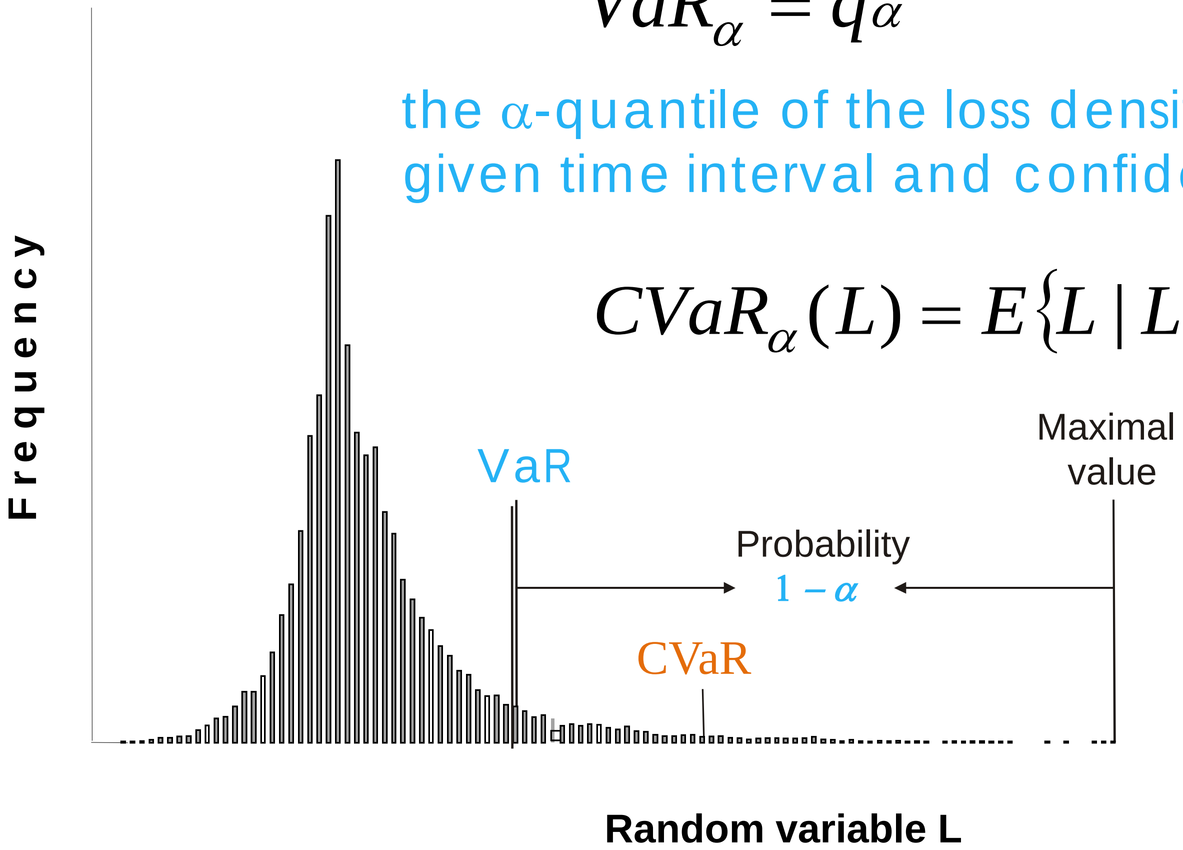
- widely used risk measures such as the ***variance*** (standard deviation) or ***VaR become unreliable***.
- ***correlation-based*** (variance-covariance) ***approaches fail*** in modelling statistical dependence.

Theory

$$VaR_\alpha = q\alpha$$

the α-quantile of the loss density function on a given time interval and confidence level (α)

$$CVaR_\alpha(L) = E\{L \mid L \geq VaR_\alpha(L)\}$$



Theory

The advantages of CVaR over VaR

- accounts for risk beyond VaR (more conservative than VaR)
- coherent in the sense of Artzner et al. (1999)
- continuous with respect to the confidence level α (VaR may be discontinuous in α)
- convex with respect to control variables
- easy to control/optimise for non-normal distributions (LP)
- consistent with mean-variance approach: for normal loss distributions optimal variance and CVaR portfolios coincide

(Rockafellar-Uryasev, 2002)

Theory

A possible solution for the second problem is using copula.

A **copula** is a function that **connects together the marginal distributions** to form the multivariate distribution:

$$H(x,y) = C(F(x),G(y))$$

where $H(x,y)$ is the joint distribution function,

$F(x)$ and $G(y)$ are the marginal distributions.

C(.) copula function above can be interpreted as a dependence structure of the random variables *X* and Y.

If the marginals are continuous, then the copula is unique (Sklar, 1959).

Theory

The advantages of copulas over correlation

- universally valid (can be applied to non-elliptical as well as elliptical distributions)
- allow one to work at two separate levels: at the first level we focus on the marginal distributions, then we consider the dependence structure (i.e. the copula)to be fitted to the marginal distributions
- make possible to fit quite different marginal distributions to each risk factor
- give a much greater range of choice over the dependence structure to be fitted to the data

(Dowd, 2005)

Simulation Study

- Two-asset portfolios have been simulated by selecting different pairs of company shares from the FTSE 100 constituents.
- Daily closing prices have been utilized for the period of 4 January 1999 – 31 December 2014.
- Real time series of returns were used to estimate input parameters , i.e. expected returns, standard deviations of returns and correlation coefficients as well as parameters of GPD marginals and those of the copulas.
- CVaR was applied as a risk measure.
- Dependence structure was modeled by using Clayton and Gumbel copulas assuming both normal and GPD marginals.
- For benchmarking the traditional portfolio selection was also applied, i.e. the simulation was also carried out with Gaussian copula and normal marginals.

Simulation Study

Simulation process

1. Generate two independent random variables from a standard uniform distribution: v_1, v_2
2. Set

$$u_1 = v_1 \qquad (i)$$

$$C(u_2 | u_1) = \frac{\partial C(u_1, u_2)}{\partial u_1} = v_2 \qquad (ii)$$

3. Solve *(ii)* for u_2
4. Use the classical inversion method to obtain random values for the intended marginals.

$$F^{-1}(u_i) = r_i \quad i = 1,2.$$

Simulation Study

Simulation process (cont.)

5. Obtain a simulated portfolio return

$$R = wr_1 + (1 - w)r_2$$

6. Repeat the steps m times (in our case $m = 10000$).
7. Determine *CVaR* based on the simulated return distribution.
8. Set a different portfolio weight (w) and determine CVaR of the portfolio (for each model 100 different portfolio allocations were considered).

(Bouyé et al., 2000; Dowd, 2005)

Results – Goodness of Fit Tests

- The hypothesis of **normally distributed** marginals can be **rejected**.
- The **Clayton and Gumbel copulas** seem to **fit better to yearly data** than to longer periods. However, Gumbel copula usually gives better results than Clayton.
- For the **GPD** one needs **5-year time span** to get a better fit (for yearly data the hypothesis of GPD marginals can be rejected).

Backtesting

Two research designs

1. Both the marginals and the different copula models are estimated based on an observation period of a year (250-250 data points).
2. For the marginals 5-year and for copula models 1-year observation period has been applied (1250 -250 data).

Performance evaluation

- The empirical CVaR has been calculated as the average of the 10 highest realized losses (with an implied 96% confidence level in the simulations).
- Both the direction of estimation (i.e. under (-), over (+)) and extent of the difference from the empirical CVaR has been considered.
- The performance of a model is regarded as „better” in case of overestimating CVaR and having a „closer” estimated CVaR to the empirical one.

Results - Backtesting

Proportion of cases* the model in row *i* performed better than the one in column *j*.

Pairwise comp.	Clayton-GPD	Gumbel-GPD	Clayton-normal	Gumbel-normal	Gaussian-normal	Ranking
Clayton-GPD		15%	16%	12%	14%	5
Gumbel-GPD	85%		21%	17%	18%	4
Clayton-normal	84%	79%		10%	15%	3
Gumbel-normal	88%	83%	90%		75%	1
Gaussian-normal	86%	82%	85%	25%		2

Notes:

* The number of all cases is 3939.

- The CVaR simulations were based on portfolios of 3 randomly chosen equity-couples (100 different portfolio weights were considered).
- The length of the estimation (observation) period was one year.
- A sliding window of one-year has been applied (13 periods can be compared for each equity couples and portfolio weight).

Results - Backtesting

Proportion of cases* the model in row *i* performed better than the one in column *j*.

Pairwise comp.	Clayton-GPD	Gumbel-GPD	Clayton-normal	Gumbel-normal	Gaussian-normal	Ranking
Clayton-GPD		13%	6%	4%	4%	5
Gumbel-GPD	87%		11%	5%	6%	4
Clayton-normal	94%	89%		6%	12%	3
Gumbel-normal	96%	95%	94%		55%	1
Gaussian-normal	96%	94%	88%	45%		2

Notes:

* The number of all cases is 2727.

- The CVaR simulations were based on portfolios of 3 randomly chosen equity-couples (100 different portfolio weights were considered).
- For the marginals 5-year and for copula models 1-year observation period has been applied.
- A sliding window of one-year has been applied (9 periods can be compared for each equity couples and portfolio weight).

Results - Backtesting

Proportion of cases* the model in row *i* performed better than the one in column *j*.

Pairwise comp.	Clayton-GPD	Gumbel-GPD	Clayton-normal	Gumbel-normal	Gaussian-normal	Ranking
Clayton-GPD		20%	13%	10%	12%	5
Gumbel-GPD	80%		17%	15%	15%	4
Clayton-normal	87%	83%		12%	15%	3
Gumbel-normal	90%	85%	88%		73%	1
Gaussian-normal	88%	85%	85%	27%		2

Notes:

* The number of all cases is 6565.

- The CVaR simulations were based on portfolios of 5 randomly chosen equity-couples (100 different portfolio weights were considered).
- The length of the estimation (observation) period was one year.
- A sliding window of one-year has been applied (13 periods can be compared for each equity couples and portfolio weight).

Results - Backtesting

Proportion of cases* the model in row *i* performed better than the one in column *j*.

Pairwise comp.	Clayton-GPD	Gumbel-GPD	Clayton-normal	Gumbel-normal	Gaussian-normal	Ranking
Clayton-GPD		20%	14%	12%	12%	5
Gumbel-GPD	80%		19%	14%	15%	4
Clayton-normal	86%	81%		7%	13%	3
Gumbel-normal	88%	86%	93%		60%	1
Gaussian-normal	88%	85%	87%	40%		2

Notes:

* The number of all cases is 4545.

- The CVaR simulations were based on portfolios of 5 randomly chosen equity-couples (100 different portfolio weights were considered).
- For the marginals 5-year and for copula models 1-year observation period has been applied.
- A sliding window of one-year has been applied (9 periods can be compared for each equity couples and portfolio weight).

Concluding Remarks

- Among the five models considered the **Gumbel copula coupled with normal marginals** had the **best performance** in all simulation modules carried out until now.
- However, it is somehow surprising that the **Gaussian copula with normal marginals** came as the **second best**.
- The research seems to confirm the idea of **supporting the use of CVaR** instead of traditional risk measures (such as the VaR as well as the variance / standard deviation) even if some assumptions of the traditional portfolio selection apply.

References

Artzner, P. – Delbaen, F. – Eber, J. M. – Heath, D. (1999): Coherent Measures of Risk, *Mathematical Finance*, Vol. 9, 203-228.

Bouyé, E. – Durrleman, V. – Nikeghbali, A. –Riboulet, G. Roncalli, T. (2000): Copulas for Finance - A Reading Guide and Some Applications , Groupe de Recherce Opérationelle, Crédit Lyonnais, Paris (http:66ssrn.com/abstract=1032533)

Dowd, K. (2005): Copulas and Coherence – Portfolio Analysis in a Non-Normal World, *Journal of Portfolio Management*, Fall 2005, 123-127.

Markowitz, H. M. (1952): Portfolio Selection (1952), *Journal of Finance*, Vol. 8, 77-91.

Rockafellar, R. T. – Uryasev, S. (2002): Optimization of Conditional Value-at-Risk, Journal of Risk, Vol. 3 (3), 21-41.

Sklar (1959):Fonctions de répartition á n dimensions et leurs marges, *Publ. Inst.Statist., Univ. Paris*, Vol. 8, 229-231.

http://CRAN.R-project.org