

R Handout: Likelihood Ratio Test

This handout will show various ways to conduct the Likelihood ratio test in practice. Suppose we are interested in the following hypothesis testing for the mean of sepal width of Virginica (from the Iris data set *Link: <https://archive.ics.uci.edu/ml/datasets/iris>* (Fisher, 1936 [1])) $H_0 : \mu = \mu_0 = 3$ vs. $H_1 : \mu = \mu_1 \neq 3$ under the assumption that the observations X_i 's are i.i.d. the normal distribution with $N(\mu, \sigma^2 = 0.1040)$. Also, let the significance level be $\alpha = 0.05$.

1. Exact Method

The likelihood

$$L(\mu; \underline{x}) = \prod_{i=1}^n (2\pi\sigma^2)^{-1/2} e^{-(x_i - \mu)^2 / (2\sigma^2)}$$

The log-likelihood

$$\ln L(\mu; \underline{x}) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \quad (1)$$

It is easy to show the m.l.e. be $\hat{\mu} = \bar{x}$. The log-likelihood ratio

$$\begin{aligned} \delta &= \ln\left(\frac{L(\mu_0; \underline{x})}{L(\hat{\mu}; \underline{x})}\right) = \ln L(\mu_0; \underline{x}) - \ln L(\hat{\mu}; \underline{x}) \\ &= -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu_0)^2 + \left(\frac{n}{2} \ln(2\pi\sigma^2) + \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \hat{\mu})^2\right) \\ &= \sum_{i=1}^n (x_i - \mu_0)^2 - \sum_{i=1}^n (x_i - \bar{x} + \bar{x} - \mu_0)^2 \\ &= \sum_{i=1}^n [(x_i - \bar{x})^2 + 2(x_i - \bar{x})(\bar{x} - \mu_0) + (\bar{x} - \mu_0)^2] \\ &= \sum_{i=1}^n [(x_i - \bar{x})^2] + n(\bar{x} - \mu_0)^2 \end{aligned}$$

So

$$\delta = -\frac{n}{2\sigma^2} (\bar{x} - \mu_0)^2$$

Because $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, $\frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} \sim N(0, 1) \Rightarrow (\frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}})^2 \sim \chi_1^2$, according to the equal-tail assumption, the size

$$P(-2\delta \leq C_1 \cup -2\delta \geq C_2 | H_0) = P(-2\delta \leq C_1 \cup -2\delta \geq C_2 | \mu = \mu_0) = \alpha$$

Under the null hypothesis, let $\Delta = -2\delta \sim \chi_1^2$. The critical region

$$\Rightarrow \{\Delta \leq 0.0010 \cup \Delta \geq 5.0239\}$$

Wilks (1938) [2] proved that, if the sample size is large enough and H_0 is true, for a nested model, Δ is asymptotically following the chi-squared distribution with the degrees of freedom equal to the difference between the degrees of freedom under the alternative hypothesis and the degrees of freedom under the null hypothesis. This result is called **Wilks' Theorem**.

2. Bootstrap Method

Suppose the analytical form of Δ is too complicated and we do not have enough sample size to reach the asymptote but we can calculate the likelihoods. We can do our test by simulating the distribution of the log-likelihood ratio.

(a) Parametric Bootstrap

We will **simulate the data in a specific distribution under the null hypothesis** and then do the test repeatedly.

Steps for our problem:

- i. **Simulate a sample from the normal distribution under the null hypothesis**
- ii. Calculate the likelihood under the null hypothesis
- iii. Calculate the likelihood under the alternative hypothesis (Notice that the m.l.e. need to be re-estimated)
- iv. Calculate and save Δ_s for the empirical distribution of Δ
- v. repeat (1) - (4) until the process reaches the maximum of simulations
- vi. Compare the Δ from the original data to the distribution of Δ

(b) Nonparametric Bootstrap

We will **resample the data from the original data set** and then do the test repeatedly.

Steps for our problem:

- i. **Resample a sample from the original data**

- ii. Calculate the likelihood under the null hypothesis
- iii. Calculate the likelihood under the alternative hypothesis (Notice that the m.l.e. need to be re-estimated)
- iv. Calculate and save Δ_s for the empirical distribution of Δ
- v. repeat (1) - (4) until the process reaches the maximum of simulations
- vi. Compare the Δ from the original data to the distribution of Δ

3. Comparison

The following is the table for the p-values from different methods

Methods	Exact Method	Parametric Bootstrap	Nonparametric Bootstrap
p-value	0.8628	0.8518	0.7312

The p-values of the exact method is highest, the second place is the one of the Parametric Bootstrap method and the lowest one is from the Nonparametric Bootstrap method.

4. Practice (Goodness-of-fit): suppose we are interested in the following hypothesis testing for deciding whether the sample of sepal width of Virginica is from the specific normal distribution with mean $\mu = 3$ and variance $\sigma^2 = 0.1$. Follow the three methods of conducting the likelihood ratio test above, what is your setting for your test and what is your conclusion? (**Hint:** the degrees of freedom for this test is 2. Also, you may need to estimate the m.l.e. of sample variance).

■References References

- [1] Fisher, R.A. (1936), *The use of multiple measurements in taxonomic problems*, Annual Eugenics, 7, Part II, 179-188.
- [2] Wilks, S. S. (1938) *The Large-Sample Distribution of the Likelihood Ratio for Testing Composite Hypotheses*, The Annals of Mathematical Statistics. **9**: 60-62.