1 Bicartesian Doubly Closed Category

Given a category $\mathcal{C}$, its presheaf category ($\widehat{\mathcal{C}} := [\mathcal{C}^{op}, Set]$) is bicartesian closed. Given a monoidal category $(\mathcal{C}, \otimes_C, I_C)$, its presheaf category is bicartesian closed and monoidal closed via the Day convolution. The monoidal product is given by:

$$(P \otimes^{Day} Q)(x) = \int^{y,z} \mathcal{C}[x, y \otimes_C z] \times P(y) \times Q(z)$$

The Day monoidal product has the universal property that any maps out of it are in bijective correspondence with a family of maps natural in x and y (Agda): [1]

$$\widehat{\mathcal{C}}[P \otimes^{Day} Q, R] \;\cong\; \widehat{\mathcal{C} \times \mathcal{C}}[P \overline{\times} Q, R \circ \otimes_C] \;\cong\; \Pi_{x,y:\, ob\ C}\ Set[P(x) \times Q(y), R(x \otimes_C y)]$$

The monoidal closed structure is given by:

$$(P \twoheadrightarrow\!\!\!* \, Q)(X) = \widehat{\mathcal{C}}[P, Q(X, -)]$$

With the universal property that the closed structure is right adjoint to the tensor (Agda):

$$\widehat{\mathcal{C}}[A \otimes_C B, C] \cong \widehat{\mathcal{C}}[A, B \twoheadrightarrow\!\!\!* \, C] \tag{1}$$

Bicartesian doubly closed categories have been used in the denotational semantics of bunched type theories [5][1][4].

2 Towards Bunched Call By Push Value with Dynamic Store

Categorical models of dynamic store use presheaf categories to model the dependence of the heap structure on a current *world* [3][6][2]. Seemingly none of these existing models attempt to combine a call by push value language with the separating type connectives, $\otimes$ and $\twoheadrightarrow\!\!\!*$, used in bunched type theories. Our investigation into possible models of such a language have run into some potential issues. To illustrate this, we will start with the model for a call by push value language with dynamic store presented in chapter 7 of Levy's thesis.

2.1 Definitions

Let $(C, \otimes_C, I_C)$ be a monoidal category, the value category be $\mathcal{V} := [C, Set]$, computation category $\mathcal{C} := [C^{op}, Set]$, and use the *standard* monad for ground dynamic store with $F : \mathcal{V} \to \mathcal{C}$ as:

$$F(A)(x) := \Sigma_{y:ob\ C}\Sigma_{f:C[x,y]}A(y)$$

and $U : \mathcal{C} \to \mathcal{V}$ as :

$$U(\underline{B})(x) := \Pi_{y:ob\ C}\Pi_{f:C[x,y]}\underline{B}(y)$$

The oblique morphisms in this model are given by families of maps:

$$\mathcal{O}[A, \underline{B}] := \Pi_{x:ob\ C}Set[A(x), \underline{B}(x)]$$

we have the following isomorphims:

$$\mathcal{V}[A, U(\underline{B})] \cong \mathcal{O}[A, \underline{B}] \cong \mathcal{C}[F(A), \underline{B}]$$

And we can attempt to define a computation separating function by:

$$(A \twoheadrightarrow\!\!\!* \, \underline{B})(x) := \Pi_{y:ob\ C}Set[A(y), \underline{B}(x \otimes_C y)]$$

2.2 Problems with an Abstract Monoidal Category

Before committing to the category of worlds used in Levy's model, we will work with an arbitrary monoidal category $(C, \otimes_C, I_C)$.

[1] here $\overline{\times}$ is the *external* product

2.2.1 Issue 1: Universal Property of Tensor for Oblique Morphisms

Let's attempt to show the following:

$$\mathcal{O}[P \otimes Q, \underline{R}] \cong \mathcal{O}\times[P \overline{\times} Q, \underline{R} \circ \otimes_C]$$

where

$$\mathcal{O}\times[P \overline{\times} Q, \underline{R} \circ \otimes_C] := \Pi_{x,y:ob\ C} Set[P(x) \times Q(y), \underline{R}(x \otimes_C y)]$$

A problem arises when trying to define the backwards map of this isomorphims. Given $m : \mathcal{O}\times[P \overline{\times} Q, \underline{R} \circ \otimes_C]$ and $x : ob\ C$, we need to define a map $Set[(P \otimes Q)(x), \underline{R}(x)]$. This is a map out of a coequalizer [2] which we can attempt to give as a map induced from:

$$(f : y \otimes_C z \to x, p : P(y), q : Q(z)) \mapsto ? : \underline{R}(x)$$

However, using the data we currently have, we can only construct

$$m(y)(z)(p, q) : \underline{R}(y \otimes_C z)$$

and since $\underline{R}$ is contravariant, we can't use $\underline{R}(f) : \underline{R}(x) \to \underline{R}(y \otimes_C z)$. This is not surprising since the proof of this universal property in the value category $\mathcal{V}[P \otimes Q, R] \cong \mathcal{V}\times[P \overline{\times} Q, R \circ \otimes_C]$ uses the functorial action of R on f (see here) [3] So by swapping the variance of R (now $\underline{R}$ since it is from the computation category) this proof should break. Seemingly, this proof won't go through when we assume a generic monoidal category C. Perhaps we can recover this property if we work with a specific concrete category?

2.2.2 Issue 2: Universal Property of the Separating Function Type

3 Issue

There is a variance issue when trying to add a **computational** separating function type to Levy's dynamic store model[3] [4][5]. Take the category of worlds to be $\mathcal{W} := FinSet_{mono}$, the value category to be $\mathcal{V} := [\mathcal{W}, Set]$ and the computation category to be $\mathcal{C} := [\mathcal{W}^{op}, Set]$. Value judgments $\Gamma \vdash_v M : A$ are denoted as morphisms in $\mathcal{V}$. Computation judgments $\Gamma \vdash_c M : B$ are denoted as families of maps $\forall (w : ob\ W) \to Set[[\![\Gamma]\!](w), [\![B]\!](w)]$. Note that we are dropping the storage part (S) of Levy's monad. The monoidal structure on $\mathcal{W}$ given by disjoint union yields a monoidal structure on $\mathcal{V}$ via the Day convolution[6].

$$(A \otimes_D B)_0(w_1) = \int^{w_2, w_3} \mathcal{W}[w_2 \otimes w_3, w_1] \times A(w_2) \times B(w_3)$$

The separating function in the **value category**$(A, B : ob\ \mathcal{V})$ is given by:

$$(A \twoheadrightarrow B)_0(w) = \mathcal{V}[[\![A]\!], [\![B]\!](w \otimes \_)]$$

And we have that:

$$\mathcal{V}[A \otimes_D B, C] \cong \mathcal{V}[A, B \twoheadrightarrow C] \tag{2}$$

The **computational** function type $(A : ob\ \mathcal{V}, B : ob\ \mathcal{C})$ is given by:

$$(A \to B)_0(w) = Set[[\![A]\!](w), [\![B]\!](w)]$$

We can try to define the **computational** separating function $(A : ob\ \mathcal{V}, B : ob\ \mathcal{C})$ as :

$$(A \twoheadrightarrow B)_0(w) = \forall (w' : ob\ W) \to Set[[\![A]\!](w'), [\![B]\!](w \otimes w')]$$

which is a contravariant functor. We should expect the following isomorpism of types(in Set?):

$$(A \otimes_D B) \to C \cong A \to B \twoheadrightarrow C$$

[2] since coends in Set can be encoded as coequalizers

[3] note the difference in variance is due to the fact this proof is for presheaves and not covariant presheaves

[4] Chapter 6

[5] The following issue exists in our setup too.

[6] *covariant* Day convolution given by taking the monoidal structure on $\mathcal{W}^{op}$ and then applying the day convolution

given by:

$$fun : ((A \otimes_D B) \to C) \to (A \to B \twoheadrightarrow C)$$
$$fun\ M\ w_1\ (a : [\![A]\!](w_1))\ w_2\ (b : [\![B]\!](w_2)) = M(w_1 \otimes w_2)(id_{w_1 \otimes w_2}, a, b)$$

$$inv : (A \to B \twoheadrightarrow C) \to ((A \otimes_D B) \to C)$$
$$inv\ M\ w_1\ (w_2, w_3, f : w_2 \otimes w_3 \to w_1, a : [\![A]\!](w_2), b : [\![B]\!](w_3)) = [\![B]\!]_1(f)(M\ w_2\ a\ w_3\ b)$$

However, the variance of $[\![B]\!]$ gives us $[\![B]\!]_1(f) : [\![B]\!](w_1) \to [\![B]\!](w_2 \otimes w_3)$ which is the opposite direction that *we want*[7].

3.1 Our Model

I was able to derive an *inverse* (likely not able to show the isomorphism) in our model, but it felt like a hack and involves an arbitrary choice. Without reproducing all the details here, the gist is the following:

$$s2p : \mathcal{V}[A \otimes_D B, A \times B]$$
$$s2p(w_1)(w_2, w_3, f : w_2 \otimes w_3 \hookrightarrow w_1, a, b) = [\![A]\!]_1(inl\ ; f)(a), [\![B]\!]_1(inr\ ; f)(b)$$

$$inv : (A \to B \twoheadrightarrow C) \to ((A \otimes_D B) \to C)$$
$$inv\ M\ w\ s = [\![B]\!]_1(inl\ or\ inr)(M\ w\ (\pi_1\ p)\ w\ (\pi_2\ p))$$
$$\quad where$$
$$\quad\quad p : [\![A \times B]\!](w)$$
$$\quad\quad p = s2p\ w\ s$$

4 A possible way forward

I'm starting to look at a weaker version of the setup in section 2.4 of [6] which is a model of SystemF_μ^{ref}. I think we had already worked out the computational separating function for an algebra model of CBPV.

References

[1] Biering, B. On the Logic of Bunched Implications.

[2] Kammar, O., Levy, P. B., Moss, S. K., and Staton, S. A monad for full ground reference cells. In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)* (June 2017), pp. 1–12.

[3] Levy, P. *Call-By-Push-Value: A Functional/Imperative Synthesis.* 01 2004.

[4] O'Hearn, P. On bunched typing. *Journal of Functional Programming 13*, 4 (July 2003), 747–796.

[5] Pym, D. J. *The Semantics and Proof Theory of the Logic of Bunched Implications*, vol. 26 of *Applied Logic Series*. Springer Netherlands, Dordrecht, 2002.

[6] Sterling, J., Gratzer, D., and Birkedal, L. Denotational semantics of general store and polymorphism, Apr. 2023. arXiv:2210.02169 [cs].

[7] Meaning this is how the isomorpism goes in (1)