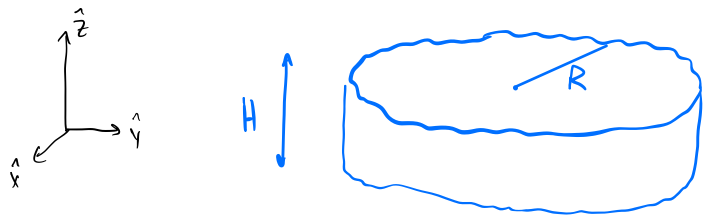


$$\rho = \frac{M}{V} = \frac{m}{\pi R^2 H}$$

$$x = s\cos\phi$$
$$y = s\sin\phi \qquad d\tau = s\,ds\,d\phi\,dz$$
$$z = z$$

$$I_{xx} = \rho\int_0^{2\pi}\int_{-H/2}^{H/2}\int_0^R \left[s^3\cos^2\phi + s^3\sin^2\phi + sz^2 - s^3\cos^2\phi\right] ds\,dz\,d\phi$$

$$= \rho\int_0^{2\pi}\int_{-H/2}^{H/2}\left[\frac{1}{4}R^4\sin^2\phi + \frac{1}{2}R^2 z^2\right] dz\,d\phi$$

$$= \rho\int_0^{2\pi}\left[\frac{1}{4}R^4\sin^2\phi\left(\frac{H}{2}\right) + \frac{1}{6}R^2\left(\frac{H}{2}\right)^3 - \frac{1}{4}R^4\sin^2\phi\left(\frac{-H}{2}\right) - \frac{1}{6}R^2\left(\frac{-H}{2}\right)^3\right] d\phi$$

$$= \rho\int_0^{2\pi}\left[\frac{1}{4}HR^4\sin^2\phi + \frac{1}{24}H^3R^2\right] d\phi \qquad \int_0^{2\pi}\sin^2\phi = \pi$$

$$= \rho\left[\frac{\pi H R^4}{4} + \frac{\pi R^2 H^3}{12}\right] \qquad \rho = \frac{m}{\pi R^2 H}$$

$$= m\left[\frac{R^2}{4} + \frac{H^2}{12}\right]$$

$$I_{xx} = \frac{1}{12}m\left[3R^2 + H^2\right]$$

Because of symmetry in $\hat{x}$ and $\hat{y}$, $I_{xx} = I_{yy}$

$$\hookrightarrow I_{yy} = I_{xx} = \frac{1}{12}m\left[3R^2 + H^2\right]$$

$$I_{zz} = \rho \int_0^{2\pi} \int_{-H/2}^{H/2} \int_0^R s\left[s^2\cos^2\phi + s^2\sin^2\phi + z^2 - z^2\right] ds\, dz\, d\phi$$

$$= \rho \int_0^{2\pi} \int_{-H/2}^{H/2} \int_0^R s^3 \, ds\, dz\, d\phi$$

$$= 2\pi H \rho \int_0^R s^3 \, ds$$

$$= 2\pi H \rho \left(\tfrac{1}{4}R^4\right) \qquad \rho = \frac{m}{\pi R^2 H}$$

$$I_{zz} = \frac{1}{2} m R^2$$

$$I_{xy} = I_{yx} = \rho \int_0^{2\pi} \int_{-H/2}^{H/2} \int_0^R s(-s\cos\phi \; s\sin\phi)\, ds\, dz\, d\phi$$

$$= -\rho \int_0^{2\pi} \int_{-H/2}^{H/2} \int_0^R \left(s^3 \sin\phi \cos\phi\right) ds\, dz\, d\phi$$

$\int_0^{2\pi} \sin\phi \cos\phi \, d\phi$ integrates over 2 periods exactly, so it $= 0$

$$I_{xy} = I_{yx} = 0$$

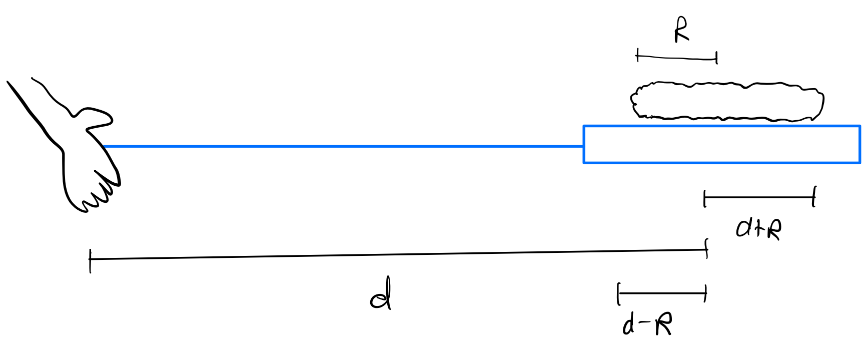
$$I_{xz} = I_{zx} = -\rho \int_0^{2\pi} \int_{-H/2}^{H/2} \int_0^R s(sz\cos\phi)\, ds\, dz\, d\phi \qquad \int_0^{2\pi} \cos\phi \, d\phi = 0$$

$$I_{xz} = I_{zx} = 0$$

$$I_{yz} = I_{zy} = -\rho \int_0^{2\pi} \int_{-H/2}^{H/2} \int_0^R s(sz\sin\phi)\, ds\, dz\, d\phi \qquad \int_0^{2\pi} \sin\phi \, d\phi = 0$$

$$\boxed{I_{yz} = I_{zy} = 0}$$

$$\therefore \; \textcircled{1} \quad \boxed{I = \begin{bmatrix} \frac{1}{12}m(3R^2+H^2) & 0 & 0 \\ 0 & \frac{1}{2}m(3R^2+H^2) & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{bmatrix}}$$



2 Points for torque: $d+R$, $d-R$

$$\tau = \tau_2 - \tau_1 = r_2 \times F - r_1 \times F = r_2 F - r_1 F$$

$$= (d+R)F - (d-R)F$$

② $\boxed{\tau = 2RF}$

Euler's Rigid-Body Equation for $\hat{x}$ (only rotating around $\hat{x}$)

$$I_{xx}\dot{\omega}_x - \cancel{(I_{yy} - I_{zz})}\omega_2\omega_3 = N_1$$

0

$$I_{xx}\dot{\omega}_x = N_1 = 2RF$$

$$\dot{\omega}_x = \frac{2RF}{I_{xx}}$$

$$\omega_x = \frac{2R}{I}\int F \cdot dt$$

$$\boxed{\omega_x = \frac{2R}{I} m V_0}$$

where $|V_0|$ is the initial velocity at $d+R$ or $d-R$

$$\omega_x = \dot{\theta}$$

$$\boxed{\theta(t) = \frac{2R}{I} m V_0 t}$$

Now for translational motion:

$$\ddot{z} = -g$$

$$\dot{z} = -gt + \dot{z}_0$$

$$z = -\frac{1}{2}gt^2 + \dot{z}_0 t + z_0$$

$z_0 = 0$ (initially on flipper)

$$z = -\frac{1}{2}gt^2 + \dot{z}_0 t$$

Find $\dot{z}_0$ with conservation of energy:

$$\frac{1}{2}m\dot{z}_0^2 = mgh$$

$$\dot{z}_0 = \sqrt{2gh}$$

$$\boxed{z(t) = -\frac{1}{2}gt^2 + \sqrt{2gh}\, t}$$

$$T = \frac{(2n+1)\pi}{\omega} = \frac{(2n+1)\pi I_{xx}}{2mRV_0}$$

Plug into $z(t)$ to find h

$$z(T) = 0 = -\frac{1}{2}gT^2 + \sqrt{2gh}\ T$$

$$\frac{1}{2}gT = \sqrt{2gh}$$

$$\frac{1}{2}g\frac{(2n+1)\pi I_{xx}}{2mRV_0} = \sqrt{2gh}$$

$$V_0 = \frac{(2n+1)I_{xx}\pi}{4mR}\sqrt{\frac{g}{2h}}$$

$$L = I_{xx}\omega_x = I_{xx}\frac{2mRV_0}{I_{xx}} = 2mRV_0$$

So now we have equations of motion that only depend on our max height h, along with how many times we want it to flip n.

- $z(t) = -\frac{1}{2}gt^2 + \sqrt{2gh}\,t$

where $I_{xx} = \frac{1}{12}m(3R^2 + H^2)$

- $\theta(t) = \omega t = \frac{2mRV_0}{I_{xx}}t$

and $n = 0, 1, 2 \ldots$

- $V_0 = \frac{(2n+1) I_{xx} \pi}{4mR}\sqrt{\frac{g}{2h}}$

- $L = 2mRV_0$

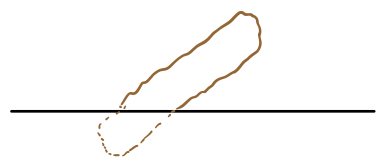
Ratio of h and n needed so burger does not hit ground before flipping completely:

$$h = \frac{1}{32} g \left[\frac{(2n+1)\pi I_{xx}}{mRV_0} \right]^2 \qquad z(t) = -\frac{1}{2} g t^2 + \sqrt{2gh}\, t$$

$$z(t) = -\frac{1}{2} g t^2 + \left[2g \frac{1}{32} g \left(\frac{(2n+1)\pi I_{xx}}{mRV_0} \right)^2 \right]^{1/2} t$$

$$= -\frac{1}{2} g t^2 + \left[\frac{g^2}{16} \left(\frac{(2n+1)\pi I_{xx}}{mRV_0} \right)^2 \right]^{1/2} t$$

$$= -\frac{1}{2} g t^2 + \frac{1}{4} g \frac{(2n+1)\pi I_{xx}}{mRV_0} t$$

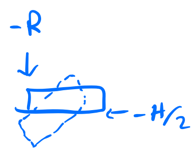


Our burger must be above $z=0$ at all points. We need to make sure the areas around the CM do not fall below 0.

But remember our CM is the center of the frame, so really no points should fall below $-H/2$

$$\vec{\lambda} = [x,\ y\cos\theta - z\sin\theta,\ y\sin\theta + z\cos\theta]$$

$\hookrightarrow \lambda_2 = y\sin\theta + z\cos\theta + z(t) > -\frac{H}{2}$ $\quad y = -R \rightarrow$



$-R\sin(\theta(t)) - \frac{H}{2}\cos(\theta(t)) + z(t) > -\frac{H}{2}$ $\quad z = -\frac{H}{2}$

$$-R\sin(\theta(t)) - \frac{H}{2}\cos(\theta(t)) - \frac{1}{2} g t^2 + \frac{1}{4} g \frac{(2n+1)\pi I_{xx}}{mRV_0} t > -\frac{H}{2}$$

Small θ, Since this would only happen in small time (near initial kick)

$$\sin\theta \simeq \theta(t), \quad \cos\theta \simeq 1 - \frac{\theta(t)^2}{2}$$

$$-R\theta(t) - \frac{H}{2}\left(1 - \frac{\theta(t)^2}{2}\right) - \frac{1}{2}gt^2 + \frac{1}{4}g\frac{(2n+1)\,n\,I_{xx}}{mRV_0}t > -\frac{H}{2}$$

$$-R\,\theta(t) + \frac{H}{4}\theta(t)^2 - \frac{1}{2}gt^2 + \frac{1}{4}g\frac{(2n+1)\,n\,I_{xx}}{mRV_0}t > 0$$

$$-R\frac{2mRV_0}{I_{xx}}t + \cancel{\frac{H}{4}\frac{4m^2R^2V_0^2}{I_{xx}^2}t^2} \cancel{-\frac{1}{2}gt^2} + \frac{1}{4}g\frac{(2n+1)\,n\,I_{xx}}{mRV_0}t > 0$$

Small t, ignore 2nd order terms

$$\frac{2mR^2V_0}{I_{xx}}t < \frac{1}{4}g\frac{(2n+1)\,n\,I_{xx}}{mRV_0}t$$

$$\frac{2n+1}{V_0^2} > \frac{8m^2R^3}{I^2gn} \qquad V_0^2 = \frac{(2n+1)^2 I_{xx}^2 n^2}{16m^2R^2}\frac{g}{2h}$$

$$\frac{32m^2R^2h}{(2n+1)\,I_{xx}^2n^2g} > \frac{8m^2R^3}{I_gn}$$

$$\boxed{\frac{h}{2n+1} > \frac{1}{4}nR} \longleftarrow$$

Condition for a perfect result

(edge will not hit the flipper)