Bogoliubov Theory

Tristan.W

December 30, 2025

Contents

1 Basic Ideas of Bogoliubov Theory

1.1 General Setup

Bogoliubov theory is a *perturbative* many-body framework for the low-energy excitations of a *weakly interacting* Bose–Einstein condensate (BEC). The logic is to start from a macroscopically occupied condensate mode and treat the remaining degrees of freedom as small quantum fluctuations.

In a three-dimensional dilute Bose gas, the small parameter controlling the weak-coupling expansion is

$$\gamma \equiv \sqrt{n a_s^3} \ll 1, \tag{1}$$

where n is the total density and a_s is the s-wave scattering length. When $\gamma \ll 1$, the condensate fraction is close to one: the condensate occupation N_0 satisfies $N_0 \sim N$, while the depletion $N_{\text{ex}} \ll N$.

Unlike hydrodynamic theory, which is formulated directly in terms of coarse-grained density and phase fields, Bogoliubov theory starts from a microscopic second-quantized Hamiltonian with a contact interaction. In a uniform volume V, we take

$$\hat{H} = \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu)\, \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \frac{g}{2V} \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \hat{a}^\dagger_{\mathbf{k}_1} \hat{a}^\dagger_{\mathbf{k}_2} \hat{a}_{\mathbf{k}_3} \hat{a}_{\mathbf{k}_4}, \tag{2}$$

with momentum conservation

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4, \tag{3}$$

and single-particle dispersion

$$\epsilon_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m}. \tag{4}$$

Here μ is the chemical potential and g is the (effective) coupling constant. The bosonic operators satisfy $[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}$.

1.2 Bogoliubov Approximation

The Bogoliubov approximation is the controlled step that reduces the interacting Hamiltonian to an effective *quadratic* model for non-condensed modes by expanding around a macroscopically occupied condensate mode.

We begin by defining explicitly the unperturbed condensate state. Consider the zero-momentum mode operator $\hat{a}_0 \equiv \hat{a}_{\mathbf{k}=\mathbf{0}}$. A convenient symmetry-breaking reference state for a condensate is the coherent state

$$|G_0\rangle \equiv e^{-\frac{N_0}{2}} \exp\!\left(\sqrt{N_0}\, \hat{a}^\dagger_0\right) |0\rangle, \tag{5}$$

where $|0\rangle$ is the vacuum of all $\hat{a}_{\mathbf{k}}$, and N_0 is a variational parameter to be identified with the condensate occupation. The normalization factor ensures $\langle G_0 | G_0 \rangle = 1$.

Condensate mode becomes classical at large N_0.

We first show that $|G_0\rangle$ is an eigenstate of $\hat{a}_0$ with eigenvalue $\sqrt{N_0}$. Define $\alpha \equiv \sqrt{N_0}$. Using the commutator $[\hat{a}_0, \hat{a}^\dagger_0] = 1$, one has the identity

$$\hat{a}_0\, e^{\alpha \hat{a}^\dagger_0} = e^{\alpha \hat{a}^\dagger_0} (\hat{a}_0 + \alpha), \tag{6}$$

which can be proven by differentiating $f(t) = e^{-t\alpha \hat{a}^\dagger_0} \hat{a}_0 e^{t\alpha \hat{a}^\dagger_0}$ with respect to t:

$$\frac{df}{dt} = e^{-t\alpha \hat{a}^\dagger_0} [\hat{a}_0, \alpha \hat{a}^\dagger_0]\, e^{t\alpha \hat{a}^\dagger_0} = \alpha, \qquad f(0) = \hat{a}_0 \quad \Rightarrow \quad f(1) = \hat{a}_0 + \alpha. \tag{7}$$

Applying this identity to $|G_0\rangle$ and using $\hat{a}_0|0\rangle = 0$ yields

$$\hat{a}_0|G_0\rangle = e^{-\frac{N_0}{2}}\hat{a}_0 e^{\alpha\hat{a}_0^\dagger}|0\rangle = e^{-\frac{N_0}{2}}e^{\alpha\hat{a}_0^\dagger}(\hat{a}_0+\alpha)|0\rangle = \alpha\,|G_0\rangle = \sqrt{N_0}\,|G_0\rangle. \tag{8}$$

Therefore,

$$\langle G_0|\hat{a}_0|G_0\rangle = \sqrt{N_0}, \qquad \langle G_0|\hat{a}_0^\dagger|G_0\rangle = \sqrt{N_0}. \tag{9}$$

Next define the condensate number operator $\hat{N}_0 \equiv \hat{a}_0^\dagger\hat{a}_0$. Since $\hat{a}_0|G_0\rangle = \sqrt{N_0}|G_0\rangle$, we have

$$\langle G_0|\hat{N}_0|G_0\rangle = N_0. \tag{10}$$

The variance is also explicit. Using the coherent-state property that all normally ordered moments factorize,

$$\langle G_0|(\hat{a}_0^\dagger)^p(\hat{a}_0)^q|G_0\rangle = (\sqrt{N_0})^{p+q}, \tag{11}$$

one finds

$$\langle\hat{N}_0^2\rangle = \langle\hat{a}_0^\dagger\hat{a}_0\hat{a}_0^\dagger\hat{a}_0\rangle = \langle\hat{a}_0^\dagger(\hat{a}_0^\dagger\hat{a}_0+1)\hat{a}_0\rangle = \langle(\hat{a}_0^\dagger)^2\hat{a}_0^2\rangle + \langle\hat{a}_0^\dagger\hat{a}_0\rangle = N_0^2 + N_0, \tag{12}$$

hence

$$\langle(\Delta\hat{N}_0)^2\rangle = N_0, \qquad \frac{\sqrt{\langle(\Delta\hat{N}_0)^2\rangle}}{\langle\hat{N}_0\rangle} = \frac{1}{\sqrt{N_0}} \to 0 \quad (N_0\to\infty). \tag{13}$$

This is the precise sense in which the condensate mode becomes classical: the relative fluctuations vanish as N_0 becomes macroscopic.

Given these identities, the Bogoliubov c-number substitution is the approximation

$$\hat{a}_0 \approx \sqrt{N_0}, \qquad \hat{a}_0^\dagger \approx \sqrt{N_0}. \tag{14}$$

It is not an operator equality; it is a controlled replacement valid for low-order fluctuation physics in the regime $N_0 \gg 1$, where condensate number fluctuations are negligible at leading order.

Expansion of the interaction term by powers of $\sqrt{N_0}$.

We now expand the interaction Hamiltonian explicitly by counting the number of zero-momentum operators. Write $\hat{H}_{\text{int}}$ as

$$\hat{H}_{\text{int}} = \frac{g}{2V}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4}\,\hat{a}_{\mathbf{k}_1}^\dagger\hat{a}_{\mathbf{k}_2}^\dagger\hat{a}_{\mathbf{k}_3}\hat{a}_{\mathbf{k}_4}, \tag{15}$$

where $\delta_{\mathbf{p},\mathbf{q}}$ is the Kronecker delta enforcing momentum conservation. Separate contributions by how many of $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4$ are zero. After the c-number replacement, each zero-momentum operator contributes a factor $\sqrt{N_0}$.

(1) All four momenta equal to zero.

$$\hat{H}_{\text{int}}^{(0)} = \frac{g}{2V}\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0 \approx \frac{g}{2V}N_0^2 = \frac{gV}{2}n_0^2, \qquad n_0 \equiv \frac{N_0}{V}. \tag{16}$$

Including the single-particle term $\sum_{\mathbf{k}}(\epsilon_{\mathbf{k}}-\mu)\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}}$, the condensate contribution to the grand-canonical energy density is

$$\frac{\mathcal{E}_0}{V} = -\mu n_0 + \frac{g}{2}n_0^2. \tag{17}$$

Minimizing with respect to n_0 gives

$$\mu = gn_0, \tag{18}$$

which we will use consistently in the quadratic theory.

(2) Exactly three zero momenta: forbidden by momentum conservation. A representative term is of the form $\hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_{\mathbf{k}}$, which would scale as $N_0^{3/2}$. However, momentum conservation forbids any term with precisely one nonzero momentum: if three momenta are zero, then $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$ forces the fourth to be zero as well. Hence there is no linear term in $\hat{a}_{\mathbf{k}\neq 0}$.

(3) Exactly two zero momenta: quadratic Hamiltonian. There are two distinct operator structures:
(i) Number-conserving terms. For example,

$$\frac{g}{2V} \sum_{\mathbf{k}\neq 0} \hat{a}_0^\dagger \hat{a}_{\mathbf{k}}^\dagger \hat{a}_0 \hat{a}_{\mathbf{k}} \approx \frac{g}{2V} \sum_{\mathbf{k}\neq 0} N_0 \, \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}. \tag{19}$$

Accounting for all permutations that produce $\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}$ gives a total coefficient gn_0.
(ii) Pairing terms. For example,

$$\frac{g}{2V} \sum_{\mathbf{k}\neq 0} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger \hat{a}_0 \hat{a}_0 \approx \frac{g}{2V} \sum_{\mathbf{k}\neq 0} N_0 \, \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger, \tag{20}$$

and the Hermitian conjugate arises analogously.

Putting everything together, and using $\mu = gn_0$ from Eq. (18), the quadratic Bogoliubov Hamiltonian for $\mathbf{k} \neq 0$ modes becomes

$$\hat{H}_{\text{Bog}} = \sum_{\mathbf{k}\neq 0} (\epsilon_{\mathbf{k}} + gn_0)\, \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{gn_0}{2} \sum_{\mathbf{k}\neq 0} \left(\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger + \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}} \right) + \text{const.} \tag{21}$$

All terms beyond quadratic order in $\hat{a}_{\mathbf{k}\neq 0}$ are discarded at this level.

Meaning of the Bogoliubov truncation

Retaining only $\hat{H}_{\text{Bog}}$ implies that Bogoliubov quasiparticles are noninteracting at this order: they have sharp dispersion and infinite lifetime. Finite lifetime and quasiparticle scattering arise from cubic and quartic terms neglected here.

(4) What the next orders look like. Terms with one condensate operator and three nonzero-momentum operators scale as $\sqrt{N_0}$ and have schematic forms

$$\sqrt{N_0}\, \hat{a}^\dagger \hat{a}^\dagger \hat{a} \qquad \text{or} \qquad \sqrt{N_0}\, \hat{a}^\dagger \hat{a} \hat{a}, \tag{22}$$

with momenta constrained by conservation. These cubic terms generate quasiparticle decay and scattering processes (Beliaev and Landau damping). Quartic terms describe residual quasiparticle–quasiparticle interactions.

1.3 Bogoliubov Transformation: Diagonalizing the Quadratic Hamiltonian

The quadratic Bogoliubov Hamiltonian couples the modes $\mathbf{k}$ and $-\mathbf{k}$ through pairing terms, so it is natural to diagonalize it in each two-mode sector $(\mathbf{k}, -\mathbf{k})$ separately. Writing

$$\hat{H}_{\text{Bog}} = \sum_{\mathbf{k}\neq 0} (\epsilon_{\mathbf{k}} + gn_0)\, \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{gn_0}{2} \sum_{\mathbf{k}\neq 0} \left(\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger + \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}} \right) + \text{const,} \tag{23}$$

each $(\mathbf{k}, -\mathbf{k})$ pair forms a closed block. One introduces new bosonic operators $\hat{\alpha}_{\mathbf{k}}$ by the (real) Bogoliubov transformation

$$\hat{a}_{\mathbf{k}} = u_{\mathbf{k}} \hat{\alpha}_{\mathbf{k}} - v_{\mathbf{k}} \hat{\alpha}_{-\mathbf{k}}^\dagger, \qquad \hat{a}_{-\mathbf{k}} = u_{\mathbf{k}} \hat{\alpha}_{-\mathbf{k}} - v_{\mathbf{k}} \hat{\alpha}_{\mathbf{k}}^\dagger, \tag{24}$$

which equivalently implies $\hat{a}^{\dagger}_{-\mathbf{k}} = -v_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}} + u_{\mathbf{k}}\hat{\alpha}^{\dagger}_{-\mathbf{k}}$. Requiring $[\hat{\alpha}_{\mathbf{k}}, \hat{\alpha}^{\dagger}_{\mathbf{k}}] = 1$ gives the canonical constraint

$$u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2 = 1. \tag{25}$$

With a suitable choice of $u_{\mathbf{k}}, v_{\mathbf{k}}$, all anomalous terms $\hat{\alpha}_{\mathbf{k}}\hat{\alpha}_{-\mathbf{k}}$ and $\hat{\alpha}^{\dagger}_{\mathbf{k}}\hat{\alpha}^{\dagger}_{-\mathbf{k}}$ cancel, and the Hamiltonian becomes a sum of independent harmonic oscillators,

$$\hat{H}_{\mathrm{Bog}} = \sum_{\mathbf{k}\neq 0} \mathcal{E}_{\mathbf{k}}\, \hat{\alpha}^{\dagger}_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}} + \frac{1}{2}\sum_{\mathbf{k}\neq 0}\Big(\mathcal{E}_{\mathbf{k}} - (\epsilon_{\mathbf{k}} + g n_0)\Big) + \mathrm{const}, \tag{26}$$

where the quasiparticle dispersion is

$$\mathcal{E}_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} + g n_0)^2 - (g n_0)^2} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2 g n_0)}. \tag{27}$$

A standard parametrization consistent with Eq. (25) is

$$u_{\mathbf{k}}^2 = \frac{1}{2}\left(\frac{\epsilon_{\mathbf{k}} + g n_0}{\mathcal{E}_{\mathbf{k}}} + 1\right), \qquad v_{\mathbf{k}}^2 = \frac{1}{2}\left(\frac{\epsilon_{\mathbf{k}} + g n_0}{\mathcal{E}_{\mathbf{k}}} - 1\right). \tag{28}$$

At this perturbative order it is consistent to use the leading-order relation between g and the physical scattering length a_s,

$$g = U = \frac{4\pi\hbar^2 a_s}{m}, \tag{29}$$

so Eq. (27) becomes $\mathcal{E}_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2 U n_0)}$.

Derivation of Eqs.(26), (27) **and** (28)

Fix a momentum pair $(\mathbf{k}, -\mathbf{k})$ and denote $A \equiv \epsilon_{\mathbf{k}} + g n_0$, $B \equiv g n_0$. Up to an additive constant, the Hamiltonian restricted to this two-mode sector can be written as

$$\hat{H}_{\mathbf{k}} = A\Big(\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}\Big) + B\Big(\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}} + \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}\Big). \tag{30}$$

Now substitute the Bogoliubov transformation Eq. (24) (and its $-\mathbf{k}$ counterpart). A short but straightforward algebra gives the structure

$$\hat{H}_{\mathbf{k}} = C_{\mathbf{k}} + \Lambda_{\mathbf{k}}\Big(\hat{\alpha}^{\dagger}_{\mathbf{k}}\hat{\alpha}_{\mathbf{k}} + \hat{\alpha}^{\dagger}_{-\mathbf{k}}\hat{\alpha}_{-\mathbf{k}}\Big) + \Delta_{\mathbf{k}}\Big(\hat{\alpha}_{\mathbf{k}}\hat{\alpha}_{-\mathbf{k}} + \hat{\alpha}^{\dagger}_{\mathbf{k}}\hat{\alpha}^{\dagger}_{-\mathbf{k}}\Big), \tag{31}$$

where the coefficients (for real $u_{\mathbf{k}}, v_{\mathbf{k}}$) are

$$\Lambda_{\mathbf{k}} = A\,(u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2) - 2B\, u_{\mathbf{k}}v_{\mathbf{k}}, \tag{32}$$

$$\Delta_{\mathbf{k}} = -2A\, u_{\mathbf{k}}v_{\mathbf{k}} + B\,(u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2), \tag{33}$$

$$C_{\mathbf{k}} = 2A\, v_{\mathbf{k}}^2 - 2B\, u_{\mathbf{k}}v_{\mathbf{k}}. \tag{34}$$

Diagonalization means eliminating anomalous terms, i.e. imposing $\Delta_{\mathbf{k}} = 0$. Using the canonical constraint $u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2 = 1$, it is convenient to introduce

$$S_{\mathbf{k}} \equiv u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 \geq 1, \qquad P_{\mathbf{k}} \equiv u_{\mathbf{k}}v_{\mathbf{k}}. \tag{35}$$

Then $(u^2 - v^2)^2 = 1$ implies $S_{\mathbf{k}}^2 - 4P_{\mathbf{k}}^2 = 1$. The condition $\Delta_{\mathbf{k}} = 0$ becomes

$$-2A\, P_{\mathbf{k}} + B\, S_{\mathbf{k}} = 0 \quad \Longrightarrow \quad P_{\mathbf{k}} = \frac{B}{2A}\, S_{\mathbf{k}}. \tag{36}$$

Plugging Eq. (36) into $S^2 - 4P^2 = 1$ yields

$$S_{\mathbf{k}}^2\left(1 - \frac{B^2}{A^2}\right) = 1 \quad \Longrightarrow \quad S_{\mathbf{k}} = \frac{A}{\sqrt{A^2 - B^2}}. \tag{37}$$

Substituting into Eq. (32) gives the diagonal coefficient

$$\Lambda_{\mathbf{k}} = A\,S_{\mathbf{k}} - 2B\,P_{\mathbf{k}} = A\,S_{\mathbf{k}} - \frac{B^2}{A}S_{\mathbf{k}} = \frac{A^2 - B^2}{A}\,S_{\mathbf{k}} = \sqrt{A^2 - B^2}. \tag{38}$$

Therefore the quasiparticle energy is

$$\mathcal{E}_{\mathbf{k}} = \sqrt{A^2 - B^2} = \sqrt{(\epsilon_{\mathbf{k}} + gn_0)^2 - (gn_0)^2} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2gn_0)}, \tag{39}$$

which is Eq. (27).
Finally, from $S_{\mathbf{k}} = u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = A/\mathcal{E}_{\mathbf{k}}$ and $u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2 = 1$, we solve for $u_{\mathbf{k}}^2$ and $v_{\mathbf{k}}^2$:

$$u_{\mathbf{k}}^2 = \frac{1}{2}\,(S_{\mathbf{k}} + 1) = \frac{1}{2}\left(\frac{A}{\mathcal{E}_{\mathbf{k}}} + 1\right), \qquad v_{\mathbf{k}}^2 = \frac{1}{2}\,(S_{\mathbf{k}} - 1) = \frac{1}{2}\left(\frac{A}{\mathcal{E}_{\mathbf{k}}} - 1\right), \tag{40}$$

i.e.

$$u_{\mathbf{k}}^2 = \frac{1}{2}\left(\frac{\epsilon_{\mathbf{k}} + gn_0}{\mathcal{E}_{\mathbf{k}}} + 1\right), \qquad v_{\mathbf{k}}^2 = \frac{1}{2}\left(\frac{\epsilon_{\mathbf{k}} + gn_0}{\mathcal{E}_{\mathbf{k}}} - 1\right), \tag{41}$$

which is Eq. (28).

1.4 Ground State and the Quasiparticle Concept

A quasiparticle in a uniform many-body system is an eigenmode with a well-defined energy–momentum dispersion. In Bogoliubov theory, the quasiparticle operator $\hat{\alpha}_{\mathbf{k}}$ is a coherent superposition of a bare particle $\hat{a}_{\mathbf{k}}$ and a bare hole $\hat{a}_{-\mathbf{k}}^\dagger$ relative to the condensate background.

Because $\mathcal{E}_{\mathbf{k}} > 0$ for all $\mathbf{k} \neq 0$, the ground state of $\hat{H}_{\text{Bog}}$ must be the quasiparticle vacuum:

$$\hat{\alpha}_{\mathbf{k}}|G\rangle = 0, \qquad \forall\,\mathbf{k} \neq 0. \tag{42}$$

Using the inverse transformation $\hat{\alpha}_{\mathbf{k}} = u_{\mathbf{k}}\hat{a}_{\mathbf{k}} + v_{\mathbf{k}}\hat{a}_{-\mathbf{k}}^\dagger$, the vacuum condition becomes

$$\left(u_{\mathbf{k}}\hat{a}_{\mathbf{k}} + v_{\mathbf{k}}\hat{a}_{-\mathbf{k}}^\dagger\right)|G\rangle = 0, \qquad \left(u_{\mathbf{k}}\hat{a}_{-\mathbf{k}} + v_{\mathbf{k}}\hat{a}_{\mathbf{k}}^\dagger\right)|G\rangle = 0. \tag{43}$$

Since $\hat{H}_{\text{Bog}}$ only couples $\mathbf{k}$ and $-\mathbf{k}$, the nonzero-momentum part of the ground state factorizes into independent two-mode sectors:

$$|G\rangle_{\mathbf{k}\neq 0} = \prod_{\mathbf{k}>0} |G_{\mathbf{k}}\rangle, \tag{44}$$

where $\mathbf{k} > 0$ means that one chooses half of momentum space to avoid double counting of $\pm\mathbf{k}$ pairs.

Fix a pair $(\mathbf{k}, -\mathbf{k})$. The constraints (43) reduce to the two-mode equations

$$\left(u\,\hat{a}_{\mathbf{k}} + v\,\hat{a}_{-\mathbf{k}}^\dagger\right)|G_{\mathbf{k}}\rangle = 0, \qquad \left(u\,\hat{a}_{-\mathbf{k}} + v\,\hat{a}_{\mathbf{k}}^\dagger\right)|G_{\mathbf{k}}\rangle = 0, \tag{45}$$

where $u \equiv u_{\mathbf{k}}$, $v \equiv v_{\mathbf{k}}$. These equations immediately suggest that the ground state cannot be the bare vacuum $|0\rangle$: the presence of $\hat{a}_{-\mathbf{k}}^\dagger$ (or $\hat{a}_{\mathbf{k}}^\dagger$) means the state must contain *correlated pairs* of $(\mathbf{k}, -\mathbf{k})$ bosons. Moreover, because the constraints treat $\mathbf{k}$ and $-\mathbf{k}$ symmetrically, the natural support is on number states $|n_{\mathbf{k}}, n_{-\mathbf{k}}\rangle$ with equal occupations $n_{\mathbf{k}} = n_{-\mathbf{k}}$. This motivates a squeezed (pair-coherent) ansatz of the form "exponential of a pair-creation operator".

With this motivation, we can do some interesting derivations and then naturally take the explicit ansatz

$$|G_{\mathbf{k}}\rangle = \mathcal{N}_{\mathbf{k}} \exp\!\left(-\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}}\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{-\mathbf{k}}^\dagger\right)|0\rangle. \tag{46}$$

Reason for ansatz (46)

Fix a momentum pair $(\mathbf{k}, -\mathbf{k})$ and write the two-mode constraints

$$\Big(u\,\hat{a}_{\mathbf{k}} + v\,\hat{a}^{\dagger}_{-\mathbf{k}}\Big)|G_{\mathbf{k}}\rangle = 0, \qquad \Big(u\,\hat{a}_{-\mathbf{k}} + v\,\hat{a}^{\dagger}_{\mathbf{k}}\Big)|G_{\mathbf{k}}\rangle = 0, \tag{47}$$

where $u \equiv u_{\mathbf{k}}$, $v \equiv v_{\mathbf{k}}$ are taken real for simplicity.

Step 1: expand in the Fock basis. Write a general two-mode state as

$$|G_{\mathbf{k}}\rangle = \sum_{n,m\geq 0} C_{n,m}\,|n,m\rangle, \qquad |n,m\rangle \equiv |n_{\mathbf{k}} = n,\ n_{-\mathbf{k}} = m\rangle. \tag{48}$$

Use

$$\begin{aligned} \hat{a}_{\mathbf{k}}|n,m\rangle &= \sqrt{n}\,|n-1,m\rangle, & \hat{a}_{-\mathbf{k}}|n,m\rangle &= \sqrt{m}\,|n,m-1\rangle, \\ \hat{a}^{\dagger}_{\mathbf{k}}|n,m\rangle &= \sqrt{n+1}\,|n+1,m\rangle, & \hat{a}^{\dagger}_{-\mathbf{k}}|n,m\rangle &= \sqrt{m+1}\,|n,m+1\rangle. \end{aligned}$$

Step 2: convert the operator constraints into coefficient recursions. Apply the first constraint in Eq. (47) to Eq. (48), and collect the coefficient of $|n,m\rangle$ (relabeling indices). One obtains, for all $n \geq 0$ and $m \geq 1$,

$$u\,\sqrt{n+1}\,C_{n+1,m} + v\,\sqrt{m}\,C_{n,m-1} = 0. \tag{49}$$

For the boundary case $m = 0$, Eq. (49) reduces to

$$u\,\sqrt{n+1}\,C_{n+1,0} = 0 \quad\Longrightarrow\quad C_{n,0} = 0 \quad (n \geq 1). \tag{50}$$

Similarly, applying the second constraint in Eq. (47) gives, for all $m \geq 0$ and $n \geq 1$,

$$u\,\sqrt{m+1}\,C_{n,m+1} + v\,\sqrt{n}\,C_{n-1,m} = 0, \tag{51}$$

and for the boundary case $n = 0$,

$$u\,\sqrt{m+1}\,C_{0,m+1} = 0 \quad\Longrightarrow\quad C_{0,m} = 0 \quad (m \geq 1). \tag{52}$$

Step 3: only the diagonal $n = m$ survives. Equation (49) can be rewritten as the recursion

$$C_{n+1,m} = -\frac{v}{u}\sqrt{\frac{m}{n+1}}\,C_{n,m-1} \qquad (m \geq 1). \tag{53}$$

Iterating Eq. (53) decreases m by 1 each time, so any coefficient with $m > n$ is eventually related to some $C_{0,\ell}$ with $\ell \geq 1$, which vanishes by Eq. (52). Thus $C_{n,m} = 0$ for all $m > n$. Likewise, Eq. (51) implies a recursion that decreases n by 1 each step, so any coefficient with $n > m$ is eventually related to some $C_{\ell,0}$ with $\ell \geq 1$, which vanishes by Eq. (50). Hence $C_{n,m} = 0$ for all $n > m$. Therefore $C_{n,m} \neq 0$ is possible only when $n = m$, and we may write

$$|G_{\mathbf{k}}\rangle = \sum_{n\geq 0} C_n\,|n,n\rangle, \qquad C_n \equiv C_{n,n}. \tag{54}$$

Step 4: diagonal recursion and the exponential (squeezed) form. Set $m = n+1$ in Eq. (53) to obtain

$$C_{n+1,n+1} = -\frac{v}{u}\,C_{n,n} \quad\Longrightarrow\quad C_n = C_0\left(-\frac{v}{u}\right)^n. \tag{55}$$

Using $(\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}})^n|0\rangle = n!\,|n,n\rangle$, Eq. (54) becomes

$$|G_{\mathbf{k}}\rangle = \mathcal{N}_{\mathbf{k}} \sum_{n\geq 0} \frac{1}{n!}\left(-\frac{v}{u}\,\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}}\right)^n |0\rangle = \mathcal{N}_{\mathbf{k}} \exp\!\left(-\frac{v}{u}\,\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}}\right)|0\rangle, \tag{56}$$

which explains the origin of the commonly used ansatz.

Also, to verify it directly, we use the identity

$$\hat{a}_{\mathbf{k}}\, e^{\lambda \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}}} = e^{\lambda \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}}} \left(\hat{a}_{\mathbf{k}} + \lambda \hat{a}^\dagger_{-\mathbf{k}} \right), \tag{57}$$

which follows from $[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}}] = \hat{a}^\dagger_{-\mathbf{k}}$. Choosing $\lambda = -v_{\mathbf{k}}/u_{\mathbf{k}}$ shows that Eq. (47) is satisfied.

Therefore the full Bogoliubov ground state can be written as

$$|G\rangle = \exp\!\Big(\sqrt{N_0}\, \hat{a}^\dagger_0 \Big) \prod_{\mathbf{k}>0} \exp\!\left(-\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}}\, \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}} \right) |0\rangle, \tag{58}$$

up to normalization (equivalently, one may combine the product into an exponential of a sum since different $\mathbf{k}$ sectors commute).

2 Healing Length

The Bogoliubov dispersion $\mathcal{E}_{\mathbf{k}}$ exhibits a crossover between a phonon regime at small k and a free-particle regime at large k. This crossover defines the healing length ξ.

Small k: phonons. Using $\epsilon_{\mathbf{k}} = \hbar^2 k^2/(2m)$, for $\epsilon_{\mathbf{k}} \ll U n_0$ we have

$$\mathcal{E}_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2Un_0)} \approx \hbar c\, k, \tag{59}$$

where

$$c = \sqrt{\frac{Un_0}{m}} \tag{60}$$

is the sound velocity. In this regime,

$$\frac{\epsilon_{\mathbf{k}} + Un_0}{\mathcal{E}_{\mathbf{k}}} \approx \frac{Un_0}{\hbar c k} = \frac{\sqrt{mUn_0}}{\hbar k}. \tag{61}$$

Consequently,

$$u_{\mathbf{k}}^2 \approx \frac{1}{2} \frac{Un_0}{\hbar c k}, \qquad v_{\mathbf{k}}^2 \approx \frac{1}{2} \frac{Un_0}{\hbar c k}, \tag{62}$$

showing $u_{\mathbf{k}}^2 \sim v_{\mathbf{k}}^2$ and hence strong particle–hole mixing: the excitation is collective.

Large k: particle-like excitations. For $\epsilon_{\mathbf{k}} \gg Un_0$,

$$\mathcal{E}_{\mathbf{k}} = \epsilon_{\mathbf{k}} \sqrt{1 + \frac{2Un_0}{\epsilon_{\mathbf{k}}}} \approx \epsilon_{\mathbf{k}} + Un_0, \tag{63}$$

and

$$\frac{\epsilon_{\mathbf{k}} + Un_0}{\mathcal{E}_{\mathbf{k}}} \approx 1, \qquad u_{\mathbf{k}} \to 1, \qquad v_{\mathbf{k}} \to 0. \tag{64}$$

Therefore quasiparticles become particle-like and $\hat{\alpha}_{\mathbf{k}} \to \hat{a}_{\mathbf{k}}$.

The crossover scale k_0 is estimated by equating $\hbar^2 k^2/(2m)$ and $\hbar c k$:

$$\frac{\hbar^2 k_0^2}{2m} = \hbar c k_0 \quad \Rightarrow \quad k_0 = \frac{2mc}{\hbar}. \tag{65}$$

Define the healing length as

$$\xi = \frac{\hbar}{\sqrt{2}\, mc}. \tag{66}$$

Then

$$\frac{\hbar^2}{2m\xi^2} = mc^2 = Un_0, \tag{67}$$

where we used $c^2 = Un_0/m$. This makes the physical meaning transparent: ξ is the length scale where kinetic energy $\hbar^2/(2m\xi^2)$ and interaction energy Un_0 are comparable.

Physical meaning of the healing length

The healing length ξ quantifies the competition between kinetic and interaction energies. Modes with $k \ll 1/\xi$ are dominated by interactions and hence are collective (phonons). Modes with $k \gg 1/\xi$ are dominated by kinetic energy and hence are particle-like. Equivalently, ξ sets the spatial scale over which the condensate wave function recovers from a local perturbation.

3 Lee–Huang–Yang (LHY) Correction from Bogoliubov Zero-Point Fluctuations

3.1 Bogoliubov diagonalization and the appearance of a vacuum energy

After Bogoliubov diagonalization, the Hamiltonian takes the schematic form

$$\hat{H}_{\text{Bog}} = E_{\text{MF}} + E_{\text{vac}} + \sum_{\mathbf{k}\neq 0} \mathcal{E}_{\mathbf{k}}\, \hat{\alpha}_{\mathbf{k}}^{\dagger}\hat{\alpha}_{\mathbf{k}}, \tag{68}$$

where $\hat{\alpha}_{\mathbf{k}}$ are Bogoliubov quasi-particle operators, $\mathcal{E}_{\mathbf{k}}$ is the Bogoliubov dispersion, E_{MF} is the mean-field condensate energy, and E_{vac} is the sum of zero-point energies of all quasi-particle modes. Here

$$\epsilon_{\mathbf{k}} \equiv \frac{\hbar^2 k^2}{2m}, \qquad \mathcal{E}_{\mathbf{k}} \equiv \sqrt{\epsilon_{\mathbf{k}}\,(\epsilon_{\mathbf{k}} + 2gn_0)}. \tag{69}$$

The origin of E_{vac} is the same as for a harmonic oscillator: even in the ground state there is a zero-point energy $\frac{1}{2}\hbar\omega$. In Bogoliubov theory each $\mathbf{k} \neq 0$ mode behaves as an independent harmonic oscillator with frequency $\mathcal{E}_{\mathbf{k}}/\hbar$, so the ground state accumulates an extensive vacuum contribution from summing over all modes.

From Eq. (26), one identifies the vacuum term

$$E_{\text{vac}} \equiv \frac{1}{2}\sum_{\mathbf{k}\neq 0} \left(\mathcal{E}_{\mathbf{k}} - \epsilon_{\mathbf{k}} - gn_0\right). \tag{70}$$

It is often useful to rewrite Eq. (70) in a rational form. Using the algebraic identity

$$\mathcal{E}_{\mathbf{k}} - (\epsilon_{\mathbf{k}} + gn_0) = -\frac{(gn_0)^2}{\mathcal{E}_{\mathbf{k}} + \epsilon_{\mathbf{k}} + gn_0}, \tag{71}$$

we obtain

$$E_{\text{vac}} = -\frac{1}{2}\sum_{\mathbf{k}\neq 0} \frac{(gn_0)^2}{\mathcal{E}_{\mathbf{k}} + \epsilon_{\mathbf{k}} + gn_0}. \tag{72}$$

What is the "rational form" and why use it here?

By "rational form" we mean rewriting an expression involving a square root (here $\mathcal{E}_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2gn_0)}$) into a *ratio of polynomials* in $\epsilon_{\mathbf{k}}$ and $\mathcal{E}_{\mathbf{k}}$, i.e. with no subtraction of two close quantities.

Concretely, one multiplies by the conjugate:

$$\mathcal{E}_{\mathbf{k}} - (\epsilon_{\mathbf{k}} + gn_0) = \frac{\mathcal{E}_{\mathbf{k}}^2 - (\epsilon_{\mathbf{k}} + gn_0)^2}{\mathcal{E}_{\mathbf{k}} + \epsilon_{\mathbf{k}} + gn_0}.$$

The numerator simplifies using $\mathcal{E}_{\mathbf{k}}^2 = \epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2gn_0)$, giving

$$\mathcal{E}_{\mathbf{k}} - (\epsilon_{\mathbf{k}} + gn_0) = -\frac{(gn_0)^2}{\mathcal{E}_{\mathbf{k}} + \epsilon_{\mathbf{k}} + gn_0}.$$

This manipulation is useful because (i) it exposes the large-k behavior immediately: the denominator $\sim 2\epsilon_{\mathbf{k}}$ so the integrand scales as $1/\epsilon_{\mathbf{k}}$, making the UV divergence transparent; and (ii) it avoids subtracting two nearly equal large numbers at high k (since $\mathcal{E}_{\mathbf{k}} \approx \epsilon_{\mathbf{k}} + gn_0$), which is cleaner algebraically and numerically.

Divergence of the vacuum energy and why renormalization is needed.

To see the ultraviolet behavior, consider large $|\mathbf{k}|$ where $\epsilon_{\mathbf{k}} \gg gn_0$. Then

$$\mathcal{E}_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}\,(\epsilon_{\mathbf{k}} + 2gn_0)} = \epsilon_{\mathbf{k}}\sqrt{1 + \frac{2gn_0}{\epsilon_{\mathbf{k}}}} \approx \epsilon_{\mathbf{k}} + gn_0 - \frac{(gn_0)^2}{2\epsilon_{\mathbf{k}}} + O\!\left(\epsilon_{\mathbf{k}}^{-2}\right). \tag{73}$$

Plugging this into Eq. (70) yields

$$\mathcal{E}_{\mathbf{k}} - \epsilon_{\mathbf{k}} - gn_0 \approx -\frac{(gn_0)^2}{2\epsilon_{\mathbf{k}}}, \tag{74}$$

so that

$$E_{\mathrm{vac}} \sim -\frac{1}{4}\sum_{\mathbf{k}\neq 0}\frac{(gn_0)^2}{\epsilon_{\mathbf{k}}}. \tag{75}$$

Since $\epsilon_{\mathbf{k}} = \hbar^2k^2/(2m)$,

$$\sum_{\mathbf{k}}\frac{1}{\epsilon_{\mathbf{k}}} \sim \sum_{\mathbf{k}}\frac{m}{\hbar^2k^2}, \tag{76}$$

which is ultraviolet divergent.

This divergence is not physical; it is an artifact of modeling the interaction as a zero-range contact potential with a bare coupling. The correct low-energy physics is fixed by the two-body s-wave scattering length a_s, and renormalization is the procedure that replaces the unphysical bare parameter by a_s, rendering observables finite.

3.2 Renormalization and a manifestly finite expression for the LHY energy

We now compute the ground-state energy density at $T = 0$ to this order. In Bogoliubov theory,

$$\frac{\mathcal{E}}{V} = \frac{gn^2}{2} + \frac{1}{2V}\sum_{\mathbf{k}\neq 0}\left(\mathcal{E}_{\mathbf{k}} - \epsilon_{\mathbf{k}} - gn\right), \qquad \mathcal{E}_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}\,(\epsilon_{\mathbf{k}} + 2gn)}, \tag{77}$$

where we used $n_0 \simeq n$ at weak coupling (this will be justified by the depletion estimate later).

A standard way to implement renormalization at this order is to add and subtract the counterterm $(gn)^2/(2\epsilon_{\mathbf{k}})$ inside the sum:

$$\frac{\mathcal{E}}{V} = \frac{gn^2}{2} + \frac{1}{2V}\sum_{\mathbf{k}\neq 0}\left(\mathcal{E}_{\mathbf{k}} - \epsilon_{\mathbf{k}} - gn + \frac{(gn)^2}{2\epsilon_{\mathbf{k}}}\right). \tag{78}$$

The bracket behaves as $O(\epsilon_{\mathbf{k}}^{-2})$ at large k, hence the sum is ultraviolet convergent. At this perturbative order we may set

$$g = \frac{4\pi\hbar^2 a_s}{m}. \tag{79}$$

3.3 Evaluating the convergent integral and the final LHY formula

Equation (78) gives the beyond-mean-field correction as a convergent momentum integral. One may evaluate it explicitly in three dimensions by replacing the sum with an integral. The final ground-state energy density to leading order beyond mean-field is

$$\frac{\mathcal{E}}{V} = \frac{gn^2}{2} + \frac{\mathcal{E}_{\rm LHY}}{V}, \tag{80}$$

with the Lee–Huang–Yang correction

$$\frac{\mathcal{E}_{\rm LHY}}{V} = \frac{gn^2}{2}\,\frac{128}{15\sqrt{\pi}}\,\sqrt{na_s^3}. \tag{81}$$

Physical interpretation of the LHY term.

The mean-field energy $\frac{gn^2}{2}$ is "classical" in the sense that it comes from replacing the condensate mode by a *c*-number and keeping only the Hartree energy of the macroscopically occupied state. In contrast, the LHY correction arises from the quantum zero-point motion of the Bogoliubov modes: even in the ground state, the quasiparticle oscillators contribute $\frac{1}{2}\mathcal{E}_{\mathbf{k}}$ per mode, and their collective effect produces the leading correction beyond mean-field.

The small expansion parameter is the gas parameter na_s^3. When $na_s^3 \ll 1$, the LHY term is parametrically smaller than the mean-field energy, consistent with Bogoliubov theory as a controlled weak-coupling expansion.

4 Quantum Depletion and Self-Consistency of Bogoliubov Theory

4.1 Definition of quantum depletion and its evaluation in the Bogoliubov ground state

Bogoliubov theory is a perturbative approach around the noninteracting Bose condensate: the unperturbed state places almost all particles in the $\mathbf{k} = 0$ mode. Once interactions are turned on, particles are scattered into finite-momentum states even at zero temperature. The total number of particles outside the condensate due to interactions is called the *quantum depletion*.

The Bogoliubov ground state $|G\rangle$ is defined as the vacuum of quasi-particles,

$$\hat{\alpha}_{\mathbf{k}}|G\rangle = 0 \qquad (\mathbf{k} \neq 0), \tag{82}$$

but it is *not* the vacuum of the original atomic operators $\hat{a}_{\mathbf{k}}$. Using the Bogoliubov transformation Eq. (24), the depletion number is

$$N_{dp} = \left\langle G \left| \sum_{\mathbf{k}\neq 0} \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \right| G \right\rangle = \sum_{\mathbf{k}\neq 0} v_{\mathbf{k}}^2, \tag{83}$$

and hence the depletion density $n_{dp} \equiv N_{dp}/V$ is

$$n_{dp} = \frac{1}{V}\sum_{\mathbf{k}\neq 0} v_{\mathbf{k}}^2. \tag{84}$$

In the thermodynamic limit, $\frac{1}{V}\sum_{\mathbf{k}} \to \int \frac{d^3k}{(2\pi)^3}$, so

$$n_{dp} = \int \frac{d^3k}{(2\pi)^3}\, v_{\mathbf{k}}^2 = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2\, v_k^2. \tag{85}$$

To express the final result in hydrodynamic variables, recall

$$c = \sqrt{\frac{Un_0}{m}}, \qquad \xi \equiv \frac{\hbar}{\sqrt{2mUn_0}} = \frac{\hbar}{\sqrt{2}\,mc}. \tag{86}$$

A standard evaluation in three dimensions yields

$$n_{dp} = \frac{1}{6\sqrt{2}\,\pi^2}\,\frac{1}{\xi^3} = \frac{1}{3\pi^2}\left(\frac{mc}{\hbar}\right)^3 \propto \frac{1}{\xi^3}. \tag{87}$$

Derivation of Eq. (87)

Step 1: Write the closed form of v_k^2.
From Eq. (28) (with $g \to U$), one has

$$v_{\mathbf{k}}^2 = \frac{1}{2}\left(\frac{\epsilon_{\mathbf{k}} + Un_0}{\mathcal{E}_{\mathbf{k}}} - 1\right), \qquad \mathcal{E}_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}\,(\epsilon_{\mathbf{k}} + 2Un_0)}, \qquad \epsilon_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m}. \tag{88}$$

We use these relations to simplify the thermodynamic limit expression,

$$n_{dp} = \int \frac{d^3k}{(2\pi)^3}\, v_{\mathbf{k}}^2 = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2\, v_k^2. \tag{89}$$

Step 2: Certificate the convergence of the integral
∘ *UV(large k).* For $\epsilon_{\mathbf{k}} \gg Un_0$,

$$\mathcal{E}_{\mathbf{k}} = \epsilon_{\mathbf{k}} + Un_0 - \frac{(Un_0)^2}{2\epsilon_{\mathbf{k}}} + O(\epsilon_{\mathbf{k}}^{-2}),$$

so Eq. (88) implies

$$v_k^2 \sim \frac{(Un_0)^2}{4\epsilon_{\mathbf{k}}^2} \propto k^{-4}.$$

Hence $k^2 v_k^2 \sim k^{-2}$, which is integrable at infinity.
∘ *IR (small k).* For $k\xi \ll 1$, $\mathcal{E}_{\mathbf{k}} \approx \hbar ck$, and Eq. (88) gives $v_k^2 \sim \frac{mc}{2\hbar k} \propto k^{-1}$. Therefore $k^2 v_k^2 \sim k$, which is integrable at $k \to 0$ in 3D.

Step 3: Do dimensionless reduction. Let $\mu \equiv Un_0$ and define the healing length

$$\xi \equiv \frac{\hbar}{\sqrt{2m\mu}} = \frac{\hbar}{\sqrt{2}\,mc}, \qquad c = \sqrt{\frac{\mu}{m}} = \sqrt{\frac{Un_0}{m}}. \tag{90}$$

Then $\mu = \hbar^2/(2m\xi^2)$. Introduce the dimensionless variable $q \equiv k\xi$. One has

$$\epsilon_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} = \mu q^2, \qquad \mathcal{E}_{\mathbf{k}} = \mu\sqrt{q^2(q^2+2)}, \qquad v_k^2 = \frac{1}{2}\left(\frac{q^2+1}{q\sqrt{q^2+2}} - 1\right).$$

Substituting $k = q/\xi$ into Eq. (89) gives

$$n_{dp} = \frac{1}{4\pi^2\,\xi^3}\int_0^\infty dq\, q^2\left(\frac{q^2+1}{q\sqrt{q^2+2}} - 1\right). \tag{91}$$

Step 4: evaluating the dimensionless integral. Use $q = \sqrt{2}\sinh t$. Then $\sqrt{q^2+2} = \sqrt{2}\cosh t$ and

$$\frac{q^2+1}{q\sqrt{q^2+2}} = \frac{1+2\sinh^2 t}{2\sinh t\cosh t} = \frac{\cosh 2t}{\sinh 2t} = \coth 2t, \qquad dq = \sqrt{2}\cosh t\, dt.$$

Hence

$$\int_0^\infty dq\, q^2 \left(\frac{q^2+1}{q\sqrt{q^2+2}} - 1 \right) = \int_0^\infty (2\sinh^2 t)\,(\coth 2t - 1)\,(\sqrt{2}\cosh t)\, dt.$$

Using

$$\coth 2t - 1 = \frac{\cosh 2t - \sinh 2t}{\sinh 2t} = \frac{e^{-2t}}{\sinh 2t} = \frac{e^{-2t}}{2\sinh t\cosh t},$$

the integrand collapses and we get

$$I = \sqrt{2}\int_0^\infty \sinh t\, e^{-2t}\, dt = \frac{\sqrt{2}}{2}\int_0^\infty (e^{-t} - e^{-3t})\, dt = \frac{\sqrt{2}}{3}.$$

Therefore

$$n_{dp} = \frac{1}{4\pi^2\xi^3}\cdot\frac{\sqrt{2}}{3} = \frac{1}{6\sqrt{2}\,\pi^2}\,\frac{1}{\xi^3}.$$

Step 5: rewrite in terms of c. Using $\xi = \hbar/(\sqrt{2}\, mc)$ from Eq. (90),

$$\frac{1}{\xi^3} = \left(\frac{\sqrt{2}\, mc}{\hbar}\right)^3 = 2\sqrt{2}\left(\frac{mc}{\hbar}\right)^3,$$

so

$$n_{dp} = \frac{1}{6\sqrt{2}\,\pi^2}\cdot 2\sqrt{2}\left(\frac{mc}{\hbar}\right)^3 = \frac{1}{3\pi^2}\left(\frac{mc}{\hbar}\right)^3 \propto \frac{1}{\xi^3}, \tag{92}$$

which is Eq. (87).

Physical meaning of the healing length from quantum depletion. Equation (87) implies that the number of depleted atoms in a volume ξ^3 is of order unity:

$$n_{dp}\,\xi^3 \sim O(1). \tag{93}$$

Thus, ξ sets the characteristic length scale over which interactions "repair" the condensate against density perturbations: within a healing volume the depletion becomes appreciable, while on scales much larger than ξ the condensate behaves as a coherent hydrodynamic medium.

4.2 Self-consistency condition and the weakly interacting regime

Bogoliubov theory assumes that the condensate remains the dominant component, i.e. $N_{dp} \ll N$, or equivalently $n_{dp} \ll n$. This can be quantified by evaluating n_{dp}/n in terms of the gas parameter na_s^3.

Using $U = 4\pi\hbar^2 a_s/m$ and $n_0 \simeq n$ in the weakly interacting regime, one finds

$$\frac{n_{dp}}{n} = \frac{8}{3\sqrt{\pi}}\left(na_s^3\right)^{1/2}. \tag{94}$$

Therefore, when $na_s^3 \ll 1$,

$$\frac{n_{dp}}{n} \ll 1, \tag{95}$$

and the assumption $n_0 \simeq n$ is self-consistent. This small parameter is the same one that controls the LHY correction, so both the energy expansion and the depletion expansion are consistently organized by na_s^3.

5 Beliaev–Landau Damping: Finite Lifetime of Bogoliubov Quasi-Particles

5.1 Beyond the quadratic Hamiltonian: interaction between quasi-particles

At the Bogoliubov level, the Hamiltonian is truncated to quadratic order in the fluctuation operators $\hat{a}_{\mathbf{k}\neq 0}$, and the quasi-particles $\hat{\alpha}_{\mathbf{k}}$ are exact eigenmodes with infinite lifetime. However, once higher-order terms are included, quasi-particles interact and acquire a finite lifetime.

Keeping the leading cubic terms by replacing one operator in the original quartic interaction with the condensate c-number $\sqrt{N_0}$, one obtains an effective interaction among non-condensed modes:

$$V_{int} = \frac{g\sqrt{N_0}}{V} \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \left(\hat{a}^\dagger_{\mathbf{k}_1} \hat{a}^\dagger_{\mathbf{k}_2} \hat{a}_{\mathbf{k}_3} + \hat{a}^\dagger_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2} \hat{a}_{\mathbf{k}_3} \right), \tag{96}$$

where momentum conservation is implicit (the condensate carries $\mathbf{k} = 0$). This is the leading term that couples three fluctuation operators and is responsible for decay and scattering of quasi-particles.

5.2 Four types of three-quasi-particle processes and their physical meaning

Rewriting $\hat{a}$ in terms of $\hat{\alpha}$ using the Bogoliubov transformation Eq. (24), the interaction V_{int} generates four structurally distinct types of terms:

$$\text{(i) } \hat{\alpha}^\dagger\hat{\alpha}^\dagger\hat{\alpha}, \qquad \text{(ii) } \hat{\alpha}^\dagger\hat{\alpha}\hat{\alpha}, \qquad \text{(iii) } \hat{\alpha}^\dagger\hat{\alpha}^\dagger\hat{\alpha}^\dagger, \qquad \text{(iv) } \hat{\alpha}\hat{\alpha}\hat{\alpha}. \tag{97}$$

They have the following physical interpretations.

Process (i), $\hat{\alpha}^\dagger\hat{\alpha}^\dagger\hat{\alpha}$, describes the decay of one quasi-particle into two quasi-particles. At zero temperature, when the initial state contains a single excited quasi-particle and no thermal background, this is the dominant on-shell decay channel. The associated damping mechanism is called *Beliaev damping* (Beliaev damping).

Process (ii), $\hat{\alpha}^\dagger\hat{\alpha}\hat{\alpha}$, describes the inverse merging (or scattering) process where two quasi-particles combine into one. This channel requires a pre-existing population of quasi-particles, hence it becomes relevant at finite temperature in the presence of thermally excited modes. The corresponding damping mechanism is called *Landau damping* (Landau damping).

Processes (iii) and (iv), which simultaneously create or annihilate three quasi-particles, do not satisfy energy conservation for on-shell real processes starting from a low-energy state. Nevertheless, they contribute virtually in higher-order perturbation theory (e.g., as second-order energy shifts). A useful conceptual summary is that Beliaev and Landau damping originate from on-shell processes that generate an imaginary part of the self-energy (finite lifetime), whereas off-shell processes contribute primarily to the real part (energy renormalization).

A Bogoliubov Transformation

A.1 General Description

In many-body quantum physics one frequently encounters Hamiltonians that are quadratic in creation and annihilation operators. Such quadratic models arise as effective theories of small fluctuations around a mean-field state (e.g., superfluids and superconductors), as linearized theories of collective modes, and as exactly solvable limits of interacting problems. The Bogoliubov transformation is the canonical tool for diagonalizing *bosonic* quadratic Hamiltonians by mixing annihilation and creation operators in a controlled way.

What problem does the Bogoliubov transformation solve?. A generic bosonic quadratic Hamiltonian contains pairing terms of the form $\hat{a}^\dagger\hat{a}^\dagger$ and $\hat{a}\hat{a}$. These terms obstruct diagonalization by ordinary unitary rotations in the space of annihilation operators alone, because they couple the "particle" sector to the "hole" sector. The Bogoliubov strategy is to enlarge the linear space by considering the Nambu spinor that contains both $\hat{a}$ and $\hat{a}^\dagger$, and then perform a linear *canonical* transformation that preserves the bosonic commutation relations while eliminating the pairing structure in the Hamiltonian.

When does the procedure work as a genuine quasiparticle diagonalization?. For a *stable* bosonic quadratic Hamiltonian—meaning that the energy is bounded from below and the linearized dynamics does not exhibit exponentially growing modes—one can find a Bogoliubov transformation that maps the Hamiltonian to a sum of decoupled harmonic oscillators,

$$\hat{H} = \sum_{j=1}^{N} \hbar\omega_j \left(\hat{\alpha}_j^\dagger\hat{\alpha}_j + \frac{1}{2}\right) + \text{const}, \qquad \omega_j \geq 0, \tag{98}$$

where $\hat{\alpha}_j$ are bosonic quasiparticle operators.

A.2 Bogoliubov Transformation and the Symplectic Structure

Nambu spinor and the commutation matrix Σ. Define the Nambu vector

$$\hat{\boldsymbol{\xi}} = (\hat{a}_1, \ldots, \hat{a}_N, \hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger)^\mathsf{T}. \tag{99}$$

The canonical bosonic commutation relations are

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \qquad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0. \tag{100}$$

These relations can be packaged as

$$[\hat{\xi}_\mu, \hat{\xi}_\nu^\dagger] = \Sigma_{\mu\nu}, \qquad \Sigma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{101}$$

Allowed linear transformations: the canonical (symplectic) condition. Consider a linear transformation

$$\hat{\boldsymbol{\xi}} = T\hat{\boldsymbol{\eta}}, \qquad \hat{\boldsymbol{\eta}} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_N, \hat{\alpha}_1^\dagger, \ldots, \hat{\alpha}_N^\dagger)^\mathsf{T}. \tag{102}$$

Demanding that the transformed operators satisfy the same algebra implies

$$T^\dagger \Sigma T = \Sigma. \tag{103}$$

This is not a unitary condition; it is a Σ-preserving condition, and is equivalent to a complex symplectic structure in phase space.

Two-mode example and the basic constraint $|u|^2 - |v|^2 = 1$. For a single bosonic mode (or a fixed momentum pair), the Bogoliubov form

$$\hat{a} = u\hat{\alpha} + v\hat{\alpha}^\dagger, \qquad \hat{a}^\dagger = v^*\hat{\alpha} + u^*\hat{\alpha}^\dagger \tag{104}$$

and $[\hat{a}, \hat{a}^\dagger] = 1$ imply

$$|u|^2 - |v|^2 = 1. \tag{105}$$

In the general N-mode case, write T in blocks,

$$T = \begin{pmatrix} U & V \\ V^* & U^* \end{pmatrix}, \qquad \hat{\boldsymbol{a}} = U\hat{\boldsymbol{\alpha}} + V\hat{\boldsymbol{\alpha}}^\dagger, \tag{106}$$

then $T^\dagger \Sigma T = \Sigma$ is equivalent to

$$U^\dagger U - V^\dagger V = I, \qquad U^\mathsf{T} V - V^\mathsf{T} U = 0. \tag{107}$$

A.3 General Form of a Bosonic Quadratic Hamiltonian

A bosonic quadratic Hamiltonian is the most general Hermitian operator containing at most two creation/annihilation operators:

$$\hat{H} = \sum_{ij} \hat{a}_i^\dagger h_{ij} \hat{a}_j \; + \; \frac{1}{2} \sum_{ij} \Big(\hat{a}_i^\dagger \Delta_{ij} \hat{a}_j^\dagger + \hat{a}_i \Delta_{ij}^* \hat{a}_j \Big) \; + \; \text{const.} \tag{108}$$

Hermiticity implies $h = h^\dagger$. Since creation operators commute, only the symmetric part of Δ contributes, so one may take

$$\Delta = \Delta^\mathsf{T}. \tag{109}$$

Nambu representation and the block matrix $\mathcal{H}$. Introduce the Nambu vector $\hat{\boldsymbol{\xi}}$ and define the $2N \times 2N$ matrix

$$\mathcal{H} = \begin{pmatrix} A & B \\ B^\dagger & A^\mathsf{T} \end{pmatrix}, \qquad A \equiv h, \quad B \equiv \Delta. \tag{110}$$

Then

$$\hat{H} = \frac{1}{2} \hat{\boldsymbol{\xi}}^\dagger \mathcal{H} \hat{\boldsymbol{\xi}} + \text{const.} \tag{111}$$

The appearance of A^T ensures correct normal ordering up to an additive constant:

$$\hat{\boldsymbol{a}} A^\mathsf{T} \hat{\boldsymbol{a}}^\dagger = \hat{\boldsymbol{a}}^\dagger A \hat{\boldsymbol{a}} + \operatorname{Tr}(A). \tag{112}$$

A.4 Outline of the Diagonalization Procedure

There are two complementary viewpoints: (i) eliminate pairing terms by choosing U, V, (ii) solve a BdG eigenproblem from Heisenberg equations.

Eliminate pairing terms. Substitute $\hat{\boldsymbol{a}} = U\hat{\boldsymbol{\alpha}} + V\hat{\boldsymbol{\alpha}}^\dagger$ into $\hat{H}$ and choose U, V so that all terms proportional to $\hat{\alpha}\hat{\alpha}$ and $\hat{\alpha}^\dagger\hat{\alpha}^\dagger$ vanish. This is often intuitive in 2×2 examples.

Solve a BdG eigenproblem. Using the commutation structure, one can show

$$i\hbar \partial_t \hat{\boldsymbol{\xi}} = [\hat{\boldsymbol{\xi}}, \hat{H}] = \Sigma \mathcal{H} \hat{\boldsymbol{\xi}}. \tag{113}$$

Define $\mathcal{L} \equiv \Sigma\mathcal{H}$. Normal-mode frequencies follow from

$$\mathcal{L} \boldsymbol{\varphi}_j = \hbar\omega_j \boldsymbol{\varphi}_j. \tag{114}$$

For a stable quadratic Hamiltonian, the spectrum contains real $\pm\omega_j$, and one can choose positive-frequency eigenvectors normalized with respect to the Σ-metric,

$$\boldsymbol{\varphi}_i^\dagger \Sigma \boldsymbol{\varphi}_j = \delta_{ij}. \tag{115}$$

From these eigenvectors one constructs T (equivalently U, V), which automatically satisfies $T^\dagger \Sigma T = \Sigma$ and diagonalizes $\hat{H}$.

What stability means in practice. Diagonalization into bona fide quasiparticles requires real nonnegative ω_j. Complex eigenvalues signal dynamical instability; zero modes $\omega_j = 0$ require special care and typically correspond to free-particle canonical degrees of freedom rather than oscillators.

Final diagonal form and physical interpretation. When the construction succeeds, one obtains

$$\hat{H} = \sum_{j=1}^{N} \hbar\omega_j \left(\hat{\alpha}_j^\dagger \hat{\alpha}_j + \frac{1}{2} \right) + \text{const}, \qquad \omega_j \geq 0. \tag{116}$$

The ground state is the quasiparticle vacuum $\hat{\alpha}_j |G\rangle = 0$, which is generically a squeezed Gaussian state in terms of the original $\hat{a}_i$ operators. This is the physical origin of quantum depletion and vacuum-energy shifts.