1 Object Language

1.1 Raw Terms

The untyped syntax for CBPV OSum, add separating connectives, remove error.

Value Types	A	$::=$	X
		$\mid$	Unit
		$\mid$	Case A
		$\mid$	OSum
		$\mid$	$A \times A$
		$\mid$	$A * A$
		$\mid$	$\exists X.A$
		$\mid$	$U\underline{B}$
Computation Types	$\underline{B}$	$::=$	$A \to \underline{B}$
		$\mid$	$A \twoheadrightarrow \underline{B}$
		$\mid$	$\forall X.\underline{B}$
		$\mid$	FA
Values	V	$::=$	x
		$\mid$	tt
		$\mid$	σ
		$\mid$	$\mathrm{inj}_V V$
		$\mid$	(V, V)
		$\mid$	$(V * V)$
		$\mid$	pack (A, V) as $\exists X.A$
		$\mid$	thunk M
Computations	M	$::=$	$\lambda x \colon A.M$
		$\mid$	MV
		$\mid$	$\alpha x \colon A.M$
		$\mid$	$M@V$
		$\mid$	$\Lambda X.M$
		$\mid$	$M[A]$
		$\mid$	ret V
		$\mid$	force V
		$\mid$	$\mathrm{newcase}_A x; M$
		$\mid$	match V with V $\{$ inj $x.M \| N$ $\}$
		$\mid$	let $(x, x) = V; M$
		$\mid$	let $(x * x) = V; M$
		$\mid$	unpack $(X, x) = V; M$
Value Context	Γ	$::=$	$\cdot$
		$\mid$	$\Gamma, x \colon A$
		$\mid$	$\Gamma * x \colon A$
Type Context	Δ	$::=$	$\cdot$
		$\mid$	Δ, X

1.2 Typed Terms

$$\frac{}{\Delta; \Gamma, x\colon A \vdash_v x : A}$$

$$\frac{}{\Delta; \Gamma * x\colon A \vdash_v x : A}$$

$$\frac{}{\Delta; \Gamma \vdash_v \mathrm{tt} : \mathrm{Unit}}$$

$$\frac{\Delta; \Gamma \vdash_v \sigma : \mathrm{Case} A \qquad \Delta; \Gamma \vdash_v V : A}{\Delta; \Gamma \vdash_v \mathrm{inj}_\sigma V : \mathrm{OSum}}$$

$$\frac{\Delta; \Gamma \vdash_v V_1 : A_1 \qquad \Delta; \Gamma \vdash_v V_2 : A_2}{\Delta; \Gamma \vdash_v (V_1, V_2) : A_1 \times A_2}$$

$$\frac{\Delta; \Gamma_1 \vdash_v V_1 : A_1 \qquad \Delta; \Gamma_2 \vdash_v V_2 : A_2}{\Delta; \Gamma_1 * \Gamma_2 \vdash_v (V_1 * V_2) : A_1 * A_2}$$

$$\frac{\Delta; \Gamma \vdash_v V : A[A'/X]}{\Delta; \Gamma \vdash_v \mathrm{pack}(A', V) \text{ as } \exists X.A : \exists X.A}$$

$$\frac{\Delta; \Gamma \vdash_c M : \underline{B}}{\Delta; \Gamma \vdash_v \mathrm{thunk}\ M : U\underline{B}}$$

$$\frac{\Delta; \Gamma, x : A \vdash_c M : \underline{B}}{\Delta; \Gamma \vdash_c \lambda x : A.M : A \to \underline{B}}$$

$$\frac{\Delta; \Gamma \vdash_c M : A \to \underline{B} \qquad \Delta; \Gamma \vdash_v N : A}{\Delta; \Gamma \vdash_c MN : \underline{B}}$$

$$\frac{\Delta; \Gamma * x : A \vdash_c M : \underline{B}}{\Delta; \Gamma \vdash_c \alpha x : A.M : A \mathrel{-\!\!*} \underline{B}}$$

$$\frac{\Delta; \Gamma_1 \vdash_c M : A \mathrel{-\!\!*} \underline{B} \qquad \Delta; \Gamma_2 \vdash_v N : A}{\Delta; \Gamma_1 * \Gamma_2 \vdash_c M@N : \underline{B}}$$

$$\frac{\Delta, X; \Gamma \vdash_c M : \underline{B}}{\Delta; \Gamma \vdash_c \Lambda X.M : \forall X.\underline{B}}$$

$$\frac{\Delta; \Gamma \vdash_c\colon M : \forall X.\underline{B} \qquad \Delta \vdash A}{\Delta; \Gamma \vdash_c M[A] : \underline{B}[A/X]}$$

$$\frac{\Delta; \Gamma \vdash_v V : A}{\Delta; \Gamma \vdash_c \mathrm{ret}\ V : FA}$$

$$\frac{\Delta;\Gamma \vdash_v V : U\underline{B}}{\Delta;\Gamma \vdash_c \text{: force } V : \underline{B}}$$

$$\frac{\Delta;\Gamma * (\sigma : \mathrm{Case}A) \vdash_c M : \underline{B} \qquad \Delta \vdash A}{\Delta;\Gamma \vdash_c \text{newcase}_A x; M : \underline{B}}$$

$$\frac{\Delta;\Gamma \vdash_v V : \mathrm{OSum} \qquad \Delta;\Gamma \vdash_v \sigma : \mathrm{Case}\ A \qquad \Delta;\Gamma, x : A \vdash M : \underline{B} \qquad \Delta;\Gamma \vdash_c N : \underline{B}}{\Delta;\Gamma \vdash_c \text{match } V \text{ with } \sigma\{ \text{ inj } x.M \| N\} : \underline{B}}$$

$$\frac{\Delta;\Gamma \vdash_v V : A_1 \times A_2 \qquad \Delta;\Gamma, x : A_1, y : A_2 \vdash_c M : \underline{B}}{\Delta;\Gamma \vdash_c \text{let } (x, y) = V; M : \underline{B}}$$

$$\frac{\Delta;\Gamma \vdash_v V : A_1 * A_2 \qquad \Delta;\Gamma * x : A_1 * y : A_2 \vdash_c M : \underline{B}}{\Delta;\Gamma \vdash_c \text{let } (x * y) = V; M : \underline{B}}$$

$$\frac{\Delta \vdash \underline{B} \qquad \Delta;\Gamma \vdash_v V : \exists X.A \qquad \Delta, X;\Gamma, x : A \vdash_c M : \underline{B}}{\Delta;\Gamma \vdash_c \text{: unpack}(X, x) = V; M : \underline{B}}$$

2 Meta Language

2.1 Raw Formulas

$$\begin{array}{llll}
\text{Formula} & \phi, \psi & ::= & t =_A u \\
& & | & R(t, u) \\
& & | & \phi \implies \psi \\
& & | & \phi \wedge \psi \\
& & | & \phi \vee \psi \\
& & | & \exists x\colon A.\phi \\
& & | & \exists X.\phi \\
& & | & \exists \underline{X}.\phi \\
& & | & \exists R\colon \mathit{Rel}[A, B].\phi \\
& & | & \forall x\colon A.\phi \\
& & | & \forall X.\phi \\
& & | & \forall \underline{X}.\phi \\
& & | & \forall R\colon \mathit{Rel}[A, B].\phi
\end{array}$$

2.2 Typed Formulas

Propositions, or well-formed formulas, use a term environment Γ, type environment Δ and relation environment Θ. The typing judgement for Propositions is $\Delta;\Gamma;\Theta \vdash_p P$. There are value relations and computation relations.

$$\frac{\Delta;\Gamma \vdash_{\_} t : A \qquad \Delta;\Gamma \vdash_{\_} u : A}{\Delta;\Gamma;\Theta \vdash_p t =_A u} \text{ for } v, c$$

$$\frac{\Delta;\Gamma \vdash_{\_} t : A \qquad \Delta;\Gamma \vdash_{\_} u : B \qquad R : \mathit{Rel}_{\_}[A, B] \in \Theta}{\Delta;\Gamma;\Theta \vdash_p R(t, u)} \text{ for } v, c$$

$$\frac{\Delta;\Gamma;\Theta \vdash_p \phi \qquad \Delta;\Gamma;\Theta \vdash_p \psi}{\Delta;\Gamma;\Theta \vdash_p \phi\Box\psi} \; \Box \in \{\wedge, \vee, \implies\}$$

what about *? Something like exists fresh

$$\frac{\Delta;\Gamma, x : A|\Theta \vdash_p \phi}{\Delta;\Gamma;\Theta \vdash_p \exists x : A.\phi}$$

$$\frac{\Delta;\Gamma;\Theta \vdash_p \phi}{\Delta;\Gamma;\Theta \vdash_p \exists X.\phi} \; X \notin FV(\Delta,\Gamma,\Theta)$$

$$\frac{\Delta;\Gamma;\Theta \vdash_p \phi}{\Delta;\Gamma;\Theta \vdash_p \exists \underline{X}.\phi} \; \underline{X} \notin FV(\Delta,\Gamma,\Theta)$$

$$\frac{\Delta;\Gamma;\Theta, R : Rel\_[A,B] \vdash_p \phi}{\Delta;\Gamma;\Theta \vdash_p \exists R : Rel\_[A,B].\phi} \; \text{for } \{v,c\}$$

$$\frac{\Delta;\Gamma, x : A|\Theta \vdash_p \phi}{\Delta;\Gamma;\Theta \vdash_p \forall x : A.\phi}$$

$$\frac{\Delta;\Gamma;\Theta \vdash_p \phi}{\Delta;\Gamma;\Theta \vdash_p \forall X.\phi} \; X \notin FV(\Delta,\Gamma,\Theta)$$

$$\frac{\Delta;\Gamma;\Theta \vdash_p \phi}{\Delta;\Gamma;\Theta \vdash_p \forall \underline{X}.\phi} \; \underline{X} \notin FV(\Delta,\Gamma,\Theta)$$

$$\frac{\Delta;\Gamma;\Theta, R : Rel\_[A,B] \vdash_p \phi}{\Delta;\Gamma;\Theta \vdash_p \forall R : Rel\_[A,B].\phi} \; \text{for } \{v,c\}$$

2.3 Typed Relations

Relations are of the form $(x : A, y : B).\phi$ where ϕ is a proposition that can use x, y. The typing judgement for relations is $\Delta;\Gamma;\Theta \vdash_r (x : A, y : B).\phi : Rel\_[A,B]$. The body of the relation is a proposition. Here we pay attention to the difference between value and computation relations.

again, what about *?

$$\frac{\Delta;\Gamma, x : A \vdash_v t : C \qquad \Delta;\Gamma, y : B \vdash_v u : C}{\Delta;\Gamma;\Theta \vdash_r (x : A, y : B).t =_C u : Rel_v[A,B]}$$

secretly inserting stoup

$$\frac{\Delta;\Gamma|x : \underline{A} \vdash_c t : \underline{C} \qquad \Delta;\Gamma|y : \underline{B} \vdash_c u : \underline{C}}{\Delta;\Gamma;\Theta \vdash_r (x : \underline{A}, y : \underline{B}).t =_{\underline{C}} u : Rel_c[\underline{A},\underline{B}]}$$

Given some $x : A$ and $y : B$ the terms t, u are related by R, thus we have a relation on A, B. Think of these like a lambda abstraction over two parameters. If the body is related, we can

$$\frac{\Delta; \Gamma, x : A \vdash_v t : C \qquad \Delta; \Gamma, y : B \vdash_v u : D}{\Delta; \Gamma; \Theta, R : Rel_v[C, D] \vdash_r (x : A, y : B).R(t, u) : Rel_v[A, B]}$$

$$\frac{\Delta; \Gamma | x : \underline{A} \vdash_c t : \underline{C} \qquad \Delta; \Gamma | y : \underline{B} \vdash_c u : \underline{D}}{\Delta; \Gamma; \Theta, \underline{R} : Rel_c[\underline{C}, \underline{D}] \vdash_r (x : \underline{A}, y : \underline{B}).\underline{R}(t, u) : Rel_c[\underline{A}, \underline{B}]}$$

What is the intuition here? This rule is in figure 5 of the PE logic paper.

$$\frac{\Delta; \Gamma; \Theta \vdash_p \phi \qquad \Delta; \Gamma; \Theta \vdash_r (x : A, y : B).\psi : Rel_v A, B}{\Delta; \Gamma; \Theta \vdash_r (x : A, y : B).\phi \implies \psi : Rel_v[A, B]} \text{ also } c \text{ version}$$

?

$$\frac{\Delta; \Gamma; \Theta \vdash ? \qquad \Delta; \Gamma; \Theta \vdash ?}{\Delta; \Gamma; \Theta \vdash_r (x : A, y : B).\phi \wedge \psi : Rel_v[A, B]} \text{ also } c \text{ version}$$

$$\frac{\Delta; \Gamma, z : C; \Theta \vdash_r (x : A, y : B).\phi : Rel_v[A, B]?}{\Delta; \Gamma; \Theta \vdash_r (x : A, y : B).\forall(z : C).\phi : Rel_v[A, B]} \text{ also } c \text{ version}$$

$$\frac{\Delta, X; \Gamma; \Theta \vdash_r (x : A, y : B).\phi : Rel_v[A, B]?}{\Delta; \Gamma; \Theta \vdash_r (x : A, y : B).\forall X.\phi : Rel_v[A, B]} \text{ also } c \text{ version}$$

$$\frac{\Delta, X; \Gamma; \Theta, R : Rel_{\mathbf{n}}[C, D] \vdash_r (x : A, y : B).\phi : Rel_{\mathbf{m}}[A, B]}{\Delta; \Gamma; \Theta \vdash_r (x : A, y : B).\forall(R : Rel_{\mathbf{n}}[C, D]).\phi : Rel_{\mathbf{m}}[A, B]} \ \mathbf{n}, \mathbf{m} \in \{v, c\}$$

Analogous versions for $\exists$ connectives.

2.4 Deduction Rules

The judgement for deduction sequence are of the form $\Delta; \Gamma; \Theta; \Phi \vdash_d \psi$ where Δ is a type environment, Γ is a term environment, Θ is a relation environment, Θ is a proposition environment, and ψ is a proposition. like term intro and elim, but without proof terms also for computations?

$$\frac{\Delta; \Gamma \vdash_v t : A}{\Delta; \Gamma; \Theta; \Phi \vdash_d t =_A t} \text{ refl}$$

$$\frac{\Delta; \Gamma; \Theta; \Phi \vdash_d t =_A u \qquad \Delta; \Gamma; \Theta; \Phi \vdash_d \phi[t/x]}{\Delta; \Gamma; \Theta; \Phi \vdash_d \phi[u/x]} \text{ subst}$$

$$\frac{\Delta; \Gamma; \Theta; \Phi, \phi \vdash_d \psi}{\Delta; \Gamma; \Theta; \Phi \vdash_d \phi \implies \psi} \implies \text{Intro}$$

$$\frac{\Delta;\Gamma;\Theta;\Phi \vdash_d \phi \implies \psi \qquad \Delta;\Gamma;\Theta;\Phi \vdash_d \phi}{\Delta;\Gamma;\Theta;\Phi \vdash_d \psi} \implies \text{Elim}$$

and familiar rules for logical and ($\wedge$)

$$\frac{\Delta;\Gamma,(x:A);\Theta;\Phi \vdash_d \phi}{\Delta;\Gamma;\Theta;\Phi \vdash_d \forall(x:A).\phi} \forall \text{ Term Intro}$$

$$\frac{\Delta;\Gamma;\Theta;\Phi \vdash_d \forall(x:A).\phi \qquad \Delta;\Gamma \vdash_v t:A}{\Delta;\Gamma;\Theta;\Phi \vdash_d \phi[t/x]} \forall \text{ Term Elim}$$

$$X \notin FV(..), \text{ also for } c \ \frac{\Delta,X;\Gamma;\Theta;\Phi \vdash_d \phi}{\Delta;\Gamma;\Theta;\Phi \vdash_d \forall X.\phi} \forall \text{ Type Intro}$$

$$\text{also for } c \ \frac{\Delta;\Gamma;\Theta;\Phi \vdash_d \forall X.\phi \qquad \Delta \vdash A}{\Delta;\Gamma;\Theta;\Phi \vdash_d \phi[A/X]} \forall \text{ Type Elim}$$

$$\text{also for } c \ \frac{\Delta;\Gamma;\Theta,R:Rel_v[A,B];\Phi \vdash_d \phi}{\Delta;\Gamma;\Theta;\Phi \vdash_d \forall(R:Rel_v[A,B]).\phi} \forall \text{ Rel Intro}$$

$$\text{also for } c \ \frac{\Delta;\Gamma;\Theta;\Phi \vdash_d \forall(R:Rel_v[A,B]).\phi \qquad \Delta;\Gamma;\Theta \vdash_r (x:A,y:B).\psi : Rel_v[A,B]}{\Delta;\Gamma;\Theta;\Phi \vdash_d \phi[\psi[t/x,u/y]/R(t,u)]} \forall \text{ Rel Elim}$$

2.5 Axioms & Axiom Schemas

2.5.1 Congruences

$$\frac{\Delta;\Gamma,(x:A) \vdash_c t,u:\underline{B} \qquad \Delta;\Gamma,(x:A);\Theta;\Phi \vdash_d t = u}{\Delta;\Gamma;\Theta;\Phi \vdash_d (\lambda(x:A).t) =_{A\to\underline{B}} (\lambda(x:A).u)} \ x \notin FV(\Phi)$$

2.5.2 Beta / Eta Laws

2.5.3 Parametricity