Ising Model

Configurational energy or Hamiltonian for array of spins

$$E(\{S_i\}) = -h\sum_i S_i - J\sum_{<ij>} S_i S_j$$

<> is the sum over nearest neighbours. There are z nearest neighbours per site, and $\frac{Nz}{2}$ nearest neighbours total for N spin sites

1 Mean Field Approximation

energy contributions with S_j only is

$$\epsilon(S_j) = -hS_j - JS_j\sum_k^{\pm nn} S_k$$

then replace every S_k with their *mean* values

$$\epsilon_{mf}(S_j) \approx -hS_j - JS_j\sum_k^{\pm nn} \langle S_k\rangle$$

m is the magnetisation order parameter and = the mean spin

$$\epsilon_{mf}(S_j) = -(h + Jzm)S_j = -h_{mf}S_j$$

which gives single spin Boltzmann distribution

$$p(S_j) = \frac{e^{-\beta\epsilon_{mf}(S_i)}}{\sum_{S_j=\pm 1} e^{-\beta\epsilon_{mf}(S_j)}} = \frac{e^{\beta h_{mf}S_j}}{e^{\beta h_{mf}} + e^{-\beta h_{mf}}}$$

2 Consistency Condition

although m is the average spin, it must be recovered by using the derived probability distribution, which gives the *mean field equation*

$$m = \sum_{S_j=\pm 1} p(S_j)S_j = \frac{e^{-\beta\epsilon_{mf}} - e^{\beta h_{mf}}}{e^{\beta h_{mf}} + e^{-\beta h_{mf}}} = \tanh(\beta h + \beta Jzm)$$

for $h = 0$, and using $\tanh(x) \approx x - \frac{x^3}{3}$ for small x

$$m = \beta Jzm + \mathcal{O}(m^3)$$

solutions with $|m| > 0$ appear when the gradient of the tanh function at the origin is greater than 1, which gives critical temperature

$$T_c = \frac{zJ}{k}$$

for $T > T_c$ only have $m = 0$, while for $T < T_c$ have ferromagnetic phase with $\pm|m|$

Critical Behaviour: $T \approx T_c$ and $h = 0$

$$m = \tanh\left(m\frac{T_c}{T}\right) \approx m\frac{T_c}{T} - \frac{m^3}{3}\left(\frac{T_c}{T}\right)^3$$

implies $m = 0$ or

$$m^2 = 3\left(\frac{T}{T_c}\right)^3\left(\frac{T_c}{T} - 1\right)$$

define $t = \frac{T-T_c}{T_c}$

$$m^2 = 3(1+t)^3\left(\frac{1}{1+t} - 1\right) \sim -3t$$

for $h \neq 0$ expand to first order in h

$$m = m\frac{T_c}{T} + \beta h - \frac{m^3}{3}\left(\frac{T_c}{T}\right)^3$$

susceptibility χ is found by taking derivative w.r.t h

$$\chi = \chi\frac{T_c}{T} + \beta - \chi m^2\left(\frac{T_c}{T}\right)^3$$

which gives the critical behaviour

$$\chi \sim \frac{\beta_c}{t} \qquad \text{for } T > T_c$$

$$\chi \sim \frac{\beta_c}{2|t|} \qquad \text{for } T < T_c$$
