The Phase transition of dimer Eye Model

Tristan Wang

November 1, 2024

1 Introduction to the Model

I named this model as dimer Eye Model because the simplification of its structure looks like an eye.

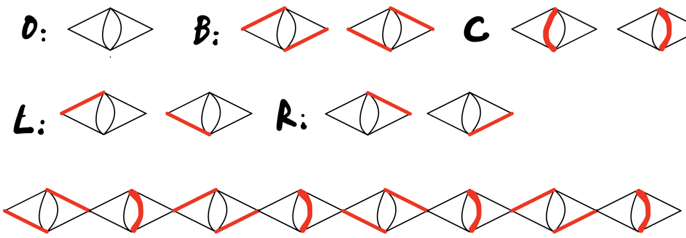


Figure 1: Eye Model (Eye version)

One can also draw it as a bowknot.

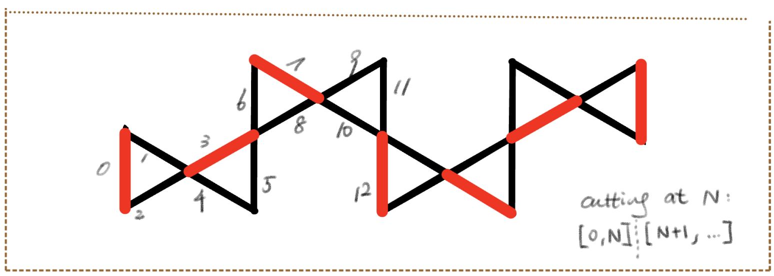


Figure 2: Eye Model (bowknot version)

I may refer to both of the versions in the following discussion, out of convenience.

Our PXP Hamitonian is:

$$H = \Omega \sum_i X_i + \Delta \sum_i Z_i + V \sum_{\langle i,j \rangle} \hat{n}_i \hat{n}_j$$

2 Observation of Entanglement Entropy

2.1 Novel behavior of the Entanglement Entropy

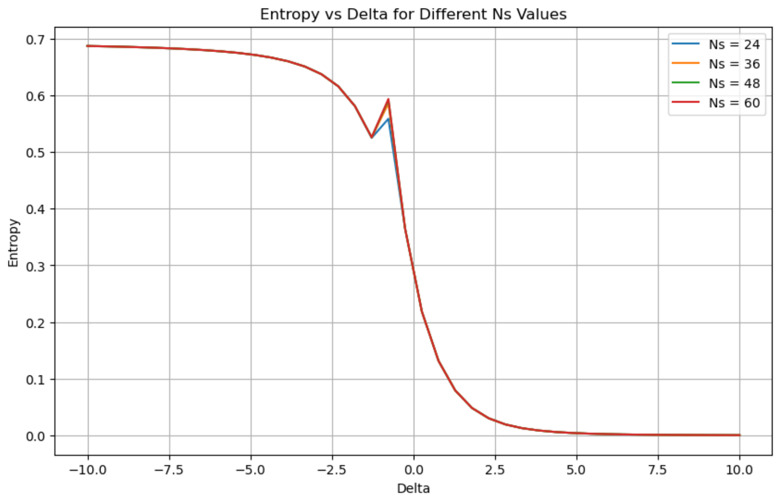
First let me define some parameters:

- N_s: the number of links in the system (That is the size of our spin model)
- $N_p = 6$: Bond dimension
- $\Omega = 1.0$: The Rabi frequency
- Δ: The detuning

I chose **Ns = {24,36,48,60}**,cutting at N = 12 (for some reason that I will elaborate later) , and the plot of the entanglement entropy is shown below. The entanglement is measured at the middle point of the system.

It can be seen from the plot that the entanglement entropy goes up, and then down, and then up again. Finally it is saturated at a certain value, which is close to $ln2$.

It's really strange because one may expect that the curve to go up and then down, finally converge at $ln2$. At the point where $\Delta = 10$, the ground state of the system is a product state, with all the spin down (equivalently all the Rydberg atoms staying at their ground state). As Δ increases, there are some

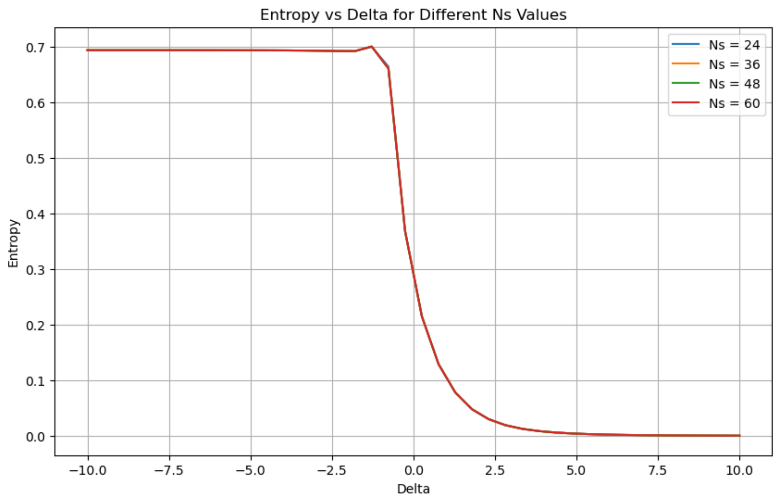


excitations in the system, and they behave like a gas of excited states. At the point where $\Delta = -10$, the ground state of the system should be very close to a dimer model, with only local entanglement. Thus the (maximum) entanglement is expected to be $ln2$. When Δ is smaller that -10, the entanglement should be larger than $ln2$, because of the appearance of defects, which can act as a domain wall and move like gas.

What I want to argue is that the argument of 'Domain Wall Gas is actually wrong'. And the rest this article is to give an explanation of the entropy behavior.

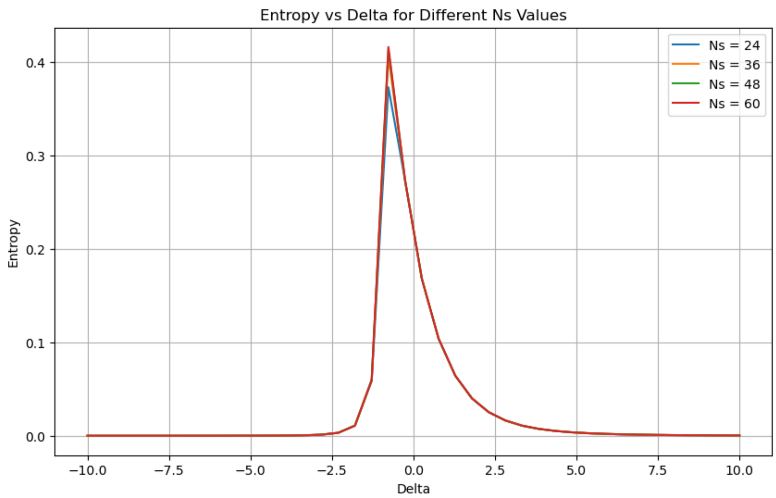
2.2 More about the Entanglement Entropy

We can then see two more examples of the entanglement entropy, with different cutting points. Here is the plot of cutting those systems at $N = 6$.



There is almost no difference between the cases of different system sizes, which is good. We do expect the entanglement entropy to be somewhat local.

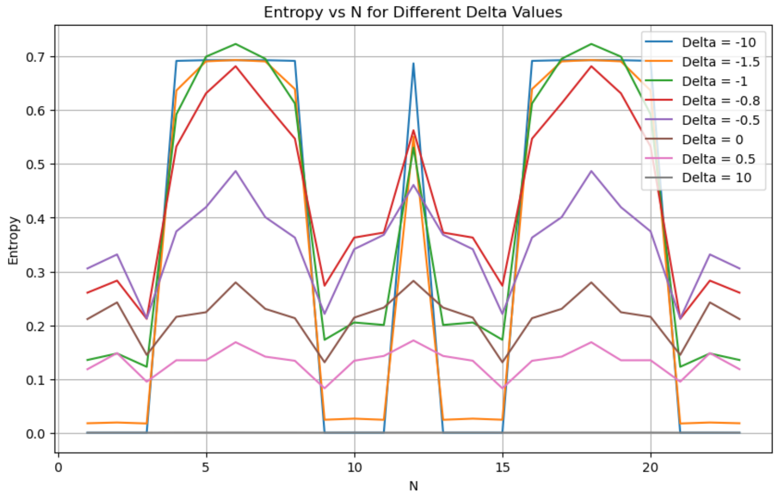
And here is the plot of cutting those systems at $N = 10$.



A trivial conclusion can be that: The entanglement entropy is a function of the cutting point N (and that of the order parameter Δ of course).

It's good to fix our system size (Here I chose $N_s = 24$) and see how the entanglement entropy behaves with different cutting points.

Here is what is happening:



The phenomenon can be concluded as:

- At the region where we expect the system to be a 'gas' of excited states, the entanglement entropy increases at every site. There is a difference between the sites, but the difference is not very large.
- At the region where we expect the system to be a 'dimer' model, the entanglement entropy is very local. At some places the entanglement entropy is zero, and at some places the entanglement entropy is close to $ln2$.

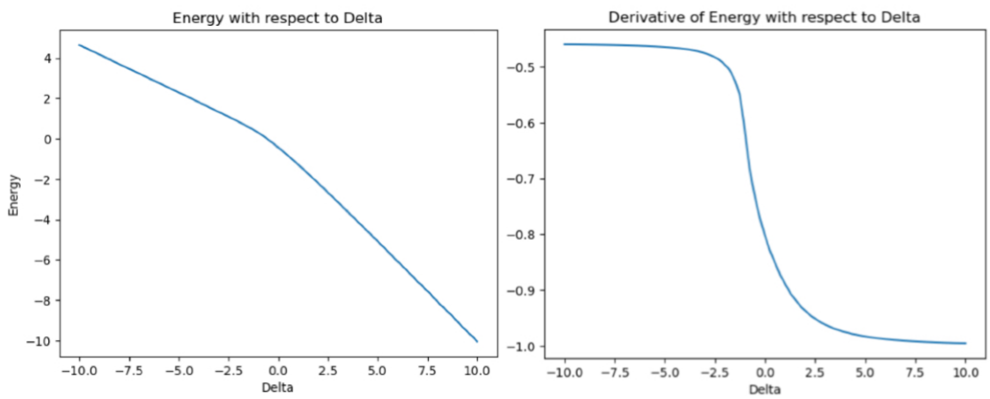
We will stop our discussion of entropy here temporarily, and move on to the next section. We will go back to talk about the structure of the entanglement entropy later.

3 Other parameters helping us to understand the strange curve at N = 12

3.1 The Energy of the System

If entropy behaves like the plot of $N = 12$, the system should have gone through a first order phase transition because of the 'discontinuity'(the range where the entropy sharply goes down).

Here is the plot of the energy per site of the system ($Ns = 48$). Also here is it's derivative with respect to Δ.



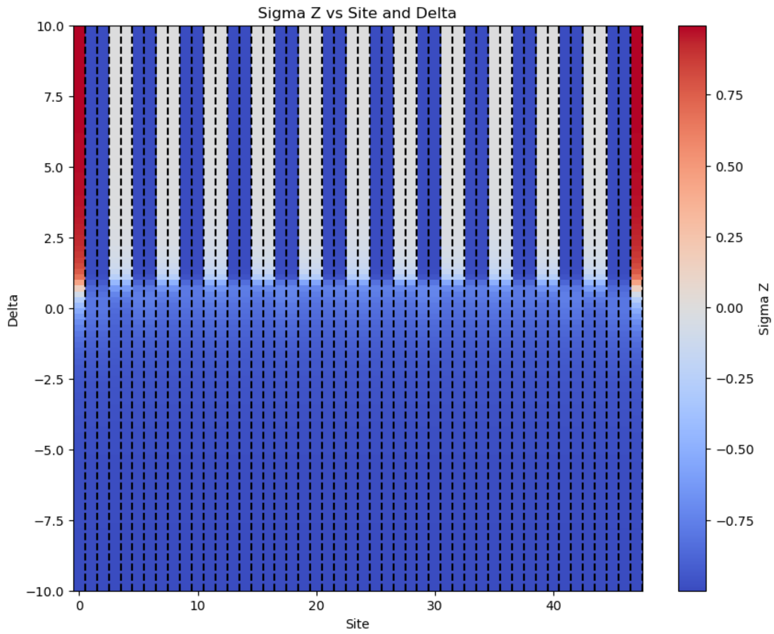
The derivative of the energy seems to be the shape of a ladder. Correspondingly, there is a discontinuity in the curve of energy. That is an indication of a first order phase transition. However, the ladder like curve of derivative is also smoother than a real step function.

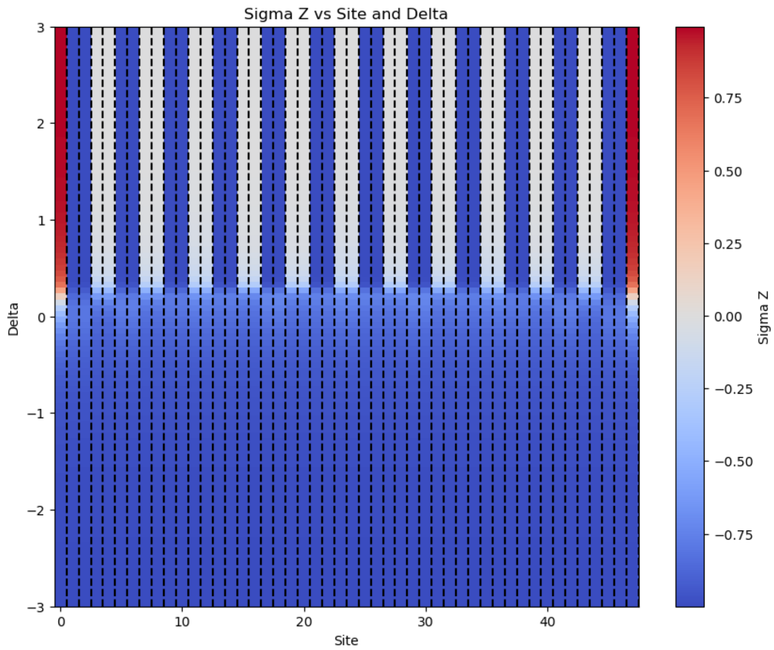
At some cutting point the entanglement entropy behaves like a first order phase transition, and somewhere it behaves like a second order phase transition. It corresponds to the middle-range behavior of the energy.

3.2 The expectation of σ_z^i

Here is the plot of the expectation of σ_z^i of the system ($Ns = 48$),as a function of its position i and the detuning Δ. I have drawn them between $\Delta = 10$ and $\Delta = -10$, and there is also a version between $\Delta = 3$ and $\Delta = -3$.

(There is actually a mistake of there signs, but it's not vert important.)





Things can be seen:

- At the boundary of the system, and after Δ is smaller that 0, the expectation of σ_z^i is very close to 1.0, fixed. This breaking of symmetry is expected, since during our VMPS Algorithm, we are always optimizing the site on the boundary first, so there will be a spin up here definitely.
- As Δ is approaching the phase transition point(a number slightly less than 0), the expectation of σ_z^i appears a uniform growth at each site.
- After the phase transition point, the expectation of σ_z^i of each site goes two different ways. Some go to 0 and others go to -1. And there is a spatial periodicity.

To the limit of $\Delta = -10$, it's easy to understand the spatial periodicity. As mentioned, the boundary of the system is fixed to be a spin up(excited state). So there can be a possible excitation (of 50% possibility) every 2 sites.

See the plot below, which illustrates the above argument.

One thing that I noticed is that all the sites are quickly assigned to there appropriate values(0 or -1).

We are more and more close to the truth! Naturally we will plot the expectation of σ_z^i with respect to Δ, as well as its variance. (We don't consider the two spins on the boundary, because they are fixed to

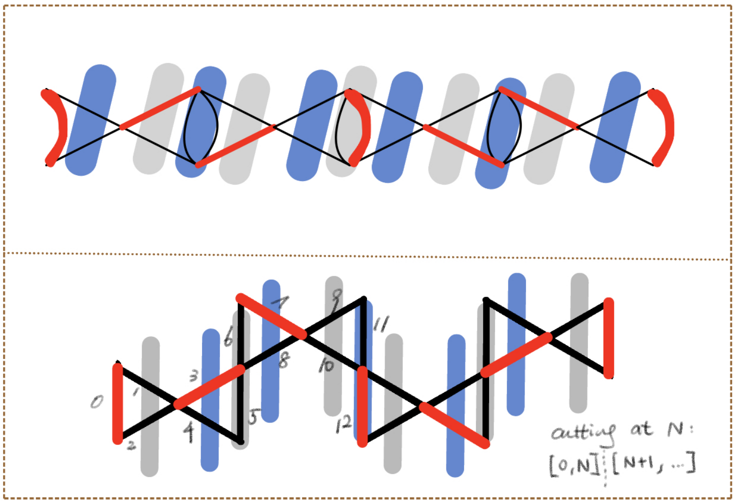


Figure 3: Periodic behavior of σ_z^i

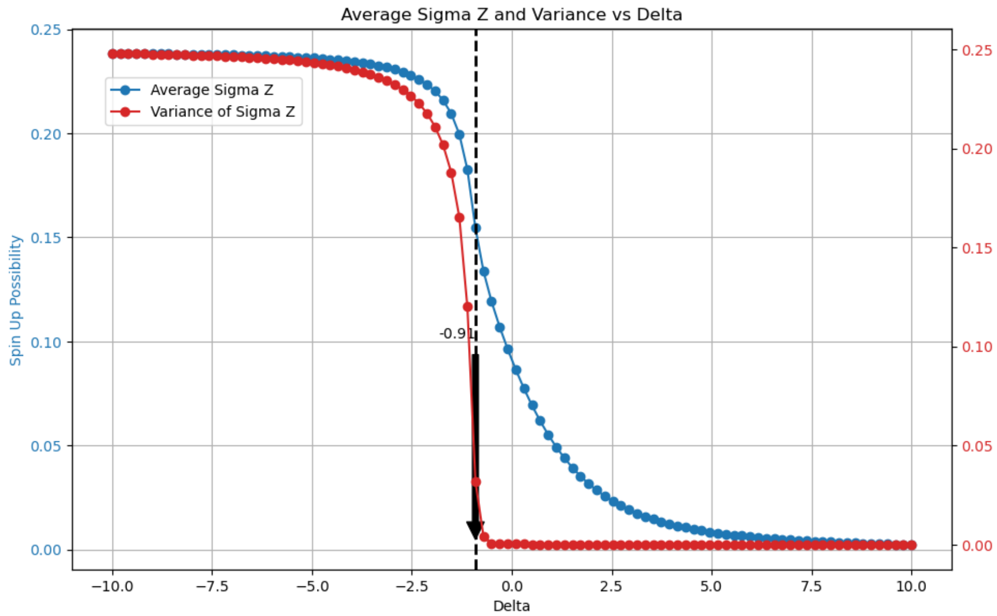


Figure 4: $\dfrac{\langle \sigma^z \rangle + 1}{2}, \quad \mathrm{Var}(\sigma^z)$ with respect to Δ

be spin up.) It can be seen that the variance of σ_z^i experiences a sharp increase at the around $\Delta = -0.91$, soaring from 0 to some number around 0.1. Then it quickly goes beyond 0.2 and gradually get to its final value.

The theoretical value is (The number of blue and grey (-1 and 0) sites are not equal, so there's a small deviation from 0.25):

$$\frac{\langle \sigma^z \rangle + 1}{2} = 0.2391$$

$$\mathrm{Var}(\sigma^z) = 0.2495$$

The value at $\Delta = -10$ is:

$$\frac{\langle \sigma^z \rangle + 1}{2} = 0.2383$$

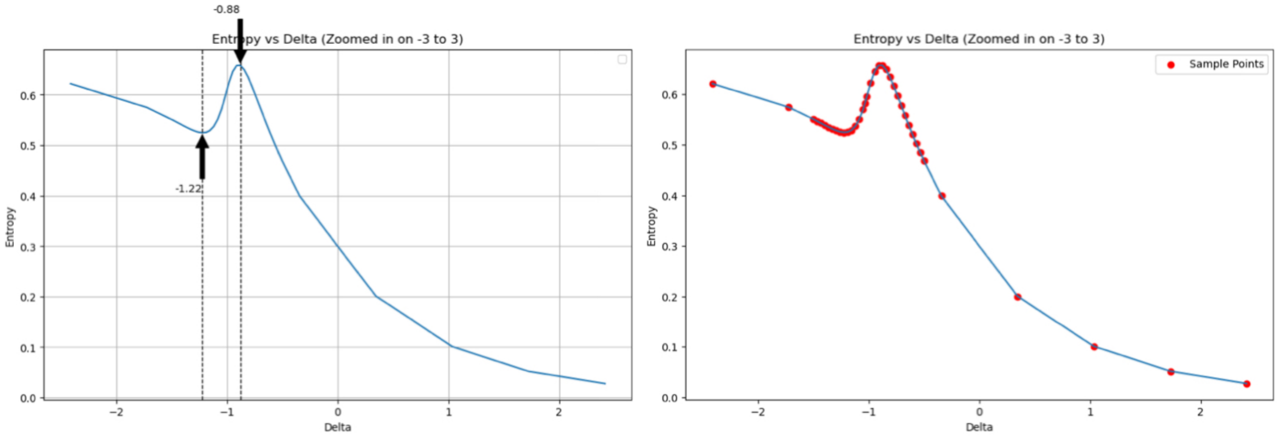
$$\mathrm{Var}(\sigma^z) = 0.2479$$

So what happens according to our plot? The expectation (possibility of spin up) $\langle \sigma^z \rangle$ indicates how many spin-ups are there in the system, or more accurately, the density of the spin-ups. The variance, $\mathrm{Var}(\sigma^z)$, indicates how uniform is the distribution of the spin-ups. When $\mathrm{Var}(\sigma^z)$ is close to 0, there is a equal possibility of spin-ups at each site. When $\mathrm{Var}(\sigma^z)$ is close to 0.25, the distribution is very non-uniform. They can only show up at where they are forced to be.

4 Go back to Entanglement Entropy

In this part we will zoom into the detailed structure of the entanglement entropy.

First let's plot a detailed plot of the entanglement entropy at the point where $\Delta \in [-3, 3], N = 12$.



What is argued below is what applies to the case of $N = 12$.

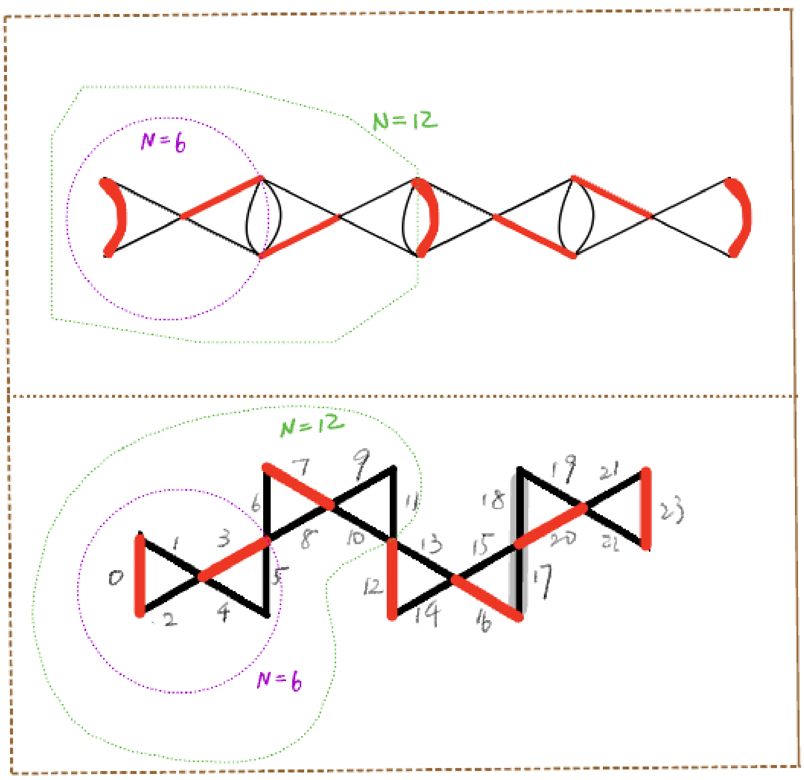
When $\Delta > \ -0.88$, the density of spin-ups is increasing, while the variance is close to 0, meaning that they are rather free to be anywhere they want(under the restriction of no-neighboring excitations).

When $\Delta \ \in [-1.22, -0.88]$, the spin-ups are quickly assigned to some particular sites, losing there freedom. The effect of losing freedom is obviously larger than the effect of density increase, so entanglement entropy falls down.

When $\Delta < -1.22$, the spin-ups are well assigned to some particular sites, and the density of spin ups is slowing increasing. Since all the possible sites are already determined, a defect can only appear at determined places. So there is no domain wall moving like gas. As I will prove, in this case, the appearance of defects will only lower the entanglement entropy. So as Δ decreases, there are less and less defects, and the entanglement entropy will go up.

4.1 Theoretical discuss of the Entanglement Entropy

We only talk about the behavior of the entanglement entropy at the cutting point $N = 12$ and $N = 6$, and $\Delta \ < -1.3$.



4.1.1 The case of $N = 6$

We consider a dual pair as our unit cell, and a unit cell has 3 possible states:

$$|\uparrow\rangle\,,|\downarrow\rangle\,,|0\rangle$$

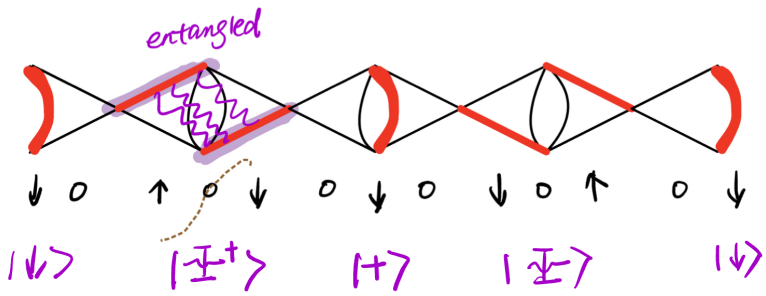
For example, in the picture below, the state of such a dimer model is:

$$|\downarrow 0 \uparrow 0 \downarrow 0 \downarrow 0 \downarrow 0 \uparrow 0 \downarrow\rangle$$

This is, actually, one component of the actual state. Those $|0\rangle$ are on the unassigned sites, which means they are forbidden to be excited. So, we can just cancel them out to simplify. The actual state is,

$$|\downarrow\rangle\,|\Psi^+\rangle\,|+\rangle\,|\Psi^+\rangle\,|\downarrow\rangle$$

When one is cutting at site 6, he/she is actually cutting inside a $|\Psi^+\rangle$, and the entanglement entropy is $ln2$.

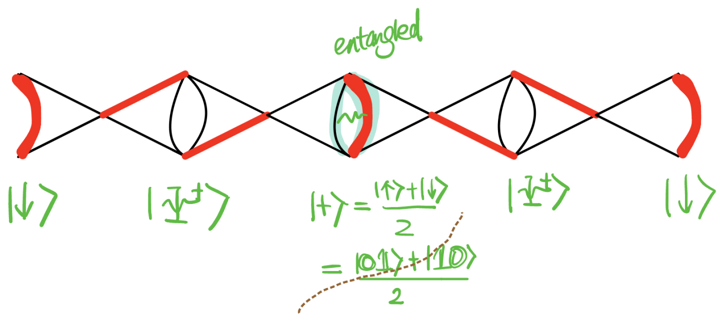


4.1.2 The case of $N = 12$

Now we are cutting inside the unit $|+\rangle$. So, insider the unit, we note $|\uparrow\rangle = |\mathbb{IO}\rangle$, while $|\downarrow\rangle = |\mathbb{OI}\rangle$. Thus, the state of the system is

$$|\downarrow\rangle\,|\Psi^+\rangle\,\frac{|\mathbb{IO}\rangle + |\mathbb{OI}\rangle}{2}\,|\Psi^+\rangle\,|\downarrow\rangle$$

Then we cut in the middle. The entanglement entropy is $ln2$.



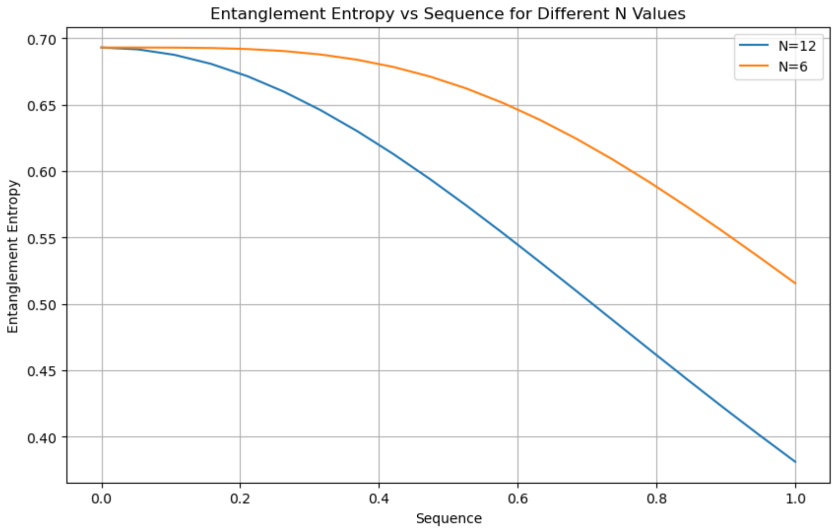
4.1.3 Perturbation because of defects

When there is possibility of defects, the entanglement entropy will be lowered, as shown below.

```
def calculate_entanglement_entropy(coefficients, n, m):
    coefficients = np.array(coefficients, dtype=np.complex128)
    coefficients /= np.linalg.norm(coefficients)
    matrix = coefficients.reshape(n, m)
    U, s, Vh = svd(matrix)
    singular_values_squared = s**2
    entanglement_entropy = -np.sum(singular_values_squared * np.log(singular_values_squared, where=(

    return entanglement_entropy

sequence = np.linspace(0, 1, 20)
list_N12 = []
for x in sequence:
    list_N12.append(calculate_entanglement_entropy([0, 1, 1, x], 2, 2))

list_N6 = []
for x in sequence:
    list_N6.append(calculate_entanglement_entropy([0, 1, x, 1, 0, x, 0, x, x**2], 3, 3))
```

Here x represents the possibility of defects.

The curve is as expected! First, as there are less and less defects, the entanglement entropy goes up. This explains why the curve increases at the region where $\Delta < -1.3$. Also, at cutting site $N = 6$, more sensitive to defects, so the curve of the corresponding entropy appears an obvious up-going. However, at small possibility of defects, the entropy at $N = 6$ is almost unaffected, thus the corresponding curve is almost flat.



4.2 General description of the phenomenon

When Δ is greater than approximately -0.8, as Δ decreases, the ground state of the system gradually evolves away from a trivial spin-down product state, slowly generating some spin-up gas. At this stage, the spin-up gas is free and can appear almost anywhere. As Δ continues to decrease, the density of the spin-up gas increases, leading to an increase in the entanglement entropy.

When Δ is between approximately -0.8 and -1.3, the system undergoes a rather dramatic phase transition. The positions of the spin-ups are no longer arbitrary but are rapidly confined to specific locations determined by the boundary conditions. The reduction in the freedom of the spin-up states leads to a sharp decrease in entanglement entropy. For certain cutting positions, the entanglement entropy directly decreases to near zero, indicating that there is no correlation between the parts of the system divided at these positions, and thus no entanglement entropy. For other partitions(cuttings), the entanglement entropy decreases to a finite value, indicating that the parts of the system are still correlated. The entanglement is localized near the boundaries and, due to the presence of defects, is less than $\ln 2$.

As Δ continues to decrease, the number of defects decreases, and the entanglement entropy at these locations gradually increases ,at different rates (depending on the nature of the partition points), until it reaches $\ln 2$.

One interesting thing is that the entanglement at different places is different. Some addictive plots are shown below.

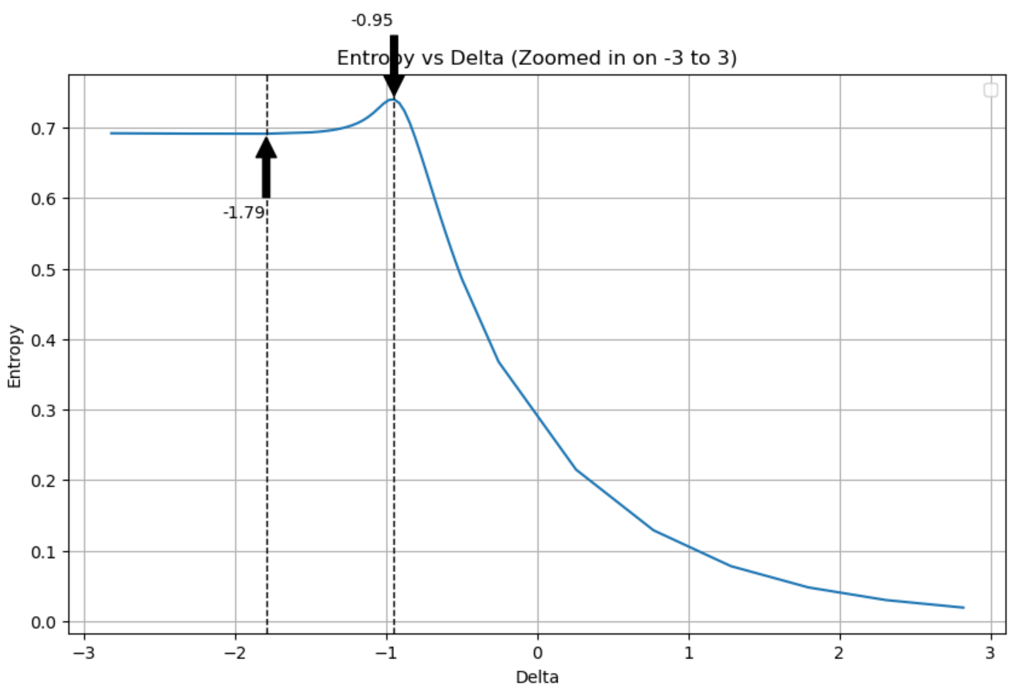


Figure 5: Phase transition at $N = 6$

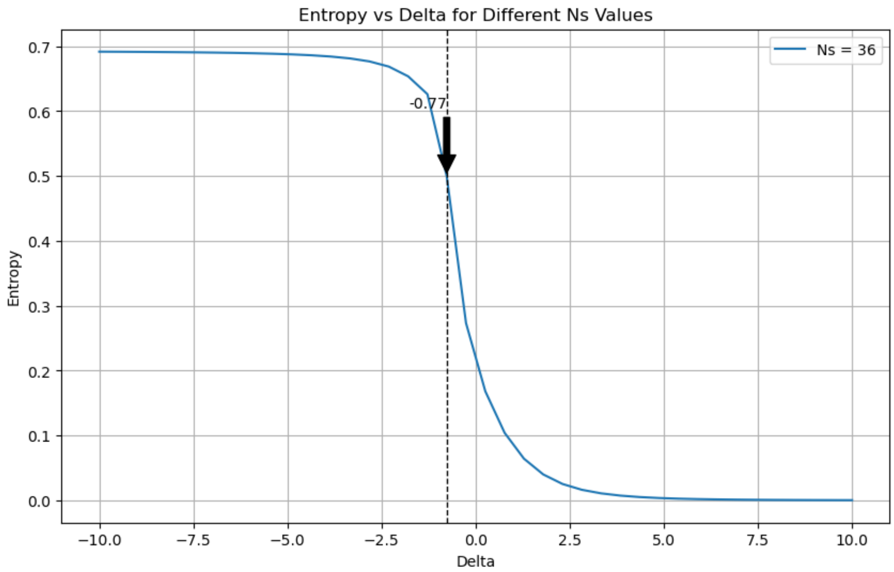


Figure 6: Phase transition at $N = 16$

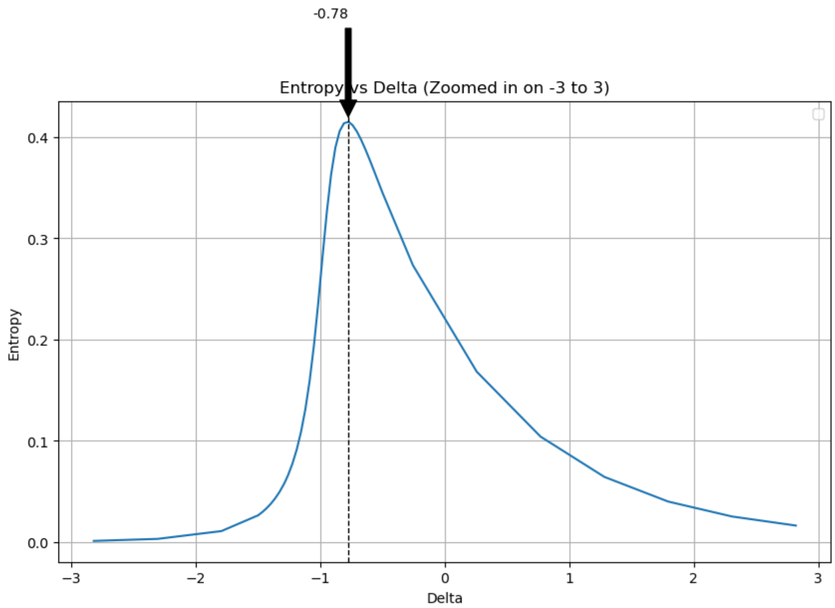


Figure 7: Phase transition at $N = 10$

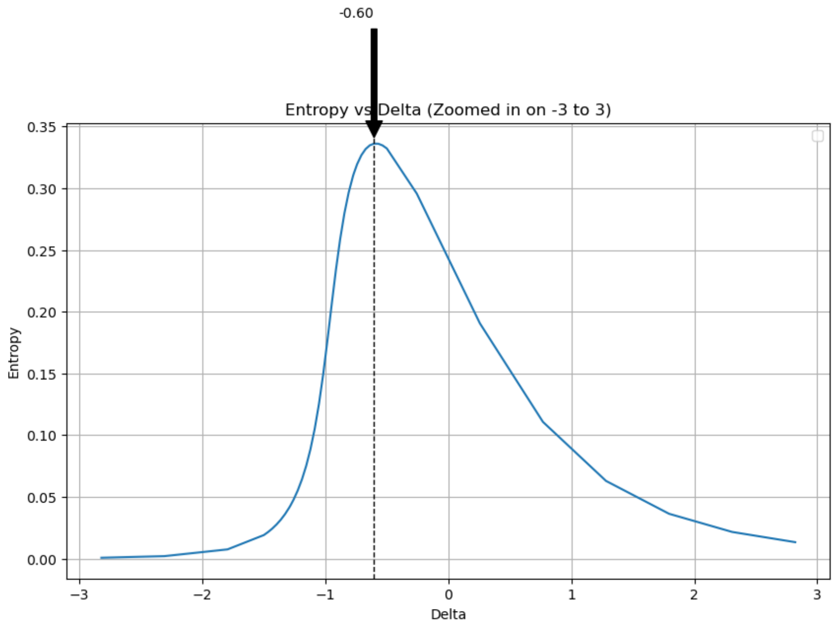


Figure 8: Phase transition at $N = 2$

5 Outlook

Is there QSL in this model?

1. Maybe there is long-range entanglement in the range that $\Delta \in [\ -1.3,\ 0.8]$. It can be seen from previous pictures that the maximum of entropy lis larger that $ln2$, which indicates the system size of entanglement must be larger than two. Also, the freedom in this region is somewhere between very strict restriction and totally free, which may indicate QSL.

2. I want to think of the question from the perspective of information theory. Consider the perfect dimer model at $\Delta = -10$. Since we now have a boundary condition because of our VMPS Algorithm, we already now which unit(remember that a unit is a pair of dual links/spins) is allowed to have a spin-up ,and which is not. So if you are told that there is a spin-up in a unit, you don't get any information about another unit which is far away. Whether there is a spin-up in that unit is already determined......by the boundary condition.

But what if we cancel out the boundary condition? Practically we can change where we start our VMPS Algorithm, and the boundary condition will be different. However, what is given is always the ground state of the system. What we are going to do next is to make a superposition of states with different boundary conditions. Then, when I again tell you that there is a spin-up in a unit, you will get the information about what boundary condition we are at, and then you will know whether it is possible for an unit far away to have a spin-up.

That is, when you are given the information of a unit, you immediately know the information about anywhere near or far away—-even the system! You suddenly kill half of the Hilbert Space, and the state must be in the other half

6 Result

I talked about this model with Prof. Erich Mueller. The conclusion is this model has just local entanglement, and there is no QSL in this model.

It is similar to a one-dimensional checkerboard state, with Z2 symmetry.

It's interesting, anyway.