1 Syntax

1.1 Judgements

$$\Gamma \text{ Vctx}$$
$$\Theta \text{ Rctx}$$
$$\Gamma \vdash A \text{ VType}$$
$$\Gamma \vdash \underline{B} \text{ CType}$$
$$\Gamma \vdash R \text{ RType}$$
$$\Gamma \vdash x : A$$
$$\Gamma | \Delta \vdash x : \underline{B}$$
$$\Gamma ; \Theta \vdash \rho : R$$
$$\Gamma ; \Theta \vdash \phi : \mathbf{prop}$$
$$\Gamma ; \Theta | \Phi \vdash \psi$$

Remark. *No stoup needed for the CType judgement*

Remark. *Univ is our judgement for universe types.*

Remark. *Following PE logic, drop the stoup from prop, relation, and derivation judgements.*

1.2 Contexts

$$\frac{}{\cdot \text{ Vctx}}$$

$$\frac{\Gamma \text{ Vctx} \qquad \Gamma \vdash A \text{ VType}}{\Gamma, x : A \text{ Vctx}}$$

$$\frac{\Gamma \text{ Vctx} \qquad A \text{ VType}}{\Gamma, A \text{ Vctx}}$$

$$\frac{\Gamma \text{ Vctx} \qquad \underline{A} \text{ CType}}{\Gamma, \underline{A} \text{ Vctx}}$$

$$\frac{}{\cdot \text{ RCtx}}$$

$$\frac{\Theta \text{ Rctx} \qquad \Gamma \vdash R \text{ RType}}{\Theta, \rho : R \text{ Rctx}}$$

1.3 Types

Definition 1. *Value Types*

$$A := X \mid \text{Unit} \mid \text{OSum} \mid \text{Case } A \mid A \times A \mid \exists X.A \mid \exists \underline{X}.A \mid U\underline{B}$$

$$\frac{}{\Gamma \vdash \text{Unit VType}}\ \text{Unit-F}$$

$$\frac{}{\Gamma \vdash \text{OSum VType}}\ \text{OSum-F}$$

$$\frac{\Gamma \vdash A \text{ VType}}{\Gamma \vdash \text{Case } A \text{ VType}}\ \text{Case-F}$$

Remark. *Without guarded recursion, we limit case symbols to be of a restricted set of value types.*

$$\frac{\Gamma \vdash A \text{ VType} \qquad \Gamma \vdash A' \text{ VType}}{\Gamma \vdash A \times A' \text{ VType}}\ \times\text{-F}$$

$$\frac{\Gamma, X \vdash A \text{ VType}}{\Gamma \vdash \exists X.A \text{ VType}}\ \exists_V\text{-F}$$

$$\frac{\Gamma, \underline{X} \vdash A \text{ VType}}{\Gamma \vdash \exists \underline{X}.A \text{ VType}}\ \exists_C\text{-F}$$

Remark. *Note that quantification is impredicative.*

$$\frac{\Gamma \vdash \underline{B} \text{ CType}}{\Gamma \vdash U\underline{B} \text{ VType}}\ U\text{-F}$$

Definition 2. *Computation types*

$$\underline{B} := \underline{X} \mid A \to \underline{B} \mid \forall X.\underline{B} \mid \forall \underline{X}.\underline{B} \mid FA$$

$$\frac{\Gamma \vdash A \text{ VType} \qquad \Gamma \vdash \underline{B} \text{ Ctype}}{\Gamma \vdash A \to \underline{B} \text{ Ctype}}\ \to\text{-F}$$

$$\frac{\Gamma, X \vdash \underline{B} \text{ CType}}{\Gamma \vdash \forall X.\underline{B} \text{ CType}}\ \forall_V\text{-F}$$

$$\frac{\Gamma, \underline{X} \vdash \underline{B} \text{ CType}}{\Gamma \vdash \forall \underline{X}.\underline{B} \text{ CType}}\ \forall_C\text{-F}$$

$$\frac{\Gamma \vdash A \text{ VType}}{\Gamma \vdash FA \text{ CType}}\ F\text{-F}$$

TODO: Introduction and Elimination rules for CBPV OSum, routine

Definition 3. *Logic Types*

$$\frac{}{\Gamma \vdash \textbf{prop}\ \text{VType}}\ \textbf{prop}\text{-F}$$

Question. *Do we have a duplicate/separate logic for computation propositions? That is, a Hyperdoctrine on $\mathcal{C}_{\mathcal{T}}$? Or, do we factor a computation logic through the hyperdoctrine on the value cateory? PE logic seems to choose the latter, but they are using a subobject interpretation and we need a more general hyperdoctrine interpretation.*

$$\frac{\Gamma \vdash A\ \ \mathit{VType} \qquad \Gamma \vdash B\ \ \mathit{VType}}{\Gamma \vdash \mathit{Rel}_V[A, B]\ \ \mathit{RType}}\ \mathit{Rel}_V\text{-}F$$

$$\frac{\Gamma \vdash \underline{A}\ \ \mathit{CType} \qquad \Gamma \vdash \underline{B}\ \ \mathit{CType}}{\Gamma \vdash \mathit{Rel}_C[A, B]\ \ \mathit{RType}}\ \mathit{Rel}_C\text{-}F$$

Definition 4. *Propositions*

Remark. *Brushing over any distinction between value and computation propositions for the moment. Plausible usages for computation propositions highlighted in blue.*

$$\begin{aligned}
\phi := {} & \top \mid \bot \mid (t =_A u) \mid (t =_{\underline{B}} u) \mid R(t, u) \mid \underline{R}(t, u) \\
& \mid \phi \implies \psi \mid \phi \wedge \psi \mid \phi \vee \psi \\
& \mid\ \forall (x : A).\phi \mid \forall X, \phi \mid \forall \underline{X}, \phi \mid \forall (R : \mathit{Rel}_V[A, B]), \phi \mid \forall (\underline{R} : \mathit{Rel}_C[\underline{A}, \underline{B}]), \phi \\
& \mid\ \exists (x : A).\phi \mid \exists X, \phi \mid \exists \underline{X}, \phi \mid \exists (R : \mathit{Rel}_V[A, B]), \phi \mid \exists (\underline{R} : \mathit{Rel}_C[\underline{A}, \underline{B}]), \phi
\end{aligned}$$

$$\frac{}{\Gamma; \Theta \vdash \top : \textbf{prop}}$$

$$\frac{}{\Gamma; \Theta \vdash \bot : \textbf{prop}}$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma; \Theta \vdash t =_A u : \textbf{prop}}$$

$$\frac{\Gamma | \Delta \vdash t : \underline{B} \qquad \Gamma | \Delta \vdash u : \underline{B}}{\Gamma; \Theta \vdash t =_{\underline{B}} u : \textbf{prop}}$$

Question. *Assuming we denote* **prop** *as an internal heyting alebra in the value category, how are we denoting equality of computation types? $=$ is interpreted as right adjoint to $\mathcal{P}(\Delta)$ where $\Delta : \mathcal{V}[X, X \times X]$*

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B \qquad \Gamma; \Theta \vdash R : \mathit{Rel}_V[A, B]}{\Gamma; \Theta \vdash R(t, u) : \textbf{prop}}$$

$$\frac{\Gamma|\Delta \vdash t : \underline{A} \qquad \Gamma|\Delta \vdash u : \underline{B} \qquad \Gamma;\Theta \vdash \underline{R} : Rel_C[\underline{A}, \underline{B}]}{\Gamma;\Theta \vdash \underline{R}(t,u) : \mathbf{prop}}$$

$$\frac{\Gamma;\Theta \vdash \phi : \mathbf{prop} \qquad \Gamma;\Theta \vdash \psi : \mathbf{prop}}{\Gamma;\Theta \vdash \phi \,\square\, \psi \mathbf{prop}} \;\square \in \{\implies, \wedge, \vee\}$$

$$\frac{\Gamma, x : A;\Theta \vdash \phi : \mathbf{prop}}{\Gamma;\Theta \vdash \mathcal{Q}(x : A), \phi : \mathbf{prop}} \;\mathcal{Q} \in \{\forall, \exists\}$$

$$\frac{\Gamma;\Theta \vdash \phi : \mathbf{prop}}{\Gamma;\Theta \vdash \mathcal{Q}\mathcal{X}, \phi : \mathbf{prop}} \;\mathcal{Q} \in \{\forall, \exists\}, \mathcal{X} \in \{X, \underline{X}\}, \mathcal{X} \notin FV(\Gamma;\Theta)$$

$$\frac{\Gamma;\Theta, R \vdash \mathcal{Q}(R : Rel_*[A, B] : \mathbf{prop})}{\Gamma;\Theta \vdash \mathcal{Q}(R : Rel_*[A, B]), \phi : \mathbf{prop}} \;\mathcal{Q} \in \{\forall, \exists\}, * \in \{V, C\}$$

Definition 5. *Relations*

$$\frac{\Gamma, x : A, y : B;\Theta \vdash \phi : \mathbf{prop}}{\Gamma;\Theta \vdash (x : A, y : B).\phi : Rel_V[A, B]}$$

Question. *Definable relations seem sufficient for value relations? How about computation relations? We don't have the stoup in our* **prop** *formation judgement (like so:* $\Gamma;\Theta|\Delta \vdash \phi : \mathbf{prop}$*) AND* **prop** *"should" be interpreted as the internal HA in* $\mathcal{V}$*. The definable computation relation would be something like* $\Gamma;\Theta|(x : \underline{A} \times \underline{B}) \vdash \phi : \underline{\mathbf{prop}}$*. Instead, explicitly define the computation relation formation rules.*

$$\frac{\Gamma|x : \underline{A} \vdash t : \underline{C} \qquad \Gamma|y : \underline{B} \vdash u : \underline{C}}{\Gamma;\Theta \vdash (x : \underline{A}, y : \underline{B}).t =_{\underline{C}} u : Rel_C[A, B]}$$

$$\frac{\Gamma|x : \underline{A} \vdash t : \underline{A'} \qquad \Gamma|y : \underline{B} \vdash u : \underline{B'}}{\Gamma;\Theta, \underline{R} : Rel_C[A', B'] \vdash (x : \underline{A}, y : \underline{B}).R(t, u) : Rel_C[A, B]}$$

etc..

Definition 6. *Deduction rules*

$$\frac{}{\Gamma;\Theta|\Phi \vdash \top} \;\top\text{-}I$$

$$\frac{\Gamma \vdash x : A}{\Gamma;\Theta|\Phi \vdash x =_A x} \;=_A\text{-}I$$

$$\frac{\Gamma|\Delta \vdash x : \underline{A}}{\Gamma, \Delta;\Theta|\Phi \vdash x =_{\underline{A}} x} \;=_{\underline{A}}\text{-}I$$

Remark. *Here we need to be careful with equality of computation terms. The PE logic states there is an equivalence* $\Gamma; \Theta|\Delta \vdash t =_{\underline{A}} u \equiv \Gamma, \Delta; \Theta|- \vdash t =_{\underline{A}} u$ *because of the "faithfullness of the forgetful functor* U*" in their model. Check this in our model.*

$$\frac{\Gamma; \Theta|\Phi \vdash t =_A u \qquad \Gamma; \Theta|\Phi \vdash \phi[t/x]}{\Gamma; \Theta|\Phi \vdash \phi[u/x]} =_A\text{-}E$$

Question. *Lawvere style mate rules for quantification and equality?*

$$\frac{\Gamma; \Theta|\Phi, \phi \vdash \psi}{\Gamma; \Theta|\Phi \vdash \phi \implies \psi} \implies\text{-}I$$

$$\frac{\Gamma; \Theta|\Phi \vdash \phi \implies \psi \qquad \Gamma; \Theta|\Phi \vdash \phi}{\Gamma; \Theta|\Phi \vdash \psi} \implies\text{-}E$$

$$\frac{\Gamma; \Theta|\Phi \vdash \phi \qquad \Gamma; \Theta|\Phi \vdash \psi}{\Gamma; \Theta|\Phi \vdash \phi \wedge \psi} \wedge\text{-}I$$

$$\frac{\Gamma; \Theta|\Phi \vdash \phi \wedge \psi}{\Gamma; \Theta|\Phi \vdash \phi} \wedge\text{-}E_1$$

$$\frac{\Gamma; \Theta|\Phi \vdash \phi \wedge \psi}{\Gamma; \Theta|\Phi \vdash \psi} \wedge\text{-}E_2$$

$$\frac{\Gamma; \Theta|\Phi \vdash \phi}{\Gamma; \Theta|\Phi \vdash \phi \vee \psi} \vee\text{-}I_1$$

$$\frac{\Gamma; \Theta|\Phi \vdash \psi}{\Gamma; \Theta|\Phi \vdash \phi \vee \psi} \vee\text{-}I_2$$

$$\frac{\Gamma; \Theta|\Phi, \phi \vdash \xi \qquad \Gamma; \Theta|\Phi, \psi \vdash \xi}{\Gamma; \Theta|\Phi, \phi \vee \psi \vdash \xi} \vee\text{-}E$$

$$\frac{\Gamma, x : A; \Theta|\Phi \vdash \phi}{\Gamma; \Theta|\Phi \vdash \forall(x : A), \phi} \forall_{vtm}\text{-}I,\ x \notin FV(\Phi)$$

$$\frac{\Gamma; \Theta|\Phi \vdash \forall(x : A), \phi \qquad \Gamma \vdash t : A}{\Gamma; \Theta|\Phi \vdash \phi[t/x]} \forall_{vtm}\text{-}E$$

$$\frac{\Gamma; \Theta|\Phi \vdash \phi}{\Gamma; \Theta|\Phi \vdash \forall X, \phi} \forall_{vty}\text{-}I,\ X \notin FV(\Gamma, \Theta, \Phi)$$

$$\frac{\Gamma; \Theta|\Phi \vdash \forall X, \phi \qquad A\ \ VType}{\Gamma; \Theta|\Phi \vdash \phi[A/X]} \forall_{vty}\text{-}E$$

$$\frac{\Gamma; \Theta | \Phi \vdash \phi}{\Gamma; \Theta | \Phi \vdash \forall \underline{X}, \phi} \; \forall_{cty}\text{-}I, \; \underline{X} \notin FV(\Gamma, \Theta, \Phi)$$

$$\frac{\Gamma; \Theta | \Phi \vdash \forall \underline{X}, \phi \qquad \underline{A} \;\; CType}{\Gamma; \Theta | \Phi \vdash \phi[\underline{A}/\underline{X}]} \; \forall_{cty}\text{-}E$$

$$\frac{\Gamma; \Theta, R : Rel_V[A, B] | \Phi \vdash \phi}{\Gamma; \Theta | \Phi \vdash \forall (R : Rel_V[A, B]), \phi} \; \forall_{vrel}\text{-}I$$

$$\frac{\Gamma; \Theta | \Phi \vdash \forall (R : Rel_V[A, B]), \phi \qquad \Gamma; \Theta \vdash (x : A, y : B).\psi : Rel_V[A, B]}{\Gamma; \Theta | \Phi \vdash \phi[(\psi[t/x, u/y])/R(t, u)]} \; \forall v_r el\text{-}E$$

$$\frac{\Gamma; \Theta, \underline{R} : Rel_C[\underline{A}, \underline{B}] | \Phi \vdash \phi}{\Gamma; \Theta | \Phi \vdash \forall (\underline{R} : Rel_C[\underline{A}, \underline{B}]), \phi} \; \forall_{crel}\text{-}I$$

$$\frac{\Gamma; \Theta | \Phi \vdash \forall (\underline{R} : Rel_C[\underline{A}, \underline{B}]), \phi \qquad \Gamma; \Theta \vdash (x : \underline{A}, y : \underline{B}).\psi : Rel_C[\underline{A}, \underline{B}]}{\Gamma; \Theta | \Phi \vdash \phi[(\psi[t/x, u/y])/\underline{R}(t, u)]} \; \forall crel\text{-}E$$

TODO: deduction rules for existentials, routine
TODO: computation rules and term equalities, routine (except for)
TODO: definition of the substitution for relations into types

2 Semantics

2.1 PE Logic

PE logic uses an algebra model of CBPV with a subobject interpretation of the logic. The forgetful functor U is faithful

$$U_{X,Y} : C[X, Y] \hookrightarrow V[UX, UY]$$

so it is injective on homsets.

Because U preserves limits, every monomorphism $\underline{A} \to \underline{B}$ in C is mapped to a monomorphism $UA \to UB$ in V. Thus

$$\forall (\underline{X} : ob\ C), \; Sub_C(\underline{X}) \to Sub_V(U\underline{X})$$

is an order embedding

$$x \leq y \iff f(x) \leq f(y)$$

Value types,A, are interpreted as a set $V[\![A]\!]$ and computation types,$\underline{A}$, are interpreted as algebras $C[\![\underline{A}]\!]$. In PE logic, every computation type is also a value type. Thus it is given two interpretations and the relation between the interpretations is given by the equation $U(C[\![\underline{A}]\!]) = V[\![\underline{A}]\!]$.