

# Bose Hydrodynamic Theory

Tristan.W

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# 1 From Pseudo-potential Model to the Gross–Pitaevskii Equation

## 1.1 Recap: the pseudo-potential model

Starting from the pseudo-potential model introduced earlier (cf. the standard Huang–Yang/Fermi pseudo-potential description of short-range  $s$ -wave interactions), an interacting many-boson system in an external potential is modeled by the Hamiltonian

$$\hat{\mathcal{H}} = \sum_{i=1}^N \left( \frac{\hat{\mathbf{p}}_i^2}{2m} + V(\mathbf{r}_i) \right) + \sum_{i < j} \frac{4\pi\hbar^2 a_s}{m} \delta(\mathbf{r}_{ij}) \frac{\partial}{\partial r_{ij}} (r_{ij} \cdot). \quad (1)$$

Here  $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$  and  $r_{ij} \equiv |\mathbf{r}_{ij}|$ . The term  $V(\mathbf{r})$  denotes a one-body external potential (e.g. a trapping potential), and  $a_s$  is the  $s$ -wave scattering length characterizing low-energy two-body collisions.

■ **Remark on what the pseudo-potential encodes.** The operator

$$\delta(\mathbf{r}) \frac{\partial}{\partial r} (r \cdot) \quad (2)$$

should not be read as a naive “contact potential” only. Its role is to encode, in an effective zero-range way, the correct short-distance boundary condition of the *two-body* wave function: for low-energy  $s$ -wave scattering in three dimensions, the relative-coordinate wave function has a universal short-distance structure controlled by  $a_s$ . In other words, the pseudo-potential is designed so that when it acts on physically relevant wave functions (possibly with non-trivial short-distance behavior), it reproduces the correct low-energy scattering amplitude and enforces the appropriate boundary condition at  $r \rightarrow 0$ .

■ **Remark on the scope of the condensate wave-function ansatz.** A conceptual tension arises if one evaluates (1) using a simple Bose-condensed many-body wave function: such a wave function is typically *regular* when two particles approach each other, and therefore does not exhibit the required two-body short-range singularity (often discussed as a  $1/r$ -type divergence in the relative coordinate in the zero-range idealization).

This is not a paradox, but a reminder of what the condensate ansatz is actually capturing. The condensate trial wave function is built to reproduce the approximate form of the one-body density matrix (one-body reduced density matrix)

$$\rho(\mathbf{r}, \mathbf{r}') \equiv \langle \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}') \rangle, \quad (3)$$

by retaining only the *macroscopically occupied* single-particle mode and discarding all non-macroscopically occupied modes. Equivalently, the trial wave function is meant to describe *low-energy, long-wavelength* physics, with characteristic wavelengths much larger than the interparticle spacing. It is therefore natural that it fails to encode short-range/high-energy physics at length scales comparable to, or shorter than, the interparticle spacing.

As a result, any theory derived from this assumption should be understood as an effective description valid only on length scales larger than the interparticle spacing (and, in practice, for sufficiently dilute gases where  $na_s^3 \ll 1$ ).

## 1.2 Gross–Pitaevskii equation

■ **Energy functional from the Hartree condensate ansatz.**

Assume a Bose-condensed (Hartree) product-state ansatz in first quantization:

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \prod_{i=1}^N \psi(\mathbf{r}_i), \quad \int d^3\mathbf{r} |\psi(\mathbf{r})|^2 = 1. \quad (4)$$

We decompose the Hamiltonian into one-body and two-body parts:

$$\hat{\mathcal{H}} = \sum_{i=1}^N \hat{h}_0(i) + \sum_{i<j} \hat{V}_{ij}, \quad \hat{h}_0(i) = \frac{\hat{\mathbf{p}}_i^2}{2m} + V(\mathbf{r}_i), \quad \hat{V}_{ij} = \frac{4\pi\hbar^2 a_s}{m} \delta(\mathbf{r}_{ij}) \frac{\partial}{\partial r_{ij}} (r_{ij} \cdot). \quad (5)$$

For the one-body part, using separability of (4) and the normalization of  $\psi$ , one finds

$$\begin{aligned} \langle \Psi | \sum_{i=1}^N \hat{h}_0(i) | \Psi \rangle &= \sum_{i=1}^N \int d^3\mathbf{r} \Psi^* \hat{h}_0(i) \Psi \\ &= N \int d^3\mathbf{r} \psi^*(\mathbf{r}) \left( -\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) \right) \psi(\mathbf{r}). \end{aligned} \quad (6)$$

For the two-body pseudo-potential term, symmetry implies that each pair contributes equally. Hence

$$\langle \Psi | \sum_{i<j} \hat{V}_{ij} | \Psi \rangle = \frac{N(N-1)}{2} \langle \Psi | \hat{V}_{12} | \Psi \rangle, \quad (7)$$

so it suffices to evaluate  $\langle \Psi | \hat{V}_{12} | \Psi \rangle$  explicitly. Substituting the product form (4) gives

$$\begin{aligned} \langle \Psi | \hat{V}_{12} | \Psi \rangle &= \frac{4\pi\hbar^2 a_s}{m} \int d^3\mathbf{r} \Psi^* \delta(\mathbf{r}_{12}) \frac{\partial}{\partial r_{12}} (r_{12} \Psi) \\ &= \frac{4\pi\hbar^2 a_s}{m} \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 \left( \prod_{i=3}^N d^3\mathbf{r}_i |\psi(\mathbf{r}_i)|^2 \right) \psi^*(\mathbf{r}_1) \psi^*(\mathbf{r}_2) \delta(\mathbf{r}_1 - \mathbf{r}_2) \frac{\partial}{\partial r_{12}} [r_{12} \psi(\mathbf{r}_1) \psi(\mathbf{r}_2)]. \end{aligned} \quad (8)$$

The integrations over  $\mathbf{r}_3, \dots, \mathbf{r}_N$  produce 1 due to  $\int |\psi|^2 = 1$ . Moreover, under the condensate ansatz the wave function is regular as  $r_{12} \rightarrow 0$ , so the pseudo-potential acts as a “contact evaluation”:

$$\left. \frac{\partial}{\partial r_{12}} [r_{12} F(\mathbf{r}_1, \mathbf{r}_2)] \right|_{r_{12}=0} = F(\mathbf{r}, \mathbf{r}), \quad \text{for } F \text{ regular at } r_{12} \rightarrow 0 \text{ and } \mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}. \quad (9)$$

Using (9) inside (8) (on the support of  $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ ) yields

$$\begin{aligned} \langle \Psi | \hat{V}_{12} | \Psi \rangle &= \frac{4\pi\hbar^2 a_s}{m} \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 \psi^*(\mathbf{r}_1) \psi^*(\mathbf{r}_2) \delta(\mathbf{r}_1 - \mathbf{r}_2) \psi(\mathbf{r}_1) \psi(\mathbf{r}_2) \\ &= \frac{4\pi\hbar^2 a_s}{m} \int d^3\mathbf{r} |\psi(\mathbf{r})|^4. \end{aligned} \quad (10)$$

Combining (6) and (10), the total energy per particle takes the standard mean-field form

$$\frac{\mathcal{E}}{N} = \int d^3\mathbf{r} \left[ \psi^*(\mathbf{r}) \left( -\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) \right) \psi(\mathbf{r}) + \frac{N-1}{2} \frac{4\pi\hbar^2 a_s}{m} |\psi(\mathbf{r})|^4 \right]. \quad (11)$$

### ■ Variational principle: why minimize the energy?

At zero temperature and fixed particle number, the equilibrium state of a closed quantum system is its ground state. The Rayleigh–Ritz variational principle states that for any normalized trial state  $|\Phi\rangle$ ,

$$\langle \Phi | \hat{H} | \Phi \rangle \geq E_0, \quad (12)$$

where  $E_0$  is the exact ground-state energy. Once we restrict ourselves to the Hartree condensate manifold (4), minimizing (11) means: *within this constrained class of Bose-condensed states, find the most stable configuration*. The minimizer provides the best approximation to the true ground state *within the chosen ansatz*.

### GP theory as a classical field theory (mean-field viewpoint)

The Gross–Pitaevskii (GP) framework treats the condensate order parameter  $\psi(\mathbf{r}, t)$  as a complex classical field: microscopic quantum operators are replaced by their macroscopic expectation values in a coherent, symmetry-breaking state. The nonlinearity  $U|\psi|^2\psi$  is the mean-field encoding of two-body interactions: each particle experiences an effective potential proportional to the local density created by all others.

#### ■ Functional variation and Euler–Lagrange equation.

We now minimize the energy functional under the normalization constraint. Introduce a Lagrange multiplier  $\mu$  and define

$$\mathcal{F}[\psi, \psi^*] = \mathcal{E}[\psi, \psi^*] - \mu \left( \int d^3\mathbf{r} |\psi(\mathbf{r})|^2 - N \right). \quad (13)$$

The stationary condition is  $\delta\mathcal{F} = 0$ .

Formally, we write

$$\delta\mathcal{F} = \int d^3\mathbf{r} \left( \frac{\delta\mathcal{F}}{\delta\psi} \delta\psi + \frac{\delta\mathcal{F}}{\delta\psi^*} \delta\psi^* \right), \quad (14)$$

and demand the coefficients of  $\delta\psi$  and  $\delta\psi^*$  vanish separately.

#### Wirtinger calculus and varying $\psi$ vs. $\psi^*$

In variational problems with complex fields, it is convenient (and standard) to treat  $\psi$  and  $\psi^*$  as independent variables when taking functional derivatives. This is the infinite-dimensional analog of Wirtinger derivatives in complex analysis for non-analytic functions  $F(\psi, \psi^*)$ . Equivalently, one may view the complex function space as a real vector space of twice the dimension (real and imaginary parts), and vary those independently. The equation obtained from  $\delta/\delta\psi$  is the complex conjugate of the one obtained from  $\delta/\delta\psi^*$ , so it is sufficient to write one of them explicitly.

We now compute the variation term by term. From (11), the variation of the one-body part with respect to  $\psi^*$  is

$$\delta_{\psi^*} \left[ \int d^3\mathbf{r} \psi^* \left( -\frac{\hbar^2 \nabla^2}{2m} + V \right) \psi \right] = \int d^3\mathbf{r} \delta\psi^* \left( -\frac{\hbar^2 \nabla^2}{2m} + V \right) \psi. \quad (15)$$

For the interaction term, using  $|\psi|^4 = (\psi^* \psi)^2$ ,

$$\delta_{\psi^*} |\psi|^4 = \delta_{\psi^*} (\psi^* \psi)^2 = 2(\psi^* \psi) \psi \delta\psi^* = 2|\psi|^2 \psi \delta\psi^*, \quad (16)$$

hence

$$\delta_{\psi^*} \left[ \int d^3\mathbf{r} |\psi|^4 \right] = \int d^3\mathbf{r} 2|\psi|^2 \psi \delta\psi^*. \quad (17)$$

Finally, for the constraint term,

$$\delta_{\psi^*} \left[ -\mu \int d^3\mathbf{r} |\psi|^2 \right] = -\mu \int d^3\mathbf{r} \psi \delta\psi^*. \quad (18)$$

Collecting terms and using the arbitrariness of  $\delta\psi^*$ , the stationary condition  $\delta\mathcal{F} = 0$  yields

$$\left( -\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) \right) \psi(\mathbf{r}) + \frac{4\pi\hbar^2 a_s}{m} (N-1) |\psi(\mathbf{r})|^2 \psi(\mathbf{r}) = \mu \psi(\mathbf{r}). \quad (19)$$

It is customary to absorb the particle number into the field by redefining  $\psi \mapsto \sqrt{N}\psi$ , so that  $n(\mathbf{r}) = |\psi(\mathbf{r})|^2$  directly becomes the condensate density and  $(N - 1) \approx N$  for large  $N$ . Defining

$$U \equiv \frac{4\pi\hbar^2 a_s}{m}, \quad (20)$$

one arrives at the standard time-independent Gross–Pitaevskii equation

$$\left(-\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r})\right) \psi(\mathbf{r}) + U|\psi(\mathbf{r})|^2 \psi(\mathbf{r}) = \mu \psi(\mathbf{r}). \quad (21)$$

Physically, the nonlinear term  $U|\psi|^2\psi$  represents the mean-field interaction: each particle experiences an effective potential proportional to the local density of other particles.

The dynamical (time-dependent) version is obtained by promoting the stationary variational condition to Hamiltonian field dynamics, resulting in

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = \left(-\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r})\right) \psi(\mathbf{r}, t) + U|\psi(\mathbf{r}, t)|^2 \psi(\mathbf{r}, t). \quad (22)$$

Equation (22) is often referred to as a nonlinear Schrödinger equation: it differs from the single-particle Schrödinger equation precisely through the density-dependent nonlinear term encoding interactions in a condensate with macroscopic occupation.

## 2 Hydrodynamic Equations

### 2.1 Madelung decomposition

Any complex scalar field  $\psi(\mathbf{r}, t)$  can be written in polar (amplitude–phase) form,

$$\psi(\mathbf{r}, t) = \sqrt{n(\mathbf{r}, t)} e^{i\theta(\mathbf{r}, t)}, \quad (23)$$

where the amplitude  $\sqrt{n}$  is nonnegative and  $\theta$  is a real phase. This is sometimes called the Madelung decomposition (or Madelung transformation) when used to rewrite wave dynamics into fluid-like equations.

In the GP setting,  $n(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2$  has the direct interpretation as the condensate particle density (mean-field density). The phase  $\theta(\mathbf{r}, t)$  carries the information about coherent flow. One defines the superfluid velocity field

$$\mathbf{v}_s(\mathbf{r}, t) = \frac{\hbar}{m} \nabla \theta(\mathbf{r}, t). \quad (24)$$

In regions where  $\psi \neq 0$  and  $\theta$  is single-valued,  $\mathbf{v}_s$  is locally irrotational,  $\nabla \times \mathbf{v}_s = \mathbf{0}$ . Quantized vortices appear when the phase winds by  $2\pi\ell$  around a closed loop; then  $\theta$  cannot be globally single-valued, and the vortex core typically has  $n \rightarrow 0$  so that (23) remains well-defined.

### 2.2 Hydrodynamic equations from the time-dependent GP equation

We substitute (23) into the time-dependent GP equation (22). A useful intermediate identity is the Laplacian of  $\psi$ :

$$\begin{aligned} \nabla^2 \psi &= \nabla^2 (\sqrt{n} e^{i\theta}) \\ &= e^{i\theta} \left[ \nabla^2 \sqrt{n} - \sqrt{n} (\nabla \theta)^2 + i (2 \nabla \sqrt{n} \cdot \nabla \theta + \sqrt{n} \nabla^2 \theta) \right]. \end{aligned} \quad (25)$$

After substituting and dividing out the common phase factor  $e^{i\theta}$ , the GP equation splits into its real and imaginary parts, which become the hydrodynamic equations.

■ **Real part: “Newton” (Euler-like) equation.** The real part yields an equation for  $\partial_t \theta$ :

$$\hbar \frac{\partial \theta}{\partial t} = - \left[ -\frac{\hbar^2}{2m} \frac{1}{\sqrt{n}} \nabla^2 \sqrt{n} + \frac{1}{2} m v_s^2 + V(\mathbf{r}) + U n \right], \quad (26)$$

where  $v_s^2 \equiv \mathbf{v}_s \cdot \mathbf{v}_s$ . Taking a gradient and using  $\mathbf{v}_s = \frac{\hbar}{m} \nabla \theta$  gives

$$m \frac{\partial \mathbf{v}_s}{\partial t} = -\nabla \left[ -\frac{\hbar^2}{2m} \frac{1}{\sqrt{n}} \nabla^2 \sqrt{n} + \frac{1}{2} m v_s^2 + V(\mathbf{r}) + U n \right]. \quad (27)$$

The first term in the bracket,

$$-\frac{\hbar^2}{2m} \frac{1}{\sqrt{n}} \nabla^2 \sqrt{n}, \quad (28)$$

is called the *quantum pressure*. The other terms correspond respectively to kinetic energy density, trapping potential energy, and interaction energy (mean-field “pressure”). Equation (27) is the force-balance law for the superfluid: the gradient of an effective energy/enthalpy acts as a force, analogous to  $F = ma$  in classical mechanics.

■ **Imaginary part: continuity equation.** The imaginary part gives

$$\frac{\partial \sqrt{n}}{\partial t} = -\frac{\hbar}{2m} (2\nabla \sqrt{n} \cdot \nabla \theta + \sqrt{n} \nabla^2 \theta). \quad (29)$$

Multiplying by  $2\sqrt{n}$  and using  $\mathbf{v}_s = \frac{\hbar}{m} \nabla \theta$ , one obtains

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}_s) = 0. \quad (30)$$

This is the continuity equation: the local density changes only via the net flux of the current  $n\mathbf{v}_s$  through the boundary of a small volume element.

Equations (27) and (30) together form the zero-temperature superfluid hydrodynamic equations derived from the GP mean-field theory.

### 2.3 Remark: hydrodynamics versus Bose–Einstein condensation

It is tempting to identify hydrodynamic behavior with Bose–Einstein condensation (BEC), since an interacting BEC naturally leads to the hydrodynamic description above. However, hydrodynamics as a *phenomenology* is more general: it is an effective description valid when only a few coarse-grained fields (such as density, velocity, and pressure) are needed to close the dynamics, because microscopic degrees of freedom relax rapidly compared with macroscopic evolution.

■ **Key viewpoint.** The more fundamental prerequisite for hydrodynamics is efficient local equilibration (or, more broadly, sufficiently rapid momentum exchange and relaxation) so that one can meaningfully define local fluid variables and write closed equations for them. Whether the system possesses a condensate order parameter is an additional, stronger structural statement. In particular, strong interactions can generate hydrodynamic behavior even without BEC: in normal dense fluids hydrodynamics arises from frequent collisions, while in dilute ultracold gases strong interactions can be engineered via a Feshbach resonance, leading to hydrodynamic flow signatures even when the temperature is not low enough to establish superfluidity.

### 3 Linearization and Sound Velocity

#### 3.1 Near-equilibrium behavior of a uniform system

It is instructive to start from a *uniform* condensate, i.e. set the external potential  $V(\mathbf{r}) = 0$ . Conceptually, the uniform case is the cleanest setting in which translation invariance eliminates inhomogeneous background profiles, so that any nontrivial dynamics can be unambiguously attributed to genuine collective fluctuations rather than to the geometry of the trap. Moreover, many experimentally relevant phenomena (sound propagation, critical velocity, long-wavelength modes) are controlled by the long-distance physics that is well approximated by a locally uniform description.

To study *near-equilibrium* (hence low-energy) dynamics, one first identifies the equilibrium (steady) state and then considers small deviations around it. For a weakly interacting Bose condensate described by the Gross–Pitaevskii (GP) energy functional

$$E[\psi] = \int d^3\mathbf{r} \left[ \frac{\hbar^2}{2m} |\nabla\psi|^2 + \frac{U}{2} |\psi|^4 \right],$$

a spatially uniform configuration  $\psi(\mathbf{r}) = \sqrt{n_0} e^{i\theta_0}$  is favored: the gradient term penalizes spatial variations, while the interaction term depends only on the local density.

##### ■ Ground state of the uniform system from energy minimization

In the uniform case, the equilibrium configuration is obtained by minimizing  $E[\psi]$  at fixed particle number  $N = \int d^3\mathbf{r} |\psi|^2$ . Introducing a Lagrange multiplier  $\mu$  and minimizing  $E - \mu \int |\psi|^2$  gives the stationary GP equation

$$\left( -\frac{\hbar^2 \nabla^2}{2m} + U|\psi|^2 \right) \psi = \mu \psi.$$

A uniform solution  $\psi = \sqrt{n_0} e^{i\theta_0}$  makes  $\nabla^2 \psi = 0$  and yields  $\mu = Un_0$ . In hydrodynamic variables  $\psi = \sqrt{n} e^{i\theta}$ , the superfluid velocity is

$$\mathbf{v}_s = \frac{\hbar}{m} \nabla \theta,$$

so a uniform phase  $\theta(\mathbf{r}) = \theta_0$  implies  $\mathbf{v}_s = 0$ . Thus, the ground state of the uniform system has constant density  $n = n_0$  and vanishing flow  $\mathbf{v}_s = 0$ .

##### ■ Linearized small fluctuations

We now consider small fluctuations about this equilibrium:

$$n(\mathbf{r}, t) = n_0 + \delta n(\mathbf{r}, t), \quad \mathbf{v}_s(\mathbf{r}, t) = \delta \mathbf{v}(\mathbf{r}, t),$$

where  $\delta n$  and  $\delta \mathbf{v}$  are assumed to be small. “Near equilibrium” means the fluctuation amplitudes are small enough that the hydrodynamic equations can be linearized, keeping only leading order in  $\delta n$  and  $\delta \mathbf{v}$ . “Low energy” (equivalently, long wavelength) means we keep only the leading order in gradients (leading order in  $|\mathbf{k}|$  in Fourier space) and drop higher-order derivative terms, such as those originating from the quantum pressure.

#### 3.2 Linearized hydrodynamic equations

We start from the zero-temperature superfluid hydrodynamic equations derived from the time-dependent GP equation in Madelung form. They consist of the continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}_s) = 0,$$

and the Euler/Newton-type equation

$$m \frac{\partial \mathbf{v}_s}{\partial t} = -\nabla \left[ -\frac{\hbar^2}{2m} \frac{1}{\sqrt{n}} \nabla^2 \sqrt{n} + \frac{1}{2} m v_s^2 + U n \right],$$

where the first term in brackets is the *quantum pressure*.

■ **Linearized continuity equation** Substituting  $n = n_0 + \delta n$  and  $\mathbf{v}_s = \delta \mathbf{v}$  into the continuity equation gives

$$\frac{\partial(n_0 + \delta n)}{\partial t} + \nabla \cdot ((n_0 + \delta n) \delta \mathbf{v}) = 0.$$

Keeping only linear terms and dropping the quadratic contribution  $\nabla \cdot (\delta n \delta \mathbf{v})$ , one obtains

$$\frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \delta \mathbf{v} = 0. \quad (31)$$

■ **Linearized Newton equation (acoustic limit)** Near equilibrium, the term  $\frac{1}{2} m v_s^2$  is second order in  $\delta \mathbf{v}$  and is dropped. In the long-wavelength (low- $k$ ) regime, we also neglect the quantum pressure contribution at leading order, keeping only the interaction-driven “pressure” gradient. This gives

$$m \frac{\partial \delta \mathbf{v}}{\partial t} = -U \nabla \delta n. \quad (32)$$

#### Why the quantum pressure can be neglected at long wavelength

Linearizing the quantum pressure term around  $n_0$  yields a contribution schematically of the form

$$-\nabla \left[ -\frac{\hbar^2}{2m} \frac{1}{\sqrt{n}} \nabla^2 \sqrt{n} \right] \sim \frac{\hbar^2}{4m} \nabla (\nabla^2 \delta n),$$

which involves three spatial derivatives. In Fourier space, this contributes  $\propto k^3 \delta n_{\mathbf{k}}$ , whereas the interaction term  $-\nabla(U n)$  contributes  $\propto k \delta n_{\mathbf{k}}$ . Therefore, in the  $k \rightarrow 0$  (long-wavelength) limit, the quantum-pressure correction is higher order in  $k$  and can be neglected to obtain the pure acoustic regime. More precisely, keeping quantum pressure restores the Bogoliubov dispersion with an additional  $k^4$  term in  $\omega^2$ .

#### Detailed derivation:

Consider the quantum pressure term

$$\nabla \left[ \frac{\hbar^2}{2m} \frac{1}{\sqrt{n}} \nabla^2 \sqrt{n} \right], \quad (33)$$

and expand it around a uniform background density

$$n(\mathbf{r}) = n_0 + \delta n(\mathbf{r}), \quad |\delta n| \ll n_0, \quad (34)$$

First, expand the square root:

$$\sqrt{n} = \sqrt{n_0} \left( 1 + \frac{\delta n}{2n_0} - \frac{(\delta n)^2}{8n_0^2} + \dots \right). \quad (35)$$

Keeping only terms linear in  $\delta n$ , we obtain

$$\nabla^2 \sqrt{n} \simeq \sqrt{n_0} \frac{1}{2n_0} \nabla^2 \delta n. \quad (36)$$

Similarly,

$$\frac{1}{\sqrt{n}} \simeq \frac{1}{\sqrt{n_0}} \left(1 - \frac{\delta n}{2n_0}\right), \quad (37)$$

so that to the lowest order

$$\frac{1}{\sqrt{n}} \nabla^2 \sqrt{n} \simeq \frac{1}{2n_0} \nabla^2 \delta n. \quad (38)$$

Substituting this result back into the quantum pressure term yields

$$\nabla \left[ \frac{\hbar^2}{2m} \frac{1}{\sqrt{n}} \nabla^2 \sqrt{n} \right] \sim -\frac{\hbar^2}{4mn_0} \nabla (\nabla^2 \delta n), \quad (39)$$

which is the linearized form of the quantum pressure contribution around a uniform background density.

### ■ Locked density and velocity (phase) fluctuations: one collective mode

Equations (31) and (32) imply that density fluctuations  $\delta n$  and velocity (hence phase-gradient) fluctuations  $\delta \mathbf{v}$  do not represent two independent low-energy degrees of freedom. Instead, they are dynamically tied to each other: once  $\delta n$  is specified,  $\delta \mathbf{v}$  is determined (up to consistency with boundary conditions).

To make this explicit, consider a plane-wave ansatz

$$\delta n(\mathbf{r}, t), \delta \mathbf{v}(\mathbf{r}, t) \propto e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}.$$

Then (32) gives

$$-im\omega \delta \mathbf{v} = -iU\mathbf{k} \delta n \quad \Rightarrow \quad \delta \mathbf{v} = \frac{U}{m\omega} \mathbf{k} \delta n, \quad (40)$$

so  $\delta \mathbf{v}$  is longitudinal (parallel to  $\mathbf{k}$ ) and its amplitude is fixed by  $\delta n$ . Substituting (40) into (31) yields the dispersion relation and eliminates the apparent redundancy of variables. In this sense,  $\delta n$  and  $\delta \mathbf{v}$  are “locked together” and constitute a *single* low-energy collective excitation (the sound/phonon mode).

A useful analogy is provided by electromagnetic waves: one may treat  $\mathbf{E}$  and  $\mathbf{B}$  as two fields, but Maxwell’s equations lock them into a fixed relationship for a propagating plane wave. The number of true propagating degrees of freedom is counted by independent polarizations rather than by the number of fields written down.

Combining (31) and (32) yields a closed equation for  $\delta n$ . Taking a time derivative of (31) and using (32) gives

$$\frac{\partial^2 \delta n}{\partial t^2} = -n_0 \nabla \cdot \frac{\partial \delta \mathbf{v}}{\partial t} = -n_0 \nabla \cdot \left( -\frac{U}{m} \nabla \delta n \right) = \frac{Un_0}{m} \nabla^2 \delta n. \quad (41)$$

Defining the sound velocity

$$c \equiv \sqrt{\frac{Un_0}{m}}, \quad (42)$$

we obtain the wave equation

$$\frac{\partial^2 \delta n}{\partial t^2} = c^2 \nabla^2 \delta n. \quad (43)$$

### Wave equation versus diffusion equation

A prototypical *wave equation* for an observable  $W$  has the structure

$$\frac{\partial^2 W}{\partial t^2} = c^2 \nabla^2 W,$$

with a constant propagation speed  $c$ . If  $W$  obeys such an equation, a localized spatial profile can propagate while largely preserving its shape: the “center” of the profile moves through space at speed  $c$ . This is the mathematical reason why waves can transmit information over distance.

In contrast, a *diffusion equation* has one time derivative and two spatial derivatives:

$$\frac{\partial W}{\partial t} = D \nabla^2 W,$$

where  $D$  is the diffusion constant. A localized profile governed by diffusion typically broadens and smears out over time. Heat transport and Brownian motion in a normal state are standard examples of diffusive dynamics.

### 3.3 Phonons as low-energy excitations

The appearance of the wave equation

$$\frac{\partial^2 \delta n}{\partial t^2} = c^2 \nabla^2 \delta n \quad (44)$$

encodes the existence of a propagating collective mode. Physically, it reflects a balance between *inertia* (mass density resisting acceleration) and an *elastic restoring force* (here provided by interaction-induced “pressure”: compressing the fluid costs interaction energy  $\propto Un^2$ , producing a force that tends to restore uniform density). The normal modes of such a system are plane waves

$$\delta n(\mathbf{r}, t) \propto e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)},$$

with linear dispersion

$$\omega = c|\mathbf{k}|. \quad (45)$$

These are precisely *sound waves*. Upon quantization, each normal mode becomes a harmonic oscillator with energy quanta  $\hbar\omega$ ; the corresponding quasiparticles are the familiar *phonons*.

#### Experimental determination of the sound velocity

A common experimental protocol is to use a focused laser beam to imprint a localized density dip (or hump) near the center of a condensate, then monitor the subsequent evolution of the density profile in time. In the sound-wave regime, the profile propagates outward with little distortion, and tracking its motion allows one to extract the sound velocity  $c$ . Repeating the measurement for different background densities  $n_0$  tests the mean-field prediction  $c = \sqrt{Un_0/m}$ .

#### ■ Goldstone-mode viewpoint

Phonons in a superfluid are not merely “ordinary” sound; at the deepest level they are the gapless collective modes associated with the spontaneous breaking of a continuous symmetry. In the GP description, the condensate order parameter  $\psi = \sqrt{n} e^{i\theta}$  selects a definite global phase, signaling spontaneous breaking of the global  $U(1)$  phase-rotation symmetry. Goldstone’s theorem then implies the existence of a gapless mode corresponding to slow spatial and temporal variations of this phase.

In particular, the velocity field is the phase gradient,  $\mathbf{v}_s = (\hbar/m)\nabla\theta$ . A long-wavelength phonon can therefore be viewed as a gentle spatial twist of  $\theta(\mathbf{r})$ , accompanied (through the continuity equation) by a small density modulation. In the extreme long-wavelength limit  $\mathbf{k} \rightarrow 0$ , the excitation becomes a uniform rotation of the condensate phase. Since condensate states with different global phases are degenerate at the mean-field level, the corresponding

excitation energy vanishes as  $|\mathbf{k}| \rightarrow 0$ , consistent with the gapless (Goldstone) nature of the phonon.

## 4 Superfluidity and Critical Velocity

### 4.1 Superfluidity from a finite velocity of low-energy excitations

A striking consequence of having a linearly dispersing, gapless phonon as the unique lowest-energy excitation is the emergence of frictionless flow below a certain velocity. The key physical idea is energetic: dissipation requires the ability to transfer kinetic energy into internal excitations of the condensate. If excitations can only be created above a kinematic threshold, then sufficiently slow motion cannot lose energy by emitting excitations.

To make this concrete, consider an impurity of mass  $m_0$  moving through a condensate with initial velocity  $\mathbf{v}_i$ . Friction (dissipation) occurs when the impurity can scatter to a final velocity  $\mathbf{v}_f$  while creating an excitation in the condensate that carries momentum  $\mathbf{q}$ .

One should also comment on the role of *Galilean invariance*: in a Galilean-invariant system, “a moving impurity in a stationary fluid” is equivalent to “a moving fluid past a stationary impurity” by a change of inertial frame. If Galilean invariance is broken (e.g. by a lattice or spin-orbit coupling), these situations are not equivalent and distinct critical velocities may arise. In what follows we focus on the simplest case: a moving impurity inside a static BEC.

### 4.2 Derivation from energy and momentum conservation

Assume that the only relevant low-lying excitation is the phonon with dispersion  $\mathcal{E}(\mathbf{q}) = c|\mathbf{q}|$ . Momentum conservation in the scattering process gives

$$m_0\mathbf{v}_i = m_0\mathbf{v}_f + \mathbf{q}, \quad (46)$$

and energy conservation gives

$$\frac{m_0 v_i^2}{2} = \frac{m_0 v_f^2}{2} + c|\mathbf{q}|. \quad (47)$$

Using  $\mathbf{v}_f = \mathbf{v}_i - \mathbf{q}/m_0$  from (46) and substituting into (47), one finds

$$\mathbf{v}_i \cdot \mathbf{q} - c|\mathbf{q}| = \frac{q^2}{2m_0}. \quad (48)$$

The right-hand side is nonnegative. However, if  $|\mathbf{v}_i| < c$ , then for any nonzero  $\mathbf{q}$ ,

$$\mathbf{v}_i \cdot \mathbf{q} \leq |\mathbf{v}_i| |\mathbf{q}| < c|\mathbf{q}|, \quad (49)$$

so the left-hand side of (48) is strictly negative, making the equation impossible to satisfy. Therefore, when the impurity moves slower than the sound velocity, it cannot create a phonon while conserving energy and momentum, and hence it cannot dissipate energy through this channel.

Consequently, if the impurity velocity satisfies  $|\mathbf{v}_i| < c$ , there is no friction for its motion. In a Galilean-invariant setting this is equivalent to saying that a fluid flowing with speed smaller than  $c$  past a static obstacle experiences no dissipation. This phenomenon is called *superfluidity*. The maximal flow speed below which frictionless motion is possible is called the *superfluid critical velocity*  $v_c$ . In the present phonon-dominated regime, one finds  $v_c = c$ .

#### Experimental signature of the critical velocity

Soon after BEC was achieved experimentally, a critical velocity was observed by scanning a focused laser beam back and forth through the condensate, thereby acting as a moving

impurity. After a fixed scan time the system is allowed to re-thermalize, and the increase of the thermal fraction is measured. Below a threshold scan speed, little heating is observed; above it, the thermal fraction grows with the scan velocity. The collapse of data taken at different scan amplitudes and frequencies onto a near-universal curve provides clear evidence for a critical velocity.

### 4.3 Landau criterion: from linear dispersion to the general case

The above argument can be generalized beyond phonons and beyond linear dispersion. For an isotropic quasiparticle excitation spectrum  $\mathcal{E}(|\mathbf{q}|)$ , energy–momentum conservation implies that dissipation becomes possible only if the flow (or impurity) speed exceeds

$$v_c = \min_{\mathbf{q} \neq 0} \left( \frac{\mathcal{E}(|\mathbf{q}|)}{|\mathbf{q}|} \right). \quad (50)$$

This is the *Landau criterion* for the superfluid critical velocity. The physical content is transparent: the quantity  $\mathcal{E}(q)/q$  is the minimal “energy cost per unit momentum” that the fluid can absorb in the form of excitations. If the flow speed is smaller than this minimum, no excitation can be created without violating conservation laws.

■ **Noninteracting BEC: quadratic dispersion and zero critical velocity** The Landau criterion also clarifies why interactions are essential for superfluidity. For a noninteracting Bose gas, the Hamiltonian in momentum space is

$$\hat{H} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}, \quad \epsilon_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m}. \quad (51)$$

The ground state places all particles in the  $\mathbf{k} = 0$  mode. An elementary excitation consists of promoting a particle from  $\mathbf{k} = 0$  to some  $\mathbf{k} \neq 0$ , which costs precisely the single-particle kinetic energy,

$$\mathcal{E}(k) = \epsilon_{\mathbf{k}} - \epsilon_0 = \frac{\hbar^2 k^2}{2m}. \quad (52)$$

This spectrum is manifestly quadratic at small  $k$ . Consequently,

$$\frac{\mathcal{E}(k)}{\hbar k} = \frac{\hbar k}{2m} \longrightarrow 0 \quad (k \rightarrow 0),$$

so the minimum of  $\mathcal{E}(q)/q$  is zero and the Landau criterion gives  $v_c = 0$ . In other words, a noninteracting BEC is *not* a superfluid: Bose condensation alone does not guarantee frictionless flow. Superfluidity emerges from the combination of condensation and interactions, which reshape the low-energy spectrum from quadratic single-particle excitations to linear, collective phonons.

## 5 Thomas–Fermi Distribution

We now include an external trapping potential  $V(\mathbf{r})$  and ask what the zero-temperature hydrodynamic theory predicts for the equilibrium density profile of an interacting condensate. The key simplification is the Thomas–Fermi (TF) approximation: we first neglect the quantum pressure term, obtain an analytic equilibrium solution, and then check that this neglect is self-consistent in the regime of large atom number.

The hydrodynamic equations follow from the time-dependent Gross–Pitaevskii (GP) equation after the Madelung decomposition  $\psi = \sqrt{n}e^{i\theta}$ , with superfluid velocity  $\mathbf{v}_s = (\hbar/m)\nabla\theta$ . The

“Newton equation” for  $\mathbf{v}_s$  has the schematic structure

$$m \frac{\partial \mathbf{v}_s}{\partial t} = -\nabla \left[ V(\mathbf{r}) + U n(\mathbf{r}, t) + \frac{1}{2} m v_s^2 + (\text{quantum pressure}) \right]. \quad (53)$$

At equilibrium we expect no flow, hence we impose

$$\mathbf{v}_s(\mathbf{r}) = 0. \quad (54)$$

Neglecting quantum pressure, equilibrium requires that the bracket be spatially constant; this constant is identified as the global chemical potential  $\mu$ . Therefore the equilibrium density satisfies

$$V(\mathbf{r}) + U n_0(\mathbf{r}) = \mu. \quad (55)$$

Equivalently,

$$n_0(\mathbf{r}) = \frac{\mu - V(\mathbf{r})}{U} \quad (\mu > V(\mathbf{r})), \quad (56)$$

and, since density cannot be negative,

$$n_0(\mathbf{r}) = 0 \quad (\mu < V(\mathbf{r})). \quad (57)$$

Here  $\mu$  is the chemical potential (chemical potential),  $U$  is the contact interaction strength (contact interaction) given by  $U = 4\pi\hbar^2 a_s/m$ , and  $\mathbf{v}_s$  is the superfluid velocity.

The boundary of the condensate cloud is where the density just vanishes. Denoting this boundary point by  $\mathbf{R}$  (with radius  $R = |\mathbf{R}|$ ), we have

$$\mu = V(\mathbf{R}). \quad (58)$$

This relation is not an additional assumption: it is simply the statement that the TF profile is supported on the region where  $\mu - V(\mathbf{r}) \geq 0$ . In what follows we focus on a prototypical example, the *harmonic trap*.

### ■ A harmonic trap produces an inverted parabola

The crucial reason is that in TF theory the density is an *affine* function of the trapping potential:  $n_0(\mathbf{r}) \propto \mu - V(\mathbf{r})$ . If  $V(\mathbf{r})$  grows quadratically with  $r$ , then  $n_0(r)$  must decrease quadratically with  $r$ , hence an inverted parabola.

For a three-dimensional isotropic harmonic trap

$$V(\mathbf{r}) = \frac{1}{2} m \omega_0^2 r^2, \quad (59)$$

the TF profile becomes explicitly

$$n_0(r) = \frac{1}{U} \left( \mu - \frac{1}{2} m \omega_0^2 r^2 \right), \quad 0 \leq r \leq R, \quad (60)$$

and  $n_0(r) = 0$  for  $r > R$ . Using the boundary condition  $\mu = V(R)$ , one finds

$$\mu = \frac{1}{2} m \omega_0^2 R^2, \quad (61)$$

so the density can also be written as

$$n_0(r) = \frac{m \omega_0^2}{2U} (R^2 - r^2), \quad 0 \leq r \leq R, \quad (62)$$

which makes the “inverted parabola” character completely manifest.

### ■ Determining $\mu$ (or $R$ ) from number conservation

The global chemical potential  $\mu$  is not free: it is fixed by the total particle number

$$\int d^3\mathbf{r} n_0(\mathbf{r}) = N, \quad (63)$$

where the integral is restricted to the condensate region  $r \leq R$ . Substituting the isotropic TF profile,

$$N = \int_0^R 4\pi r^2 dr \frac{1}{U} \left( \mu - \frac{1}{2} m \omega_0^2 r^2 \right) = \frac{4\pi}{U} \left( \mu \frac{R^3}{3} - \frac{1}{2} m \omega_0^2 \frac{R^5}{5} \right). \quad (64)$$

Eliminating  $\mu$  using  $\mu = \frac{1}{2} m \omega_0^2 R^2$  gives

$$N = \frac{4\pi}{U} \cdot \frac{1}{2} m \omega_0^2 R^5 \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{4\pi m \omega_0^2}{15U} R^5, \quad (65)$$

i.e.

$$4\pi m \omega_0^2 R^5 = 15UN, \quad (66)$$

and therefore

$$R \propto N^{1/5}. \quad (67)$$

This scaling already hints at the physical mechanism: in the TF regime the cloud size grows slowly with  $N$  because increasing  $N$  also increases the interaction energy, which resists compression.

### ■ Self-consistency of neglecting quantum pressure

The TF approximation ignores the quantum pressure, which is ultimately a kinetic-energy cost associated with spatial variation of the density. A quick scaling estimate shows that it becomes negligible at large  $N$ . Using  $R \propto N^{1/5}$ , the typical density scale is  $n \sim N/R^3$ , and the interaction energy scales as

$$E_{\text{int}} \sim UnN \sim U \frac{N^2}{R^3} \propto UN^{7/5}. \quad (68)$$

The trapping energy scales as

$$E_{\text{trap}} \sim N\langle V \rangle \sim NR^2 \propto N^{7/5}. \quad (69)$$

Thus interaction and trap energies have the same  $N$ -dependence and can balance each other to determine  $R$ . In contrast, the quantum-pressure contribution is controlled by gradients, with a typical scale  $\nabla \sim 1/R$ , giving

$$E_{\text{qp}} \sim N \frac{\hbar^2}{2mR^2} \propto N^{3/5}. \quad (70)$$

Hence for sufficiently large  $N$ ,  $E_{\text{qp}}$  is parametrically smaller than both  $E_{\text{int}}$  and  $E_{\text{trap}}$ , validating the TF approximation. This also explains why the TF profile is very different from the noninteracting case: when  $U \rightarrow 0$ , the interaction term cannot balance the trap, and the equilibrium profile must instead come from balancing kinetic energy against the trap, producing a Gaussian density distribution.

### ■ Further remarks

Once the TF background  $n_0(\mathbf{r})$  is established, one can study low-energy collective modes  $\delta n(\mathbf{r}, t)$  on top of it. For fluctuations with wavelength much shorter than the cloud size ( $k \gg 1/R$ ), the system behaves locally like a uniform condensate and supports sound propagation with a *local* sound speed  $c(\mathbf{r}) = \sqrt{Un_0(\mathbf{r})/m}$ , consistent with the local density approximation. For global modes with wavelength comparable to  $R$ , the inhomogeneity becomes essential and the

excitation spectrum becomes discrete; in a harmonic trap one finds families of surface modes (with density variations concentrated near the boundary) and compressional modes (including a breathing mode that oscillates the cloud size). These collective frequencies differ from the noninteracting harmonic-oscillator spectrum, providing a sharp manifestation of interaction effects.

## 6 \* Finite Temperature Effects: Two-Fluid Hydrodynamics

We now move beyond  $T = 0$  and discuss finite-temperature superfluids. At zero temperature, the single-component hydrodynamics tied to the condensate order parameter yields a sound velocity of the form

$$c = \sqrt{\frac{Un_0}{m}}, \quad (71)$$

where  $n_0$  is the condensate density. If one naively extrapolates this expression to finite temperature by interpreting  $n_0$  as the condensate density that vanishes at the transition temperature  $T_c$ , the formula would predict  $c \rightarrow 0$  as  $T \rightarrow T_c$ . This is incompatible with the basic observation that ordinary fluids and gases above  $T_c$  still support sound propagation with a finite velocity; thus the zero-temperature hydrodynamic description is missing essential finite-temperature degrees of freedom.

The correct finite-temperature hydrodynamic theory for a superfluid is the *two-fluid hydrodynamics*, originally formulated by Landau in the nondissipative limit. The central physical picture is that the fluid separates into two interpenetrating components:

The **superfluid component** corresponds to the macroscopically occupied mode and carries no entropy; the **normal component** is made of thermal excitations and carries all the entropy.

The two components can move with different velocities,  $\mathbf{v}_s$  and  $\mathbf{v}_n$ , and their coupled dynamics generically produces two propagating modes: one predominantly a density/pressure wave (first sound), and the other predominantly an entropy/temperature wave (second sound). In general the two are hybridized rather than perfectly separated.

In the nondissipative two-fluid theory, the hydrodynamic variables include the superfluid density  $n_s$ , the normal density  $n_n$ , the total density  $n = n_s + n_n$ , the superfluid velocity  $\mathbf{v}_s$ , the normal velocity  $\mathbf{v}_n$ , and the entropy density  $S$ . They obey the coupled set of equations

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = 0, \quad (72)$$

$$\frac{\partial S}{\partial t} + \nabla \cdot (S\mathbf{v}_n) = 0, \quad (73)$$

$$\frac{\partial \mathbf{v}_s}{\partial t} = -\nabla \left( \mu + \frac{1}{2}v_s^2 \right), \quad (74)$$

$$\frac{\partial \mathbf{j}_i}{\partial t} + \frac{\partial \Pi_{ij}}{\partial x_j} = 0, \quad (75)$$

where the total current is

$$\mathbf{j} = n_s \mathbf{v}_s + n_n \mathbf{v}_n, \quad (76)$$

and the momentum flux density tensor is

$$\Pi_{ij} = P\delta_{ij} + n_s v_{si} v_{sj} + n_n v_{ni} v_{nj}. \quad (77)$$

A noteworthy feature is the entropy equation: in the nondissipative limit, entropy is conserved and is transported only by the normal component, hence it flows with  $\mathbf{v}_n$ . Solving the coupled equations reveals two sound-like modes below  $T_c$ . As one approaches  $T_c$  from below, one mode continuously connects to the ordinary density wave that survives above  $T_c$ , while the other mode becomes predominantly an entropy (heat) wave; across  $T_c$  this heat mode changes character and becomes diffusive rather than propagating, in sharp contrast to the superfluid phase where heat can propagate as a wave.