Categorical Logic

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1 Simply Typed Logics

Here we present the setup for a logic of simply typed lambda calculus term equality.

1.1 Syntax

$$
\begin{array}{llcl}
\text{Types} & \alpha & ::= & \text{Unit} \\
 & & \mid & \alpha \times \alpha \\
 & & \mid & \alpha \to \alpha \\
\text{Terms} & M & ::= & tt \\
 & & \mid & (M, N) \\
 & & \mid & \pi_1 M \\
 & & \mid & \pi_2 M \\
 & & \mid & \lambda(x\colon \alpha).M \\
 & & \mid & MN
\end{array}
$$

With the usual typing rules, structural rules, and beta/eta equalities.

1.2 Classifying Category

This category represents the syntax of the simply typed lambda calculus. (not just β/η equivalence? conversion can have α renaming or congruence rules?)

$$
\begin{aligned}
\text{Objects} &:= \text{Types, } \alpha\text{, constructed by the grammar} \\
\text{Morphisms} &:= \alpha \to \beta \text{ are equivalence classes } [M] \text{ of terms } x : \alpha \vdash M : \beta \\
&\quad \text{with one free variable(Equivalence relative to } \beta/\eta \text{ laws)} \\
\text{Identity } \alpha &:= \text{equivalence class of term } x : \alpha \vdash x : \alpha \\
\text{Composition } fg &:= \text{ is substitution. Say } f \text{ contains } x : \alpha \vdash M : \beta \\
&\quad \text{and g contains } y : \beta \vdash N : \gamma \text{ then } x : \alpha \vdash N[M/y] : \gamma \text{ in g} \circ \text{f.}
\end{aligned}
$$

This category, denoted here by $\mathcal{CL}(\lambda_{1,\times,\to})$, is cartesian closed.

1.3 Term Model

A **model** of the term language in a cartesian closed category $\mathcal{C}$ is a functor from the classifying category into $\mathcal{C}$

$$M : \mathcal{CL}(\lambda_{1,\times,\to}) \to \mathcal{C}$$

1.4 Base Logic Syntax

We now turn out attention to a logic where the only proposition is equality of simply typed lambda calculus terms. We have two judgements:

$$\Gamma \vdash \phi \text{ Prop}$$

which states ϕ is a proposition. And

$$\Gamma | \Phi \vdash \psi$$

which is a proof derivation judgement. There are a few structural and context rules for proof derivations

- Axiom
- Identity
- Cut
- weakening (for prop)
- contraction (for prop)
- exchange (for prop)
- weakening (for term context)
- contraction (for term context)
- exchange (for term context)
- substitution

1.5 Equational Logic Syntax

Here are the rules for the equality proposition. For the first judgement, ϕ is generated from a boring bnf grammar:

$\phi \quad ::= \quad M =_\alpha M'$

There is one formation rule for propositions

$$\frac{\Gamma \vdash M : \alpha \qquad \Gamma \vdash M' : \alpha}{\Gamma \vdash M =_\alpha M' \text{ Prop}}$$

You have the expected rules for $=$.

$$\frac{\Gamma \vdash M : \alpha}{\Gamma|\Phi \vdash M =_\alpha M} \text{ refl}$$

$$\frac{\Gamma|\Phi \vdash M =_\alpha M'}{\Gamma|\Phi \vdash M' =_\alpha M} \text{ sym}$$

$$\frac{\Gamma|\Phi \vdash M =_\alpha N \qquad \Gamma|\Phi \vdash N =_\alpha P}{\Gamma|\Phi \vdash M =_\alpha P} \text{ trans}$$

$$\frac{\Gamma|\Phi \vdash M =_\alpha M' \qquad \Gamma, x : \alpha \vdash N : \beta}{\Gamma|\Phi \vdash N[M/x] =_\beta N[M'/x]} \text{ replace}$$

The four previous rules are equivalent to the following two. For proof, see page 180 of Jacobs.

$$\frac{\Gamma \vdash M : \alpha \qquad \Delta, x : \alpha, \Delta' \mid \overrightarrow{N} =_{\overrightarrow{\beta}} \overrightarrow{N'} \vdash L =_\gamma L'}{\Delta, \Gamma, \Delta' \mid \overrightarrow{N[M/x]} =_{\overrightarrow{\beta}} \overrightarrow{N'[M/x]} \vdash L[M/x] =_\gamma L'[M/x]} \text{ substitution}$$

$$\frac{\Gamma, x : \alpha|\Phi \vdash N[x/y] =_\beta N'[x/y]}{\Gamma, x : \alpha, y : \alpha|\Phi, x =_\alpha y \vdash N =_\beta N'} \text{ Lawvere Equality, =-mate}$$

Additionally, we also have that definitional equality / conversion is contained in propositional equality.

$$\frac{\Gamma \vdash M : \alpha \qquad \Gamma \vdash M' : \alpha \qquad \Gamma \vdash M = M' : \alpha}{\Gamma \vdash M =_\alpha M'}$$

1.6 Preliminaries for the Logic Semantics

Def. Preorder: A set X with a binary relation $R \subseteq X \times X$ that is reflexive and transitive.
Def. Partially Ordered Set (Poset): A set X with a binary relation $R :\subseteq X \times X$ that is reflexive, transitive, and anti-symmetric.
Any preorder can be regarded as a thin category where $(x, y) \in R$ is regarded as the existence of a unique morphism from x to y.
Any preorder can be turned into a partial order via a *posetal reflection*. Let $(A, \leq)$ be a preorder. We construct a poset $(A/\cong, \leq')$ where

$$a \cong b \iff a \leq b \wedge b \leq a$$

and

$$[a] \leq' [b] \iff a \leq b$$

The quotient *enforces* the anti-symmetry condition.
For a category $\mathcal{C}$ and an object, $X : ob\, \mathcal{C}$, we can take the category $Mono(\mathcal{C})$ to be a full subcategory $\mathcal{C}/X$ which consists only of monomorphisms. This

category can be regarded as a preorder.
For a category $\mathcal{C}$ and an object, $X : ob\, \mathcal{C}$, we can take the *posetal reflection* of $Mono(X)$ and regard it as a category $Sub(X)$. This category consists of the subobjects of X in $\mathcal{C}$.
Miscelaneous facts:

- binary products + pullbacks $\implies$ equalizers
- every equalizer is a monomorphism
- pullbacks preserve monomorphisms
- Every category with pullbacks of monomorphisms has a contravariant functor $Sub : C^{op} \to Pos$ to the category of posets called the subobject poset functor, making it into a hyperdoctrine.

1.7 Basics of Set Based Logic Semantics

Assume we have a set based model, $F : \mathcal{CL}(\lambda_{1,\times,\to}) \to Set$, for our simply typed lambda calculus. We need to interpret our two equational logic judgments:

$$\Gamma \vdash \phi \text{ Prop}$$
$$\Gamma|\Phi \vdash \phi$$

For a context Γ, the proposition $\Gamma \vdash \phi$ will be interpreted as a subset $X \subseteq [\![\Gamma]\!]_F$. Propositional context, $\Phi = \phi \wedge \psi \wedge ..$, will be interpreted as a conjunction of propositions where a conjunction of propositions $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$ is:

$$[\![\phi \wedge \psi]\!]_F = [\![\phi]\!]_F \cap [\![\psi]\!]_F$$

We can drop F from the denotation subscript when the model is obvious. For any term context Γ, we have the poset $(\mathcal{P}([\![\Gamma]\!]_F), \leq)$ where $\mathcal{P}([\![\Gamma]\!]_F)$ is the powerset of $[\![\Gamma]\!]_F$. The ordering is given by subset inclusion, that is:

$$[\![\phi]\!] \leq [\![\psi]\!] \iff [\![\phi]\!] \subseteq [\![\psi]\!]$$

Checking this is a poset.. we clearly have that the relation is reflexive since $X \subseteq X$ Additionally, it is also transitive since subset inclusion is transitive. Finally, we have that $\leq$ is antisymmetric since

$$X \subseteq Y \wedge Y \subseteq X \implies X = Y$$

Thus we will interpret $[\![\Gamma|\Phi \vdash \phi]\!] = [\![\Phi]\!] \leq [\![\phi]\!]$ which means that either $[\![\Phi]\!] \subseteq [\![\phi]\!]$ or not. Our derivations are *proof irrelevant* since there is at most one term representing if a derivation is inhabited.

1.8 Set Based Semantics for Propositional Equality

Our equational logic only has one logical connective, $M =_\alpha M'$ propositional equality. To interpret this proposition, we use equalizers. Equalizers in Set are rather simple. Given $\Gamma \vdash M : \alpha$ and $\Gamma \vdash M' : \alpha$, we have the equalizer:

$$Eq([\![M]\!], [\![M']\!]) = \{x : [\![\Gamma]\!] \mid [\![M]\!](x) = [\![M']\!](x)\}$$

Remember that terms are denoted as morphisms $[\![\Gamma \vdash M : \alpha]\!] : Set[[\![\Gamma]\!], [\![\alpha]\!]]$ or functions in Set. The equalizer of terms M and M' is just the subset of $[\![\Gamma]\!]$ for which these functions return equal values in $[\![\alpha]\!]$.

1.9 Demonstration of Propositional Equality Logic

In our logic, we should be able to prove the sequent

$$x : \alpha, y : \alpha, z : \alpha | x =_\alpha y, y =_\alpha z \vdash x =_\alpha z$$

via the transitivity rule. Lets check that this is sound w.r.t the set based denotation. We have that $[\![\alpha]\!]$ is some set and $[\![\Gamma]\!] = [\![\alpha]\!] \times [\![\alpha]\!] \times [\![\alpha]\!]$. The terms x, y, z just projections out of the context.

$$[\![x]\!] : [\![\Gamma]\!] \to [\![\alpha]\!] = \pi_1$$
$$[\![y]\!] : [\![\Gamma]\!] \to [\![\alpha]\!] = \pi_2$$
$$[\![z]\!] : [\![\Gamma]\!] \to [\![\alpha]\!] = \pi_3$$

We have that:

$$[\![x =_\alpha y]\!] = Eq(x, y) = \{\gamma : [\![\Gamma]\!] \mid \pi_1(\gamma) = \pi_2(\gamma)\}$$
$$[\![y =_\alpha z]\!] = Eq(y, z) = \{\gamma : [\![\Gamma]\!] \mid \pi_2(\gamma) = \pi_3(\gamma)\}$$
$$[\![x =_\alpha z]\!] = Eq(x, z) = \{\gamma : [\![\Gamma]\!] \mid \pi_1(\gamma) = \pi_3(\gamma)\}$$

and

$$(x =_\alpha y \wedge y =_\alpha z) = [\![x =_\alpha y]\!] \cap [\![y =_\alpha z]\!] = \{\gamma : [\![\Gamma]\!] \mid \pi_1(\gamma) = \pi_2(\gamma) = \pi_3(\gamma)\}$$

Our proposition considered true in the model if:

$$[\![x =_\alpha y]\!] \cap [\![y =_\alpha z]\!] \subseteq [\![x =_\alpha z]\!]$$

The LHS consists of all tuples of the form

$$(M, M, M) : [\![\alpha]\!] \times [\![\alpha]\!] \times [\![\alpha]\!]$$

where all components of the tuple are equal. While the RHS consists of all tuples of the form

$$(M, N, M) : [\![\alpha]\!] \times [\![\alpha]\!] \times [\![\alpha]\!]$$

where only the first and third components are required to be equal. Thus, there is clearly a subset inclusion from the LHS to the RHS.

1.10 Changing Contexts

We have sequent $x : \alpha \mid \cdot \vdash x =_\alpha x$ via reflexivity. We can apply weakening for terms to this sequent to obtain a proof of $x : \alpha, y : \beta \mid \cdot \vdash x =_\alpha x$. How is this operation justified in the set based semantics? Notice that the first sequent is in the poset *over* $[\![\alpha]\!]$ while the second sequent is in the poset *over* $[\![\alpha]\!] \times [\![\beta]\!]$.

We need a way to *change basis*. This is the role of substitution functors. Recall that we can regard preorders(refl,trans) as categories if we interpret $a \leq_R b$ as the existence of a unique morphism between objects a, b. Furthermore, a poset(refl,trans,anti-sym) can be regarded as a category where $a \leq_R b$ and $b \leq_R b$ implies $a \cong b$. [1] We can then regard a poset *over* $[\![\alpha]\!]$ as a category $Sub([\![\alpha]\!])$. For any morphism

$$f : [\![\alpha]\!] \to [\![\beta]\!]$$

we have an induced functor

$$f^* : \text{Functor } Sub([\![\beta]\!]) \; Sub([\![\alpha]\!])$$

given by

$$\begin{aligned} f_0^*(X \subseteq [\![\beta]\!]) &= \{a : [\![\alpha]\!] \mid f(a) \in X\} \subseteq [\![\alpha]\!] \\ f_1^*(X \leq_{[\![\beta]\!]} Y) &= \{a : [\![\alpha]\!] \mid f(a) \in X\} \leq_{[\![\alpha]\!]} \{a : [\![\alpha]\!] \mid f(a) \in Y\} \end{aligned}$$

We can check that

$$\begin{aligned} f_1^*(X \leq_{[\![\beta]\!]} X) &= \{a : [\![\alpha]\!] \mid f(a) \in X\} \leq_{[\![\alpha]\!]} \{a : [\![\alpha]\!] \mid f(a) \in X\} \\ &= id_{f_0^*(X)} \end{aligned}$$

and

$$\begin{aligned} f_1^*(X \leq_{[\![\beta]\!]} Y; Y \leq_{[\![\beta]\!]} Z) &= f_1^*(X \leq_{[\![\beta]\!]} Z) \\ &\quad \text{b.c. at most one morphism and } X \subseteq Y \wedge Y \subseteq Z \implies X \subseteq Z \\ &= \{a : [\![\alpha]\!] \mid f(a) \in X\} \leq_{[\![\alpha]\!]} \{a : [\![\alpha]\!] \mid f(a) \in Z\} \\ &= f_1^*(X \leq_{[\![\beta]\!]} Y); f_1^*(Y \leq_{[\![\beta]\!]} Z) \end{aligned}$$

Lets consider a special case

$$\begin{aligned} &\pi : [\![\alpha]\!] \times [\![\beta]\!] \to [\![\alpha]\!] \\ &\pi(a, b) = a \end{aligned}$$

This induces a substitution functor $\pi^* : \text{Functor } Sub([\![\alpha]\!]) \; Sub([\![\alpha]\!] \times [\![\beta]\!])$ which we can use to transport our proof over $[\![\alpha]\!]$ to be over $[\![\alpha]\!] \times [\![\beta]\!]$.

[1] Since there is at most one morphisms between objects, this isomorphism is forced

1.11 Eq as left adjoint to δ^*

1.12 TODO: explain the total category $Sub(\mathcal{C})$ and how Functor $Sub(\mathcal{C})$ $\mathcal{C}$ is a fibration

1.13 Regular Logic Syntax

Equational logic only has one type of proposition, propositional equality. We will now focus on a logic with four connectives:

$$\begin{array}{lll} \phi & ::= & M =_\alpha M' \\ & | & \top \\ & | & \phi \wedge \phi' \\ & | & \exists (x\colon \alpha).\phi \end{array}$$

This is a subset of first order logic called *regular logic*. In addition to the base logic structural rules, we have the following:

1.13.1 Proposition Formation Rules

$$\frac{\Gamma \vdash M : \alpha \qquad \Gamma \vdash M' : \alpha}{\Gamma \vdash M =_\alpha M' \text{ Prop}}$$

$$\frac{}{\Gamma \vdash \top \text{ Prop}}$$

$$\frac{\Gamma \vdash \phi \text{ Prop} \qquad \Gamma \vdash \psi \text{ Prop}}{\Gamma \vdash \phi \wedge \psi \text{ Prop}}$$

$$\frac{\Gamma, x : \alpha \vdash \phi \text{ Prop}}{\Gamma \vdash \exists (x : \alpha).\phi \text{ Prop}}$$

1.13.2 Derivation Rules

$$\frac{}{\Gamma | \Phi \vdash \top}\ \top\text{-Intro}$$

$$\frac{\Gamma | \Phi \vdash \phi \qquad \Gamma | \Phi \vdash \psi}{\Gamma | \Phi \vdash \phi \wedge \psi}\ \wedge\text{-Intro}$$

$$\frac{\Gamma | \Phi \vdash \phi \wedge \psi}{\Gamma | \Phi \vdash \phi}\ \wedge\text{-Elim}_1$$

$$\frac{\Gamma | \Phi \vdash \phi \wedge \psi}{\Gamma | \Phi \vdash \psi}\ \wedge\text{-Elim}_2$$

Equality and existential propositions have special, bidirectional, *mate* rules. These combined introduction/elimination rules are a reflection of adjunctions used in the semantics.
[2]

[2]See page 225 Jacobs for the 4 rules =-mate replaces and the 2 rules ∃-mate replaces

$$\frac{\Gamma, x:\alpha|\Phi \vdash \phi[x/y]}{\Gamma, x:\alpha, y:\alpha|\Phi, x =_\alpha y \vdash \phi} \text{=-mate}$$

$$\frac{\Gamma|\exists(x:\alpha).\phi \vdash \psi}{\Gamma, x:\alpha|\phi \vdash \psi} \exists\text{-mate}$$

1.14 Semantics of Regular Logic

A sequent, $\Gamma|\Phi \vdash \phi$, is *valid* if

$$[\![\Gamma \vdash \Phi]\!] \leq_{[\![\Gamma]\!]} [\![\Gamma \vdash \phi]\!]$$

and a proposition, $\Gamma \vdash \phi$ is *valid* if the sequent $\Gamma|\top \vdash \phi$ is valid.

$$[\![\Gamma]\!] \leq_\Gamma [\![\Gamma \vdash \phi]\!]$$

We need a semantic interpretation of our new logical connectives.

$$\begin{aligned}
[\![\Gamma \vdash M =_\alpha M']\!] &= Eq_{[\![\Gamma]\!]}([\![\Gamma \vdash M : \alpha]\!], [\![\Gamma \vdash M' : \alpha]\!]) \\
[\![\Gamma \vdash \top]\!] &= [\![\Gamma]\!] \\
[\![\Gamma \vdash \phi \wedge \psi]\!] &= [\![\Gamma \vdash \phi]\!] \wedge [\![\Gamma \vdash \psi]\!] \\
[\![\Gamma \vdash \exists(x:\alpha).\phi]\!] &= \exists_{[\![\Gamma]\!],[\![\alpha]\!]}([\![\Gamma, x:\alpha \vdash \phi]\!])
\end{aligned}$$

The terms $Eq_{[\![\Gamma]\!],[\![\alpha]\!]}$ and $\exists_{[\![\Gamma]\!],[\![\alpha]\!]}$ need a bit more explaination.

1.14.1 Adjoints to Substitution Functors

As mentioned in section 1.10, any morphism $f : [\![\alpha]\!] \to [\![\beta]\!]$ induces a substitution functor $f^* : \text{Functor}(Sub([\![\beta]\!]))(Sub([\![\alpha]\!]))$. We've already seen the weakening substitution functor π^* induced from

$$\begin{aligned}
&\pi : [\![\alpha]\!] \times [\![\beta]\!] \to [\![\alpha]\!] \\
&\pi(a, b) = a
\end{aligned}$$

Another important functor is the contraction functor, δ^*, induced by

$$\begin{aligned}
&\delta : [\![\alpha]\!] \to [\![\alpha]\!] \times [\![\alpha]\!] \\
&\delta(a) = (a, a)
\end{aligned}$$

The operation Eq is defined to be left adjoint to δ^*.

$$Eq \dashv \delta^*$$

and

$$\exists \dashv \pi^*$$

Lets expand these out to see what they mean. We'll start with equality. move this to section 1.11 We have $\delta_\Gamma : [\![\Gamma]\!] \to [\![\Gamma]\!] \times [\![\Gamma]\!]$. The statement $Eq \dashv \delta^*$ elaborates to the following isomorphism of hom sets for any $X :\ Sub([\![\Gamma]\!])$ and $Y : ob\ Sub([\![\Gamma]\!] \times [\![\Gamma]\!])$.

$$Sub([\![\Gamma]\!] \times [\![\Gamma]\!])[Eq(X), Y] \cong Sub([\![\Gamma]\!])[X, \delta^*(Y)]$$

The definition of $Eq(X)$ for which this holds is

$$Eq(X) = \{(\gamma, \gamma') : [\![\Gamma]\!] \times [\![\Gamma]\!] \mid \gamma = \gamma' \wedge \gamma \in X\}$$

In our set based poset models, the hom set adjunction elaborates to

$$Eq(X) \subseteq_{[\![\Gamma]\!] \times [\![\Gamma]\!]} Y \iff X \subseteq_{[\![\Gamma]\!]} \delta^*(Y)$$

This corresponds to the =-mate rule above. (renaming X as Φ and Y as ϕ)

$$\frac{\Gamma, x : \alpha | \Phi \vdash \phi[x/y]}{\Gamma, x : \alpha, y : \alpha | \Phi, x =_\alpha y \vdash \phi} =\text{-mate}$$

Now for $\exists$. We have $\pi_{\Gamma,\alpha} : [\![\Gamma]\!] \times [\![\alpha]\!] \to [\![\Gamma]\!]$. The statement $\exists \dashv \pi^*$ elaborates to the following isomorphism of hom sets for any $X : ob\ Sub([\![\Gamma]\!] \times [\![\alpha]\!])$ and $Y : ob\ Sub([\![\Gamma]\!])$

$$Sub([\![\Gamma]\!])[\exists(X), Y] \cong Sub([\![\Gamma]\!] \times [\![\alpha]\!])[X, \pi^*(Y)]$$

The definition of $\exists$ for which this holds is

$$\exists(X) = \{\gamma : [\![\Gamma]\!] \mid \exists(a : \alpha).(\gamma, a) \in X\}$$

In out set based poset models, the homset adjunction elaborates to

$$\exists(X) \subseteq_{[\![\Gamma]\!]} Y \iff X \subseteq_{[\![\Gamma]\!] \times [\![\alpha]\!]} \pi^*(Y)$$

This corresponds to the $\exists$-mate rule above. (renaming X as ϕ and Y as ψ)

$$\frac{\Gamma | \exists(x : \alpha).\phi \vdash \psi}{\Gamma, x : \alpha | \phi \vdash \psi} \exists\text{-mate}$$

1.15 Demonstration of Regular Logic