# Spinor BEC

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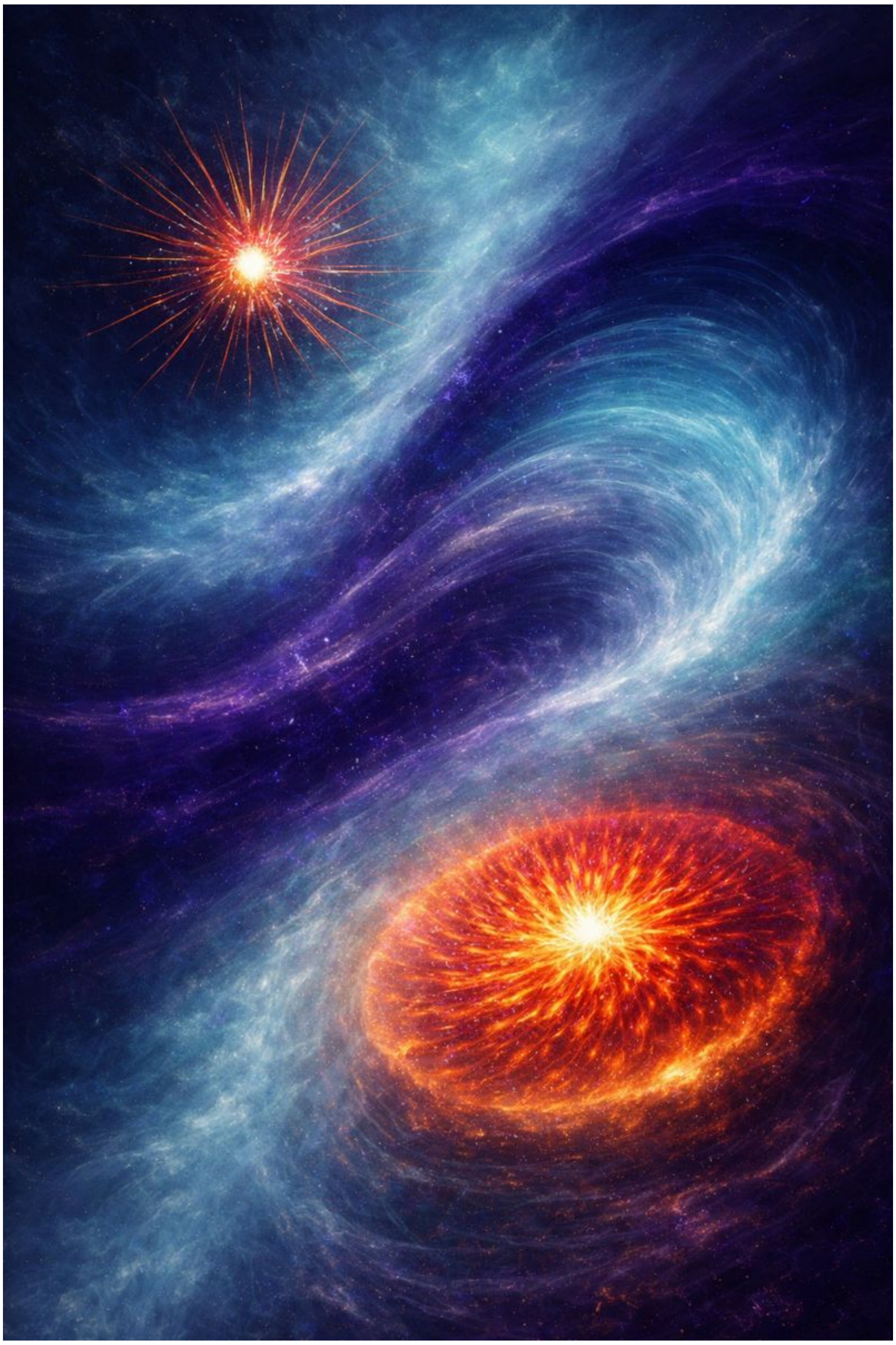

# Contents

# 1 Majorana Stellar Representation for Spin-$F$

**Why we need a new geometric representation beyond the Bloch sphere.**

For a spin-$\frac{1}{2}$ state, a normalized two-component spinor

$$\zeta = \begin{pmatrix} \zeta_{1/2} \\ \zeta_{-1/2} \end{pmatrix}, \qquad \zeta^\dagger \zeta = 1,$$

has two real degrees of freedom after modding out the overall phase, and is faithfully represented by a point on the Bloch sphere $S^2$.

For a general spin-$F$ state, however, the wavefunction in the $|F, m\rangle$ basis is a $(2F + 1)$-component normalized spinor

$$|\psi\rangle = \sum_{m=-F}^{F} \psi_m |F, m\rangle, \qquad \sum_{m=-F}^{F} |\psi_m|^2 = 1.$$

After removing the overall phase degree of freedom and the normalization constraint, the state has $2(2F + 1) - 2 = 4F$ real degrees of freedom. A single point on $S^2$ is no longer sufficient. The Majorana stellar representation (MSR) provides a geometric description in terms of $2F$ points on $S^2$.

**Physical picture: "a spin-$F$ state as $2F$ spin-$\frac{1}{2}$ directions"**

The Majorana representation rewrites a general spin-$F$ state as a *symmetrized* product of $2F$ spin-$\frac{1}{2}$ spinors. Each spin-$\frac{1}{2}$ spinor determines a direction on the Bloch sphere. Thus, a spin-$F$ state is encoded by a constellation of $2F$ points on $S^2$ (counting multiplicity).

**Schwinger-boson representation of $SU(2)$.**

Introduce two bosonic modes $\hat{a}, \hat{b}$ with

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1, \qquad [\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^\dagger] = 0.$$

Define spin operators

$$\hat{F}_x = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}), \qquad \hat{F}_y = \frac{1}{2\mathrm{i}}(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}), \qquad \hat{F}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}).$$

One checks directly that $[\hat{F}_\mu, \hat{F}_\nu] = \mathrm{i}\,\epsilon_{\mu\nu\lambda} \hat{F}_\lambda$.

**Remark: why Schwinger bosons are natural here**

A spin-$F$ irrep is realized as the subspace with fixed total boson number $\hat{N} = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} = 2F$. The $|F, m\rangle$ basis corresponds to distributing $2F$ bosons between $\hat{a}$ and $\hat{b}$.

**Basis states $|F, m\rangle$ as bosonic Fock states.**

Let $|0\rangle$ be the vacuum of $\hat{a}, \hat{b}$. Then the normalized spin basis states are

$$|F, m\rangle = \frac{1}{\sqrt{(F + m)!(F - m)!}} (\hat{a}^\dagger)^{F+m} (\hat{b}^\dagger)^{F-m} |0\rangle.$$

This satisfies $\hat{F}_z |F, m\rangle = m|F, m\rangle$ and $\hat{\mathbf{F}}^2 |F, m\rangle = F(F + 1)|F, m\rangle$.

**From a spinor to a homogeneous polynomial.**

To avoid confusion with later Euler angles $(\alpha, \beta, \gamma)$, we use $(x, y)$ as polynomial variables. Given amplitudes $\{\psi_m\}$, define the degree-$2F$ homogeneous polynomial in two commuting variables $(x, y)$:

$$P_\psi(x, y) \equiv \sum_{m=-F}^{F} \frac{\psi_m}{\sqrt{(F+m)!(F-m)!}} x^{F+m} y^{F-m}. \tag{1}$$

The corresponding quantum state is obtained by the substitution $(x, y) \mapsto (\hat{a}^\dagger, \hat{b}^\dagger)$:

$$|\psi\rangle = P_\psi(\hat{a}^\dagger, \hat{b}^\dagger)|0\rangle. \tag{2}$$

Thus, the spin state is equivalently encoded by a homogeneous polynomial.

**Majorana factorization – stars and constellation.**

Over $\mathbb{C}$, any homogeneous degree-$2F$ polynomial in two variables factorizes into linear factors. Therefore, up to an overall (physically irrelevant) normalization,

$$P_\psi(x, y) = C \prod_{i=1}^{2F} \left(u_i x + v_i y\right), \qquad (u_i, v_i) \in \mathbb{C}^2,\ C \in \mathbb{C}^\times. \tag{3}$$

This factorization leads to the following precise definitions.

- **Majorana star.** A single factor $(u_i x + v_i y)$ defines a ray $[u_i : v_i] \in \mathbb{CP}^1$, i.e. $(u_i, v_i) \sim \lambda(u_i, v_i)$ with $\lambda \neq 0$. Choose a representative spin-$\frac{1}{2}$ spinor

$$\chi_i = \frac{1}{\sqrt{|u_i|^2 + |v_i|^2}} \begin{pmatrix} u_i \\ v_i \end{pmatrix},$$

  and map it to a point on the Bloch sphere via the Hopf/Bloch map

$$\hat{\mathbf{n}}_i \equiv \chi_i^\dagger \boldsymbol{\sigma} \chi_i = \left( \frac{2\mathrm{Re}(u_i^* v_i)}{|u_i|^2 + |v_i|^2}, \frac{2\mathrm{Im}(u_i^* v_i)}{|u_i|^2 + |v_i|^2}, \frac{|u_i|^2 - |v_i|^2}{|u_i|^2 + |v_i|^2} \right) \in S^2. \tag{4}$$

  This $\hat{\mathbf{n}}_i$ is the *star*.

- **Majorana constellation.** The unordered multiset of all $2F$ stars (counting multiplicity) is the constellation:

$$C(\psi) \equiv \{\hat{\mathbf{n}}_1, \ldots, \hat{\mathbf{n}}_{2F}\}. \tag{5}$$

**A minimal "how-to"**

**Step 1:** form $P_\psi(x, y)$ from $\{\psi_m\}$ via Eq. (1).
**Step 2:** factor $P_\psi(x, y)$ over $\mathbb{C}$ into $2F$ linear factors as in Eq. (3).
**Step 3:** for each factor, normalize $\chi_i \propto (u_i, v_i)^\top$ and compute $\hat{\mathbf{n}}_i = \chi_i^\dagger \boldsymbol{\sigma} \chi_i$ via Eq. (4).

We then briefly comment on the uniqueness of MSR. The constellation is defined up to permutations of the $2F$ factors. Repeated factors produce coincident stars (multiplicity $> 1$). Overall scaling of each $(u_i, v_i)$ is irrelevant because $[u_i : v_i] \in \mathbb{CP}^1$ is projective; the global phase of $|\psi\rangle$ is also irrelevant and does not affect $C(\psi)$.

**How symmetries act – rotations and time reversal.**

Two key properties make MSR powerful:

- Under a spin rotation $\hat{U} \in SU(2)$, each spin-$\frac{1}{2}$ spinor $\chi_i$ transforms as $\chi_i \mapsto \hat{U}\chi_i$, hence each star rotates rigidly:

$$\hat{U} : \ \hat{\mathbf{n}}_i \mapsto R\,\hat{\mathbf{n}}_i, \qquad R \in SO(3).$$

  Therefore the whole constellation rotates as a rigid body.

- Under time reversal $\hat{\mathcal{T}}$ (complex conjugation plus a $\pi$-rotation in spin space), the Bloch vector flips sign:

$$\hat{\mathcal{T}}: \ \hat{\mathbf{n}}_i \mapsto -\hat{\mathbf{n}}_i,$$

so the constellation undergoes central inversion $C(\psi) \mapsto -C(\psi)$.

**Physical use: reading off isotropy (stabilizer) groups**

A state is invariant (up to a global phase) under a symmetry operation iff its constellation is invariant *as a set* under the corresponding geometric action on $S^2$. Group-theory questions about the spinor become geometry of point sets on a sphere.

**Examples (spin-1 with explicit derivation; spin-2 for intuition).**
**(i) Spin-1 ($F = 1$): two stars from a quadratic polynomial.**

A general spin-1 state $|\psi\rangle = \psi_1|1,1\rangle + \psi_0|1,0\rangle + \psi_{-1}|1,-1\rangle$ corresponds to

$$P_\psi(x,y) = \frac{\psi_1}{\sqrt{2!0!}}x^2 + \frac{\psi_0}{\sqrt{1!1!}}xy + \frac{\psi_{-1}}{\sqrt{0!2!}}y^2 = \frac{\psi_1}{\sqrt{2}}x^2 + \psi_0 xy + \frac{\psi_{-1}}{\sqrt{2}}y^2.$$

It is often convenient to clear the harmless overall factor $\sqrt{2}$ and write equivalently

$$\widetilde{P}_\psi(x,y) \equiv \sqrt{2}\,P_\psi(x,y) = \psi_1 x^2 + \sqrt{2}\,\psi_0 xy + \psi_{-1}y^2.$$

Now factor $\widetilde{P}_\psi$ into two linear factors:

$$\widetilde{P}_\psi(x,y) = C\,(u_1x + v_1y)(u_2x + v_2y).$$

A direct procedure is: if $\psi_1 \neq 0$, choose representatives with $u_1 = u_2 = 1$, i.e.

$$\widetilde{P}_\psi(x,y) \propto (x + r_1y)(x + r_2y),$$

and matching coefficients gives

$$r_1 + r_2 = \frac{\sqrt{2}\,\psi_0}{\psi_1}, \qquad r_1r_2 = \frac{\psi_{-1}}{\psi_1}.$$

Hence $r_{1,2}$ are the two solutions of the quadratic equation

$$r^2 - \frac{\sqrt{2}\,\psi_0}{\psi_1}r + \frac{\psi_{-1}}{\psi_1} = 0,$$

so each star can be represented by the spin-$\frac{1}{2}$ spinor $\chi_i \propto (1, r_i)^{\mathsf{T}}$, and its direction is $\hat{\mathbf{n}}_i = \chi_i^\dagger \boldsymbol{\sigma}\chi_i$.

Three important special cases are immediate:

- $|1,1\rangle$ ($\psi_1 = 1, \psi_0 = \psi_{-1} = 0$):

$$\widetilde{P} = x^2 = (x + 0\cdot y)^2 \quad \Rightarrow \quad \chi_1 = \chi_2 = \begin{pmatrix}1\\0\end{pmatrix},$$

two coincident stars at the north pole.

- $|1,-1\rangle$ ($\psi_{-1} = 1, \psi_1 = \psi_0 = 0$):

$$\widetilde{P} = y^2 = (0\cdot x + y)^2 \quad \Rightarrow \quad \chi_1 = \chi_2 = \begin{pmatrix}0\\1\end{pmatrix},$$

two coincident stars at the south pole.

- $|1,0\rangle$ ($\psi_0 = 1, \psi_1 = \psi_{-1} = 0$):

$$\widetilde{P} = \sqrt{2}\,xy = (x)(y) \quad \Rightarrow \quad \chi_1 = \begin{pmatrix}1\\0\end{pmatrix},\ \chi_2 = \begin{pmatrix}0\\1\end{pmatrix},$$

two antipodal stars (north and south), giving a “headless” nematic axis $\mathbf{n} \sim -\mathbf{n}$.

**Physical meaning (spin-1)**

For $F = 1$, the constellation always consists of two points on $S^2$. Coincident stars correspond to a spin-coherent (fully polarized) state. An antipodal pair corresponds to a polar/nematic configuration where only an unoriented director is well-defined. Generic states give two distinct, non-antipodal points and interpolate between these limits.

**(ii) Spin-2 ($F = 2$): four stars.**

Spin-2 states correspond to degree-4 homogeneous polynomials and hence to four stars on $S^2$. Spin-coherent states have four coincident stars; more structured phases can yield highly symmetric constellations (e.g. tetrahedral-like patterns), making discrete stabilizer subgroups visually immediate.

# 2 Spin-1 Spinor BEC: Mean-Field Energy Functional and Phases

## 2.1 Microscopic interaction and the spinor order parameter

**Spin-dependent contact interaction for spin-1 bosons.**

We consider bosonic atoms with single-particle hyperfine spin $f = 1$ (three Zeeman components $m = 1, 0, -1$). At ultracold temperatures, collisions are dominated by $s$-wave scattering. Spin rotation symmetry at (approximately) zero magnetic field implies that scattering is diagonal in the *total spin* $F_{\text{tot}}$ channel of two atoms.

For two spin-1 bosons, the symmetric spin channels allowed in $s$-wave are $F_{\text{tot}} = 0, 2$ (the $F_{\text{tot}} = 1$ spin wavefunction is antisymmetric and thus excluded for identical bosons in $s$-wave). Denote the corresponding scattering lengths by $a_0$ and $a_2$.

A standard way to write the effective contact interaction is

$$\hat{V}(\mathbf{r}_{12}) = \frac{4\pi\hbar^2}{m}\Big(a^{(n)} + a^{(s)}\,\hat{\mathbf{F}}_1 \cdot \hat{\mathbf{F}}_2\Big)\delta(\mathbf{r}_{12}), \tag{6}$$

where $\hat{\mathbf{F}}$ are the spin-1 matrices acting on the internal space of a single atom. The coefficients are related to $(a_0, a_2)$ by

$$a^{(n)} = \frac{a_0 + 2a_2}{3}, \qquad a^{(s)} = \frac{a_2 - a_0}{3}. \tag{7}$$

Equivalently, one often defines coupling constants

$$c_0 = \frac{4\pi\hbar^2}{m}a^{(n)}, \qquad c_2 = \frac{4\pi\hbar^2}{m}a^{(s)}. \tag{8}$$

**Remark: from channel projectors to $a^{(n)} + a^{(s)}\,\hat{\mathbf{F}}_1 \cdot \hat{\mathbf{F}}_2$**

Here $\hat{\mathbf{F}}_1 \cdot \hat{\mathbf{F}}_2$ is a two-body operator acting on the tensor-product spin space $\mathcal{H}_{\text{spin}} = \mathbb{C}^3 \otimes \mathbb{C}^3$, where

$$\hat{\mathbf{F}}_1 = \hat{\mathbf{F}} \otimes \mathbb{I}, \qquad \hat{\mathbf{F}}_2 = \mathbb{I} \otimes \hat{\mathbf{F}}, \qquad \hat{\mathbf{F}}_1 \cdot \hat{\mathbf{F}}_2 = \sum_{\mu=x,y,z} \hat{F}_\mu \otimes \hat{F}_\mu.$$

Spin-rotation invariance implies that the $s$-wave contact interaction is diagonal in the total-spin channel of the pair, so the most basic form is

$$\hat{V} = \frac{4\pi\hbar^2}{m}\,\delta(\mathbf{r}_{12})\Big(a_0\,\hat{P}_0 + a_2\,\hat{P}_2\Big),$$

where $\hat{P}_F$ is the projector onto the total-spin-$F$ subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$. For identical spin-1 bosons in the $s$-wave channel, only the symmetric sectors $F = 0, 2$ are allowed; on this restricted space

$$\hat{P}_F^2 = \hat{P}_F, \qquad \hat{P}_0\hat{P}_2 = 0, \qquad \hat{P}_0 + \hat{P}_2 = \mathbb{I}.$$

To construct $\hat{P}_0, \hat{P}_2$, use a rotationally invariant operator with distinct eigenvalues on different total-spin sectors. A convenient choice is

$$\hat{X} \equiv \hat{\mathbf{F}}_1 \cdot \hat{\mathbf{F}}_2 = \frac{1}{2}\Big(\hat{\mathbf{F}}_{\text{tot}}^2 - \hat{\mathbf{F}}_1^2 - \hat{\mathbf{F}}_2^2\Big), \qquad \hat{\mathbf{F}}_{\text{tot}} = \hat{\mathbf{F}}_1 + \hat{\mathbf{F}}_2.$$

For spin-1, $\hat{\mathbf{F}}_1^2 = \hat{\mathbf{F}}_2^2 = 2$, hence

$$\hat{X} = \frac{1}{2}\Big(\hat{\mathbf{F}}_{\text{tot}}^2 - 4\Big).$$

On a state with definite total spin $F$, $\hat{X}$ is a scalar:

$$\hat{X} \to \frac{1}{2}\Big(F(F+1) - 4\Big) = \begin{cases} -2, & F = 0, \\ 1, & F = 2. \end{cases}$$

Since only two sectors are present, each projector must be a linear function of $\hat{X}$. Imposing that $\hat{P}_0$ acts as 1 on $F = 0$ and 0 on $F = 2$ (and vice versa for $\hat{P}_2$) gives

$$\hat{P}_0 = \frac{1 - \hat{X}}{3}, \qquad \hat{P}_2 = \frac{2 + \hat{X}}{3},$$

valid within the $F = 0, 2$ sector. Substituting yields

$$a_0\hat{P}_0 + a_2\hat{P}_2 = \frac{a_0 + 2a_2}{3} + \frac{a_2 - a_0}{3}\,\hat{X} = a^{(n)} + a^{(s)}\,\hat{\mathbf{F}}_1 \cdot \hat{\mathbf{F}}_2,$$

which reproduces Eq. (7).

**Second-quantized Hamiltonian (field-operator form).**

Let $\hat{\Psi}_m(\mathbf{r})$ annihilate a boson in Zeeman component $m \in \{1, 0, -1\}$. Define the density and spin density operators

$$\hat{n}(\mathbf{r}) = \sum_m \hat{\Psi}_m^\dagger(\mathbf{r})\hat{\Psi}_m(\mathbf{r}), \qquad \hat{\mathbf{S}}(\mathbf{r}) = \sum_{m,m'} \hat{\Psi}_m^\dagger(\mathbf{r})\,\mathbf{F}_{mm'}\,\hat{\Psi}_{m'}(\mathbf{r}).$$

Then a widely used microscopic Hamiltonian is

$$\hat{H} = \int d^3\mathbf{r}\left[\sum_m \hat{\Psi}_m^\dagger\left(-\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r})\right)\hat{\Psi}_m + \frac{c_0}{2}\hat{n}^2 + \frac{c_2}{2}\hat{\mathbf{S}}^2\right]. \tag{9}$$

This form makes the physical content explicit: $c_0$ controls density-density repulsion, $c_2$ controls the energetic preference for local spin alignment.

**Short recap: second quantization of one-body spin operators**

A one-body operator $\hat{O}^{(1)} = \sum_{m,m'} |m\rangle\, O_{mm'}\, \langle m'|$ is promoted to

$$\hat{O} = \int d^3r \sum_{m,m'} \hat{\Psi}_m^\dagger(\mathbf{r})\, O_{mm'}\, \hat{\Psi}_{m'}(\mathbf{r}).$$

Applying this to $\hat{\mathbf{F}}$ yields the local spin density operator $\hat{\mathbf{S}}(\mathbf{r})$ used in Eq. (9).

### 2.2 Mean-field ansatz and the meaning of $\zeta$

**Condensate order parameter as a classical spinor field.**

In mean-field theory, we replace $\hat{\Psi}_m(\mathbf{r})$ by a $c$-number condensate wavefunction $\psi_m(\mathbf{r})$. Collect them into a spinor

$$\boldsymbol{\psi}(\mathbf{r}) = \begin{pmatrix} \psi_1(\mathbf{r}) \\ \psi_0(\mathbf{r}) \\ \psi_{-1}(\mathbf{r}) \end{pmatrix}, \qquad \int d^3\mathbf{r}\, \boldsymbol{\psi}^\dagger(\mathbf{r})\boldsymbol{\psi}(\mathbf{r}) = N.$$

A convenient decomposition is

$$\boldsymbol{\psi}(\mathbf{r}) = \sqrt{n(\mathbf{r})}\, e^{\mathrm{i}\theta(\mathbf{r})}\, \zeta(\mathbf{r}), \qquad \zeta^\dagger(\mathbf{r})\zeta(\mathbf{r}) = 1. \tag{10}$$

Here $n(\mathbf{r})$ is the total density, $\theta(\mathbf{r})$ is the overall condensate phase, and $\zeta(\mathbf{r})$ is the normalized internal (spinor) wavefunction.

**Physical meaning of $\zeta$: "internal state per particle"**

At each point $\mathbf{r}$, $\zeta(\mathbf{r})$ encodes the relative populations and relative phases among the three components $m = 1, 0, -1$. The overall amplitude $\sqrt{n}$ and the global phase $e^{\mathrm{i}\theta}$ have been factored out. In this sense, $\zeta$ is the local internal quantum state of a typical condensed particle.

**Spin-1 matrices and how to compute $\langle \mathbf{F} \rangle$.**

We use the standard spin-1 representation of $SO(3)$ generators:

$$F_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Given a normalized spinor $\zeta$, define the local spin expectation

$$\langle \mathbf{F} \rangle = \zeta^\dagger \mathbf{F}\, \zeta, \qquad |\langle \mathbf{F} \rangle|^2 = \sum_{\mu = x,y,z} \left(\zeta^\dagger F_\mu \zeta\right)^2. \tag{11}$$

**Worked examples: plugging in spin-1 spinors (ferromagnetic vs. polar)**

We illustrate how to compute the spin expectation value $\langle \mathbf{F} \rangle = \zeta^\dagger \mathbf{F}\, \zeta$ by two explicit spin-1 examples in the $(m = 1, 0, -1)$ basis, using

$$F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad F_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad F_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}.$$

**Example 1 (ferromagnetic):** $\zeta = (1, 0, 0)^\mathsf{T}$. This is the pure $m = 1$ component. We have

$$\zeta^\dagger F_z \zeta = (1, 0, 0)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1.$$

Next,

$$F_x\zeta = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \zeta^\dagger F_x \zeta = (1, 0, 0) \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0,$$

and similarly $F_y\zeta = \frac{1}{\sqrt{2}}(0, \mathrm{i}, 0)^\top$ gives $\zeta^\dagger F_y \zeta = 0$. Therefore

$$\langle \mathbf{F} \rangle = (0, 0, 1), \qquad |\langle \mathbf{F} \rangle| = 1,$$

showing full spin polarization.

**Example 2 (polar):** $\zeta = (0, 1, 0)^\top$**.** This is the pure $m = 0$ component. First,

$$\zeta^\dagger F_z \zeta = (0, 1, 0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0.$$

Moreover,

$$F_x\zeta = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \zeta^\dagger F_x \zeta = (0, 1, 0) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0,$$

and

$$F_y\zeta = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i} \\ 0 \\ \mathrm{i} \end{pmatrix} \quad \Rightarrow \quad \zeta^\dagger F_y \zeta = (0, 1, 0) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i} \\ 0 \\ \mathrm{i} \end{pmatrix} = 0.$$

Hence

$$\langle \mathbf{F} \rangle = (0, 0, 0), \qquad |\langle \mathbf{F} \rangle| = 0,$$

which characterizes the polar (nematic) state: it has no net magnetization even though it is not a scalar.

## 2.3 Deriving the mean-field energy functional (detailed)

**Start from the field-theory Hamiltonian and take the mean-field expectation.**

Replace $\hat{\Psi}_m \to \psi_m$. The kinetic+trap part becomes

$$E_{\mathrm{sp}} = \int d^3\mathbf{r}\, \boldsymbol{\psi}^\dagger(\mathbf{r}) \left( -\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) \right) \boldsymbol{\psi}(\mathbf{r}).$$

For interactions:

$$E_{\mathrm{int}} = \int d^3\mathbf{r} \left[ \frac{c_0}{2} n^2(\mathbf{r}) + \frac{c_2}{2} \mathbf{S}^2(\mathbf{r}) \right],$$

where now

$$n(\mathbf{r}) = \boldsymbol{\psi}^\dagger \boldsymbol{\psi}, \qquad \mathbf{S}(\mathbf{r}) = \boldsymbol{\psi}^\dagger \mathbf{F}\, \boldsymbol{\psi}.$$

**Rewrite interaction energy in terms of $\zeta$ and $\langle \mathbf{F} \rangle$.**

Using Eq. (10),

$$\mathbf{S}(\mathbf{r}) = \boldsymbol{\psi}^\dagger \mathbf{F}\, \boldsymbol{\psi} = n(\mathbf{r})\, \zeta^\dagger(\mathbf{r}) \mathbf{F}\, \zeta(\mathbf{r}) = n(\mathbf{r})\, \langle \mathbf{F} \rangle(\mathbf{r}),$$

so

$$\mathbf{S}^2(\mathbf{r}) = n^2(\mathbf{r})\, |\langle \mathbf{F} \rangle(\mathbf{r})|^2.$$

**Final mean-field energy functional.**

Putting pieces together,

$$E[\boldsymbol{\psi}] = \int d^3\mathbf{r} \left[ \boldsymbol{\psi}^\dagger \left( -\frac{\hbar^2 \nabla^2}{2m} + V \right) \boldsymbol{\psi} + \frac{c_0}{2} n^2 + \frac{c_2}{2} n^2 |\langle \mathbf{F} \rangle|^2 \right]. \tag{12}$$

■ **Two energy scales.**
The density interaction $\propto c_0 n^2$ penalizes density depletion, while the spin interaction $\propto c_2 n^2 |\langle \mathbf{F} \rangle|^2$ selects the internal spin structure. Typically $|c_2| \ll c_0$, meaning spin textures can vary on longer length scales than density.

## 2.4 Uniform system: phase distinction and physical meaning

**Assume a uniform condensate.**
Set $V(\mathbf{r}) = 0$, $n(\mathbf{r}) = n_0$ constant, and ignore gradients. Then the (interaction) energy density is

$$\mathcal{E} = \frac{c_0}{2} n_0^2 + \frac{c_2}{2} n_0^2 |\langle \mathbf{F} \rangle|^2. \tag{13}$$

With repulsive density interaction $c_0 > 0$, stability is ensured. The remaining question is: what value of $|\langle \mathbf{F} \rangle|$ minimizes the spin-dependent term?

### 2.4.1 Ferromagnetic phase: $c_2 < 0$

**Energy minimization and a reference spinor.**
For a uniform condensate, the interaction energy density is

$$\mathcal{E} = \frac{c_0}{2} n_0^2 + \frac{c_2}{2} n_0^2 |\langle \mathbf{F} \rangle|^2. \tag{14}$$

If $c_2 < 0$, the system lowers energy by *maximizing* $|\langle \mathbf{F} \rangle|$. For spin-1 one has the bound $|\langle \mathbf{F} \rangle| \le 1$, hence the minimum occurs at

$$|\langle \mathbf{F} \rangle| = 1. \tag{15}$$

A convenient reference ("north-pole") spinor is

$$\zeta_{\mathrm{FM}}^{(0)} = (1, 0, 0)^{\mathsf{T}} \equiv |1, 1\rangle. \tag{16}$$

**Generate the full ferromagnetic family by symmetry.**
At (approximately) zero magnetic field, the microscopic symmetry is

$$\mathcal{G} = U(1)_{\mathrm{gauge}} \times SO(3)_{\mathrm{spin}}, \tag{17}$$

acting on the condensate spinor as

$$\zeta \;\mapsto\; e^{\mathrm{i}\theta}\, U(R)\, \zeta, \qquad R \in SO(3). \tag{18}$$

In the spin-1 representation, a standard Euler-angle parametrization is

$$U(R) = e^{-\mathrm{i}\alpha F_z}\, e^{-\mathrm{i}\beta F_y}\, e^{-\mathrm{i}\gamma F_z}, \qquad (\alpha, \beta, \gamma) \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi). \tag{19}$$

Therefore all ferromagnetic minima are generated from $\zeta_{\mathrm{FM}}^{(0)}$ as

$$\zeta_{\mathrm{FM}}(\theta; \alpha, \beta, \gamma) = e^{\mathrm{i}\theta}\, e^{-\mathrm{i}\alpha F_z}\, e^{-\mathrm{i}\beta F_y}\, e^{-\mathrm{i}\gamma F_z}\, \zeta_{\mathrm{FM}}^{(0)}. \tag{20}$$

**Spin–gauge locking: why one Euler angle is redundant.**
The key observation is that $\zeta_{\mathrm{FM}}^{(0)}$ is an eigenvector of $F_z$ with eigenvalue $+1$:

$$F_z\, \zeta_{\mathrm{FM}}^{(0)} = +\, \zeta_{\mathrm{FM}}^{(0)}. \tag{21}$$

Hence the last $z$-rotation produces only an overall phase on this reference state,

$$e^{-\mathrm{i}\gamma F_z}\,\zeta_{\mathrm{FM}}^{(0)} = e^{-\mathrm{i}\gamma}\,\zeta_{\mathrm{FM}}^{(0)}. \tag{22}$$

Substituting Eq. (22) into Eq. (20), we see that $\gamma$ can be absorbed into the gauge phase:

$$\zeta_{\mathrm{FM}}(\theta;\alpha,\beta,\gamma) = e^{\mathrm{i}(\theta-\gamma)}\,e^{-\mathrm{i}\alpha F_z}\,e^{-\mathrm{i}\beta F_y}\,\zeta_{\mathrm{FM}}^{(0)}. \tag{23}$$

Define the physically relevant gauge phase $\tilde{\theta} \equiv \theta - \gamma$. Then a canonical parametrization of the ferromagnetic manifold is

$$\zeta_{\mathrm{FM}}(\tilde{\theta},\alpha,\beta) = e^{\mathrm{i}\tilde{\theta}}\,e^{-\mathrm{i}\alpha F_z}\,e^{-\mathrm{i}\beta F_y}\,\zeta_{\mathrm{FM}}^{(0)}. \tag{24}$$

Evaluating the spin-1 rotation matrices gives the explicit components

$$\zeta_{\mathrm{FM}}(\tilde{\theta},\alpha,\beta) = e^{\mathrm{i}\tilde{\theta}}\begin{pmatrix} e^{-\mathrm{i}\alpha}\cos^2\frac{\beta}{2} \\ \frac{1}{\sqrt{2}}\sin\beta \\ e^{\mathrm{i}\alpha}\sin^2\frac{\beta}{2} \end{pmatrix}, \qquad |\langle \mathbf{F}\rangle| = 1. \tag{25}$$

**Magnetization direction n in terms of Euler angles.**

The ferromagnetic state is fully polarized, so $\langle \mathbf{F}\rangle$ is a unit vector. For the parametrization in Eq. (24), the spin points along the rotated $z$-axis:

$$\mathbf{n} \equiv \langle \mathbf{F}\rangle = R(\alpha,\beta)\,\hat{\mathbf{z}} = (\sin\beta\cos\alpha,\ \sin\beta\sin\alpha,\ \cos\beta), \qquad |\mathbf{n}| = 1. \tag{26}$$

Here $R(\alpha,\beta) \equiv R_z(\alpha)R_y(\beta)$ is the corresponding $SO(3)$ rotation (the final $R_z(\gamma)$ has been removed by spin–gauge locking).

**Order-parameter manifold (zero field).**

Because of Eq. (22), a combined transformation $(\theta, R_z(\theta))$ leaves the reference state invariant:

$$e^{\mathrm{i}\theta}\,U(R_z(\theta))\,\zeta_{\mathrm{FM}}^{(0)} = e^{\mathrm{i}\theta}\,e^{-\mathrm{i}\theta}\,\zeta_{\mathrm{FM}}^{(0)} = \zeta_{\mathrm{FM}}^{(0)}. \tag{27}$$

Thus the stabilizer (isotropy group) is a $U(1)$ subgroup, and the degenerate manifold is

$$\mathcal{M}_{\mathrm{FM}}^{(0)} \simeq \frac{U(1)\times SO(3)}{U(1)} \simeq SO(3). \tag{28}$$

**Physical meaning: what is ordered in the ferromagnetic phase?**

The condensate maximizes the local spin length: $|\langle \mathbf{F}\rangle| = 1$, so the order parameter contains a *direction* $\mathbf{n}$ in spin space. Because a rotation about $\mathbf{n}$ is equivalent to a gauge shift (spin–gauge locking), the superfluid phase is not independent of spin rotations: the low-energy manifold reduces to rigid 3D rotations, i.e. $SO(3)$.

### 2.4.2 Polar (antiferromagnetic) phase: $c_2 > 0$

**Energy minimization and a reference spinor.**

If $c_2 > 0$, Eq. (14) is minimized by

$$|\langle \mathbf{F}\rangle| = 0. \tag{29}$$

A convenient reference spinor is

$$\zeta_{\mathrm{P}}^{(0)} = (0,1,0)^{\mathsf{T}} \equiv |1,0\rangle. \tag{30}$$

**Generate the full polar family and define the nematic director n.**

All polar minima are obtained via

$$\zeta_{\mathrm{P}}(\theta; R) = e^{\mathrm{i}\theta} U(R) \zeta_{\mathrm{P}}^{(0)}. \tag{31}$$

Using Euler angles as in Eq. (19),

$$\zeta_{\mathrm{P}}(\theta; \alpha, \beta, \gamma) = e^{\mathrm{i}\theta} e^{-\mathrm{i}\alpha F_z} e^{-\mathrm{i}\beta F_y} e^{-\mathrm{i}\gamma F_z} \zeta_{\mathrm{P}}^{(0)}.$$

Now the crucial difference from the FM case is that

$$F_z \zeta_{\mathrm{P}}^{(0)} = 0, \qquad \Longrightarrow \qquad e^{-\mathrm{i}\gamma F_z} \zeta_{\mathrm{P}}^{(0)} = \zeta_{\mathrm{P}}^{(0)}, \tag{32}$$

so the last Euler angle $\gamma$ is *trivial* (it does not even produce a phase). Therefore one may parametrize polar states by $(\theta, \alpha, \beta)$ with

$$\zeta_{\mathrm{P}}(\theta; \alpha, \beta) = e^{\mathrm{i}\theta} e^{-\mathrm{i}\alpha F_z} e^{-\mathrm{i}\beta F_y} \zeta_{\mathrm{P}}^{(0)}. \tag{33}$$

The polar order is not a magnetization direction, but a *nematic director*:

$$\mathbf{n} \equiv R(\alpha, \beta)\, \hat{\mathbf{z}} = (\sin\beta\cos\alpha,\ \sin\beta\sin\alpha,\ \cos\beta) \in S^2, \tag{34}$$

where again $R(\alpha, \beta) = R_z(\alpha) R_y(\beta)$. Evaluating Eq. (33) gives an explicit representative in terms of $\mathbf{n}$:

$$\zeta_{\mathrm{P}}(\mathbf{n}, \theta) = e^{\mathrm{i}\theta} \begin{pmatrix} -\frac{1}{\sqrt{2}}(n_x - \mathrm{i} n_y) \\ n_z \\ \frac{1}{\sqrt{2}}(n_x + \mathrm{i} n_y) \end{pmatrix}, \qquad \mathbf{n} \in S^2, \qquad \langle \mathbf{F} \rangle = 0. \tag{35}$$

**$Z_2$ identification and the order-parameter manifold.**

A $\pi$-rotation that flips $\mathbf{n} \to -\mathbf{n}$ also produces an overall minus sign on the spinor. For instance,

$$e^{-\mathrm{i}\pi F_y} \zeta_{\mathrm{P}}^{(0)} = -\zeta_{\mathrm{P}}^{(0)}. \tag{36}$$

This implies the redundancy

$$\zeta_{\mathrm{P}}(-\mathbf{n}, \theta) = -\zeta_{\mathrm{P}}(\mathbf{n}, \theta) = \zeta_{\mathrm{P}}(\mathbf{n}, \theta + \pi), \tag{37}$$

so the physical identification is $(\theta, \mathbf{n}) \sim (\theta + \pi, -\mathbf{n})$. Therefore

$$\mathcal{M}_{\mathrm{P}}^{(0)} \simeq \frac{U(1) \times S^2}{Z_2}, \qquad (\theta, \mathbf{n}) \sim (\theta + \pi, -\mathbf{n}). \tag{38}$$

■ **Phase distinction at zero field.**
Ferromagnetic ($c_2 < 0$): $|\langle \mathbf{F} \rangle| = 1$, $\mathcal{M}_{\mathrm{FM}}^{(0)} \simeq SO(3)$.
Polar ($c_2 > 0$): $|\langle \mathbf{F} \rangle| = 0$, $\mathcal{M}_{\mathrm{P}}^{(0)} \simeq (U(1) \times S^2)/Z_2$.

### 2.5 Quadratic Zeeman effect (as preparation for defects)

We include the leading single-particle Zeeman terms

$$\hat{H}_Z = \int d^3\mathbf{r}\, \boldsymbol{\psi}^\dagger \big(hF_z + qF_z^2\big) \boldsymbol{\psi}. \tag{39}$$

Within a fixed magnetization sector (conserved under spin-exchange collisions), the linear term $\propto hF_z$ is a constant shift, so the spin structure is selected mainly by the quadratic term $qF_z^2$.

For a normalized spinor $\zeta$, since $F_z^2 = \mathrm{diag}(1,0,1)$,

$$\langle F_z^2\rangle \equiv \zeta^\dagger F_z^2 \zeta = |\zeta_1|^2 + |\zeta_{-1}|^2. \tag{40}$$

Hence $q > 0$ favors minimizing $\langle F_z^2\rangle$, i.e. suppressing the $m = \pm 1$ population in favor of $m = 0$. Operationally, we minimize $q\langle F_z^2\rangle$ within the zero-field manifolds $\mathcal{M}_{\mathrm{FM}}^{(0)}$ and $\mathcal{M}_{\mathrm{P}}^{(0)}$.

■ **Ferromagnetic phase ($c_2 < 0$): $q > 0$ selects an in-plane FM state and reduces $\mathcal{M}$**
Substituting Eq. (25) into Eq. (40), one finds

$$\langle F_z^2\rangle_{\mathrm{FM}} = 1 - \frac{1}{2}\sin^2\beta. \tag{41}$$

Therefore, for $q > 0$ the minimum occurs at $\beta = \pi/2$, and the ferromagnetic spinor reduces to the in-plane form

$$\zeta_{\mathrm{FM}}^{(\mathrm{eq})}(\alpha,\theta) = e^{\mathrm{i}\theta}\begin{pmatrix} \frac{1}{2}e^{-\mathrm{i}\alpha} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2}e^{\mathrm{i}\alpha} \end{pmatrix}. \tag{42}$$

Its magnetization lies in the $xy$-plane:

$$\langle \mathbf{F}\rangle = (\cos\alpha, \sin\alpha, 0), \tag{43}$$

so $\alpha$ is the azimuthal spin angle. The quadratic Zeeman term singles out the $z$-axis, leaving two independent continuous angles $(\theta,\alpha)$, hence

$$\mathcal{M}_{\mathrm{FM}}(q > 0) \simeq U(1)_{\mathrm{gauge}} \times U(1)_{\mathrm{spin}}. \tag{44}$$

■ **Polar phase ($c_2 > 0$): $q > 0$ pins the director and reduces $\mathcal{M}$**
For the polar family, using Eq. (35) and Eq. (40) gives

$$\langle F_z^2\rangle_{\mathrm{P}} = 1 - n_z^2. \tag{45}$$

Thus $q > 0$ is minimized by $n_z^2 = 1$, i.e. $\mathbf{n} = \pm\hat{\mathbf{z}}$. Using the $Z_2$ identification in Eq. (38), this pins the polar spinor to

$$\zeta_{\mathrm{P}}^{(\text{easy-axis})}(\theta) = e^{\mathrm{i}\theta}(0,1,0)^{\mathsf{T}}, \tag{46}$$

and the remaining degeneracy is the gauge phase alone:

$$\mathcal{M}_{\mathrm{P}}(q > 0) \simeq U(1). \tag{47}$$

**Remark: which manifold to use for defects**

When $q > 0$ is appreciable, the relevant low-energy manifolds are Eqs. (44) and (47). For defects that rely on the full polar manifold $\mathcal{M}_{\mathrm{P}}^{(0)} = (U(1) \times S^2)/Z_2$, one should assume $q$ is negligible on the length scales of interest (or consider regimes where the director is not pinned).

# 3 Topological Excitations in Spinor Condensates

## 3.1 General principle: homotopy classification

**Defects as maps from boundaries to the order-parameter manifold.**

If the order parameter takes values in $\mathcal{M}$, then a defect is characterized by how the order parameter winds when one encircles (or encloses) the defect:

- In 2D, a point vortex is classified by $\pi_1(\mathcal{M})$ (maps from a loop $S^1$ to $\mathcal{M}$).
- In 3D, a point defect (monopole) is classified by $\pi_2(\mathcal{M})$ (maps from a surrounding sphere $S^2$ to $\mathcal{M}$).

**Remark: defects vs textures**

A defect concerns a mapping defined on a boundary (e.g. $S^1$ or $S^2$) that cannot be contracted without crossing a singularity. A texture concerns a mapping defined on the entire space; to classify it topologically one imposes a boundary condition at infinity and compactifies $\mathbb{R}^d$ to $S^d$, leading to classification by $\pi_d(\mathcal{M})$.

### 3.2 Spin vortex in the (in-plane) ferromagnetic condensate

In the ferromagnetic phase with $q > 0$, the quadratic Zeeman term selects the in-plane representative Eq. (42). Thus the low-energy order parameter is parametrized by two angles:

$$\theta(\mathbf{r}) \in U(1)_{\text{gauge}}, \qquad \alpha(\mathbf{r}) \in U(1)_{\text{spin}}.$$

**Mass vortex vs spin vortex.**

■ **Mass vortex.**
Take $\theta = \kappa\varphi$ (spatial azimuthal angle $\varphi$), $\alpha = \text{const}$. This produces the usual circulating mass current.

■ **Spin vortex.**
Take $\alpha = \kappa\varphi$, $\theta = \text{const}$. This produces a winding of the spin direction, but no net mass current.

**Why a spin vortex carries no mass circulation (explicit cancellation)**

Write $\psi_m = \sqrt{n}\, e^{i\theta}\zeta_m$ as in Eq. (10). For the in-plane FM spinor Eq. (42) with $\theta = \text{const}$ and $\alpha = \kappa\varphi$, the component phases are

$$\phi_{+1} = -\alpha, \qquad \phi_0 = 0, \qquad \phi_{-1} = +\alpha,$$

and the weights are $n_{+1} : n_0 : n_{-1} = \frac{1}{4} : \frac{1}{2} : \frac{1}{4}$. The mass current density is proportional to the weighted phase gradients,

$$\mathbf{j} \propto \sum_m n_m \nabla\phi_m = \frac{1}{4}\nabla(-\kappa\varphi) + \frac{1}{2}\nabla(0) + \frac{1}{4}\nabla(\kappa\varphi) = 0,$$

so the two counter-circulating $\pm 1$ components cancel exactly.

### 3.3 What happens at the core of a spin vortex? Three canonical scenarios

**Why a core is needed.**
A vortex implies gradients $|\nabla\varphi| \sim 1/r$, so the kinetic energy density scales like $\sim 1/r^2$. Without a short-distance regularization, the energy diverges as $r \to 0$. The system resolves this by modifying density and/or internal spin structure near the core.

**Two healing lengths.**
For a uniform background density $n_0$, the density and spin healing lengths can be estimated as

$$\xi_n = \frac{\hbar}{\sqrt{2mc_0n_0}}, \qquad \xi_s = \frac{\hbar}{\sqrt{2m|c_2|n_0}}. \tag{48}$$

Typically $|c_2| \ll c_0$, so $\xi_s \gg \xi_n$: it is cheaper to deform the spin structure than to deplete density.

**(i) Empty-core vortex**

**Density depletion regularizes the core.**

Let the total density vanish at the core: $n(r) \to 0$ as $r \to 0$. Then the kinetic-energy divergence is cut off by the disappearance of the superfluid amplitude.

**Energy cost: density interaction**

Depleting density costs the dominant density interaction $\sim c_0(n - n_0)^2$, so the core size is typically $\sim \xi_n$.

**(ii) Polar-core vortex**

**Keep density but rotate into the polar manifold near the core.**

An instructive ansatz is

$$\zeta(r, \varphi) = e^{i\theta_0} \begin{pmatrix} \frac{1}{2} f(r) e^{-i\kappa\varphi} \\ \frac{1}{\sqrt{2}} \sqrt{2 - f^2(r)} \\ \frac{1}{2} f(r) e^{+i\kappa\varphi} \end{pmatrix}, \qquad f(r) \to 1 \ (r \gg \xi_s), \quad f(r) \to 0 \ (r \to 0).$$

Far away ($f = 1$) this is ferromagnetic (in-plane). At the core ($f = 0$) it becomes polar $(0, 1, 0)^{\mathsf{T}}$. Density can remain approximately constant, while the spin length collapses near the core.

**Energy cost: spin interaction**

Leaving the ferromagnetic manifold near the core costs spin interaction energy $\sim |c_2| n_0^2$, so the relevant core size is $\sim \xi_s$.

**(iii) Mermin–Ho vortex**

**Stay ferromagnetic everywhere by "escaping" out of the equator.**

A classic ferromagnetic texture is

$$\zeta(r, \varphi) = \begin{pmatrix} \cos^2 \frac{\beta(r)}{2} \\ e^{i\kappa\varphi} \frac{1}{\sqrt{2}} \sin \beta(r) \\ e^{2i\kappa\varphi} \sin^2 \frac{\beta(r)}{2} \end{pmatrix}, \qquad \beta(r) \to 0 \ (r \to 0), \quad \beta(r) \to \frac{\pi}{2} \ (r \to \infty).$$

At long distance, the spin winds around the equator (like a spin vortex). Near the core, $\beta(r) \to 0$ moves the spin toward the north pole, avoiding an in-plane singularity while remaining within the ferromagnetic manifold.

**Energy cost: quadratic Zeeman (for $q > 0$)**

This texture keeps $|\langle \mathbf{F} \rangle| = 1$ everywhere, avoiding spin-interaction cost. However, it tilts spins out of the equator, producing a quadratic Zeeman penalty when $q > 0$.

■ **Core energetics at a glance.**
Empty core: pay density interaction, core size $\sim \xi_n$.
Polar core: pay spin interaction, core size $\sim \xi_s$.
Mermin–Ho: stay ferromagnetic, but pay quadratic Zeeman for out-of-plane spin.

### 3.4 Half-vortex (half-quantum vortex) in the polar condensate

**Why $\pi$ phase winding can be allowed (when $\mathcal{M}_{\mathrm{P}}^{(0)}$ applies).**

In the polar phase at (approximately) zero quadratic Zeeman field, the order-parameter manifold is Eq. (38):

$$\mathcal{M}_{\mathrm{P}}^{(0)} = \frac{U(1)\times S^2}{Z_2}, \qquad (\theta,\mathbf{n}) \sim (\theta+\pi,-\mathbf{n}).$$

Hence a loop may implement $\Delta\theta = \pi$ provided $\mathbf{n}$ flips sign simultaneously.

**A canonical half-vortex ansatz and explicit single-valuedness check.**

Consider

$$\Psi(\varphi) = e^{\mathrm{i}\varphi/2}\begin{pmatrix} -\frac{e^{-\mathrm{i}\alpha}}{\sqrt{2}}\sin(\varphi/2) \\ \cos(\varphi/2) \\ \frac{e^{\mathrm{i}\alpha}}{\sqrt{2}}\sin(\varphi/2) \end{pmatrix}, \qquad \varphi : 0 \to 2\pi.$$

Then $e^{\mathrm{i}(\varphi+2\pi)/2} = -e^{\mathrm{i}\varphi/2}$ and also $\sin((\varphi+2\pi)/2) = -\sin(\varphi/2)$, $\cos((\varphi+2\pi)/2) = -\cos(\varphi/2)$, so the spinor part acquires another minus sign. The product is invariant:

$$\Psi(\varphi+2\pi) = (-1)\times(-1)\,\Psi(\varphi) = \Psi(\varphi).$$

**Physical picture: half circulation bound to a nematic flip**

A half-vortex is a combined defect where the gauge phase and the nematic director jointly wind so that the full order parameter returns to itself in $(U(1)\times S^2)/Z_2$.

### 3.5 Monopole versus skyrmion: defects vs nonsingular textures

**(i) Monopole (3D point defect): classification by $\pi_2(\mathcal{M})$**

Because the polar order parameter contains an $S^2$ director, $\pi_2(S^2) = \mathbb{Z}$, allowing monopoles. In contrast, $\pi_2(SO(3)) = 0$, so the ferromagnetic manifold does not support a stable topological monopole.

A prototypical monopole is the hedgehog $\mathbf{n}(\mathbf{r}) = \hat{\mathbf{r}} = \mathbf{r}/r$. The gradient scale is $|\nabla\mathbf{n}| \sim 1/r$, so the stiffness energy density behaves as $\propto 1/r^2$, implying a singular core as $r \to 0$ (typically resolved by density depletion on scale $\xi_n$). The total energy grows linearly with system size $R$: $E_{\mathrm{mono}} \sim R$.

**Remark: why there is no BKT-like transition for monopoles in 3D**

In 2D, vortex energy $\sim \log R$ competes with entropy $\sim \log R$, enabling a finite-temperature unbinding transition (BKT). For 3D monopoles, energy grows as $\sim R$, overwhelming entropy at large $R$.

**(ii) Skyrmion (2D nonsingular texture): classification by $\pi_2(\mathcal{M})$ after compactification**

To define a topological invariant for a smooth texture on the plane, impose a boundary condition that the order parameter approaches the same value at spatial infinity. Then $\mathbb{R}^2$ is compactified to $S^2$, and textures are classified by $\pi_2(\mathcal{M})$.

A standard skyrmion in the polar director field is

$$\mathbf{n}(r,\varphi) = \big(\sin\beta(r)\cos\varphi,\ \sin\beta(r)\sin\varphi,\ \cos\beta(r)\big), \qquad \beta(0)=0,\ \beta(\infty)=\pi,$$

which covers $S^2$ once and carries unit topological charge, yet is smooth everywhere.

■ **Same $\pi_2(S^2)$, different physics.**
Monopole: a defect defined by a nontrivial map on a surrounding $S^2$; requires a singular core. Skyrmion: a texture defined on the whole compactified space $S^2$; can be nonsingular everywhere.

### 3.6 Beyond spin-1: non-Abelian defects (conceptual remark)

**Combining gauge winding with discrete spin symmetries.**

In higher-spin condensates (e.g. spin-2 with tetrahedral order), the spin part may have a discrete point-group symmetry. One can build vortices by combining a gauge-phase twist with a nontrivial element of the discrete spin symmetry. If $\pi_1(\mathcal{M})$ becomes non-Abelian, defect fusion becomes order-dependent.

**Remark: what "non-Abelian defect" means operationally**

If two defects are characterized by group elements $g_1, g_2$ in a non-Abelian group, then bringing them together yields a combined defect $g_1 g_2$, which differs from $g_2 g_1$.