# Dynamics

## Relative Motion for Rotating Frame

$$\underline{v} = \frac{d\underline{r}}{dt} = \underline{v}_{o'} + \underline{v}_r + \underline{w} \times \underline{r}$$

$$\frac{d\underline{v}}{dt} = \underline{a}_r + 2\underline{\omega} \times \underline{v}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{a}_{o'} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

Polar Coordinates:

$$\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$$

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$$

Binet's Formula:
For $\underline{a} \times \underline{r} = 0$ we have $a_\theta = 0 \implies r^2\dot{\theta} = c$ which gives

$$a_r = -\frac{c^2}{r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right]$$

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## Gravity

$$m\underline{a} = -\frac{GmM}{r^2}\underline{\hat{r}} = -\frac{m\mu}{r^2}\underline{\hat{r}}$$

Total Mechanical Energy:

$$\boldsymbol{E = \frac{1}{2}mv^2 - \frac{m\mu}{r}}$$

Angular Momentum:

$$\underline{L} = \underline{r} \times m\underline{v}$$

$$|\underline{L}| = mr^2\dot{\theta} \implies \text{conserved}$$

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# Keplerian Motion

## Two-Body Dynamics

Gravitational Forces:

$$m_i\underline{\ddot{r}}_i = \pm\frac{Gm_im_j}{r^2}\underline{\hat{r}}$$

Centre-of-Mass:

$$\underline{r}_C = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1 + m_2}$$

Sum of forces acting on centre of mass $= 0 \implies \underline{r}_C = \underline{r}_0 + \underline{r}'t$
Approximate $\mu \approx Gm_1$ gives:

$$\boldsymbol{\underline{\ddot{r}} = -\frac{\mu}{r^2}\underline{\hat{r}}}$$

Total energy is conserved, take $\underline{\ddot{r}} \cdot \underline{r}$ to prove $(d/dt)E = 0$, since:

$$\frac{d}{dt}\left(\frac{1}{2}v^2\right) = \underline{\ddot{r}} \cdot \underline{\dot{r}} = \frac{d}{dt}\left(\frac{G(m_1 + m_2)}{r}\right)$$

## Equations of Motion

Use chain rule and $r^2\dot{\theta} = l$ to get:

$$\frac{dr}{dt} = \frac{l}{r^2}\frac{dr}{d\theta}$$

Change variable to $u = 1/r$:

$$\frac{dr}{dt} = -l\frac{du}{d\theta}$$

Gives:

$$\frac{d^2r}{dt^2} = -l^2u^2\frac{d^2u}{d\theta^2} \qquad \text{and} \qquad -r\dot{\theta}^2 = -l^2u^3$$

Equations of motion in polar coordinates are:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \qquad \text{and} \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

Transforming for $u$:

$$-l^2u^2\frac{d^2u}{d\theta^2} - l^2u^3 = -\mu u^2$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{l^2}$$

Finally change to $z = u - \mu/l^2$ to get:

$$\frac{d^2z}{d\theta^2} + z = 0 \implies z = A\cos(\theta - \theta_0)$$

Which gives polar equation of conic section for $r$:

$$r(\theta) = \frac{l^2/\mu}{1 + \frac{Al^2}{\mu}\cos(\theta)}$$

The energy equation is :

$$\boldsymbol{E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)}$$

Which can be solved by integrating w.r.t $r$, changing variable from $t$ to $\theta$, and substituting $U(r) = m\mu/r$. This gives:

$$r = \frac{p}{\pm 1 + e\cos(\theta - \theta_0)}$$

where

$$\boldsymbol{p = \frac{c^2}{|\mu|} = a(1 - e^2)} \qquad \text{and} \qquad \boldsymbol{e = \sqrt{\frac{2Ec^2}{m\mu^2} + 1}}$$

so r describes an ellipse with semimajor axis $a$ and eccentricity $e$, pericentre at $\theta = 0$ and apocentre at $\theta = \pi$:

$$r = \frac{a(1 - e^2)}{1 + e\cos(\theta - \theta_0)}$$

# Orbital Parameters

$$a = \frac{r_p + r_a}{2} \qquad \text{and} \qquad e = \frac{r_a - r_p}{r_a + r_p}$$

so

$$r_p = a(1-e) \qquad \text{and} \qquad r_a = a(1+e)$$

Comparing equations for $r$:

$$\boldsymbol{l^2 = \mu a(1-e^2)}$$

$$v_p = \sqrt{\frac{\mu(1+e)}{a(1-e)}} \qquad \text{and} \qquad v_a = \sqrt{\frac{\mu(1-e)}{a(1+e)}}$$

Conservation of energy:

$$\boldsymbol{E = -\frac{\mu}{2a} \implies v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

## Kepler's Laws

1st Law: Orbits are elliptical with attracting body at focus of ellipse

2nd Law: Radial vector sweeps out equal areas in equal time

$$\implies T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

3rd Law: Orbital period squared $\propto$ semi-major axis cubed:

## Kepler's Equation

Mean motion

$$n = \sqrt{\frac{\mu}{a^3}}$$

Mean anomoly

$$M = n(t - \tau)$$

Kepler's equation is

$$M = E - e\sin E$$

and use trig identity to get $\theta$ from $E$

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$$

This must be solved iteratively, start with guess $E_0 = M$, then find correction

$$\Delta E = \frac{M - E_0 + e\sin E_0}{1 - e\cos E_0}$$

Then repeat, using $E_i = E_{i-1} + \Delta E_{i-1}$ until $\Delta E_i \approx 0$

## Position and Velocity from Orbital Elements

From orbit equation

$$\boldsymbol{\dot{r} = \frac{a(1-e^2)}{(1+e\cos\theta)^2} e\sin\theta\dot{\theta} = r\frac{e\sin\theta}{1+e\cos\theta}\dot{\theta}}$$

Flight path angle

$$\tan\gamma = \frac{v_r}{v_\theta} = \frac{\dot{r}}{r\dot{\theta}} = \frac{e\sin\theta}{1+e\cos\theta}$$

To get in plane position and velocity at time $t$, solve Kepler's equation for $\tau$ time of passage from pericentre, then compute orbit radius and thus cartesian positions.

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

Velocity components:

$$\boldsymbol{v_r = \frac{r^2 e\sin\theta}{p}\dot{\theta}} \qquad \text{and} \qquad \boldsymbol{v_\theta = \frac{l}{r} = r\dot{\theta}}$$

Gives cartesian velocities:

$$\dot{x} = -\sqrt{\frac{\mu}{p}}\sin\theta \qquad \text{and} \qquad \dot{y} = \sqrt{\frac{\mu}{p}}(e + \cos\theta)$$

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# Orbit Transfers

## Hohmann Transfer

Minimum energy transfer between two circular, coplanar orbits is an ellipse with semimajor axis $a = \frac{1}{2}(r_1 + r_2)$. The change in velocities required are the difference between the velocities of the transfer ellipse at pericentre and apocentre, and the velocities of the circular orbits at $r_1$ and $r_2$

$$\Delta v_1 = \sqrt{\frac{2\mu}{r_1} - \frac{2\mu}{r_1 + r_2}} - \sqrt{\frac{\mu}{r_1}}$$

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{2\mu}{r_1 + r_2}}$$

## Parabolic Transfer

For a parabolic transfer where the orbiting body escapes circular orbit at $r_1$ to infinity, and is then captured from infinity into orbit at $r_2$

$$\Delta v_1 = (\sqrt{2} - 1)\sqrt{\frac{\mu}{r_1}}$$

$$\Delta v_2 = (\sqrt{2} - 1)\sqrt{\frac{\mu}{r_2}}$$

## Single Impulse Manoeuvres

Raise apocentre:

$$\Delta v = \sqrt{-\frac{\mu}{a_2} + \frac{2\mu}{r_p}} - \sqrt{-\frac{\mu}{a_1} + \frac{2\mu}{r_p}}$$

Change inclination:

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos \Delta i}$$

If $v_1 = v_2$ then

$$\Delta v = 2v \sin\left(\frac{\Delta i}{2}\right)$$

Multiple impulse manoeuvres can be used to reduce required delta v, by first raising apocentre and performing multiple burns.

### Patched Conic

In Earth-centred frame, velocity to escape is:

$$\Delta v_{esc} = \sqrt{\frac{2\mu_{Earth}}{r_p} + v_{\infty,1}^2} - \sqrt{\frac{2\mu_{Earth}}{r_p} - \frac{\mu_{Earth}}{a_p}}$$

In sun centre frame, $v_{\infty,1}$ must match the first $\Delta v$ for transfer. E.g. $\Delta v_1$ of Hohmann transfer, and similar for capture.

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## Rocket Equation

Specific impulse

$$I_{sp} = \frac{F}{\dot{m} g_0}$$

Effective exhaust velocity

$$c = I_{sp} g_0$$

Rocket equation

$$(m - dm)dv = -dm v_e$$

$$\Delta v = v_e \ln \frac{m_i}{m_f}$$

$$m_f = m_i e^{-\frac{\Delta v}{v_e}}$$

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## Aerobraking

$$D = \frac{1}{2}\rho C_D A v^2$$

$$L = \frac{1}{2}\rho C_L A v^2$$

Flight path angle $\gamma$ is measured to horizontal

$$m\frac{dv}{dt} = -D - mg\sin\gamma$$

$$mv\frac{d\gamma}{dt} = L - mg\cos\gamma$$

Atmosphere density is exponential model, relative to a scale height $H$ and base density $\rho_0$

$$\rho(h) = \rho_0 \exp\left[-\frac{h}{H}\right]$$

For aerocapture, assume gravity terms negligible

$$\frac{1}{v}\frac{dv}{d\gamma} = -\frac{D}{L} = -\frac{C_D}{C_L}$$

Integrate to get

$$\ln\left(\frac{v_2}{v_1}\right) = -\frac{C_D}{C_L}(\gamma_2 - \gamma_1)$$

$$\Delta v = v_1\left(1 - \exp\left[2\frac{C_D}{C_L}\gamma_1\right]\right)$$

### Vertical Dynamics

$$\frac{dh}{dt} = v\sin\gamma = \frac{dh}{d\gamma}\frac{d\gamma}{dt} = \frac{dh}{d\gamma}\frac{L}{mv}$$

Substituting for $L$ and using density function

$$\frac{dh}{d\gamma} = \frac{2m\sin\gamma}{\rho_0 C_L A}\exp\left[\frac{h}{H}\right]$$

Integrate to find $h_{min}$

$$\int_{h_1}^{h_{min}} \exp\left(-\frac{h}{H}\right) = \frac{2m}{\rho_0 C_L A}\int_{\gamma_1}^{0} \sin\gamma d\gamma$$

Note that $h_1 >> H$ and so

$$h_{min} = H \ln\left(\frac{\rho_0 C_L A H}{2m(1 - \cos\gamma_1)}\right)$$

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## Important Things

### 1) Derive Orbital Energy

#### - Derive Energy for 2-Body Problem

$$E = T + V$$

$$E = \frac{1}{2}m_1 v^2 + V(r)$$

$$\underline{F} = -\frac{m\mu}{r^2}\hat{\underline{r}} = -\underline{\nabla}V(r)$$

$$\implies V(r) = -\frac{m\mu}{r}$$

$$E = \frac{1}{2}mv^2 - \frac{m\mu}{r}$$

#### - Demonstrate Energy Constant

Take time derivative of $1/2v^2$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\underline{r}}\cdot\dot{\underline{r}}\right) = \ddot{\underline{r}}\cdot\dot{\underline{r}}$$

Take dot product of equation of motion with velocity vector

$$\ddot{\underline{r}}\cdot\dot{\underline{r}} - \frac{G(m_1+m_2)}{r^2}(\hat{\underline{r}}\cdot\dot{\underline{r}})$$

$$\implies \frac{d}{dt}\left(\frac{1}{2}v^2\right) = -\frac{G(m_1+m_2)\dot{r}}{r^2} = \frac{d}{dt}\left(\frac{G(m_1+m_2)}{r}\right)$$

$$\implies \frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{G(m_1+m_2)}{r}\right) = 0$$

This constant of motion is the energy

$$E = \frac{1}{2}v^2 - \frac{G(m_1+m_2)}{r}$$

## 2) Energy from Orbit Geometry

$$r = \frac{p}{\pm 1 + e\cos(\theta-\theta_0)}$$

$$p = \frac{c^2}{\mu} = \frac{r^4\dot{\theta}^2}{\mu} = a(1-e^2) \qquad \text{and} \qquad e = \sqrt{\frac{2Ec^2}{m\mu^2}+1}$$

$$a(1-e^2) = \frac{c^2}{\mu} = a\left(1 - \frac{2Ec^2}{m\mu^2}+1\right)$$

$$\implies E = -\frac{m\mu}{2a}$$

$$E = \frac{-m\mu}{2a} = \frac{1}{2}mv^2 - \frac{m\mu}{r}$$

$$\implies v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

## 3) Demonstrate Angular Momentum is Constant

Definition of Angular Momentum

$$\underline{L} = \underline{r} \times m\dot{\underline{r}}$$

Take cross of position vector with equation of motion

$$\underline{r} \times \ddot{\underline{r}} = -\frac{\mu}{r^2}(\underline{r}\times\hat{\underline{r}}) = 0$$

Take time derivative of angular momentum

$$\frac{d}{dt}(\underline{r}\times\dot{\underline{r}}) = \dot{\underline{r}}\times\dot{\underline{r}} + \underline{r}\times\ddot{\underline{r}} = 0$$

$\implies$ Angular momentum conserved

## 4) Kepler's Laws

1. Orbits are elliptical with attracting body at focus of ellipse
2. Radial vector sweeps out equal areas in equal time
3. Orbital period squared is $\propto$ semi-major axis cubed

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Mean motion

$$n = \sqrt{\frac{\mu}{a^3}}$$

## 5) Equation of Orbit

$$r = \frac{p}{1+e\cos\theta}$$

$$p = \frac{c^2}{|\mu|} = a(1-e^2) \qquad \text{and} \qquad e = \sqrt{\frac{2Ec^2}{m\mu^2}+1}$$

## 6) Minimum Aerobraking Altitude

$$L = mv\frac{d\gamma}{dt} \qquad \text{neglecting gravity}$$

$$\rho(h) = \rho_0\exp\left[-\frac{h}{H}\right]$$

$$\frac{dh}{dt} = v\sin\gamma = \frac{dh}{d\gamma}\frac{d\gamma}{dt} = \frac{dh}{d\gamma}\frac{L}{mv}$$

Substituting for $L$ and using density function

$$\frac{dh}{d\gamma} = \frac{2m\sin\gamma}{\rho_0 C_L A}\exp\left[\frac{h}{H}\right]$$

Integrate to find $h_{min}$

$$\int_{h_1}^{h_{min}} \exp\left(-\frac{h}{H}\right) = \frac{2m}{\rho_0 C_L A}\int_{\gamma_1}^{0}\sin\gamma d\gamma$$

Note that $h_1 >> H$ and so

$$h_{min} = H\ln\left(\frac{\rho_0 C_L A H}{2m(1-\cos\gamma_1)}\right)$$

## 7) Rocket Equation

Conservation of linear momentum, momentum gained by rocket = −momentum of exhaust gas

$$(m-dm)dv = -dmv_e$$

Note that $dmdv \approx 0$

$$mdv = -v_e dm$$

$$\Delta v = c\ln\frac{mi}{mf}$$

$$m_f = m_i e^{-\frac{\Delta v}{c}}$$

Where $c$ is effective exhaust velocity

$$c = I_{sp}g_0$$

## 8) Δv Manoeuvres

Find difference between velocity of initial orbit and velocity of target orbit using

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

True for Hohmann, raising/lowering pericentre/apocentre, escapes etc. For change of orbital plane, consider geometry and change in angle of velocity vector. Change in inclination is out of plane burn, change of $\omega$ is radial burn
In general

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos(\Delta\phi)}$$

Where $\phi$ is the change in angle of the velocity vector. For $v_1 = v_2$

$$\Delta v = 2v\sin\left(\frac{\Delta\phi}{2}\right)$$

To find $\Delta\phi$ consider geometry, direction of burn, flight path angle etc.