1 Object Language

1.1 Raw Terms

The untyped syntax for CBPV OSum, add separating connectives, remove error.
Add $\forall \underline{X}.\underline{B}$?

Value Types	A	$::=$	X
		$\mid$	Unit
		$\mid$	Case A
		$\mid$	OSum
		$\mid$	$A \times A$
		$\mid$	$A * A$
		$\mid$	$\exists X.A$
		$\mid$	$U\underline{B}$
Computation Types	$\underline{B}$	$::=$	$A \to \underline{B}$
		$\mid$	$A \twoheadrightarrow\!\!* \; \underline{B}$
		$\mid$	$\forall X.\underline{B}$
		$\mid$	FA
Values	V	$::=$	x
		$\mid$	tt
		$\mid$	σ
		$\mid$	$\mathrm{inj}_V V$
		$\mid$	(V, V)
		$\mid$	$(V * V)$
		$\mid$	pack (A, V) as $\exists X.A$
		$\mid$	thunk M
Computations	M	$::=$	$\lambda x\colon A.M$
		$\mid$	MV
		$\mid$	$\alpha x\colon A.M$
		$\mid$	$M@V$
		$\mid$	$\Lambda X.M$
		$\mid$	$M[A]$
		$\mid$	ret V
		$\mid$	$x \leftarrow M; N$
		$\mid$	force V
		$\mid$	$\mathrm{newcase}_A x; M$
		$\mid$	match V with V { inj $x.M \| N$ }
		$\mid$	let $(x, x) = V; M$
		$\mid$	let $(x * x) = V; M$
		$\mid$	unpack $(X, x) = V; M$
Value Context	Γ	$::=$	$\cdot$
		$\mid$	$\Gamma, x\colon A$
		$\mid$	$\Gamma * x\colon A$
Type Context	Ξ	$::=$	$\cdot$
		$\mid$	Ξ, X

1.2 Typed Terms

$$\frac{}{\Xi; \Gamma, x \colon A \vdash_v x : A}$$

$$\frac{}{\Xi; \Gamma * x \colon A \vdash_v x : A}$$

$$\frac{}{\Xi; \Gamma \vdash_v \mathrm{tt} : \mathrm{Unit}}$$

$$\frac{\Xi; \Gamma \vdash_v \sigma : \mathrm{Case} A \qquad \Xi; \Gamma \vdash_v V : A}{\Xi; \Gamma \vdash_v \mathrm{inj}_\sigma V : \mathrm{OSum}}$$

$$\frac{\Xi; \Gamma \vdash_v V_1 : A_1 \qquad \Xi; \Gamma \vdash_v V_2 : A_2}{\Xi; \Gamma \vdash_v (V_1, V_2) : A_1 \times A_2}$$

$$\frac{\Xi; \Gamma_1 \vdash_v V_1 : A_1 \qquad \Xi; \Gamma_2 \vdash_v V_2 : A_2}{\Xi; \Gamma_1 * \Gamma_2 \vdash_v (V_1 * V_2) : A_1 * A_2}$$

$$\frac{\Xi; \Gamma \vdash_v V : A[A'/X]}{\Xi; \Gamma \vdash_v \mathrm{pack}(A', V) \text{ as } \exists X.A : \exists X.A}$$

$$\frac{\Xi; \Gamma \vdash_c M : \underline{B}}{\Xi; \Gamma \vdash_v \mathrm{thunk}\ M : U\underline{B}}$$

$$\frac{\Xi; \Gamma, x : A \vdash_c M : \underline{B}}{\Xi; \Gamma \vdash_c \lambda x : A.M : A \to \underline{B}}$$

$$\frac{\Xi; \Gamma \vdash_c M : A \to \underline{B} \qquad \Xi; \Gamma \vdash_v N : A}{\Xi; \Gamma \vdash_c MN : \underline{B}}$$

$$\frac{\Xi; \Gamma * x : A \vdash_c M : \underline{B}}{\Xi; \Gamma \vdash_c \alpha x : A.M : A \twoheadrightarrow \underline{B}}$$

$$\frac{\Xi; \Gamma_1 \vdash_c M : A \twoheadrightarrow \underline{B} \qquad \Xi; \Gamma_2 \vdash_v N : A}{\Xi; \Gamma_1 * \Gamma_2 \vdash_c M@N : \underline{B}}$$

$$\frac{\Xi, X; \Gamma \vdash_c M : \underline{B}}{\Xi; \Gamma \vdash_c \Lambda X.M : \forall X.\underline{B}}$$

$$\frac{\Xi; \Gamma \vdash_c : M : \forall X.\underline{B} \qquad \Xi \vdash A}{\Xi; \Gamma \vdash_c M[A] : \underline{B}[A/X]}$$

$$\frac{\Xi; \Gamma \vdash_v V : A}{\Xi; \Gamma \vdash_c \mathrm{ret}\ V : FA}$$

$$\frac{\Xi;\Gamma \vdash_c M : FA \qquad \Xi;\Gamma, x : A \vdash_c N : \underline{B}}{\Xi;\Gamma \vdash_c x \leftarrow M; N : \underline{B}}$$

$$\frac{\Xi;\Gamma \vdash_v V : U\underline{B}}{\Xi;\Gamma \vdash_c \text{: force } V : \underline{B}}$$

$$\frac{\Xi;\Gamma * (\sigma : \mathrm{Case}A) \vdash_c M : \underline{B} \qquad \Xi \vdash A}{\Xi;\Gamma \vdash_c \mathrm{newcase}_A x; M : \underline{B}}$$

$$\frac{\Xi;\Gamma \vdash_v V : \mathrm{OSum} \qquad \Xi;\Gamma \vdash_v \sigma : \mathrm{Case}\ A \qquad \Xi;\Gamma, x : A \vdash M : \underline{B} \qquad \Xi;\Gamma \vdash_c N : \underline{B}}{\Xi;\Gamma \vdash_c \text{match } V \text{ with } \sigma\{ \text{ inj } x.M \| N\} : \underline{B}}$$

$$\frac{\Xi;\Gamma \vdash_v V : A_1 \times A_2 \qquad \Xi;\Gamma, x : A_1, y : A_2 \vdash_c M : \underline{B}}{\Xi;\Gamma \vdash_c \text{let } (x, y) = V; M : \underline{B}}$$

$$\frac{\Xi;\Gamma \vdash_v V : A_1 * A_2 \qquad \Xi;\Gamma * x : A_1 * y : A_2 \vdash_c M : \underline{B}}{\Xi;\Gamma \vdash_c \text{let } (x * y) = V; M : \underline{B}}$$

$$\frac{\Xi \vdash \underline{B} \qquad \Xi;\Gamma \vdash_v V : \exists X.A \qquad \Xi, X;\Gamma, x : A \vdash_c M : \underline{B}}{\Xi;\Gamma \vdash_c \text{: unpack}(X, x) = V; M : \underline{B}}$$

2 Logic

2.1 Judgments

The relation environment, Θ in PE logic contains both value and computation relations. How does this work in the semantics when value relations are denoted as objects of $Sub_{\mathcal{V}}(A \times B)$ for $Rel_{\mathcal{V}}[A, B]$ and computation relations are denoted as objects of $Sub_{\mathcal{C}}(\underline{A} \times \underline{B})$ for $Rel_{\mathcal{V}}[\underline{A}, \underline{B}]$? Maybe we have separate relation environments?

$$\begin{aligned}
&\Xi;\Gamma;\Theta \vdash \phi \text{ VProp}\\
&\Xi;\Gamma;\Theta \vdash (x : A, y : B).\phi : Rel_{\mathcal{V}}[A, B]\\
&\Xi;\Gamma;\Theta|\Phi \vdash \phi\\
&\Xi;\Gamma;\Delta;\Theta;\Omega \vdash \underline{\phi} \text{ CProp}\\
&\Xi;\Gamma;\Delta;\Theta;\Omega \vdash (x : \underline{A}, y : \underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]\\
&\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\underline{\Psi} \vdash \underline{\phi}
\end{aligned}$$

for type environment Ξ, term environment Γ, stoup Δ, value relation environment Θ, computation relation environment Ω, value proposition environment Φ, and computation proposition environment $\underline{\Psi}$.

2.2 Formation Rules

2.2.1 Value Propositions

$$\phi := \top | \phi \wedge \phi | t =_A u | R(t,u)$$

$$\frac{}{\Xi; \Gamma; \Theta \vdash \top \text{ VProp}}$$

$$\frac{\Xi; \Gamma; \Theta \vdash \phi \text{ VProp} \qquad \Xi; \Gamma; \Theta \vdash \psi \text{ VProp}}{\Xi; \Gamma; \Theta \vdash \phi \wedge \psi \text{ VProp}}$$

$$\frac{\Xi; \Gamma \vdash_v t : A \qquad \Xi; \Gamma \vdash_v u : A}{\Xi; \Gamma; \Theta \vdash t =_A u \text{ VProp}}$$

$$\frac{\Xi; \Gamma \vdash_v t : A \qquad \Xi; \Gamma \vdash_v u : B \qquad R : Rel_{\mathcal{V}}[A,B] \in \Theta}{\Xi; \Gamma; \Theta \vdash R(t,u) \text{ VProp}}$$

2.2.2 Computation Propositions

$$\underline{\psi} := \underline{\top} | \underline{\psi} \wedge \underline{\psi} | t =_{\underline{B}} u | \phi \implies \underline{\psi} | \underline{R}(t,u) | \forall (x : A).\underline{\psi} | \forall X.\underline{\psi} | \forall \underline{X}.\underline{\psi} | \forall (R : Rel_{\mathcal{V}}[A,B]).\underline{\psi} | \forall (R : Rel_{\mathcal{C}}[\underline{A},\underline{B}]).\underline{\psi}$$

$$\frac{}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\top} \text{ CProp}}$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\phi} \text{ CProp} \qquad \Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\phi} \wedge \underline{\psi} \text{ CProp}}$$

$$\frac{\Xi; \Gamma \vdash_c t : \underline{B} \qquad \Xi; \Gamma \vdash_c u : \underline{B}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash t =_{\underline{B}} u \text{ CProp}}$$

$$\frac{\Xi; \Gamma; \Theta \vdash \phi \text{ VProp} \qquad \Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \phi \implies \underline{\psi} \text{ CProp}}$$

$$\frac{\Xi; \Gamma \vdash_c t : \underline{A} \qquad \Xi; \Gamma \vdash_c u : \underline{B} \qquad \underline{R} : Rel_{\mathcal{C}}[\underline{A},\underline{B}] \in \Omega}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{R}(t,u) \text{ CProp}}$$

$$\frac{\Xi; \Gamma, x : A; \Delta; \Theta; \Omega \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \forall (x : A).\underline{\psi} \text{ CProp}}$$

$$\frac{\Xi, X; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \forall X.\underline{\psi} \text{ CProp}}$$

$$\frac{\Xi, \underline{X}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \forall \underline{X}.\underline{\psi} \text{ CProp}}$$

$$\frac{\Xi; \Gamma; \Delta; \Theta, R; \Omega \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \forall R.\underline{\psi} \text{ CProp}}$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega, \underline{R} \vdash \underline{\psi} \text{ CProp}}{\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \forall \underline{R}.\underline{\psi} \text{ CProp}}$$

2.2.3 Value Relations

$$\frac{\Xi;\Gamma,x:A\vdash_v t:C \qquad \Xi;\Gamma,y:B\vdash_v u:C}{\Xi;\Gamma;\Theta\vdash (x:A,y:B).t =_C u : Rel_{\mathcal{V}}[A,B]}$$

$$\frac{\Xi;\Gamma,x:A\vdash_v t:C \qquad \Xi;\Gamma,y:B\vdash_v u:D}{\Xi;\Gamma;\Theta,R:Rel_{\mathcal{V}}[C,D]\vdash (x:A,y:B).R(t,u) : Rel_{\mathcal{V}}[A,B]}$$

Introduction Rule

$$\frac{\Xi;\Gamma,x:A,y:B;\Theta\vdash\phi}{\Xi;\Gamma;\Theta\vdash (x:A,y:B).\phi : Rel_{\mathcal{V}}[A,B]}$$

2.2.4 Computation Relations

Something seems off including the stoup, Δ, in the computation relation judgment..

$$\frac{\Xi;\Gamma|x:\underline{A}\vdash_c t:\underline{C} \qquad \Xi;\Gamma|y:\underline{B}\vdash_c u:\underline{C}}{\Xi;\Gamma;\Delta;\Theta;\Omega\vdash (x:\underline{A},y:\underline{B}).t =_{\underline{C}} u : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}$$

$$\frac{\Xi;\Gamma|x:\underline{A}\vdash_c t:\underline{C} \qquad \Xi;\Gamma|y:\underline{B}\vdash_c u:\underline{D}}{\Xi;\Gamma;\Delta;\Theta;\Omega,\underline{R}:Rel_{\mathcal{C}}[\underline{C},\underline{D}]\vdash (x:\underline{A},y:\underline{B}).\underline{R}(t,u) : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}$$

$$\frac{\Xi;\Gamma;\Theta\vdash\phi\ \text{VProp} \qquad \Xi;\Gamma;\Delta;\Theta;\Omega\vdash (x:\underline{A},y:\underline{B}).\underline{\psi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}{\Xi;\Gamma;\Delta;\Theta;\Omega\vdash (x:\underline{A},y:\underline{B}).\phi\implies\underline{\psi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}$$

$$\frac{\Xi;\Gamma,z:C;\Delta;\Theta;\Omega\vdash (x:\underline{A},y:\underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}{\Xi;\Gamma;\Delta;\Theta;\Omega\vdash (x:\underline{A},y:\underline{B}).\forall(z:C).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}$$

$$\frac{\Xi,X;\Gamma;\Delta;\Theta;\Omega\vdash (x:\underline{A},y:\underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}{\Xi;\Gamma;\Delta;\Theta;\Omega\vdash (x:\underline{A},y:\underline{B}).\forall X.\underline{\phi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}$$

$$\frac{\Xi,\underline{X};\Gamma;\Delta;\Theta;\Omega\vdash (x:\underline{A},y:\underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}{\Xi;\Gamma;\Delta;\Theta;\Omega\vdash (x:\underline{A},y:\underline{B}).\forall \underline{X}.\underline{\phi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta,R:Rel_{\mathcal{V}}[A,B];\Omega\vdash (x:\underline{A},y:\underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}{\Xi;\Gamma;\Delta;\Theta;\Omega\vdash (x:\underline{A},y:\underline{B}).\forall (R:Rel_{\mathcal{V}}[A,B]).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega,\underline{R}:Rel_{\mathcal{C}}[\underline{A},\underline{B}]\vdash (x:\underline{A},y:\underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}{\Xi;\Gamma;\Delta;\Theta;\Omega\vdash (x:\underline{A},y:\underline{B}).\forall (\underline{R}:Rel_{\mathcal{C}}[\underline{A},\underline{B}]).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}]}$$

Introduction Rule. Compare this to the value relation introduction rule.. the stoup has at most one computation..

$$\frac{\Xi;\Gamma;x:\underline{B};\Theta;\Omega\vdash\underline{\phi}}{\Xi;\Gamma;\Delta;\Theta;\Omega\vdash ?? : Rel_{\mathcal{C}}[?,?]}$$

2.3 Derivation Rules

2.3.1 Values

$$\frac{}{\Xi; \Gamma; \Theta | \Phi \vdash \top}$$

$$\frac{\Xi; \Gamma; \Theta | \Phi \vdash \phi \qquad \Xi; \Gamma; \Theta | \Phi \vdash \psi}{\Xi; \Gamma; \Theta | \Phi \vdash \phi \wedge \psi} \text{ I-}\wedge$$

$$\frac{\Xi; \Gamma; \Theta | \Phi \vdash \phi \wedge \psi}{\Xi; \Gamma; \Theta | \Phi \vdash \phi} \text{ E1-}\wedge$$

$$\frac{\Xi; \Gamma; \Theta | \Phi \vdash \phi \wedge \psi}{\Xi; \Gamma; \Theta | \Phi \vdash \psi} \text{ E2-}\wedge$$

$$\frac{\Xi; \Gamma \vdash_v t : A}{\Xi; \Gamma; \Theta | \Phi \vdash t =_A t}$$

$$\frac{\Xi; \Gamma; \Theta | \Phi \vdash t =_A u \qquad \Xi; \Gamma; \Theta | \Phi \vdash \phi[t/x]}{\Xi; \Gamma; \Theta | \Phi \vdash \phi[u/x]}$$

Rel

$$\frac{\Xi; \Gamma; \Theta | \Phi \vdash \qquad \Xi; \Gamma; \Theta | \Phi \vdash}{\Xi; \Gamma; \Theta | \Phi \vdash}$$

2.3.2 Computation

$$\frac{}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\top}}$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi} \qquad \Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\psi}}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi} \wedge \underline{\psi}} \text{ I-}\wedge$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi} \wedge \underline{\psi}}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi}} \text{ E1-}\wedge$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi} \wedge \underline{\psi}}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\psi}} \text{ E2-}\wedge$$

$$\frac{\Xi; \Gamma \vdash_c t : \underline{B}}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash t =_{\underline{B}} t}$$

$$\frac{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash t =_{\underline{B}} u \qquad \Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi}[t/x]}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \underline{\phi}[u/x]}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi,\phi;\Psi\vdash\underline{\psi}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\phi\implies\underline{\psi}}\text{ I-}\implies$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\phi\implies\underline{\psi}\qquad\Xi;\Gamma;\Theta|\Phi\vdash\phi}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\psi}}\text{ E-}\implies$$

Rel?

$$\frac{\Xi;\Gamma,x:A;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\phi}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(x:A).\underline{\phi}}\text{ I-}\forall\text{ term , FV constraint?}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(x:A).\underline{\phi}\qquad\Xi;\Gamma\vdash_v t:A}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\phi}[t/x]}\text{ E-}\forall\text{ term}$$

$$\frac{\Xi,X;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\phi}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall X.\underline{\phi}}\text{ I-}\forall\text{ vtype , FV constraint?}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall X.\underline{\phi}\qquad\Xi\vdash A}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\phi}[A/X]}\text{ E-}\forall\text{ vtype}$$

$$\frac{\Xi,\underline{X};\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\phi}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall\underline{X}.\underline{\phi}}\text{ I-}\forall\text{ ctype , FV constraint?}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall\underline{X}.\underline{\phi}\qquad\Xi\vdash\underline{A}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\underline{\phi}[\underline{A}/\underline{X}]}\text{ E-}\forall\text{ ctype}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta,(R:Rel_{\mathcal{V}}[A,B]);\Omega|\Phi;\Psi\vdash\underline{\phi}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(R:Rel_{\mathcal{V}}[A,B]).\underline{\phi}}\text{ I-}\forall\text{ vrel}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(R:Rel_{\mathcal{V}}[A,B]).\underline{\phi}\qquad\Xi;\Gamma;\Theta,\vdash(x:A,y:B).\psi:Rel_{\mathcal{V}}[A,B]}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(R:Rel_{\mathcal{V}}[A,B]).\underline{\phi}[\psi[t/x,u/y]/R(t,u)]}\text{ E-}\forall\text{ vrel}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta,(\underline{R}:Rel_{\mathcal{C}}[\underline{A},\underline{B}]);\Omega|\Phi;\Psi\vdash\underline{\phi}}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(\underline{R}:Rel_{\mathcal{C}}[\underline{A},\underline{B}]).\underline{\phi}}\text{ I-}\forall\text{ crel}$$

$$\frac{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(R:Rel_{\mathcal{C}}[\underline{A},\underline{B}]).\underline{\phi}\qquad\Xi;\Gamma;\Delta;\Theta;\Omega,\vdash(x:\underline{A},y:\underline{B}).\underline{\psi}:Rel_{\mathcal{C}}[\underline{A},\underline{B}]}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\forall(R:Rel_{\mathcal{C}}[\underline{A},\underline{B}]).\underline{\phi}[\underline{\psi}[t/x,u/y]/R(t,u)]}\text{ E-}\forall\text{ crel}$$

2.3.3 Congruences

$$\frac{\Xi;\Gamma\vdash_c t:\underline{B}\qquad\Xi;\Gamma\vdash_c u:\underline{B}\qquad\Xi;\Gamma,x:A;\Delta;\Theta;\Omega|\Phi;\Psi\vdash t=u}{\Xi;\Gamma;\Delta;\Theta;\Omega|\Phi;\Psi\vdash\lambda(x:A).t=\lambda(x:A).u}\text{ cong-}\lambda$$

2.4 Axioms

Beta/Eta/(parametricity schema?)

2.5 Logical Interpretation of Types

Let $\boldsymbol{X}$ and $\underline{\boldsymbol{X}}$ be vectors of value type and computation type variables of length n. Let $\boldsymbol{\rho}$ be a vector of value relations $\boldsymbol{\rho} : Rel_{\mathcal{V}}[\boldsymbol{\alpha}, \boldsymbol{\alpha'}]$ where $\Xi; \Gamma; \Theta \vdash \rho_i : Rel_{\mathcal{V}}[\alpha_i, \alpha'_i]$ for all $i \in 1..n$. Let $\underline{\boldsymbol{\rho}}$ be a vector of computation relations $\underline{\boldsymbol{\rho}} : Rel_{\mathcal{C}}[\underline{\boldsymbol{\beta}}, \underline{\boldsymbol{\beta'}}]$ where $\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\rho_i} : Rel_{\mathcal{C}}[\underline{\beta_i}, \underline{\beta'_i}]$ for all $i \in 1..n$.
Let A be a **value type** with $FTV(A) \in \{\boldsymbol{X}, \underline{\boldsymbol{X}}\}$. Define:

$$A[\boldsymbol{\rho}/\boldsymbol{X}, \underline{\boldsymbol{\rho}}/\underline{\boldsymbol{X}}] : Rel_{\mathcal{V}}[A[\boldsymbol{\alpha}/\boldsymbol{X}, \underline{\boldsymbol{\beta}}/\underline{\boldsymbol{X}}], A[\boldsymbol{\alpha'}/\boldsymbol{X}, \underline{\boldsymbol{\beta'}}/\underline{\boldsymbol{X}}]]$$

by induction on A.

$$\begin{aligned}
X_i[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \rho_i \\
\mathrm{Unit}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= (x : Unit, y : Unit).x =_{Unit} y \\
\mathrm{Case}A[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= (x : (\mathrm{Case}\ A)[\boldsymbol{\alpha}/\boldsymbol{X}, \underline{\boldsymbol{\beta}}/\underline{\boldsymbol{X}}], y : (\mathrm{Case}\ A)[\boldsymbol{\alpha'}/\boldsymbol{X}, \underline{\boldsymbol{\beta'}}/\underline{\boldsymbol{X}}]). \\
&\quad \text{think exists?} \\
\mathrm{OSum}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= (\mathrm{inj}_{\sigma_x} v_x : \mathrm{OSum}, \mathrm{inj}_{\sigma_y} v_y : \mathrm{OSum}). \\
&\quad \exists (R : Rel_{\mathcal{V}}[\sigma_x, \sigma_y]). \\
A \times A'[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= ((x, y) : A \times A'[\boldsymbol{C}/\boldsymbol{X}, \underline{\boldsymbol{C}}/\underline{\boldsymbol{X}}], (x', y') : A \times A'[\boldsymbol{C'}/\boldsymbol{X}, \underline{\boldsymbol{C'}}/\underline{\boldsymbol{X}}]). \\
&\quad A[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}](x, x') \wedge A'[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}](y, y') \\
A * A'[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \text{similar to product?} \\
\exists X.A[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \text{standard} \\
U\underline{B}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \text{related thunks?}
\end{aligned}$$

Let $\underline{B}$ be a **computation type** with $FTV(\underline{B}) \in \{\boldsymbol{X}, \underline{\boldsymbol{X}}\}$. Define:

$$\underline{B}[\boldsymbol{\rho}/\boldsymbol{X}, \underline{\boldsymbol{\rho}}/\underline{\boldsymbol{X}}] : Rel_{\mathcal{C}}[\underline{B}[\boldsymbol{C}/\boldsymbol{X}, \underline{\boldsymbol{C}}/\underline{\boldsymbol{X}}], \underline{B}[\boldsymbol{C'}/\boldsymbol{X}, \underline{\boldsymbol{C'}}/\underline{\boldsymbol{X}}]]$$

by induction on $\underline{B}$.

$$\begin{aligned}
A \to \underline{B}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \text{Add weakest precondition to the proposition syntax to interpret this?} \\
A \twoheadrightarrow\!\!* \underline{B}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \\
\forall X.\underline{B}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \\
FA[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \text{Could be encoded if we add } \forall \underline{X}.\underline{B}?
\end{aligned}$$

3 Goal

Try to prove

$$
\begin{aligned}
&\vdash_d \forall A\ \underline{B}.\\
&\quad \forall (y : \mathrm{OSum}).\\
&\quad \forall (f : U(\forall X.\mathrm{Case}\ X \twoheadrightarrow \mathrm{OSum} \to FX)).\\
&\quad \forall (k\ k' : U(A \to \underline{B})).\\
&\quad \mathrm{newcase}_A \sigma; x \leftarrow (!f)[A]\sigma y; (!k)x\\
&\quad\ =\\
&\quad \mathrm{newcase}_A \sigma; x \leftarrow (!f)[A]\sigma y; (!k')x
\end{aligned}
$$

4 V2 Simplification

Removing Contexts via Universes and Ω.

4.1 Universes a la Coquand

To remove the type context, Ξ, we can add two value types $Val, Comp$. There is a new judgment form

$$\Gamma \vdash A\ \mathit{small}$$

where A can be a value or computation type. Val and $Comp$ are not *small*, but the other types are. We have $\Gamma \vdash Unit\ \mathit{small}$ and

$$\frac{\Gamma \vdash A\ \mathit{small} \qquad \Gamma \vdash A'\ \mathit{small}}{\Gamma \vdash A \times A'\ \mathit{small}}$$

but **NOT** $\Gamma \vdash Val\ \mathit{small}$ or $\Gamma \vdash Comp\ \mathit{small}$.
Any *small* value type has a *code* in Val.

$$\frac{\Gamma \vdash A\ \mathit{small}}{\Gamma \vdash \lceil ddd \rceil : Val}$$

And any *code* in Val is *small*

$$\frac{\Gamma \vdash A : Val}{\Gamma \vdash \lfloor A \rfloor\ \mathit{small}}$$

These *quoting* ($\lceil \_ \rceil$) and *unquoting* ($\lfloor \_ \rfloor$) operations form an isomorphism. (between what? Val and ??) That is to say, we have the two equations

$$\frac{\Gamma \vdash A\ \mathit{small}}{\Gamma \vdash A = \lfloor \lceil A \rceil \rfloor}$$

$$\frac{\Gamma \vdash A : Val}{\Gamma \vdash A = \lceil \lfloor A \rfloor \rceil}$$

To have an impredicative universe, we take the rule

$$\frac{\Gamma, \lceil X \rceil : Val \vdash A \ small}{\Gamma \vdash \forall X.A \ small}$$

Q: By removing the type context and using universes, types (or codes in Val) are now impacted by bunched connectives? (see bunched polymorphism paper) And what about having $\phi :\ Prop$ also in Γ?

5 TODO/Questions

- Do we add the type $\forall \underline{X}.\underline{B}$ to the object lang (yes) and then encode FA ?
- Finalize the judgment forms, what contexts do they actually need to be displayed over? Do we need to split the relation and proposition contexts into distinct value/computation contexts?
- Check that the classification of logical connectives makes sense (value prop vs comp prop) (yes)
- yesDoes the body of a value relation need to be value propositions? (same question for computation relation)
- What is the introduction rule for computation relations (definable computation relations)?
- Denotation of value/computation propositions
- Understand the operation $\oslash$ and its laws
- Denotation of value/computation derivations
- Define the operation $\oslash^*$ and find its laws
- Finish the known relational interpretation of types
- Attempt the relational interpretation of our new types cant internalize some of the interpretations, ex Case
- Write up the beta and eta deduction rules
- Check the correctness of the relational interpretation of types. (By proving Reynold's Identity Extension Lemma?)
- PE Logic denotes the collection of computation relations, $Rel_{\mathcal{C}}[\underline{A}, \underline{B}]$, by $Sub_{\mathcal{C}}(\underline{A} \times \underline{B})$. However, they never define $\underline{A} \times \underline{B}$ or state that it is a derivable type. we have cartesian product of computation types