Functional Minimisation

Can derive Lagrange's equation from minimising the functional:

$$J[y] = \int_a^b F(\{y(x)\}, \{\dot{y}(x)\}, x) dx$$

Let $y(x) \to y(x) + \epsilon\eta(x)$, $\dot{y}(x) \to \dot{y}(x) + \epsilon\dot{\eta}(x)$ with $\eta(a) = \eta(b) = 0$

F is at a minimum with $\epsilon = 0$, we therefore have:

$$\left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} = 0$$

$$\frac{dJ}{d\epsilon} = \int_a^b \frac{dF}{d\epsilon} dx = \int_a^b \left(\frac{\partial F}{\partial y}\frac{dy}{d\epsilon} + \frac{\partial F}{\partial \dot{y}}\frac{d\dot{y}}{d\epsilon}\right) dx$$

Using:

$$\frac{dy}{d\epsilon} = \eta, \frac{d\dot{y}}{d\epsilon} = \dot{\eta}$$

We have:

$$0 = \int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial \dot{y}}\dot{\eta}\right) dx$$

Using integration by parts wrt. x on $\frac{\partial F}{\partial \dot{y}}\dot{\eta}$ we get:

$$0 = \int_a^b \left(\frac{\partial F}{\partial y}\eta - \eta\frac{d}{dx}\frac{\partial F}{\partial \dot{y}}\right) dx + \left[\frac{\partial F}{\partial \dot{y}}\eta\right]_a^b$$

Which becomes (using $\eta(a) = \eta(b) = 0$)

$$0 = \int_a^b \eta\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial \dot{y}}\right) dx$$

With η being any general small displacement, only having requirements at points a, b this means

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial \dot{y}} = 0$$

Which is Lagrange's equation.

Beltrami Identity

Beltrami's Identity is a simplification of Lagrange's equation that can be used when F isn't specifically dependent on x, $F = F(\{q(x)\}, \{\dot{q}(x)\})$

Start with:

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial \dot{y}}$$

Multiply by $\dot{y}$:

$$\dot{y}\frac{\partial F}{\partial y} = \dot{y}\frac{d}{dx}\frac{\partial F}{\partial \dot{y}}$$

Take the derivative of F wrt. x for subbing in:

$$\frac{dF}{dx} = \frac{\partial F}{\partial y}\dot{y} + \frac{\partial F}{\partial \dot{y}}\ddot{y} + \frac{\partial F}{\partial x}$$

Rearrange:

$$\dot{y}\frac{\partial F}{\partial y} = \frac{dF}{dx} - \frac{\partial F}{\partial \dot{y}}\ddot{y} - \frac{\partial F}{\partial x}$$

Sub this in to the left half of the second equation:

$$\frac{dF}{dx} - \frac{\partial F}{\partial \dot{y}}\ddot{y} - \frac{\partial F}{\partial x} = \dot{y}\frac{d}{dx}\frac{\partial F}{\partial \dot{y}}$$

The term on the right hand side can be given by:

$$\dot{y}\frac{d}{dx}\frac{\partial F}{\partial \dot{y}} = \frac{d}{dx}\left(\dot{y}\frac{\partial F}{\partial \dot{y}}\right) - \frac{\partial F}{\partial \dot{y}}\ddot{y}$$

Subbing this in gives:

$$\frac{dF}{dx} - \frac{\partial F}{\partial x} = \frac{d}{dx}\left(\dot{y}\frac{\partial F}{\partial \dot{y}}\right)$$

Since $\frac{\partial F}{\partial x} = 0$, we get:

$$\frac{dF}{dx} - \frac{d}{dx}\left(\dot{y}\frac{\partial F}{\partial \dot{y}}\right) = 0$$

Which can be integrated to give:

$$F - \dot{y}\left(\frac{\partial F}{\partial \dot{y}}\right) = C$$

Where C is a constant.