Dynamics

Relative Motion for Rotating Frame

$$\underline{v} = \frac{d\underline{r}}{dt} = \underline{v}_{o'} + \underline{v}_r + \underline{w} \times \underline{r}$$

$$\frac{d\underline{v}}{dt} = \underline{a}_r + 2\underline{\omega} \times \underline{v}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{a}_{o'} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

Polar Coordinates:

$$\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$$

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$$

Binet's Formula:
For $\underline{a} \times \underline{r} = 0$ we have $a_\theta = 0 \implies r^2\dot{\theta} = c$ which gives

$$a_r = -\frac{c^2}{r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right]$$

Gravity

$$m\underline{a} = -\frac{GmM}{r^2}\hat{\underline{r}} = -\frac{m\mu}{r^2}\hat{\underline{r}}$$

Total Mechanical Energy:

$$E = \frac{1}{2}mv^2 - \frac{m\mu}{r}$$

Angular Momentum:

$$\underline{L} = \underline{r} \times m\underline{v}$$

$$|\underline{L}| = mr^2\dot{\theta} \implies \text{conserved}$$

Keplerian Motion

Two-Body Dynamics

Gravitational Forces:

$$m_i\ddot{\underline{r}}_i = \pm\frac{Gm_im_j}{r^2}\hat{\underline{r}}$$

Centre-of-Mass:

$$\underline{r}_C = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1 + m_2}$$

Sum of forces acting on centre of mass $= 0 \implies \underline{r}_C = \underline{r}_0 + \underline{r}'t$
Approximate $\mu \approx Gm_1$ gives:

$$\ddot{\underline{r}} = -\frac{\mu}{r^2}\hat{\underline{r}}$$

Total energy is conserved, take $\ddot{\underline{r}} \cdot \underline{r}$ to prove $(d/dt)E = 0$, since:

$$\frac{d}{dt}\left(\frac{1}{2}v^2\right) = \ddot{\underline{r}} \cdot \dot{\underline{r}} = \frac{d}{dt}\left(\frac{G(m_1 + m_2)}{r}\right)$$

Equations of Motion

Use chain rule and $r^2\dot{\theta} = l$ to get:

$$\frac{dr}{dt} = \frac{h}{r2}\frac{dr}{d\theta}$$

Change variable to $u = 1/r$:

$$\frac{d}{dt} = -h\frac{du}{d\theta}$$

Gives:

$$\frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2} \qquad \text{and} \qquad -r\dot{\theta}^2 = -h^2u^3$$

Equations of motion in polar coordinates are:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \qquad \text{and} \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

Transforming for u:

$$-l^2u^2\frac{d^2u}{d\theta^2} - l^2u^3 = -\mu u^2$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{l^2}$$

Finally change to $z = u - \mu/l^2$ to get:

$$\frac{d^2z}{d\theta^2} + z = 0 \implies z = A\cos(\theta - \theta_0)$$

Which gives polar equation of conic section for r:

$$r(\theta) = \frac{l^2/\mu}{1 + \frac{Al^2}{\mu}\cos(\theta)}$$

The energy equation is :

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

Which can be solved by integrating w.r.t r, changing variable from t to θ, and substituting $U(r) = m\mu/r$. This gives:

$$r = \frac{p}{\pm 1 + e\cos(\theta - \theta_0)}$$

where

$$p = \frac{c^2}{|\mu|} = a(1 - e^2) \qquad \text{and} \qquad e = \sqrt{\frac{2Ec^2}{m\mu^2} + 1}$$

so r describes an ellipse with semimajor axis a and eccentricity e, pericentre at $\theta = 0$ and apocentre at $\theta = \pi$:

$$r = \frac{a(1 - e^2)}{1 + e\cos(\theta - \theta_0)}$$
