Supplementary Materials for
"A Dynamic Additive and Multiplicative Effects Model with Application to the United Nations Voting Behaviors"

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1 Metropolis-Hastings algorithm for GP parameters

For Gaussian process parameters—variance parameter τ and length-scale parameter κ, we use the Metropolis-Hastings algorithm with a proposal density Q being the multivariate Gaussian distribution, with a diagonal covariance matrix—i.e., $\text{diag}(\sigma^2_{Q1}, \sigma^2_{Q2})$. Given the proposal variance $\sigma^2_Q = (\sigma^2_{Q1}, \sigma^2_{Q2})$, we sample the new values τ' and κ' from

$$(\tau', \kappa') \sim \exp(MVN_2(\log(\tau, \kappa), \sigma^2_Q I_2)),$$

where we sample from the mean $\log(\tau, \kappa)$ and take exponentiation since both τ and κ have to be positive. Under the symmetric proposal distribution as above, we cancel out Q-ratio and then accept the new proposed value (τ', κ') with probability equal to:

$$\text{Acceptance Probability} = \begin{cases} \frac{P(\tau^{x\prime}, \kappa^{x\prime} | x(\cdot), a_x, b_x, \gamma)}{P(\tau^{x}, \kappa^{x} | x(\cdot), a_x, b_x, \gamma)} & \text{if} < 1 \\ 1 \quad \text{else} \end{cases}, \tag{1}$$

where x is the T-length vector of interest (e.g. $\boldsymbol{\beta}_p, \boldsymbol{\theta}, \boldsymbol{d}$), τ has a prior $\tau \sim \mathcal{IG}(a_x, b_x)$, and κ has a prior $\kappa \sim$ half-cauchy(γ). If a sample from $\mathcal{U}(0, 1)$ is less than the acceptance probability, we accept the proposed value. Otherwise, we reject.

Below are the derivation of acceptance ratio for each of the variables.

1. $(\tau^\beta_p, \kappa^\beta_p)$, for $p = 1, \ldots, P$:

$$\begin{aligned}\frac{P(\tau^{\beta\prime}_p, \kappa^{\beta\prime}_p | \boldsymbol{\beta}_p, a_\beta, b_\beta, \gamma)}{P(\tau^{\beta}_p, \kappa^{\beta}_p | \boldsymbol{\beta}_p, a_\beta, b_\beta, \gamma)} &= \frac{P(\tau^{\beta\prime}_p, \kappa^{\beta\prime}_p, \boldsymbol{\beta}_p | a_\beta, b_\beta, \gamma)}{P(\tau^{\beta}_p, \kappa^{\beta}_p, \boldsymbol{\beta}_p | a_\beta, b_\beta, \gamma)} \\ &= \frac{P(\tau^{\beta\prime}_p | a_\beta, b_\beta) P(\kappa^{\beta\prime}_p | \gamma) P(\boldsymbol{\beta}_p | \tau^{\beta\prime}_p, \kappa^{\beta\prime}_p)}{P(\tau^{\beta}_p | a_\beta, b_\beta) P(\kappa^{\beta}_p | \gamma) P(\boldsymbol{\beta}_p | \tau^{\beta}_p, \kappa^{\beta}_p)},\end{aligned} \tag{2}$$

2. $(\tau^\theta, \kappa^\theta)$

$$\begin{aligned}\frac{P(\tau^{\theta\prime}, \kappa^{\theta\prime} | \boldsymbol{\theta}, a_\theta, b_\theta, \gamma)}{P(\tau^{\theta}, \kappa^{\theta} | \boldsymbol{\theta}, a_\theta, b_\theta, \gamma)} &= \frac{P(\tau^{\theta\prime}, \kappa^{\theta\prime}, \boldsymbol{\theta} | a_\theta, b_\theta, \gamma)}{P(\tau^{\theta}, \kappa^{\theta}, \boldsymbol{\theta} | a_\theta, b_\theta, \gamma)} \\ &= \frac{P(\tau^{\theta\prime} | a_\theta, b_\theta) P(\kappa^{\theta\prime} | \gamma) \prod_{i=1}^N P(\boldsymbol{\theta}_i | \tau^{\theta\prime}, \kappa^{\theta\prime})}{P(\tau^{\theta} | a_\theta, b_\theta) P(\kappa^{\theta} | \gamma) \prod_{i=1}^N P(\boldsymbol{\theta}_i | \tau^{\theta}, \kappa^{\theta})},\end{aligned} \tag{3}$$

3. (τ^d_r, κ^d_r), for $r = 1, \ldots, R$:

$$\begin{aligned}\frac{P(\tau^{d\prime}_r, \kappa^{d\prime}_r | \boldsymbol{d}_r, a_d, b_d, \gamma)}{P(\tau^{d}_r, \kappa^{d} | \boldsymbol{d}_r, a_d, b_d, \gamma)} &= \frac{P(\tau^{d\prime}_r, \kappa^{d\prime}_r, \boldsymbol{d}_r | a_d, b_d, \gamma)}{P(\tau^{d}_r, \kappa^{d}_r, \boldsymbol{d}_r | a_d, b_d, \gamma)} \\ &= \frac{P(\tau^{d\prime}_r | a_d, b_d) P(\kappa^{d\prime}_r | \gamma) P(\boldsymbol{d}_r | \tau^{d\prime}_r, \kappa^{d\prime}_r)}{P(\tau^{d}_r | a_d, b_d) P(\kappa^{d}_r | \gamma) P(\boldsymbol{d}_r | \tau^{d}_r, \kappa^{d}_r)}.\end{aligned} \tag{4}$$

2 Proofs on Posterior Computation

2.1 Noise error variance σ_e^2

$$
\begin{aligned}
&P(\sigma_e^2|\mathbf{Y}, a_\sigma, b_\sigma) \propto P(\mathbf{Y}|\mathbf{X}, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{d}, \boldsymbol{u}, \sigma_e^2) \times P(\sigma_e^2|a_\sigma, b_\sigma) \\
&\propto \prod_{t=1}^{T}\prod_{i>j}(\sigma_e^2)^{-\frac{1}{2}}\exp\Big\{-\frac{1}{2\sigma_e^2}||y_{ij}^t - (\sum_{p=1}^{P}\beta_p^t X_{ijp}^t + \theta_i^t + \theta_j^t + \boldsymbol{u}_i^{t\prime}\mathbf{D}^t\boldsymbol{u}_j^t)||^2\Big\} \times (\sigma_e^2)^{-a_\sigma-1}\exp\Big\{\frac{1}{\sigma_e^2}b_\sigma\Big\} \\
&= (\sigma_e^2)^{-\frac{T}{2}\cdot\frac{N(N-1)}{2}-a_\sigma-1} \times \exp\Big\{-\frac{1}{\sigma_e^2}\Big(\frac{1}{2}\sum_{t=1}^{T}\sum_{i>j}||y_{ij}^t - (\sum_{p=1}^{P}\beta_p^t X_{ijp}^t + \theta_i^t + \theta_j^t + \boldsymbol{u}_i^{t\prime}\mathbf{D}^t\boldsymbol{u}_j^t)||^2 + b_\sigma\Big)\Big\} \\
&\sim \mathcal{IG}\big(\frac{T\cdot N(N-1)}{4} + a_\sigma, \quad \frac{1}{2}\sum_{t=1}^{T}\sum_{i>j}(E_{ij}^t)^2 + b_\sigma\big)
\end{aligned} \tag{5}
$$

2.2 Fixed effect coeffient β_p

$$
\begin{aligned}
&P(\boldsymbol{\beta}_p|\mathbf{Y}, \mathbf{X}, \kappa_p^\beta, \tau_p^\beta) \propto P(\mathbf{Y}|\mathbf{X}, \boldsymbol{\beta}_p, \boldsymbol{\beta}_{[-p]}, \boldsymbol{\theta}, \boldsymbol{d}, \boldsymbol{u}, \sigma_e^2) \times P(\boldsymbol{\beta}_p|\kappa_p^\beta, \tau_p^\beta) \\
&\propto \prod_{i>j}\exp\Big\{-\frac{1}{2\sigma_e^2}||\mathbf{E}_{ij[-p]} - \mathbf{X}_{ijp}\boldsymbol{\beta}_p||^2\Big\} \times \exp\Big\{-\frac{1}{2}\big(\boldsymbol{\beta}_p'(\tau_p^\beta c_p^\beta)^{-1}\boldsymbol{\beta}_p\big)\Big\} \\
&\qquad \text{where } \mathbf{E}_{ij[-p]} = \{E_{ij[-p]}^t\}_{t=1}^T \text{ (with } E_{ij[-p]}^t = E_{ij}^t + \beta_p^t X_{ijp}^t\text{) and } \mathbf{X}_{ijp} = \{X_{ijp}^t\}_{t=1}^T \\
&\propto \exp\Big\{-\frac{1}{2\sigma_e^2}\Big(\sum_{i>j} -2(\mathbf{E}_{ij[-p]}\mathbf{X}_{ijp})'\boldsymbol{\beta}_p + \boldsymbol{\beta}_p'\big(\text{diag}(\sum_{i>j}\mathbf{X}_{ijp}^2)\big)\boldsymbol{\beta}_p\Big)\Big\} \times \exp\Big\{-\frac{1}{2}\big(\boldsymbol{\beta}_p'(\tau_p^\beta c_p^\beta)^{-1}\boldsymbol{\beta}_p\big)\Big\} \\
&\propto \exp\Big\{-\frac{1}{2}\Big(\boldsymbol{\beta}_p'\big((\tau_p^\beta c_p^\beta)^{-1} + \frac{\text{diag}(\sum_{i>j}\mathbf{X}_{ijp}^2)}{\sigma_e^2}\big)\boldsymbol{\beta}_p - \frac{2}{\sigma_e^2}\big(\sum_{i>j}(\mathbf{E}_{ij[-p]}\mathbf{X}_{ijp})'\boldsymbol{\beta}_p\big)\Big)\Big\} \\
&\sim \mathcal{N}_T\big(\tilde{\mu}_{\beta_p}, \tilde{\Sigma}_{\beta_p}\big), \\
&\text{where } \tilde{\Sigma}_{\beta_p} = \Big((\tau_p^\beta c_p^\beta)^{-1} + \frac{\text{diag}\big(\{\sum_{i>j}X_{ijp}^{t2}\}_{t=1}^T\big)}{\sigma_e^2}\Big)^{-1} \text{ and } \tilde{\mu}_{\beta_p} = \Big(\frac{\{\sum_{i>j}(E_{ij[-p]}^t X_{ijp}^t)\}_{t=1}^T}{\sigma_e^2}\Big)\tilde{\Sigma}_{\beta_p}.
\end{aligned} \tag{6}
$$

2.3 Additive random effect θ_i

$$
\begin{aligned}
&p(\boldsymbol{\theta}_i|\mathbf{Y}, \kappa^\theta, \tau^\theta) \propto \prod_{i=i, j\neq i} p(\mathbf{Y}|\mathbf{X}, \boldsymbol{\beta}, \boldsymbol{\theta}_i, \boldsymbol{\theta}_{[-i]}, \boldsymbol{d}, \boldsymbol{u}, \sigma_e^2) \times p(\boldsymbol{\theta}_i|\kappa^\theta, \tau^\theta) \\
&\propto \prod_{i=i, j\neq i}\exp\Big\{-\frac{1}{2\sigma_e^2}||\mathbf{E}_{ij[-i]} - \boldsymbol{\theta}_i||^2\Big\} \times \exp\Big\{-\frac{1}{2}\big(\boldsymbol{\theta}_i'(\tau^\theta c^\theta)^{-1}\boldsymbol{\theta}_i\big)\Big\} \\
&\qquad \text{where } \mathbf{E}_{ij[-i]} = \{E_{ij[-i]}^t\}_{t=1}^T \text{ with } E_{ij[-i]}^t = E_{ij}^t + \boldsymbol{\theta}_i^t \\
&\propto \exp\Big\{-\frac{1}{2\sigma_e^2}\Big(\sum_{i=i, j\neq i} -2(\mathbf{E}_{ij[-i]})'\boldsymbol{\theta}_i + \boldsymbol{\theta}_i'\big(\sum_{i=i, j\neq i} I_T\big)\boldsymbol{\theta}_i\Big)\Big\} \times \exp\Big\{-\frac{1}{2}\big(\boldsymbol{\theta}_i'(\tau^\theta c^\theta)^{-1}\boldsymbol{\theta}_i\big)\Big\} \\
&\propto \exp\Big\{-\frac{1}{2}\Big(\boldsymbol{\theta}_i'\big((\tau^\theta c^\theta)^{-1} + \frac{(N-1)I_T}{\sigma_e^2}\big)\boldsymbol{\theta}_i - \frac{2}{\sigma_e^2}\big(\sum_{i=i, j\neq i}(\mathbf{E}_{ij[-i]})'\boldsymbol{\theta}_i\big)\Big)\Big\} \\
&\sim \mathcal{N}_T\big(\tilde{\mu}_{\theta_i}, \tilde{\Sigma}_{\theta_i}\big), \\
&\text{where } \tilde{\Sigma}_{\theta_i} = \Big((\tau^\theta c^\theta)^{-1} + \frac{(N-1)I_T}{\sigma_e^2}\Big)^{-1} \text{ and } \tilde{\mu}_{\theta_i} = \Big(\frac{\{\sum_{i=i, j\neq i}E_{ij[-i]}^t\}_{t=1}^T}{\sigma_e^2}\Big)\tilde{\Sigma}_{\theta_i}.
\end{aligned} \tag{7}
$$

2.4 Multiplicative random effect d_r

$$
\begin{aligned}
&P(\boldsymbol{d}_r|\mathbf{Y},\tau_r^d,\kappa_r^d) \propto P(\mathbf{Y}|\mathbf{X},\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{d}_r,\boldsymbol{d}_{[-r]},\boldsymbol{u},\sigma_e^2) \times P(\boldsymbol{d}_r|\tau_r^d,\kappa_r^d)\\
&\propto \prod_{i>j}\exp\Big\{-\frac{1}{2\sigma_e^2}||\mathbf{E}_{ij[-r]}-\boldsymbol{u}_{ir}'\boldsymbol{d}_r\boldsymbol{u}_{jr}||^2\Big\}\times\exp\Big\{-\frac{1}{2}\big(\boldsymbol{d}_r'(\tau_r^d c_r^d)^{-1}\boldsymbol{d}_r\big)\Big\}\\
&\qquad \text{where } \mathbf{E}_{ij[-r]}=\{E_{ij[-r]}^t\}_{t=1}^T \text{ (with } E_{ij[-r]}^t=E_{ij}^t+u_{ir}^t{}'d_r^t u_{ir}^t) \text{ and } \boldsymbol{u}_{ir}'\boldsymbol{d}_r\boldsymbol{u}_{jr}=\{u_{ir}^t{}'d_r^t u_{ir}^t\}_{t=1}^T\\
&\propto \exp\Big\{-\frac{1}{2\sigma_e^2}\Big(\sum_{i>j}-2(\mathbf{E}_{ij[-r]}\boldsymbol{u}_{ir}\boldsymbol{u}_{jr})'\boldsymbol{d}_r+\boldsymbol{d}_r'\big(\mathrm{diag}(\sum_{i>j}(\boldsymbol{u}_{ir}\boldsymbol{u}_{jr})^2)\big)\boldsymbol{d}_r\Big)\Big\}\times\exp\Big\{-\frac{1}{2}\big(\boldsymbol{d}_r'(\tau_r^d c_r^d)^{-1}\boldsymbol{d}_r\big)\Big\}\\
&\propto \exp\Big\{-\frac{1}{2}\Big(\boldsymbol{d}_r'\big((\tau_r^d c_r^d)^{-1}+\frac{\mathrm{diag}(\sum_{i>j}(\boldsymbol{u}_{ir}\boldsymbol{u}_{jr})^2)}{\sigma_e^2}\big)\boldsymbol{d}_r-\frac{2}{\sigma_e^2}\big(\sum_{i>j}(\mathbf{E}_{ij[-r]}\boldsymbol{u}_{ir}\boldsymbol{u}_{jr})'\boldsymbol{d}_r\big)\Big)\Big\}\\
&\sim \mathcal{N}_T\big(\tilde{\mu}_{d_r},\tilde{\Sigma}_{d_r}\big)\\
&\text{where } \tilde{\Sigma}_{d_r}=\Big((\tau_r^d c_r^d)^{-1}+\frac{\mathrm{diag}\big(\{\sum_{i>j}(u_{ir}^t u_{jr}^t)^2\}_{t=1}^T\big)}{\sigma_e^2}\Big)^{-1} \text{ and } \tilde{\mu}_{d_r}=\Big(\frac{\{\sum_{i>j}(E_{ij[-r]}^t u_{ir}^t u_{jr}^t)\}_{t=1}^T}{\sigma_e^2}\Big)\tilde{\Sigma}_{d_r}).
\end{aligned}
\tag{8}
$$

2.5 Multiplicative random effect u_i^t

2.5.1 Variance parameter τ_{rt}^u

$$
\begin{aligned}
P(\tau_{rt}^u|\boldsymbol{u}_r^t,a_u,b_u) &\propto \prod_{i=1}^N P(u_{ir}^t|\tau_{rt}^u r)\times P(\tau_{rt}^u|a_u,b_u)\\
&\propto \prod_{i=1}^N|\tau_{rt}^u|^{-\frac{1}{2}}\exp\Big\{-\frac{1}{2\tau_{rt}^u}\big(u_{ir}^{t\prime}u_{ir}^t\big)\Big\}\times(\tau_{rt}^u)^{-a_u-1}\exp\Big\{-\frac{1}{\tau_{rt}^u}b_u\Big\}\\
&\propto (\tau_{rt}^u)^{-\frac{N}{2}-a_u-1}\exp\Big\{-\frac{1}{\tau_{rt}^u}\Big(\frac{1}{2}\sum_{i=1}^N(u_{ir}^t)^2+b_u\Big)\Big\}\\
&\sim \mathcal{IG}(\frac{N}{2}+a_u,\quad \frac{1}{2}\sum_{i=1}^N(u_{ir}^t)^2+b_u).
\end{aligned}
\tag{9}
$$

2.5.2 Latent vector u_i^t

$$
\begin{aligned}
&P(\boldsymbol{u}_i^t|\mathbf{Y},\boldsymbol{\tau}_t^u) \propto P(\mathbf{Y}|\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{d},\boldsymbol{u}_i^t,\boldsymbol{u}_{[-ti]},\sigma_e^2)\times P(\boldsymbol{u}_i^t|\boldsymbol{\tau}_t^u)\\
&\propto \prod_{i=i,j\neq i}\exp\Big\{-\frac{1}{2\sigma_e^2}||E_{ij[-u]}^t-\boldsymbol{u}_j^t{}'\mathbf{D}^t\boldsymbol{u}_i^t||^2\Big\}\times\exp\Big\{-\frac{1}{2}\big(\boldsymbol{u}_i^{t\prime}(\boldsymbol{\tau}_t^u)^{-1}\boldsymbol{u}_i^t\big)\Big\}\\
&\qquad \text{where } E_{ij[-u]}^t=E_{ij}^t+\boldsymbol{u}_i^{t\prime}\mathbf{D}^t\boldsymbol{u}_j^t\\
&\propto \exp\Big\{-\frac{1}{2\sigma_e^2}\Big(\sum_{i=i,j\neq i}-2(E_{ij[-u]}^t\boldsymbol{u}_j^t{}'\mathbf{D}^t)\boldsymbol{u}_i^t+\boldsymbol{u}_i^{t\prime}\big(\sum_{j\neq i}\mathbf{D}^t\boldsymbol{u}_j^t\boldsymbol{u}_j^t{}'\mathbf{D}^t\big)\boldsymbol{u}_i^t\Big)\Big\}\times\exp\Big\{-\frac{1}{2}\big(\boldsymbol{u}_i^{t\prime}(\boldsymbol{\tau}_t^u)^{-1}\boldsymbol{u}_i^t\big)\Big\}\\
&\propto \exp\Big\{-\frac{1}{2}\Big(\boldsymbol{u}_i^{t\prime}\big((\boldsymbol{\tau}_t^u)^{-1}+\frac{\sum_{j\neq i}\mathbf{D}^t\boldsymbol{u}_j^t\boldsymbol{u}_j^t{}'\mathbf{D}^t}{\sigma_e^2}\big)\boldsymbol{u}_i^t-\frac{2}{\sigma_e^2}\big(\sum_{i=i,j\neq i}(E_{ij[-u]}^t\boldsymbol{u}_j^t{}'\mathbf{D}^t)'\boldsymbol{u}_i^t\Big)\Big\}\\
&\sim \mathcal{N}_R\big(\tilde{\mu}_{u_i^t},\tilde{\Sigma}_{u_i^t}\big),\\
&\text{where } \tilde{\Sigma}_{u_i^t}=\Big((\boldsymbol{\tau}_t^u)^{-1}+\frac{\sum_{j\neq i}\mathbf{D}^t\boldsymbol{u}_j^t\boldsymbol{u}_j^t{}'\mathbf{D}^t}{\sigma_e^2}\Big)^{-1} \text{ and } \tilde{\mu}_{u_i^t}=\Big(\frac{\sum_{i=i,j\neq i}(E_{ij[-u]}^t\boldsymbol{u}_j^t{}'\mathbf{D}^t)'}{\sigma_e^2}\Big)\tilde{\Sigma}_{u_i^t}.
\end{aligned}
\tag{10}
$$