Logic of Closeness Revision

Challenging relations in social networks*

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Abstract. In social epistemology and dynamic epistemic logic (DEL), the study of belief revision and opinion dynamics in social networks has recently gained increasing attention. Our social contacts affect the way we form our opinions about the world. However, in many real life situations we can also observe the dual effect: people's opinions may also play a role in the evolution of the network's structure. In this paper, we present a complete logic that models the dynamical changes in the agents' network relations with respect to opinion exchange. We make use of a 2×2 coordination game, the "discussion game". We first focus on the simplified cases where issues are equally weighted, agents never change opinion on them, and just modify their network relations accordingly. Next, we introduce different weights on issues in order to express agents' priorities. Finally, we discuss an extension of our model that can capture more refined schemata of human interaction.

1 Introduction

After Claire met Frank, she found that this young man shared similar opinions and attitudes towards most things with her. Love therefore grew between the two. Sometimes they had quarrels, and big opinion differences almost led them to breaking up, but the same goal of achieving power always united them and gave them strength. This is the story of House of Cards in three sentences. In this paper, we will develop logical tools based on a game-theoretical framework in order to answer the following question: *how* does opinion exchange affect our closeness with our social contacts?

According to the standard approach of belief revision in DEL, agents are continuously under the influence of their network-neighbors and modify their opinions in the view of the social norm. However, as far as empirical evidence is concerned, these attempts seem defective. Consider a "stubborn" agent: she stands on firm to her opinions; once she interacts with someone, she reshapes her relationship with him. Naturally, agreement is viewed as a positive boost for a

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relationship, while conflict is a burden. Belief revision, as presented for example in Baltag et al. 2015, does not account for similar scenarios. It is precisely this gap that this paper wishes to bridge.

We should emphasize that what follows is not a marginal, case-study of some peculiar agents. For instance, any agent can be treated as a stubborn agent at a certain context, in the sense that some issues may trigger non-negotiable opinions which cannot be subject of change merely due to the diffusion of a fashion in a network. Agreement and disagreement among agents in a network can characterize the network's structure and evolution. In Section 2, a coordination game is introduced, in order to capture opinion differences between a pair of agents, on a given issue. The model of closeness revision and its update are grounded on this "discussion game". The fact that different issues attract more attention than others, depending on the agents that interact, is also incorporated in the framework of this paper. In Section 3, we present a complete dynamic logic, which combines probabilistic and qualitative logics, in order to reason about the dynamics in a network. To conclude, realistic human interactions prescribe that agents behave stubbornly or not, depending on their social environment; this variation of the model is discussed in Section 4.

2 The Model of Closeness Revision

Our setting consists of finite networks of agents and *closeness* relations among them. Closeness relations are weighted (ranging from -1, reflecting hate, to 1, reflecting total closeness) and not necessarily symmetric.[1] Next, we assume that there is a countable set of issues according to which agents re-evaluate their relations in the network. The motivating idea reads as follows: although two agents may already know each other's opinion before discussing an issue, the discussion on it can be iterated, and their relation may be re-evaluated. For instance, I may know that we disagree on the political proposals of the Republican Party of the US, but re-asserting our divergent opinions makes me upset and subsequently decreases our closeness. We define a mathematical model which represents the revision of the closeness relation according to the *measure of agreement* between agents, and include it into a logical system. The definition of *the model of closeness revision* will serve as the building block for all the illustrations that will follow.

Definition 1. *A model of closeness revision is a tuple*

$$M = \langle A, C, I, O \rangle$$

where A is a set of agents, $C : A \times A \to [-1, 1]$, with $C(a, b)$ interpreted as the closeness relation between agents a and b, I is a set of issues, and $O : A \times I \to [0, 1]$, with $O(a, i)$ interpreted as the opinion of agent a on issue i.

[1] We drop symmetry, even if it is very commonly used in other works on the topic (Liu, Seligman, and Girard 2014, Baltag et al. 2015, Christoff, Hansen, and Proietti 2014), as not-necessarily symmetric relations seem to better capture cases where feelings between agents are not mutual.

Note that we allow $C(a,a)$, which can be read as "closeness to oneself", to take any value in $[-1,1]$. We will further assume that closeness to oneself is unaltered, not affected by any issue-induced revision. The latter assumption reflects the idea of one's stable relationship to herself, and modelling different personality types will not concern us for the purposes of this paper.
We denote **C** the class of models of closeness revision.

2.1 The Discussion Game

In the model of closeness revision, any agent a holds an opinion towards any given issue i, denoted as $O(a,i)$. When a pair of agents a, b discusses about the issue i, what is of our interest is the degree on which $O(a,i)$ and $O(b,i)$ deviate from each other. Assume for simplicity that every agent announces her opinion truthfully during the discussion. Opinions' divergence transforms agents' closeness afterwards.

Discussion entails interaction between agents.[2] The satisfaction that two agents a and b gain from a discussion on the issue i progresses with respect to their agreement on i. Thus, we can view the discussion on i as a coordination game, and interpret agents' pure opinions towards the issue i as their strategies in the *discussion game*. Let AG and DG be the pure strategies that express agreement and disagreement on i respectively.

i	AG	DG
AG	1;1	-1;-1
DG	-1;-1	1;1

Then, take $(O(a,i)AG, (1-O(a,i))DG)$, with $O(a,i) \in [0,1]$ to be the mixed strategy of agent a over agreement and disagreement on i. Consider, for example, that i is the issue "proposals of the Republican party". $O(a,i) = 0$ tells us that agent a does not like such proposals at all, having probability zero to agree on them. $O(a,i) = \frac{1}{2}$ suggests that agent a is equally expected to agree or disagree with the proposals and yields out an indifferent state of opinion. $O(a,i) = 1$ indicates that agent a fully agrees with the proposals.

		$O(b,i)$	$1-O(b,i)$
	i	AG	DG
$O(a,i)$	AG	1;1	-1;-1
$1-O(a,i)$	DG	-1;-1	1;1

[2] It is common to consider discussion as a strategic situation where cooperation is preferred. Approaches that involve competition between agents who discuss will not be considered in this paper, but are suggested for further research.

The expected utility of agents a and b in this game is interpreted as their *measure of agreement* on i[3]: $V^i(O_a, O_b) := O_aO_b - O_a(1 - O_b) - (1 - O_a)O_b + (1 - O_a)(1 - O_b)$. To simplify the formulation we will write $V^i(a, b)$ instead of $V^i(O_a, O_b)$. We will see that cooperation between agents in the discussion game will increase their closeness after playing it.

2.2 Model Update

Once we have the model, we want to update it to capture the dynamics in social interaction. We make use of the 2×2 discussion game to define our update function.

Definition 2. *The update of the model $M = \langle A, C, I, O\rangle$ over an issue $i \in I$ is the model $M^i = \langle A, C^i, I, O\rangle$, where C^i is given by the following formula.*

$$C^i(a, b) = \frac{C(a, b) + V^i(a, b)}{2}$$

and $C^i(a, a) = C(a, a)$.

The updating function captures the impact of the measure of agreement on the agents' closeness. We will motivate the use of this function with a number of real life examples.[4]

Example 1. **Discussion with an indifferent agent.**
Consider agents Alice (a) and Bob (b) who argue on the proposals of the Republican party. In particular, the issue is: the party's proposal on military expenditure (m). Suppose that the agents are very close to each other, that is, $C(a, b) = C(b, a) = C(a, a) = C(b, b) = 1$. However, agent a strongly supports the party's policy of military procurement whereas agent b is indifferent. Formally: $O(a, m) = 1$, $O(b, m) = 0.5$. According to our model, the measure of agreement is $V^m(a, b) = 0$, and subsequently the agents' closeness will be $C^m(a, b) = C^m(b, a) = 0.5$.

The above example reflects the scenario where an agent is indifferent on a thorny political issue, and this can indeed be proven to be harmful for her relationship with a strong supporter of this issue.

Example 2. **Hostile agents get closer when they agree...**
On the contrary, let agents Claire (c) and Dan (d) be such that $C(c, c) = C(d, d) = 1$ and $C(c, d) = -0.6$, $C(d, c) = -1$. They, too, engage in a political conversation over the issue m with both agreeing on it, having $O(c, m) = 1$ and $O(d, m) = 0.9$. According to our model, the measure of agreement is $V^m(c, d) = 0.8$ and their revised closeness is $C^m(c, d) = 0.1$ and $C^m(d, c) = -0.1$.

[3] Where for simplicity we write O_a instead of $O(a, i)$ and O_b instead of $O(b, i)$.

[4] We should note that in this framework –and throughout the paper in general– we assume that agents perform their revisions simultaneously. Therefore, an agent's judgment is only affected by the previous-stage data and not by the possible shifts other agents make in the current stage.

Therefore, although the agents were initially hostile, their strong political agreement brought them closer and made them provisionally more tolerant towards each other.

Example 3. **...But not for long.**
Next, assume that Claire and Dan keep discussing about Republican proposals, introducing the issue of corporate tax (t). Suppose that $O(c,t) = 0.2$ and $O(d,t) = 1$. Then, their measure of agreement is $V^t(c,d) = -0.6$ and now their revised closeness will be $C^t(c,d) = -0.25$ and $C^t(d,c) = -0.35$.

Overall, the two agents' divergent opinions on the second political issue decreased the shaky closeness they acquired after their first agreement. Triggered by this example and talking in general terms, predictions and insights on the long-term behavior of a network can be accommodated once the particular model is employed.[5]

Example 4. **The order of the discussed issues matters.**
Finally, suppose that Claire and Dan discussed the same issues presented before, but discussion on the corporate tax preceded the one on military spending. Our model prescribes that, after the first round of discussion, the revised closeness will be $C^t(c,d) = -0.6, C^t(d,c) = -0.8$, and after the second round of discussion, $C^m(c,d) = 0.1$ and $C^m(d,c) = 0$.

Example 4 illustrates the following

Proposition 1. *Closeness revision is order-dependent.*

Indeed, the fluctuations of a relationship can be reasonably accounted in terms of the alternations of agreement and disagreement over time. Specifically, in real life scenarios, the impact that the discussion of an issue can have on a relationship does not only depend on the issue itself, but also on the context in which the discussion takes place (the issues that have been discussed before, etc.)

Of course, once we have the discussion games in our toolbox, the updating attempt is not unique. Depending on the scenario that is modeled, additional constraints can be established and the updated closeness might also be calculated in a different manner. In Section 4, we propose a refinement of our model update.

2.3 Weighted Issues

It is also reasonable to consider the priority that an agent gives to a specific issue. We expect that the more important an issue is for an agent a, the more it affects a's relations. We represent agent a's priority over issues by adding a weighting function into our model, $W_a : I \to [0,1]$, where $W_a(i) = 0$ reflects no importance, and $W_a(i) = 1$ reflects the highest priority. Therefore, the agents'

[5] General results on networks' evolution using the framework of this paper are open for further investigation.

payoffs in the discussion game may differ depending on the weights. For example, if $O(a,i) = O(b,i) = 1$ but $W_a(i) > W_b(i)$, we expect that agent a will get higher subjective utility by agreeing with b, because the issue i is more important to her. Overall, the payoffs of the discussion game express the "amount of satisfaction" for each agent after the discussion.

		$O(b,i)$	$1-O(b,i)$
	i	AG	DG
$O(a,i)$	AG	W_a;W_b	-W_a;-W_b
$1-O(a,i)$	DG	-W_a;-W_b	W_a;W_b

A model of Closeness Revision with weighted issues and its update can be defined accordingly.

Definition 3. *A model of Closeness Revision with weights is a tuple* $M = \langle A, C, I, O, (W_a)_{a\in A}\rangle$.

Let us now consider the following example.

Example 5. **Discussion between agents with different priorities.**
Alice (a) is very close to Ben (b), she supports the Republican policy on military expenditure and this also constitutes one of her top priorities. On the contrary, Ben disagrees with it, but he places military concerns low in his agenda. Formally, take: $C(a,b) = C(b,a) = C(a,a) = C(b,b) = 1$, $O(a,m) = 1$, $O(b,m) = 0$, $W_a(m) = 1$ and $W_b(m) = 0$. The measure of agreement is $V^m(a,b) = -1$. According to the model with weighted issues: $C^m(a,b) = 0$ and $C^m(b,a) = 0.5$.

In other words, following the update, Alice becomes utterly distant to Ben, due to Ben's disagreement on an issue that is so essential for her. Yet Ben, despite slightly shifting away from Alice, still regards her relatively close.

Hopefully the above example convinced the reader that adding weights in the model is a step closer to the idea of imitating real life scenarios.

3 The Logic of Closeness Revision

In this section, we present a complete dynamic logic to capture the notions that have been described so far. The logic is based on the model in Definition 1 and its update in Definition 2, and is inspired by techniques used in logics for reasoning about probability (Fagin, Halpern, and Megiddo 1990; Van Benthem, Gerbrandy, and Kooi 2009).

3.1 Syntax and Semantics

Definition 4. *Let A be a finite set and I be a countable set.*
The set $\mathcal{T}$ of terms contains the sets of constants $\{C_{ab} : a, b \in A\}$, $\{O_{ai} : a \in A, i \in I\}$ and $\{V_{abi} : a, b \in A, i \in I\}$.

For $q_1, \ldots, q_n \in \mathcal{T}$ and $a_1, \ldots, a_k, c \in \mathbf{Z}$, the set $\mathcal{A}$ contains atoms of the form $\alpha_1 q_1 + \ldots + \alpha_k q_k \geq c$.
Let $\Phi := \mathcal{T} \cup \mathcal{A}$ be the set of all primitive propositions.
The Language of Closeness Revision $\mathcal{L}_{CR}$ is defined as follows:

$$p \in \Phi \mid \neg\phi \mid \phi \wedge \phi \mid [i]\phi$$

Intuitively, the set $\mathcal{T}$ indicates facts about the agents' closeness and opinions, whereas the set $\mathcal{A}$ suggests numerical inequalities between the values representing closeness and opinions. The $[i]$ modality is interpreted as in standard dynamic epistemic logic (Van Ditmarsch, Der Hoek, and Kooi 2007; Van Benthem and Liu 2007): we evaluate $[i]\phi$ as true "today" if and only if ϕ is true "tomorrow" after the revision induced by issue i.
The symbols $\vee$, $\rightarrow$, $-$, $\leq$, $>$, $<$, $=$ are defined in the usual way. For example the formula $q = c$ stands as abbreviation for $(q \geq c) \wedge ((-1)q \geq -c)$. Moreover, a formula with rational numbers such as $q > \frac{1}{5}$ can be expressed by $5q > 1$. So, we can always allow rational numbers in our formulas as abbreviations for the formula that can be obtained by clearing the denominators.

Definition 5. *Let $M = \langle A, C, I, O \rangle$ be a model of closeness revision as defined in Definition 1. The interpretation q^M of terms q in M is defined as follows: $C^M_{ab} := C(a,b)$, $O^M_{ai} := O(a,i)$ and $V^M_{abi} := V^i(a,b)$.*

Given a model $M = \langle A, C, I, O \rangle$, the truth clauses for $\mathcal{L}_{CR}$ are the following.

- $M \vDash \alpha_1 q_1 + \ldots + \alpha_k q_k \geq c$ iff $\alpha_1 q^M_1 + \ldots + \alpha_k q^M_k \geq c$
- $M \vDash \neg\phi$ iff $M \nvDash \phi$
- $M \vDash \phi \wedge \psi$ iff $M \vDash \phi$ and $M \vDash \psi$
- $M \vDash [i]\phi$ iff $M^i \vDash \phi$

Abbreviations We introduce the following abbreviations $t^{[i]}$ in order to capture (in the logical language) the values of terms $t \in \mathcal{T}$ after the revision with issue i, according to Definition 2.

- $O^{[i]}_{aj} := O_{aj}$, for any $j \in I$
- $V^{[i]}_{abj} := V_{abj}$, for any $j \in I$
- $C^{[i]}_{ab} := \frac{1}{2}C_{ab} + \frac{1}{2}V_{abi}$, for $a \neq b$
- $C^{[i]}_{aa} := C_{aa}$

3.2 Complete Axiomatization

The system L_{CR} that we present divides nicely into three parts, which deal respectively with propositional reasoning, reasoning about linear inequalities and reasoning about dynamics. We obtain a complete axiomatization of the logic for the models of closeness revision and their updates, by using the standard technique of reduction laws from DEL (Van Ditmarsch, Der Hoek, and Kooi 2007; Blackburn, De Rijke, and Venema 2002).

Definition 6. *The following axiom system is sound and complete with respect to the class of models* $\mathbf{C}$.

All instances of valid formulas for propositional logic	*Prop*
All instances of valid formulas for linear inequalities	*Ineq*
$0 \leq O_{ai} \leq 1$	*Bound O*
$-1 \leq C_{abi} \leq 1$	*Bound C*
$0 \leq V_{abi} \leq 1$	*Bound V*
$O_{ai} = v \wedge O_{bi} = w \to V_{abi} = u$ *for all* $v, w, u \in [0,1]$ *s.t.* $vw - v(1-w) - (1-v)w + (1-v)(1-w) = u$	*Cor.* O, V
$[i]((\sum_{m=1}^{k} a_m q_m) \geq c) \leftrightarrow ((\sum_{m=1}^{k} a_m q_m^{[i]}) \geq c)$ *for all* $k \in \mathbf{N}$	*Red.Ax.Ineq*
$[i](\phi \wedge \psi) \leftrightarrow [i]\phi \wedge [i]\psi$	*Red.Ax.* $\wedge$
$[i]\neg\phi \leftrightarrow \neg[i]\phi$	*Red.Ax.* $\neg$
From ϕ *and* $\phi \to \psi$, *infer* ψ	*Modus Ponens*

The static part of the logic consists of the axioms of propositional logic Prop, the axiom Ineq, the Bounding axioms for opinion, closeness and measure of agreement, the correlation axiom between O and V, and the rule of Modus Ponens. In order to deal with the dynamic part of the logic, we need rules which reduce formulas that contain the $[i]$ modality to formulas without it. This is possible, as all the information required to determine the updated model M^i is present in the model M before the update. The reduction laws are trivial in all cases apart from those involving atoms of the form $\alpha_1 q_1 + \ldots + \alpha_k q_k \geq c$, for $a_1, \ldots, a_k, c \in \mathbf{Z}$. By making use of the abbreviations presented before, the axiom Red.Ax.Ineq encodes the numerical changes on the terms' values.

Theorem 1 (Completeness). *For any* $\phi \in \mathcal{L}_{CR}$, *we have that*

$$\vDash_{\mathbf{C}} \phi \quad \textit{iff} \quad \vdash_{L_{CR}} \phi.$$

Proof. Soundness: Let $M = \langle A, C, I, O \rangle$ be an arbitrary model, with $a \in A$ and $i \in I$. The axiom Ineq is easily checked to be true on M, as the updates are on atoms, so sophisticated checks whether we can stay inside the language of linear inequalities are not required. Then, the formulas $0 \leq O_{ai} \leq 1$, $-1 \leq C_{abi} \leq 1$ and $0 \leq V_{abi} \leq 1$ are satisfied, by the way the model is defined. Definition 1

combined with the definition of the measure of agreement can also verify that the Cor. O, V axiom is satisfied. Soundness of Red.Ax.$\wedge$ and Red.Ax.$\neg$ can be shown by induction on the structure of the formulas.
Completeness: Fagin, Halpern, and Megiddo 1990 provide us with a complete axiomatization of all valid formulas about linear inequalities. The axioms Bound O, Bound C and Bound V guarantee that the numerical bounds on opinion, closeness and measure of agreement are provable in our system. Moreover, the axiom Cor. O, V ensures that the correlation between agent's opinions and their measure of agreement, as defined in our framework, is provable.
Finally, we can translate the dynamic part of the language into its static part using the reduction laws given above. Then, the proof goes in the standard way (Van Ditmarsch, Der Hoek, and Kooi 2007). □

3.3 Safe Friends, Future Friends and Dangerous Issues

We now present some supplementary definitions that support putting the logical framework in practical context.

Given a certain friendship threshold θ_F we say that:
Agent b is agent a's *friend* (F_{ab}) whenever $C(a,b)$ is above θ_F. Therefore $M \vDash F_{ab}$ iff $M \vDash C_{ab} \geq \theta_F$. We call a and b friends whenever $M \vDash F_{ab}$ and $M \vDash F_{ba}$.
Agent b is a's *safe friend* for issue i (SF^i_{ab}) whenever b is a's friend and the revision induced by i cannot break this friendship, that is $M \vDash SF^i_{ab}$ iff $M \vDash F_{ab} \wedge [i]C_{ab} \geq \theta_F$. We call a and b safe friends whenever a is b's safe friend and b is a's safe friend.
Agent a is a *future friend* of agent b given issue i (FF^i_{ab}) whenever b is not a's friend, yet after the revision induced by i friendship is established. Formally, $M \vDash FF^i_{ab}$ iff $M \vDash \neg F_{ab} \wedge [i]C_{ab} \geq \theta_F$. We call a and b future friends whenever $M \vDash FF^i_{ab}$ and $M \vDash FF^i_{ba}$.
An issue i is *dangerous* for a's friendship with b (D^i_{ab}) whenever F_{ab} is true before the update induced by i, but not after, namely $M \vDash D^i_{ab}$ iff $M \vDash F_{ab} \wedge [i]\neg F_{ab}$.

When we combine the previous framework with the notions above, we can observe that further validities hold.

- $[i]F_{ab} \leftrightarrow SF^i_{ab} \vee FF^i_{ab}$: b is a's friend after discussing i iff b is a's safe friend for issue i or i establishes a future friendship.
- $SF^i_{ab} \vee FF^i_{ab} \vee [i]\neg F_{ab}$: given issue i, if b is neither safe nor future friend with a, then b is not going to be a's friend after discussing i.
- $F_{ab} \wedge V_{abi} \geq C_{ab} \rightarrow [i]F_{ab}$: if b is already a's friend and the value of agreement on issue i is higher than the closeness value of a and b, then discussing i will not break the friendship.

4 Model with Updating Threshold

In this section, a variation of the model of closeness revision is provided, with modifications required to deal with potential challenges. We will enrich the model update of Definition 2 adding a further threshold condition on the update function.

Even if so far we claimed that all agents *can be considered* stubborn with respect to certain issues, it is still plausible to argue that not all agents *are* always stubborn. Agreement and disagreement affect people's relationships in different degrees. In this part of the paper we consider that only agents who have enough social closeness can afford being stubborn.[6] If being stubborn can lead an agent to be socially isolated, this agent is more conservative about updating her closeness with her social contacts. In other words, if updating closeness will result in social isolation, agents do not perform the update.

Definition 7. *A model of closeness revision with threshold is a tuple*

$$M_\theta = \langle A, C, I, O, \theta \rangle$$

where A , C, I, O as in Definition 1 and $\theta \in [0,1]$ is a threshold for revising closeness.

The threshold for revising closeness represents the level of closeness that an agent does not feel comfortable to go below. Consequently, agents will behave stubbornly and keep revising only when their closeness level is above the threshold θ.

Definition 8. *The update of the model of closeness revision with threshold $M_\theta = \langle A, C, I, O, \theta \rangle$ over an issue $i \in I$ is the model $M_\theta^i = \langle A, C^i, I, O, \theta \rangle$, where C^i is given by the following formula.*

$$C^i(a,b) = \begin{cases} \frac{C(a,b)+V^i(a,b)}{2} & \text{if } \frac{\sum_{d\in A} C(a,d)}{|A|} \geq \theta \text{ or } C(a,b) < V^i(a,b), \text{ and } a \neq b \\ C(a,b) & \text{else} \end{cases}$$

The condition $\frac{\sum_{d\in A} C(a,d)}{|A|} \geq \theta$ or $C(a,b) < V^i(a,b)$ says that for agent a to revise her closeness with agent b over the issue i, she needs to be safe enough to be stubborn (expressed by the formula $\frac{\sum_{d\in A} C(a,d)}{|A|} \geq \theta$), or if she revises, she will increase the value of her "social closeness" (reflected by the formula $C(a,b) < V^i(a,b)$ as follows: $\frac{C(a,b)+V^i(a,b)}{2} > C(a,b) \Leftrightarrow C(a,b) + V^i(a,b) > 2C(a,b) \Leftrightarrow C(a,b) < V^i(a,b)$).

Example 6. **Who behaves stubbornly after all?**
Suppose that Claire and Dan are the only agents in the network, and they discuss the Republican proposal on military expenditure (m). Their closeness relations are represented by the values: $C(c,c) = C(d,d) = 1$, $C(c,d) = 0.5$ and $C(d,c) = 0.1$. Suppose that their opinions are $O(c,m) = 1$ and $O(d,m) = 0.5$,

[6] Sufficiency of social closeness will be captured by a threshold condition. For the purposes of this paper, the threshold will be uniform for all the agents. However, different thresholds can be added to express different agents' tendency to stubbornness.

that is, Claire strongly agrees on the issue, while Dan is indifferent. It follows that their measure of agreement on m is $V^m(c,d) = 0$. Let the threshold for revising closeness be $\theta = 0.6$. According to the model of closeness revision with threshold:

- For Claire, the condition for closeness revision is satisfied, as $\frac{C(c,c)+C(c,d)}{2} = 0.75 > 0.6$. This means that Claire feels confident enough to behave stubbornly. Therefore, she will revise her closeness with Dan, having $C^m(c,d) = 0.25$ after the discussion.
- For Dan, however, the condition for closeness revision is not satisfied, as $\frac{C(d,d)+C(d,c)}{2} = 0.55 < 0.6$ and $C(d,c) = 0.1 > 0 = V^m(c,d)$. This means that Dan does not have enough social closeness, so he does not revise his relationships. Therefore, his closeness with Claire remains the same $C^m(d,c) = C(d,c) = 0.1$ after the discussion.

Overall, this scenario demonstrates how two different agents may behave stubbornly or not, according to their social closeness at the moment of a discussion.

5 Conclusion and Further Research

To sum up, in this paper we use a game-theoretical approach to build a dynamical model that is able to represent the interactive revision of both agents' opinions and agents' relations in a social network. Measures of agreement and disagreement are used to define the revision dynamics of these two main dimensions considered here. Specifically, the interaction between agents who discuss is expressed by the discussion game that captures agreement and disagreement on a specific issue under consideration. We finally introduce a sound and complete axiom system for models of closeness revision.

A first limitation of our framework concerns an implicit assumption: the willingness of agents to cooperate, reflected by the discussion game. One could reasonably argue that her closeness with someone may increase not only in situations of agreement, but also in cases of *constructive disagreement*. This is an intriguing issue to reflect on, even though a counter-argument would support that the number of people in a network who would appreciate a disagreement as fruitful is so small that becomes insignificant. Still in the direction of our design choices, an objection can be raised regarding the difference between the *quantitative* and the *qualitative* side of an opinion's expression. The function O captures the former, while the latter is ignored. In the presented framework, an agent merely announces the content of her opinion, that is the degree on which she agrees with the discussed issue. However, real life examples suggest that the *strength* of opinions plays a principal role in human interactions, too. Defining opinion as a twofold notion, with both a quantitative and a qualitative part, and modifying the revising conditions accordingly, would be a natural and appealing extension of our model. Some other questions that deserve further investigation are: Firstly, concerning the game-theoretical part: How can we strengthen the connection of our framework with games? General results on network studies can provide more insights into the topic of networks' dynamical changes triggered by agents' discussions. Moreover, techniques from evolutionary game theory could be proven to be useful in analyzing how profitable human interactions of certain kind are for a society, or for a social network. Secondly, on the logical part: How can we extend the Logic of Closeness Revision to capture more refined schemata of interaction, as for instance the one presented in Section 4? In real life scenarios, it is also possible to observe the combined action of two different dynamics between connected agents: opinions affecting relations and relations affecting opinions. Agents in different contexts may behave stubbornly or revise their opinions instead. Furthermore, the model of closeness revision does not take into account the level of information that agents have about the opinions of the others in their network. So, in which way could our logic evolve, if we add epistemic -indistinguishability- relations for agents? To conclude with a philosophically oriented concern: Is it possible, in logical and mathematical terms, to value the future of such a complicated concept, as a human relationship?

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