1 Object Language

1.1 Raw Terms

The untyped syntax for CBPV OSum, add separating connectives, remove error.
Add $\forall \underline{X}.\underline{B}$?

$$
\begin{array}{llcl}
\text{Value Types} & A & ::= & X \\
 & & | & \text{Unit} \\
 & & | & \text{Case } A \\
 & & | & \text{OSum} \\
 & & | & A \times A \\
 & & | & A * A \\
 & & | & \exists X.A \\
 & & | & U\underline{B} \\
\text{Computation Types} & \underline{B} & ::= & A \to \underline{B} \\
 & & | & A\underline{B} \\
 & & | & \forall X.\underline{B} \\
 & & | & FA \\
\text{Values} & V & ::= & x \\
 & & | & tt \\
 & & | & \sigma \\
 & & | & \text{inj}_V V \\
 & & | & (V, V) \\
 & & | & (V * V) \\
 & & | & \text{pack } (A, V) \text{ as } \exists X.A \\
 & & | & \text{thunk } M \\
\text{Computations} & M & ::= & \lambda x \colon A.M \\
 & & | & MV \\
 & & | & \alpha x \colon A.M \\
 & & | & M@V \\
 & & | & \Lambda X.M \\
 & & | & M[A] \\
 & & | & \text{ret } V \\
 & & | & x \leftarrow M; N \\
 & & | & \text{force } V \\
 & & | & \text{newcase}_A x; M \\
 & & | & \text{match V with V } \{ \text{ inj } x.M \| N \ \} \\
 & & | & \text{let } (x, x) = V; M \\
 & & | & \text{let } (x * x) = V; M \\
 & & | & \text{unpack } (X, x) = V; M \\
\text{Value Context} & \Gamma & ::= & \cdot \\
 & & | & \Gamma, x \colon A \\
 & & | & \Gamma * x \colon A \\
\text{Type Context} & \Xi & ::= & \cdot \\
 & & | & \Xi, X
\end{array}
$$

1.2 Typed Terms

$$\frac{}{\Xi; \Gamma, x\colon A \vdash_v x : A}$$

$$\frac{}{\Xi; \Gamma * x\colon A \vdash_v x : A}$$

$$\frac{}{\Xi; \Gamma \vdash_v : \text{Unit}}$$

$$\frac{\Xi; \Gamma \vdash_v \sigma : \text{Case}A \qquad \Xi; \Gamma \vdash_v V : A}{\Xi; \Gamma \vdash_v \sigma V : \text{OSum}}$$

$$\frac{\Xi; \Gamma \vdash_v V_1 : A_1 \qquad \Xi; \Gamma \vdash_v V_2 : A_2}{\Xi; \Gamma \vdash_v (V_1, V_2) : A_1 \times A_2}$$

$$\frac{\Xi; \Gamma_1 \vdash_v V_1 : A_1 \qquad \Xi; \Gamma_2 \vdash_v V_2 : A_2}{\Xi; \Gamma_1 * \Gamma_2 \vdash_v (V_1 * V_2) : A_1 * A_2}$$

$$\frac{\Xi; \Gamma \vdash_v V : A[A'/X]}{\Xi; \Gamma \vdash_v \text{pack}(A', V) \text{ as } \exists X.A : \exists X.A}$$

$$\frac{\Xi; \Gamma \vdash_c M : \underline{B}}{\Xi; \Gamma \vdash_v \text{thunk } M : U\underline{B}}$$

$$\frac{\Xi; \Gamma, x : A \vdash_c M : \underline{B}}{\Xi; \Gamma \vdash_c \lambda x : A.M : A \to \underline{B}}$$

$$\frac{\Xi; \Gamma \vdash_c M : A \to \underline{B} \qquad \Xi; \Gamma \vdash_v N : A}{\Xi; \Gamma \vdash_c MN : \underline{B}}$$

$$\frac{\Xi; \Gamma * x : A \vdash_c M : \underline{B}}{\Xi; \Gamma \vdash_c \alpha x : A.M : A\underline{B}}$$

$$\frac{\Xi; \Gamma_1 \vdash_c M : A\underline{B} \qquad \Xi; \Gamma_2 \vdash_v N : A}{\Xi; \Gamma_1 * \Gamma_2 \vdash_c M@N : \underline{B}}$$

$$\frac{\Xi, X; \Gamma \vdash_c M : \underline{B}}{\Xi; \Gamma \vdash_c \Lambda X.M : \forall X.\underline{B}}$$

$$\frac{\Xi; \Gamma \vdash_c : M : \forall X.\underline{B} \qquad \Xi \vdash A}{\Xi; \Gamma \vdash_c M[A] : \underline{B}[A/X]}$$

$$\frac{\Xi; \Gamma \vdash_v V : A}{\Xi; \Gamma \vdash_c \text{ret } V : FA}$$

$$\frac{\Xi;\Gamma \vdash_c M : FA \qquad \Xi;\Gamma, x : A \vdash_c N : \underline{B}}{\Xi;\Gamma \vdash_c x \leftarrow M; N : \underline{B}}$$

$$\frac{\Xi;\Gamma \vdash_v V : U\underline{B}}{\Xi;\Gamma \vdash_c\text{: force } V : \underline{B}}$$

$$\frac{\Xi;\Gamma * (\sigma : \text{Case}A) \vdash_c M : \underline{B} \qquad \Xi \vdash A}{\Xi;\Gamma \vdash_c \text{newcase}_A x; M : \underline{B}}$$

$$\frac{\Xi;\Gamma \vdash_v V : \text{OSum} \qquad \Xi;\Gamma \vdash_v \sigma : \text{Case } A \qquad \Xi;\Gamma, x : A \vdash M : \underline{B} \qquad \Xi;\Gamma \vdash_c N : \underline{B}}{\Xi;\Gamma \vdash_c \text{match } V \text{ with } \sigma\{ \text{ inj } x.M \| N\} : \underline{B}}$$

$$\frac{\Xi;\Gamma \vdash_v V : A_1 \times A_2 \qquad \Xi;\Gamma, x : A_1, y : A_2 \vdash_c M : \underline{B}}{\Xi;\Gamma \vdash_c \text{let } (x, y) = V; M : \underline{B}}$$

$$\frac{\Xi;\Gamma \vdash_v V : A_1 * A_2 \qquad \Xi;\Gamma * x : A_1 * y : A_2 \vdash_c M : \underline{B}}{\Xi;\Gamma \vdash_c \text{let } (x * y) = V; M : \underline{B}}$$

$$\frac{\Xi \vdash \underline{B} \qquad \Xi;\Gamma \vdash_v V : \exists X.A \qquad \Xi, X;\Gamma, x : A \vdash_c M : \underline{B}}{\Xi;\Gamma \vdash_c\text{: unpack}(X, x) = V; M : \underline{B}}$$

2 Logic

2.1 Judgments

The relation environment, Θ in PE logic contains both value and computation relations. How does this work in the semantics when value relations are denoted as objects of $Sub_{\mathcal{V}}(A \times B)$ for $Rel_{\mathcal{V}}[A, B]$ and computation relations are denoted as objects of $Sub_{\mathcal{C}}(\underline{A} \times \underline{B})$ for $Rel_{\mathcal{V}}[\underline{A}, \underline{B}]$? Maybe we have separate relation environments?

$$\Xi;\Gamma;\Theta \vdash \phi \text{ VProp}$$
$$\Xi;\Gamma;\Theta \vdash (x : A, y : B).\phi : Rel_{\mathcal{V}}[A, B]$$
$$\Xi;\Gamma;\Theta|\Phi \vdash \phi$$
$$\Xi;\Gamma;\Delta;\Theta;\Omega \vdash \underline{\phi} \text{ CProp}$$
$$\Xi;\Gamma;\Delta;\Theta;\Omega \vdash (x : \underline{A}, y : \underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]$$
$$\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\underline{\Psi} \vdash \underline{\phi}$$

for type environment Ξ, term environment Γ, stoup Δ, value relation environment Θ, computation relation environment Ω, value proposition environment Φ, and computation proposition environment $\underline{\Psi}$.

2.2 Formation Rules

2.2.1 Value Propositions

$$\phi := \top | \phi \wedge \phi | t =_A u | R(t,u)$$

$$\frac{}{\Xi; \Gamma; \Theta \vdash \top \text{ VProp}}$$

$$\frac{\Xi; \Gamma; \Theta \vdash \phi \text{ VProp} \qquad \Xi; \Gamma; \Theta \vdash \psi \text{ VProp}}{\Xi; \Gamma; \Theta \vdash \phi \wedge \psi \text{ VProp}}$$

$$\frac{\Xi; \Gamma \vdash_v t : A \qquad \Xi; \Gamma \vdash_v u : A}{\Xi; \Gamma; \Theta \vdash t =_A u \text{ VProp}}$$

$$\frac{\Xi; \Gamma \vdash_v t : A \qquad \Xi; \Gamma \vdash_v u : B \qquad R : Rel_{\mathcal{V}}[A, B] \in \Theta}{\Xi; \Gamma; \Theta \vdash R(t,u) \text{ VProp}}$$

2.2.2 Computation Propositions

$$\underline{\psi} := \underline{\top} | \underline{\psi} \underline{\wedge} \underline{\psi} | t =_{\underline{B}} u | \phi \implies \underline{\psi} | \underline{R}(t,u) | \forall (x : A).\underline{\psi} | \forall X.\underline{\psi} | \forall \underline{X}.\underline{\psi} | \forall (R : Rel_{\mathcal{V}}[A, B]).\underline{\psi} | \forall (R : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]).\underline{\psi}$$

$$\frac{}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\top} \text{ CProp}}$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\phi} \text{ CProp} \qquad \Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\phi} \wedge \underline{\psi} \text{ CProp}}$$

$$\frac{\Xi; \Gamma \vdash_c t : \underline{B} \qquad \Xi; \Gamma \vdash_c u : \underline{B}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash t =_{\underline{B}} u \text{ CProp}}$$

$$\frac{\Xi; \Gamma; \Theta \vdash \phi \text{ VProp} \qquad \Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \phi \implies \underline{\psi} \text{ CProp}}$$

$$\frac{\Xi; \Gamma \vdash_c t : \underline{A} \qquad \Xi; \Gamma \vdash_c u : \underline{B} \qquad \underline{R} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}] \in \Omega}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{R}(t,u) \text{ CProp}}$$

$$\frac{\Xi; \Gamma, x : A; \Delta; \Theta; \Omega \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \forall (x : A).\underline{\psi} \text{ CProp}}$$

$$\frac{\Xi, X; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \forall X.\underline{\psi} \text{ CProp}}$$

$$\frac{\Xi, \underline{X}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \forall \underline{X}.\underline{\psi} \text{ CProp}}$$

$$\frac{\Xi; \Gamma; \Delta; \Theta, R; \Omega \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \forall R.\underline{\psi} \text{ CProp}}$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega, \underline{R} \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \forall \underline{R}.\underline{\psi} \text{ CProp}}$$

2.2.3 Value Relations

$$\frac{\Xi; \Gamma, x : A \vdash_v t : C \qquad \Xi; \Gamma, y : B \vdash_v u : C}{\Xi; \Gamma; \Theta \vdash (x : A, y : B).t =_C u : Rel_{\mathcal{V}}[A, B]}$$

$$\frac{\Xi; \Gamma, x : A \vdash_v t : C \qquad \Xi; \Gamma, y : B \vdash_v u : D}{\Xi; \Gamma; \Theta, R : Rel_{\mathcal{V}}[C, D] \vdash (x : A, y : B).R(t, u) : Rel_{\mathcal{V}}[A, B]}$$

Introduction Rule

$$\frac{\Xi; \Gamma, x : A, y : B; \Theta \vdash \phi}{\Xi; \Gamma; \Theta \vdash (x : A, y : B).\phi : Rel_{\mathcal{V}}[A, B]}$$

2.2.4 Computation Relations

Something seems off including the stoup, Δ, in the computation relation judgment..

$$\frac{\Xi; \Gamma | x : \underline{A} \vdash_c t : \underline{C} \qquad \Xi; \Gamma | y : \underline{B} \vdash_c u : \underline{C}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash (x : \underline{A}, y : \underline{B}).t =_{\underline{C}} u : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}$$

$$\frac{\Xi; \Gamma | x : \underline{A} \vdash_c t : \underline{C} \qquad \Xi; \Gamma | y : \underline{B} \vdash_c u : \underline{D}}{\Xi; \Gamma; \Delta; \Theta; \Omega, \underline{R} : Rel_{\mathcal{C}}[\underline{C}, \underline{D}] \vdash (x : \underline{A}, y : \underline{B}).\underline{R}(t, u) : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}$$

$$\frac{\Xi; \Gamma; \Theta \vdash \phi \text{ VProp} \qquad \Xi; \Gamma; \Delta; \Theta; \Omega \vdash (x : \underline{A}, y : \underline{B}).\underline{\psi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash (x : \underline{A}, y : \underline{B}).\phi \implies \underline{\psi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}$$

$$\frac{\Xi; \Gamma, z : C; \Delta; \Theta; \Omega \vdash (x : \underline{A}, y : \underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash (x : \underline{A}, y : \underline{B}).\forall (z : C).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}$$

$$\frac{\Xi, X; \Gamma; \Delta; \Theta; \Omega \vdash (x : \underline{A}, y : \underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash (x : \underline{A}, y : \underline{B}).\forall X.\underline{\phi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}$$

$$\frac{\Xi, \underline{X}; \Gamma; \Delta; \Theta; \Omega \vdash (x : \underline{A}, y : \underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash (x : \underline{A}, y : \underline{B}).\forall \underline{X}.\underline{\phi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}$$

$$\frac{\Xi; \Gamma; \Delta; \Theta, R : Rel_{\mathcal{V}}[A, B]; \Omega \vdash (x : \underline{A}, y : \underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash (x : \underline{A}, y : \underline{B}).\forall (R : Rel_{\mathcal{V}}[A, B]).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega, \underline{R} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}] \vdash (x : \underline{A}, y : \underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash (x : \underline{A}, y : \underline{B}).\forall (\underline{R} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}$$

Introduction Rule. Compare this to the value relation introduction rule.. the stoup has at most one computation..

$$\frac{\Xi; \Gamma; x : \underline{B}; \Theta; \Omega \vdash \underline{\phi}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash ?? : Rel_{\mathcal{C}}[?, ?]}$$

2.3 Derivation Rules

2.3.1 Values

$$\frac{}{\Xi; \Gamma; \Theta | \Phi \vdash \top}$$

$$\frac{\Xi; \Gamma; \Theta | \Phi \vdash \phi \qquad \Xi; \Gamma; \Theta | \Phi \vdash \psi}{\Xi; \Gamma; \Theta | \Phi \vdash \phi \wedge \psi} \text{ I-}\wedge$$

$$\frac{\Xi; \Gamma; \Theta | \Phi \vdash \phi \wedge \psi}{\Xi; \Gamma; \Theta | \Phi \vdash \phi} \text{ E1-}\wedge$$

$$\frac{\Xi; \Gamma; \Theta | \Phi \vdash \phi \wedge \psi}{\Xi; \Gamma; \Theta | \Phi \vdash \psi} \text{ E2-}\wedge$$

$$\frac{\Xi; \Gamma \vdash_v t : A}{\Xi; \Gamma; \Theta | \Phi \vdash t =_A t}$$

$$\frac{\Xi; \Gamma; \Theta | \Phi \vdash t =_A u \qquad \Xi; \Gamma; \Theta | \Phi \vdash \phi[t/x]}{\Xi; \Gamma; \Theta | \Phi \vdash \phi[u/x]}$$

Rel

$$\frac{\Xi; \Gamma; \Theta | \Phi \vdash \qquad \Xi; \Gamma; \Theta | \Phi \vdash}{\Xi; \Gamma; \Theta | \Phi \vdash}$$

2.3.2 Computation

$$\frac{}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\top}}$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi} \qquad \Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\psi}}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi} \wedge \underline{\psi}} \text{ I-}\wedge$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi} \wedge \underline{\psi}}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi}} \text{ E1-}\wedge$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi} \wedge \underline{\psi}}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\psi}} \text{ E2-}\wedge$$

$$\frac{\Xi; \Gamma \vdash_c t : \underline{B}}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash t =_{\underline{B}} t}$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash t =_{\underline{B}} u \qquad \Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi}[t/x]}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi}[u/x]}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi,\phi;\Psi\vdash\underline{\psi}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\phi\implies\underline{\psi}}\text{ I-}\implies$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\phi\implies\underline{\psi}\qquad\Xi;\Gamma;\Theta|\Phi\vdash\phi}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\psi}}\text{ E-}\implies$$

Rel?

$$\frac{\Xi;\Gamma,x:A;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\phi}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(x:A).\underline{\phi}}\text{ I-}\forall\text{ term , FV constraint?}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(x:A).\underline{\phi}\qquad\Xi;\Gamma\vdash_v t:A}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\phi}[t/x]}\text{ E-}\forall\text{ term}$$

$$\frac{\Xi,X;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\phi}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall X.\underline{\phi}}\text{ I-}\forall\text{ vtype , FV constraint?}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall X.\underline{\phi}\qquad\Xi\vdash A}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\phi}[A/X]}\text{ E-}\forall\text{ vtype}$$

$$\frac{\Xi,\underline{X};\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\phi}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall\underline{X}.\underline{\phi}}\text{ I-}\forall\text{ ctype , FV constraint?}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall\underline{X}.\underline{\phi}\qquad\Xi\vdash\underline{A}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\phi}[\underline{A}/\underline{X}]}\text{ E-}\forall\text{ ctype}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta,(R:Rel_{\mathcal{V}}[A,B]);\Omega|\Phi;\Psi\vdash\underline{\phi}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(R:Rel_{\mathcal{V}}[A,B]).\underline{\phi}}\text{ I-}\forall\text{ vrel}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(R:Rel_{\mathcal{V}}[A,B]).\underline{\phi}\qquad\Xi;\Gamma;\Theta,\vdash(x:A,y:B).\psi:Rel_{\mathcal{V}}[A,B]}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(R:Rel_{\mathcal{V}}[A,B]).\underline{\phi}[\psi[t/x,u/y]/R(t,u)]}\text{ E-}\forall\text{ vrel}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta,(\underline{R}:Rel_{\mathcal{C}}[\underline{A},\underline{B}]);\Omega|\Phi;\Psi\vdash\underline{\phi}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(\underline{R}:Rel_{\mathcal{C}}[\underline{A},\underline{B}]).\underline{\phi}}\text{ I-}\forall\text{ crel}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(R:Rel_{\mathcal{C}}[\underline{A},\underline{B}]).\underline{\phi}\qquad\Xi;\Gamma;\Delta;\Theta;\Omega,\vdash(x:\underline{A},y:\underline{B}).\underline{\psi}:Rel_{\mathcal{C}}[\underline{A},\underline{B}]}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(R:Rel_{\mathcal{C}}[\underline{A},\underline{B}]).\underline{\phi}[\underline{\psi}[t/x,u/y]/R(t,u)]}\text{ E-}\forall\text{ crel}$$

2.3.3 Congruences

$$\frac{\Xi;\Gamma\vdash_c t:\underline{B}\qquad\Xi;\Gamma\vdash_c u:\underline{B}\qquad\Xi;\Gamma,x:A;\Delta;\Theta;\Omega|\Phi;\Psi\vdash t=u}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\lambda(x:A).t=\lambda(x:A).u}\text{ cong-}\lambda$$

2.4 Axioms

Beta/Eta/(parametricity schema?)

2.5 Logical Interpretation of Types

Let $\boldsymbol{X}$ and $\underline{\boldsymbol{X}}$ be vectors of value type and computation type variables of length n. Let $\boldsymbol{\rho}$ be a vector of value relations $\boldsymbol{\rho} : Rel_{\mathcal{V}}[\boldsymbol{\alpha}, \boldsymbol{\alpha'}]$ where $\Xi; \Gamma; \Theta \vdash \rho_i : Rel_{\mathcal{V}}[\alpha_i, \alpha'_i]$ for all $i \in 1..n$. Let $\underline{\boldsymbol{\rho}}$ be a vector of computation relations $\underline{\boldsymbol{\rho}} : Rel_{\mathcal{C}}[\underline{\boldsymbol{\beta}}, \underline{\boldsymbol{\beta'}}]$ where $\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\rho_i} : Rel_{\mathcal{C}}[\underline{\beta_i}, \underline{\beta'_i}]$ for all $i \in 1..n$.
Let A be a **value type** with $FTV(A) \in \{\boldsymbol{X}, \underline{\boldsymbol{X}}\}$. Define:

$$A[\boldsymbol{\rho}/\boldsymbol{X}, \underline{\boldsymbol{\rho}}/\underline{\boldsymbol{X}}] : Rel_{\mathcal{V}}[A[\boldsymbol{\alpha}/\boldsymbol{X}, \underline{\boldsymbol{\beta}}/\underline{\boldsymbol{X}}], A[\boldsymbol{\alpha'}/\boldsymbol{X}, \underline{\boldsymbol{\beta'}}/\underline{\boldsymbol{X}}]]$$

by induction on A.

$$\begin{aligned}
X_i[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \rho_i \\
\text{Unit}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= (x : Unit, y : Unit).x =_{Unit} y \\
\text{Case}A[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= (x : (\text{Case } A)[\boldsymbol{\alpha}/\boldsymbol{X}, \underline{\boldsymbol{\beta}}/\underline{\boldsymbol{X}}], y : (\text{Case } A)[\boldsymbol{\alpha'}/\boldsymbol{X}, \underline{\boldsymbol{\beta'}}/\underline{\boldsymbol{X}}]). \\
&\quad \text{think exists?} \\
\text{OSum}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= (\sigma_x v_x : \text{OSum}, \sigma_y v_y : \text{OSum}). \\
&\quad \exists (R : Rel_{\mathcal{V}}[\sigma_x, \sigma_y]). \\
A \times A'[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= ((x, y) : A \times A'[\boldsymbol{C}/\boldsymbol{X}, \underline{\boldsymbol{C}}/\underline{\boldsymbol{X}}], (x', y') : A \times A'[\boldsymbol{C'}/\boldsymbol{X}, \underline{\boldsymbol{C'}}/\underline{\boldsymbol{X}}]). \\
&\quad A[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}](x, x') \wedge A'[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}](y, y') \\
A * A'[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \text{similar to product?} \\
\exists X.A[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \text{standard} \\
U\underline{B}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \text{related thunks?}
\end{aligned}$$

Let $\underline{B}$ be a **computation type** with $FTV(\underline{B}) \in \{\boldsymbol{X}, \underline{\boldsymbol{X}}\}$. Define:

$$\underline{B}[\boldsymbol{\rho}/\boldsymbol{X}, \underline{\boldsymbol{\rho}}/\underline{\boldsymbol{X}}] : Rel_{\mathcal{C}}[\underline{B}[\boldsymbol{C}/\boldsymbol{X}, \underline{\boldsymbol{C}}/\underline{\boldsymbol{X}}], \underline{B}[\boldsymbol{C'}/\boldsymbol{X}, \underline{\boldsymbol{C'}}/\underline{\boldsymbol{X}}]]$$

by induction on $\underline{B}$.

$$\begin{aligned}
A \to \underline{B}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \text{Add weakest precondition to the proposition syntax to interpret this?} \\
A\underline{B}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \\
\forall X.\underline{B}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \\
FA[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \text{Could be encoded if we add } \forall \underline{X}.\underline{B}\text{?}
\end{aligned}$$

3 Goal

Try to prove

$$\begin{aligned}
&\vdash_d \forall A\ \underline{B}.\\
&\quad \forall (y : \text{OSum}).\\
&\quad \forall (f : U(\forall X.\text{Case } X\text{OSum} \to FX)).\\
&\quad \forall (k\ k' : U(A \to \underline{B})).\\
&\quad \text{newcase}_A \sigma; x \leftarrow (!f)[A]\sigma y; (!k)x\\
&\quad =\\
&\quad \text{newcase}_A \sigma; x \leftarrow (!f)[A]\sigma y; (!k')x
\end{aligned}$$

4 V2 Simplification

Removing Contexts via Universes and Ω.

4.1 Universes a la Coquand

To remove the type context, Ξ, we can add two value types $Val, Comp$. There is a new judgment form

$$\Gamma \vdash A\ small$$

where A can be a value or computation type. Val and $Comp$ are not *small*, but the other types are. We have $\Gamma \vdash Unit\ small$ and

$$\frac{\Gamma \vdash A\ small \qquad \Gamma \vdash A'\ small}{\Gamma \vdash A \times A'\ small}$$

but **NOT** $\Gamma \vdash Val\ small$ or $\Gamma \vdash Comp\ small$.
Any *small* value type has a *code* in Val.

$$\frac{\Gamma \vdash A\ small}{\Gamma \vdash \lceil ddd \rceil : Val}$$

And any *code* in Val is *small*

$$\frac{\Gamma \vdash A : Val}{\Gamma \vdash \lfloor A \rfloor\ small}$$

These *quoting* ($\lceil \_ \rceil$) and *unquoting* ($\lfloor \_ \rfloor$) operations form an isomorphism. (between what? Val and ??) That is to say, we have the two equations

$$\frac{\Gamma \vdash A\ small}{\Gamma \vdash A = \lfloor \lceil A \rceil \rfloor}$$

$$\frac{\Gamma \vdash A : Val}{\Gamma \vdash A = \lceil \lfloor A \rfloor \rceil}$$

To have an impredicative universe, we take the rule

$$\frac{\Gamma, \lceil X \rceil : Val \vdash A\ small}{\Gamma \vdash \forall X.A\ small}$$

Q: By removing the type context and using universes, types (or codes in Val) are now impacted by bunched connectives? (see bunched polymorphism paper) And what about having $\phi :\ Prop$ also in Γ?

5 TODO/Questions

- Do we add the type $\forall \underline{X}.\underline{B}$ to the object lang (yes) and then encode FA ?
- Finalize the judgment forms, what contexts do they actually need to be displayed over? Do we need to split the relation and proposition contexts into distinct value/computation contexts?
- Check that the classification of logical connectives makes sense (value prop vs comp prop) (yes)
- yesDoes the body of a value relation need to be value propositions? (same question for computation relation)
- What is the introduction rule for computation relations (definable computation relations)?
- Denotation of value/computation propositions
- Understand the operation $\oslash$ and its laws
- Denotation of value/computation derivations
- Define the operation $\oslash^*$ and find its laws
- Finish the known relational interpretation of types
- Attempt the relational interpretation of our new types cant internalize some of the interpretations, ex Case
- Write up the beta and eta deduction rules
- Check the correctness of the relational interpretation of types. (By proving Reynold's Identity Extension Lemma?)
- PE Logic denotes the collection of computation relations, $Rel_{\mathcal{C}}[\underline{A}, \underline{B}]$, by $Sub_{\mathcal{C}}(\underline{A} \times \underline{B})$. However, they never define $\underline{A} \times \underline{B}$ or state that it is a derivable type. we have cartesian product of computation types