Supplementary Materials for
"Dynamic Additive and Multiplicative Effects (DAME) Model with Application to the United Nations Voting Behaviors"

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1 Metropolis-Hastings algorithm for GP parameters

For Gaussian process parameters—variance parameter τ and length-scale parameter κ, we use the Metropolis-Hastings algorithm with a proposal density Q being the truncated Normal distribution with support $(0, \infty)$. Given the proposal distribution variance $\sigma^2_Q = (\sigma^2_{Q1}, \sigma^2_{Q2})$, we sample the new values τ' and κ' from

$$\begin{aligned} \tau' &\sim \text{truncnorm}(\tau, \sigma^2_{Q1}) \\ \kappa' &\sim \text{truncnorm}(\kappa, \sigma^2_{Q2}). \end{aligned} \tag{1}$$

We then accept the new proposed value (τ', κ') with probability equal to:

$$\text{Acceptance Probability} = \begin{cases} \frac{P(\tau|\tau')P(\kappa|\kappa')P(\tau^{x\prime}, \kappa^{x\prime}|x(\cdot), a_x, b_x, \gamma)}{P(\tau'|\tau)P(\kappa'|\kappa)P(\tau^{x}, \kappa^{x}|x(\cdot), a_x, b_x, \gamma)} & \text{if} < 1 \\ 1 & \text{else} \end{cases}, \tag{2}$$

where x is the T-length vector of interest (e.g. β_p, θ, d), τ has a prior $\tau \sim \text{IG}(a_x, b_x)$, and κ has a prior $\kappa \sim \text{half-cauchy}(\gamma)$. If a sample from Uniform(0,1) is less than the acceptance probability, we accept the proposed value. Otherwise, we reject. For any x, the probability from Q-ratio is $\frac{P(\tau|\tau')P(\kappa|\kappa')}{P(\tau'|\tau)P(\kappa'|\kappa)}$—i.e., ratio of truncated normal densities—while the posterior part $\frac{P(\tau^{x\prime}, \kappa^{x\prime}|x(\cdot), a_x, b_x, \gamma)}{P(\tau^{x}, \kappa^{x}|x(\cdot), a_x, b_x, \gamma)}$ may vary. Below are the derivation of acceptance ratio for each of the variables.

1. $(\tau_p^\beta, \kappa_p^\beta)$, for $p = 1, \ldots, P$:

$$\begin{aligned} \frac{P(\tau_p^{\beta\prime}, \kappa_p^{\beta\prime}|\beta_p, a_\beta, b_\beta, \gamma)}{P(\tau_p^{\beta}, \kappa_p^{\beta}|\beta_p, a_\beta, b_\beta, \gamma)} &= \frac{P(\tau_p^{\beta\prime}, \kappa_p^{\beta\prime}, \beta_p|a_\beta, b_\beta, \gamma)}{P(\tau_p^{\beta}, \kappa_p^{\beta}, \beta_p|a_\beta, b_\beta, \gamma)} \\ &= \frac{P(\tau_p^{\beta\prime}|a_\beta, b_\beta)P(\kappa_p^{\beta\prime}|\gamma)P(\beta_p|\tau_p^{\beta\prime}, \kappa_p^{\beta\prime})}{P(\tau_p^{\beta}|a_\beta, b_\beta)P(\kappa_p^{\beta}|\gamma)P(\beta_p|\tau_p^{\beta}, \kappa_p^{\beta})}, \end{aligned} \tag{3}$$

2. $(\tau^\theta, \kappa^\theta)$

$$\begin{aligned} \frac{P(\tau^{\theta\prime}, \kappa^{\theta\prime}|\boldsymbol{\theta}, a_\theta, b_\theta, \gamma)}{P(\tau^{\theta}, \kappa^{\theta}|\boldsymbol{\theta}, a_\theta, b_\theta, \gamma)} &= \frac{P(\tau^{\theta\prime}, \kappa^{\theta\prime}, \boldsymbol{\theta}|a_\theta, b_\theta, \gamma)}{P(\tau^{\theta}, \kappa^{\theta}, \boldsymbol{\theta}|a_\theta, b_\theta, \gamma)} \\ &= \frac{P(\tau^{\theta\prime}|a_\theta, b_\theta)P(\kappa^{\theta\prime}|\gamma)\prod_{i=1}^N P(\theta_i|\tau^{\theta\prime}, \kappa^{\theta\prime})}{P(\tau^{\theta}|a_\theta, b_\theta)P(\kappa^{\theta}|\gamma)\prod_{i=1}^N P(\theta_i|\tau^{\theta}, \kappa^{\theta})}, \end{aligned} \tag{4}$$

3. (τ_r^d, κ_r^d), for $r = 1, \ldots, R$:

$$\begin{aligned} \frac{P(\tau_r^{d\prime}, \kappa_r^{d\prime}|d_r, a_d, b_d, \gamma)}{P(\tau_r^{d}, \kappa^{d}|d_r, a_d, b_d, \gamma)} &= \frac{P(\tau_r^{d\prime}, \kappa_r^{d\prime}, d_r|a_d, b_d, \gamma)}{P(\tau_r^{d}, \kappa_r^{d}, d_r|a_d, b_d, \gamma)} \\ &= \frac{P(\tau_r^{d\prime}|a_d, b_d)P(\kappa_r^{d\prime}|\gamma)P(d_r|\tau_r^{d\prime}, \kappa_r^{d\prime})}{P(\tau_r^{d}|a_d, b_d)P(\kappa_r^{d}|\gamma)P(d_r|\tau_r^{d}, \kappa_r^{d})}. \end{aligned} \tag{5}$$

2 Proofs on Posterior Computation

2.1 Noise error variance σ_e^2

$$\begin{aligned}
&P(\sigma_e^2|\mathbf{Y}, a_\sigma, b_\sigma) \propto P(\mathbf{Y}|\mathbf{X}, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{d}, \boldsymbol{u}, \sigma_e^2) \times P(\sigma_e^2|a_\sigma, b_\sigma) \\
&\propto \prod_{t=1}^{T}\prod_{i>j}(\sigma_e^2)^{-\frac{1}{2}}\exp\Big\{-\frac{1}{2\sigma_e^2}||y_{ij}^t - (\sum_{p=1}^{P}\beta_p^t X_{ijp}^t + \theta_i^t + \theta_j^t + u_i^{t\prime} D^t u_j^t)||^2\Big\} \times (\sigma_e^2)^{-a_\sigma - 1}\exp\Big\{\frac{1}{\sigma_e^2}b_\sigma\Big\} \\
&= (\sigma_e^2)^{-\frac{T}{2}\cdot\frac{N(N-1)}{2} - a_\sigma - 1} \times \exp\Big\{-\frac{1}{\sigma_e^2}\Big(\frac{1}{2}\sum_{t=1}^{T}\sum_{i>j}||y_{ij}^t - (\sum_{p=1}^{P}\beta_p^t X_{ijp}^t + \theta_i^t + \theta_j^t + u_i^{t\prime} D^t u_j^t)||^2 + b_\sigma\Big)\Big\} \\
&\sim \mathrm{IG}\big(\frac{T\cdot N(N-1)}{4} + a_\sigma, \quad \frac{1}{2}\sum_{t=1}^{T}\sum_{i>j}(E_{ij}^t)^2 + b_\sigma\big)
\end{aligned} \tag{6}$$

2.2 Fixed effect β_p

$$\begin{aligned}
&P(\beta_p|\mathbf{Y}, \mathbf{X}, \kappa_p^\beta, \tau_p^\beta) \propto P(\mathbf{Y}|\mathbf{X}, \beta_p, \boldsymbol{\beta}_{[-p]}, \boldsymbol{\theta}, \boldsymbol{d}, \boldsymbol{u}, \sigma_e^2) \times P(\beta_p|\kappa_p^\beta, \tau_p^\beta) \\
&\propto \prod_{i>j}\exp\Big\{-\frac{1}{2\sigma_e^2}||E_{ij[-p]} - X_{ijp}\beta_p||^2\Big\} \times \exp\Big\{-\frac{1}{2}\big(\beta_p'(\tau_p^\beta c_p^\beta)^{-1}\beta_p\big)\Big\} \\
&\qquad \text{where } E_{ij[-p]} = \{E_{ij[-p]}^t\}_{t=1}^T \text{ (with } E_{ij[-p]}^t = E_{ij}^t + \beta_p^t X_{ijp}^t) \text{ and } X_{ijp} = \{X_{ijp}^t\}_{t=1}^T \\
&\propto \exp\Big\{-\frac{1}{2\sigma_e^2}\Big(\sum_{i>j} -2(E_{ij[-p]}X_{ijp})'\beta_p + \beta_p'\big(\mathrm{diag}(\sum_{i>j}X_{ijp}^2)\big)\beta_p\Big)\Big\} \times \exp\Big\{-\frac{1}{2}\big(\beta_p'(\tau_p^\beta c_p^\beta)^{-1}\beta_p\big)\Big\} \\
&\propto \exp\Big\{-\frac{1}{2}\Big(\beta_p'\big((\tau_p^\beta c_p^\beta)^{-1} + \frac{\mathrm{diag}(\sum_{i>j}X_{ijp}^2)}{\sigma_e^2}\big)\beta_p - \frac{2}{\sigma_e^2}\big(\sum_{i>j}(E_{ij[-p]}X_{ijp})'\beta_p\big)\Big)\Big\} \\
&\sim \mathrm{MVN}_T\big(\tilde{\mu}_{\beta_p}, \tilde{\Sigma}_{\beta_p}\big), \\
&\text{where } \tilde{\Sigma}_{\beta_p} = \Big((\tau_p^\beta c_p^\beta)^{-1} + \frac{\mathrm{diag}\big(\{\sum_{i>j}X_{ijp}^{t2}\}_{t=1}^T\big)}{\sigma_e^2}\Big)^{-1} \text{ and } \tilde{\mu}_{\beta_p} = \Big(\frac{\{\sum_{i>j}(E_{ij[-p]}^t X_{ijp}^t)\}_{t=1}^T}{\sigma_e^2}\Big)\tilde{\Sigma}_{\beta_p}.
\end{aligned} \tag{7}$$

2.3 Additive random effect θ_i

$$\begin{aligned}
&p(\theta_i|\mathbf{Y}, \kappa^\theta, \tau^\theta) \propto \prod_{i=i, j\neq i} p(\mathbf{Y}|\mathbf{X}, \boldsymbol{\beta}, \theta_i, \boldsymbol{\theta}_{[-i]}, \boldsymbol{d}, \boldsymbol{u}, \sigma_e^2) \times p(\theta_i|\kappa^\theta, \tau^\theta) \\
&\propto \prod_{i=i, j\neq i}\exp\Big\{-\frac{1}{2\sigma_e^2}||E_{ij[-i]} - \theta_i||^2\Big\} \times \exp\Big\{-\frac{1}{2}\big(\theta_i'(\tau^\theta c^\theta)^{-1}\theta_i\big)\Big\} \\
&\qquad \text{where } E_{ij[-i]} = \{E_{ij[-i]}^t\}_{t=1}^T \text{ with } E_{ij[-i]}^t = E_{ij}^t + \theta_i^t \\
&\propto \exp\Big\{-\frac{1}{2\sigma_e^2}\Big(\sum_{i=i, j\neq i} -2(E_{ij[-i]})'\theta_i + \theta_i'\big(\sum_{i=i, j\neq i} I_T\big)\theta_i\Big)\Big\} \times \exp\Big\{-\frac{1}{2}\big(\theta_i'(\tau^\theta c^\theta)^{-1}\theta_i\big)\Big\} \\
&\propto \exp\Big\{-\frac{1}{2}\Big(\theta_i'\big((\tau^\theta c^\theta)^{-1} + \frac{(N-1)I_T}{\sigma_e^2}\big)\theta_i - \frac{2}{\sigma_e^2}\big(\sum_{i=i, j\neq i}(E_{ij[-i]})'\theta_i\big)\Big)\Big\} \\
&\sim \mathrm{MVN}_T\big(\tilde{\mu}_{\theta_i}, \tilde{\Sigma}_{\theta_i}\big), \\
&\text{where } \tilde{\Sigma}_{\theta_i} = \Big((\tau^\theta c^\theta)^{-1} + \frac{(N-1)I_T}{\sigma_e^2}\Big)^{-1} \text{ and } \tilde{\mu}_{\theta_i} = \Big(\frac{\{\sum_{i=i, j\neq i}E_{ij[-i]}^t\}_{t=1}^T}{\sigma_e^2}\Big)\tilde{\Sigma}_{\theta_i}.
\end{aligned} \tag{8}$$

2.4 Multiplicative random effect d_r

$$
\begin{aligned}
&P(d_r|\mathbf{Y},\tau_r^d,\kappa_r^d) \propto P(\mathbf{Y}|\mathbf{X},\boldsymbol{\beta},\boldsymbol{\theta},d_r,\boldsymbol{d}_{[-r]},\boldsymbol{u},\sigma_e^2)\times P(d_r|\tau_r^d,\kappa_r^d)\\
&\propto \prod_{i>j}\exp\Big\{-\frac{1}{2\sigma_e^2}||E_{ij[-r]}-u_{ir}'d_r u_{jr}||^2\Big\}\times\exp\Big\{-\frac{1}{2}\big(d_r'(\tau_r^d c_r^d)^{-1}d_r\big)\Big\}\\
&\qquad\text{where } E_{ij[-r]}=\{E_{ij[-r]}^t\}_{t=1}^T \text{ (with } E_{ij[-r]}^t=E_{ij}^t+u_{ir}^t{}'d_r^t u_{ir}^t) \text{ and } u_{ir}'d_r u_{jr}=\{u_{ir}^t{}'d_r^t u_{ir}^t\}_{t=1}^T\\
&\propto \exp\Big\{-\frac{1}{2\sigma_e^2}\Big(\sum_{i>j}-2(E_{ij[-r]}u_{ir}u_{jr})'d_r+d_r'\big(\text{diag}(\sum_{i>j}(u_{ir}u_{jr})^2)\big)d_r\Big)\Big\}\times\exp\Big\{-\frac{1}{2}\big(d_r'(\tau_r^d c_r^d)^{-1}d_r\big)\Big\}\\
&\propto \exp\Big\{-\frac{1}{2}\Big(d_r'\big((\tau_r^d c_r^d)^{-1}+\frac{\text{diag}(\sum_{i>j}(u_{ir}u_{jr})^2)}{\sigma_e^2}\big)d_r-\frac{2}{\sigma_e^2}\big(\sum_{i>j}(E_{ij[-r]}u_{ir}u_{jr})'d_r\big)\Big)\Big\}\\
&\sim \text{MVN}_T\big(\tilde{\mu}_{d_r},\tilde{\Sigma}_{d_r}\big)\\
&\text{where } \tilde{\Sigma}_{d_r}=\Big((\tau_r^d c_r^d)^{-1}+\frac{\text{diag}\big(\{\sum_{i>j}(u_{ir}^t u_{jr}^t)^2\}_{t=1}^T\big)}{\sigma_e^2}\Big)^{-1} \text{ and } \tilde{\mu}_{d_r}=\Big(\frac{\{\sum_{i>j}(E_{ij[-r]}^t u_{ir}^t u_{jr}^t)\}_{t=1}^T}{\sigma_e^2}\Big)\tilde{\Sigma}_{d_r}).
\end{aligned} \tag{9}
$$

2.5 Multiplicative random effect u_i^t

2.5.1 Variance parameter τ_{rt}^u

$$
\begin{aligned}
P(\tau_{rt}^u|\boldsymbol{u}_r^t,a_u,b_u) &\propto \prod_{i=1}^N P(u_{ir}^t|\tau_{rt}^u r)\times P(\tau_{rt}^u|a_u,b_u)\\
&\propto \prod_{i=1}^N|\tau_{rt}^u|^{-\frac{1}{2}}\exp\Big\{-\frac{1}{2\tau_{rt}^u}\big(u_{ir}^{t\prime}u_{ir}^t\big)\Big\}\times(\tau_{rt}^u)^{-a_u-1}\exp\Big\{-\frac{1}{\tau_{rt}^u}b_u\Big\}\\
&\propto (\tau_{rt}^u)^{-\frac{N}{2}-a_u-1}\exp\Big\{-\frac{1}{\tau_{rt}^u}\Big(\frac{1}{2}\sum_{i=1}^N(u_{ir}^t)^2+b_u\Big)\Big\}\\
&\sim \text{IG}(\frac{N}{2}+a_u,\quad \frac{1}{2}\sum_{i=1}^N(u_{ir}^t)^2+b_u).
\end{aligned} \tag{10}
$$

2.5.2 Latent vector u_i^t

$$
\begin{aligned}
&P(u_i^t|\mathbf{Y},\boldsymbol{\tau}_t^u) \propto P(\mathbf{Y}|\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{d},u_i^t,\boldsymbol{u}_{[-ti]},\sigma_e^2)\times P(u_i^t|\boldsymbol{\tau}_t^u)\\
&\propto \prod_{i=i,j\neq i}\exp\Big\{-\frac{1}{2\sigma_e^2}||E_{ij[-u]}^t-u_j^t{}'D^t u_i^t||^2\Big\}\times\exp\Big\{-\frac{1}{2}\big(u_i^{t\prime}(\boldsymbol{\tau}_t^u)^{-1}u_i^t\big)\Big\}\\
&\qquad\text{where } E_{ij[-u]}^t=E_{ij}^t+u_i^{t\prime}D^t u_j^t\\
&\propto \exp\Big\{-\frac{1}{2\sigma_e^2}\Big(\sum_{i=i,j\neq i}-2(E_{ij[-u]}^t u_j^t{}'D^t)u_i^t+u_i^{t\prime}\big(\sum_{j\neq i}D^t u_j^t u_j^t{}'D^t\big)u_i^t\Big)\Big\}\times\exp\Big\{-\frac{1}{2}\big(u_i^{t\prime}(\boldsymbol{\tau}_t^u)^{-1}u_i^t\big)\Big\}\\
&\propto \exp\Big\{-\frac{1}{2}\Big(u_i^{t\prime}\big((\boldsymbol{\tau}_t^u)^{-1}+\frac{\sum_{j\neq i}D^t u_j^t u_j^t{}'D^t}{\sigma_e^2}\big)u_i^t-\frac{2}{\sigma_e^2}\big(\sum_{i=i,j\neq i}(E_{ij[-u]}^t u_j^t{}'D^t)'u_i^t\Big)\Big\}\\
&\sim \text{MVN}_R\big(\tilde{\mu}_{u_i^t},\tilde{\Sigma}_{u_i^t}\big),\\
&\text{where } \tilde{\Sigma}_{u_i^t}=\Big((\boldsymbol{\tau}_t^u)^{-1}+\frac{\sum_{j\neq i}D^t u_j^t u_j^t{}'D^t}{\sigma_e^2}\Big)^{-1} \text{ and } \tilde{\mu}_{u_i^t}=\Big(\frac{\sum_{i=i,j\neq i}(E_{ij[-u]}^t u_j^t{}'D^t)'}{\sigma_e^2}\Big)\tilde{\Sigma}_{u_i^t}.
\end{aligned} \tag{11}
$$