Greedy Algorithm

Contents

Greedy Algorithm

- Making the locally optimal choice at each stage with the intent of finding a global optimum
  ("Greedy Algorithm", Wikipedia)

Advantage

- Often easy to implement
- Often efficient on time complexity

Disadvantage

- Often the wrong way.
- Even if it is correct method, often difficult to prove validness.

Famous Problems

- Solution of greedy problems depends on each problems.

Example:

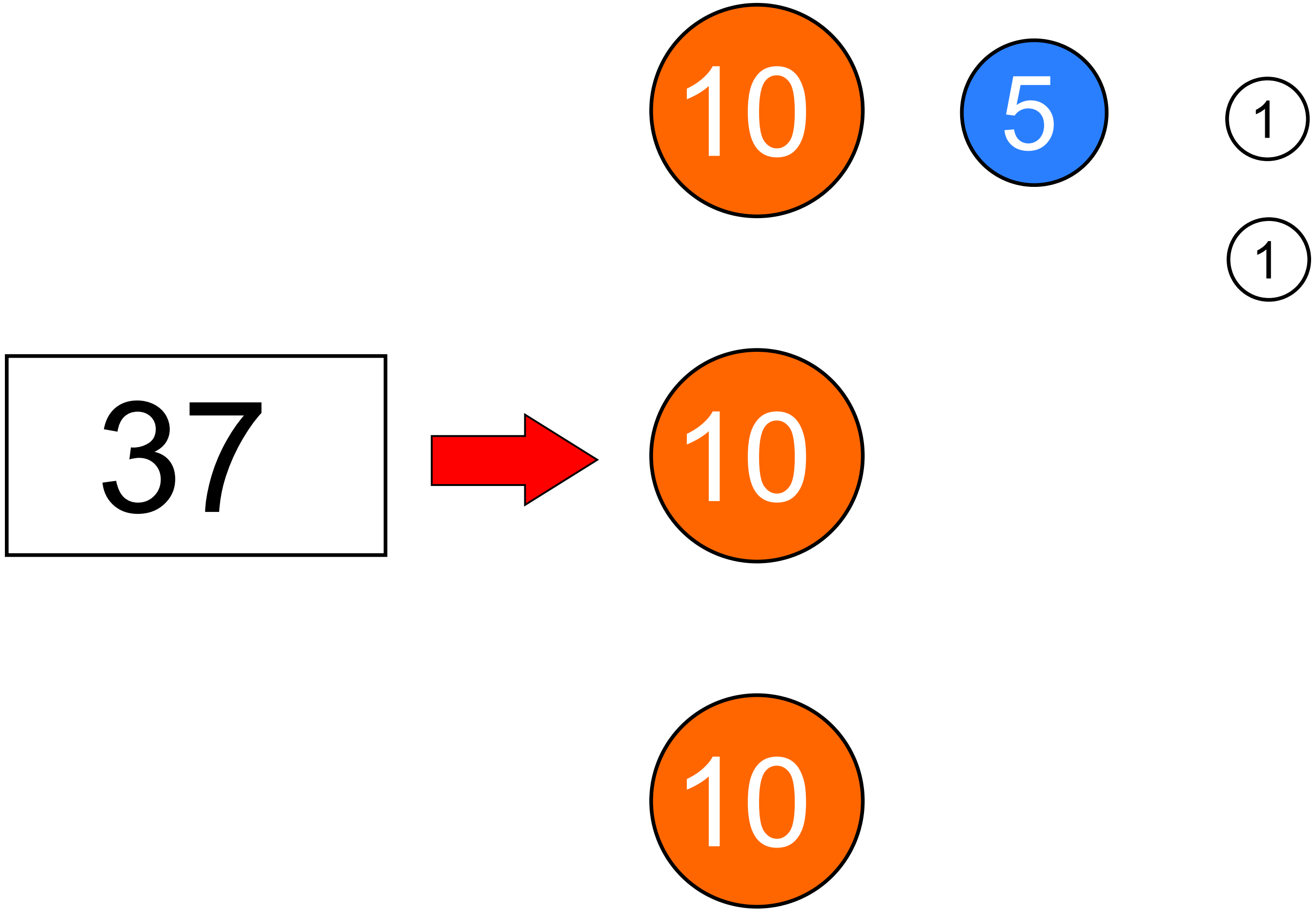
- Coin Changing Problem (Special Case)
- Interval Scheduling

Coin-making Problem (Special Case)

- Japanese: コインの両替問題 (Coin Changing Problem)
- This problem may not be useful in competitive programming because of narrowness of application.

Problem Statement

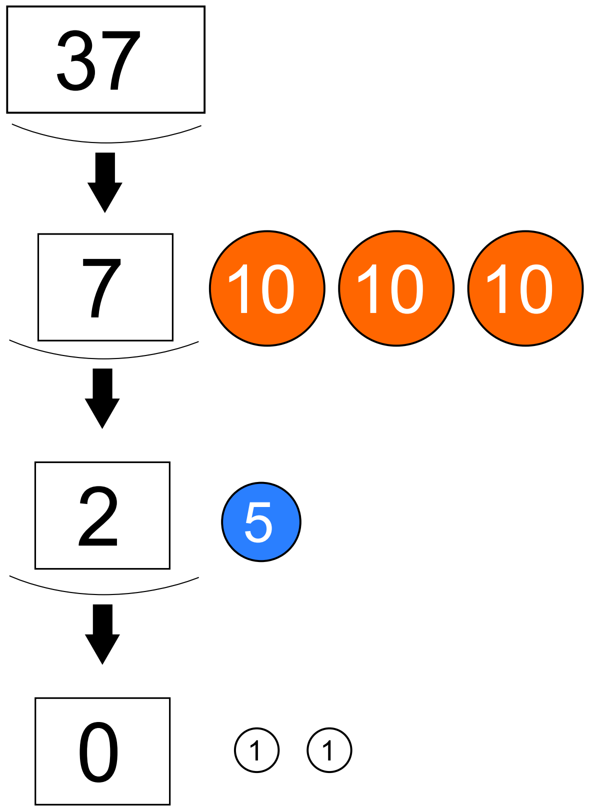
There are coins: 1yen, 5yen, and 10yen.
Minimize the number of coin used to pay K yen.


37
10
5
1
1
10
10

Solution: Choose maximum yen greedly.

- Choose 10yen as many as possible: $N/10$ coins.
  $\Rightarrow$ Remainder is $N' = N\%10$.
- Choose 5yen as many as possible: $N'/5$ coins.
  $\Rightarrow$ Remainder is $N'' = N'\%5$.
- Choose 1yen as many as possible: $N''/1$ coins.
  $\Rightarrow$ Remainder should be 0.

The answer is $(N/10 + N'/5 + N''/1)$.



Condition to solve by Greedy

- **You don't have to remember this condition.**

Here, i-th coin is c_i, and coins are sorted in ascending order, and $H(x)$ is defined this:

$$H(x) = \text{minimum number of coins used to pay } x \text{ yen}$$

Then, this is true, we can solve this problem by Greedy:

$$\forall j (1 \leq j \leq N-1),\ H(\delta_j) < p_j - 1$$
$$(\text{There is a unique } p_j,\, \delta_j \text{ such that } c_{j+1} = p_j c_j - \delta_j)$$

This problem is a specialization of Unbounded Knapsack Problem (I don't check the proof...)
硬貨の問題が貪欲法で解けるための条件

```cpp
// vc: list of coins, K: coins to pay
int coin_making(vector<int>& vc, int K) {
  sort(vc.begin(), vc.end(), greater<int>());
  int ret = 0;
  for (auto& c : cv) {
    ret += K / c;
    K %= c;
  }
  return ret;
}
```

General Coin-making Problem

- cannot be solved by Geedy.
- can often be solved by DP.

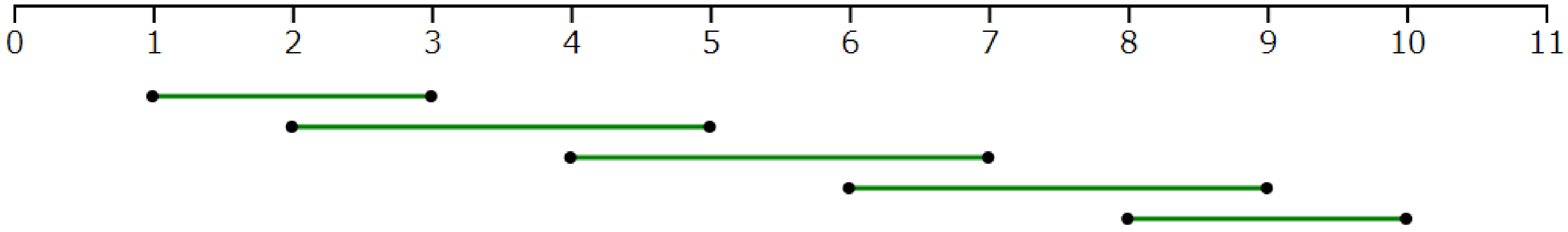
Interval Scheduling

Problem Statement:

There are N tasks. i-th tasks run from s_i to t_i.

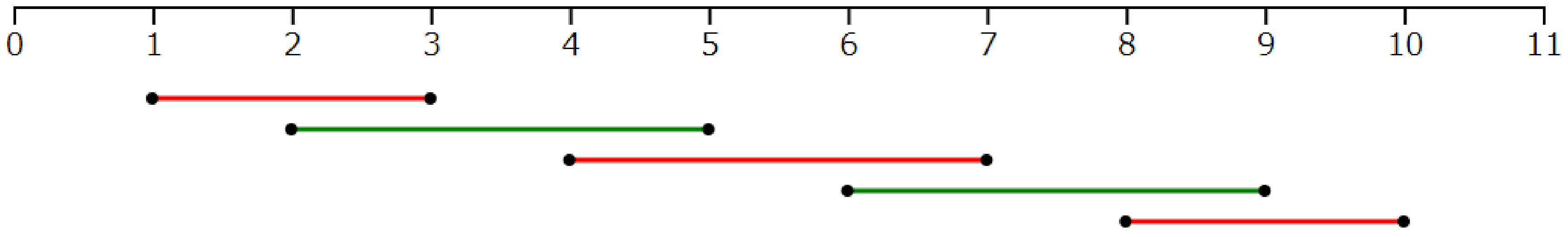
Task-i and task-j is **disjoint** if two are not overlap as time interval.

Maximize the size of disjoint set of tasks.



$$N = 5$$

$$(s_i, t_i) = (1,3), (2,5), (4,7), (6,9), (8,10)$$



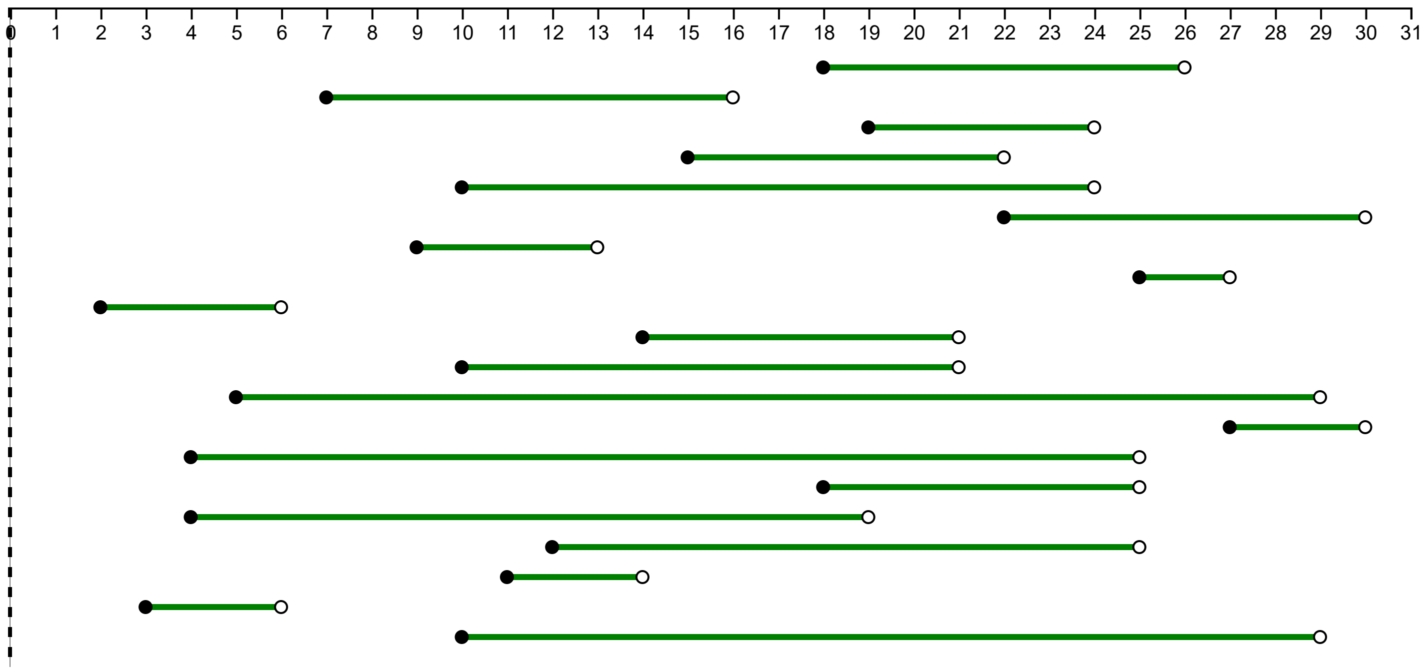
$$N = 5$$

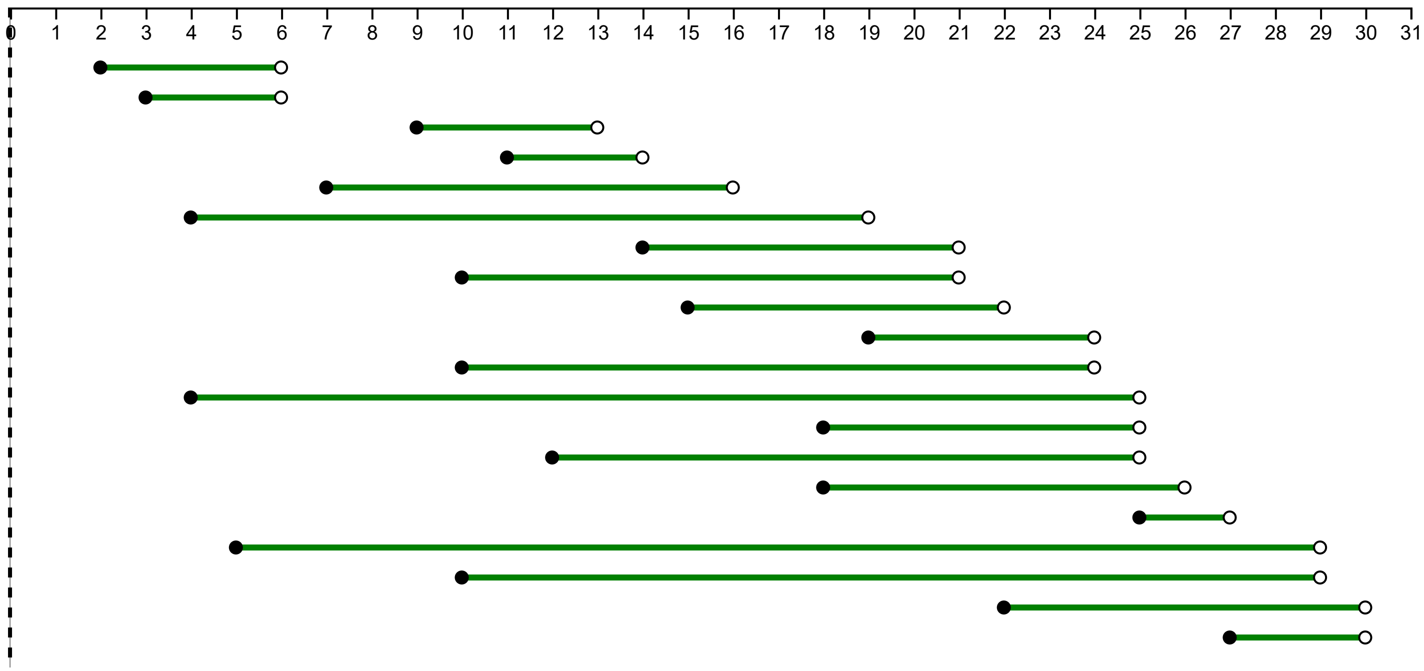
$$(s_i, t_i) = (1,3), (2,5), (4,7), (6,9), (8,10)$$

$$\text{ans} = 3$$

Intuition

- If you choose the tasks which has earlier finishing time, tasks you can choose will increase after the step.


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31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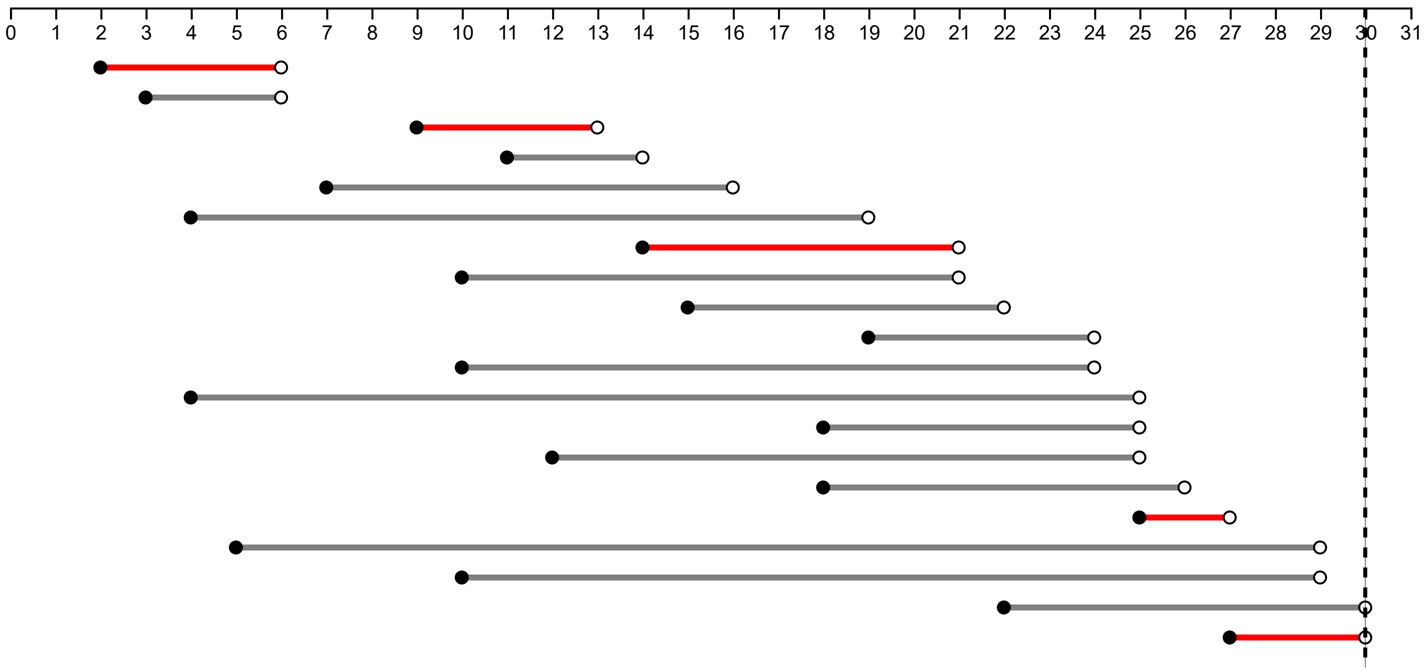

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31

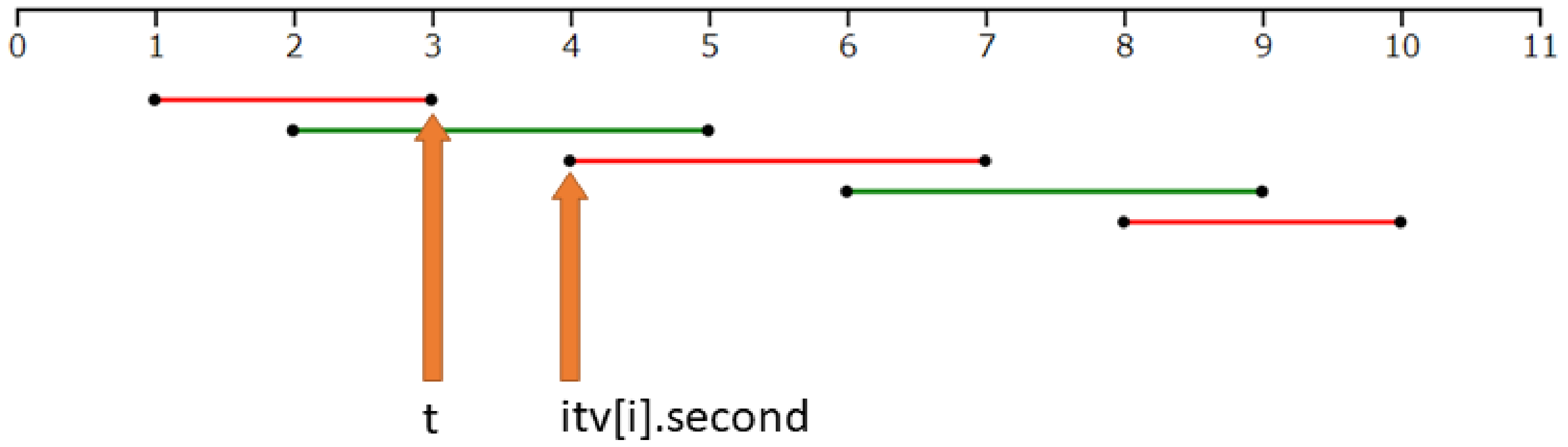
- In fact, this idea is true.
- Appendix of this slide describes this proof.

```cpp
int interval_scheduling(int N, vector<int>& s, vector<int>& t) {
  vector<pair<int, int>> itv;
  for (int i = 0; i < N; i++) {
    itv.push_back(make_pair(t[i], s[i])); // Note the order: not (s[i], t[i])!
  }
  sort(itv.begin(), itv.end());

  // t: finished time chosen last
  int ans = 0, t = 0;
  for (int i = 0; i < N; i++) {
    if (t <= itv[i].second) {
      ans++;
      t = itv[i].first;
    }
  }
  return ans;
}
```

Addition: Meaning of

```
...
  if (t < itv[i].second) {
    ans++;
    t = itv[i].first;
  }
...
```



Greedy Idea

- Idea: Problems about lexical order
- Idea: Problems about "Choose 'strict' things for each step"
- Idea: when you think validness of greedy:
  "Replacement of 'usual thing' for 'optical thing' doesn't get things worse" - said by *drken*:
  More detail describes Appendix of this slide.

Idea: Problems about lexical order

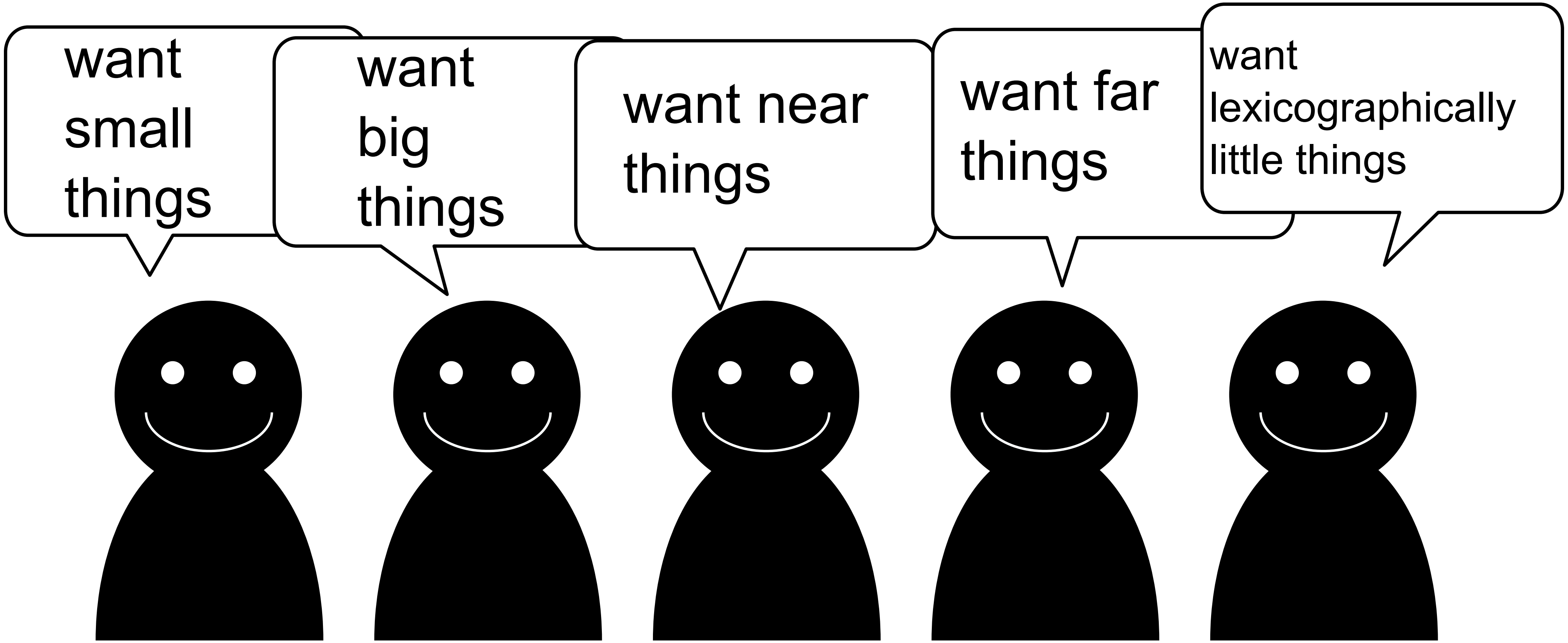
Problems asked a lexicographiclly minimum string is often the idea of greedy: you choose as lexicographiclly little character as possible.

Idea: Problems about "Choose 'strict' things for each step"

Example:

for each step,

- choose as a small/little thing as possible
- choose as a big/large thing as possible
- choose as a near thing as possible
- choose as a far thing as possible
- choose as lexicographiclly little character as possible.

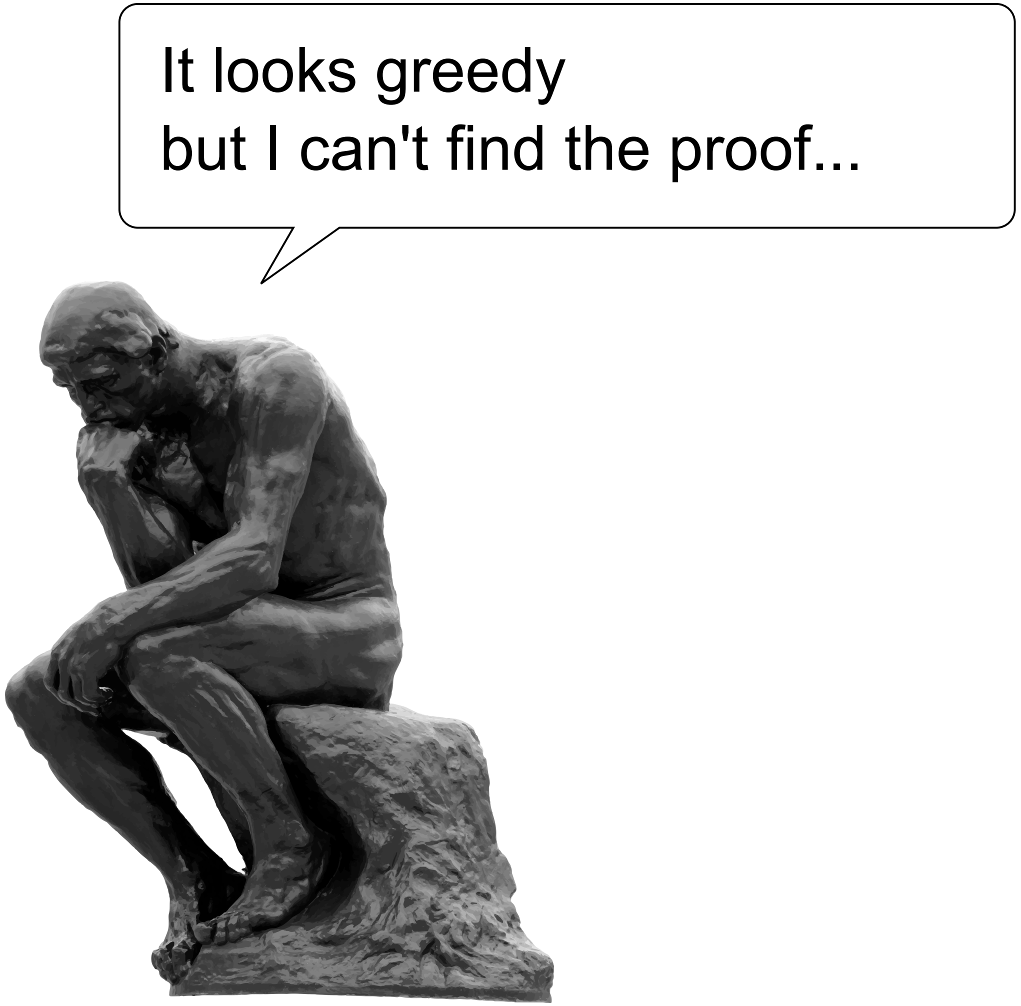


- For that reason, sorting is often good strategy.

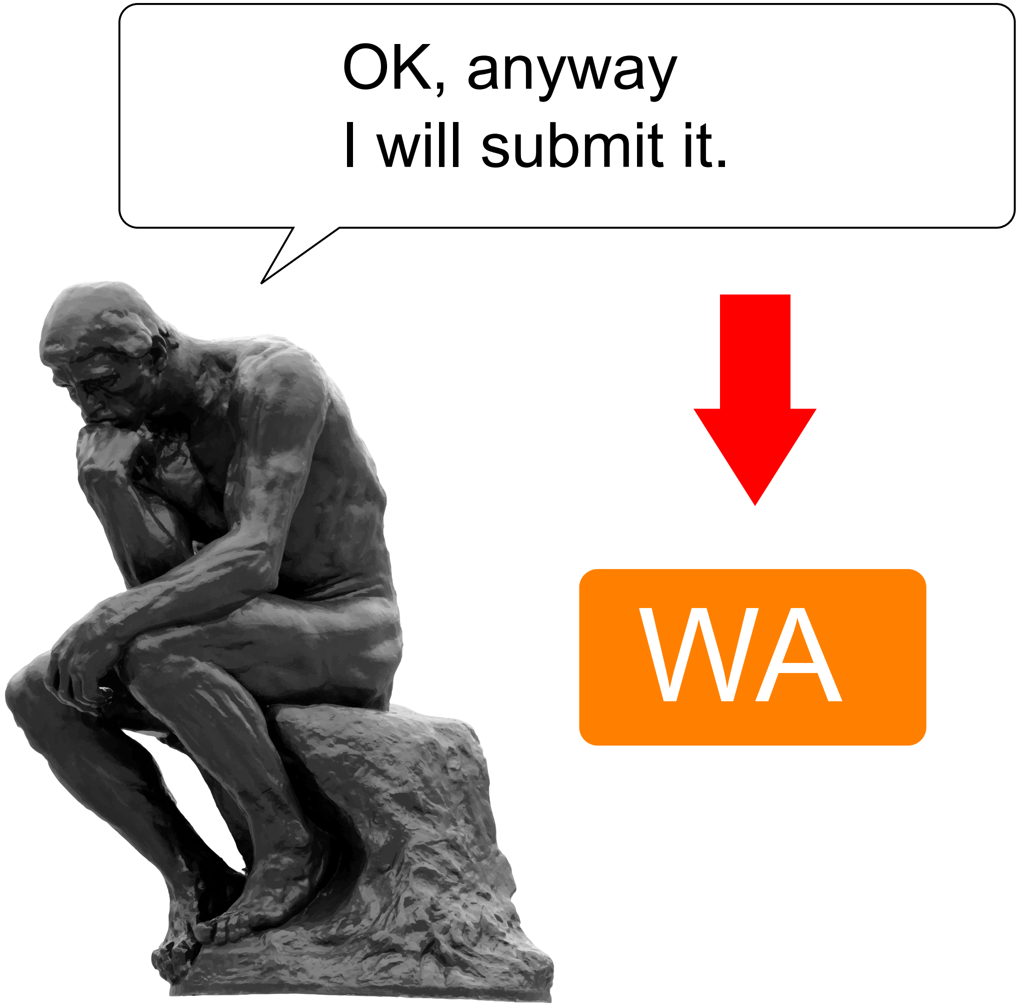
Proof of validness

- Some typical problems should be in your mind (e.g. interval scheduling)

- In a contest, you shouldn't have to prove the validness strictly.
  You shouldn't spend so much time to prove that the contest is over.



- But you should think rough proof.



- If you have an idea of greedy method in a contest whitch is difficult to prove, it is good to try to make **counterexample**.
  If you cannot find it, this method may be true.

Famous Greedy Algorithm

This algorithm choose as a small thing as possible

- Dijkstra Algorithm (Shortest Path)
- Prim and Kruskal Algorithm (Minimum Spanning Tree)
- Ford Fulkerson Algorithm (Maximum Flow)

Appendix

Proof of validness of greedy method on Interval Scheduling

- Understanding this proof may not contribute to your growness on competitive programming.
- But it is interesting.

Outline

1. Formal definition of Interval Scheduling
2. Having optimal substructure
3. Includion of the earliest finishing task in the answer.

This proof is based on "Introduction to Algorithms (CLRS)."

Formal definition of Interval Scheduling

Input: Set of intervals: $S_i = \{a_1, a_2, \cdots, a_n\}$

$$a_i := \text{right-open interval } [s_i, t_i).$$

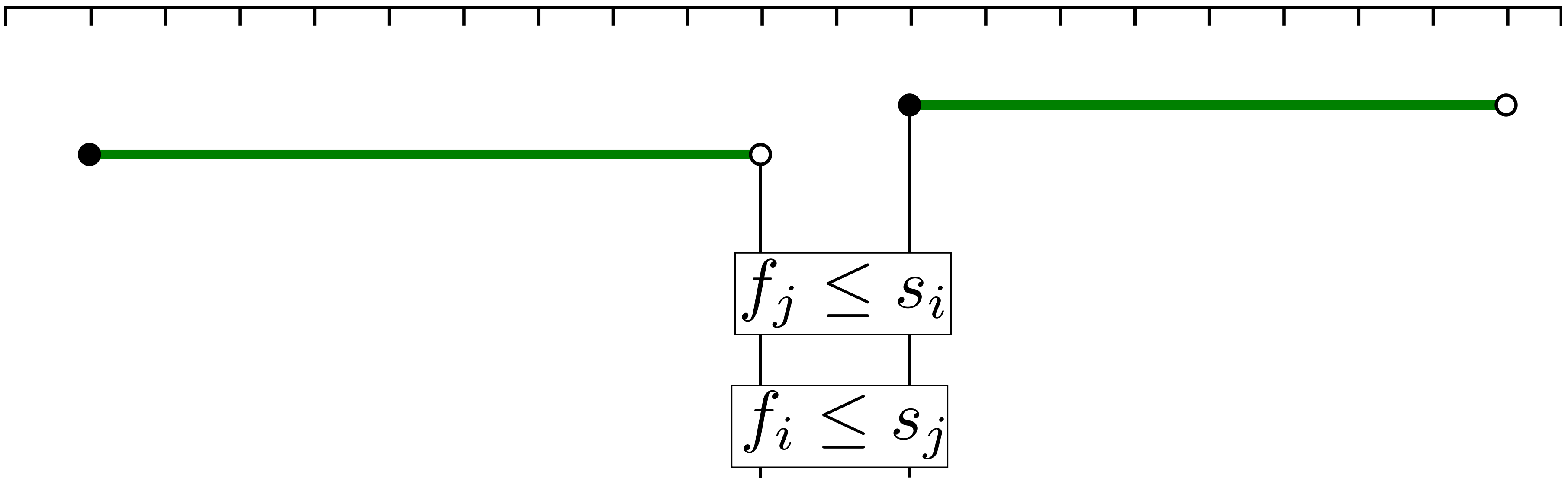
$$a_i \text{ and } a_j \text{ is disjoint } \overset{def}{\Leftrightarrow} f_i \leq s_j \vee f_j \leq s_i$$

Output: Maxium size set of mutually disjoint tasks

Supplement:

$$a_i \text{ and } a_j \text{ is disjoint } \overset{def}{\Leftrightarrow} f_i \leq s_j \vee f_j \leq s_i$$

In other words, two intervals are not overlapped.



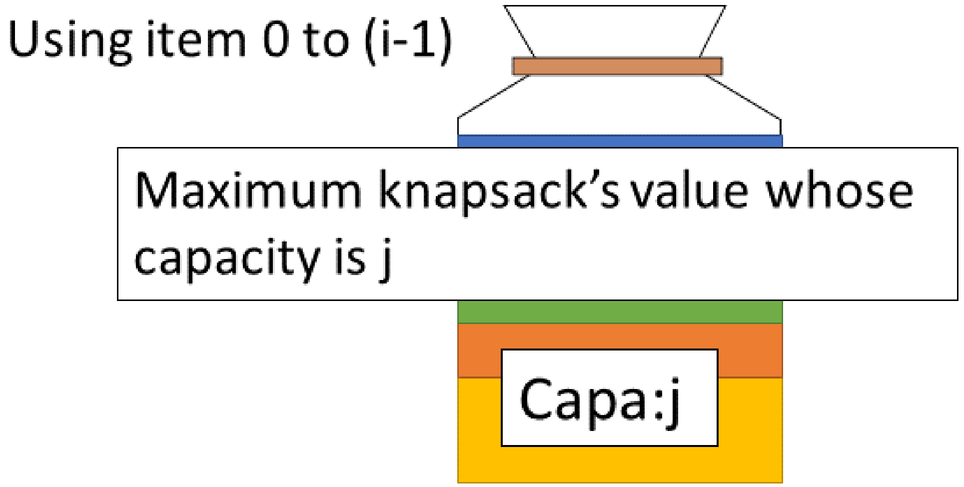
Optimal substructure of the Interval Scheduling

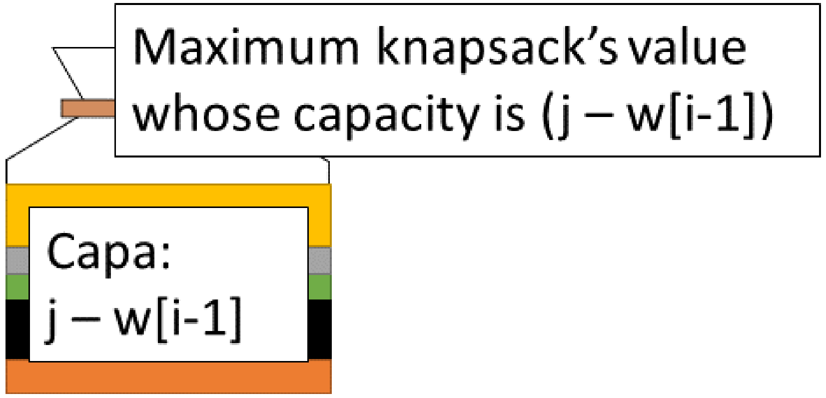
What is optimal substructure ?

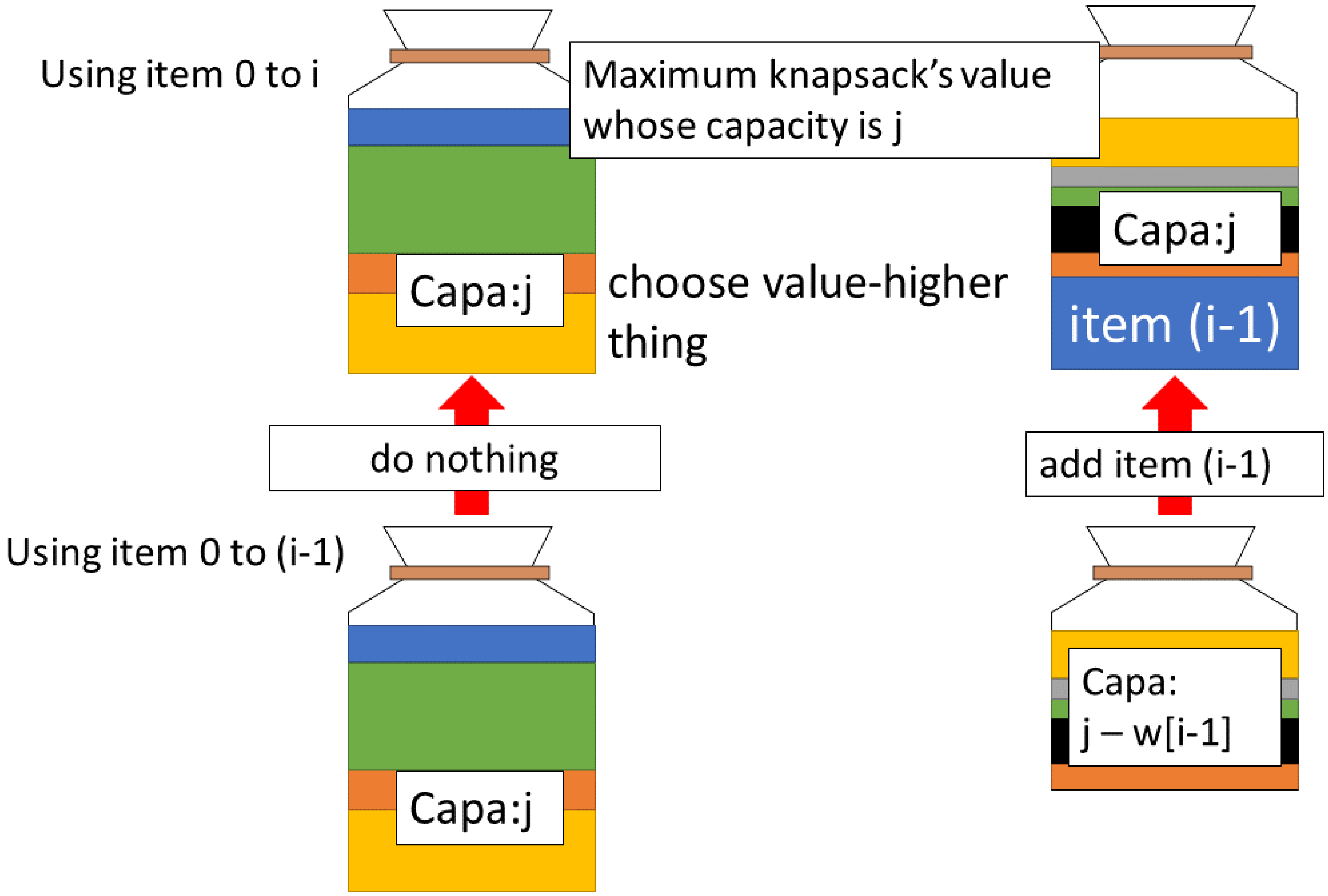
In computer science, a problem is said to have optimal substructure if an optimal solution can be constructed from optimal solutions of its subproblems.
("Optimal substructure", Wikipedia)

Example: Knapsack Problem

We already know how to maximize these knapsack's value (In other words, we already know optimal problems of subproblems.)






Using item 0 to i
Maximum knapsack's value whose capacity is j
Capa:j
choose value-higher thing
Capa:j
item (i-1)
do nothing
add item (i-1)
Using item 0 to (i-1)
Capa:j
Capa: j – w[i-1]

Optimal substructure of the Interval Scheduling

$$S_{ij} := \{a_n \mid f_i \leq s_n \land f_n < s_j\}$$

$$A_{ij} := \text{the answer of Interval Scheduling for } S_{ij}$$

then if $a_k \in A_{ij}$,

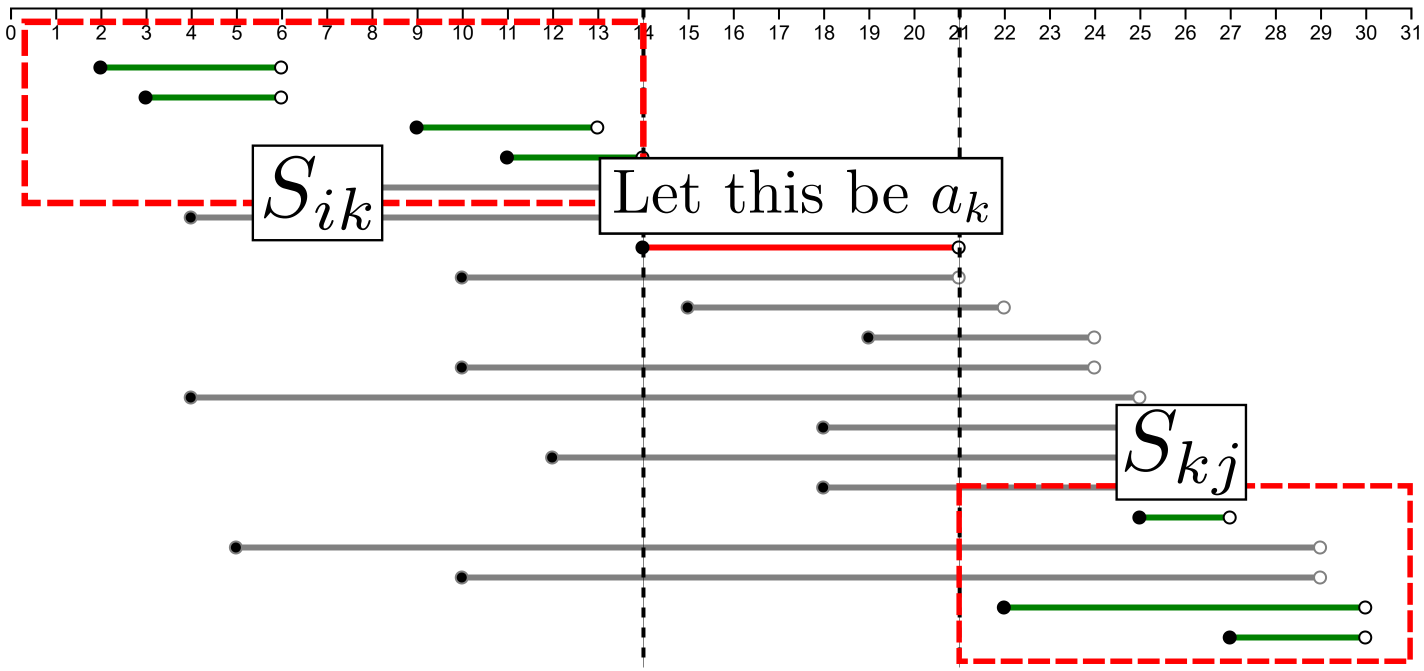
$$A_{ik} := A_{ij} \cap S_{ik}$$

$$A_{kj} := A_{ij} \cap S_{kj}$$

$$A_{ij} = A_{ik} \cup \{a_k\} \cup A_{kj}$$

$$\therefore |A_{ij}| = |A_{ik}| + 1 + |A_{kj}|$$

In other words, A_{ij} consists in A_{ik} and A_{kj}.



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S_{ik}
Let this be a_k
S_{kj}

Lem1. A_{ik} is the answer of Interval Scheduling for S_{ik}

Proof:

$$A_{ik} := A_{ij} \cap S_{ik} \Rightarrow A_{ik} \subset S_{ik}$$

If we could find A'_{ik} such that $A'_{ik}, \subset S_{ik} \wedge |A'_{ik}| > |A_{ik}|$

Note that $A'_{ik} \cap A_{kj} = \emptyset$, then

$$|A'_{ik}| + 1 + |A_{kj}| > |A_{ik}| + 1 + |A_{kj}| = |A_{ij}|$$

That cntradicts the assumution that A_{ij} is an optimal solutions.

We can also prove in the same way that A_{kj} is the answer of Interval Scheduling for S_{kj}.

3. Includion of the earliest finishing task in the answer.

Lem2.

$$S_k := \text{no empty subproblems}$$

$$a_m := \text{an activity in } S_k \text{ with the earliest finish time}$$

$$A_k := \text{the answer of Interval Scheduling for } S_k$$

then, $a_m \in A_k$.

Proof: Let a_j be the activity in A_k with the earliest finish time.

If $a_j = a_m$, we are done.

If $a_j \neq a_m$, let a_j be replaced to a_m :

$$A'_k = A_k - \{a_j\} \cup \{a_m\}$$

The tasks in A'_k are disjoint.

$\because A_k$ is disjoint,

The definition of a_j,

$f_m \leq f_j$.

$|A'_k| = |A_k|$, so we substitute A'_k for A_k.

In conclusion, $a_m \in A_k$.

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In conclusion, $a_m \in A_k$.

Idea: Replacement of 'usual thing' for 'optical thing' doesn't get things worse

Note

- Idea: "Replacement of 'usual thing' for 'optical thing' doesn't things worse" is said by *drken*:
  Ref: AtCoder 版！蟻本 (初級編)
  Ref: AtCoder AGC 029 B - Powers of two (600 点)

Excercise

- AGC001A - BBQ Easy
- ABC076C - Dubious Document 2
- ABC103D - Islands War

Answer

AGC001A

- You should choose two minimum numbers from $\{L_1, L_2, \cdots, L_{2N}\}$ and make pair.
  Because minimum number has an effect on score.
- In sammary, let's sort $\{L_i\}$ and make pair in asceding order.

```cpp
#include <iostream>
#include <vector>
#include <algorithm>

using namespace std;

int main()
{
  int N; cin >> N;
  vector<int> L(2*N);
  for (int i = 0; i < 2*N; i++) cin >> L[i];
  sort(L.begin(), L.end());
  int ans = 0;
  for (int i = 0; i < 2*N; i += 2) {
    ans += L[i];
  }
  cout << ans << endl;
  return 0;
}
```

ABC076C

- You want to place 'a' at as much left as possible.
- Searching substring T in S from back to front.
- Making small function makes your implement easy.
- If you want to search from back, reverse makes your implement easy.

```cpp
#include <iostream>
#include <algorithm>
#include <string>

using namespace std;

bool check(string& S, string& T, int idx)
{
  for (int i = 0; i < T.size(); i++) {
    if (i + idx >= S.size()) return false;
    if (S[i + idx] != T[i] && S[i + idx] != '?') return false;
  }
  return true;
}

int main()
{
  string S, T;
  cin >> S >> T;
  reverse(S.begin(), S.end());
  reverse(T.begin(), T.end());

  for (int i = 0; i < S.size(); i++) {
    if (check(S, T, i)) {
      for (int j = 0; j < T.size(); j++) {
        S[i + j] = T[j];
      }
      for (int i = 0; i < S.size(); i++) {
        if (S[i] == '?') S[i] = 'a';
      }
      reverse(S.begin(), S.end());
      cout << S << endl;
      return 0;
    }
  }
  cout << "UNRESTORABLE" << endl;

  return 0;
}
```

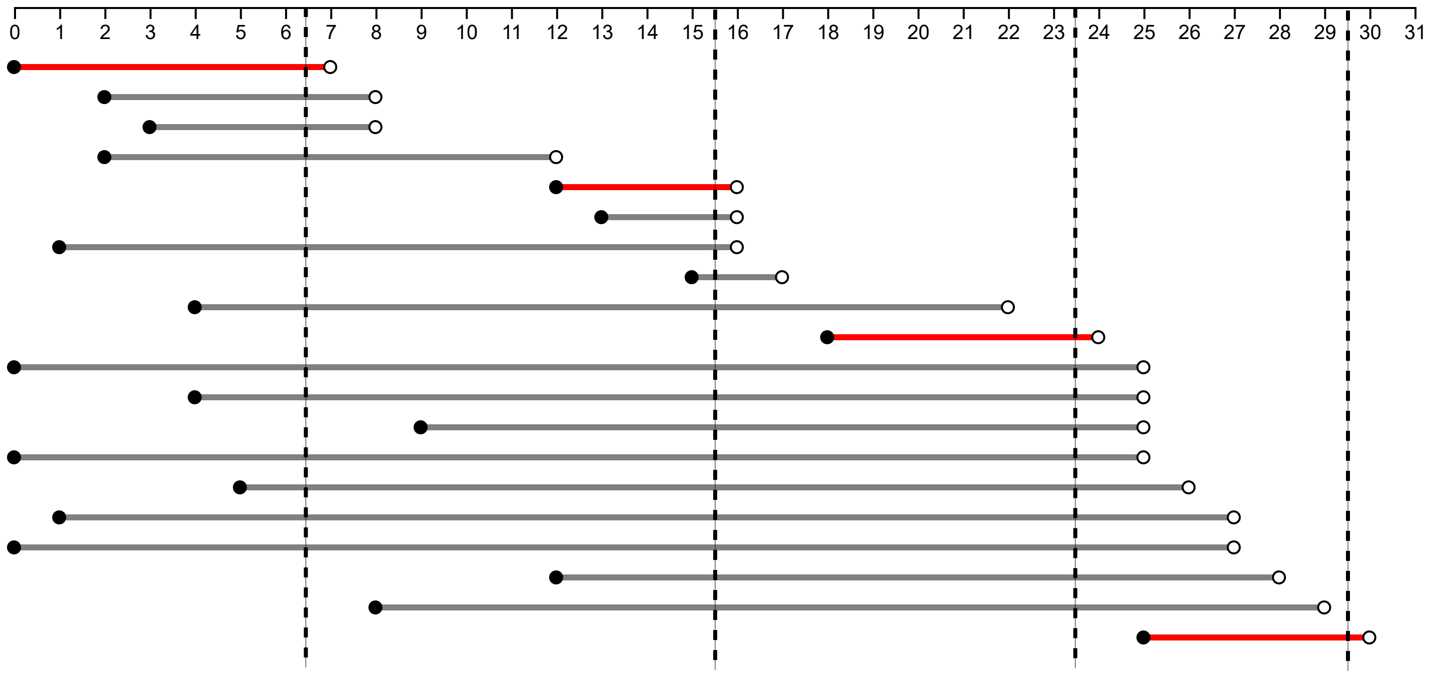
ABC103D

Reduction to Interval Scheduling

- It is no use choosing two intervals which are mutually overlapped.
  If you remove one interval, another interval need not be removed.
  So, you only choose mutually disjoint intervals.
- If a interval is not overlapped of all other intervals, it should be chosen.
  $\Rightarrow$ It looks Interval Scheduling...

- Here, the intervals are sored in the accending order on b_i.
  Let A be an answer of Interval Scheduling.
- If an element in A is removed, a request told in problem statement is not satisfied.
- If an element not in A is added, it is overlapped to some intervals in A.
- If a interval is (a_i, b_i), you only remove the bridge which links $(b_i - 1)$ to b_i.
- So, $|A|$ is problem's answer.

Example: Dotted line shows positions of removing bridge.



```cpp
#include <iostream>
#include <vector>
#include <algorithm>

using namespace std;

int interval_scheduling(int N, vector<int>& vs, vector<int>& vt) {
  vector<pair<int, int>> itv;
  for (int i = 0; i < N; i++) {
    itv.push_back(make_pair(vt[i], vs[i]));
  }
  sort(itv.begin(), itv.end());

  int ans = 0, t = 0;
  for (int i = 0; i < N; i++) {
    if (t <= itv[i].second) {
      ans++;
      t = itv[i].first;
    }
  }
  return ans;
}

int main()
{
  int N, M; cin >> N >> M;
  vector<int> s(M), t(M);
  for (int i = 0; i < M; i++) cin >> s[i] >> t[i];
  cout << interval_scheduling(M, s, t) << endl;
  return 0;
}
```