Dynamics

Relative Motion for Rotating Frame

$$\underline{v} = \frac{d\underline{r}}{dt} = \underline{v}_{o'} + \underline{v}_r + \underline{w} \times \underline{r}$$

$$\frac{d\underline{v}}{dt} = \underline{a}_r + 2\underline{\omega} \times \underline{v}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{a}_{o'} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

Polar Coordinates:

$$\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$$

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$$

Binet's Formula:
For $\underline{a} \times \underline{r} = 0$ we have $a_\theta = 0 \implies r^2\dot{\theta} = c$ which gives

$$a_r = -\frac{c^2}{r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right]$$

Gravity

$$m\underline{a} = -\frac{GmM}{r^2}\underline{\hat{r}} = -\frac{m\mu}{r^2}\underline{\hat{r}}$$

Total Mechanical Energy:

$$\boldsymbol{E = \frac{1}{2}mv^2 - \frac{m\mu}{r}}$$

Angular Momentum:

$$\underline{L} = \underline{r} \times m\underline{v}$$

$$|\underline{L}| = mr^2\dot{\theta} \implies \text{conserved}$$

Keplerian Motion

Two-Body Dynamics

Gravitational Forces:

$$m_i\underline{\ddot{r}}_i = \pm\frac{Gm_im_j}{r^2}\underline{\hat{r}}$$

Centre-of-Mass:

$$\underline{r}_C = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1 + m_2}$$

Sum of forces acting on centre of mass $= 0 \implies \underline{r}_C = \underline{r}_0 + \underline{r}'t$
Approximate $\mu \approx Gm_1$ gives:

$$\boldsymbol{\underline{\ddot{r}} = -\frac{\mu}{r^2}\underline{\hat{r}}}$$

Total energy is conserved, take $\underline{\ddot{r}} \cdot \underline{r}$ to prove $(d/dt)E = 0$, since:

$$\frac{d}{dt}\left(\frac{1}{2}v^2\right) = \underline{\ddot{r}} \cdot \underline{\dot{r}} = \frac{d}{dt}\left(\frac{G(m_1 + m_2)}{r}\right)$$

Equations of Motion

Use chain rule and $r^2\dot{\theta} = l$ to get:

$$\frac{dr}{dt} = \frac{l}{r^2}\frac{dr}{d\theta}$$

Change variable to $u = 1/r$:

$$\frac{dr}{dt} = -l\frac{du}{d\theta}$$

Gives:

$$\frac{d^2r}{dt^2} = -l^2u^2\frac{d^2u}{d\theta^2} \quad \text{and} \quad -r\dot{\theta}^2 = -l^2u^3$$

Equations of motion in polar coordinates are:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \quad \text{and} \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

Transforming for u:

$$-l^2u^2\frac{d^2u}{d\theta^2} - l^2u^3 = -\mu u^2$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{l^2}$$

Finally change to $z = u - \mu/l^2$ to get:

$$\frac{d^2z}{d\theta^2} + z = 0 \implies z = A\cos(\theta - \theta_0)$$

Which gives polar equation of conic section for r:

$$r(\theta) = \frac{l^2/\mu}{1 + \frac{Al^2}{\mu}\cos(\theta)}$$

The energy equation is :

$$\boldsymbol{E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)}$$

Which can be solved by integrating w.r.t r, changing variable from t to θ, and substituting $U(r) = m\mu/r$. This gives:

$$r = \frac{p}{\pm 1 + e\cos(\theta - \theta_0)}$$

where

$$\boldsymbol{p = \frac{c^2}{|\mu|} = a(1 - e^2)} \quad \text{and} \quad \boldsymbol{e = \sqrt{\frac{2Ec^2}{m\mu^2} + 1}}$$

so r describes an ellipse with semimajor axis a and eccentricity e, pericentre at $\theta = 0$ and apocentre at $\theta = \pi$:

$$r = \frac{a(1 - e^2)}{1 + e\cos(\theta - \theta_0)}$$

Orbital Parameters

$$a = \frac{r_p + r_a}{2} \quad \text{and} \quad e = \frac{r_a - r_p}{r_a + r_p}$$

so

$$r_p = a(1-e) \quad \text{and} \quad r_a = a(1+e)$$

Comparing equations for r:

$$\boldsymbol{l^2 = \mu a(1-e^2)}$$

$$v_p = \sqrt{\frac{\mu(1+e)}{a(1-e)}} \quad \text{and} \quad v_a = \sqrt{\frac{\mu(1-e)}{a(1+e)}}$$

Conservation of energy:

$$\boldsymbol{E = -\frac{\mu}{2a} \implies v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Kepler's Laws

1st Law: Orbits are elliptical with attracting body at focus of ellipse

2nd Law: Radial vector sweeps out equal areas in equal time

$$\implies T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

3rd Law: Orbital period squared $\propto$ semi-major axis cubed:

Kepler's Equation

Mean motion

$$n = \sqrt{\frac{\mu}{a^3}}$$

Mean anomoly

$$M = n(t - \tau)$$

Kepler's equation is

$$M = E - e\sin E$$

and use trig identity to get θ from E

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$$

This must be solved iteratively, start with guess $E_0 = M$, then find correction

$$\Delta E = \frac{M - E_0 + e\sin E_0}{1 - e\cos E_0}$$

Then repeat, using $E_i = E_{i-1} + \Delta E_{i-1}$ until $\Delta E_i \approx 0$

Position and Velocity from Orbital Elements

From orbit equation

$$\boldsymbol{\dot{r} = \frac{a(1-e^2)}{(1+e\cos\theta)^2} e\sin\theta\dot{\theta} = r\frac{e\sin\theta}{1+e\cos\theta}\dot{\theta}}$$

Flight path angle

$$\tan\gamma = \frac{v_r}{v_\theta} = \frac{\dot{r}}{r\dot{\theta}} = \frac{e\sin\theta}{1+e\cos\theta}$$

To get in plane position and velocity at time t, solve Kepler's equation for τ time of passage from pericentre, then compute orbit radius and thus cartesian positions.

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Velocity components:

$$\boldsymbol{v_r = \frac{r^2 e\sin\theta}{p}\dot{\theta}} \quad \text{and} \quad \boldsymbol{v_\theta = \frac{l}{r} = r\dot{\theta}}$$

Gives cartesian velocities:

$$\dot{x} = -\sqrt{\frac{\mu}{p}}\sin\theta \quad \text{and} \quad \dot{y} = \sqrt{\frac{\mu}{p}}(e + \cos\theta)$$

Orbit Transfers

Hohmann Transfer

Minimum energy transfer between two circular, coplanar orbits is an ellipse with semimajor axis $a = \frac{1}{2}r_1 + r_2$. The change in velocities required are the difference between the velocities of the transfer ellipse at pericentre and apocentre, and the velocities of the circular orbits at r_1 and r_2

$$\Delta v_1 = \sqrt{\frac{2\mu}{r_1} - \frac{2\mu}{r_1 + r_2}} - \sqrt{\frac{\mu}{r_1}}$$

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{2\mu}{r_1 + r_2}}$$

Parabolic Transfer

For a parabolic transfer where the orbiting body escapes circular orbit at r_1 to infinity, and is then captured from infinity into orbit at r_2

$$\Delta v_1 = (\sqrt{2} - 1)\sqrt{\frac{\mu}{r_1}}$$

$$\Delta v_2 = (\sqrt{2} - 1)\sqrt{\frac{\mu}{r_2}}$$

Single Impulse Manoeuvres

Raise apocentre:

$$\Delta v = \sqrt{-\frac{\mu}{a_2} + \frac{2\mu}{r_p}} - \sqrt{-\frac{\mu}{a_1} + \frac{2\mu}{r_p}}$$

Change inclination:

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos \Delta i}$$

If $v_1 = v_2$ then

$$\Delta v = 2v \sin\left(\frac{\Delta i}{2}\right)$$

Multiple impulse manoeuvres can be used to reduce required delta v, by first raising apocentre and performing multiple burns.

Patched Conic

In Earth-centred frame, velocity to escape is:

$$\Delta v_{esc} = \sqrt{\frac{2\mu_{Earth}}{r_p} + v_{\infty,1}^2} - \sqrt{\frac{2\mu_{Earth}}{r_p} - \frac{\mu_{Earth}}{a_p}}$$

In sun centre frame, $v_{\infty,1}$ must match the first Δv for transfer. E.g. Δv_1 of Hohmann transfer, and similar for capture.

Rocket Equation

Specific impulse

$$I_{sp} = \frac{F}{\dot{m} g_0}$$

Effective exhaust velocity

$$c = I_{sp} g_0$$

Rocket equation

$$(m - dm)dv = -dm v_e$$

$$\Delta v = v_e \ln \frac{m_i}{m_f}$$

$$m_f = m_i e^{-\frac{\Delta v}{v_e}}$$

Aerobraking

$$D = \frac{1}{2}\rho C_D A v^2$$

$$L = \frac{1}{2}\rho C_L A v^2$$

Flight path angle γ is measured to horizontal

$$m\frac{dv}{dt} = -D - mg \sin\gamma$$

$$mv\frac{d\gamma}{dt} = L - mg\cos\gamma$$

Atmosphere density is exponential model, relative to a scale height H and base density ρ_0

$$\rho(h) = \rho_0 \exp\left[-\frac{h}{H}\right]$$

For aerocapture, assume gravity terms negligible

$$\frac{1}{v}\frac{dv}{d\gamma} = -\frac{D}{L} = -\frac{C_D}{C_L}$$

Integrate to get

$$\ln\left(\frac{v_2}{v_1}\right) = -\frac{C_D}{C_L}(\gamma_2 - \gamma_1)$$

$$\Delta v = v_1\left(1 - \exp\left[2\frac{C_D}{C_L}\gamma_1\right]\right)$$

Vertical Dynamics

$$\frac{dh}{dt} = v\sin\gamma = \frac{dh}{d\gamma}\frac{d\gamma}{dt} = \frac{dh}{d\gamma}\frac{L}{mv}$$

Substituting for L and using density function

$$\frac{dh}{d\gamma} = \frac{2m\sin\gamma}{\rho_0 C_L A}\exp\left[\frac{h}{H}\right]$$

Integrate to find h_{min}

$$\int_{h_1}^{h_{min}} \exp\left(-\frac{h}{H}\right) = \frac{2m}{\rho_0 C_L A}\int_{\gamma_1}^{0} \sin\gamma d\gamma$$

Note that $h_1 >> H$ and so

$$h_{min} = H \ln\left(\frac{\rho_0 C_L A H}{2m(1-\cos\gamma_1)}\right)$$

Important Things

1) Derive Orbital Energy

- Derive Energy for 2-Body Problem

$$E = T + V$$

$$E = \frac{1}{2}m_1 v^2 + V(r)$$

$$\underline{F} = -\frac{m\mu}{r^2}\underline{\hat{r}} = -\underline{\nabla}V(r)$$

$$\implies V(r) = -\frac{m\mu}{r}$$

$$E = \frac{1}{2}mv^2 - \frac{m\mu}{r}$$

- Demonstrate Energy Constant

Take time derivative of $1/2v^2$

$$\frac{d}{dt}\left(\frac{1}{2}\underline{\dot{r}}\cdot\underline{\dot{r}}\right) = \underline{\ddot{r}}\cdot\underline{\dot{r}}$$

Take dot product of equation of motion with velocity vector

$$\underline{\ddot{r}}\cdot\underline{\dot{r}} - \frac{G(m_1+m_2)}{r^2}(\underline{\hat{r}}\cdot\underline{\dot{r}})$$

$$\implies \frac{d}{dt}\left(\frac{1}{2}v^2\right) = -\frac{G(m_1+m_2)\dot{r}}{r^2} = \frac{d}{dt}\left(\frac{G(m_1+m_2)}{r}\right)$$

$$\implies \frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{G(m_1+m_2)}{r}\right) = 0$$

This constant of motion is the energy

$$E = \frac{1}{2}v^2 - \frac{G(m_1+m_2)}{r}$$

2) Energy from Orbit Geometry

$$r = \frac{p}{\pm 1 + e\cos(\theta - \theta_0)}$$

$$p = \frac{c^2}{\mu} = \frac{r^4\dot{\theta}^2}{\mu} = a(1-e^2) \qquad \text{and} \qquad e = \sqrt{\frac{2Ec^2}{m\mu^2}+1}$$

$$a(1-e^2) = \frac{c^2}{\mu} = a\left(1 - \frac{2Ec^2}{m\mu^2}+1\right)$$

$$\implies E = -\frac{m\mu}{2a}$$

$$E = \frac{-m\mu}{2a} = \frac{1}{2}mv^2 - \frac{m\mu}{r}$$

$$\implies v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

3) Demonstrate Angular Momentum is Constant

Definition of Angular Momentum

$$\underline{L} = \underline{r} \times m\underline{\dot{r}}$$

Take cross of position vector with equation of motion

$$\underline{r} \times \underline{\ddot{r}} = -\frac{\mu}{r^2}(\underline{r} \times \underline{\hat{r}}) = 0$$

Take time derivative of angular momentum

$$\frac{d}{dt}(\underline{r} \times \underline{\dot{r}}) = \underline{\dot{r}} \times \underline{\dot{r}} + \underline{r} \times \underline{\ddot{r}} = 0$$

$\implies$ Angular momentum conserved

4) Kepler's Laws

1. Orbits are elliptical with attracting body at focus of ellipse
2. Radial vector sweeps out equal areas in equal time
3. Orbital period squared is $\propto$ semi-major axis cubed

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Mean motion

$$n = \sqrt{\frac{\mu}{a^3}}$$

5) Equation of Orbit

$$r = \frac{p}{1+e\cos\theta}$$

$$p = \frac{c^2}{|\mu|} = a(1-e^2) \qquad \text{and} \qquad e = \sqrt{\frac{2Ec^2}{m\mu^2}+1}$$

6) Minimum Aerobraking Altitude

$$L = mv\frac{d\gamma}{dt} \qquad \text{neglecting gravity}$$

$$\rho(h) = \rho_0 \exp\left[-\frac{h}{H}\right]$$

$$\frac{dh}{dt} = v\sin\gamma = \frac{dh}{d\gamma}\frac{d\gamma}{dt} = \frac{dh}{d\gamma}\frac{L}{mv}$$

Substituting for L and using density function

$$\frac{dh}{d\gamma} = \frac{2m\sin\gamma}{\rho_0 C_L A}\exp\left[\frac{h}{H}\right]$$

Integrate to find h_{min}

$$\int_{h_1}^{h_{min}} \exp\left(-\frac{h}{H}\right) = \frac{2m}{\rho_0 C_L A}\int_{\gamma_1}^{0} \sin\gamma d\gamma$$

Note that $h_1 >> H$ and so

$$h_{min} = H\ln\left(\frac{\rho_0 C_L A H}{2m(1-\cos\gamma_1)}\right)$$

7) Rocket Equation

Conservation of linear momentum, momentum gained by rocket = −momentum of exhaust gas

$$(m-dm)dv = -dmv_e$$

Note that $dmdv \approx 0$

$$mdv = -v_e dm$$

$$\Delta v = c\ln\frac{mi}{mf}$$

$$m_f = m_i e^{-\frac{\Delta v}{c}}$$

Where c is effective exhaust velocity

$$c = I_{sp}g_0$$

8) Δv Manoeuvres

Find difference between velocity of initial orbit and velocity of target orbit using

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

True for Hohmann, raising/lowering pericentre/apocentre, escapes etc. For change of orbital plane, consider geometry and change in angle of velocity vector. Change in inclination is out of plane burn, change of ω is radial burn
In general

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos(\Delta\phi)}$$

Where ϕ is the change in angle of the velocity vector. For $v_1 = v_2$

$$\Delta v = 2v\sin\left(\frac{\Delta\phi}{2}\right)$$

To find $\Delta\phi$ consider geometry, direction of burn, flight path angle etc.