Dynamics

Relative Motion for Rotating Frame

$$\underline{v} = \frac{d\underline{r}}{dt} = \underline{v}_{o'} + \underline{v}_r + \underline{w} \times \underline{r}$$

$$\frac{d\underline{v}}{dt} = \underline{a}_r + 2\underline{\omega} \times \underline{v}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{a}_{o'} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

Polar Coordinates:

$$\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$$

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$$

Binet's Formula:
For $\underline{a} \times \underline{r} = 0$ we have $a_\theta = 0 \implies r^2\dot{\theta} = c$ which gives

$$a_r = -\frac{c^2}{r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right]$$

Gravity

$$m\underline{a} = -\frac{GmM}{r^2}\hat{\underline{r}} = -\frac{m\mu}{r^2}\hat{\underline{r}}$$

Total Mechanical Energy:

$$E = \frac{1}{2}mv^2 - \frac{m\mu}{r}$$

Angular Momentum:

$$\underline{L} = \underline{r} \times m\underline{v}$$

$$|\underline{L}| = mr^2\dot{\theta} \implies \text{conserved}$$

Keplerian Motion

Two-Body Dynamics

Gravitational Forces:

$$m_i\ddot{\underline{r}}_i = \pm\frac{Gm_im_j}{r^2}\hat{\underline{r}}$$

Centre-of-Mass:

$$\underline{r}_C = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1 + m_2}$$

Sum of forces acting on centre of mass $= 0 \implies \underline{r}_C = \underline{r}_0 + \underline{r}'t$
Approximate $\mu \approx Gm_1$ gives:

$$\ddot{\underline{r}} = -\frac{\mu}{r^2}\hat{\underline{r}}$$

Total energy is conserved, take $\ddot{\underline{r}} \cdot \underline{r}$ to prove $(d/dt)E = 0$, since:

$$\frac{d}{dt}\left(\frac{1}{2}v^2\right) = \ddot{\underline{r}} \cdot \dot{\underline{r}} = \frac{d}{dt}\left(\frac{G(m_1+m_2)}{r}\right)$$

Equations of Motion

Use chain rule and $r^2\dot{\theta} = l$ to get:

$$\frac{dr}{dt} = \frac{h}{r2}\frac{dr}{d\theta}$$

Change variable to $u = 1/r$:

$$\frac{d}{dt} = -h\frac{du}{d\theta}$$

Gives:

$$\frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2} \qquad \text{and} \qquad -r\dot{\theta}^2 = -h^2u^3$$

Equations of motion in polar coordinates are:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \qquad \text{and} \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

Transforming for u:

$$-l^2u^2\frac{d^2u}{d\theta^2} - l^2u^3 = -\mu u^2$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{l^2}$$

Finally change to $z = u - \mu/l^2$ to get:

$$\frac{d^2z}{d\theta^2} + z = 0 \implies z = A\cos(\theta - \theta_0)$$

Which gives polar equation of conic section for r:

$$r(\theta) = \frac{l^2/\mu}{1 + \frac{Al^2}{\mu}\cos(\theta)}$$

The energy equation is :

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

Which can be solved by integrating w.r.t r, changing variable from t to θ, and substituting $U(r) = m\mu/r$. This gives:

$$r = \frac{p}{\pm 1 + e\cos(\theta - \theta_0)}$$

where

$$p = \frac{c^2}{|\mu|} = a(1 - e^2) \qquad \text{and} \qquad e = \sqrt{\frac{2Ec^2}{m\mu^2} + 1}$$

so r describes an ellipse with semimajor axis a and eccentricity e, pericentre at $\theta = 0$ and apocentre at $\theta = \pi$:

$$r = \frac{a(1 - e^2)}{1 + e\cos(\theta - \theta_0)}$$

Orbital Parameters

$$a = \frac{r_p + r_a}{2} \qquad \text{and} \qquad e = \frac{r_a - r_p}{r_a + r_p}$$

so

$$r_p = a(1-e) \qquad \text{and} \qquad r_a = a(1+e)$$

Comparing equations for r:

$$l^2 = \mu a(1-e^2)$$

$$v_p = \sqrt{\frac{\mu(1+e)}{a(1-e)}} \qquad \text{and} \qquad v_a = \sqrt{\frac{\mu(1-e)}{a(1+e)}}$$

Conservation of energy:

$$E = -\frac{\mu}{2a} \implies v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

Kepler's Laws

1st Law: Orbits are elliptical with attracting body at focus of ellipse

2nd Law: Radial vector sweeps out equal areas in equal time

$$\implies T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

3rd Law: Orbital period squared $\propto$ semi-major axis cubed:

Kepler's Equation

Mean motion

$$n = \sqrt{\frac{\mu}{a^3}}$$

Mean anomoly

$$M = n(t-\tau)$$

Kepler's equation is

$$M = E - e\sin E$$

and use trig identity to get θ from E

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$$

This must be solved iteratively, start with guess $E_0 = M$, then find correction

$$\Delta E = \frac{M - E_0 + e\sin E_0}{1 - e\cos E_0}$$

Then repeat, using $E_i = E_{i-1} + \Delta E_{i-1}$ until $\Delta E_i \approx 0$

Position and Velocity from Orbital Elements

From orbit equation

$$\dot{r} = \frac{a(1-e^2)}{(1+e\cos\theta)^2} e\sin\theta\dot{\theta} = r\frac{e\sin\theta}{1+e\cos\theta}\dot{\theta}$$

Flight path angle

$$\tan\gamma = \frac{v_r}{v_\theta} = \frac{\dot{r}}{r\dot{\theta}} = \frac{e\sin\theta}{1+e\cos\theta}$$

To get in plane position and velocity at time t, solve Kepler's equation for τ time of passage from pericentre, then compute orbit radius and thus cartesian positions.

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

Velocity components:

$$v_r = \frac{r^2 e\sin\theta}{p}\dot{\theta} \qquad \text{and} \qquad v_\theta = \frac{l}{r} = r\dot{\theta}$$

Gives cartesian velocities:

$$\dot{x} = -\sqrt{\frac{\mu}{p}}\sin\theta \qquad \text{and} \qquad \dot{y} = \sqrt{\frac{\mu}{p}}(e + \cos\theta)$$