

Supplementary Materials for “Dynamic Additive and Multiplicative Effects (DAME) Model with Application to the United Nations Voting Behaviors”

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1 Metropolis-Hastings algorithm for GP parameters

For Gaussian process parameters—variance parameter τ and length-scale parameter κ , we use the Metropolis-Hastings algorithm with a proposal density Q being the multivariate Gaussian distribution, with a diagonal covariance matrix—i.e., $\text{diag}(\sigma_{Q1}^2, \sigma_{Q2}^2)$. Given the proposal variance $\sigma_Q^2 = (\sigma_{Q1}^2, \sigma_{Q2}^2)$, we sample the new values τ' and κ' from

$$(\tau', \kappa') \sim \exp(MVN_2(\log(\tau, \kappa), \sigma_Q^2 I_2)),$$

where we sample from the mean $\log(\tau, \kappa)$ and take exponentiation since both τ and κ have to be positive. Under the symmetric proposal distribution as above, we cancel out Q-ratio and then accept the new proposed value (τ', κ') with probability equal to:

$$\text{Acceptance Probability} = \begin{cases} \frac{P(\tau^{x'}, \kappa^{x'} | x(\cdot), a_x, b_x, \gamma)}{P(\tau^x, \kappa^x | x(\cdot), a_x, b_x, \gamma)} & \text{if } < 1 \\ 1 & \text{else} \end{cases}, \quad (1)$$

where x is the T -length vector of interest (e.g. β_p, θ, d), τ has a prior $\tau \sim \text{IG}(a_x, b_x)$, and κ has a prior $\kappa \sim \text{half-cauchy}(\gamma)$. If a sample from $\text{Uniform}(0,1)$ is less than the acceptance probability, we accept the proposed value. Otherwise, we reject.

Below are the derivation of acceptance ratio for each of the variables.

1. $(\tau_p^\beta, \kappa_p^\beta)$, for $p = 1, \dots, P$:

$$\begin{aligned} \frac{P(\tau_p^{\beta'}, \kappa_p^{\beta'} | \beta_p, a_\beta, b_\beta, \gamma)}{P(\tau_p^\beta, \kappa_p^\beta | \beta_p, a_\beta, b_\beta, \gamma)} &= \frac{P(\tau_p^{\beta'}, \kappa_p^{\beta'}, \beta_p | a_\beta, b_\beta, \gamma)}{P(\tau_p^\beta, \kappa_p^\beta, \beta_p | a_\beta, b_\beta, \gamma)} \\ &= \frac{P(\tau_p^{\beta'} | a_\beta, b_\beta) P(\kappa_p^{\beta'} | \gamma) P(\beta_p | \tau_p^{\beta'}, \kappa_p^{\beta'})}{P(\tau_p^\beta | a_\beta, b_\beta) P(\kappa_p^\beta | \gamma) P(\beta_p | \tau_p^\beta, \kappa_p^\beta)}, \end{aligned} \quad (2)$$

2. $(\tau^\theta, \kappa^\theta)$

$$\begin{aligned} \frac{P(\tau^{\theta'}, \kappa^{\theta'} | \theta, a_\theta, b_\theta, \gamma)}{P(\tau^\theta, \kappa^\theta | \theta, a_\theta, b_\theta, \gamma)} &= \frac{P(\tau^{\theta'}, \kappa^{\theta'}, \theta | a_\theta, b_\theta, \gamma)}{P(\tau^\theta, \kappa^\theta, \theta | a_\theta, b_\theta, \gamma)} \\ &= \frac{P(\tau^{\theta'} | a_\theta, b_\theta) P(\kappa^{\theta'} | \gamma) \prod_{i=1}^N P(\theta_i | \tau^{\theta'}, \kappa^{\theta'})}{P(\tau^\theta | a_\theta, b_\theta) P(\kappa^\theta | \gamma) \prod_{i=1}^N P(\theta_i | \tau^\theta, \kappa^\theta)}, \end{aligned} \quad (3)$$

3. (τ_r^d, κ_r^d) , for $r = 1, \dots, R$:

$$\begin{aligned} \frac{P(\tau_r^{d'}, \kappa_r^{d'} | d_r, a_d, b_d, \gamma)}{P(\tau_r^d, \kappa_r^d | d_r, a_d, b_d, \gamma)} &= \frac{P(\tau_r^{d'}, \kappa_r^{d'}, d_r | a_d, b_d, \gamma)}{P(\tau_r^d, \kappa_r^d, d_r | a_d, b_d, \gamma)} \\ &= \frac{P(\tau_r^{d'} | a_d, b_d) P(\kappa_r^{d'} | \gamma) P(d_r | \tau_r^{d'}, \kappa_r^{d'})}{P(\tau_r^d | a_d, b_d) P(\kappa_r^d | \gamma) P(d_r | \tau_r^d, \kappa_r^d)}. \end{aligned} \quad (4)$$

2 Proofs on Posterior Computation

2.1 Noise error variance σ_e^2

$$\begin{aligned}
P(\sigma_e^2 | \mathbf{Y}, a_\sigma, b_\sigma) &\propto P(\mathbf{Y} | \mathbf{X}, \boldsymbol{\beta}, \boldsymbol{\theta}, \mathbf{d}, \mathbf{u}, \sigma_e^2) \times P(\sigma_e^2 | a_\sigma, b_\sigma) \\
&\propto \prod_{t=1}^T \prod_{i>j} (\sigma_e^2)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2\sigma_e^2} \|y_{ij}^t - (\sum_{p=1}^P \beta_p^t X_{ijp}^t + \theta_i^t + \theta_j^t + u_i^{t'} D^t u_j^t)\|^2 \right\} \times (\sigma_e^2)^{-a_\sigma-1} \exp \left\{ \frac{1}{\sigma_e^2} b_\sigma \right\} \\
&= (\sigma_e^2)^{-\frac{T}{2} \cdot \frac{N(N-1)}{2} - a_\sigma - 1} \times \exp \left\{ -\frac{1}{\sigma_e^2} \left(\frac{1}{2} \sum_{t=1}^T \sum_{i>j} \|y_{ij}^t - (\sum_{p=1}^P \beta_p^t X_{ijp}^t + \theta_i^t + \theta_j^t + u_i^{t'} D^t u_j^t)\|^2 + b_\sigma \right) \right\} \\
&\sim \text{IG} \left(\frac{T \cdot N(N-1)}{4} + a_\sigma, \frac{1}{2} \sum_{t=1}^T \sum_{i>j} (E_{ij}^t)^2 + b_\sigma \right)
\end{aligned} \tag{5}$$

2.2 Fixed effect β_p

$$\begin{aligned}
P(\beta_p | \mathbf{Y}, \mathbf{X}, \kappa_p^\beta, \tau_p^\beta) &\propto P(\mathbf{Y} | \mathbf{X}, \beta_p, \boldsymbol{\beta}_{[-p]}, \boldsymbol{\theta}, \mathbf{d}, \mathbf{u}, \sigma_e^2) \times P(\beta_p | \kappa_p^\beta, \tau_p^\beta) \\
&\propto \prod_{i>j} \exp \left\{ -\frac{1}{2\sigma_e^2} \|E_{ij[-p]} - X_{ijp} \beta_p\|^2 \right\} \times \exp \left\{ -\frac{1}{2} (\beta_p' (\tau_p^\beta c_p^\beta)^{-1} \beta_p) \right\} \\
&\quad \text{where } E_{ij[-p]} = \{E_{ij[-p]}^t\}_{t=1}^T \text{ (with } E_{ij[-p]}^t = E_{ij}^t + \beta_p^t X_{ijp}^t) \text{ and } X_{ijp} = \{X_{ijp}^t\}_{t=1}^T \\
&\propto \exp \left\{ -\frac{1}{2\sigma_e^2} \left(\sum_{i>j} -2(E_{ij[-p]} X_{ijp})' \beta_p + \beta_p' (\text{diag}(\sum_{i>j} X_{ijp}^2)) \beta_p \right) \right\} \times \exp \left\{ -\frac{1}{2} (\beta_p' (\tau_p^\beta c_p^\beta)^{-1} \beta_p) \right\} \\
&\propto \exp \left\{ -\frac{1}{2} \left(\beta_p' ((\tau_p^\beta c_p^\beta)^{-1} + \frac{\text{diag}(\sum_{i>j} X_{ijp}^2)}{\sigma_e^2}) \beta_p - \frac{2}{\sigma_e^2} (\sum_{i>j} (E_{ij[-p]} X_{ijp})' \beta_p) \right) \right\} \\
&\sim \text{MVN}_T(\tilde{\mu}_{\beta_p}, \tilde{\Sigma}_{\beta_p}), \\
&\quad \text{where } \tilde{\Sigma}_{\beta_p} = \left((\tau_p^\beta c_p^\beta)^{-1} + \frac{\text{diag}(\{\sum_{i>j} X_{ijp}^2\}_{t=1}^T)}{\sigma_e^2} \right)^{-1} \text{ and } \tilde{\mu}_{\beta_p} = \left(\frac{\{\sum_{i>j} (E_{ij[-p]} X_{ijp})\}_{t=1}^T}{\sigma_e^2} \right) \tilde{\Sigma}_{\beta_p}.
\end{aligned} \tag{6}$$

2.3 Additive random effect θ_i

$$\begin{aligned}
p(\theta_i | \mathbf{Y}, \kappa^\theta, \tau^\theta) &\propto \prod_{i=i, j \neq i} p(\mathbf{Y} | \mathbf{X}, \boldsymbol{\beta}, \theta_i, \boldsymbol{\theta}_{[-i]}, \mathbf{d}, \mathbf{u}, \sigma_e^2) \times p(\theta_i | \kappa^\theta, \tau^\theta) \\
&\propto \prod_{i=i, j \neq i} \exp \left\{ -\frac{1}{2\sigma_e^2} \|E_{ij[-i]} - \theta_i\|^2 \right\} \times \exp \left\{ -\frac{1}{2} (\theta_i' (\tau^\theta c^\theta)^{-1} \theta_i) \right\} \\
&\quad \text{where } E_{ij[-i]} = \{E_{ij[-i]}^t\}_{t=1}^T \text{ with } E_{ij[-i]}^t = E_{ij}^t + \theta_i^t \\
&\propto \exp \left\{ -\frac{1}{2\sigma_e^2} \left(\sum_{i=i, j \neq i} -2(E_{ij[-i]} \theta_i)' + \theta_i' (\sum_{i=i, j \neq i} I_T) \theta_i \right) \right\} \times \exp \left\{ -\frac{1}{2} (\theta_i' (\tau^\theta c^\theta)^{-1} \theta_i) \right\} \\
&\propto \exp \left\{ -\frac{1}{2} \left(\theta_i' ((\tau^\theta c^\theta)^{-1} + \frac{(N-1)I_T}{\sigma_e^2}) \theta_i - \frac{2}{\sigma_e^2} (\sum_{i=i, j \neq i} (E_{ij[-i]} \theta_i)') \right) \right\} \\
&\sim \text{MVN}_T(\tilde{\mu}_{\theta_i}, \tilde{\Sigma}_{\theta_i}), \\
&\quad \text{where } \tilde{\Sigma}_{\theta_i} = \left((\tau^\theta c^\theta)^{-1} + \frac{(N-1)I_T}{\sigma_e^2} \right)^{-1} \text{ and } \tilde{\mu}_{\theta_i} = \left(\frac{\{\sum_{i=i, j \neq i} E_{ij[-i]}^t\}_{t=1}^T}{\sigma_e^2} \right) \tilde{\Sigma}_{\theta_i}.
\end{aligned} \tag{7}$$

2.4 Multiplicative random effect d_r

$$\begin{aligned}
P(d_r | \mathbf{Y}, \tau_r^d, \kappa_r^d) &\propto P(\mathbf{Y} | \mathbf{X}, \boldsymbol{\beta}, \boldsymbol{\theta}, d_r, \mathbf{d}_{[-r]}, \mathbf{u}, \sigma_e^2) \times P(d_r | \tau_r^d, \kappa_r^d) \\
&\propto \prod_{i>j} \exp\left\{-\frac{1}{2\sigma_e^2} \|E_{ij[-r]} - u'_{ir} d_r u_{jr}\|^2\right\} \times \exp\left\{-\frac{1}{2} (d'_r (\tau_r^d c_r^d)^{-1} d_r)\right\} \\
&\quad \text{where } E_{ij[-r]} = \{E_{ij[-r]}^t\}_{t=1}^T \text{ (with } E_{ij[-r]}^t = E_{ij}^t + u_{ir}^{t'} d_r^t u_{jr}^t \text{)} \text{ and } u'_{ir} d_r u_{jr} = \{u_{ir}^{t'} d_r^t u_{jr}^t\}_{t=1}^T \\
&\propto \exp\left\{-\frac{1}{2\sigma_e^2} \left(\sum_{i>j} -2(E_{ij[-r]} u_{ir} u_{jr})' d_r + d'_r (\text{diag}(\sum_{i>j} (u_{ir} u_{jr})^2)) d_r\right)\right\} \times \exp\left\{-\frac{1}{2} (d'_r (\tau_r^d c_r^d)^{-1} d_r)\right\} \\
&\propto \exp\left\{-\frac{1}{2} \left(d'_r ((\tau_r^d c_r^d)^{-1} + \frac{\text{diag}(\sum_{i>j} (u_{ir} u_{jr})^2)}{\sigma_e^2}) d_r - \frac{2}{\sigma_e^2} \left(\sum_{i>j} (E_{ij[-r]} u_{ir} u_{jr})' d_r\right)\right)\right\} \\
&\sim \text{MVN}_T(\tilde{\mu}_{d_r}, \tilde{\Sigma}_{d_r}) \\
&\quad \text{where } \tilde{\Sigma}_{d_r} = \left((\tau_r^d c_r^d)^{-1} + \frac{\text{diag}(\{\sum_{i>j} (u_{ir}^t u_{jr}^t)^2\}_{t=1}^T)}{\sigma_e^2}\right)^{-1} \text{ and } \tilde{\mu}_{d_r} = \left(\frac{\{\sum_{i>j} (E_{ij[-r]}^t u_{ir}^t u_{jr}^t)\}_{t=1}^T}{\sigma_e^2}\right) \tilde{\Sigma}_{d_r}.
\end{aligned} \tag{8}$$

2.5 Multiplicative random effect u_i^t

2.5.1 Variance parameter τ_{rt}^u

$$\begin{aligned}
P(\tau_{rt}^u | \mathbf{u}_r^t, a_u, b_u) &\propto \prod_{i=1}^N P(u_{ir}^t | \tau_{rt}^u) \times P(\tau_{rt}^u | a_u, b_u) \\
&\propto \prod_{i=1}^N |\tau_{rt}^u|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\tau_{rt}^u} (u_{ir}^{t'} u_{ir}^t)\right\} \times (\tau_{rt}^u)^{-a_u-1} \exp\left\{-\frac{1}{\tau_{rt}^u} b_u\right\} \\
&\propto (\tau_{rt}^u)^{-\frac{N}{2}-a_u-1} \exp\left\{-\frac{1}{\tau_{rt}^u} \left(\frac{1}{2} \sum_{i=1}^N (u_{ir}^t)^2 + b_u\right)\right\} \\
&\sim \text{IG}\left(\frac{N}{2} + a_u, \frac{1}{2} \sum_{i=1}^N (u_{ir}^t)^2 + b_u\right).
\end{aligned} \tag{9}$$

2.5.2 Latent vector u_i^t

$$\begin{aligned}
P(u_i^t | \mathbf{Y}, \boldsymbol{\tau}_t^u) &\propto P(\mathbf{Y} | \boldsymbol{\beta}, \boldsymbol{\theta}, \mathbf{d}, u_i^t, \mathbf{u}_{[-ti]}, \sigma_e^2) \times P(u_i^t | \boldsymbol{\tau}_t^u) \\
&\propto \prod_{i=i, j \neq i} \exp\left\{-\frac{1}{2\sigma_e^2} \|E_{ij[-u]}^t - u_j^{t'} D^t u_i^t\|^2\right\} \times \exp\left\{-\frac{1}{2} (u_i^{t'} (\boldsymbol{\tau}_t^u)^{-1} u_i^t)\right\} \\
&\quad \text{where } E_{ij[-u]}^t = E_{ij}^t + u_i^{t'} D^t u_j^t \\
&\propto \exp\left\{-\frac{1}{2\sigma_e^2} \left(\sum_{i=i, j \neq i} -2(E_{ij[-u]}^t u_j^{t'} D^t) u_i^t + u_i^{t'} \left(\sum_{j \neq i} D^t u_j^t u_j^{t'} D^t\right) u_i^t\right)\right\} \times \exp\left\{-\frac{1}{2} (u_i^{t'} (\boldsymbol{\tau}_t^u)^{-1} u_i^t)\right\} \\
&\propto \exp\left\{-\frac{1}{2} \left(u_i^{t'} ((\boldsymbol{\tau}_t^u)^{-1} + \frac{\sum_{j \neq i} D^t u_j^t u_j^{t'} D^t}{\sigma_e^2}) u_i^t - \frac{2}{\sigma_e^2} \left(\sum_{i=i, j \neq i} (E_{ij[-u]}^t u_j^{t'} D^t)' u_i^t\right)\right)\right\} \\
&\sim \text{MVN}_R(\tilde{\mu}_{u_i^t}, \tilde{\Sigma}_{u_i^t}), \\
&\quad \text{where } \tilde{\Sigma}_{u_i^t} = \left((\boldsymbol{\tau}_t^u)^{-1} + \frac{\sum_{j \neq i} D^t u_j^t u_j^{t'} D^t}{\sigma_e^2}\right)^{-1} \text{ and } \tilde{\mu}_{u_i^t} = \left(\frac{\sum_{i=i, j \neq i} (E_{ij[-u]}^t u_j^{t'} D^t)' u_i^t}{\sigma_e^2}\right) \tilde{\Sigma}_{u_i^t}.
\end{aligned} \tag{10}$$