Bayesian Inference on Parametric Zombie Survival Model

Bomin Kim

May 7, 2017

Abstract

In this paper, we implement Bayesian Inference on the new split population survival model, that explicitly models the misclassification probability of failure (vs. right censored) events. This includes two parametric survival models (Exponential and Weibull) and (possibly) Cox proportional hazards regression model.

Review on Parametric Zombie Survival Model

Likelihood function

Recall from Ben's "Parametric Zombie Survival Model" that the probability of misclassification (that is, subset of non-censured failure outcomes that are being misclassified) is

$$\alpha = \Pr(C_i = 1|\widetilde{C}_i = 0). \tag{1}$$

The unconditional density is thus given by the combination of an observation's misclassification probability and its probability of experiencing an actual failure conditional on not being misclassified,

$$\alpha_i + (1 - \alpha_i) * f(t_i) \tag{2}$$

And the unconditional survival function is therefore

$$(1 - \alpha_i) * S(t_i), \tag{3}$$

where

$$\alpha_i = \frac{\exp(\mathbf{Z}\gamma)}{1 + \exp(\mathbf{Z}\gamma)}. \tag{4}$$

The likeihood function of the Parametric Zombie Survival Model is defined as

$$L = \prod_{i=1}^{N} [\alpha_i + (1 - \alpha_i) f(t_i|\mathbf{X}, \boldsymbol{\beta})]^{C_i} [(1 - \alpha_i) S(t_i|\mathbf{X},\boldsymbol{\beta})]^{1-C_i} \tag{5}$$

And the log likelihood is

$$lnL = \sum_{i=1}^{N} \{C_i \ln[\alpha_i + (1 - \alpha_i) f(t_i|\mathbf{X}, \boldsymbol{\beta})] + (1 - C_i) \ln[(1 - \alpha_i) S(t_i|\mathbf{X},\boldsymbol{\beta})]\}. \tag{6}$$

Bayesian Analysis of Parametric Zombie Survival Models

Exponential

For exponential survival model, the density function and survival function are

$$\begin{aligned} f(t_i|X_i, \boldsymbol{\beta}) &= \exp(X_i\boldsymbol{\beta})\exp(-\exp(X_i\boldsymbol{\beta})t_i) \\ S(t_i|X_i, \boldsymbol{\beta}) &= \exp(-\exp(X_i\boldsymbol{\beta})t_i). \end{aligned} \tag{7}$$

Then, the likelihood function of Exponential Zombie survival model is

$$L(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \prod_{i=1}^{N}[\alpha_i + (1-\alpha_i)\exp(X_i\boldsymbol{\beta})\exp(-\exp(X_i\boldsymbol{\beta})t_i)]^{C_i}[(1-\alpha_i)\exp(-\exp(X_i\boldsymbol{\beta})t_i)]^{1-C_i} \tag{8}$$

where X_i is the i^{th} row of the covariate matrix $\mathbf{X}$.

In the exponential survival model, we assume the prior of $\boldsymbol{\beta} = \{\beta_1, ..., \beta_{p_1}\}$ as

$$\boldsymbol{\beta} \sim \mathrm{MVN}_{p_1}(\boldsymbol{\mu}_\beta, \Sigma_\beta), \tag{9}$$

thus the conditional posterior distribution for $\boldsymbol{\beta}$ parameters is given by

$$\pi(\boldsymbol{\beta}|\mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\gamma}) \propto L(\boldsymbol{\beta}|\mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\gamma}) \times \pi(\boldsymbol{\beta}|\boldsymbol{\mu}_\beta, \Sigma_\beta). \tag{10}$$

Moreover, we can also assign mutivariate Normal prior to $\boldsymbol{\gamma} = \{\gamma_1, ..., \gamma_{p_2}\}$,

$$\boldsymbol{\gamma} \sim \mathrm{MVN}_{p_2}(\boldsymbol{\mu}_\gamma, \Sigma_\gamma), \tag{11}$$

and the corresponding conditional posterior distribution of $\boldsymbol{\gamma}$ becomes

$$\pi(\boldsymbol{\gamma}|\mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\beta}) \propto L(\boldsymbol{\gamma}|\mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\beta}) \times \pi(\boldsymbol{\gamma}|\boldsymbol{\mu}_\gamma, \Sigma_\gamma). \tag{12}$$

Weibull

If the survival time t has a Weibull distribution of $W(t|\lambda, X_i\boldsymbol{\beta})$, the density function and survival function are

$$\begin{aligned} f(t_i|\lambda, X_i, \boldsymbol{\beta}) &= \lambda t_i^{\lambda-1}\text{exp}(X_i\boldsymbol{\beta} - \text{exp}(X_i\boldsymbol{\beta})t_i^{\lambda}) \\ S(t_i|\lambda, X_i, \boldsymbol{\beta}) &= \text{exp}(-\text{exp}(X_i\boldsymbol{\beta})t_i^{\lambda}), \end{aligned} \tag{13}$$

which shows that $\lambda = 1$ reduces to Exponential survival model, which is a well-known property. The likelihood function of Weibull Zombie survival model is

$$L(\lambda, \boldsymbol{\beta}, \boldsymbol{\gamma}) = \prod_{i=1}^{N}[\alpha_i + (1-\alpha_i)\lambda t_i^{\lambda-1}\text{exp}(X_i\boldsymbol{\beta} - \text{exp}(X_i\boldsymbol{\beta})t_i^{\lambda})]^{C_i}[(1-\alpha_i)\text{exp}(-\text{exp}(X_i\boldsymbol{\beta})t_i^{\lambda})]^{1-C_i}. \tag{14}$$

For the two parameters λ and $\boldsymbol{\beta} = \{\beta_1, ..., \beta_{p_1}\}$, we assign prior to each parameter as

$$\begin{aligned} \lambda &\sim \text{Gamma}(a_\lambda, b_\lambda) \\ \boldsymbol{\beta} &\sim \text{MVN}_{p_1}(\boldsymbol{\mu}_\beta, \Sigma_\beta), \end{aligned} \tag{15}$$

where the conditional distribution for λ and $\boldsymbol{\beta}$ parameters are given by

$$\begin{aligned} \pi(\lambda|\mathbf{C}, \boldsymbol{\alpha}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\beta}, \boldsymbol{\gamma}) &\propto L(\lambda|\mathbf{C}, \boldsymbol{\alpha}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \times \pi(\lambda|a_\lambda, b_\lambda) \\ \pi(\boldsymbol{\beta}|\mathbf{C}, \boldsymbol{\alpha}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\gamma}, \lambda) &\propto L(\boldsymbol{\beta}|\mathbf{C}, \boldsymbol{\alpha}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\gamma}, \lambda) \times \pi(\boldsymbol{\beta}|\boldsymbol{\mu}_\beta, \Sigma_\beta). \end{aligned} \tag{16}$$

Same as Exponential case, we can assign mutivariate Normal prior to $\boldsymbol{\gamma} = \{\gamma_1, ..., \gamma_{p_2}\}$,

$$\boldsymbol{\gamma} \sim \text{MVN}_{p_2}(\boldsymbol{\mu}_\gamma, \Sigma_\gamma), \tag{17}$$

and the corresponding conditional posterior distribution becomes

$$\pi(\boldsymbol{\gamma}|\mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\beta}, \lambda) \propto L(\boldsymbol{\gamma}|\mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\beta}, \lambda) \times \pi(\boldsymbol{\gamma}|\boldsymbol{\mu}_\gamma, \Sigma_\gamma). \tag{18}$$

Markov chain Monte Carlo (MCMC) algorithm

Since we do not have any informative prior for parametric Zombie survival model, we can follow the common approach and specify our hyperparameters as below:

$$\begin{aligned}
&\boldsymbol{\mu}_\beta = \mathbf{0}, \Sigma_\beta \sim \mathrm{IW}(\mathbf{I}_{p_1}, \nu_1), \\
&\boldsymbol{\mu}_\gamma = \mathbf{0}, \Sigma_\gamma \sim \mathrm{IW}(\mathbf{I}_{p_2}, \nu_2), \\
&a_\lambda = b_\lambda = 0.001,
\end{aligned} \tag{19}$$

where we use hierarchical Bayesian modeling to esitmate Σ_β and Σ_γ using Inverse-Wishart distribution. Note that if this step seems to be unnecessary, we can instead simply fix those such as $\Sigma_\beta = \Sigma_\gamma = 10^4\mathbf{I}$.

In the survival regression setting, closed forms for the posterior distribution of $\boldsymbol{\beta}$ (as well as $\boldsymbol{\gamma}$ in our model) are generally not available (which is also our cases: Equation (10), (12), (16), (18)), and therefore one needs to use numerical integration or Markov chain Monte Carlo (MCMC) methods. Here we will use MCMC methods with following update scheme:

Step 0. Choose an arbitrary starting point $\boldsymbol{\beta}_0, \boldsymbol{\gamma}_0$, and λ_0 (if Weibull) and set $i = 0$.

Step 1. Update $\Sigma_\beta \sim \pi(\Sigma_\beta|\boldsymbol{\beta}_i)$ and $\Sigma_\gamma \sim \pi(\Sigma_\gamma|\boldsymbol{\gamma}_i)$ using Gibbs sampler.

Step 2. Update $\boldsymbol{\beta} \sim \pi(\boldsymbol{\beta}|\mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\gamma}, \lambda, \boldsymbol{\mu}_\beta, \Sigma_\beta)$ and $\boldsymbol{\gamma} \sim \pi(\boldsymbol{\gamma}|\mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\beta}, \lambda, \boldsymbol{\mu}_\gamma, \Sigma_\gamma)$ using Metropolis-Hastings algorithm or slice sampler.

Step 2′. If Weibull, update $\lambda \sim \pi(\lambda|\mathbf{C}, \boldsymbol{\alpha}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \boldsymbol{\beta}, \boldsymbol{\gamma}, a_\lambda, b_\lambda)$ using Metropolis-Hastings algorithm or slice sampler.

Step 3. Set $i = i + 1$, and go to Step 1.

The closed form of $\pi(\Sigma_\beta|\boldsymbol{\beta}_i)$ and $\pi(\Sigma_\gamma|\boldsymbol{\gamma}_i)$ in Step 1 are derived in APPENDIX A. Detailed procedure of Metropolis-Hastings algorithm or slice sampler will be illustrated later.

APPENDIX

APPENDIX A: Derivation of the full conditional distributions

1. Σ_β:

$$
\begin{aligned}
\pi(\Sigma_\beta|\boldsymbol{\beta}) &\propto \pi(\boldsymbol{\beta}|\boldsymbol{\mu}_\beta = \mathbf{0}, \Sigma_\beta) \times \pi(\Sigma_\beta) \\
&\propto |\Sigma_\beta|^{-\frac{1}{2}} \exp\Big\{ -\frac{1}{2}\Big(\boldsymbol{\beta}'\Sigma_\beta^{-1}\boldsymbol{\beta}\Big)\Big\} \times |\Sigma_\beta|^{-\frac{\nu_1+p_1+1}{2}} \exp\Big\{ -\frac{1}{2}\mathrm{tr}\Big(\mathbf{I}_{p_1}\Sigma_\beta^{-1}\Big)\Big\} \\
&= |\Sigma_\beta|^{-\frac{1+\nu_1+p_1+1}{2}} \exp\Big\{ -\frac{1}{2}\mathrm{tr}\Big(\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_\beta^{-1}\Big)\Big\} \times \exp\Big\{ -\frac{1}{2}\mathrm{tr}\Big(\mathbf{I}_{p_1}\Sigma_\beta^{-1}\Big)\Big\} \\
&= |\Sigma_\beta|^{-\frac{1+\nu_1+p_1+1}{2}} \exp\Big\{ -\frac{1}{2}\mathrm{tr}\Big((\boldsymbol{\beta}\boldsymbol{\beta}' + \mathbf{I}_{p_1})\Sigma_\beta^{-1}\Big)\Big\} \\
&\sim \mathrm{IW}(\boldsymbol{\beta}\boldsymbol{\beta}' + \mathbf{I}_{p_1}, \quad 1 + \nu_1)
\end{aligned}
$$

2. Σ_γ:

$$
\begin{aligned}
\pi(\Sigma_\gamma|\boldsymbol{\gamma}) &\propto \pi(\boldsymbol{\gamma}|\boldsymbol{\mu}_\gamma = \mathbf{0}, \Sigma_\gamma) \times \pi(\Sigma_\gamma) \\
&\propto |\Sigma_\gamma|^{-\frac{1}{2}} \exp\Big\{ -\frac{1}{2}\Big(\boldsymbol{\gamma}'\Sigma_\gamma^{-1}\boldsymbol{\gamma}\Big)\Big\} \times |\Sigma_\gamma|^{-\frac{\nu_1+p_2+1}{2}} \exp\Big\{ -\frac{1}{2}\mathrm{tr}\Big(\mathbf{I}_{p_2}\Sigma_\gamma^{-1}\Big)\Big\} \\
&= |\Sigma_\gamma|^{-\frac{1+\nu_2+p_2+1}{2}} \exp\Big\{ -\frac{1}{2}\mathrm{tr}\Big(\boldsymbol{\gamma}\boldsymbol{\gamma}'\Sigma_\gamma^{-1}\Big)\Big\} \times \exp\Big\{ -\frac{1}{2}\mathrm{tr}\Big(\mathbf{I}_{p_2}\Sigma_\gamma^{-1}\Big)\Big\} \\
&= |\Sigma_\gamma|^{-\frac{1+\nu_2+p_2+1}{2}} \exp\Big\{ -\frac{1}{2}\mathrm{tr}\Big((\boldsymbol{\gamma}\boldsymbol{\gamma}' + \mathbf{I}_{p_2})\Sigma_\gamma^{-1}\Big)\Big\} \\
&\sim \mathrm{IW}(\boldsymbol{\gamma}\boldsymbol{\gamma}' + \mathbf{I}_{p_2}, \quad 1 + \nu_2)
\end{aligned}
$$