Newtonian Mechanics

N1 - *If no forces act on a body, it remains at rest or moves with constant velocity:* $\dot{\underline{v}} = 0$

N2 - $\dot{\underline{p}} = \underline{F}$

N3 - $\underline{F}_{ab} = -\underline{F}_{ba}$

$$\underline{L} \equiv \underline{r} \times \underline{p}$$

$$W_{BA} \equiv \int_A^B \underline{F} \cdot d\underline{r} = T_B - T_A$$

Orbits (cylindrical polars):

$$\underline{e_r} = \cos\phi \underline{i} + \sin\phi \underline{j}$$

$$\underline{e_\phi} = -\sin\phi \underline{i} + \cos\phi \underline{j}$$

$$\dot{\underline{r}} = \dot{r}\underline{e_r} + r\dot{\phi}\underline{e_\phi}$$

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r)$$

Newton to Lagrange

Holonomic constraint is an algebraic relation between coordinates:

$$f(\underline{r_a}, \underline{r_b}, ..., \underline{r_N}; t) = 0$$

For system with N cartesian cordinates x_i, M constraints, and $3N - M$ generalised coordinates q_i, and $x_i = x_i(\{q\}, t)$

Virtual displacement:

$$\delta x_i = \sum_i \frac{\partial x_i}{\partial q_j}\delta q_j + 0$$

Generalised forces:

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j}$$

for a function $f = f(\{q\}, \{\dot{q}\}, t)$

$$df = \sum_j \frac{\partial f}{\partial q_j} + \sum_j \frac{\partial f}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial f}{\partial t}dt$$

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for a function $f = f(\{q\}, \{\cancel{\dot{q}}\}, t)$
cancellation of dots

$$\frac{\partial \dot{f}}{\partial \dot{q}_j} = \frac{\partial f}{\partial q_j}$$

commuting derivatives:

$$\frac{d}{dt}\left(\frac{\partial f}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{df}{dt}\right)$$

Lagrange's equations (general form):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

or

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

where $\mathcal{L}(\{q\}, \{\dot{q}\}.t) = T(\{q\}.\{\dot{q}\}, t) - V(\{q\}, t)$

Calculus of Variations

Euler's Equation: for functional of the form

$$I[y(s)] = \int_a^b F(y(s), y'(s), s)ds$$

$$\frac{\partial F}{\partial y} - \frac{d}{ds}\left(\frac{\partial F}{\partial y'}\right) = 0$$

with first integrals

$$\frac{\partial F}{\partial y'} = \text{constant} \qquad \text{if F does not depend on y}$$

$$y'\frac{\partial F}{\partial y'} - F = \text{constant} \qquad \text{if F does not depend on s}$$

Hamilton's Principle:

$$\delta S = 0 \quad \text{for the } \textit{action} \quad S = \int_{t_1}^{t_2} \mathcal{L}dt$$

Energy Function

The energy function (h) is usually conserved, whereas the energy ($E = T + V$) might not be. h is given by:

$$h = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \sum_i p_i\dot{q}_i - L$$

h and L are the legendre transforms of each other on the variables p_i and $\dot{q}_i$:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \dot{q}_i = \frac{\partial h}{\partial p_i}$$

Relativistic Particle

The lagrangian for a free relativistic particle can be found by minimising the action S, then requiring $\frac{\partial L}{\partial \dot{q}_i} = \gamma m \dot{q}_i$

$$S = \int_{\tau_1}^{\tau_2} \epsilon d\tau = \epsilon \int_{t_1}^{t_2} \frac{dt}{\gamma(\dot{q})} = \epsilon$$

Finding ϵ from relativistic momentum:

$$p_i = \gamma m \dot{q}_i = \frac{\partial L}{\partial \dot{q}_i} = -\frac{\epsilon\gamma\dot{q}_i}{c^2}$$

And we have $\epsilon = -mc^2$, so the lagrangian is:

$$L = -\frac{mc^2}{\gamma}$$