

## Newtonian Mechanics

**N1** - If no forces act on a body, it remains at rest or moves with constant velocity:  $\dot{\underline{v}} = 0$

**N2** -  $\underline{\dot{p}} = \underline{F}$

**N3** -  $\underline{F}_{ab} = -\underline{F}_{ba}$

$$\underline{L} \equiv \underline{r} \times \underline{p}$$

$$W_{BA} \equiv \int_A^B \underline{F} \cdot d\underline{r} = T_B - T_A$$

Orbits (cylindrical polars):

$$\underline{e}_r = \cos \phi \underline{i} + \sin \phi \underline{j}$$

$$\underline{e}_\phi = -\sin \phi \underline{i} + \cos \phi \underline{j}$$

$$\dot{\underline{r}} = \dot{r} \underline{e}_r + r \dot{\phi} \underline{e}_\phi$$

$$E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + V(r)$$

## Newton to Lagrange

Holonomic constraint is an algebraic relation between coordinates:

$$f(\underline{r}_a, \underline{r}_b, \dots, \underline{r}_N; t) = 0$$

For system with N cartesian coordinates  $x_i$ , M constraints, and  $3N - M$  generalised coordinates  $q_i$ , and  $x_i = x_i(\{q\}, t)$

Virtual displacement:

$$\delta x_i = \sum_j \frac{\partial x_i}{\partial q_j} \delta q_j + 0$$

Generalised forces:

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j}$$

for a function  $f = f(\{q\}, \{\dot{q}\}, t)$

$$df = \sum_j \frac{\partial f}{\partial q_j} + \sum_j \frac{\partial f}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial f}{\partial t} dt$$

for a function  $f = f(\{q\}, \{\dot{q}\}, t)$   
cancellation of dots

$$\frac{\partial \dot{f}}{\partial \dot{q}_j} = \frac{\partial f}{\partial q_j}$$

commuting derivatives:

$$\frac{d}{dt} \left( \frac{\partial f}{\partial q_j} \right) = \frac{\partial}{\partial q_j} \left( \frac{df}{dt} \right)$$

Lagrange's equations (general form):

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$$

or

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

where  $\mathcal{L}(\{q\}, \{\dot{q}\}, t) = T(\{q\}, \{\dot{q}\}, t) - V(\{q\}, t)$

## Calculus of Variations

Euler's Equation: for functional of the form

$$I[y(s)] = \int_a^b F(y(s), y'(s), s) ds$$

$$\frac{\partial F}{\partial y} - \frac{d}{ds} \left( \frac{\partial F}{\partial y'} \right) = 0$$

with first integrals

$$\frac{\partial F}{\partial y'} = \text{constant} \quad \text{if } F \text{ does not depend on } y$$

$$y' \frac{\partial F}{\partial y'} - F = \text{constant} \quad \text{if } F \text{ does not depend on } s$$

Hamilton's Principle:

$$\delta S = 0 \quad \text{for the action} \quad S = \int_{t_1}^{t_2} \mathcal{L} dt$$

## Energy Function

The energy function ( $h$ ) is usually conserved, whereas the energy ( $E = T + V$ ) might not be.  $h$  is given by:

$$h = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = \sum_i p_i \dot{q}_i - L$$

$h$  and  $L$  are the legendre transforms of each other on the variables  $p_i$  and  $\dot{q}_i$ :

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \dot{q}_i = \frac{\partial h}{\partial p_i}$$

## Relativistic Particle

The lagrangian for a free relativistic particle can be found by minimising the action  $S$ , then requiring  $\frac{\partial L}{\partial \dot{q}_i} = \gamma m \dot{q}_i$

$$S = \int_{\tau_1}^{\tau_2} \epsilon d\tau = \epsilon \int_{t_1}^{t_2} \frac{1}{dt/d\tau} dt = \epsilon \int_{t_1}^{t_2} \frac{dt}{\gamma(\dot{q})}$$

Finding  $\epsilon$  from relativistic momentum:

$$p_i = \gamma m \dot{q}_i = \frac{\partial L}{\partial \dot{q}_i} = -\frac{\epsilon \gamma \dot{q}_i}{c^2}$$

And we have  $\epsilon = -mc^2$ , so the lagrangian is:

$$L = -\frac{mc^2}{\gamma}$$