# Point process modeling for directed interaction networks

Patrick O. Perry
*Stern School of Business, New York University, USA*
Patrick J. Wolfe
*Department of Statistical Science, University College London, UK*

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**Summary.** Network data often take the form of repeated interactions between senders and receivers tabulated over time. A primary question to ask of such data is which traits and behaviors are predictive of interaction. To answer this question, a model is introduced for treating directed interactions as a multivariate point process: a Cox multiplicative intensity model using covariates that depend on the history of the process. Consistency and asymptotic normality are proved for the resulting partial-likelihood-based estimators under suitable regularity conditions, and an efficient fitting procedure is described. Multicast interactions—those involving a single sender but multiple receivers—are treated explicitly. The resulting inferential framework is then employed to model message sending behavior in a corporate e-mail network. The analysis gives a precise quantification of which static shared traits and dynamic network effects are predictive of message recipient selection.

*Keywords:* Cox proportional hazards model; Network data analysis; Partial likelihood inference; Point processes

## 1. Introduction

Much effort has been devoted to the statistical analysis of network data; see Jackson (2008), Goldenberg et al. (2009), and Kolaczyk (2009) for recent overviews. Often network observables comprise counts of interactions between individuals or groups tabulated over time. Communications networks give rise to *directed* interactions: phone calls, text messages, or e-mails exchanged amongst a given set of individuals over a specific time period (Tyler et al., 2005; Eagle and Pentland, 2006). Specific examples of repeated interactions from other types of networks include the following: Fowler's (2006) study of legislators authoring and cosponsoring bills (a collaboration network); Mckenzie and Rapoport's (2007) study of families migrating between communities in Mexico (a migration network); Sundaresan, Fischoff, Dushoff, and Rubenstein's (2007) study of zebras congregating at locations in their habitat (an animal association network); and Papachristos's (2009) study of gangs in Chicago murdering members of rival factions (an organized crime network).

In this article, we consider partial-likelihood-based inference for general directed interaction data in the presence of covariates. We first develop asymptotic theory for the case in which interactions are strictly pairwise, and then generalize our results to the multiple-receiver (multicast) case; we also provide efficient algorithms for partial likelihood maximization in these settings. Our main assumption on the covariates is that they be predictable, which allows them to vary with time and potentially depend on the past.

*Address for correspondence:* Patrick O. Perry, Information, Operations, and Management Sciences Department, Stern School of Business, New York University, 44 West 4th St, New York, NY 10012, USA
E-mail: pperry@stern.nyu.edu

The interaction data we consider comprise a set of triples, with triple $(t, i, j)$ indicating that at time $t$, directed interaction $i \to j$ took place—for instance, individual $i$ sent a message to individual $j$. Given such a set of triples, a primary modeling goal lies in determining which characteristics and behaviors of the senders and receivers are predictive of interaction. In this vein, three important questions stand out:

**Homophily** Is there evidence of homophily (an increased rate of interaction among similar individuals)? To what degree is a shared attribute predictive of heightened interaction?

**Network Effects** To what extent are past interaction behaviors predictive of future ones? If we observe interactions $i \to h$ and $h \to j$, are we more likely to see the interaction $i \to j$?

**Multiplicity** How should multiple-receiver interactions of the type $i \to \{j_1, j_2, \dots, j_L\}$ be modeled? What are the implications of treating these as $L$ separate pairwise interactions?

The issues of homophily, network effects, and their interactions arise frequently in the networks literature; see, e.g., McPherson et al. (2001); Butts (2008); Aral et al. (2009); Snijders et al. (2010), and references contained therein. Multiplicity has largely been ignored in this context, however, with notable exceptions including Lunagómez et al. (2009) for graphical models, and Shafiei and Chipman (2010) for network modeling.

In the remainder of this article, we provide a modeling framework and computationally efficient partial likelihood inference procedures to facilitate analysis of these questions. We employ a Cox proportional intensity model incorporating both static and history-dependent covariates to address the first of these two questions, and a parametric bootstrap to address the third. Section 2 presents our point process model for directed pairwise interactions, along with the resultant inference procedures. Section 3 establishes consistency and asymptotic normality of the corresponding maximum partial likelihood estimator, and Section 4 extends our framework to the case of multiple-receiver interactions. Section 5 employs this framework to model message sending behavior in a corporate e-mail network. Section 6 evaluates the strength of homophily and network effects in explaining these data, and Section 7 concludes the main body of the article. Appendices A–C contain respectively implementation details and technical results from Sections 3 and 4. The supplementary material provides comparative analyses based on related network models in the literature.

## 2. A point process model and partial likelihood inference

Every interaction process can be encoded by a multivariate counting measure. For sender $i$, receiver $j$, and positive time $t$, define

$$N_t(i, j) = \#\{\text{directed interactions } i \to j \text{ in time interval } [0, t]\}.$$

For technical reasons, assume that $N_0(i, j) = 0$ and that $N_t(i, j)$ is adapted to a stochastic basis of $\sigma$-algebras $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. Then, $N_t(i, j)$ is a local submartingale, so by the Doob-Meyer decomposition, there exists a predictable increasing process $\Lambda_t(i, j)$, null at zero, such that $N_t(i, j) - \Lambda_t(i, j)$ is an $\mathcal{F}_t$-local martingale. Under mild conditions—the most important of which is that no two interactions happen simultaneously—there exists a predictable continuous process $\lambda_t(i, j)$ such that $\Lambda_t(i, j) = \int_0^t \lambda_s(i, j)\, ds$. (In practical applications, simultaneous events exist and are an annoyance; Efron (1977) handles simultaneity through an ad-hoc adjustment, while Broström (2002) adds a discrete component to $\Lambda$.) The process $\lambda$ is known as the stochastic intensity of $N$. Heuristically,

$$\lambda_t(i, j)\, dt = \mathbb{P}\{\text{interaction } i \to j \text{ occurs in time interval } [t, t + dt)\}.$$

We will model $N$ through $\lambda$ using a version of the Cox (1972) proportional intensity model.

Let $\mathcal{I}$ be a set of senders and $\mathcal{J}$ be a (not necessarily disjoint) set of receivers. For each sender $i$, let $\bar{\lambda}_t(i)$ be a non-negative predictable process called the baseline intensity of sender $i$; let $\mathcal{J}_t(i)$ be a predictable finite subset of $\mathcal{J}$ called the receiver set of sender $i$. For each sender-receiver pair $(i, j)$, let $x_t(i, j)$ be a predictable locally bounded vector of covariates in $\mathbb{R}^p$. Let $\beta_0$ be an unknown vector of coefficients in $\mathbb{R}^p$. For the remainder of this section, assume that each interaction has a single receiver.

Given a multivariate counting process $N$ on $\mathbb{R}_+ \times \mathcal{I} \times \mathcal{J}$, we model its stochastic intensity as

$$\lambda_t(i, j) = \bar{\lambda}_t(i) \cdot \exp\{\beta_0^{\mathrm{T}} x_t(i, j)\} \cdot 1\{j \in \mathcal{J}_t(i)\}. \tag{1}$$

This model posits that sender $i$ in $\mathcal{I}$ interacts with receiver $j$ in $\mathcal{J}_t(i)$ at a baseline rate $\bar{\lambda}_t(i)$ modulated up or down according to the pair's covariate vector, $x_t(i, j)$. As Efron (1977) notes, the specific parametric form for the multiplier $\exp\{\beta_0^{\mathrm{T}} x_t(i, j)\}$ is not central to the theoretical analysis, but this choice is amenable to computation and gives the parameter vector $\beta_0$ a straightforward interpretation. Butts (2008), Vu et al. (2011a), and Vu et al. (2011b) used variants of this model to analyze repeated directed actions within social settings.

The form of (1) is deceptively simple but remains flexible enough to be useful in practice. The model allows for homophily and group level effects via inclusion of covariates of the form "$1\{i$ and $j$ belong to the same group$\}$," where "group" is some observable trait like ethnicity, gender, or age group. Its real strength, though, is that $x_t(i, j)$ is allowed to be *any* predictable process, in particular $x_t(i, j)$ can depend on the history of interactions. To model reciprocation and transitivity in the interactions (with $\mathcal{I} = \mathcal{J}$), for example, choose appropriate values for $\Delta_k$ and include relevant covariates in $x_t(i, j)$:

$$1\{\text{interaction } j \to i \text{ occurred in } [t - \Delta_k, t)\}$$

and

$$1\{\text{for some } h, \text{ interactions } i \to h \text{ and } h \to j \text{ occurred in } [t - \Delta_k, t)\}.$$

Any process measurable with respect to the predictable $\sigma$-algebra is a valid covariate; this excludes only covariates depending on the future or the immediate present. In Section 5.2 we detail specific covariates suitable for measuring homophily and network effects.

Also note that despite presuming $\mathcal{I}$ and $\mathcal{J}$ to be fixed, our analysis allows senders and receivers to enter and leave the study during the observation period. The effective number of senders at time $t$ is the set of $i$ such that $\bar{\lambda}_t(i) \neq 0$, which potentially varies with time. Likewise, the effective number of receivers is controlled through $\mathcal{J}_t(i)$.

Following Cox (1975), we treat the baseline rate $\bar{\lambda}_t(i)$ as a nuisance parameter and estimate the coefficient vector $\beta_0$ using a partial likelihood. Specifically, let $(t_1, i_1, j_1), \ldots, (t_n, i_n, j_n)$ be the sequence of observed interactions. The inference procedure is motivated by decomposing the full likelihood, $L$, as

$$\begin{aligned}
&L(t_1, i_1, j_1, t_2, i_2, j_2, \ldots, t_n, i_n, j_n) \\
&\quad = L(t_1, i_1)\, L(j_1 | t_1, i_1)\, L(t_2, i_2 | t_1, i_1, j_1)\, L(j_2 | t_2, i_2, t_1, i_1, j_1) \\
&\qquad \cdots L(t_n, i_n | t_{n-1}, i_{n-1}, j_{n-1}, \ldots t_1, i_1, j_1)\, L(j_n | t_n, i_n, t_{n-1}, i_{n-1} \ldots t_1, i_1, j_1) \\
&\quad = \Big[L(t_1, i_1)\, L(t_2, i_2 | t_1, i_1, j_1) \cdots L(t_n, i_n | t_{n-1}, i_{n-1}, j_{n-1}, \ldots t_1, i_1, j_1)\Big] \\
&\qquad \cdot \Big[L(j_1 | t_1, i_1)\, L(j_2 | t_2, i_2, t_1, i_1, j_1) \cdots L(j_n | t_n, i_n, t_{n-1}, i_{n-1} \ldots t_1, i_1, j_1)\Big];
\end{aligned}$$

the term comprised of the product of conditional likelihoods of $j_1, \ldots, j_n$ is dubbed a partial likelihood. In continuous time, the log partial likelihood at time $t$, evaluated at $\beta$, is

$$\log PL_t(\beta) = \sum_{t_m \leq t} \Big\{ \beta^{\mathrm{T}} x_{t_m}(i_m, j_m) - \log \big[ \sum_{j \in \mathcal{J}_{t_m}(i_m)} \exp\{\beta^{\mathrm{T}} x_{t_m}(i_m, j)\} \big] \Big\}. \tag{2}$$

In Section 3, we prove under suitable regularity conditions that the maximizer of $\log PL_t(\cdot)$ is a consistent estimator of $\beta_0$ as $t$ increases.

The function $\log PL_t(\cdot)$ is concave, and so can be maximized via Newton's method or a gradient-based optimization approach (Nocedal and Wright, 2006). These methods require one or both of the first two derivatives of $\log PL_t(\cdot)$, which can be expressed in terms of weighted means and covariances of the covariates. The weights are

$$w_t(\beta, i, j) = \exp\{\beta^{\mathrm{T}} x_t(i, j)\} \cdot 1\{j \in \mathcal{J}_t(i)\}, \tag{3a}$$

$$W_t(\beta, i) = \sum_{j \in \mathcal{J}_t(i)} w_t(\beta, i, j). \tag{3b}$$

The inner sum in $\log PL_t(\beta)$ is $W_{t_m}(\beta, i_m)$. The function $\log W_t(\cdot, i)$ has gradient $E_t(\cdot, i)$ and Hessian $V_t(\cdot, i)$, given by

$$E_t(\beta, i) = \frac{1}{W_t(\beta, i)} \sum_{j \in \mathcal{J}_t(i)} w_t(\beta, i, j)\, x_t(i, j), \tag{4a}$$

$$V_t(\beta, i) = \frac{1}{W_t(\beta, i)} \sum_{j \in \mathcal{J}_t(i)} w_t(\beta, i, j) \Big[ x_t(i, j) - E_t(\beta, i) \Big]^{\otimes 2}, \tag{4b}$$

where $a^{\otimes 2} = a \otimes a = aa^{\mathrm{T}}$. Consequently, the gradient and negative Hessian of $\log PL_t(\cdot)$ are

$$U_t(\beta) = \nabla \big[ \log PL_t(\beta) \big] = \sum_{t_m \leq t} x_{t_m}(i_m, j_m) - E_{t_m}(\beta, i_m), \tag{5a}$$

$$I_t(\beta) = -\nabla^2 \big[ \log PL_t(\beta) \big] = \sum_{t_m \leq t} V_{t_m}(\beta, i_m). \tag{5b}$$

We call $U_t(\beta_0)$ the unnormalized score and $I_t(\beta_0)$ the observed information matrix.

Note the dependence of these terms on time-varying covariates, which precludes using sufficient statistics and introduces additional complexity in maximizing $\log PL_t(\cdot)$. For most large interaction datasets, existing computational routines for handling Cox models (e.g., the function `coxph` from the `survival` package for R (Therneau and Lumley, 2009)) will not suffice. In Appendix A, we describe a customized method for maximizing $\log PL_t(\cdot)$ that exploits sparsity in $x_t(i, j)$.

## 3. Consistency of maximum partial likelihood inference

Under the model of Section 2, the maximum partial likelihood estimator (MPLE) is a natural estimate of $\beta_0$; the inverse Hessian of $\log PL_t(\cdot)$ evaluated at the MPLE is a natural estimate of its covariance matrix. We now give conditions under which these estimators are consistent.

In the sampling regime where observation time $t$ is fixed and the number of senders $|\mathcal{I}|$ increases, Andersen and Gill's (1982) consistency proof for the Cox proportional hazards model in survival analysis extends to cover model (1). This setting is natural in the context of clinical trial data, where $\mathcal{I}$ corresponds to the set of patients under study, but does not meet the requirements typical of interaction data. For most interaction data we cannot control $\mathcal{I}$ and $\mathcal{J}$, and the only way to collect more data is to increase the observation time. Cox (1972, 1975) outlines a proof for general MPLE consistency that applies to our sampling regime, but his argument is heuristic; Wong's (1986) treatment is more rigorous but does not cover continuous or time-varying covariates. The general interaction data sampling regime warrants a new consistency proof.

Our proof of consistency relies on rescaling time to make the interaction times uniform. To this end, define marginal processes $N_t(i) = \sum_{j \in \mathcal{J}} N_t(i,j)$ and $N_t = \sum_{i \in \mathcal{I}} N_t(i)$; also note that $t_n = \sup\{t : N_t < n\}$ is a stopping time and let $\mathcal{F}_{t_n}$ be the $\sigma$-algebra of events prior to $t_n$. The main idea of the proof is to change time from the original scale to a scale on which $t_n - t_{n'}$ is proportional to $n - n'$.

### *3.1. Assumptions*

Let $\mathcal{B}$ be a neighborhood of $\beta_0$. For a vector, $a$, let $\|a\|$ denote its Euclidean norm; for a matrix, $A$, let $\|A\|$ denote its spectral norm, equal to the largest eigenvalue of $(A^{\mathrm{T}}A)^{1/2}$. We require the following assumptions:

A1. **The covariates are uniformly square-integrable.** That is,

$$\mathbb{E}\left[\sup_{t,i,j} \|x_t(i,j)\|^2\right] \text{ is bounded.}$$

A2. **The integrated covariance function is well behaved.** When $\beta \in \mathcal{B}$ and $\alpha \in [0,1]$, as $n \to \infty$, then with respect to the covariance function $\Sigma_\alpha(\beta)$ we have that

$$\frac{1}{n} \sum_{i \in \mathcal{I}} \int_0^{t_{\lfloor \alpha n \rfloor}} V_s(\beta, i)\, W_s(\beta, i)\, \bar{\lambda}_s(i)\, ds \xrightarrow{P} \Sigma_\alpha(\beta).$$

A3. **The interaction arrival times are finite.** For each $n$,

$$\mathbb{P}\{t_n < \infty\} = 1.$$

A4. **The variance function is equicontinuous.** More precisely,

$$\left\{V_{t_n}(\cdot, i) : n \geq 1, i \in \mathcal{I}\right\} \text{ is an equicontinuous family of functions.}$$

These technical assumptions are similar to those of Andersen and Gill (1982), who investigate specific settings in which their assumptions hold. Note that when $\|x_t(i,j)\|$ is bounded and Assumption A3 is in force, the remaining assumptions follow.

### *3.2. Main results*

Assumptions A1–A4 imply that the MPLE is consistent and asymptotically Gaussian, as shown by the following two theorems.

Theorem 3.1. *Let $N$ be a multivariate counting process with stochastic intensity as given in* (1)*, with true parameter vector $\beta_0$. Let $t_n$ be the sequence of interaction times, and set $U_t(\beta)$ and $I_t(\beta)$ to be the gradient and negative Hessian of the log partial likelihood function as given respectively in* (5a) *and* (5b)*. If assumptions A1–A2 hold, then as $n \to \infty$:*

*(a)* $n^{-1/2}\, U_{t_{\lfloor \alpha n \rfloor}}(\beta_0)$ *converges weakly to a Gaussian process on* $[0,1]$ *with covariance function* $\Sigma_\alpha(\beta_0)$*;*

*(b) if assumptions A3–A4 also hold, then for any consistent estimator* $\hat\beta_n$ *of* $\beta_0$*, we have that*

$$\sup_{\alpha \in [0,1]} \left\| \tfrac{1}{n} I_{t_{\lfloor \alpha n \rfloor}}(\hat\beta_n) - \Sigma_\alpha(\beta_0) \right\| \xrightarrow{P} 0.$$

We don't actually require convergence of the whole sample path, but it turns out to be just as much effort to prove as convergence of the endpoint. Consistency is a direct consequence of Theorem 3.1.

Theorem 3.2. *Let* $N$ *be a multivariate counting process with stochastic intensity as given in* (1)*, with true parameter vector* $\beta_0$*. Let the log partial likelihood,* $\log PL_t(\cdot)$*, be as defined in* (2)*. Let* $t_n$ *be the sequence of interaction times.*

*Assume that for* $\beta$ *in a neighborhood of* $\beta_0$ *that* $-\frac{1}{n}\nabla^2[\log PL_{t_n}(\beta)] \xrightarrow{P} \Sigma_1(\beta)$*, where* $\Sigma_1(\cdot)$ *is locally Lipschitz and with smallest eigenvalue bounded away from zero. If* $\hat\beta_n$ *maximizes* $\log PL_{t_n}(\cdot)$ *and conclusion (a) of Theorem 3.1 holds, then the following are true as* $n \to \infty$*:*

*(a)* $\hat\beta_n$ *is a consistent estimator of* $\beta_0$*;*

*(b)* $\sqrt{n}\,(\hat\beta_n - \beta_0)$ *converges weakly to a mean-zero Gaussian random variable with covariance* $[\Sigma_1(\beta_0)]^{-1}$.

We prove Theorems 3.1 and 3.2 in Appendix B.

## 4. Multicast interactions

In Sections 2 and 3, we have assumed that each interaction involves a single sender and a single receiver. The model and corresponding asymptotic theory are sufficient to cover strictly pairwise directed interactions (e.g., phone calls), but they do not describe interactions that can involve multiple receivers (e.g., e-mail messages). We call an interaction involving a single sender and possibly multiple receivers a multicast interaction.

In practice, multicast interactions are typically treated in an ad-hoc manner via duplication—for example, interaction $i \to \{j_1, j_2, j_3\}$ gets recorded as three separate pairwise interactions $i \to j_1$, $i \to j_2$, and $i \to j_3$—giving rise to approximate likelihood and inference. In this section we explore the implications of using this approximate likelihood in the multicast setting. In particular we show it to be closely related to an extension of our model for directed pairwise interactions, and that the bias introduced by such an approximation can be quantified and in certain cases corrected.

To this end, we introduce an extension of the model to the multicast setting. Let $\mathcal{I}$, $\mathcal{J}$, $\mathcal{J}_t(i)$, $x_t(i,j)$, and $\beta_0$ be as in Section 2. For each sender $i$ and positive integer $L$, let $\bar\lambda_t(i;L)$ be a non-negative predictable process called the baseline $L$-receiver intensity of sender $i$. Let $(t_1, i_1, J_1), \ldots, (t_n, i_n, J_n)$ be the sequence of observed multicast interactions, with tuple $(t, i, J)$ indicating that at time $t$, sender $i$ interacted with receiver set $J$. For a set $J$, let $|J|$ denote its cardinality.

Consider a model for multicast interactions where the rate of interaction between sender $i$ and receiver set $J$ is

$$\lambda_t(i, J) = \bar\lambda_t(i; |J|) \cdot \exp\Big\{ \sum_{j \in J} \beta_0^{\mathrm{T}} x_t(i,j) \Big\} \cdot \prod_{j \in J} 1\{ j \in \mathcal{J}_t(i) \}. \tag{6}$$

The log partial likelihood at time $t$, evaluated at $\beta$, is

$$\log PL_t(\beta) = \sum_{t_m \leq t} \Big\{ \sum_{j \in J_m} \beta^{\mathrm{T}} x_{t_m}(i_m, j) - \log \big[ \sum_{\substack{J \subseteq \mathcal{J}_{t_m}(i_m) \\ |J| = |J_m|}} \exp \big\{ \sum_{j \in J} \beta^{\mathrm{T}} x_{t_m}(i_m, j) \big\} \big] \Big\}. \tag{7}$$

Suppose instead of using the multicast model, we use duplication to get pairwise interactions from the original multicast data. If we use the model of (1) for the pairwise data and ignore ties in the interaction times, we obtain an approximate partial likelihood:

$$\log \widetilde{PL}_t(\beta) = \sum_{t_m \leq t} \Big\{ \sum_{j \in J_m} \beta^{\mathrm{T}} x_{t_m}(i_m, j) - |J_m| \log \big[ \sum_{j \in \mathcal{J}_{t_m}(i_m)} \exp\{ \beta^{\mathrm{T}} x_{t_m}(i_m, j) \} \big] \Big\}. \tag{8}$$

We claim $\log \widetilde{PL}_t(\beta)$ approximates $\log PL_t(\beta)$. Heuristically, replacing the sum over all sets of size $|J_m|$ in (7) with a sum over all multisets of size $|J_m|$ (i.e., allowing duplicate elements from $\mathcal{J}_{t_m}(i_m)$), observe

$$\begin{aligned} \log \big[ \sum_{\substack{J \subseteq \mathcal{J}_{t_m}(i_m) \\ |J| = |J_m|}} \exp \big\{ \sum_{j \in J} \beta^{\mathrm{T}} x_{t_m}(i_m, j) \big\} \big] &\approx \log \big[ \big( \sum_{j \in \mathcal{J}_{t_m}(i_m)} \exp \big\{ \beta^{\mathrm{T}} x_{t_m}(i_m, j) \big\} \big)^{|J_m|} \big] \\ &= |J_m| \log \big[ \sum_{j \in \mathcal{J}_{t_m}(i_m)} \exp \big\{ \beta^{\mathrm{T}} x_{t_m}(i_m, j) \big\} \big]. \end{aligned}$$

In this sense, $\log PL_t(\beta) \approx \log \widetilde{PL}_t(\beta)$. Section 4.1 makes this statement more precise, and Section 4.2 analyzes the bias introduced by maximizing $\log \widetilde{PL}_t(\beta)$ in lieu of $\log PL_t(\beta)$.

### 4.1. Approximation error

Define the receiver set growth sequence

$$G_n = \sum_{t_m \leq t_n} \frac{1\{|J_m| > 1\}}{|\mathcal{J}_{t_m}(i_m)|}. \tag{9}$$

This sequence plays a critical role in bounding the error introduced by replacing $\log PL$ with $\log \widetilde{PL}$. Note that when $|\mathcal{J}_{t_m}(i_m)|$ is constant $G_n$ has linear growth, but when $|\mathcal{J}_{t_m}(i_m)|$ increases, $G_n$ often has sublinear growth. For example, the Cauchy-Schwartz inequality gives

$$G_n \leq \sqrt{n} \cdot \Bigg[ \sum_{t_m \leq t_n} \frac{1\{|J_m| > 1\}}{|\mathcal{J}_{t_m}(i_m)|^2} \Bigg]^{1/2},$$

so if $|\mathcal{J}_{t_m}(i_m)| / \sqrt{m} \to \infty$, then $G_n = \mathcal{O}(\sqrt{n})$. Theorem 4.1 (proved in Appendix C) bounds the approximation error in terms of $G_n$.

THEOREM 4.1. *Let $(t_m, i_m, J_m)$ be a sequence of observations from a multivariate point processes with intensity as given in (6). Assume that $\sup_t \|x_t(i,j)\|$ and $\sup_m |J_m|$ are bounded in probability. If $\log PL$ and $\log \widetilde{PL}$ are as defined in (7–8), and $G_n$ is as defined in (9), then for $\beta$ in a neighborhood of $\beta_0$,*

$$\Big\| \nabla [\log PL_{t_n}(\beta)] - \nabla [\log \widetilde{PL}_{t_n}(\beta)] \Big\| = \mathcal{O}_P(G_n),$$

*and*

$$\Big\| \nabla^2 [\log PL_{t_n}(\beta)] - \nabla^2 [\log \widetilde{PL}_{t_n}(\beta)] \Big\| = \mathcal{O}_P(G_n).$$

### 4.2. Bias correction from the approximate partial likelihood

When we use ad-hoc duplication, we are performing approximate inference under the multicast model of (6). In practice, even if we explicitly want to use the multicast model, computing the partial likelihood of (7) involves an intractable combinatorial sum, so we may resort to using the approximation instead. Maximizing $\log \widetilde{PL}_t(\cdot)$ instead of $\log PL_t(\cdot)$ introduces bias in the estimate of $\beta_0$. Theorem 4.2 (proved in Appendix C) bounds the bias.

Theorem 4.2. *Under the setup of Theorem 4.1, let $\hat{\beta}_n$ maximize $\log PL_{t_n}(\cdot)$ and let $\tilde{\beta}_n$ maximize $\log \widetilde{PL}_{t_n}(\cdot)$. Suppose for all $n$ that the Hessian $\frac{1}{n}\nabla^2[\log \widetilde{PL}_{t_n}(\cdot)]$ is uniformly locally Lipschitz and with smallest eigenvalue bounded away from zero in a neighborhood of $\hat{\beta}_n$. If $G_n/n \xrightarrow{P} 0$, then*

$$\|\tilde{\beta}_n - \hat{\beta}_n\| = \mathcal{O}_P(G_n/n).$$

That $\hat{\beta}_n$ is a consistent estimator of $\beta_0$ follows directly from the theory in Section 3, since the multicast case can be considered as a special case of the single receiver case: Consider the product $\mathcal{I} \times \mathbb{N}_+$ as the sender set, and the power set $\mathcal{P}(\mathcal{J})$ as the receiver set. For sender $(i, L)$, the process $\bar{\lambda}(i; L)$ is then the baseline send intensity, and $\{J \subseteq \mathcal{J}_t(i) : |J| = L\}$ is the receiver set; for sender-receiver pair $\big((i, L), J\big)$, vector $\sum_{j \in J} x_t(i, j)$ is the covariate vector. Consistency of the MPLE now follows from Theorem 3.2.

Suppose the true MPLE, $\hat{\beta}_n$, is a $\sqrt{n}$-consistent estimate of $\beta_0$. (Theorem 3.2 gives sufficient conditions.) Theorem 4.2 says that if $|\mathcal{J}_{t_m}(i_m)|$ grows fast enough to make $G_n$ smaller than $\mathcal{O}_{\mathrm{P}}(\sqrt{n})$, then the approximate MPLE, $\tilde{\beta}_n$, is *also* $\sqrt{n}$-consistent. Moreover, if $\sqrt{n}(\hat{\beta}_n - \beta_0)$ is asymptotically Gaussian, then $\sqrt{n}(\tilde{\beta}_n - \beta_0)$ is asymptotically Gaussian with the same covariance matrix but possibly a different mean. Under enough regularity, $-\frac{1}{n}[\nabla^2 \log \widetilde{PL}_{t_n}(\tilde{\beta}_n)]$ consistently estimates the limiting covariance of $\sqrt{n}(\tilde{\beta}_n - \beta_0)$. To get the mean, we use a parametric bootstrap as follows.

Assume that the conditions of Theorem 4.2 hold. The residual $\tilde{\beta}_n - \beta_0$ depends continuously on $\beta_0$ and the covariate process $x_t(i, j)$. Since $\tilde{\beta}_n$ is a consistent estimator of $\beta_0$, we can estimate the bias in $\tilde{\beta}_n$ via a parametric bootstrap. We generate a bootstrap replicate dataset $\{(t_m, i_m, J_m^{(r)})\}$ by drawing $J_m^{(r)}$, a random subset of $\mathcal{J}_{t_m}(i_m)$ with size $|J_m|$ whose elements are drawn proportional to $w_{t_m}(\tilde{\beta}_n, i_m, \cdot)$. We then get a bootstrap approximate MPLE, $\tilde{\beta}_n^{(r)}$, by maximizing $\widetilde{PL}_{t_n}^{(r)}$, where

$$\log \widetilde{PL}_t^{(r)}(\beta) = \sum_{t_m \leq t} \Big\{ \sum_{j \in J_m^{(r)}} \beta^{\mathrm{T}} x_{t_m}(i_m, j) - |J_m^{(r)}| \log \big[ \sum_{j \in \mathcal{J}_{t_m}(i_m)} \exp\{\beta^{\mathrm{T}} x_{t_m}(i_m, j)\}\big] \Big\}.$$

Note that $x_t(i, j)$ is determined from the original dataset, not the bootstrap dataset. For each bootstrap replicate, we get a residual $\tilde{\beta}_n^{(r)} - \tilde{\beta}_n$. With $R$ bootstrap replicates, we estimate the bias by

$$\widehat{\text{bias}} = \frac{1}{R} \sum_{r=1}^{R} \tilde{\beta}_n^{(r)} - \tilde{\beta}_n.$$

We adjust for estimator bias by replacing $\tilde{\beta}_n$ with $\tilde{\beta}_n - \widehat{\text{bias}}$.

### 4.3. Simulation

We show a simulation study to empirically verify the result of Theorem 4.2. In the study, we have one sender, and a receiver count $|\mathcal{J}|$ ranging from 32 to 1000. Each receiver was assigned a

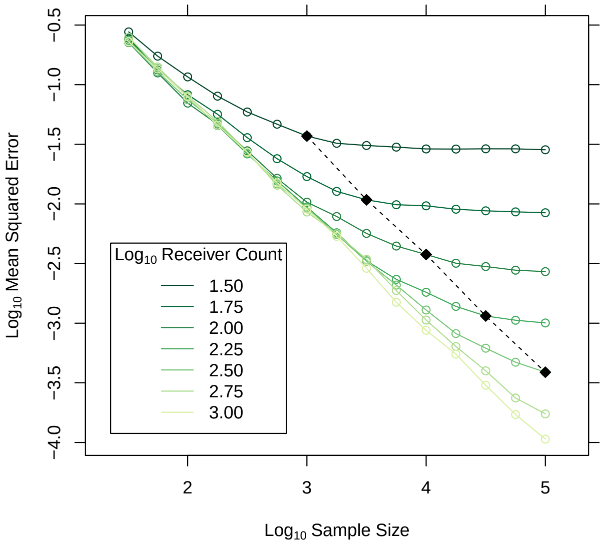


**Fig. 1.** Multicast coefficient estimation error with approximate MPLE. Receiver count $|\mathcal{J}|$ is equal to the square root of sample size $n$ along the dashed line.

constant covariate vector $x(j)$ whose elements were independent Bernouli random variables with success probability $\frac{1}{2}$. The components of the true coefficient vector $\beta$ were drawn independently from the standard Normal distribution.

We chose sample sizes $n$ ranging from 32 to 100,000. For each receiver count $|\mathcal{J}|$, we drew $n$ multicast messages, with the receiver set $J_m$ for message $m$ determined as follows: we determined the size, $|J_m|$, by drawing from a geometric distribution with success probability $p = 0.4$, so that $\mathbb{P}\{|J_m| = L\} = (1-p)^{L-1}\,p$ for $L \geq 1$; once $|J_m|$ was determined, we chose among all receiver sets with cardinality $|J_m|$, with $\mathbb{P}\{J_m = J\} \propto \exp\{\sum_{j \in J} \beta^{\mathrm{T}} x(j)\}$. Once we generated the message data, we computed $\tilde{\beta}$ by maximizing the approximate log partial likelihood analogous to (8). Finally, we computed $\|\beta - \tilde{\beta}\|$.

We repeated this procedure for 100 random replicates at each receiver count and sample size, and computed the mean squared error of $\tilde{\beta}$ by averaging the value of $\|\beta - \tilde{\beta}\|^2$ over all replicates. Figure 1 displays the results. From the spacings of the asymptotes of the solid lines in the figure, we can see that if $|\mathcal{J}|$ does not grow with $n$, then the error $\|\beta - \tilde{\beta}\|^2$ is roughly $\mathcal{O}(|\mathcal{J}|^{-2})$ for large $n$; strictly speaking, the assumptions of Theorem 4.2 do not hold in this scenario since $G_n = \mathcal{O}_{\mathrm{P}}(n/|\mathcal{J}|)$, but nevertheless the theorem predicts an error rate of $\mathcal{O}(|\mathcal{J}|^{-2})$. For the Theorem 4.2 to apply, we require that $|\mathcal{J}|$ grow with $n$. From the slope of the dashed line in Fig. 1, we can see that if $|\mathcal{J}| = \sqrt{n}$, then $\|\beta - \tilde{\beta}\|^2$ is roughly $\mathcal{O}_{\mathrm{P}}(n^{-1})$; this agrees with the theorem, since $G_n = \sqrt{n}$ in this situation.

## 5. Fitting the model to a corporate e-mail network

Recall from Section 1 that, given a set of interaction data triples $(t, i, j)$, a primary modeling goal lies in determining which characteristics and behaviors of the senders and receivers are predictive of interaction. The modeling and inference framework introduced above enables us to directly address these concerns, as we now demonstrate through the analysis of a corporate e-mail network consisting of a large subset of the e-mail messages sent within the Enron corporation between 1998 and 2002. These e-mail interaction data give rise to the following questions:

**Homophily** To what extent are traits shared between individuals (gender, department, or seniority) predictive of interaction behaviors?

**Network Effects** To what extent are dyadic or even triadic network effects, as characterized by past interaction behaviors, relevant to predicting future interaction behaviors?

We undertake our analysis using the multicast proportional intensity modeling framework developed in Sections 2 and 3 above, employing both static covariates reflecting actor traits, as well as dynamic covariates capturing network effects. The bootstrap technique introduced in Section 4 for multicast interactions is then used to reduce bias in the estimated effects, as well as to demonstrate that our asymptotic approximations are reasonable in this data modeling regime. We conclude this section with a discussion of the goodness of fit of our model in this setting, before turning our attention in Section 6 to an evaluation of the strength of homophily and network effects in explaining these data.

### *5.1. Data and methods*

Our example analysis uses publicly available data from the Enron e-mail corpus (Cohen, 2009), a large subset of the e-mail messages sent within the Enron corporation between 1998 and 2002, and made public as the result of a subpoena by the U.S. Federal Energy Regulatory Commission during an investigation into fraudulent accounting practices. We analyze the dataset compiled by Zhou et al. (2007), comprising 21,635 messages sent among 156 employees between November 13, 1998 and June 21, 2002, along with the genders, seniorities, and departments of these employees.

Approximately 30% of these messages have more than one recipient across their `To`, `CC`, and `BCC` fields, with a few messages having more than fifty recipients. In the subsequent analysis, we exclude messages with more than 5 recipients—a subjectively-chosen cutoff that avoids e-mails sent *en masse* to large groups.

We model these data using the multicast proportional intensity model of Section 4, with $\mathcal{I} = \mathcal{J} = \{1, 2, \dots, 156\}$ and $\mathcal{J}_t(i) = \mathcal{I} \setminus \{i\}$, and with static and dynamic covariates described in the next section. We fit the model by first maximizing the approximate log partial likelihood $\log \widetilde{PL}_t(\beta)$ of (8), and then employing a parametric bootstrap to estimate and correct the resultant bias in parameter estimates. We calculate standard errors using the corresponding asymptotic theory. In the setting of this example, the interaction count is high, so the asymptotic framework developed in Sections 3 and 4 is natural. The main violation of assumptions A1–A4 is that our covariates (described in Section 5.2) may in principle be unbounded; nevertheless, bootstrap calculations (described in Section 5.3) show that the asymptotic approximations we employ remain reasonable in this regime.

We wrote custom software in the C programming language to fit the model using Newton's method. Our implementation exploits structure in the covariates to make the computational complexity of the fitting procedure roughly linear in the number of messages and the number of actors. Appendix A describes the fitting procedure in detail. It took approximately 20 minutes to fit the full model using a standard desktop computer with a 2.4 GHz processor and 4GB of RAM. Each bootstrap replicate took approximately 10 minutes to generate and fit, using the original estimate as a starting point for the fitting algorithm. Most of the complexity in the fitting procedure is due to the inclusion of triadic covariates as described below; including only dyadic covariates reduces the fitting time to approximately 1 minute.

| Variate | Characteristic of actor $i$ | Count |
|---|---|---|
| $L(i)$ | member of the Legal department | 25 |
| $T(i)$ | member of the Trading department | 60 |
| $J(i)$ | seniority is Junior | 82 |
| $F(i)$ | gender is Female | 43 |

**Fig. 2.** Actor-specific traits, with counts of how many of the 156 actors share each trait

## 5.2. *Covariates*

The goal of our investigation is to assess the predictive ability of actor traits and network effects. To this end, we choose covariates that encode these traits and effects. Each covariate is encoded as a component of the time-varying dyad-dependent vector $x_t(i,j)$, which is linked to the rate of interaction between sender $i$ and receiver $j$ via the multicast proportional intensity model of (1).

### 5.2.1. *Static covariates to measure homophily and group-level effects*

Consider first those actor traits that do not vary with time: the actors' genders, departments, and seniorities. We encode the traits of actor $i$ and their second-order interactions using 9 actor-dependent binary (0/1) variables, as described in Fig. 2.

We encode all 20 identifiable first-order interactions between the traits of sender $i$ and receiver $j$ as components of $x_t(i,j)$. We do this by using variates of the form $Y(j)$ and $X(i) \cdot Y(j)$, where $X$ and $Y$ are chosen from the list of 4 actor-dependent variates ($L$, $T$, $J$, $F$). We also include 4 receiver-specific covariates of the form $1 \cdot Y(j)$. We cannot identify the coefficients for covariates of the form $X(i) \cdot 1$; if a component of $x_t(i,j)$ is the same for all values of $j$, then the corresponding component of $\beta$ will not be identifiable since the product of the two can be absorbed into $\bar{\lambda}_t(i)$ without changing the likelihood.

We measure homophily by way of the estimated coefficients for covariates of the form $X(i) \cdot X(j)$. For example, if the sum of the coefficients of $1 \cdot J(j)$ and $J(i) \cdot J(j)$ is large and positive, this tells us that Junior employees exhibit homophily in their choice of message recipients.

### 5.2.2. *Dynamic covariates to measure network effects*

Static effects are useful for determining which traits are predictive of the relative rate of interaction between sender $i$ and receiver $j$, but they do not shed light on network effects. Therefore, we are also interested in the predictive relevance of the dynamic network behaviors described in Fig. 3. The first two behaviors (**send** and **receive**) are "dyadic," involving exactly two actors, while the last four (**2-send**, **2-receive**, **sibling**, and **cosibling**) are "triadic," involving exactly three actors.

To measure first-order dependence of message exchange behavior on these network effects, we introduce binary indicators for all 6 effects as components of $x_t(i,j)$. These indicators depend on the sender $i$, the receiver, $j$, and the history of the process at the current time $t$. By the shorthand notation $1\{\textbf{send}\}$, we denote the indicator variable depending on sender $i$, receiver $j$, and the current time, $t$, which indicates if $i$ has sent $j$ a message before time $t$, with the remaining notations ($1\{\textbf{receive}\}$, $1\{\textbf{2-receive}\}$, etc.) defined similarly.

To measure higher-order time dependence, we introduce additional covariates of the following form. We partition the interval $[-\infty, t)$ into $K = 7$ sub-intervals:

$$[-\infty, t) = [t - \Delta_K, t - \Delta_{K-1}) \cup [t - \Delta_{K-1}, t - \Delta_{K-2}) \cup \cdots \cup [t - \Delta_1, t - \Delta_0)$$

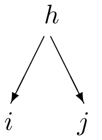
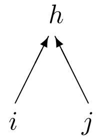
| | | |
|---|---|---|
| **send** | $i \longrightarrow j$ | $i$ has sent $j$ a message in the past |
| **receive** | $i \longleftarrow j$ | $i$ has received a message from $j$ in the past |
| **2-send** | $i \longrightarrow h \longrightarrow j$ | there exists an actor $h$ such that $i$ has sent $h$ a message and $h$ has sent $j$ a message in the past |
| **2-receive** | $i \longleftarrow h \longleftarrow j$ | there exists an actor $h$ such that $i$ has received a message from $h$, and $h$ has received a message from $j$ |
| **sibling** |  | there exists an actor $h$ such that $h$ has sent $i$ and $j$ messages in the past |
| **cosibling** |  | there exists an actor $h$ such that $h$ has received messages from $i$ and $j$ |

**Fig. 3.** Dynamic covariates to measure network effects

where $\infty = \Delta_K > \Delta_{K-1} > \cdots > \Delta_1 > \Delta_0 = 0$ and "$t - \infty$" is defined to be $-\infty$. Specifically, we set $\Delta_k = (7.5 \text{ minutes}) \times 4^k$ for $k = 1, \ldots, K-1$ so that for $k$ in this range $\Delta_k$ takes the values 30 minutes, 2 hours, 8 hours, 32 hours, 5.33 days, and 21.33 days.

Define the half-open interval $I_t^{(k)} = [t - \Delta_k, t - \Delta_{k-1})$. For $k = 1, \ldots, K$ we define the dyadic effects

$$\begin{aligned}\mathbf{send}_t^{(k)}(i,j) &= \#\{i \to j \text{ in } I_t^{(k)}\},\\ \mathbf{receive}_t^{(k)}(i,j) &= \#\{j \to i \text{ in } I_t^{(k)}\};\end{aligned}$$

for sender $i$, such that these covariates measure the number of messages sent to, and respectively received by, receiver $j$ in time interval $I_t^{(k)}$.

The dyadic effects have been defined in the manner above to enable easy interpretation of the corresponding coefficients. To illustrate this, for $k = 1, \ldots, K$, suppose that $\beta_k$ is the coefficient corresponding to $\mathbf{send}_t^{(k)}(i,j)$. If we observe the message $i \to j$ at time $t$, then for future time $t'$ in the interval $(t, t + \Delta_1]$, the rate $\lambda_{t'}(i,j)$ will be multiplied be the factor $e^{\beta_1}$; for $t'$ in the interval $(t + \Delta_1, t + \Delta_2]$, the rate will be multiplied by $e^{\beta_2}$; this continues similarly, with the rate being multiplied by $e^{\beta_k}$ whenever $t' \in (t + \Delta_{k-1}, t + \Delta_k]$; equivalently, when $\Delta_{k-1} < t' - t \leq \Delta_k$. Thus, the coefficients $\beta_1, \ldots, \beta_K$ measure the effect of a "send event" and how this effect decays over time. We expect that $\beta_k$ will decrease as $k$ increases, but we do not enforce this constraint on the estimation procedure.

The triadic effects involve pairs of messages. For $k = 1, \ldots, K$ and $l = 1, \ldots, K$ we define the triadic effects

$$\begin{aligned}
\textbf{2-send}_t^{(k,l)}(i,j) &= \sum_{h \neq i,j} \#\{i \to h \text{ in } I_t^{(k)}\} \cdot \#\{h \to j \text{ in } I_t^{(l)}\}, \\
\textbf{2-receive}_t^{(k,l)}(i,j) &= \sum_{h \neq i,j} \#\{h \to i \text{ in } I_t^{(k)}\} \cdot \#\{j \to h \text{ in } I_t^{(l)}\}, \\
\textbf{sibling}_t^{(k,l)}(i,j) &= \sum_{h \neq i,j} \#\{h \to i \text{ in } I_t^{(k)}\} \cdot \#\{h \to j \text{ in } I_t^{(l)}\}, \\
\textbf{cosibling}_t^{(k,l)}(i,j) &= \sum_{h \neq i,j} \#\{i \to h \text{ in } I_t^{(k)}\} \cdot \#\{j \to h \text{ in } I_t^{(l)}\}.
\end{aligned}$$

For sender $i$ and receiver $j$, the covariate $\textbf{2-send}_t^{(k,l)}(i,j)$ counts the pairs of messages such that for some $h$ distinct from $i$ and $j$, message $i \to h$ occurred in interval $I_t^{(k)}$ and message $h \to j$ occurred in interval $I_t^{(l)}$; the other covariates behave similarly.

As with the dyadic effects, the triadic effects are designed so that their coefficients have a straightforward interpretation. However, since triadic effects involve pairs of messages, the interpretation is a bit more involved. We illustrate with the $\textbf{2-send}_t^{(k,l)}(i,j)$ covariate having coefficient $\beta_{k,l}$ for $k = 1, \ldots, K$ and $l = 1, \ldots, K$. Take $i$ and $j$ to be two actors. Suppose at time $t$ we observe the message $h \to j$. At this point, we look through the history of the process for all messages of the form $i \to h$; when paired with the original $h \to j$ message, each of these defines a "2-send event." The other 2-send events are defined as follows: if at time $s$ we observe the message $i \to h$, then we enumerate all observed messages $h \to j$ in the history of the process; when each of these is paired with the original $i \to h$ event it constitutes a 2-send event. A pair $(s, t)$ can be associated with each 2-send event, where $s$ is the time of the $i \to h$ message and $t$ is the time of the $h \to j$ message. At time $t'$ after $s$ and $t$, the existence of the 2-send event causes the sending rate $\lambda_{t'}(i,j)$ to be multiplied by the factor $e^{\beta_{k,l}}$, where $t' \in (s + \Delta_{k-1}, s + \Delta_k]$ and $t' \in (t + \Delta_{l-1}, t + \Delta_l]$. We expect $\beta_{k,l}$ to decrease as $k$ and $l$ increase, though again we do not enforce this constraint in the fitting procedure.

As previously noted, Butts (2008) used a variant of the proportional intensity model to capture interaction behavior in social settings. As such, a correspondence can be drawn between certain of the covariates in Butts (2008) and those outlined above. If we set $K = 1$, then the $\textbf{send}_t$ covariate is equivalent to an unnormalized version of Butts' persistence covariate, and the sum $(\textbf{send}_t + \textbf{receive}_t)$ becomes an unnormalized version of Butts' preferential attachment covariate. For the triadic effects, Butts' OTP, ITP, ISP, and OSP covariates are analogous to the **2-send**, **2-receive**, **sibling**, and **cosibling** covariates, although the exact definitions differ slightly. (For example, $\mathrm{OTP}_t(i,j)$ is defined as $\sum_h \min[\#\{i \to h \text{ in } (-\infty, t)\}, \#\{h \to j \text{ in } (-\infty, t)\}]$.) The versions of these covariates that we have introduced above, however, are designed to enable a more precise quantification of the time-dependence of network effects, as well as a more straightforward interpretation of the corresponding coefficients. In related models, Vu et al. (2011a,b) use similar covariates, except that they do not partition $[-\infty, t)$ into sub-intervals.

### 5.3. Bootstrap bias correction

Given the model specification, data, and covariates outlined above, we can estimate the parameter vector $\beta_0$ under the approximate log partial likelihood of (8). Recall that the results of Section 4 bound the bias resulting from this approximate MPLE procedure as a function of the growth rate of the recipient set $\mathcal{J}$ over time. Here, treating the set $\mathcal{J}$ of 156 Enron employees as constant, the resultant bias is of order $1/|\mathcal{J}|$—and, since $|\mathcal{J}| = 156$ is on the order of the square root

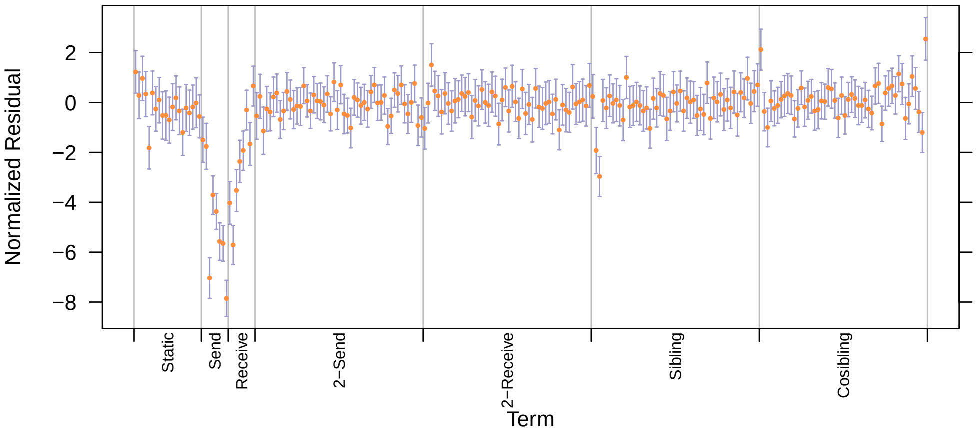


**Fig. 4.** Enron bootstrap residuals. Summary of bootstrap residuals for estimated coefficients using the Enron dataset, normalized by estimated standard errors. The points (orange) show the means, and the error bars (purple) show one standard deviation. Coefficients are grouped by model term.

of the number 21,365 of messages in the dataset, we can correct this bias using the parametric bootstrap outlined at the end of Section 4.

Fig. 4 summarizes the corresponding bootstrap residuals (from 500 replicates) for each component of the estimated parameter vector $\beta_0$; we can see from this figure that treating messages with multiple recipients as multiple single-recipient messages introduces bias on the order of the standard error for most of the coefficients. There is a pronounced negative bias in coefficient estimates for the dyadic effects, which is representative of a more general phenomenon. Sparsity in the components of $x_t(i, j)$ (when considered as a function of $j$), when combined with high values of the corresponding entries $\beta$, leads to negative bias in the coefficient estimates when there are messages with multiple recipients. The approximation in (7) is worst when for some $j^*$, weight $w_{t_m}(i_m, j^*)$ far exceeds all other values of $w_{t_m}(i_m, j)$, so that $w_{t_m}(i_m, j^*) \approx W_{t_m}(i_m)$; when $|J_m|$ is large, the maximum of $\widetilde{\mathrm{PL}}$ will avoid this situation by shrinking $\beta$ where $x_{t_m}(i_m, j)$ is sparse. The dyadic covariates are particularly sparse, so the estimates for their coefficients are particularly vulnerable to this bias.

Besides correcting for bias, the bootstrap simulations give us confidence that the asymptotic approximations are reasonable. The simulated standard errors are very close to those predicted by the theory, despite the norm $\|x_t(i, j)\|_2$ being potentially unbounded, contrary to the assumptions of Theorem 3.1.

## 5.4. Goodness of fit

Figure 5 details an ad-hoc analysis of deviance for the fitted model, showing how the approximate deviance (twice the approximate log partial likelihood) behaves as we add consecutive terms to the model. Group-level (static) effects account for 15% of the null deviance and network effects account for 37%. The most dramatic decrease in the residual deviance comes from introducing the "Send" terms into the model; with only 8 degrees of freedom, they are able to account for 33% of the null deviance. The full model accounts for 52% of the null deviance.

The residual deviance for the full model is approximately 4.8 times the residual degrees of freedom, and so an ad-hoc adjustment for this over-dispersion is to multiply the calculated

| **Term** | **Df** | **Deviance** | **Resid. Df** | **Resid. Dev** |
|---|---|---|---|---|
| Null | | | 32261 | 325412 |
| Static | 20 | 50365 | 32241 | 275047 |
| Send | 8 | 107942 | 32233 | 167105 |
| Receive | 8 | 5919 | 32225 | 161186 |
| Sibling | 50 | 3601 | 32175 | 157585 |
| 2-Send | 50 | 516 | 32125 | 157069 |
| Cosibling | 50 | 1641 | 32075 | 155428 |
| 2-Receive | 50 | 158 | 32025 | 155270 |

**Fig. 5.** Ad-hoc analysis of deviance for the Enron model. Residual deviance is defined as twice the approximate negative log partial likelihood from (8). The “Static” term contains the group level effects, and the other terms contain the network effects.

standard errors by $\sqrt{4.8} \approx 2.2$.

Note, however, that the residual deviance by itself is not adequate as a goodness-of-fit measure, as it depends only on the estimated coefficients (see Section 4.4.5 of McCullagh and Nelder (1989) for discussion of a related problem for logistic regression with sparse data). To shed more light on how well the model fits these data, we use a normalized version of the martingale residual from Therneau et al. (1990), which we call a Pearson residual. Specifically, given $\hat{\beta}$, we define

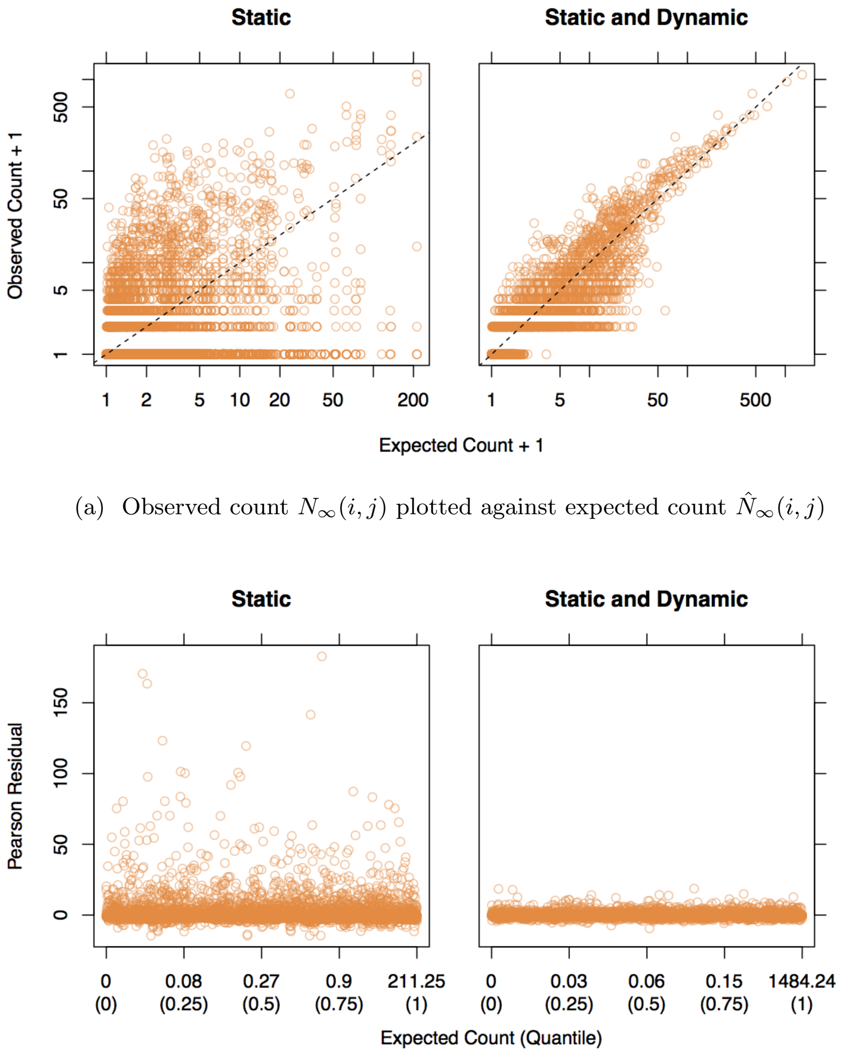
$$\hat{N}_t(i,j) = \sum_{t_m \leq t} \frac{w_{t_m}(\hat{\beta}, i, j)}{W_{t_m}(\hat{\beta}, i)} 1\{i_m = i\},$$

which is the expected number of $i \to j$ events given the estimated model, with $\int \bar{\lambda}_t(i)\,dt$ estimated by the Breslow (1974) estimate $\int W_t(\hat{\beta}, i)^{-1} \sum_j dN_{i,j}(t)$. The martingale residual analogous to that of Therneau et al. (1990) is then defined as $N_t(i,j) - \hat{N}_t(i,j)$; we normalize this quantity by an estimate of its standard deviation to get a “Pearson” residual: $(N_t(i,j) - \hat{N}_t(i,j))/\{\hat{N}_t(i,j)\}^{1/2}$.

Fig. 6a shows a plot of $N_\infty(i,j)$ versus $\hat{N}_\infty(i,j)$ for two different models. In the “static” model, we only include the static covariates, while in the full (“static and dynamic”) model, we also include all six types of network covariates. The fit for the static model is poor. For instance, it repeatedly predicts up to 200 $i \to j$ events where we only observed 1 or 2; likewise, the model predicts 1 or fewer events where we observed up to 20. For the full model, which includes the dynamic covariates to account for network effects, the fit is much better, with the relationship between observed and expected interaction counts being roughly linear.

Fig. 6b shows the Pearson residuals. For the full model, more than 95% are less than 1.21 in absolute value, and the maximum absolute residual is 18.7. In contrast, the 95% quantile for the absolute residuals in the static model is at 3.5, and the maximum absolute residual is 182.7. The sum of squares if the residuals ($X^2$) is 17281 in the full model, over 34 times lower than that for the static model (596253). We don’t know what a “reasonable” value for $X^2$ is; an ad-hoc degrees of freedom calculation suggests that for the full number this should be roughly equal to $23944 = 156 \cdot 155 - (20 + 2 \cdot 8 + 4 \cdot 50)$, which suggests that the full model is too aggressive. The bootstrap simulations confirm this, with 17055 being 5.6 standard deviances below the mean value $X^2$ for the bootstrap replicates.

For a more parsimonious model, we might drop most of the triadic effects. Indeed, the model which only uses dyadic effects has a $X^2$ value of 21094. However, at this stage we desire a model with the lowest possible bias, and also wish to acquire estimates for all of the network effects.



(a) Observed count $N_\infty(i,j)$ plotted against expected count $\hat{N}_\infty(i,j)$

(b) Pearson residual $(N_\infty(i,j) - \hat{N}_\infty(i,j))/\{\hat{N}_\infty(i,j)\}^{1/2}$ vs. expected count

**Fig. 6.** Goodness of fit plots for two models

## 6. Evaluating the strength of homophily and network effects

Given the model fitting procedure and results described above, we may now evaluate the strength of homophily and network effects in predicting the interaction behavior observed in our data.

### *6.1. Assessing evidence for homophily in the Enron data*

The analyses of Section 5 above have established that our multicast proportional intensity model with chosen covariates is reasonably accurate in describing message recipient selection, conditional on the sender and the history of the process. Thus, we are justified in using the estimated coefficients from the model to assess the predictive ability of the corresponding covariates.

| | Receiver | | | |
|---|---|---|---|---|
| Sender | L | T | J | F |
| 1 | -0.91<br>(0.04) | -0.36<br>(0.04) | -0.34<br>(0.04) | 0.04<br>(0.03) |
| L | 0.63<br>(0.05) | 0.28<br>(0.05) | 0.22<br>(0.04) | 0.15<br>(0.04) |
| T | 0.32<br>(0.07) | 0.43<br>(0.05) | 0.27<br>(0.05) | -0.07<br>(0.05) |
| J | 0.06<br>(0.05) | 0.28<br>(0.04) | 0.37<br>(0.03) | -0.13<br>(0.03) |
| F | 0.59<br>(0.05) | -0.21<br>(0.05) | -0.09<br>(0.04) | 0.15<br>(0.03) |

**Fig. 7.** Estimated coefficients and standard errors for group-level covariates of the form $X(i) \cdot Y(j)$, where $i$ is the sender, $j$ is the receiver, and $X(i)$ and $Y(j)$ are given in the row and column headings; dark coefficients are significant (via Wald test) at level $10^{-3}$.

Our first task is to gauge the predictive strength of homophily. To this end, Fig. 7 shows the estimated group-level coefficients for our model. Notably, homophily is evident for all almost all main effects (Department, Seniority, and Gender): the estimated coefficients of $L(j)$, $T(j)$, and $J(j)$ are all negative, while the sum of the estimated coefficients of $F(j)$ and $F(i) \cdot F(j)$ is positive. Negative homophily is evidenced in that the sum of the coefficients for $L(j)$ and $L(i) \cdot L(j)$ is negative. The coefficient of $F(j)$ and the sum of the coefficients for $T(j)$ and $T(i) \cdot T(j)$; and $J(j)$ and $J(i) \cdot J(j)$ are not significant.

Taking Gender as an example, the way the homophily effect manifests is as follows: if $i$ is a Female sending a message at time $t$, and person $j$ is identical to person $j'$ except for Gender, then $i$ is more likely to send to the similarly-gendered individual. The relative rate is $\exp(0.04 + 0.15) \approx 1.2$. The characterization for other types of homophily is similar.

Conspicuously, the only example of negative homophily is when the sender $i$ is in the Legal department. In this case, if person $j$ is identical to person $j'$ except for Department, then $i$ is more likely to send to an individual in a different department. The relative rates for the three departments are $\exp(0.63 - 0.91) \approx 0.76$ for the Legal department, $\exp(0.28 - 0.36) \approx 0.92$ for the Trading department, and $\exp(0) = 1$ for any Other department.

Were we interested only in homophily, we might be tempted to forgo the proportional intensity model of (1), and instead perform a contingency table analysis. The supplementary material explores this approach in detail. The major shortcoming of the contingency table approach is that it assumes that the messages are independent, which leads to bias in the parameter estimates.

### *6.2. Evaluating the importance of network effects*

In Section 6.1 we established that homophily was predictive of sending behavior, even after accounting for network effects. We now investigate the characteristics of these network effects and establish which of these effects are of greatest importance.

To begin our analysis, Fig. 8 shows the estimated coefficients for the network indicator effects, giving a crude picture of the predictive importance of each network effect. The estimated coefficients are all positive, indicating that network effects strengthen the ties between individu-

| Variate | $1\{\textbf{send}\}$ | $1\{\textbf{receive}\}$ | $1\{\textbf{2-send}\}$ | $1\{\textbf{2-receive}\}$ | $1\{\textbf{sibling}\}$ | $1\{\textbf{cosibling}\}$ |
|---|---|---|---|---|---|---|
| Coefficient | 3.26 | 0.97 | 0.67 | 0.01 | 1.06 | 0.09 |
| (SE) | (0.03) | (0.02) | (0.05) | (0.04) | (0.05) | (0.04) |

**Fig. 8.** Estimated coefficients for network indicator effects

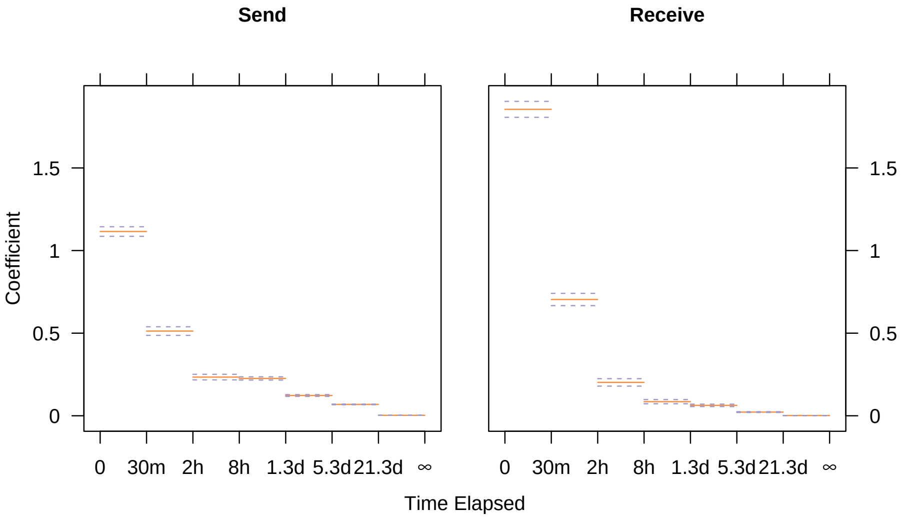


**Fig. 9.** Estimated coefficients for dyadic effects, with standard errors

als. The estimated coefficient for $1\{\textbf{send}\}$ is over three times larger than the other coefficients, agreeing with the general notion that one is most likely to do today the things one did yesterday. The next tier of indicator effects comprises $1\{\textbf{receive}\}$, $1\{\textbf{sibling}\}$, and $1\{\textbf{2-send}\}$, whose estimated coefficients range from 0.67 to 1.06. Two triadic effects, $1\{\textbf{2-receive}\}$ and $1\{\textbf{cosibling}\}$, are not significantly predictive of sending behavior.

The estimated coefficients for the recency-dependent covariates, shown in Figs. 9 and 10, give a more complete picture of network effects. Firstly, we can see that dyadic effects persist for over three weeks from the time a message is sent. The decay of the estimated coefficients is roughly exponential in the time elapsed, corresponding to a super-exponential decay in the relative sending rate. For 30 minutes after $i$ sends a message to $j$, our estimated model predicts that the rate at which $i$ sends to $j$ will be multiplied by $\exp(1.11) \approx 3.05$, and the rate at which $j$ sends to $i$ will be multiplied by $\exp(1.85) \approx 6.39$; then, between 30 minutes and 2 hours, the rates will be multiplied by $\exp(0.51) \approx 1.67$ and $\exp(0.70) \approx 2.02$, respectively; this proceeds similarly until after 21.3 days, when the rates will be multiplied by $\exp(0.003) \approx 1.002$ and $\exp(0.002) \approx 1.002$.

Comparing the coefficients for $\textbf{send}_t^{(k)}$ with those of $\textbf{receive}_t^{(k)}$ we see that the latter are higher for $k \leq 2$, while the former are higher for $k > 2$. The corresponding intuition is that if $A$ is sending a message up to two hours after receiving a message from $B$, then $A$ is likely to respond to $B$, but after that, $A$ is more likely to send to an individual whom $A$ e-mailed at the time of receiving $B$'s original message (provided $B$ and this other individual are identical in all other respects). The time window during which reciprocation is more important than past habit is less than 8 hours.

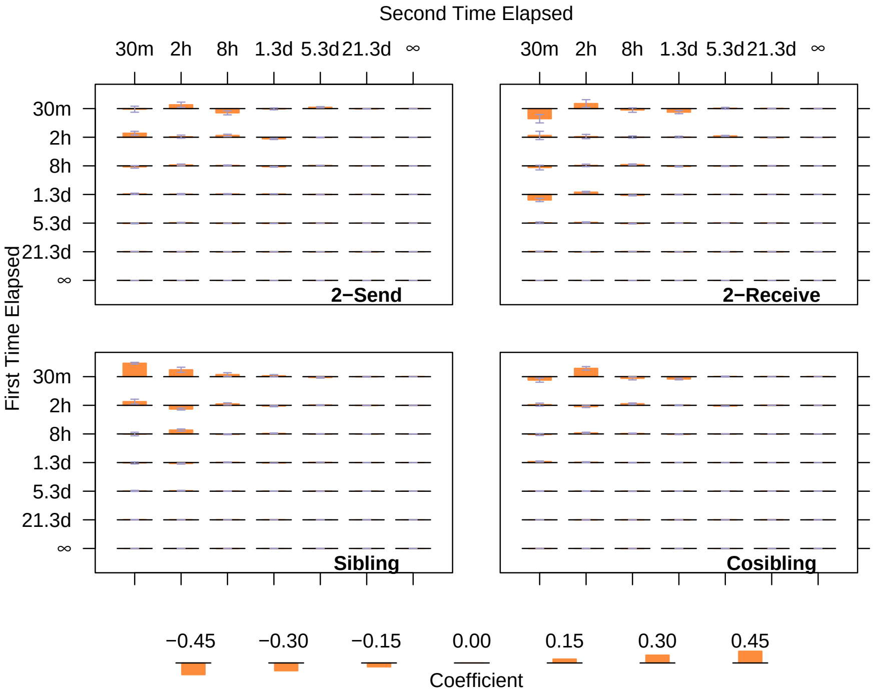


**Fig. 10.** Estimated coefficients for triadic effects, with standard errors

From Fig. 10, we can see that the triadic effects are in general less pronounced and are much more short-lived than the dyadic effects. About 86% of the estimated coefficients are within 3 standard errors of 0; even those that are significantly nonzero mostly lie between $-0.05$ and $+0.05$. The exceptions are the coefficients for $\textbf{sibling}_t^{(1,1)}$ (0.51), $\textbf{sibling}_t^{(2,2)}$ ($-0.14$), $\textbf{sibling}_t^{(3,2)}$ (0.15), $\textbf{cosibling}_t^{(1,2)}$ (0.32), $\textbf{2-receive}_t^{(4,1)}$ ($-0.21$), and $\textbf{2-receive}_t^{(4,2)}$ (0.09). We may interpret these coefficients as follows:

**sibling** If $B$ sent $A$ and $C$ messages in the last 30 minutes or between two and eight hours ago, then $A$ and $C$ are more likely to send messages to each other; however, if $B$ sent $A$ and $C$ messages between 30 minutes and two hours ago, then $A$ and $C$ are less likely to send messages to each other.

**cosibling** If $A$ sent a message to $B$ in the last 30 minutes, and $C$ sent a message to $B$ between 30 minutes and two hours ago, then $A$ will send to $B$ at a higher rate.

**2-receive** If $A$ sent a message to $B$ in the last 30 minutes, and $B$ sent a message to $C$ between 8 hours and 32 hours ago, then $C$ will send to $A$ at a lower rate; if, however, the message from $A$ to $B$ was sent between 30 minutes and two hours ago, then $C$ will send to $A$ at a higher rate.

Given the emphasis on transitivity in the networks literature, it may at first seem disconcerting that most of the estimated coefficients for the time-dependent triadic effects are found to be insignificant in this analysis. However, one must bear in mind that, except for messages sent to them directly, individuals likely have no knowledge of their colleagues' e-mail activities, and therefore there is no reason why this activity should directly affect sending behaviors. Any predictive power the triadic effects have, then, must be due to correlation with exogenous factors. In this light, it is not surprising that the triadic effects are small and have small time horizons.

The results above provide a detailed view of the ways in which network effects can manifest themselves in data. The supplementary material contains comparative analyses based on an actor-oriented model and an exponential random graph model. (See Snijders et al. (2010) and Anderson et al. (1999), respectively, for detailed surveys.) These analyses further bolster our confidence in the results of this section.

## 7. Conclusion

Our analysis of the Enron corpus in Sections 5 and 6 above has demonstrated the ways in which static and dynamic effects manifest themselves in e-mail communication networks, and we expect similar conclusions to hold broadly for other types of directed interaction data. Relative to alternatives such as contingency table analyses, actor-oriented network models, and exponential random graph models, an advantage of our approach lies in its ability to model the given data directly, rather than in an aggregated form. We are able to adjust for network effects to get more reliable estimates of homophily, and by using continuous-time information we get precise quantification on the time-dependent behavior of the network effects.

In this work, our focus has been on the coefficient vector $\beta$. We have used partial likelihood for its estimation, enabling us to treat each sender-specific baseline intensity $\bar{\lambda}_t(i)$ as a nuisance parameter. Were we to use the model for prediction, we would need to estimate baseline intensities; this could be done using a Nelson-Aalen estimator as in Andersen et al. (1993).

The foundation of our work is Cox's (1972) proportional intensity model and partial likelihood theory, tools which he first introduced almost forty years ago and which have been significantly developed since then (Cox, 1975; Fleming and Harrington, 1991; Andersen et al., 1993; Martinussen and Scheike, 2006; Cook and Lawless, 2007). These tools are used extensively in the context of survival analysis, but require further development for use in modeling interaction data. In this vein, we have extended the associated theory in two directions: first, we have provided results that are asymptotic in time rather than in the size of the population under study; and second, we have shown that treating multicast interactions via duplication leads to bias in the parameter estimates (which can in turn be corrected in certain regimes).

We find the proportional intensity model with time-varying covariates to be particularly useful for modeling repeated directed interactions. The model is simple, flexible, and well established, and it facilitates investigation into which traits and behaviors are predictive of interaction.

## Acknowledgement

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## A. Implementation

To compute the maximum partial likelihood estimator, we use Newton's method as described in Boyd and Vandenberghe (2004). This requires an efficient algorithm for computing the gradient and Hessian of the log partial likelihood. For simplicity, we describe the case of strictly pairwise interactions with no ties in the interaction times. We use the notation from Section 2, with the model from (1) and the partial likelihood from (2). Recall that $x_t(i,j)$ is in $\mathbb{R}^p$. Assume that $|\mathcal{I}| = I$ and $|\mathcal{J}| = J$.

Suppose $(t_1, i_1, j_1), \ldots, (t_n, i_n, j_n)$ is the sequence of observed interactions. Set $n(i) = \#\{i_m : i_m = i\}$. The partial likelihood factors into a product of terms, one for each sender:

$$PL_t(\beta) = \prod_{i \in \mathcal{I}} PL_t(\beta, i), \qquad PL_t(\beta, i) = \prod_{\substack{t_m \leq t, \\ i_m = i}} \frac{w_{t_m}(\beta, i, j_m)}{W_{t_m}(\beta, i)}.$$

This factorization allows us to compute $\log PL_t(\beta)$ and its derivatives by computing the sender-specific terms in parallel and then adding them together.

The gradient and Hessian of the sender-specific log partial likelihood are respectively

$$\nabla[\log PL_t(\beta, i)] = \sum_{\substack{t_m \leq t, \\ i_m = i}} x_{t_m}(i, j_m) - \sum_{\substack{t_m \leq t, \\ i_m = i}} E_{t_m}(\beta, i) \tag{10a}$$

$$-\nabla^2[\log PL_t(\beta, i)] = \sum_{\substack{t_m \leq t, \\ i_m = i}} V_{t_m}(\beta, i), \tag{10b}$$

where $E_t(\beta, i)$ and $V_t(\beta, i)$ are as defined in (5a) and (5b). When $x_t(i,j)$ is constant over time, sufficient statistics for $\beta$ imply that these formulae simplify. Otherwise, computing the first two derivatives of $\log PL_{t_n}(\beta)$ necessitates iterating over all messages, potentially requiring time $\mathcal{O}(n\, J\, p^2)$. For small- to medium-sized datasets, this is manageable, but for large network datasets it can become prohibitive. In the sequel we show how to exploit sparsity to drastically reduce the computation time.

### A.1. Initial values

We will need to compute $W_0(\beta, i)$, $w_0(\beta, i, j)$, $E_0(\beta, i)$, and $V_0(\beta, i)$ for all values of $i$ and $j$. In the worst case, doing so will take $\mathcal{O}(I\, J\, p^2)$. However, often the senders belong to a small number, $\bar{I} \ll I$ of groups such that if $i$ and $i'$ are in the same group, then the corresponding values of $W_0$, $\pi_0$, $E_0$, and $V_0$ are the same, reducing the total complexity to $\mathcal{O}(\bar{I}\, J\, p^2)$. The remaining complexity estimates assume that the initial values have all been pre-computed.

### A.2. Exploiting sparsity

We first decompose $x$ into its static (non-time-varying) and dynamic parts as follows:

$$x_t(i,j) = x_0(i,j) + \Delta x_t(i,j). \tag{11}$$

Typically, we can quickly compute the dynamic part $\Delta x_t(i,j)$ at each observed message time by incrementally updating it. Further, $\Delta x_t(i,j)$, is zero for most $(i,j)$ pairs—often $\Delta x_t(i,j)$ is zero unless $i$ and $j$ have a common acquaintance or they have interacted in the past. For convenience, set $\mathcal{J}_0(i) = \mathcal{J}$. Let

$$\bar{\mathcal{J}}(i) = \{j \in \mathcal{J} : \ j \in \mathcal{J}_t(i) \text{ and } \Delta x_t(i,j) \neq 0 \text{ for some } t\ \} \cup \{j \in \mathcal{J} : \ j \notin \mathcal{J}_t(i) \text{ for some } t\ \}.$$

For fixed $t$ and $i$, assume that computing $\Delta x_t(i,j)$ for all values of $j$ takes amortized time $\mathcal{O}(d\bar{J})$.

Since $\mathcal{J}_0(i) = \mathcal{J}$, we have that

$$\begin{aligned} w_t(\beta, i, j) &= w_0(\beta, i, j) \cdot \exp\{\beta^{\mathrm{T}} \Delta x_t(i, j)\} \cdot 1\{j \in \mathcal{J}_t(i)\} \\ &= w_0(\beta, i, j) + \Delta w_t(i, j); \\ W_t(\beta, i) &= W_0(\beta, i) + \sum_{j \in \bar{\mathcal{J}}(i)} \Delta w_t(i, j); \end{aligned}$$

where

$$\Delta w_t(i, j) = w_0(\beta, i, j)[\exp\{\beta^{\mathrm{T}} \Delta x_t(i, j)\} 1\{j \in \mathcal{J}_t(i)\} - 1];$$

here we have used that $\Delta w_t(i, j)$ is zero unless $j \in \bar{\mathcal{J}}(i)$. Write

$$\pi_t(\beta, i, j) = \frac{w_t(\beta, i, j)}{W_t(\beta, i)};$$

then, defining

$$\gamma_t(i) = \frac{W_0(\beta, i)}{W_t(\beta, i)}, \qquad \Delta\pi_t(\beta, i, j) = \frac{\Delta w_t(\beta, i, j)}{W_t(\beta, i)},$$

we can express $\pi_t(\beta, i, j)$ as follows:

$$\pi_t(\beta, i, j) = \gamma_t(i) \pi_0(\beta, i, j) + \Delta\pi_t(\beta, i, j).$$

Moreover, given the initial values $W_0(\beta, i)$ and $w_0(\beta, i, j)$, we can efficiently keep track of $\gamma_t(i)$ and $\Delta\pi_t(\beta, i, j)$: for any $i$ and $t$, it takes amortized time $\mathcal{O}(\bar{J} dp)$ to evaluate $\gamma_t(i)$ and all values of $\Delta\pi_t(i, j)$ as $j$ varies.

### A.3. Computing the gradient

In evaluating the gradient of the log partial likelihood as given by (10a), the sum $\sum_m x_{t_m}(i, j_m)$ can be computed in time $\mathcal{O}(n\, p)$, while the computationally expensive term is $\sum_m E_{t_m}(\beta, i_m)$. In the sequel we show how to exploit sparsity in $x$ to reduce the associated computational overhead.

To simplify the notation, we suppress the dependence of all quantities on $\beta$ and $i$. Consider $\pi_t$ and $\Delta\pi_t$ to be vectors of length $J$, and write

$$\pi_t = \gamma_t \pi_0 + \Delta\pi_t.$$

Also, let $X_t = X_t(i)$ and $\Delta X_t = \Delta X_t(i)$ be the $J \times p$ matrices whose $j$th rows are $x_t(i, j)$ and $\Delta x_t(i, j)$, respectively, so that

$$X_t = X_0 + \Delta X_t.$$

Using these expressions, we obtain

$$E_t = X_t^{\mathrm{T}} \pi_t = \gamma_t E_0 + X_0^{\mathrm{T}} \Delta\pi_t + \Delta X_t^{\mathrm{T}} \pi_t,$$

and thus,

$$\sum_{\substack{m \\ i_m = i}} E_{t_m} = \Big( \sum_{\substack{m \\ i_m = i}} \gamma_{t_m} \Big) E_0 + X_0^{\mathrm{T}} \Big( \sum_{\substack{m \\ i_m = i}} \Delta\pi_{t_m} \Big) + \sum_{\substack{m \\ i_m = i}} \Delta X_{t_m}^{\mathrm{T}} \pi_{t_m}.$$

Taking advantage of the sparsity in $\Delta X_t$ and $\Delta\pi_t$, computing the three sums on the right hand side takes time $\mathcal{O}\big(n(i)\, \bar{J}\, d\, p\big)$. Once the sums are known, the multiplication $\Big( \sum \gamma_{t_m} \Big) E_0$ takes time $\mathcal{O}(p)$, and the multiplication $X_0^{\mathrm{T}} \big( \sum \Delta\pi_{t_m} \big)$ takes time $\mathcal{O}(\bar{J}\, p)$. Thus, we can compute $\sum_{\substack{m \\ i_m = i}} E_{t_m}$ in time $\mathcal{O}\big(n(i)\, \bar{J}\, d\, p\big)$. Computing these terms separately for each $i$ and then summing over all $i$ to get the total gradient requires time $\mathcal{O}(n\, \bar{J}\, d\, p + I\, p)$.

### *A.4. Computing the Hessian*

Computing the Hessian according to (10b) proceeds similarly to the case of the gradient. We need to efficiently compute the sum $\sum_m V_{t_m}(\beta, i_m)$; while a naive computation requires time $\mathcal{O}(n\,J\,p^2)$, this can be significantly improved by exploiting sparsity in $x_t(i,j)$.

To this end, define $\Pi_t(\beta,i)$ to be the $J \times J$ diagonal matrix with $[\Pi_t(\beta,i)]_{jj} = \pi_t(\beta,i,j)$, and set $\Delta\Pi_t(\beta,i) = \Pi_t(\beta,i) - \Pi_0(\beta,i)$. Suppressing the dependence on $\beta$ and $i$, we have

$$\begin{aligned} V_t &= X_t^{\mathrm{T}}[\Pi_t - \pi_t\pi_t^{\mathrm{T}}]X_t \\ &= X_0^{\mathrm{T}}[\Pi_t - \pi_t\pi_t^{\mathrm{T}}]X_0 \;+\; \Delta X_t^{\mathrm{T}}[\Pi_t - \pi_t\pi_t^{\mathrm{T}}]X_0 \\ &\quad + \; X_0^{\mathrm{T}}[\Pi_t - \pi_t\pi_t^{\mathrm{T}}]\Delta X_t \;+\; \Delta X_t^{\mathrm{T}}[\Pi_t - \pi_t\pi_t^{\mathrm{T}}]\Delta X_t. \end{aligned}$$

The first of these terms reduces to

$$\begin{aligned} X_0^{\mathrm{T}}[\Pi_t - \pi_t\pi_t^{\mathrm{T}}]X_0 = \gamma_t V_0 \;+\; \gamma_t(1-\gamma_t)E_0E_0^{\mathrm{T}} \;-\; E_0(\gamma_t\Delta\pi_t)^{\mathrm{T}}X_0^{\mathrm{T}} \\ -\; X_0(\gamma_t\Delta\pi_t)E_0^{\mathrm{T}} \;+\; X_0^{\mathrm{T}}[\Delta\Pi_t - \Delta\pi_t\Delta\pi_t^{\mathrm{T}}]X_0, \end{aligned}$$

and the second can be expressed as

$$\Delta X_t^{\mathrm{T}}[\Pi_t - \pi_t\pi_t^{\mathrm{T}}]X_0 = (\gamma_t\Delta X_t\pi_t)E_0^{\mathrm{T}} \;+\; \Delta X_t^{\mathrm{T}}[\Pi_t + \pi_t\Delta\pi_t^{\mathrm{T}}]X_0.$$

The third term is the transpose of the second; the fourth does not simplify.

To compute the sum $\sum_{i_m=i}^{m} V_{t_m}$, we only accumulate sums of terms that change with time: $\gamma_t$, $\Delta\pi_t$, $\gamma_t(1-\gamma_t)$, $\gamma_t\Delta\pi_t$, $\Delta\pi_t\Delta\pi_t^{\mathrm{T}}$, $\gamma_t\Delta X_t\pi_t$, $\Delta X_t^{\mathrm{T}}[\Pi_t + \pi_t\Delta\pi_t^{\mathrm{T}}]$, and $\Delta X_t^{\mathrm{T}}[\Pi_t - \pi_t\pi_t^{\mathrm{T}}]\Delta X_t$. Doing so takes time $\mathcal{O}(\bar{J}\,d\,p^2)$ for each time increment. As with the gradient computation, we compute the sums separately for each $i$ and then sum over all $i$, so that the total computation time is $\mathcal{O}(n\,\bar{J}\,d\,p^2 + I\,p^2)$.

### *A.5. Total computation time*

To perform one Newton step in maximization of the log partial likelihood of (2), we must first compute the gradient and Hessian of the log partial likelihood at the current value of $\beta$, and then compute the inverse of the Hessian and its product with the gradient. Once we have the Hessian, computing its inverse takes time $\mathcal{O}(p^3)$. Typically, it takes $\mathcal{O}(1)$ Newton steps to compute the maximum of a convex function (the constant is often below 30). The key factors in determining the computation time using the factors laid out above are $\bar{I}$, $\bar{J}$, and $d$:

- The value of $\bar{I}$ depends on the structure of $x_0(i,j)$. Specifically, $\bar{I}$ is equal to the number of distinct values of the matrix $X_0(i)$ as $i$ varies. For the Enron data, we have that $\bar{I} = 12$: each sender belongs to one of 12 groups determined by group (L/T/O), seniority (J/S), and gender (F/M), and so the matrix $X_0(i)$ depends only on the group of $i$.
- The value of $\bar{J}$ depends on the sparsity of $x_t(i,j)$. If $x_t(i,j)$ includes only dyadic network effects, then $\bar{J}$ will typically be of size $\mathcal{O}(1)$ or $\mathcal{O}(J^\alpha)$ for a fractional value $\alpha$; when we add triadic effects, this size will typically grow to at most $\mathcal{O}(J^{2\alpha})$.
- The value of $d$ depends on further structure in $x_t(i,j)$. In our implementation, $d = \mathcal{O}(1)$ for dyadic effects and $d = \mathcal{O}(\bar{J})$ for triadic effects.

The total computational cost per Newton step is thus $\mathcal{O}(\bar{I}\,J\,p^2 + n\,\bar{J}\,d\,p^2 + I\,p^2 + p^3)$, with the significance of this expression being that it is nearly linear in $I$, $J$, and $n$. Thus, the algorithm scales naturally to large datasets.

## B. Results from Section 3

### B.1. Proof of Theorem 3.1

Observe that the process $N_t(i,j)$ has compensator $\Lambda_t(i,j) = \int_0^t \lambda_s(i,j)\,ds$; similarly, processes $N_t(i)$ and $N_t$ have compensators $\Lambda_t(i) = \sum_{j \in \mathcal{J}} \Lambda_t(i,j)$ and $\Lambda_t = \sum_{i \in \mathcal{I}} \Lambda_t(i)$. Correspondingly, define local martingales $M_t(i,j) = N_t(i,j) - \Lambda_t(i,j)$, $M_t(i) = N_t(i) - \Lambda_t(i)$, and $M_t = N_t - \Lambda_t$; also define

$$H_t(i,j) = x_t(i,j) - E_t(\beta_0, i),$$

where $E_t(\beta, i)$ is as defined in (4a).

As observed by Andersen and Gill (1982), the score function $U_t(\cdot)$ evaluated at $\beta_0$ has a simple representation in terms of these processes:

$$U_t(\beta_0) = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \int_0^t H_s(i,j)\,dN_s(i,j) = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \int_0^t H_s(i,j)\,dM_s(i,j),$$

since $\sum_{j \in \mathcal{J}} \int_0^t H_s(i,j)\,d\Lambda_s(i,j) = 0$. Since by Assumption A1, $x$ is uniformly bounded, $H$ is as well. Each term in the sum above is thus locally square integrable, with predictable covariation

$$\begin{aligned}
\Big\langle \int H_s(i,j)\,dM_s(i,j),\ \int H_s(i',j')\,dM_s(i',j') \Big\rangle_t& \\
&= \int_0^t H_s(i,j) \otimes H_s(i',j')\,d\big\langle M(i,j), M(i',j') \big\rangle_s \\
&= \int_0^t \big[H_s(i,j)\big]^{\otimes 2}\,d\Lambda_s(i,j) \cdot 1\{i = i', j = j'\}
\end{aligned}$$

(Fleming and Harrington, 1991, Thm. 2.4.3). There exists a sequence of stopping times localizing all $M(i,j)$ simultaneously, so $U(\beta_0)$ is locally square integrable with predictable variation

$$\big\langle U(\beta_0) \big\rangle_t = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \int_0^t \big[H_s(i,j)\big]^{\otimes 2}\,d\Lambda_s(i,j) = \sum_{i \in \mathcal{I}} \int_0^t V_s(\beta_0, i)\,d\Lambda_s(i). \tag{12}$$

Now we rescale time. For each positive $n$ define a discretized time-scaled version of the score that is right-continuous with limits from the left. The process is defined for times $\alpha$ in $[0,1]$; between times in $[\frac{k}{n}, \frac{k+1}{n})$, it takes the value $U_{t_k}$; i.e.,

$$\tilde{U}_\alpha^{(n)}(\beta) = U_{t_{\lfloor \alpha n \rfloor}}(\beta). \tag{13}$$

Part *(a)*: Lemma B.1 shows that $\tilde{U}_\alpha^{(n)}(\beta_0)$ is a square-integrable martingale adapted to $\tilde{\mathcal{F}}_\alpha^{(n)} = \mathcal{F}_{t_{\lfloor \alpha n \rfloor}}$, the $\sigma$-algebra of events prior to $t_{\lfloor \alpha n \rfloor}$. Since it only depends on values at jump times, the quadratic variation of $\tilde{U}^{(n)}(\beta_0)$ at time $\alpha$ is equal to the quadratic variation of $U(\beta_0)$ at time $t_{\lfloor \alpha n \rfloor}$. Therefore, since quadratic and predictable variation have the same limit when it exists (Rebolledo, 1980, Prop. 1), assumption A2 implies that $\langle \frac{1}{\sqrt{n}} \tilde{U}^{(n)}(\beta_0) \rangle_\alpha \xrightarrow{P} \Sigma_\alpha(\beta_0)$. Lemma B.2 in turn verifies that $\frac{1}{\sqrt{n}} \tilde{U}^{(n)}(\beta_0)$ satisfies a Lindeberg condition necessary for the application of Rebolledo's (1980) Martingale Central Limit Theorem. Thus the process converges in distribution to a Gaussian process with covariance function $\Sigma_\alpha(\beta_0)$ as claimed.

Part *(b)*: Recalling $M_t(i) = N_t(i) - \Lambda_t(i)$, combine (5b) and (12) to obtain the relation

$$\sum_i \int_0^{t_{\lfloor \alpha n \rfloor}} V_s(\beta_0, i)\,dM_s(i) = I_{t_{\lfloor \alpha n \rfloor}}(\beta_0) - \big\langle \tilde{U}^{(n)}(\beta_0) \big\rangle_\alpha. \tag{14}$$

When $\alpha \in [0,1]$, a repeated application of the triangle inequality to

$$\left\| \tfrac{1}{n} I_{t_{\lfloor \alpha n \rfloor}}(\hat{\beta}_n) - \tfrac{1}{n}\big( I_{t_{\lfloor \alpha n \rfloor}}(\beta_0) - I_{t_{\lfloor \alpha n \rfloor}}(\beta_0)\big) - \Sigma_\alpha(\beta_0) \right\|$$

using the relation of (14) yields

$$\left\| \tfrac{1}{n} I_{t_{\lfloor \alpha n \rfloor}}(\hat{\beta}_n) - \Sigma_\alpha(\beta_0) \right\| \leq \left\| \frac{1}{n} \sum_i \int_0^{t_{\lfloor \alpha n \rfloor}} \{ V_s(\hat{\beta}_n, i) - V_s(\beta_0, i)\}\, dN_s(i) \right\| \\ + \left\| \frac{1}{n} \sum_i \int_0^{t_{\lfloor \alpha n \rfloor}} V_s(\beta_0, i)\, dM_s(i) \right\| + \left\| \frac{1}{n} \sum_i \int_0^{t_{\lfloor \alpha n \rfloor}} V_s(\beta_0, i)\, d\Lambda_s(i) - \Sigma_\alpha(\beta_0) \right\|.$$

We show that all three terms converge to zero in probability. The first term above is uniformly bounded by $\sup_{n',i} \|V_{t_{n'}}(\hat{\beta}_n, i) - V_{t_{n'}}(\beta_0, i)\|$, which converges to zero since $\hat{\beta}_n \xrightarrow{P} \beta_0$ by hypothesis of the theorem and $\{V_{t_{n'}}(\cdot, i)\}$ is an equicontinuous family by assumption A4. Lemma B.3 proves, as a consequence of assumption A3 and Lenglart's (1977) Inequality, that the second term converges to zero uniformly in $\alpha$. The third term converges to zero by assumption A2, thereby concluding the proof.

### *B.2. Supporting lemmas for Theorem 3.1*

LEMMA B.1. *Using the notation of Theorem 3.1, under assumption A1 the process* $\tilde{U}_\alpha^{(n)}(\beta_0)$ *from* (13) *is a square-integrable martingale adapted to* $\tilde{\mathcal{F}}_\alpha^{(n)} = \mathcal{F}_{t_{\lfloor \alpha n \rfloor}}$.

PROOF. The conditional expectation property holds provided $\mathbb{E}[U_{t_n}(\beta_0) \,|\, \mathcal{F}_{t_{n-1}}] = U_{t_{n-1}}(\beta_0)$. Define $K = \sup_{t,i,j} \|x_t(i,j)\|$. Note that $\|H_t(i,j)\| \leq 2K$. Thus,

$$\|U_{t \wedge t_n}(\beta_0)\| \leq 2K\big(N_{t\wedge t_n} + \Lambda_{t \wedge t_n}\big),$$
$$\mathbb{E}\left[\sup_t \|U_{t\wedge t_n}(\beta_0)\|^2\right] \leq 8 \cdot \big(\mathbb{E}K^2\big)^{1/2} \cdot \big(\mathbb{E}N_{t_n}^2 + \mathbb{E}\Lambda_{t_n}^2\big)^{1/2}.$$

By assumption A1, $\mathbb{E}K^2$ is finite, and by construction, $N_{t_n}$ is bounded. Since $N_{t\wedge t_n}$ is a counting process, $\mathbb{E}\Lambda_{t_n}^2$ is finite, too (this follows from results in Section 2.3 of Fleming and Harrington (1991)). Thus, $U_{t\wedge t_n}(\beta_0)$ is uniformly integrable. The Optional Sampling Theorem now applies to give the conditional expectation property of $\tilde{U}^{(n)}(\beta_0)$. For square integrability, note $\sup_{1\leq m \leq n} \mathbb{E}\|U_{t_m}\|^2 \leq \mathbb{E}\left[\sup_t \|U_{t\wedge t_n}(\beta_0)\|^2\right]$.

LEMMA B.2. *Using the notation of Theorem 3.1, under assumption A1, the Lindeberg condition for Rebolledo's (1980) Central Limit Theorem is satisfied: for any positive* $\varepsilon$,

$$\frac{1}{n} \sum_{i,j} \int_0^{t_n} \|H_s(i,j)\|^2\, 1\{\|H_s(i,j)\| > \sqrt{n}\varepsilon\}\, d\Lambda_s(i,j) \xrightarrow{P} 0.$$

PROOF. With $K = \sup_{t,i,j} \|x_t(i,j)\|$ as above, the integral is bounded by $4\,K^2\, 1\{n^{-1/2}K > \varepsilon/2\} \cdot \frac{\Lambda_{t_n}}{n}$. Since $\mathbb{E}K^2 < \infty$ by assumption A1, the first term converges to zero in probability. Since $\mathbb{E}\Lambda_{t_n} = \mathbb{E}N_{t_n} = n$, the product of the two also converges to zero in probability. Thus, the Lindeberg condition is satisfied.

LEMMA B.3. *Using the notation of Theorem 3.1, under assumptions A1 and A3 we have that* $\left\| \frac{1}{n} \sum_i \int_0^{t_{\lfloor \alpha n \rfloor}} V_s(\beta_0, i)\, dM_s(i) \right\| \xrightarrow{P} 0$ *uniformly in* $\alpha$.

PROOF. Lenglart's (1977) Inequality and assumption A3 imply that for any positive $\rho$ and $\delta$,

$$\mathbb{P}\Big\{ \sup_{t \in [0,t_n]} \Big\| \frac{1}{n} \sum_i \int_0^t V_s(\beta_0, i)\, dM_s(i) \Big\| \geq \rho \Big\} \leq \frac{\delta}{\rho^2} + \mathbb{P}\Big\{ \frac{1}{n^2} \sum_i \int_0^{t_n} \|V_s(\beta_0, i)\|^2 \, d\Lambda_s(i) \geq \delta \Big\}.$$

(see Fleming and Harrington (1991, Cor. 3.4.1) for a related proof). As in the proof of Lemma B.1, set $K = \sup_{t,i,j} \|x_t(i,j)\|$. The sum is bounded by $\frac{16K^4}{n} \cdot \frac{\Lambda_{t_n}}{n}$. Since $n^{-1/2} K^2 \xrightarrow{P} 0$ by assumption A1 and $\mathbb{E}\Lambda_{t_n} = n$, the right-hand side of the inequality converges to $\frac{\delta}{\rho^2}$. Since $\delta$ is arbitrary, the right-hand side must converge to zero.

## *B.3. Proof of Theorem 3.2*

We follow Haberman's (1977) approach to proving consistency, which relies on Kantorovich's (1948) analysis of Newton's method. Tapia (1971) gives an elementary proof of the Kantorovich Theorem. We state a weak form of the result as a lemma.

LEMMA B.4 (KANTOROVICH THEOREM). *Let $P(x) = 0$ be a general system of nonlinear equations, where $P$ is a map between two Banach spaces. Let $P'(x)$ denote the Jacobian (Fréchet differential) of $P$ at $x$, assumed to exist in $D_0$, a convex open neighborhood of $x_0$. Assume that*

*(a)* $\|[P'(x_0)]^{-1}\| \leq B$,

*(b)* $\|[P'(x_0)]^{-1} P(x_0)\| \leq \eta$,

*(c)* $\|P'(x) - P'(y)\| \leq K\|x - y\|$, *for all $x$ and $y$ in $D_0$,*

*with $h = BK\eta \leq \frac{1}{2}$.*

*Let $\Omega_* = \{x : \|x - x_0\| \leq 2\eta\}$. If $\Omega_* \subset D_0$, then the Newton iterates, $x_{k+1} = x_k - [P'(x_k)]^{-1} P(x_k)$, are well defined, remain in $\Omega_*$, and converge to $x^*$ in $\Omega_*$ such that $P(x^*) = 0$. In addition,*

$$\|x^* - x_k\| \leq \frac{\eta}{h} \frac{(2h)^{2^k}}{2^k}, \qquad k = 0, 1, 2, \ldots.$$

PROOF (THEOREM 3.2). Set $U_t(\cdot)$ and $I_t(\cdot)$ to be the gradient and negative Hessian of the log partial likelihood, as defined in (5a–5b). Since $I_t(\beta)$ is a sum of rank-one matrices with positive weights, it is positive semi-definite, and $\log PL_t(\cdot)$ is a concave function. By the assumption that the smallest eigenvalue of $\Sigma_1(\cdot)$ is bounded away from zero in a neighborhood of $\beta_0$, for $n$ sufficiently large, if $\log PL_t(\cdot)$ has a local maximum in that neighborhood then it must be the unique global maximum.

We find the local maximum by applying Newton's method to the gradient of $\frac{1}{n} \log PL_{t_n}(\cdot)$, taking $\beta_0$ as the initial iterate. Define $Z_n = -[\frac{1}{n} I_{t_n}(\beta_0)]^{-1} [\frac{1}{n} U_{t_n}(\beta_0)]$. The first Newton iterate, $\beta_{n,1}$, is equal to $\beta_0 - Z_n$. Part (b) of Theorem 3.1 and the assumptions of the theorem imply $[\frac{1}{n} I_{t_n}(\beta_0)]^{-1}$ exists for $n$ large enough, so that $Z_n$ is well-defined. Moreover, Part (a) of Theorem 3.1 and Slutsky's Theorem imply $Z_n \xrightarrow{P} 0$ and $\sqrt{n}\, Z_n \xrightarrow{d} \mathcal{N}(0, [\Sigma_1(\beta_0)]^{-1})$.

Now we may apply Kantorovich's Theorem to bound $\|\hat{\beta}_n - \beta_0\|$ and $\|\hat{\beta}_n - \beta_{n,1}\|$ as follows. By assumption, there exists a neighborhood of $\beta_0$, say $D_0$, and finite $K$ and $B$, such that $\|\frac{1}{n} I_{t_n}(\beta) - \frac{1}{n} I_{t_n}(\beta')\| \leq K\|\beta - \beta'\|$ and $\|\frac{1}{n}[I_{t_n}(\beta_0)]^{-1}\| \leq B$ for $\beta, \beta' \in D_0$. Define $\eta_n = \|Z_n\|$ and $h_n = BK\eta_n$, noting that $h_n$ and $\eta_n$ are size $\mathcal{O}_{\mathrm{P}}(n^{-1/2})$. Thus, for $n$ large enough,

(a) $\|\hat{\beta}_n - \beta_0\| \leq 2\, \eta_n \xrightarrow{P} 0$,

(b) $\sqrt{n}\, \|\hat{\beta}_n - (\beta_0 - Z_n)\| \leq 2\sqrt{n}\, \eta_n\, h_n \xrightarrow{P} 0$.

Thus, $\hat{\beta}_n \xrightarrow{P} \beta_0$, and $\sqrt{n}(\hat{\beta}_n - \beta_0)$ and $\sqrt{n}\, Z_n$ converge weakly to the same limit.

## C. Results from Section 4

### C.1. Proof of Theorem 4.1

Proof (Theorem 4.1). When $J \subseteq \mathcal{J}_t(i)$, set $X_t(i, J) = \sum_{j \in J} x_t(i, j)$ and $w_t(\beta, i, J) = \exp\{\beta^{\mathrm{T}} X_t(i, J)\}$. As a slight abuse of notation, when $j$ is an element of $\mathcal{J}_t(i)$, take "$w_t(\beta, i, j)$" to mean $w_t(\beta, i, \{j\})$. Define weights

$$W_t(\beta, i; L) = \sum_{\substack{J \subseteq \mathcal{J}_t(i), \\ |J| = L}} w_t(\beta, i, J), \qquad \widetilde{W}_t(\beta, i; L) = \Big[ \sum_{j \in \mathcal{J}_t(i)} w_t(\beta, i, j) \Big]^L,$$

and note that the approximation error in $\log \widetilde{PL}_t(\beta)$ comes from replacing $W$ with $\widetilde{W}$.

The gradients of the weights are

$$E_t(\beta, i; L) = \nabla \big[ \log W_t(\beta, i; L) \big] = \frac{1}{W_t(\beta, i; L)} \sum_{\substack{J \subseteq \mathcal{J}_t(i), \\ |J| = L}} w_t(i, J)\, X_t(i, J),$$

$$\widetilde{E}_t(\beta, i; L) = \nabla \big[ \log \widetilde{W}_t(\beta, i; L) \big] = L \cdot \frac{\sum_{j \in \mathcal{J}_t(i)} w_t(\beta, i, j)\, x_t(i, j)}{\sum_{j \in \mathcal{J}_t(i)} w_t(\beta, i, j)}.$$

The second is the expectation of $\sum_{l=1}^{L} x_t(i, j_l)$ when $j_1, \dots, j_L$ are drawn independently and identically from $\mathcal{J}_t(i)$ with weights $w_t(\beta, i, \cdot)$; the first is the same expectation, conditional on the event that $j_1, \dots, j_L$ are all unique. Let $\tilde{\mathbb{P}}_{t,\beta,i;L}$ and $\mathbb{P}_{t,\beta,i;L}$ denote the two probability laws for $j_1, \dots, j_L$, and let $\tilde{\mathbb{E}}_{t,\beta,i;L}$ and $\mathbb{E}_{t,\beta,i;L}$ denote expectations with respect to them, so that $E_t(\beta, i; L) = \mathbb{E}_{t,\beta,i;L}\big[\sum_{l=1}^{L} x_t(i, j_l)\big]$ and $\widetilde{E}_t(\beta, i; L) = \tilde{\mathbb{E}}_{t,\beta,i;L}\big[\sum_{l=1}^{L} x_t(i, j_l)\big]$.

The bound on $\nabla[\log PL_{t_n}(\beta)] - \nabla[\log \widetilde{PL}_{t_n}(\beta)]$ derives from a bound on $E_t(\beta, i; L) - \widetilde{E}_t(\beta, i; L)$. Write

$$E_t(\beta, i; L) - \widetilde{E}_t(\beta, i; L) = \mathbb{E}_{t,\beta,i;L}\Big[\sum_{l=1}^{L} x_t(i, j_l)\Big] - \tilde{\mathbb{E}}_{t,\beta,i;L}\Big[\sum_{l=1}^{L} x_t(i, j_l)\Big].$$

We define probability law $\mathbb{P}^*_{t,\beta,i;L}$ and associated random variables $j_1, \dots, j_L$ and $\tilde{\jmath}_1, \dots, \tilde{\jmath}_L$, such that marginally $j_1, \dots, j_L$ are distributed according to $\mathbb{P}_{t,\beta,i;L}$ and $\tilde{\jmath}_1, \dots, \tilde{\jmath}_L$ are distributed according to $\tilde{\mathbb{P}}_{t,\beta,i;L}$, but the variables are coupled to have nontrivial chance of agreeing. Then,

$$\begin{aligned} \Big\| E_t(\beta, i; L) - \widetilde{E}_t(\beta, i; L) \Big\| &= \Big\| \mathbb{E}^*_{t,\beta,i;L}\Big[ \sum_{l=1}^{L} x_t(i, j_l) - \sum_{l=1}^{L} x_t(i, \tilde{\jmath}_l) \Big] \Big\| \\ &\leq 2L \cdot \Big[ \sup_{j \in \mathcal{J}_t(i)} \|x_t(i, j)\| \Big] \cdot \mathbb{P}^*_{t,\beta,i;L}\Big\{ (j_1, \dots, j_L) \neq (\tilde{\jmath}_1, \dots, \tilde{\jmath}_L) \Big\} \end{aligned}$$

The coupling is as follows:

(a) Draw $(\tilde{\jmath}_1, \dots, \tilde{\jmath}_L)$ according to $\tilde{\mathbb{P}}_{t,\beta,i;L}$.

(b) If $(\tilde{\jmath}_1, \dots, \tilde{\jmath}_L)$ are all unique, set $(j_1, \dots, j_L) = (\tilde{\jmath}_1, \dots, \tilde{\jmath}_L)$, otherwise draw $(j_1, \dots, j_L)$ independently according to $\mathbb{P}_{t,\beta,i;L}$.

With $K = \sup_{j \in \mathcal{J}_t(i)} \|x_t(i, j)\|$, Lemma C.1 shows

$$\mathbb{P}^*_{t,\beta,i;L}\Big\{ (j_1, \dots, j_L) \neq (\tilde{\jmath}_1, \dots, \tilde{\jmath}_L) \Big\} \leq \binom{L}{2} \cdot \frac{\exp\{4K\,\|\beta\|\}}{|\mathcal{J}_t(i)|}.$$

The resulting bound on $\|\nabla[\log PL_t(\beta)] - \nabla[\log \widetilde{PL}_t(\beta)]\|$ now follows by expressing

$$\nabla\big[\log \widetilde{PL}_t(\beta)\big] - \nabla\big[\log PL_t(\beta)\big] = \sum_{t_m \leq t} E_{t_m}(\beta, i_m; |J_m|) - \widetilde{E}_{t_m}(\beta, i_m; |J_m|).$$

Using $\big\|E_t(\beta, i; L) - \widetilde{E}_t(\beta, i; L)\big\| \leq K\, L^2\,(L-1)\, \frac{\exp\{4K\,\|\beta\|\}}{|\mathcal{J}_t(i)|}$, we get

$$\Big\|\nabla\big[\log \widetilde{PL}_t(\beta)\big] - \nabla\big[\log PL_t(\beta)\big]\Big\| \leq K \exp\{4K\|\beta\|\} \cdot \sum_{t_m \leq t} \frac{|J_m|^2(|J_m|-1)}{|\mathcal{J}_{t_m}(i_m)|}.$$

We get the final bound for the gradients by replacing the numerators of the summands with $\sup_m |J_m|$.

Using the same methods, Lemma C.2 derives the bound on the difference in Hessians.

## *C.2. Supporting lemmas for Theorem 4.1*

LEMMA C.1. *Using the notation and assumptions of Theorem 4.1,*

$$\mathbb{P}^*_{t,\beta,i;L}\Big\{(j_1, \ldots, j_L) \neq (\tilde{\jmath}_1, \ldots, \tilde{\jmath}_L)\Big\} \leq \binom{L}{2} \cdot \frac{\exp\{4K\,\|\beta\|\}}{|\mathcal{J}_t(i)|},$$

*where* $K = \sup_t \|x_t(i,j)\|$.

PROOF. The left hand side is bounded by the probability that the samples $\tilde{\jmath}_1, \ldots, \tilde{\jmath}_L$ are all unique, which can be bounded by

$$\sum_{k<l} \mathbb{P}^*_{t,\beta,i;L}\{\tilde{\jmath}_k = \tilde{\jmath}_l\} = \binom{L}{2} \sum_{j \in \mathcal{J}_t(i)} \Big[\frac{w_t(\beta,i,j)}{\sum_{j' \in \mathcal{J}_t(i)} w_t(\beta,i,j')}\Big]^2.$$

Note $\exp\{-K\,\|\beta\|\} \leq w_t(\beta,i,j) \leq \exp\{K\|\,\beta\|\}$, so that

$$\sum_{j \in \mathcal{J}_t(i)} \Big[\frac{w_t(\beta,i,j)}{\sum_{j' \in \mathcal{J}_t(i)} w_t(\beta,i,j')}\Big]^2 \leq \frac{\exp\{4K\,\|\beta\|\}}{|\mathcal{J}_t(i)|}.$$

LEMMA C.2. *Using the notation and assumptions of Theorem 4.1,*

$$\Big\|\nabla^2\big[\log \widetilde{PL}_t(\beta)\big] - \nabla^2\big[\log PL_t(\beta)\big]\Big\| \leq 2K^2 \exp\{4K\|\beta\|\} \cdot \sum_{t_m \leq t} \frac{|J_m|^3\,(|J_m|-1)}{|\mathcal{J}_{t_m}(i_m)|}.$$

PROOF. The argument is similar to the bound on the difference in gradients in the proof of Theorem 4.1. The Hessians of the weights are

$$V_t(\beta, i; L) = \nabla^2\big[\log W_t(\beta, i; L)\big] = \frac{1}{W_t(\beta, i; L)} \sum_{\substack{J \subseteq \mathcal{J}_t(i), \\ |J| = L}} w_t(\beta, i, J)\Big[X_t(i, J) - E_t(\beta, i; L)\Big]^{\otimes 2},$$

$$\widetilde{V}_t(\beta, i; L) = \nabla^2\big[\log \widetilde{W}_t(\beta, i; L)\big] = L \cdot \frac{\sum_{j \in \mathcal{J}_t(i)} w_t(\beta,i,j)\Big[x_t(i,j) - \frac{1}{L}\widetilde{E}_t(\beta,i;L)\Big]^{\otimes 2}}{\sum_{j \in \mathcal{J}_t(i)} w_t(\beta,i,j)}.$$

The first is the covariance matrix of $\sum_{l=1}^{L} x_t(i, j_l)$ under $\mathbb{P}_{t,\beta,i;L}$; the second is the covariance matrix of the same quantity under $\widetilde{\mathbb{P}}_{t,\beta,i;L}$. The result follows in the same manner as in the proof of Theorem 4.1. The relevant intermediate bound is

$$\Big\|V_t(\beta, i; L) - \widetilde{V}_t(\beta, i; L)\Big\| \leq 2\,K^2\,L^3\,(L-1)\,\frac{\exp\{4K\,\|\beta\|\}}{|\mathcal{J}_t(i)|}.$$

### C.3. Proof of Theorem 4.2

Proof (Theorem 4.2). We know that Newton's method applied to $\frac{1}{n}\log\widetilde{PL}_{t_n}(\cdot)$ converges to $\tilde{\beta}_n$ after sufficiently many iterations. We employ $\hat{\beta}_n$ as the initial iterate and use the Kantorovich Theorem (Lemma B.4) to bound $\|\tilde{\beta}_n - \hat{\beta}_n\|$.

In the notation of the lemma, $P(\cdot)$ is the gradient of $\frac{1}{n}\log\widetilde{PL}_{t_n}(\cdot)$ and $P'(\cdot)$ is its Hessian. The conditions of Theorem 4.2 imply assumptions (a) and (c) hold uniformly in $n$ for some finite $B$ and $K$. Set

$$\eta_n = \Big\| \Big[\nabla^2\big[\tfrac{1}{n}\log\widetilde{PL}_{t_n}(\hat{\beta}_n)\big]\Big]^{-1}\Big[\nabla\big[\tfrac{1}{n}\log\widetilde{PL}_{t_n}(\hat{\beta}_n)\big]\Big]\Big\|$$

and set $h_n = BK\eta_n$. Since $\nabla\big[\log PL_{t_n}(\hat{\beta}_n)\big] = 0$, Theorem 4.1 and the boundedness of the inverse Hessian imply $\eta_n = \mathcal{O}_{\mathrm{P}}(G_n/n)$. Therefore, for $n$ large enough,

$$\|\tilde{\beta}_n - \hat{\beta}_n\| \leq \frac{\eta_n}{h}\frac{(2h)^{2^0}}{2^0} = 2\eta_n = \mathcal{O}_{\mathrm{P}}(G_n/n).$$

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# Point process modeling for directed interaction networks: Supplementary material

Patrick O. Perry
*Stern School of Business, New York University, USA*
Patrick J. Wolfe
*Department of Statistical Science, University College London, UK*

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## 1. A comparative analysis based on contingency tables

Were we interested only in homophily, we might be tempted to forgo the proportional intensity model of (1) from the main text, and instead perform a contingency table analysis. However, as we now describe, such an analysis leads to very different conclusions about the predictive strength of homophily in our data.

For example, suppose that we are interested in testing for seniority-based homophily. Ignoring network effects and other dependency, we might model $\mathbb{P}\{i \to j \mid i\}$, the probability of employee $j$ being the recipient of a message given that employee $i$ is the sender, by way of a multinomial logit model:

$$\mathbb{P}\{i \to j \mid i\} \propto \exp\{\beta_J J(j) + \beta_{JJ} J(i) J(j)\}.$$

In this setting, Junior-Junior homophily would manifest in a positive value of $\beta_J + \beta_{JJ}$ and Senior-Senior homophily would manifest in a negative value of $\beta_J$.

Since there are $n_J = 82$ Junior executives and $n_S = 74$ Senior executives, and since the sender and receiver of a message are distinct, we have that

$$\begin{aligned}\mathbb{P}\{i \to j \mid i, J(i) = 1\} &= \frac{e^{(\beta_J + \beta_{JJ}) J(j)}}{(n_J - 1) e^{\beta_J + \beta_{JJ}} + n_S} \\ \mathbb{P}\{i \to j \mid i, J(i) = 0\}, &= \frac{e^{\beta_J J(j)}}{n_J e^{\beta_J} + (n_S - 1)}.\end{aligned}$$

In turn, we compute the corresponding maximum likelihood coefficient estimates using the entries of a 2 × 2 table that counts the number of messages exchanged between each group:

| | Receiver | |
|---|---|---|
| Sender | Junior | Senior |
| Junior | 7972 | 5833 |
| Senior | 3977 | 14479 |

The resultant estimates are $\hat{\beta}_J = -1.4$ and $\hat{\beta}_{JJ} = 1.6$, with (Wald) standard errors of about 0.02 for each. Indeed, these are exactly the estimated coefficients we would obtain if we were to use

*Address for correspondence:* Patrick O. Perry, Information, Operations, and Management Sciences Department, Stern School of Business, New York University, 44 West 4th St, New York, NY 10012, USA
E-mail: pperry@stern.nyu.edu

| | Receiver | | | |
|---|---|---|---|---|
| Sender | L | T | J | F |
| 1 | -1.48<br>(0.04) | -1.74<br>(0.03) | -1.83<br>(0.03) | -0.25<br>(0.03) |
| L | 3.70<br>(0.04) | 0.48<br>(0.05) | 0.23<br>(0.03) | -0.22<br>(0.03) |
| T | 1.06<br>(0.06) | 1.92<br>(0.04) | 1.11<br>(0.04) | -0.21<br>(0.04) |
| J | -0.12<br>(0.04) | 1.36<br>(0.04) | 1.70<br>(0.03) | 0.25<br>(0.03) |
| F | 0.87<br>(0.04) | -0.58<br>(0.04) | 0.07<br>(0.03) | 0.83<br>(0.03) |

**Fig. 1.** Estimated coefficients and standard errors for the contingency table-based analysis of Section 1; dark coefficients are significant (via Wald test) at level $10^{-3}$.

the proportional intensity model $\lambda_t(i,j) = \bar{\lambda}_t(i) \exp\{\beta_J J(j) + \beta_{JJ} J(i) J(j)\}$, a result that holds more generally for non-time-varying covariates.

One problem with this analysis is that we have marginalized over the other covariates (Gender and Department), potentially introducing a Simpson's paradox. This issue is easily rectified, however, by introducing covariates for senders and receivers; Fig. 1 above shows the resulting coefficient estimates.

The far more important problem is that this analysis implicitly assumes each message to be independent and identically distributed, conditional on the sender of the message. This assumption is blatantly false: common sense tells us that if Junior $A$ sends a message to Junior $B$, then the next time $B$ sends a message, $B$ is more likely to choose $A$ as a recipient. Any homophily effect present in these interaction data is thus likely to be exaggerated by reciprocation and other network effects. Indeed, comparing the contingency table-based estimates in Fig. 1 above with the estimates from the proportional intensity model with network effects in Fig. 7 from the main text, we can see that the coefficient estimates are much higher when we don't adjust for network effects. Thus even in cases where network effects themselves are not the object of primary interest, it is important to account for them when making inferences about the predictive strength of other covariates.

## 2. Comparative analyses using actor-oriented and exponential random graph models

A number of dynamic network models exist in the literature, including Hanneke, Fu, and Xing's (2010) exponential random graph model with time-varying coefficients, and Kolar, Song, Ahmed, and Xing's (2010) time-varying stochastic block model. Alternative approaches explicitly based on point processes, but excluding network effects and other covariate information, include that of Malmgren et al. (2009), who model activity at the level of the individual using a hidden Markov model, and of Heard et al. (2010), who work at the level of the dyad, assuming a piecewise-constant interaction rate. The closest match to our approach is given by the actor-oriented model of Snijders (2001, 2005), which we now detail in Section 2.1 below. Then, in Section 2.2, we provide a comparison to a static network analysis based on an exponential random graph model.

### *2.1. Actor-oriented model analysis*

The actor-oriented model is designed for a sequence of snapshots $G_1$, $G_2, \ldots$, of network activity, where each $G_t$ is an $I \times J$ binary matrix representing pairwise connectivity between actors at time $t$. The model is best suited for ties that persist in time, not instantaneous events; it treats the sequence of networks as a first-order Markov chain, with the distribution of $G_t$ determined by $G_{t-1}$. Actors are assumed to change their ties between times $t-1$ and $t$ to maximize a stochastic utility function that depends on characteristics of the overall network. Essentially, given that the network is in state $G$, and that actor $i$ is allowed to make a change, he will change his link to actor $j$ according to

$$p(j|i, G) \propto \exp\Big\{ \sum_{s=1}^{S} \beta_s T_s\big(G(i \rightsquigarrow j)\big) + \sum_{s=1}^{S'} \beta'_s T'_s(G \setminus \{i \to j\})\Big\},$$

where $T_s$ and $T'_s$ are network statistics; $\beta_s$ and $\beta'_s$ are unknown coefficients; $G(i \rightsquigarrow j)$ denotes the network obtained either by adding link $i \to j$ if it is absent or removing link $i \to j$ if it is present; $G \setminus \{i \to j\}$ denotes the network obtained by removing $i \to j$ if it is present. Additionally, the rate at which actor $i$ changes ties between observation times $t-1$ and $t$ is given by $\lambda(i)$, specified via another parametric model. (See Snijders et al. (2010) for a more thorough introduction.) The change probability function $p(j|i, G)$ plays a similar role to the multiplier $\exp\{x_t(i, j)^{\mathrm{T}}\beta\}$ in the proportional intensity model from the main text, and the change rate function $\lambda(i)$ plays a similar role to the baseline intensity $\bar{\lambda}_t(i)$.

For purposes of comparison, we specified a change probability model $p(j|i, G)$ with network statistics analogous to those used in Section 5.2 from the main text, and then we estimated coefficient sets $\beta$ and $\beta'$ analogous to the coefficients in the proportional intensity model. We used the `RSiena` package (Ripley and Snijders, 2011) to specify and fit the actor-oriented model after binning the Enron e-mail interaction data at regular intervals to obtain network snapshots $G_1$, $G_2$, and $G_3$. Here, $G_t(i, j) = 1$ if message $i \to j$ was observed in period $t$, and $G_t(i, j) = 0$ otherwise; periods 1–3 correspond to consecutive four-month periods in the year 2001. The subset we looked at contains 60% of the messages in the Enron corpus. We chose this particular subset and temporal resolution partially for computational reasons, but also to make the network change statistics (Jaccard coefficients) within the range recommended by `RSiena` (near 0.3). Approximately 2 hours' time was required to fit the model.

We included the following terms, chosen to mimic the covariates detailed in Section 5.2 from the main text:

(a) Outdegree/density (`out`). This statistic counts the number of outgoing ties; it is analogous to our $\bar{\lambda}_t(i)$, except that the rate is the same for each sender.

(b) Group-level edge effects (`traits`). One covariate is included for each identifiable first-order interaction of the form $X(i)Y(j)$, where $i$ is the sender and $j$ is the receiver; the covariate counts the number of edges $i \to j$ with $X(i)Y(j) = 1$. These effects correspond to the group-level effects in our model.

(c) Outdegree/density endowment (`outendow`). This statistic counts the number of deleted outgoing ties; it corresponds to the negative of the **send** term in our model.

(d) Reciprocity (`recip`). This statistic counts the number of reciprocal ties; i.e., edge sets of the form $\{i \to j, j \to i\}$. It corresponds to the **receive** term in our model.

(e) 3-cycles (`3cycle`). This statistic counts the number of cyclic triples; i.e., edge sets of the form $\{h \to i, i \to j, j \to h\}$. It corresponds to the **2-receive** term in our model.

| | Receiver | | | |
|---|---|---|---|---|
| Sender | L | T | J | F |
| 1 | -0.65 | -0.10 | 0.13 | 0.21 |
| | (0.12) | (0.07) | (0.07) | (0.09) |
| L | 0.62 | -0.28 | -0.11 | -0.16 |
| | (0.13) | (0.10) | (0.09) | (0.11) |
| T | 0.46 | 0.45 | -0.15 | -0.46 |
| | (0.16) | (0.06) | (0.07) | (0.10) |
| J | 0.00 | 0.15 | 0.10 | -0.21 |
| | (0.11) | (0.06) | (0.06) | (0.09) |
| F | 0.19 | 0.33 | 0.08 | 0.13 |
| | (0.12) | (0.06) | (0.07) | (0.08) |

(a) Trait effects (`traits`)

| Variate | `out` | `outendow` | `recip` | `3cycle` | `ttriple` |
|---|---|---|---|---|---|
| Coefficient | -1.94 | -0.94 | 2.02 | -0.26 | 0.30 |
| (SE) | (0.02) | (0.01) | (0.06) | (0.03) | (0.01) |

(b) Network effects

**Fig. 2.** Estimated effects for the actor-oriented model of Section 2.1

(f) Transitive triplets (`ttriple`). This statistic counts the number of transitive triples; i.e., edge sets of the form $\{h \to i, i \to h, h \to j\}$. It corresponds to the **2-send**, **sibling**, and **cosibling** terms in our model.

Note that after binning the interaction counts to form network snapshots as required by the actor-oriented model, it is impossible to separate the **2-send**, **sibling**, and **cosibling** effects. Further, the first-order Markov nature of the model restricts our ability to quantify the time decay of the dynamic network effects.

Figure 2 above shows the fitted coefficients for the actor-oriented model. We can see that the estimated network effects agree qualitatively with those in Fig. 8 from the main text, as outdegree/density endowment has a negative coefficient while reciprocity and transitive triplets have positive coefficients. Further, the dyadic coefficients are larger than the triadic coefficients. A discrepancy between the two models is that **2-receive** had a negligible effect in the proportional intensity model, while its analogue (`3cycle`) had a small negative effect in the actor-oriented model. One possible explanation for this discrepancy is that treating all ties as binary forces a negative bias in otherwise-unimportant network effects. With the limitation that ties are binary, when actor $i$ tries to maximize his stochastic utility, he is forbidden from reinforcing an existing $i \to j$ link; he is more likely to link to an actor $j'$ for which link $i \to j'$ is absent. To counteract this tendency, the coefficient of `3cycle` is forced to be negative.

### *2.2. Exponential random graph model analysis*

As a final comparison, we fit an exponential random graph model to our data. This class of models—one of the more popular for estimating the importance of network effects—specifies a probability distribution for a single directed graph represented by a binary matrix $G$. It supposes that $\mathbb{P}\{G = g\} \propto \exp\{\sum_{s=1}^{S} T_s(g)\}$, where $T_s(g)$ is a network statistic, for example the number of transitive triples in the graph. (See Anderson et al. (1999) for a detailed survey.)

To apply this form of model, we employed a reduction of our data to obtain a single directed graph $G$ as follows. Based on an "elbow" in the empirical cumulative distribution function of message counts $N_\infty(i, j)$ in our data, we chose a threshold of 10 sent messages and defined $G$ by

$$G(i, j) = 1\{N_\infty(i, j) \geq 10\}.$$

Next, as in our comparison to the actor-oriented model, we chose terms in the model to mimic the covariates from Section 5.2 of the main text. We used the `ergm` software package to fit the model (Handcock et al., 2011), based on a Markov chain Monte Carlo sample size of $10^5$ following a burn-in period of $10^6$ iterates. The covariates were as follows:

(a) Sender effects (`sender`). One covariate is included for each sender, measuring the outdegree of the sender. The corresponding coefficient plays the role of $\bar{\lambda}_t(i)$ in our model.

(b) Group-level edge effects (`edgecov`). One covariate is included for each identifiable first-order interaction of the form $X(i)Y(j)$, where $i$ is the sender and $j$ is the receiver; the covariate counts the number of edges $i \to j$ with $X(i)Y(j) = 1$. These effects correspond to the group-level effects in our model. We attempted to include second-order interactions as well, but were unable (for computational reasons) to fit the model with these terms.

(c) Mutuality (`mutual`). This statistic counts the number of mutual ties; i.e., edge sets of the form $\{i \to j, j \to i\}$, and corresponds to the **receive** term in our model.

(d) Cyclic triples (`ctriple`). This statistic counts the number of cyclic triples; i.e., edge sets of the form $\{h \to i, i \to j, j \to h\}$, and corresponds to the **2-receive** term in our model.

(e) Transitive triples (`ttriple`). This statistic counts the number of transitive triples; i.e., edge sets of the form $\{h \to i, i \to j, h \to j\}$. It corresponds to the **2-send**, **sibling**, and **cosibling** terms in our model.

Note that reducing the interaction data to a single directed graph has important modeling consequences. As with the case of the snapshot-based actor-oriented model detailed above, it is impossible to separate the **2-send**, **sibling**, and **cosibling** effects, and the inability to include second-order interactions between group-level effects precludes a direct comparison with the estimated group-level effects from the proportional intensity model. Further, for a single directed graph, there is no possibility of including a term corresponding to **send**, and it is impossible to quantify the time-dependence of the network effects.

Figure 3 overleaf shows the fitted coefficients for the exponential random graph model. As with the case of the actor-oriented model considered in Section 2.1 above, the estimated network effects agree qualitatively with those of Fig. 8 from the main text. Specifically, mutuality and transitive triples have positive effects, while cyclic triples have a negligible effect (in contrast to the case of Fig. 2 from Section 2.1 above).

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| | Receiver | | | |
|---|---|---|---|---|
| Sender | L | T | J | F |
| 1 | -2.61 | -1.49 | -0.87 | -0.55 |
| L | 2.60 | 0.77 | -0.09 | 0.12 |
| T | 0.63 | 0.69 | -0.60 | -0.32 |
| J | 0.13 | 1.04 | 1.15 | 0.27 |
| F | 0.25 | 0.49 | 0.49 | 0.79 |

(a) Trait effects

| Variate | `mutual` | `ctriple` | `ttriple` |
|---|---|---|---|
| Coefficient | 4.49 | 0.15 | 0.42 |
| (SE) | (0.003) | (0.290) | (0.060) |

(b) Network effects

**Fig. 3.** Estimated coefficients for the exponential random graph model of Section 2.2. The model also included a term for each sender (not shown); furthermore, standard errors returned for the group-level edge effects were nonsensical, and so are not reported.

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