Newtonian Mechanics

N1 - *If no forces act on a body, it remains at rest or moves with constant velocity:* $\dot{\underline{v}} = 0$

N2 - $\dot{\underline{p}} = \underline{F}$

N3 - $\underline{F_{ab}} = -\underline{F_{ba}}$

$$\underline{L} \equiv \underline{r} \times \underline{p}$$

$$W_{BA} \equiv \int_A^B \underline{F} \cdot d\underline{r} = T_B - T_A$$

Orbits (cylindrical polars):

$$\underline{e_r} = \cos\phi \underline{i} + \sin\phi \underline{j}$$

$$\underline{e_\phi} = -\sin\phi \underline{i} + \cos\phi \underline{j}$$

$$\dot{\underline{r}} = \dot{r}\underline{e_r} + r\dot{\phi}\underline{e_\phi}$$

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r)$$

Newton to Lagrange

Holonomic constraint is an algebraic relation between coordinates:

$$f(\underline{r_a}, \underline{r_b}, ..., \underline{r_N}; t) = 0$$

For system with N cartesian cordinates x_i, M constraints, and $3N - M$ generalised coordinates q_i, and $x_i = x_i(\{q\}, t)$

Virtual displacement:

$$\delta x_i = \sum_i \frac{\partial x_i}{\partial q_j}\delta q_j + 0$$

Generalised forces:

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j}$$

for a function $f = f(\{q\}, \{\dot{q}\}, t)$

$$df = \sum_j \frac{\partial f}{\partial q_j}dq_j + \sum_j \frac{\partial f}{\partial \dot{q}_j}d\dot{q}_j + \frac{\partial f}{\partial t}dt$$

---

for a function $f = f(\{q\}, \{\cancel{\dot{q}}\}, t)$
cancellation of dots

$$\frac{\partial \dot{f}}{\partial \dot{q}_j} = \frac{\partial f}{\partial q_j}$$

commuting derivatives:

$$\frac{d}{dt}\left(\frac{\partial f}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{df}{dt}\right)$$

Lagrange's equations (general form):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

or

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

where $\mathcal{L}(\{q\}, \{\dot{q}\}.t) = T(\{q\}.\{\dot{q}\}, t) - V(\{q\}, t)$

Calculus of Variations

Euler's Equation: for functional of the form

$$I[y(s)] = \int_a^b F(y(s), y'(s), s)ds$$

$$\frac{\partial F}{\partial y} - \frac{d}{ds}\left(\frac{\partial F}{\partial y'}\right) = 0$$

with first integrals

$\frac{\partial F}{\partial y'} = \text{constant}$ if F does not depend on y

$y'\frac{\partial F}{\partial y'} - F = \text{constant}$ if F does not depend on s

Hamilton's Principle:

$$\delta S = 0 \quad \text{for the } \textit{action} \quad S = \int_{t_1}^{t_2} \mathcal{L}dt$$

Energy Function

The energy function (h) is usually conserved, whereas the energy ($E = T + V$) might not be. h is given by:

$$h = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \sum_i p_i\dot{q}_i - L$$

time translational symmetry of the Lagrangian $\implies$ conservation of h
h and L are the legendre transforms of each other on the variables p_i and $\dot{q}_i$:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \dot{q}_i = \frac{\partial h}{\partial p_i}$$

The Hamiltonian H is numerically equal to energy function, but is a function of canonical momenta instead of $\dot{q}$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial}$$

Relativistic Particle

The lagrangian for a free relativistic particle can be found by minimising the action S, then requiring $\frac{\partial L}{\partial \dot{q}_i} = \gamma m \dot{q}_i$

$$S = \int_{\tau_1}^{\tau_2} \epsilon d\tau = \epsilon \int_{t_1}^{t_2} \frac{1}{dt/d\tau}dt = \epsilon \int_{t_1}^{t_2} \frac{dt}{\gamma(\dot{q})}$$

Finding ϵ from relativistic momentum:

$$p_i = \gamma m \dot{q}_i = \frac{\partial L}{\partial \dot{q}_i} = -\frac{\epsilon \gamma \dot{q}_i}{c^2}$$

And we have $\epsilon = -mc^2$, so the lagrangian is:

$$L = -\frac{mc^2}{\gamma}$$

Poisson Brackets

$$\{A, B\}_{PB} = \sum_i \left[\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right]$$

Rotating Rigid Bodies

For system of N particles, with constraints such that distance between each pair of particles is fixed

$$\dot{\vec{L}} = \vec{G}^{ext}$$

$$\vec{L} = \vec{J} + \vec{R} \times \vec{P}$$

Eulers theorem states *Any displacement of a rigid body with one point fixed in space can be described as a rotation about some single axis*
Principle axes are described by the three mutually perpendicular eigenvectors of the Inertia tensor
Euler's Equations of Motion are

$$G_1 = I_1 \dot{\omega_1} + (I_1 - I_2)\omega_2\omega_3$$

$$G_2 = I_2 \dot{\omega_2} + (I_1 - I_3)\omega_3\omega_1$$

$$G_3 = I_3 \dot{\omega_3} + (I_2 - I_1)\omega_1\omega_2$$

The lagrangian for a rotating symmetric top is given by:

$$\mathcal{L} = \frac{1}{2}[A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + C(\dot{\phi}\cos\theta + \dot{\psi})^2] - Mgl\cos\theta$$

With $A = I_1 = I_2$ and $C = I_3$. This won't need to be derived in an exam - it'll be given or will need to be quoted.

Conservation Laws

$$p_\psi \equiv \frac{\partial L}{\partial \dot{\psi}} = C(\dot{\psi} + \dot{\phi}\cos\theta) = I_3\omega_3$$

"spin" of top is defined as

$$n \equiv \omega_3 = (\dot{\psi} + \dot{\phi}\cos\theta)$$

$$p_\phi = A\dot{\phi}\sin^2\theta + Cn\cos\theta = L_z$$

$$p_\theta = A\dot{\theta} \qquad and \dot{p_\theta} = \frac{\partial L}{\partial \theta}$$

leads to equation of motion for θ
Lagrangian does not depend on time implies energy function is conserved, and T is kinetic in velocities so h is the same as $T + V$