Particle Quantum Statistics

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1 Particle Quantum Statistics and the Path Integral Viewpoint

1.1 What is particle quantum statistics?

Particle quantum statistics (also called exchange statistics) is the rule that specifies how a many-body wavefunction changes when *identical* particles are exchanged in space. The emphasis is not on the motion of an individual particle, but on the action of an exchange process on the *many-particle* Hilbert space, i.e., on the overall quantum state.

In three spatial dimensions, we are familiar with two possibilities. Exchanging two bosons leaves the wavefunction invariant, while exchanging two fermions multiplies the wavefunction by -1. A common introductory argument goes as follows:

> An exchange performed twice should return the system to its original configuration, so the exchange operator should square to the identity. Since the only square roots of 1 are +1 and −1, only bosons and fermions are possible. −− A folklore

This argument is not fully adequate as a fundamental classification principle. The reason is that it implicitly assumes that the space of exchange processes has only the minimal $\mathbb{Z}_2$ structure of "swap or not swap," whereas in quantum mechanics the correct classification is controlled by the topology of *paths in configuration space*. The path integral formulation makes this topology explicit, and thereby provides a precise framework for determining which exchange rules are consistent with quantum mechanics.

1.2 Review: single-particle path integral

Consider a non-relativistic particle moving in D spatial dimensions, with position $x \in \mathbb{R}^D$. A spacetime trajectory is a function $x(t)$ defined on a time interval $[t_i, t_f]$. The fundamental object encoding time evolution is the propagator

$$K(x_f, t_f; x_i, t_i) \equiv \langle x_f | \hat{U}(t_f, t_i) | x_i \rangle, \tag{1}$$

where $\hat{U}(t_f, t_i)$ is the unitary time-evolution operator.

The propagator evolves an arbitrary wavefunction $\psi(x, t) = \langle x|\psi(t)\rangle$ by

$$\langle x_f|\psi(t_f)\rangle = \int dx_i \, \langle x_f|\hat{U}(t_f, t_i)|x_i\rangle\langle x_i|\psi(t_i)\rangle. \tag{2}$$

Two structural requirements are essential for the propagator.

• *Unitarity*: $\hat{U}^\dagger\hat{U} = \mathbf{1}$, which guarantees conservation of total probability.

• *Composition law*: Evolving from t_i to an intermediate time t_m, and then from t_m to t_f, must equal evolving directly from t_i to t_f. In the position basis, the composition is written as[1]:

$$\langle x_f|\hat{U}(t_f, t_i)|x_i\rangle = \int dx_m \, \langle x_f|\hat{U}(t_f, t_m)|x_m\rangle\langle x_m|\hat{U}(t_m, t_i)|x_i\rangle. \tag{4}$$

Feynman's key observation is that one may subdivide the time interval into many short steps and repeatedly insert complete sets of position states. In the limit of infinitely fine time

[1] This is equivalently the insertion of a resolution of identity,

$$\mathbf{1} = \int dx_m \, |x_m\rangle\langle x_m|. \tag{3}$$

slicing, this yields an integral over *all* paths $x(t)$ connecting the endpoints. This can be natively written as

$$\langle x_f|\hat{U}(t_f,t_i)|x_i\rangle = \mathcal{N} \sum_{\substack{\text{paths } \mathbf{x}(t) \text{ from} \\ (\mathbf{x}_i,t_i) \text{ to } (\mathbf{x}_f,t_f)}} e^{iS[\mathbf{x}(t)]/\hbar}, \tag{5}$$

or in a formal form:

$$\langle x_f|\hat{U}(t_f,t_i)|x_i\rangle = \mathcal{N} \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t)\ \exp\Big(\frac{i}{\hbar}S[x(t)]\Big), \tag{6}$$

with normalization $\mathcal{N}$, and action

$$S[x(t)] = \int_{t_i}^{t_f} dt\ L\Big(x(t),\dot{x}(t),t\Big), \tag{7}$$

where L is the classical Lagrangian.

In the semiclassical regime (small $\hbar$), the path integral is dominated by paths for which the phase oscillates least rapidly. This is captured by a saddle-point approximation: the dominant contributions come from trajectories $x_{\text{cl}}(t)$ that extremize the action, $\delta S = 0$, which is precisely the classical principle of least action.

2 Two Identical Particles: Path Integral and Exchange Classes

2.1 Configuration space

We now turn to systems with multiple identical particles, beginning with two particles. If particles are identical, there is no physical meaning to labeling one particle as "particle 1" at x_1 and "particle 2" at x_2; the configuration "a particle at x_1 and a particle at x_2" is the same as the swapped labeling. To avoid representing the same physical configuration twice (e.g. $|x_1,x_2\rangle$ and $|x_2,x_1\rangle$), it is convenient to impose a convention for ordering coordinates (for instance, "write the leftmost particle first"[2]).

We assume a *hard-core* constraint $x_1 \neq x_2$, i.e. the particles cannot overlap. This removes the coincidence set from configuration space and prevents trajectories from passing through each other.

To study the statistics of two or more particles, it is convenient to introduce the notion of *configuration space*. By definition, the configuration space is a set (more precisely, a manifold) whose points correspond to classical configurations of the system, such as the set of particle positions.

For distinguishable particles, the configuration space is given by the Cartesian product of single-particle position spaces, with configurations related by particle labels regarded as distinct. For example, for two distinguishable particles in d dimensions,

$$C_{\text{dist}} = \{(\mathbf{r}_1,\mathbf{r}_2) \in \mathbb{R}^d \times \mathbb{R}^d \mid \mathbf{r}_1 \neq \mathbf{r}_2\}.$$

For identical particles, however, particle labels carry no physical meaning. Configurations that differ only by a permutation of particle labels must therefore be identified. As a result, the configuration space is obtained by taking the quotient with respect to the permutation group,

$$C_{\text{id}} = C_{\text{dist}}/S_2.$$

[2]This can be understood as $x_1 < x_2 < x_3 < \cdots$ in 1D. Similar conventions can be generalized to higher dimensions. Don't care too much about it – – they are just conventions.

Consequently, the configuration space for two identical particles is effectively "half" of that for distinguishable particles, reflecting the elimination of redundant labelings.[3]

The key topological insight is that when we consider *paths* through configuration space (equivalently, worldlines in spacetime), the set of all paths splits into *topologically inequivalent* classes: certain paths cannot be continuously deformed into each other while holding endpoints fixed and respecting the hard-core constraint.

• **Type of a two-particle path**

For two identical particles, we can distinguish (at a minimal level) between paths that exchange the particles and paths that do not[4]. We label the non-exchange class as TYPE +1 and the exchange class as TYPE −1. These two classes are disconnected under continuous deformations with fixed endpoints (given the non-overlap constraint). See Fig 1.

• **Composition of path types**

Paths can be composed: follow one path from i to m, then another from m to f. The induced "multiplication" of types satisfies

$$(+1)(+1) = +1, \qquad (+1)(-1) = -1, \qquad (-1)(+1) = -1, \qquad (-1)(-1) = +1. \tag{8}$$

In particular, an exchange followed by another exchange yields a net non-exchange path type.

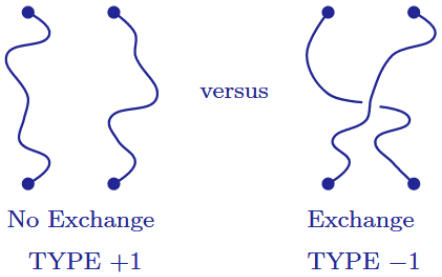

Figure 1: Types of paths between 2 identical particles

2.2 Two consistent path integrals and the meaning of statistics

The propagator between initial and final two-particle configurations can be written as a sum over all paths, weighted by $e^{iS/\hbar}$. Since the path space splits into TYPE +1 and TYPE −1, it is natural to write

$$\langle x_{1f}x_{2f}|\hat{U}(t_f,t_i)|x_{1i}x_{2i}\rangle = \mathcal{N}\left(\sum_{\text{TYPE }+1\text{ paths } i\to f} e^{iS[\text{path}]/\hbar} + \sum_{\text{TYPE }-1\text{ paths } i\to f} e^{iS[\text{path}]/\hbar}\right). \tag{9}$$

However, quantum mechanics allows a more general possibility: we may assign an additional factor to each topological class, provided it is constant within a class and consistent under composition. In particular, it is also consistent to choose

$$\langle x_{1f}x_{2f}|\hat{U}(t_f,t_i)|x_{1i}x_{2i}\rangle = \mathcal{N}\left(\sum_{\text{TYPE }+1\text{ paths } i\to f} e^{iS[\text{path}]/\hbar} - \sum_{\text{TYPE }-1\text{ paths } i\to f} e^{iS[\text{path}]/\hbar}\right). \tag{10}$$

[3]Correspondingly, the Hilbert space—where particle positions constitute one of the degrees of freedom—is also reduced relative to that of distinguishable particles.

[4]A careful reader may further wish to distinguish overcrossings from undercrossings, since in Fig. 1 they appear to be topologically inequivalent. We emphasize, however, that at this stage we have not yet specified the spatial dimension of the system, and it is therefore reasonable to keep the assumptions minimal. The purpose of this section is merely to provide an intuitive understanding that separating and counting different topological classes of paths can already lead to a self-consistent path-integral formulation.

The only difference is the relative sign assigned to the exchange class.

Supplementary derivation: why the minus sign preserves composition

Insert an intermediate time t_m and integrate over intermediate positions:

$$\langle f|\hat{U}(t_f,t_i)|i\rangle = \int dx_{1m}dx_{2m}\,\langle f|\hat{U}(t_f,t_m)|m\rangle\langle m|\hat{U}(t_m,t_i)|i\rangle.$$

If each factor is expanded according to (10), then each subpath carries a prefactor +1 if it is TYPE +1 and −1 if it is TYPE −1. When concatenating subpaths, the overall type is determined by (8). A full TYPE +1 path arises either from (+1)(+1) or (−1)(−1), in both cases yielding total prefactor (+1)(+1) = +1 or (−1)(−1) = +1. A full TYPE −1 path arises from (+1)(−1) or (−1)(+1), yielding total prefactor −1. Hence the sign assignment is stable under composition, exactly as required.

Equations (9) and (10) correspond, respectively, to bosonic and fermionic exchange behavior. Importantly, quantum mechanics itself does not select between these two options; the choice is a physical property of the particle species.

3 Many Identical Particles: Topological Classes of Paths

3.1 Goal and strategy

The goal is to determine which exchange statistics are self-consistent in quantum mechanics for N identical particles. Two ingredients are required.
⋄ First, we must characterize the topological structure of the path space through the configuration space C.
⋄ Second, we must impose consistency under composition: the assignment of statistical factors must respect the concatenation of paths so that the propagator obeys the composition law.

3.2 Configuration space for distinguishable and indistinguishable particles

For N *distinguishable* particles in D spatial dimensions, a configuration is specified by an ordered N-tuple $(x_1,\ldots,x_N) \in (\mathbb{R}^D)^N$. Imposing a hard-core condition that no two particles coincide removes the coincidence set

$$\Delta = \{(x_1,\ldots,x_N) \in (\mathbb{R}^D)^N \mid x_i = x_j \text{ for some } i \neq j\},$$

so the configuration space is $(\mathbb{R}^D)^N\backslash\Delta$.

For *indistinguishable* particles, order is unphysical. We impose an equivalence relation $\sim$ identifying all $N!$ permutations of the coordinates. The indistinguishable configuration space is then

$$C = \Big[(\mathbb{R}^D)^N\backslash\Delta\Big]/\sim . \tag{11}$$

This quotient reduces the "counting" of configurations by a factor $N!$, matching the familiar indistinguishability factor that appears in the Gibbs paradox in statistical mechanics.

The relationship between configuration space and Hilbert space should be understood as follows: a wavefunction for N identical particles is naturally a complex-valued function on C (possibly with additional internal indices), subject to appropriate regularity and boundary conditions. Different choices of statistics correspond to different consistency conditions for how the wavefunction behaves under nontrivial loops in C.

3.3 Path space, directed worldlines, and the fundamental group(oid)

A multi-particle history is specified by time-dependent coordinates $\{x_1(t), \ldots, x_N(t)\}$, interpreted with the understanding that particle labels are not physical. Geometrically, these histories can be visualized as N directed curves (worldlines) embedded in $(ND + 1)$-dimensional spacetime. We require the *"directed" condition*, which reflects that we do not allow worldlines to reverse in time. In another word, we are not yet considering pair creation/annihilation processes.

As in the two-particle case, the space of paths in C splits into topologically inequivalent components: with endpoints fixed, certain paths cannot be continuously deformed into each other without passing through forbidden configurations (e.g. particle coincidences) or violating the directed condition.

Closed paths (loops) based at a fixed configuration $c \in C$ form equivalence classes under homotopy. These classes assemble into the **first homotopy group** (also called the **fundamental group**)

$$\Pi_1(C, c), \tag{12}$$

with group multiplication given by path composition.

Strictly speaking, for propagators between *different* endpoints (initial and final configurations not equal), the most natural object is the **fundamental groupoid** of C, in which morphisms are homotopy classes of paths between arbitrary endpoints, and composition is defined only when endpoints match. For the purposes of classifying statistics, one often reduces to a group description by choosing reference paths that connect endpoints to a fixed base configuration; the physical meaning of this choice is discussed in Appendix A.

3.4 Paths in $(2 + 1)$ dimensions: the braid group

In two spatial dimensions, a path class of N particles corresponds to a **braid**. Fixing endpoints, braids may be continuously deformed without changing their topological class iff. strands are not cut through each other and do not reverse in time.

The braid group on N strands is denoted B_N. It is generated by elementary exchanges of neighboring strands. Let σ_i denote the clockwise exchange of strand i with strand $i + 1$ (for $i = 1, \ldots, N - 1$), and σ_i^{-1} the counterclockwise exchange, see Fig 2. Multiplication in B_N is given by stacking braids: performing one braid and then another corresponds to multiplying the corresponding group elements.

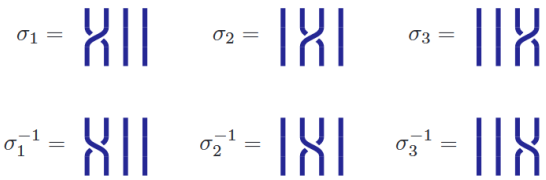

Figure 2: The three generating elements $\sigma_1, \sigma_2, \sigma_3$ of the braid group on four strands B_4, and their inverses.

A braid can be represented by a **braid word**, such as $\sigma_1\sigma_2\sigma_3^{-1}\sigma_1$, interpreted according to a fixed convention (e.g. rightmost generator acts first). A crucial point is that different braid words can represent the same braid, because the generators satisfy relations. See Fig 3 as an example. More details about the braid group can be found in Appendix B

Winding number as a braid invariant

A useful invariant for abelian statistics is the winding number $W(g)$, defined as

$$W(g) = (\#\text{ of over-crossings}) - (\#\text{ of under-crossings}), \tag{13}$$

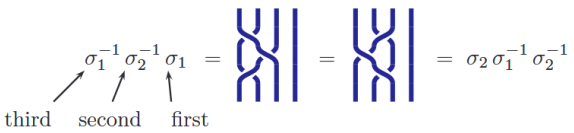

Figure 3: Different braid words can represent the same braid.

where, in a braid word representation, an over-crossing corresponds to a generator σ_i and an under-crossing to σ_i^{-1}. The winding number is invariant under continuous deformations that keep endpoints fixed and do not introduce or remove crossings; thus it depends only on the braid group element g, not on its particular word representation. Physically, $W(g)$ measures the net oriented exchange content of the braid, and it will parameterize the abelian anyon phase.

3.5 Paths in $(3+1)$ dimensions: the permutation group

In three spatial dimensions, a key geometric fact is that one-dimensional worldlines embedded in four-dimensional spacetime cannot form stable knots. Intuitively, whenever two strands appear "crossed" in a spacetime projection, one can separate them by moving one strand into the additional spatial dimension, untangling the crossing without cutting strands or violating the hard-core condition. Consequently, over-crossings and under-crossings are not topologically distinct in $(3+1)$ dimensions.

As a result, the only topological information remaining in a multi-particle history is the mapping from initial particle positions to final particle positions, i.e. a permutation of N objects. Therefore, the group of path classes is the **permutation group** S_N (also called the symmetric group).

Supplementary note: S_N as a truncation of B_N

One can view S_N as a quotient (a "truncation") of the braid group B_N obtained by identifying a double exchange with the identity. Concretely, in S_N one imposes the relation $\sigma_i^2 = \mathbf{1}$. Equivalently, $\sigma_i = \sigma_i^{-1}$, which expresses the fact that over- and under-crossings become equivalent once an extra spatial dimension is available to untangle crossings. This quotient precisely collapses the braid information down to permutations.

4 Building the Many-Particle Path Integral: Abelian Case

Suppose the position of particles uniquely determines the wavefunction, and you get the abelian case. You will understand the terminology "abelian" later, when we meet the non-abelian case later.

4.1 General form from homotopy invariance and composition

The path integral formulation sums over all spacetime histories with weights $e^{iS/\hbar}$. Quantum mechanics additionally allows prefactors depending on the *topological class* of a path. Motivated by the "two particle" discussion, we require the following two conditions.

• *Homotopy invariance*: the prefactor must be constant for all paths continuously deformable into each other with endpoints fixed.

• *Composition consistency*: when two paths are concatenated, the prefactor for the full path must equal the product of prefactors for the pieces, so that the propagator satisfies the composition law.

Let $\{x\}$ denote the unordered set of N particle coordinates. Let G be the group of path classes through configuration space, i.e. $G = \Pi_1(C)$ for closed loops, and operationally $G = S_N$ in $(3+1)$ dimensions or $G = B_N$ in $(2+1)$ dimensions. We then write the abelian path integral as

$$\langle\{x\}_f|\hat{U}(t_f,t_i)|\{x\}_i\rangle = \mathcal{N}\sum_{g\in G}\rho(g)\sum_{\text{paths}\in g}\exp\Big(\frac{i}{\hbar}S[\text{path}]\Big). \tag{14}$$

Here $\mathcal{N}$ is a normalization constant. The outer sum is over topological classes g, and the inner sum is over all paths within a fixed class. The factor $\rho(g)$ is a **scalar unitary representation** of G, i.e. a homomorphism $\rho : G \to U(1)$, so $|\rho(g)| = 1$ and

$$\rho(g_1)\rho(g_2) = \rho(g_1 g_2). \tag{15}$$

When initial and final configurations differ, one can still assign a class label g to each path by choosing a consistent procedure (e.g. a fixed viewing angle and a deformation to reference endpoints without introducing new crossings). The mathematically precise formulation uses the fundamental groupoid; see Appendix A for the conceptual picture and how such a reduction works.

Supplementary derivation: checking composition for Eq. (14)

Insert an intermediate time t_m and integrate over intermediate configurations $\{x\}_m$:

$$\langle f|\hat{U}(t_f,t_i)|i\rangle = \int d\{x\}_m\ \langle f|\hat{U}(t_f,t_m)|m\rangle\langle m|\hat{U}(t_m,t_i)|i\rangle.$$

Expanding each propagator by (14), we obtain a double sum over $g_1, g_2 \in G$ and over paths in those classes. Concatenating a path in class g_2 from $i \to m$ with a path in class g_1 from $m \to f$ produces a path from $i \to f$ in class $g_1 g_2$. The prefactor multiplies as $\rho(g_1)\rho(g_2)$, which equals $\rho(g_1 g_2)$ by (15). Therefore the full propagator retains the same form after composition, as required by quantum mechanics.

4.2 $(3+1)$ dimensions: only bosons and fermions

In $(3+1)$ dimensions, $G = S_N$. The scalar unitary representations of S_N are extremely limited.

Any one-dimensional representation is a group homomorphism $\rho : S_N \to U(1)$. Since S_N is generated by transpositions[5] τ with $\tau^2 = \mathbf{1}$, we must have

$$\rho(\tau)^2 = 1 \quad \Rightarrow \quad \rho(\tau) = \pm 1. \tag{16}$$

Moreover, all transpositions are conjugate in S_N, and a one-dimensional representation is constant on conjugacy classes. Hence either $\rho(\tau) = +1$ for all transpositions or $\rho(\tau) = -1$ for all transpositions. This yields exactly two possibilities.

If $\rho(g) = 1$ for all g, the path integral assigns no additional phase to exchanges: this is bosonic statistics.

If $\rho(g) = \text{sgn}(g) \in \{\pm 1\}$, where $\text{sgn}(g)$ is the *parity of the permutation*[6], then odd permutations contribute a minus sign: this is fermionic statistics.

Within the abelian framework of Eq. (14), there are no other distinct particle statistics in $(3+1)$ dimensions.

[5] A *transposition* is the permutation (ij) that exchanges the labels of particles i and j while leaving all others unchanged. It is an abstract group-theoretic operation, not a specification of a particular spatial exchange path.

[6] The parity of a permutation means whether the permutation is even or odd, according to how many pairwise exchanges (transpositions) are needed to build it. If odd times of transpositions are needed, the parity is odd; vice versa.

4.3 (2 + 1) dimensions: abelian anyons and fractional statistics

In (2 + 1) dimensions, $G = B_N$. The braid group admits nontrivial one-dimensional unitary representations parameterized by a continuous angle ϑ. A convenient characterization is

$$\rho(g) = \exp\Big(i\vartheta\, W(g)\Big), \tag{17}$$

where $W(g) \in \mathbb{Z}$ is the winding number of the braid g. Equivalently, one may specify the representation by sending each generator to the same phase,

$$\rho(\sigma_i) = e^{i\vartheta}, \qquad \rho(\sigma_i^{-1}) = e^{-i\vartheta}. \tag{18}$$

Thus a clockwise exchange accumulates phase $e^{i\vartheta}$, while a counterclockwise exchange accumulates $e^{-i\vartheta}$.

Special cases reproduce familiar statistics. When $\vartheta = 0$, the phase is trivial and we recover bosons. When $\vartheta = \pi$, we have $e^{i\pi} = e^{-i\pi} = -1$, so each exchange contributes a minus sign regardless of orientation, reproducing fermions.

For any other ϑ, the exchange phase is genuinely fractional, producing **abelian anyons** and fractional statistics. The key point is that quantum mechanics allows these phases because the topology of (2 + 1)-dimensional worldlines retains orientation-sensitive braiding information, encoded by B_N.

5 Nonabelian Case: Matrix Statistics and Braiding as Unitary Evolution

5.1 Motivation: degeneracy at fixed positions and beyond scalar representations

Up to this point, it may appear that specifying particle positions determines the many-body wavefunction uniquely (up to an overall phase). This need not be true. Even after fixing the particle positions $\{x\}$ and all local quantum numbers (such as spin or other internal labels), the system may retain an M-fold degeneracy. Familiar many-body examples include the toric code and other topological orders, where distinct global states can share identical local data.

From the path integral perspective, the abelian construction (14) used a one-dimensional representation of the path group G. If the state space over a fixed configuration is M-dimensional, it becomes natural and necessary to allow G to act by an M-dimensional unitary representation. This leads to nonabelian braiding statistics.

5.2 Hilbert space structure at fixed configuration

Assume that for a fixed set of particle positions $\{x\}$, the system has an M-dimensional degenerate space spanned by orthonormal basis states $|n; \{x\}\rangle$ with $n = 1, \ldots, M$. A general state at configuration $\{x\}$ is

$$|\psi_{\{x\}}\rangle = \sum_{n=1}^{M} A_n\, |n; \{x\}\rangle, \tag{19}$$

where $A_n \in \mathbb{C}$ are coefficients. Normalization imposes $\sum_n |A_n|^2 = 1$ when the basis is orthonormal. Conceptually, at fixed $\{x\}$ the wavefunction is a vector in an M-dimensional complex space.

5.3 Nonabelian path integral: matrix representation of the path group

With an internal M-dimensional degeneracy, we generalize Eq. (14) by allowing the topological prefactor to be an $M \times M$ unitary matrix $\rho(g)$. The propagator becomes a matrix element between an initial ket $|n'; \{x\}_i\rangle$ and a final bra $|n; \{x\}_f\rangle$:

$$\langle n; \{x\}_f | \hat{U}(t_f, t_i) | n'; \{x\}_i \rangle = \mathcal{N} \sum_{g \in G} [\rho(g)]_{n,n'} \sum_{\text{paths} \in g} \exp\left(\frac{i}{\hbar} S[\text{path}]\right). \tag{20}$$

Here $\rho(g) \in U(M)$ must form a representation:

$$\rho(g_1)\rho(g_2) = \rho(g_1 g_2). \tag{21}$$

The unitarity of $\rho(g)$ ensures probability conservation within the degenerate subspace.

Supplementary note: basis choices, diagonal action phases, and holonomy

Equation (20) implicitly assumes a convenient choice of basis $|n; \{x\}\rangle$ such that the dynamical weight $e^{iS[\text{path}]/\hbar}$ is diagonal in the internal indices along the path sum, and the nontrivial mixing between internal states is entirely attributed to the topological matrix factor $\rho(g)$. In well-behaved theories one can arrange such a separation by an appropriate gauge (basis) choice along configuration space, but different choices generally differ by position-dependent unitary transformations. The resulting dependence on closed loops is naturally described in terms of holonomy/monodromy: transporting the basis around a loop can produce a nontrivial unitary transformation, consistent with the representation structure of $\rho(g)$.

5.4 Adiabatic braiding and the action on the degenerate vector

Consider an experiment in which particles are moved along controlled trajectories $\{x(t)\}$, for instance by external traps. In a semiclassical (saddle-point) regime, the path integral is dominated by paths close to the chosen classical trajectories, and therefore by a single topological class $g \in G$ corresponding to that motion.

Suppose the particles are braided and returned to the same spatial configuration $\{x\}$. Since the endpoints match, the initial and final states lie in the same degenerate space spanned by $|n; \{x\}\rangle$. Writing the initial state as in (20), the effect of the braid g is that the coefficient vector transforms by the unitary matrix $\rho(g)$:

$$A_n^{(f)} = \sum_{n'} [\rho(g)]_{n,n'} A_{n'}^{(i)}. \tag{22}$$

A particle species whose exchanges act in this way is called a **nonabelian anyon**. The term "nonabelian" reflects the fact that the matrices $\rho(g)$ generally do not commute, so the order of braids matters.

Supplementary derivation

Physical setup. External controls (e.g. traps) guide the particles along a chosen trajectory $\{x(t)\}$. In an adiabatic/saddle-point regime, the path integral is dominated by paths close to this controlled motion, and therefore by a *single* topological class $g \in G$ associated with that braid.

$$\sum_{g' \in G} [\rho(g')]_{n,n'} \sum_{\text{paths} \in g'} e^{\frac{i}{\hbar} S[\text{path}]} \approx [\rho(g)]_{n,n'} \sum_{\text{paths} \in g} e^{\frac{i}{\hbar} S[\text{path}]}, \tag{23}$$

From the nonabelian propagator to a matrix action. Assume $\{x\}_f = \{x\}_i \equiv \{x\}$, so initial

and final states lie in the same M-dimensional degenerate space $\{|n;\{x\}\rangle\}_{n=1}^{M}$. Write

$$|\psi_{\{x\}}^{(i)}\rangle = \sum_{n'} A_{n'}^{(i)} |n';\{x\}\rangle, \qquad A_n^{(f)} = \sum_{n'} \langle n;\{x\}|\hat{U}(t_f,t_i)|n';\{x\}\rangle\, A_{n'}^{(i)}.$$

Using (20) and single-sector dominance(23),

$$\langle n;\{x\}|\hat{U}(t_f,t_i)|n';\{x\}\rangle \approx \mathcal{N}\,[\rho(g)]_{n,n'} \sum_{\text{paths}\in g} e^{\frac{i}{\hbar}S[\text{path}]}.$$

In the semiclassical approximation the remaining path sum in a *fixed* sector g produces a common scalar factor (independent of n, n'),

$$\mathcal{N} \sum_{\text{paths}\in g} e^{\frac{i}{\hbar}S[\text{path}]} \equiv e^{i\theta_g}.$$

Hence

$$A_n^{(f)} \approx e^{i\theta_g} \sum_{n'} [\rho(g)]_{n,n'}\, A_{n'}^{(i)}.$$

Dropping the overall phase $e^{i\theta_g}$ (irrelevant within the degenerate subspace) yields

$$A_n^{(f)} = \sum_{n'} [\rho(g)]_{n,n'}\, A_{n'}^{(i)}. \tag{24}$$

5.5 Quantum dimension and exponential growth of degeneracy

In many nonabelian anyon systems, the dimension M of the degenerate space grows exponentially with particle number N. One defines the **quantum dimension** d by the scaling relation

$$M \sim d^N, \qquad (3.11) \tag{25}$$

where $\sim$ indicates asymptotic scaling at large N. The regime $d > 1$ is characteristic of nontrivial fusion and braiding structure: adding more anyons increases the number of independent global states in a way that cannot be attributed to local degrees of freedom alone.

5.6 Braiding as quantum computation

Quantum computation can be viewed as the controlled application of unitary operations to a Hilbert space, together with initialization and measurement. Nonabelian braiding provides precisely such controlled unitaries: a braid g acts on the computational state vector A by

$$A^{(f)} = \rho(g)\, A^{(i)},$$

as in (24). If the set of matrices generated by braiding is sufficiently rich, one can implement quantum gates purely through braids, and thereby perform quantum computation in a way that is intrinsically tied to topological features of particle motion.

5.7 Parastatistics in $(3+1)$ dimensions and the role of extra constraints

A natural question is whether exotic nonabelian statistics can occur for point-like particles in $(3+1)$ dimensions. Mathematically, the permutation group S_N has higher-dimensional representations, suggesting the possibility of more complicated statistics, often referred to as **parastatistics**. However, subject to additional physical constraints, it is essentially not

possible to obtain fundamentally new particle statistics beyond bosons and fermions, possibly supplemented by internal degrees of freedom.

Two constraints are particularly important. First, one requires the ability to create and annihilate particle–antiparticle pairs and to allow processes such as fusion when particles are brought together. This enlarges the relevant structure beyond braiding alone and is naturally organized by categorical frameworks.

Second, one requires locality: operations performed far away (e.g. pair creation on a distant planet) should not affect the outcomes of local experiments here. Together these requirements severely restrict the allowable theories in $(3+1)$ dimensions and lead to the conclusion that point-like particle statistics reduce to bosonic or fermionic behavior with possible internal structure.

Despite this restriction, $(3+1)$ dimensions is not devoid of interesting topological phenomena. If one relaxes the assumption that excitations are point-like and instead considers extended objects such as loops or strings, one can encounter exotic statistical effects associated with the topology of world sheets and linking processes (for example, a loop passing through another loop, or a particle threading a loop). In such settings, the relevant topological classification transcends the simple permutation group picture.

A Fundamental groupoid and the physical meaning of "assigning a group element"

The fundamental group $\Pi_1(C, c_0)$ classifies homotopy classes of *closed* loops based at a fixed configuration $c_0 \in C$. In many physical situations, however, we are interested in propagators between *different* configurations c_i and c_f. The correct topological object that keeps track of homotopy classes of *open* paths is the **fundamental groupoid**, denoted $\Pi_1(C)$.

In the groupoid picture, the *objects* are points of C, and the *morphisms* are homotopy classes of paths between them. Composition is defined only when the endpoint of one path matches the start point of the next. The fundamental group at a basepoint is recovered as the automorphism group of that object:

$$\Pi_1(C, c_0) = \mathrm{Hom}_{\Pi_1(C)}(c_0, c_0).$$

Physically, when the initial and final configurations differ, one often still wants to label a path by an element of a fixed group G (such as S_N or B_N). This can be done after making an additional choice that plays the role of a gauge fixing.

• **Reference paths and reduction to a group element**

Choose a base configuration $c_0 \in C$. For each configuration c, choose a reference path $\eta_c : c_0 \to c$. Given an open path $\gamma : c_i \to c_f$, define a loop at c_0 by

$$\widetilde{\gamma} = \eta_{c_f}^{-1} \circ \gamma \circ \eta_{c_i}, \tag{26}$$

whose homotopy class is an element of $\Pi_1(C, c_0)$. In this way, γ is assigned a group element $g = [\widetilde{\gamma}\,] \in G$.

The **key idea** is to deform the spacetime path so that endpoints lie at reference positions without introducing new crossings. Different choices of η_c generally change g by conjugation in the group.

Reference-path dependence and conjugation

Suppose that for the same configuration c we choose two different reference paths,

$$\eta_c \qquad \text{and} \qquad \eta'_c.$$

They necessarily differ by a closed loop based at the reference configuration c_0,

$$\eta'_c = \lambda_c \circ \eta_c, \qquad \lambda_c \in \Pi_1(C, c_0).$$

Now assign a group element to the same physical path $\gamma : c_i \to c_f$ using these two choices. Using η_c, one obtains

$$g = \left[\eta_{c_f}^{-1} \circ \gamma \circ \eta_{c_i}\right].$$

Using η'_c, one obtains

$$g' = \left[(\eta'_{c_f})^{-1} \circ \gamma \circ \eta'_{c_i}\right].$$

Substituting $\eta'_c = \lambda_c \circ \eta_c$ immediately gives

$$g' = \lambda_{c_f}^{-1}\, g\, \lambda_{c_i}.$$

In the special case of a closed process, $c_i = c_f$, this reduces to

$$g' = \lambda^{-1} g \lambda,$$

showing that different choices of reference paths modify the assigned group element only by conjugation.

• **Why this subtlety is mild for abelian statistics and important for nonabelian statistics**

In the abelian case, $\rho(g) \in U(1)$ is a scalar. Conjugation does not change $\rho(g)$, so the assignment is robust against many details of the reference choice.

In the nonabelian case, $\rho(g) \in U(M)$ is matrix-valued. Changing reference paths corresponds to a change of basis in the degenerate internal space and can modify $\rho(g)$ by conjugation with a unitary matrix. This is not an inconsistency; it reflects the gauge freedom in choosing the basis $|n; \{x\}\rangle$ along configuration space. Physical predictions are invariant under such basis changes, but keeping track of them conceptually motivates the groupoid language and its connection to holonomy/monodromy.

B More on the Braid Group B_N and the Permutation Group S_N — Solution to Ex 3.1&3.2

In this section, we follow **Exercise 3.1&3.2**, further understanding the structure of the Braid Group B_N and the Permutation Group S_N.

B.1 Structure of the Braid group

We have seen that the braid group B_N is generated by the elementary exchanges of neighboring strands. More precisely, B_N admits the standard generating set $\{\sigma_i\}_{i=1}^{N-1}$, where σ_i denotes a (say, clockwise) exchange of strand i with strand $i+1$. A general braid can be represented by a *braid word*, i.e. a product of generators and their inverses. Crucially, different braid words may represent the same group element; hence, one is naturally led to ask for local relations among braid words that leave the underlying braid unchanged up to continuous deformation (with endpoints fixed, and without cutting strands through each other).

• **Far-commutativity** For $|i - j| > 1$, one has the relation

$$\sigma_i \sigma_j = \sigma_j \sigma_i. \tag{27}$$

This is geometrically evident: in a braid diagram, the crossing of σ_i occurs in a small "time window" and a small horizontal neighborhood involving only strands i and $i+1$, while σ_j occurs in a disjoint neighborhood involving only strands j and $j+1$. Since these neighborhoods do not overlap, one may continuously "push" one crossing upward/downward in time without ever passing through the other crossing. Thus $\sigma_i\sigma_j$ and $\sigma_j\sigma_i$ represent the same braid.

• **The three-strand local relation (Yang–Baxter)** A second fundamental relation (stated without proof here) is

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}. \qquad (3.12) \qquad (28)$$

A geometric proof can be found in Fig. 4: restricting attention to the three involved strands $i, i+1, i+2$, both sides describe the same topological braid. Visually, the top layer corresponds to strand i sliding to the $i+2$ position, the middle layer can be straightened so that strand $i+1$ goes to $i+1$, and the bottom layer corresponds to strand $i+2$ sliding to i, hence the equality.

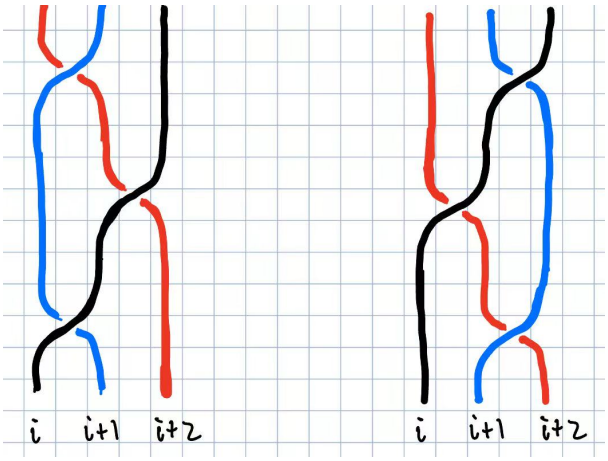

Figure 4: Geometric illustration of the Yang–Baxter braid relation.

• **Sphere Identity** If the particles live on a sphere[7], there is an additional identity:

$$\sigma_1\sigma_2\cdots\sigma_{M-2}\,\sigma_{M-1}\sigma_{M-1}\,\sigma_{M-2}\cdots\sigma_2\sigma_1 \;=\; I. \qquad (29)$$

A proof is sketched in Fig. 5; for conciseness we only draw the three-strand case. Physically, the left-hand side corresponds to moving the first particle around the remaining $M-1$ particles once and returning it to its original position. Following the red dashed guide in the figure, one can deform the worldlines along a large loop on the sphere and continuously reduce the braid to the identity.

The origin of this extra relation lies in the topological difference between the plane and the sphere. On the plane (or disk), there is a distinguished "point at infinity" (equivalently, a boundary), so a loop encircling all punctures can be topologically nontrivial: it cannot be shrunk without crossing that point at infinity. On the sphere, there is no such distinguished infinity; a loop encircling all punctures can be deformed to the complementary loop on the other side and is contractible in the appropriate configuration space picture. This removes one global topological obstruction and produces the additional constraint (29) in $B_M(S^2)$.

Note that Equation (29) only holds for spinless particles. Recall that the worldline of a particle should be thought of as a belt or a ribbon, rather than a "line". The winding shown

[7] In this case, the braid group of M strands on the sphere is written as $B_M(S^2)$. To think about braids on a sphere, it is useful to think of time as being the radial direction of the sphere.

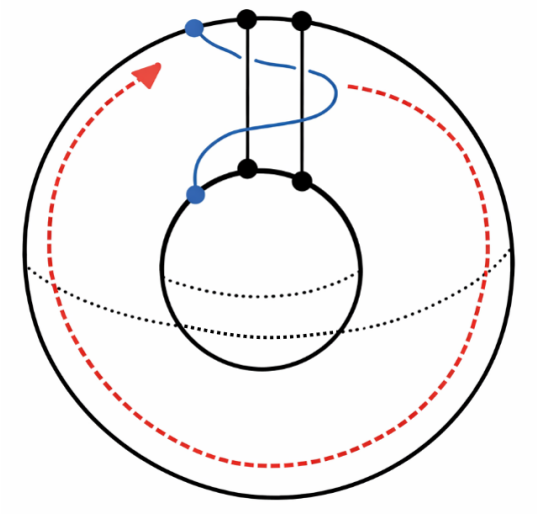

Figure 5: Geometric sketch of the sphere relation (illustrated for three strands).

in Fig. 5 is accompanied with a half-rotation of the first particle (blue line) itself. Thus, for particles with a spin, the true identity is

$$\sigma_1\sigma_2\cdots\sigma_{M-2}\,\sigma_{M-1}\sigma_{M-1}\,\sigma_{M-2}\cdots\sigma_2\sigma_1 \;=\; e^{i\Phi}. \tag{30}$$

B.2 Structure of the Permutation group S_N

Let S_M be the permutation (symmetric) group on M objects, generated by adjacent transpositions σ_i that exchange labels i and $i+1$. It is straightforward to verify that the generators of S_M obey the same *far-commutativity* relation as in the braid group: if $|i-j|>1$, then $\sigma_i\sigma_j = \sigma_j\sigma_i$, since the two transpositions act on disjoint sets of labels and therefore commute.

Moreover, the generators also satisfy the three-strand local relation (3.12). We record a direct verification below.

Verifying the three-strand relation in S_M

The relation only involves the three labels $i, i+1, i+2$, so it suffices to check that both sides act identically on these labels (all other labels are fixed). Concretely, in S_3 let

$$\sigma_1 = (1\,2), \qquad \sigma_2 = (2\,3).$$

Then

$$\sigma_1\sigma_2\sigma_1 = (1\,2)(2\,3)(1\,2) = (1\,3),$$

and similarly

$$\sigma_2\sigma_1\sigma_2 = (2\,3)(1\,2)(2\,3) = (1\,3).$$

Hence $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ in S_3, and by relabeling this implies (3.12) for general i in S_M.

We now classify the one-dimensional (scalar) unitary representations $\rho : S_M \to U(1)$. Let $a_i \equiv \rho(\sigma_i) \in U(1)$. In S_M, each generator is an involution: $\sigma_i^2 = 1$. Applying ρ and using multiplicativity,

$$1 = \rho(1) = \rho(\sigma_i^2) = \rho(\sigma_i)^2 = a_i^2,$$

so

$$a_i = \pm 1 \qquad \text{for every } i. \tag{31}$$

Next, apply ρ to the three-strand relation (3.12):

$$\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho(\sigma_{i+1}\sigma_i\sigma_{i+1}),$$

which gives

$$a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}.$$

Since these are complex numbers, they commute, and thus

$$a_i^2 a_{i+1} = a_{i+1}^2 a_i.$$

Using $a_i^2 = a_{i+1}^2 = 1$, we conclude

$$a_{i+1} = a_i \qquad \text{for all } i. \tag{32}$$

Therefore all a_i are equal: either $a_i = +1$ for every i, or $a_i = -1$ for every i. These are precisely the two scalar unitary representations of S_M: the *trivial representation* $\rho(g) = 1$ (corresponding to bosons), and the *sign representation* $\rho(g) = \mathrm{sgn}(g) \in \{\pm 1\}$ (corresponding to fermions), where $\mathrm{sgn}(g) = -1$ for odd permutations and $+1$ for even permutations.

C Ising Anyons and Majorana Fermions — Solution to Ex.3.3

Non-Abelian anyons are characterized by the fact that, even after fixing the quasiparticle positions, the quantum state is not uniquely determined: there remains an intrinsically protected degeneracy. The Ising anyon (often denoted σ) is the canonical example. In a Majorana-zero-mode realization (e.g. vortices in a 2D topological superconductor), each σ carries a localized Majorana operator.

In this section we model $2N$ Ising anyons by $2N$ Majorana operators $\{\gamma_i\}_{i=1}^{2N}$ satisfying the Majorana algebra

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \qquad \gamma_i^\dagger = \gamma_i. \tag{33}$$

We assume the anyons are well separated and non-interacting, so that the effective Hamiltonian is taken to be $H = 0$. Consequently, the entire Hilbert space is degenerate in energy, and the nontrivial content is in the braid-group action.

C.1 Ground-state degeneracy from pairing Majoranas

A convenient way to enumerate the Hilbert space is to **pair neighboring Majoranas into ordinary (Dirac) fermions**. For each $k = 1, \ldots, N$, define

$$c_k \equiv \frac{1}{2}\big(\gamma_{2k-1} + i\gamma_{2k}\big), \qquad c_k^\dagger \equiv \frac{1}{2}\big(\gamma_{2k-1} - i\gamma_{2k}\big). \tag{34}$$

These satisfy canonical fermion anticommutation relations, hence each mode k contributes a two-level system labeled by the occupation number $n_k = c_k^\dagger c_k \in \{0, 1\}$.

Derivation of $\{c_k, c_\ell\} = 0$ and $\{c_k, c_\ell^\dagger\} = \delta_{k\ell}$

Using $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$, for $k \neq \ell$ all Majoranas in c_k anticommute with those in c_ℓ, hence $\{c_k, c_\ell\} = 0$ and $\{c_k, c_\ell^\dagger\} = 0$.
For $k = \ell$, write $a \equiv 2k - 1$ and $b \equiv 2k$. Then

$$c_k = \frac{1}{2}(\gamma_a + i\gamma_b), \qquad c_k^\dagger = \frac{1}{2}(\gamma_a - i\gamma_b). \tag{35}$$

Compute

$$\begin{aligned}\{c_k, c_k^\dagger\} &= \frac{1}{4}\Big\{(\gamma_a + i\gamma_b), (\gamma_a - i\gamma_b)\Big\} \\ &= \frac{1}{4}\Big(\{\gamma_a, \gamma_a\} - i\{\gamma_a, \gamma_b\} + i\{\gamma_b, \gamma_a\} + \{\gamma_b, \gamma_b\}\Big) \\ &= \frac{1}{4}(2 + 0 + 0 + 2) = 1, \end{aligned} \tag{36}$$

and similarly

$$\begin{aligned}\{c_k, c_k\} &= \frac{1}{4}\Big\{(\gamma_a + i\gamma_b), (\gamma_a + i\gamma_b)\Big\} \\ &= \frac{1}{4}\Big(\{\gamma_a, \gamma_a\} + i\{\gamma_a, \gamma_b\} + i\{\gamma_b, \gamma_a\} - \{\gamma_b, \gamma_b\}\Big) = 0. \end{aligned} \tag{37}$$

Thus $\{c_k, c_\ell\} = 0$ and $\{c_k, c_\ell^\dagger\} = \delta_{k\ell}$.

Therefore the Hilbert space dimension is 2^N. With $H = 0$, all these states share the same energy, so the ground-state degeneracy [8] is 2^N.

C.2 Braiding operators and the braid-group relations

A convenient starting point is the anti-Hermitian Majorana bilinear

$$K_{ij} \equiv \gamma_i \gamma_j \qquad (i \neq j). \tag{38}$$

Using the Majorana algebra, one immediately finds

$$K_{ij}^\dagger = -K_{ij}, \qquad K_{ij}^2 = -1. \tag{39}$$

Thus K_{ij} behaves as an "imaginary unit" acting on the two-dimensional subspace spanned by γ_i and γ_j: exponentiating it generates a continuous $SO(2)$ rotation in the (γ_i, γ_j) plane. Indeed, using $K_{ij}^2 = -1$, we have

$$e^{\frac{\theta}{2}K_{ij}} = \cos\frac{\theta}{2} + K_{ij}\sin\frac{\theta}{2}. \tag{40}$$

Its action on Majoranas is most naturally described by conjugation. One finds

$$e^{\frac{\theta}{2}K_{ij}}\gamma_i e^{-\frac{\theta}{2}K_{ij}} = \gamma_i\cos\theta - \gamma_j\sin\theta, \tag{41}$$

$$e^{\frac{\theta}{2}K_{ij}}\gamma_j e^{-\frac{\theta}{2}K_{ij}} = \gamma_j\cos\theta + \gamma_i\sin\theta, \tag{42}$$

$$e^{\frac{\theta}{2}K_{ij}}\gamma_a e^{-\frac{\theta}{2}K_{ij}} = \gamma_a \qquad (a \neq i, j). \tag{43}$$

In particular, at $\theta = \pi/2$,

$$e^{\frac{\pi}{4}K_{ij}}\gamma_i e^{-\frac{\pi}{4}K_{ij}} = -\gamma_j, \qquad e^{\frac{\pi}{4}K_{ij}}\gamma_j e^{-\frac{\pi}{4}K_{ij}} = \gamma_i, \qquad e^{\frac{\pi}{4}K_{ij}}\gamma_a e^{-\frac{\pi}{4}K_{ij}} = \gamma_a\ (a \neq i, j), \tag{44}$$

which is analogous to an exchange between the ith and jth particle.

Motivated by this intuition, a clockwise exchange of two Majorana-carrying anyons i and j can be implemented (up to an overall exchange phase) by a $\pi/2$ rotation generated by K_{ij}. This motivates the choice

$$U_{ij} \equiv e^{i\alpha}\, e^{\frac{\pi}{4}K_{ij}} = \frac{e^{i\alpha}}{\sqrt{2}}\Big(1 + \gamma_i\gamma_j\Big). \tag{45}$$

[8] before imposing any superselection constraint

Unitarity is immediate from anti-Hermiticity:

$$U_{ij}^{\dagger} = e^{-i\alpha} e^{-\frac{\pi}{4} K_{ij}} = \frac{e^{-i\alpha}}{\sqrt{2}} \Big(1 - \gamma_i \gamma_j\Big), \qquad U_{ij} U_{ij}^{\dagger} = \mathbf{1}. \tag{46}$$

For the braid-group representation it suffices to consider nearest-neighbor generators,

$$U_i \equiv U_{i,i+1} = e^{i\alpha} e^{\frac{\pi}{4} \gamma_i \gamma_{i+1}} = \frac{e^{i\alpha}}{\sqrt{2}} \Big(1 + \gamma_i \gamma_{i+1}\Big), \qquad i = 1, \ldots, 2N-1. \tag{47}$$

To rigorously confirm that $\{U_i\}$ can be seen as the generator of a braid group, we need to check *far-commutativity* and the *Yang–Baxter relation*.
∘**Far-commutativity:** $U_i U_j = U_j U_i$ **for** $|i-j| > 1$ If $|i-j| > 1$, then the sets $\{i, i+1\}$ and $\{j, j+1\}$ are disjoint. Using the Majorana algebra, one checks that K_i commutes with K_j:

$$K_i K_j = (\gamma_i \gamma_{i+1})(\gamma_j \gamma_{j+1}) = (\gamma_j \gamma_{j+1})(\gamma_i \gamma_{i+1}) = K_j K_i, \tag{48}$$

because swapping γ_j past $\gamma_i \gamma_{i+1}$ produces two minus signs, and similarly for γ_{j+1}. Hence

$$U_i U_j = \frac{e^{2i\alpha}}{2} (1 + K_i)(1 + K_j) = \frac{e^{2i\alpha}}{2} (1 + K_j)(1 + K_i) = U_j U_i. \tag{49}$$

This is the braid relation $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| > 1$.
∘**Yang–Baxter relation:** The Yang–Baxter relation is

$$U_i U_{i+1} U_i = U_{i+1} U_i U_{i+1}. \tag{50}$$

Since both sides contain the same overall phase factor $e^{3i\alpha}$, it suffices to verify the relation for $V_i \equiv \frac{1}{\sqrt{2}}(1 + K_i)$, namely

$$(1 + K_i)(1 + K_{i+1})(1 + K_i) = (1 + K_{i+1})(1 + K_i)(1 + K_{i+1}). \tag{51}$$

We leave the detailed verification in the box below.

Verification of the Yang–Baxter relation

Let

$$K_i = \gamma_i \gamma_{i+1}, \qquad K_{i+1} = \gamma_{i+1} \gamma_{i+2}. \tag{52}$$

Then $K_i^2 = K_{i+1}^2 = -1$, and

$$K_i K_{i+1} = (\gamma_i \gamma_{i+1})(\gamma_{i+1} \gamma_{i+2}) = \gamma_i (\gamma_{i+1}^2) \gamma_{i+2} = \gamma_i \gamma_{i+2}, \tag{53}$$

$$K_{i+1} K_i = (\gamma_{i+1} \gamma_{i+2})(\gamma_i \gamma_{i+1}) = -\gamma_{i+1} \gamma_i \gamma_{i+2} \gamma_{i+1} = -\gamma_i \gamma_{i+2}. \tag{54}$$

Hence $K_i K_{i+1} = -K_{i+1} K_i$.
Expand the left-hand side of (3.12):

$$\begin{aligned} (1 + K_i)(1 + K_{i+1})(1 + K_i) &= (1 + K_i + K_{i+1} + K_i K_{i+1})(1 + K_i) \\ &= 1 + 2K_i + K_{i+1} + K_{i+1} K_i + K_i K_{i+1} + K_i K_{i+1} K_i. \end{aligned} \tag{55}$$

Using $K_{i+1} K_i = -K_i K_{i+1}$, the middle terms cancel:

$$(1 + K_i)(1 + K_{i+1})(1 + K_i) = 1 + 2K_i + K_{i+1} + K_i K_{i+1} K_i. \tag{56}$$

Similarly,

$$(1 + K_{i+1})(1 + K_i)(1 + K_{i+1}) = 1 + 2K_{i+1} + K_i + K_{i+1} K_i K_{i+1}. \tag{57}$$

It remains to compare the cubic terms. Compute

$$K_iK_{i+1}K_i = (\gamma_i\gamma_{i+1})(\gamma_{i+1}\gamma_{i+2})(\gamma_i\gamma_{i+1}) = (\gamma_i\gamma_{i+2})(\gamma_i\gamma_{i+1})$$
$$= \gamma_i\gamma_{i+2}\gamma_i\gamma_{i+1} = -(\gamma_i^2)\gamma_{i+2}\gamma_{i+1} = -\gamma_{i+2}\gamma_{i+1} = \gamma_{i+1}\gamma_{i+2} = K_{i+1}, \tag{58}$$

and likewise

$$K_{i+1}K_iK_{i+1} = (\gamma_{i+1}\gamma_{i+2})(\gamma_i\gamma_{i+1})(\gamma_{i+1}\gamma_{i+2}) = (-\gamma_i\gamma_{i+2})(\gamma_{i+1}\gamma_{i+2})$$
$$= -\gamma_i\gamma_{i+2}\gamma_{i+1}\gamma_{i+2} = +\gamma_i\gamma_{i+1}(\gamma_{i+2}^2) = K_i. \tag{59}$$

Therefore the two expanded expressions coincide:

$$1 + 2K_i + K_{i+1} + K_{i+1} = 1 + K_i + 2K_{i+1} + K_i, \tag{60}$$

which proves Eq. 51 and hence the Yang–Baxter relation.

A short summary of majorana identities

At the end of this subsection, we list (again) several useful identities about majorana operators.

$$K_i \equiv \gamma_i\gamma_{i+1}, \quad V_i = \frac{1}{\sqrt{2}}(1 + K_i) = \exp\Big(\frac{\pi}{4}K_i\Big).$$

$$[K_i, \gamma_i] = -2\gamma_{i+1}, \quad [K_i, \gamma_{i+1}] = 2\gamma_i, \quad [K_i, \gamma_k] = 0 \quad (|i-k| > 1) \tag{61}$$

$$V_i\gamma_iV_i^\dagger = -\gamma_{i+1}, \qquad V_i\gamma_{i+1}V_i^\dagger = \gamma_i, \qquad V_i\gamma_kV_i^\dagger = \gamma_k \quad (|i-k| > 1). \tag{62}$$

$$V_i^2 = K_i = \gamma_i\gamma_{i+1} \tag{63}$$

C.3 Fermion parity and the operator γ^{FIVE}

Define the fermion-parity operator

$$\gamma^{\text{FIVE}} \equiv (i)^N \gamma_1\gamma_2\cdots\gamma_{2N}. \tag{64}$$

It is often most transparent to rewrite it in terms of the Dirac fermions $c_k, c_k^\dagger$:

$$\gamma^{\text{FIVE}} = (-1)^{n_1+\cdots+n_N}, \qquad \text{where } n_i = c_i^\dagger c_i. \tag{65}$$

See the box below for a derivation.

Rewriting γ^{FIVE} as total fermion parity

For $k = 1$, the inverse relations are

$$\gamma_1 = c_1 + c_1^\dagger, \qquad \gamma_2 = -i(c_1 - c_1^\dagger). \tag{66}$$

Then

$$i\gamma_1\gamma_2 = i(c_1 + c_1^\dagger)\Big[-i(c_1 - c_1^\dagger)\Big]$$
$$= (c_1 + c_1^\dagger)(c_1 - c_1^\dagger) = -c_1c_1^\dagger + c_1^\dagger c_1 = 1 - 2c_1^\dagger c_1 = 1 - 2n_1. \tag{67}$$

Similarly, for each k,

$$i\gamma_{2k-1}\gamma_{2k} = 1 - 2n_k. \tag{68}$$

Therefore

$$\gamma^{\mathrm{FIVE}} = (i)^N \prod_{k=1}^{N} \gamma_{2k-1}\gamma_{2k} = \prod_{k=1}^{N} \left(i\gamma_{2k-1}\gamma_{2k}\right) = \prod_{k=1}^{N} (1-2n_k) = (-1)^{n_1+\cdots+n_N}. \tag{69}$$

Hence γ^{FIVE} measures the total fermion parity.

In particular, γ^{FIVE} has eigenvalues ± 1 and splits the Hilbert space into two parity sectors,

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \qquad \gamma^{\mathrm{FIVE}}|\psi_\pm\rangle = \pm|\psi_\pm\rangle, \qquad \dim \mathcal{H}_\pm = 2^{N-1}. \tag{70}$$

A crucial fact is that γ^{FIVE} is an invariant under braiding operations,

$$[\gamma^{\mathrm{FIVE}}, U_i] = 0 \quad , \forall U_i. \tag{71}$$

We leave the derivavtion in the following box.

γ^{FIVE} commutes with all braids

It suffices to show $[\gamma^{\mathrm{FIVE}}, U_i] = 0$ for generators U_i, and since U_i is a polynomial in $K_i = \gamma_i\gamma_{i+1}$, it suffices to show $[\gamma^{\mathrm{FIVE}}, K_i] = 0$.
Write

$$\gamma^{\mathrm{FIVE}} = (i)^N \gamma_1 \cdots \gamma_{i-1} \left(\gamma_i\gamma_{i+1}\right) \gamma_{i+2} \cdots \gamma_{2N}. \tag{72}$$

Because $K_i = \gamma_i\gamma_{i+1}$ commutes with every γ_k for $k \neq i, i+1$, we have[a]

$$\begin{aligned} K_i\gamma^{\mathrm{FIVE}} &= (i)^N (\gamma_1 \cdots \gamma_{i-1})\, K_i (\gamma_i\gamma_{i+1})\, (\gamma_{i+2} \cdots \gamma_{2N}) \\ &= (i)^N (\gamma_1 \cdots \gamma_{i-1})\, (\gamma_i\gamma_{i+1})^2\, (\gamma_{i+2} \cdots \gamma_{2N}), \end{aligned} \tag{73}$$

and similarly

$$\begin{aligned} \gamma^{\mathrm{FIVE}} K_i &= (i)^N (\gamma_1 \cdots \gamma_{i-1})\, (\gamma_i\gamma_{i+1})\, (\gamma_{i+2} \cdots \gamma_{2N})\, K_i \\ &= (i)^N (\gamma_1 \cdots \gamma_{i-1})\, (\gamma_i\gamma_{i+1})^2\, (\gamma_{i+2} \cdots \gamma_{2N}), \end{aligned} \tag{74}$$

so $K_i\gamma^{\mathrm{FIVE}} = \gamma^{\mathrm{FIVE}} K_i$. Hence $[\gamma^{\mathrm{FIVE}}, U_i] = 0$ for all i.

[a] Use Eq. (61),$[K_i, \gamma_i] = -2\gamma_{i+1}, \quad [K_i, \gamma_{i+1}] = 2\gamma_i, \quad [K_i, \gamma_k] = 0 \quad (|i-k| > 1)$

From the above discussion, we reach a key result that *the representation of the braiding group B_{2N} is reducible in the whole Hilbert space $\mathcal{H}$*. See the box below for a more rigorous explanation.

Representation-theoretic meaning using Schur's lemma

Let ρ be the braid-group representation on $\mathcal{H}$. If ρ were irreducible, then any operator commuting with $\rho(B_{2N})$ would have to be proportional to the identity (Schur's lemma). Since γ^{FIVE} commutes with all braid generators but is not a scalar multiple of the identity (it has eigenvalues ± 1), the representation is necessarily reducible. Concretely, $\mathcal{H}$ decomposes into invariant blocks $\mathcal{H}_\pm$ labeled by total fermion parity.

Fixing the total topological charge (equivalently, fixing total fermion parity) restricts the physically accessible non-Abelian degeneracy from 2^N to 2^{N-1}. Braiding alone cannot change the eigenvalue of γ^{FIVE}.

C.4 Four Ising anyons on the sphere: one qubit, but braiding is not universal

Consider $2N = 4$ Majoranas $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, i.e. $N = 2$ Dirac modes c_1, c_2. In many physical settings (e.g. a superconducting island) the total fermion parity is fixed. Take $\gamma^{\mathrm{FIVE}} = +1$, i.e.

$$(-1)^{n_1+n_2} = +1 \quad \Longleftrightarrow \quad n_1 + n_2 \text{ is even.} \tag{75}$$

Thus the accessible two-dimensional space is

$$\mathcal{H}_+ = \mathrm{span}\Big\{|00\rangle,\ |11\rangle\Big\}. \tag{76}$$

This is naturally identified as a single logical qubit.

Within this fixed-parity sector, Majorana bilinears realize Pauli operators (up to conventional signs). A convenient choice is

$$\hat{Z} \equiv i\gamma_1\gamma_2, \qquad \hat{X} \equiv i\gamma_2\gamma_3, \qquad \hat{Y} \equiv i\gamma_1\gamma_3. \tag{77}$$

In the basis $\{|00\rangle, |11\rangle\}$, one finds[9]

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{X} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_x, \qquad \hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y, \tag{78}$$

and $\hat{Z} = i\gamma_3\gamma_4$ acts identically on $\mathcal{H}_+$ because in the even-parity sector $n_2 = n_1$ on $\{|00\rangle, |11\rangle\}$.

The braiding generators (removing a global phase) are

$$V_i = \frac{1}{\sqrt{2}}(1 + \gamma_i\gamma_{i+1}) = \exp\Big(\frac{\pi}{4}\gamma_i\gamma_{i+1}\Big), \qquad U_i = e^{i\alpha}V_i. \tag{79}$$

Since $\gamma_i\gamma_{i+1} = -i(i\gamma_i\gamma_{i+1})$, in the fixed-parity qubit space the action is a $\pi/2$ rotation about the corresponding Pauli axis (up to a global phase):

$$U_1 \propto \exp\Big(-i\frac{\pi}{4}\hat{Z}\Big), \qquad U_2 \propto \exp\Big(-i\frac{\pi}{4}\hat{X}\Big), \qquad U_3 \propto \exp\Big(-i\frac{\pi}{4}\hat{Z}\Big). \tag{80}$$

Therefore the group generated by $\{U_1, U_2, U_3\}$ is generated by $\pi/2$ rotations about two orthogonal axes. This is a discrete subgroup of $SU(2)$ (in fact, it realizes the single-qubit Clifford group up to phases), hence it cannot equal $SU(2)$ and cannot be dense in $SU(2)$. Consequently, braiding four Ising anyons does not approximate arbitrary single-qubit unitaries.

C.5 Spherical braid relation and the exchange phase α

On the sphere there is an additional global relation in the braid group (braids on S^2); schematically,

$$\sigma_1\sigma_2\cdots\sigma_{2N-2}\,\sigma_{2N-1}^2\,\sigma_{2N-2}\cdots\sigma_2\sigma_1 = e^{i\Phi}, \tag{81}$$

where the phase Φ encodes the intrinsic spin/topological spin data.[10]

Replace $\sigma_i \mapsto U_i$ and define the corresponding operator

$$S \equiv U_1U_2\cdots U_{2N-2}\,U_{2N-1}^2\,U_{2N-2}\cdots U_2U_1. \tag{82}$$

[9] To derive these matrices, one should use the relation $\gamma_{2k-1} = c_k + c_k^\dagger, \qquad \gamma_{2k} = -i(c_k - c_k^\dagger)$, expanding these operators as products of fermion creation and annihilation operators. We leave the explicit derivation to the readers.

[10] In a topological phase, such global relations are controlled by topological spins and central elements of the mapping class group. For Ising anyons, the topological spin of σ is $\theta_\sigma = e^{i\pi/8}$, which will be consistent with the value of α derived below.

Factor out the exchange phase by writing $U_i = e^{i\alpha} V_i$ with

$$V_i \equiv \frac{1}{\sqrt{2}}(1 + \gamma_i \gamma_{i+1}). \tag{83}$$

Counting factors gives

$$S = e^{i(4N-2)\alpha} \left(V_1 V_2 \cdots V_{2N-2}\, V_{2N-1}^2\, V_{2N-2} \cdots V_1 \right). \tag{84}$$

Using $V_{2N-1}^2 = \gamma_{2N-1}\gamma_{2N}$, one can show the central identity

$$V_1 V_2 \cdots V_{2N-2}\, V_{2N-1}^2\, V_{2N-2} \cdots V_1 = e^{-i\frac{\pi N}{2}}\, \gamma^{\mathrm{FIVE}}. \tag{85}$$

Proof of Eq. (85) by induction

Define the palindromic words

$$S_m \equiv (V_1 V_2 \cdots V_m)(V_m \cdots V_2 V_1). \tag{86}$$

One proves by induction that for even $m = 2k$,

$$S_{2k} = \gamma_2 \gamma_3 \cdots \gamma_{2k+1}. \tag{87}$$

During our proof, we use repeatedly the identity $V_i^2 = K_i$ and $V_i \gamma_{i+1} = \gamma_i V_i$, see Eq. (63).
The base case $k = 1$ is

$$S_2 = V_1 V_2^2 V_1 = V_1(\gamma_2\gamma_3)V_1 = \gamma_2\gamma_3, \tag{88}$$

using $V_1\gamma_2 = \gamma_1 V_1$, $V_1^2 = \gamma_1\gamma_2$, and $[\gamma_3, V_1] = 0$.
For the induction step, write

$$S_{2k+2} = (V_1 \cdots V_{2k}) \Big(V_{2k+1} V_{2k+2}^2 V_{2k+1} \Big) (V_{2k} \cdots V_1), \tag{89}$$

note $V_{2k+2}^2 = \gamma_{2k+2}\gamma_{2k+3}$ and

$$V_{2k+1}(\gamma_{2k+2}\gamma_{2k+3})V_{2k+1} = \gamma_{2k+2}\gamma_{2k+3}, \tag{90}$$

while $\gamma_{2k+2}, \gamma_{2k+3}$ commute with $V_1, \ldots, V_{2k}$. Hence

$$S_{2k+2} = S_{2k}\,(\gamma_{2k+2}\gamma_{2k+3}) = \gamma_2 \cdots \gamma_{2k+1}\gamma_{2k+2}\gamma_{2k+3}. \tag{91}$$

Now apply this with $m = 2N - 2$ to obtain

$$(V_1 \cdots V_{2N-2})(V_{2N-2} \cdots V_1) = \gamma_2\gamma_3 \cdots \gamma_{2N-1}. \tag{92}$$

Finally,

$$\begin{aligned} V_1 \cdots V_{2N-2}\, V_{2N-1}^2\, V_{2N-2} \cdots V_1 &= (V_1 \cdots V_{2N-2})(\gamma_{2N-1}\gamma_{2N})(V_{2N-2} \cdots V_1) \\ &= \gamma_1 \Big[(V_1 \cdots V_{2N-2})(V_{2N-2} \cdots V_1) \Big] \gamma_{2N} \\ &= \gamma_1(\gamma_2 \cdots \gamma_{2N-1})\gamma_{2N} = \gamma_1\gamma_2 \cdots \gamma_{2N}. \end{aligned} \tag{93}$$

Since $e^{-i\frac{\pi N}{2}} \gamma^{\mathrm{FIVE}} = \gamma_1\gamma_2 \cdots \gamma_{2N}$, this yields Eq. (85).

Therefore,

$$S = e^{i(4N-2)\alpha}\, e^{-i\frac{\pi N}{2}}\, \gamma^{\mathrm{FIVE}}. \tag{94}$$

Working in a fixed parity sector $\gamma^{\mathrm{FIVE}} = \pm 1$, the operator S must reduce to a pure phase, identified with $e^{i\Phi}$. Requiring that α be a property of the exchange operation itself (hence independent of N) forces the N-dependent phase to cancel:

$$e^{i(4N-2)\alpha}\, e^{-i\frac{\pi N}{2}} = e^{i\Phi} \quad \text{for all } N. \tag{95}$$

This implies

$$4\alpha - \frac{\pi}{2} = 0 \quad \Longrightarrow \quad \alpha = \frac{\pi}{8}, \tag{96}$$

and then

$$e^{i\Phi} = e^{-i2\alpha} = e^{-i\frac{\pi}{4}} \quad \Longrightarrow \quad \Phi = -\frac{\pi}{4} \pmod{2\pi}. \tag{97}$$

Thus the exchange phase for Ising anyons is fixed to $\alpha = \pi/8$, consistent with the standard Ising topological spin data.

D Small numbers of anyons on a sphere — Solution to Ex.3.4

The topology of configuration space on a sphere imposes additional global constraints on braiding processes, leading to a drastic reduction of the braid group for a small number of particles.

D.1 Two particles

For two particles on the plane, the braid group is $B_2(\mathbb{R}^2) \cong \mathbb{Z}$, reflecting the fact that one particle can wind around the other an arbitrary number of times. On a sphere, however, such windings can be continuously unwound by sliding the trajectory around the back of the sphere.

Algebraically, the spherical braid group for two particles is generated by a single exchange σ_1 subject to the additional spherical relation

$$\sigma_1^2 = 1. \tag{98}$$

Therefore,

$$B_2(S^2) \cong \mathbb{Z}_2. \tag{99}$$

There are only two topologically distinct processes: the trivial one and a single exchange.

D.2 Three particles

For three particles on a sphere, the braid group is generated by σ_1, σ_2, satisfying the usual braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, together with the spherical constraint

$$\sigma_1\sigma_2^2\sigma_1 = 1. \tag{100}$$

The resulting group $B_3(S^2)$ is finite, with order 12. It is a well-known discrete group, isomorphic to the dicyclic group Dic_3 (equivalently, a semidirect product $\mathbb{Z}_4 \rtimes \mathbb{Z}_3$).

D.3 Consequences for non-Abelian statistics

Suppose that the ground state of two or three anyons on a sphere is N-fold degenerate. Braiding acts on this degenerate space through a unitary representation of the corresponding spherical braid group. Since $B_2(S^2)$ and $B_3(S^2)$ are finite discrete groups, their unitary images are necessarily finite subsets of $U(N)$.

Consequently, braiding alone cannot generate arbitrary continuous rotations in $SU(2)$ (or more generally $SU(N)$) on the degenerate ground-state manifold. At best, one obtains a finite set of discrete unitary transformations. This sharply contrasts with the planar case, where infinite braid groups may allow dense subgroups of $SU(N)$.

References

[1] S. Simon, *Quantum Topology*, Chap 3, Oxford University Press (2016).