1 Issue

There is a variance issue when trying to add a **computational** separating function type to Levy's dynamic store model[1] [1][2]. Take the category of worlds to be $\mathcal{W} := FinSet_{mono}$, the value category to be $\mathcal{V} := [\mathcal{W}, Set]$ and the computation category to be $\mathcal{C} := [\mathcal{W}^{op}, Set]$. Value judgments $\Gamma \vdash_v M : A$ are denoted as morphisms in $\mathcal{V}$. Computation judgments $\Gamma \vdash_c M : B$ are denoted as families of maps $\forall (w : ob\ W) \to Set[[\![\Gamma]\!](w), [\![B]\!](w)]$. Note that we are dropping the storage part (S) of Levy's monad. The monoidal structure on $\mathcal{W}$ given by disjoint union yields a monoidal structure on $\mathcal{V}$ via the Day convolution[3].

$$(A \otimes_D B)_0(w_1) = \int^{w_2,w_3} \mathcal{W}[w_2 \otimes w_3, w_1] \times A(w_2) \times B(w_3)$$

The separating function in the **value category**$(A, B : ob\ \mathcal{V})$ is given by:

$$(A \twoheadrightarrow B)_0(w) = \mathcal{V}[[\![A]\!], [\![B]\!](w \otimes \_)]$$

And we have that:

$$\mathcal{V}[A \otimes_D B, C] \cong \mathcal{V}[A, B \twoheadrightarrow C] \tag{1}$$

The **computational** function type $(A : ob\ \mathcal{V}, B : ob\ \mathcal{C})$ is given by:

$$(A \to B)_0(w) = Set[[\![A]\!](w), [\![B]\!](w)]$$

We can try to define the **computational** separating function $(A : ob\ \mathcal{V}, B : ob\ \mathcal{C})$ as :

$$(A \twoheadrightarrow B)_0(w) = \forall (w' : ob\ W) \to Set[[\![A]\!](w'), [\![B]\!](w \otimes w')]$$

which is a contravariant functor. We should expect the following isomorpism of types(in Set?):

$$(A \otimes_D B) \to C \cong A \to B \twoheadrightarrow C$$

given by:

$$\begin{aligned}
&fun : ((A \otimes_D B) \to C) \to (A \to B \twoheadrightarrow C)\\
&fun\ M\ w_1\ (a : [\![A]\!](w_1))\ w_2\ (b : [\![B]\!](w_2)) = M(w_1 \otimes w_2)(id_{w_1 \otimes w_2}, a, b)\\
\\
&inv : (A \to B \twoheadrightarrow C) \to ((A \otimes_D B) \to C)\\
&inv\ M\ w_1\ (w_2, w_3, f : w_2 \otimes w_3 \to w_1, a : [\![A]\!](w_2), b : [\![B]\!](w_3)) = [\![B]\!]_1(f)(M\ w_2\ a\ w_3\ b)
\end{aligned}$$

However, the variance of $[\![B]\!]$ gives us $[\![B]\!]_1(f) : [\![B]\!](w_1) \to [\![B]\!](w_2 \otimes w_3)$ which is the opposite direction that *we want*[4].

1.1 Our Model

I was able to derive an *inverse* (likely not able to show the isomorphism) in our model, but it felt like a hack and involves an arbitrary choice. Without reproducing all the details here, the gist is the following:

$$\begin{aligned}
&s2p : \mathcal{V}[A \otimes_D B, A \times B]\\
&s2p(w_1)(w_2, w_3, f : w_2 \otimes w_3 \hookrightarrow w_1, a, b) = [\![A]\!]_1(inl\ ; f)(a), [\![B]\!]_1(inr\ ; f)(b)\\
\\
&inv : (A \to B \twoheadrightarrow C) \to ((A \otimes_D B) \to C)\\
&inv\ M\ w\ s = [\![B]\!]_1(inl\ or\ inr)(M\ w\ (\pi_1\ p)\ w\ (\pi_2\ p))\\
&\quad where\\
&\qquad p : [\![A \times B]\!](w)\\
&\qquad p = s2p\ w\ s
\end{aligned}$$

[1] Chapter 6

[2] The following issue exists in our setup too.

[3] *covariant* Day convolution given by taking the monoidal structure on $\mathcal{W}^{op}$ and then applying the day convolution

[4] Meaning this is how the isomorpism goes in (1)

2 A possible way forward

I'm starting to look at a weaker version of the setup in section 2.4 of [2] which is a model of $\mathrm{SystemF}_{\mu}^{ref}$. I think we had already worked out the computational separating function for an algebra model of CBPV.

References

[1] Levy, P. *Call-By-Push-Value: A Functional/Imperative Synthesis*. 01 2004.

[2] Sterling, J., Gratzer, D., and Birkedal, L. Denotational semantics of general store and polymorphism, Apr. 2023. arXiv:2210.02169 [cs].