Dynamics

Relative Motion for Rotating Frame

$$\underline{v} = \frac{d\underline{r}}{dt} = \underline{v}_{o'} + \underline{v}_r + \underline{w} \times \underline{r}$$

$$\frac{d\underline{v}}{dt} = \underline{a}_r + 2\underline{\omega} \times \underline{v}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{a}_{o'} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

Polar Coordinates:

$$\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$$

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$$

Binet's Formula:
For $\underline{a} \times \underline{r} = 0$ we have $a_\theta = 0 \implies r^2\dot{\theta} = c$ which gives

$$a_r = -\frac{c^2}{r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right]$$

Gravity

$$m\underline{a} = -\frac{GmM}{r^2}\underline{\hat{r}} = -\frac{m\mu}{r^2}\underline{\hat{r}}$$

Total Mechanical Energy:

$$E = \frac{1}{2}mv^2 - \frac{m\mu}{r}$$

Angular Momentum:

$$\underline{L} = \underline{r} \times m\underline{v}$$

$$|\underline{L}| = mr^2\dot{\theta} \implies \text{conserved}$$

Keplerian Motion

Two-Body Dynamics

Gravitational Forces:

$$m_i\underline{\ddot{r}}_i = \pm\frac{Gm_im_j}{r^2}\underline{\hat{r}}$$

Centre-of-Mass:

$$\underline{r}_C = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1 + m_2}$$

Sum of forces acting on centre of mass $= 0 \implies \underline{r}_C = \underline{r}_0 + \underline{r}'t$
Approximate $\mu \approx Gm_1$ gives:

$$\underline{\ddot{r}} = -\frac{\mu}{r^2}\underline{\hat{r}}$$

Total energy is conserved, take $\underline{\ddot{r}} \cdot \underline{r}$ to prove $(d/dt)E = 0$, since:

$$\frac{d}{dt}\left(\frac{1}{2}v^2\right) = \underline{\ddot{r}} \cdot \underline{\dot{r}} = \frac{d}{dt}\left(\frac{G(m_1+m_2)}{r}\right)$$
