Functional Minimisation

Can derive lagrange's equations from minimising the functional:

$$J[y] = \int_a^b F(\{y(x)\}, \{\dot{y}(x)\}, x) dx$$

Let $y(x) \to y(x) + \epsilon\eta(x)$, $\dot{y}(x) \to \dot{y}(x) + \epsilon\dot{\eta}(x)$ with $\eta(a) = \eta(b) = 0$
F is at a minimum with $\epsilon = 0$, we therefore have:

$$\left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} = 0$$

$$\frac{dJ}{d\epsilon} = \int_a^b \frac{dF}{d\epsilon} dx = \int_a^b \left(\frac{\partial F}{\partial y}\frac{dy}{d\epsilon} + \frac{\partial F}{\partial \dot{y}}\frac{d\dot{y}}{d\epsilon} \right) dx$$

Using:

$$\frac{dy}{d\epsilon} = \eta, \frac{d\dot{y}}{d\epsilon} = \dot{\eta}$$

We have:

$$0 = \int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial \dot{y}}\dot{\eta} \right) dx$$

Using integration by parts wrt. x on $\frac{\partial F}{\partial \dot{y}}\dot{\eta}$ we get:

$$0 = \int_a^b$$

Beltrami Identity