Statistical Fields: A Phenomenological Introduction

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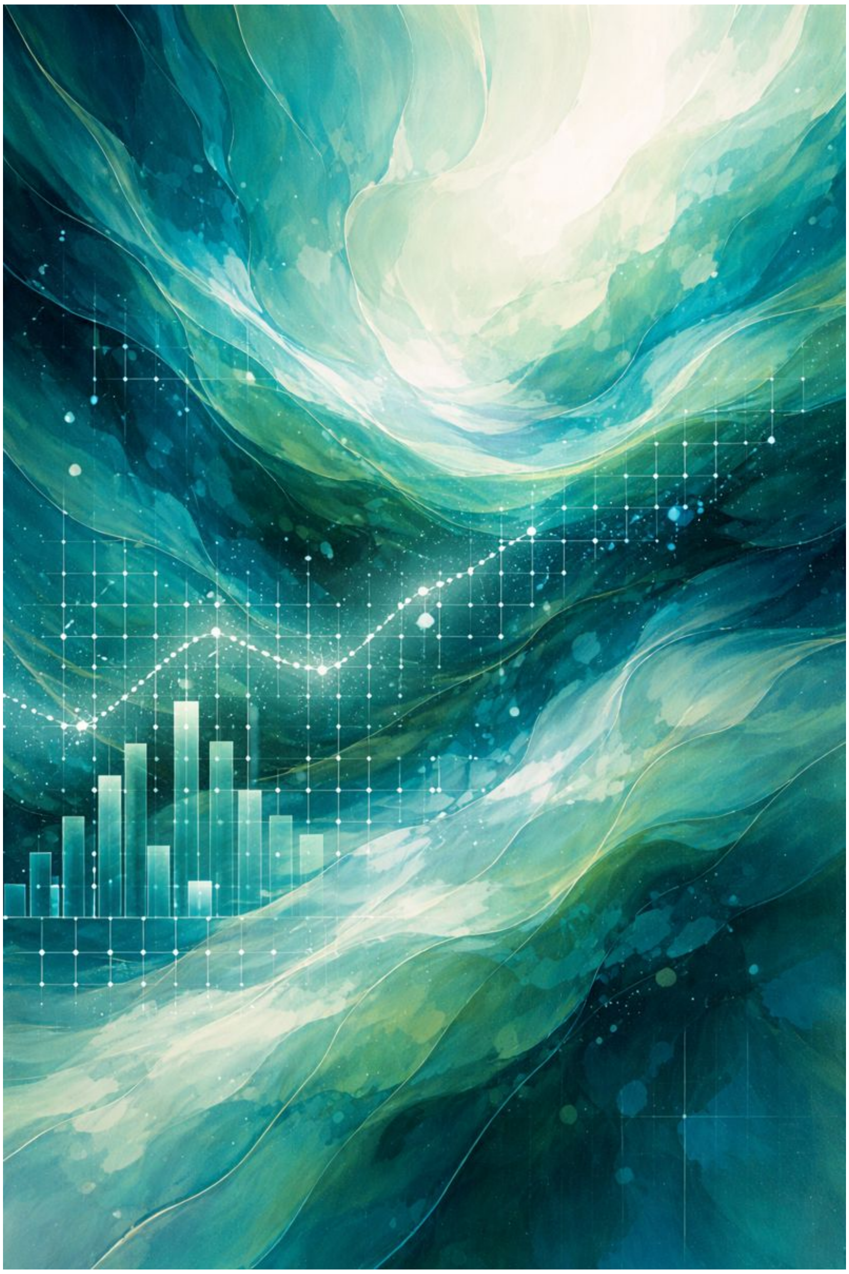

Contents

These notes reorganize the core ideas of Chapter 2 (*Statistical Fields*) in Kardar's statistical physics: why coarse-grained fields emerge near criticality, how the Landau–Ginzburg functional is constrained by symmetry and locality, and how mean-field critical behavior, Goldstone modes, and domain walls follow from saddle-point reasoning.

1 Preface

We aim for a general, system-independent language, but we will repeatedly anchor the discussion in two concrete classes of physical systems:

- magnetic systems, where the order parameter is a coarse-grained magnetization field;
- superfluids, where the order parameter is a complex scalar field whose phase supports Goldstone fluctuations.

This is not a restriction of principle; rather, these examples make the abstract construction tangible.

2 Why introduce statistical fields?

Near a continuous phase transition, thermodynamic response functions often become singular and long-wavelength fluctuations become strongly correlated. Scattering experiments directly reveal that the relevant fluctuations have wavelengths much larger than the microscopic lattice spacing, suggesting that an effective description should focus on slow, collective degrees of freedom rather than on individual atoms/spins.

From microscopic variables to a coarse-grained field. At the microscopic level, a system is described by a Hamiltonian H_{mic} depending on many discrete degrees of freedom (for instance spins $\{m_i\}$). The partition function is

$$Z = \mathrm{Tr}\left(e^{-\beta H_{\text{mic}}}\right), \tag{1}$$

where $\beta \equiv (k_B T)^{-1}$.

The key phenomenological step is to replace the huge set of microscopic variables by a *continuous order-parameter field* that varies smoothly on mesoscopic scales. For a magnetic system we introduce an n-component magnetization field $\boldsymbol{m}(\boldsymbol{x}) \in \mathbb{R}^n$ in d spatial dimensions.

$$Z(T) \equiv \mathrm{Tr}\left(e^{-\beta H_{\text{mic}}}\right) \implies Z = \int \mathcal{D}\boldsymbol{m}(\boldsymbol{x})\, \mathcal{W}[\boldsymbol{m}(\boldsymbol{x})]\,, \tag{2}$$

where $\mathcal{D}\boldsymbol{m}(\boldsymbol{x})$ denotes a functional integration measure and $\mathcal{W}[\boldsymbol{m}]$ is a coarse-grained statistical weight.

From discrete sums to functional integrals

Microscopically one has a weighted sum over configurations:

$$Z = \sum_{\{m_i\}} e^{-\beta \mathcal{H}_{\text{mic}}(\{m_i\})}. \tag{3}$$

If each m_i can be approximated as a continuous variable, the sum becomes a product of integrals:

$$Z \approx \prod_{i=1}^{N} \int \mathrm{d}m_i \; \exp[-\beta \mathcal{H}_{\text{mic}}(m_1, m_2, \ldots)]\,. \tag{4}$$

In the continuum/coarse-grained limit, the discrete set $\{m_i\}$ is replaced by a smooth field $\boldsymbol{m}(\boldsymbol{x})$, and the weight can be written in the exponential form

$$\mathcal{W}[\boldsymbol{m}(\boldsymbol{x})] = \exp[-\beta\mathcal{H}[\boldsymbol{m}(\boldsymbol{x})]], \qquad \beta\mathcal{H}[\boldsymbol{m}] \equiv -\ln\mathcal{W}[\boldsymbol{m}]. \tag{5}$$

Formally the functional measure is the continuum limit of the product measure:

$$\int \mathcal{D}\boldsymbol{m}(\boldsymbol{x}) \equiv \lim_{N\to\infty} \prod_{i=1}^{N} \int \mathrm{d}m_i. \tag{6}$$

When is the continuum field description meaningful?. The continuum description is intended for *mesoscopic* length scales. If a denotes a microscopic cutoff (e.g. lattice spacing), we require the dominant fluctuations to satisfy $\lambda \gg a$, or equivalently that the field varies slowly on the scale a. On scales comparable to a, the field description is not expected to be accurate.

The statistical field $\boldsymbol{m}(\boldsymbol{x})$ is a *coarse-grained* variable: it replaces microscopic degrees of freedom only after one commits to a scale separation $\lambda \gg a$. The resulting theory is designed to capture universal long-distance physics near criticality.

3 Landau–Ginzburg Hamiltonian

We now construct a phenomenological Hamiltonian functional $\mathcal{H}[\boldsymbol{m}]$ consistent with locality and symmetries.

3.1 Locality, analyticity, and symmetry constraints

Local functional form. A general local expansion writes the reduced Hamiltonian as an integral of a free-energy density functional:

$$\beta\mathcal{H}[\boldsymbol{m}] = \int \mathrm{d}^d x\, \Phi\Big(\boldsymbol{x}, \boldsymbol{m}(\boldsymbol{x}), \nabla\boldsymbol{m}(\boldsymbol{x}), \nabla^2\boldsymbol{m}(\boldsymbol{x}), \ldots\Big). \tag{7}$$

If microscopic interactions are short-ranged and the coarse-graining is well-defined, higher-derivative terms are typically less important at long wavelengths, so one keeps only the lowest few gradient terms.

Analyticity. After coarse-graining, one assumes Φ is analytic in $\boldsymbol{m}$ and its derivatives (within the regime of validity of the effective theory), so it can be expanded in powers of $\boldsymbol{m}$.

Symmetries. For a homogeneous isotropic system:

- **Translational invariance.** Φ does not depend explicitly on $\boldsymbol{x}$.
- **Rotational invariance in order-parameter space.** For an $O(n)$-symmetric magnet at zero field, Φ depends on $\boldsymbol{m}$ only through $\boldsymbol{m}^2 \equiv \boldsymbol{m}\cdot\boldsymbol{m}$, implying only even powers of $\boldsymbol{m}$ appear.
- **Coupling to an external field.** A (reduced) external field $\boldsymbol{h}$ couples linearly via $-\boldsymbol{h}\cdot\boldsymbol{m}$.

3.2 The Landau–Ginzburg functional

Keeping the lowest nontrivial terms consistent with the above, one arrives at the Landau–Ginzburg Hamiltonian:

$$\beta\mathcal{H}[\boldsymbol{m}] = \beta F_0 + \int \mathrm{d}^d x \left[\frac{t}{2}\,\boldsymbol{m}^2(\boldsymbol{x}) + u\left(\boldsymbol{m}^2(\boldsymbol{x})\right)^2 + \frac{K}{2}\,(\nabla\boldsymbol{m}(\boldsymbol{x}))^2 - \boldsymbol{h}\cdot\boldsymbol{m}(\boldsymbol{x}) + \cdots\right]. \tag{8}$$

Here F_0 is a background contribution dominated by short-distance physics (often irrelevant for universal properties), t is the tuning parameter that changes sign at the mean-field critical point, u stabilizes the theory at large $\|\boldsymbol{m}\|$, and K penalizes spatial inhomogeneity.

Precise meaning of gradient terms

Let $\boldsymbol{m}(\boldsymbol{x}) = (m_1(\boldsymbol{x}), \ldots, m_n(\boldsymbol{x}))$. Then

$$\boldsymbol{m}^2(\boldsymbol{x}) = \sum_{j=1}^{n} m_j(\boldsymbol{x})^2. \tag{9}$$

The gradient $\nabla \boldsymbol{m}$ is a $d \times n$ array with entries $\partial_i m_j$ (spatial index $i = 1, \ldots, d$, component index $j = 1, \ldots, n$), and

$$(\nabla \boldsymbol{m})^2 \equiv \sum_{i=1}^{d} \sum_{j=1}^{n} \left(\partial_i m_j\right)^2. \tag{10}$$

Similarly,

$$\nabla^2 \boldsymbol{m} \equiv \left(\nabla^2 m_1, \ldots, \nabla^2 m_n\right), \qquad \nabla^2 m_j \equiv \sum_{i=1}^{d} \partial_i^2 m_j, \tag{11}$$

and higher-derivative invariants can be built if needed.

Stability of the coarse-grained theory. Because $\mathcal{W}[\boldsymbol{m}] = e^{-\beta \mathcal{H}[\boldsymbol{m}]}$ must define a normalizable statistical weight, the effective functional must suppress unphysical large-field configurations. In particular,

$$u > 0 \tag{12}$$

is required for stability at large $\|\boldsymbol{m}\|$, and

$$K > 0 \tag{13}$$

ensures that sharp spatial variations cost free energy.

4 Saddle point approximation

4.1 Saddle point logic for the Landau–Ginzburg functional

Even after writing down $\mathcal{H}[\boldsymbol{m}]$, the functional integral in (2) is generally intractable. A basic approximation is to assume the integral is dominated by field configurations near minima of $\mathcal{H}[\boldsymbol{m}]$.

Uniform-field reduction and mean-field spirit. If $K > 0$, spatial gradients are penalized, so it is natural to first restrict attention to nearly uniform configurations $\boldsymbol{m}(\boldsymbol{x}) \approx \boldsymbol{m}$ (a constant vector), i.e. one keeps only the zero-momentum mode. This produces an ordinary integral:

$$Z \approx Z_{\text{uni}} = e^{-\beta F_0} \int \mathrm{d}^n \boldsymbol{m} \, \exp\!\left[-V\left(\frac{t}{2}\boldsymbol{m}^2 + u(\boldsymbol{m}^2)^2 + \cdots - \boldsymbol{h} \cdot \boldsymbol{m}\right)\right], \tag{14}$$

where V is the system volume and $\mathrm{d}^n \boldsymbol{m}$ is the measure over the n-component order parameter.

Steepest descent over the uniform mode. For large V, the remaining integral is dominated by minima of the effective potential density

$$\Psi(\boldsymbol{m}) \equiv \frac{t}{2}\boldsymbol{m}^2 + u(\boldsymbol{m}^2)^2 + \cdots - \boldsymbol{h} \cdot \boldsymbol{m}. \tag{15}$$

Correspondingly, the free energy becomes

$$\beta F_{\mathrm{MF}} \equiv -\ln Z_{\mathrm{uni}} \approx \beta F_0 + V \min_{\boldsymbol{m}} \Psi(\boldsymbol{m}). \tag{16}$$

This is the familiar Landau mean-field construction, now seen as a saddle-point evaluation of a coarse-grained statistical field theory.

A minimal saddle-point template (one variable)

For an integral of the form $I = \int \mathrm{d}x\, e^{-f(x)}$, if f has a sharp minimum at x_0, expand

$$f(x) \approx f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2, \tag{17}$$

giving

$$I \approx e^{-f(x_0)} \int \mathrm{d}x\, \exp\Big[-\frac{1}{2} f''(x_0)(x - x_0)^2\Big] = e^{-f(x_0)} \sqrt{\frac{2\pi}{f''(x_0)}}, \tag{18}$$

assuming $f''(x_0) > 0$. In field theory the same idea applies, with x replaced by a function $\boldsymbol{m}(\boldsymbol{x})$.

4.2 Mean-field critical behavior

We now analyze the minima of $\Psi(\boldsymbol{m})$ and extract standard mean-field exponents. Since $\boldsymbol{h}$ selects a direction, at the minimum $\boldsymbol{m}$ aligns with $\boldsymbol{h}$. Write $\boldsymbol{m} = m\,\hat{\boldsymbol{h}}$ and $h \equiv \|\boldsymbol{h}\|$. Then

$$\Psi(m) = \frac{t}{2} m^2 + u m^4 - h m. \tag{19}$$

The stationarity condition is

$$\frac{\partial \Psi}{\partial m} = t m + 4 u m^3 - h = 0. \tag{20}$$

Near the mean-field critical point, we take

$$t = a\,(T - T_c) + O\big((T - T_c)^2\big), \tag{21}$$

with $a > 0$.

Order parameter (magnetization). At $h = 0$, (20) reduces to $m(t + 4um^2) = 0$, giving

$$m = \begin{cases} 0, & t > 0 \;\; (T > T_c), \\ \pm\sqrt{-\dfrac{t}{4u}}, & t < 0 \;\; (T < T_c). \end{cases} \tag{22}$$

Thus $m \sim (T_c - T)^{1/2}$ for $T \to T_c^-$, and the mean-field exponent is $\beta = 1/2$.

At $t = 0$, (20) gives $4um^3 = h$, hence

$$m(t = 0) \sim \left(\frac{h}{4u}\right)^{1/3}, \tag{23}$$

so $\delta = 3$.

Longitudinal susceptibility. Define the longitudinal susceptibility $\chi_\ell \equiv \left.\dfrac{\partial m}{\partial h}\right|_{h\to 0}$. Differentiating (20) gives

$$\chi_\ell^{-1} = \left.\frac{\partial h}{\partial m}\right|_{h\to 0} = t + 12u\,\bar{m}^2, \tag{24}$$

where $\bar{m}$ is the zero-field spontaneous magnetization from (22). Therefore

$$\chi_\ell^{-1} = \begin{cases} t, & t > 0, \\ -2t, & t < 0, \end{cases} \qquad \Rightarrow \qquad \chi_\ell \propto |t|^{-1}, \tag{25}$$

so $\gamma_+ = \gamma_- = 1$. Moreover, the amplitude ratio predicted by (25) is

$$\frac{A_+}{A_-} = 2. \tag{26}$$

Specific heat. At $h = 0$, insert the minimizing m into Ψ. Using (22),

$$\Psi(\bar{m}) = \begin{cases} 0, & t > 0, \\ -\dfrac{t^2}{16u}, & t < 0. \end{cases} \tag{27}$$

Hence

$$\beta F_{\mathrm{MF}} = \beta F_0 + V\Psi(\bar{m}). \tag{28}$$

Since $t = a(T - T_c) + \cdots$, one has $\partial/\partial T \approx a\,\partial/\partial t$ near T_c. This yields a finite jump rather than a divergence in the specific heat, corresponding to the mean-field exponent $\alpha = 0$.

Mean-field exponents from the Landau free energy $\Psi(m) = \frac{t}{2}m^2 + um^4 - hm$:

$$\beta = \frac{1}{2}, \qquad \delta = 3, \qquad \gamma_+ = \gamma_- = 1, \qquad \alpha = 0, \tag{29}$$

together with the universal amplitude ratio $A_+/A_- = 2$ for the longitudinal susceptibility.

5 Spontaneous symmetry breaking and Goldstone modes

5.1 Spontaneous symmetry breaking in an $O(n)$ order parameter

Consider zero external field $\boldsymbol{h} = 0$. The functional (8) is invariant under global $O(n)$ rotations $\boldsymbol{m}(\boldsymbol{x}) \mapsto \mathcal{R}\boldsymbol{m}(\boldsymbol{x})$ with $\mathcal{R} \in O(n)$. In the ordered phase ($t < 0$), the saddle-point solution has a nonzero magnitude $\|\boldsymbol{m}\| = \bar{m}$, but its direction is not fixed by the Hamiltonian. The system *chooses* a direction in order-parameter space, producing:

- *spontaneous symmetry breaking*: the equilibrium state is not invariant under the full $O(n)$;
- *long-range order*: most local degrees of freedom align with the chosen direction.

A global rotation maps one ordered state to another energetically equivalent one.

5.2 Goldstone modes and a superfluid example

Slow rotations cost little free energy. If a uniform rotation costs no free energy, then (by continuity) a *slowly varying* rotation in space should cost only a small amount, controlled by gradients. These low-energy, long-wavelength fluctuations are Goldstone modes associated with broken continuous symmetries.

Superfluid as an $n = 2$ example. A standard coarse-grained superfluid order parameter is a complex scalar field

$$\psi(\boldsymbol{x}) = \mathrm{Re}\,\psi(\boldsymbol{x}) + i\,\mathrm{Im}\,\psi(\boldsymbol{x}) = |\psi(\boldsymbol{x})|e^{i\theta(\boldsymbol{x})}, \tag{30}$$

which can be viewed as a two-component real field $(\mathrm{Re}\,\psi, \mathrm{Im}\,\psi)$ with $O(2)$ symmetry at $h = 0$. A Landau–Ginzburg functional for ψ takes the form

$$\beta\mathcal{H}[\psi] = \beta F_0 + \int \mathrm{d}^d\boldsymbol{x} \left[\frac{K}{2}|\nabla\psi|^2 + \frac{t}{2}|\psi|^2 + u|\psi|^4 + \cdots\right]. \tag{31}$$

In the ordered phase, a saddle-point treatment fixes the amplitude $|\psi| \approx \bar{\psi}$ while leaving the phase θ as a soft mode. Plugging $\psi(\boldsymbol{x}) \approx \bar{\psi}e^{i\theta(\boldsymbol{x})}$ into (31) yields an effective phase-only theory.

$$\beta\mathcal{H} \approx \beta\mathcal{H}_0 + \frac{\bar{K}}{2}\int \mathrm{d}^d\boldsymbol{x}\,(\nabla\theta)^2, \qquad \bar{K} \equiv K\bar{\psi}^2, \tag{32}$$

where $\mathcal{H}_0$ collects the (approximately constant) amplitude contribution.

Fourier expanding θ leads to

$$\beta\mathcal{H} \approx \beta\mathcal{H}_0 + \frac{\bar{K}}{2}\sum_{\boldsymbol{q}} q^2\,|\theta(\boldsymbol{q})|^2. \tag{33}$$

This shows that long-wavelength phase fluctuations ($q \to 0$) are very soft: the *free-energy cost* scales as q^2. (A true dynamical dispersion relation requires adding time-dependent physics; the static stiffness structure above is the starting point.)

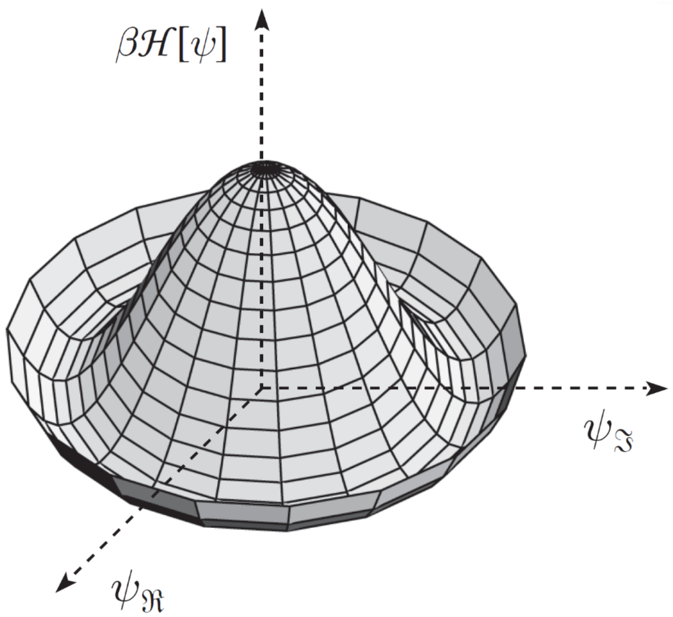

Figure 1: Phase-only (Goldstone) description of a superfluid: amplitude is fixed at the saddle point, while long-wavelength phase twists cost energy only through gradients.

Derivation of the phase-only Hamiltonian

Start from (31) and substitute $\psi(\boldsymbol{x}) = \bar{\psi}e^{i\theta(\boldsymbol{x})}$ with constant $\bar{\psi}$. Then

$$\nabla\psi = \bar{\psi}\,ie^{i\theta(\boldsymbol{x})}\nabla\theta(\boldsymbol{x}), \qquad \Rightarrow \qquad |\nabla\psi|^2 = \bar{\psi}^2(\nabla\theta)^2. \tag{34}$$

Keeping only the θ-dependent part yields (32). Writing

$$\theta(\boldsymbol{x}) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{q}} e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\,\theta(\boldsymbol{q}), \tag{35}$$

one finds

$$\int \mathrm{d}^d\boldsymbol{x}\,(\nabla\theta)^2 = \sum_{\boldsymbol{q}} q^2\,|\theta(\boldsymbol{q})|^2, \tag{36}$$

leading to (33).

6 Discrete symmetry breaking and domain walls

For a one-component (scalar) order parameter $m(\boldsymbol{x})$ at $h = 0$, the ordered phase typically has two degenerate minima $m = \pm\bar{m}$. Unlike the continuous case, one cannot connect these two states by an infinitesimal rotation; consequently, spatial regions may choose different signs, separated by a sharp but smooth interface: a domain wall.

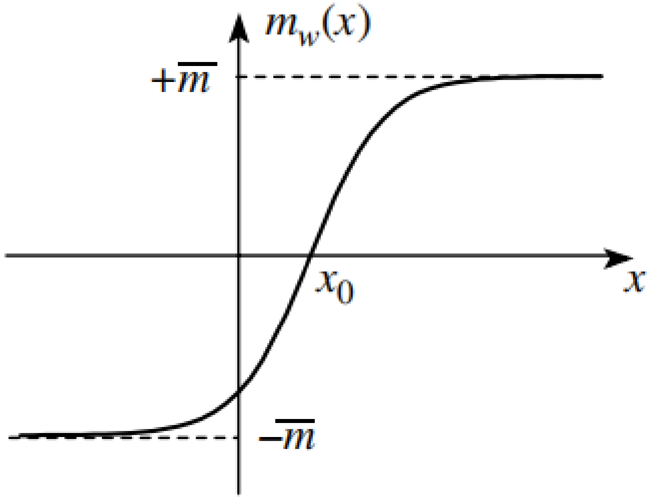

Figure 2: A domain wall interpolating between two degenerate ordered states in a scalar theory with $m = \pm\bar{m}$.

Euler–Lagrange equation for a domain wall profile. Consider $t < 0$ and $h = 0$, and look for a one-dimensional profile $m_w(x)$ with boundary conditions

$$m_w(x \to -\infty) = -\bar{m}, \qquad m_w(x \to +\infty) = +\bar{m}. \tag{37}$$

Now the gradient term in (8) is essential. Minimizing the functional yields the nonlinear differential equation

$$K\frac{\mathrm{d}^2 m_w(x)}{\mathrm{d}x^2} = t\,m_w(x) + 4u\,m_w(x)^3. \tag{38}$$

A standard solution is

$$m_w(x) = \bar{m}\,\tanh\!\Big(\frac{x - x_0}{w}\Big), \tag{39}$$

with

$$\bar{m} = \sqrt{-\frac{t}{4u}}, \qquad w = \sqrt{\frac{K}{-t}}. \tag{40}$$

Here x_0 is the domain-wall center and w is the characteristic width.

Free-energy cost (interface tension). The free-energy cost of introducing a domain wall is the difference between the inhomogeneous configuration and the uniform ground state:

$$\Delta F_w = \int \mathrm{d}x \left[\frac{K}{2} \left(\frac{\mathrm{d}m_w}{\mathrm{d}x} \right)^2 + \frac{t}{2} \left(m_w^2 - \bar{m}^2 \right) + u \left(m_w^4 - \bar{m}^4 \right) \right] A, \tag{41}$$

where A is the cross-sectional area perpendicular to the x-direction. For the profile (39), one obtains

$$\Delta F_w = -\frac{2}{3}\, t\, \bar{m}^2\, w\, A. \tag{42}$$

Domain wall in a scalar Landau–Ginzburg theory ($t < 0$, $h = 0$):

$$m_w(x) = \bar{m} \tanh\!\left(\frac{x - x_0}{w} \right), \qquad \bar{m} = \sqrt{-\frac{t}{4u}}, \qquad w = \sqrt{\frac{K}{-t}}, \tag{43}$$

and the associated free-energy cost scales as $\Delta F_w \propto |t|^{3/2} A$ near the critical point (since $w \propto |t|^{-1/2}$ and $\bar{m}^2 \propto |t|$).