

1 Untyped Syntax

value types	A	$::=$	Unit Bool Case A OSum $A \times A$ $A * A$ $U \underline{B}$	
computation types	B	$::=$	$A \rightarrow \underline{B}$ $A \multimap \underline{B}$ $F A$	
values	V	$::=$	tt true false x $\text{inj}_V V$ (V, V) $\pi_1 V$ $\pi_2 V$ $(V * V)$ $\rho_1 V$ $\rho_2 V$ $\text{thunk } M$	variable injection into the open sum separating product
computations	M	$::=$	\mathcal{O} $\text{force } V$ $\text{ret } V$ $x \leftarrow M; N$ $M N$ $M @ N$ $\lambda x: A. M$ $\alpha x: A. M$ $\text{newcase}_A x; M$ $\text{match } V \text{ with } V \{ \text{inj } x. M \mid N \}$ $\text{if } V \text{ then } M \text{ else } N$	need monad for this separating function "allocate" a new case in the open sum match on open sum
value typing context	Γ	$::=$	\emptyset $x : A$ $\Gamma; \Gamma'$ $\Gamma \mathbin{\text{\tiny ;}} \Gamma'$	cartesian product monoidal product
stoup	Θ	$::=$	\cdot $\bullet : B$	

stoup forms? could add computation products

2 Typed Syntax

This version of the syntax is a combination of the ν -calculus [?] [?] of Pitts and Stark and the affine $\alpha\lambda$ -calculus of O’Hearn [?] expressed in CBPV and with the addition of an extensible sum type. [for \(rule airity > 1\) non separating connectives, contexts don’t need to be appended\(\$\Gamma\$; \$\Delta\$ \)?](#)

2.1 Structural Rules

; admits weakening, contraction, and exchange (cartesian closed structure)
 \S admits weakening and exchange (semicartisian symmetric monoidal structure)

2.2 Value Typing

2.2.1 Unit

$$\frac{}{\Gamma \vdash_v \text{tt} : \text{Unit}} \text{Intro}$$

2.2.2 Bool

$$\frac{}{\Gamma \vdash_v \text{true} : \text{Bool}} \text{Intro}_1$$

$$\frac{}{\Gamma \vdash_v \text{false} : \text{Bool}} \text{Intro}_2$$

$$\frac{\Gamma \vdash_v V : \text{Bool} \quad \Gamma \mid \cdot \vdash_c M_1 : B \quad \Gamma \mid \cdot \vdash_c M_2 : B}{\Gamma \mid \cdot \vdash_c \text{if } V \text{ then } M_1 \text{ else } M_2 : B} \text{Elim}$$

2.2.3 Var

$$\frac{}{x : A \vdash_v x : A} \text{Intro}$$

2.2.4 OSum

$$\frac{\Gamma \vdash_v V_c : \text{Case A} \quad \Gamma \vdash_v V : A}{\Gamma \vdash_v \text{inj}_{V_c} V : \text{OSum}} \text{Intro}$$

$$\frac{\Gamma \vdash_v V_T : \text{Case A} \quad \Gamma \vdash_v V : \text{OSum} \quad \Gamma; (x : A) \mid \cdot \vdash_c M : B \quad \Gamma \mid \cdot \vdash_c N : B}{\Gamma \mid \cdot \vdash_c \text{match } V_T \text{ with } V \{ \text{inj } x.M \mid N \}} \text{Elim}$$

2.2.5 Value Product

$$\frac{\Gamma \vdash_v N : A_1 \quad \Gamma \vdash_v M : A_2}{\Gamma \vdash_v (N, M) : A_1 \times A_2} \text{Intro}$$

$$\frac{\Gamma \vdash_v P : A_1 \times A_2}{\Gamma \vdash_v \pi_1 P : A_1} \text{Elim}_1$$

$$\frac{\Gamma \vdash_v P : A_1 \times A_2}{\Gamma \vdash_v \pi_2 P : A_2} \text{Elim}_2$$

2.2.6 Separating Product

we have these because the monoidal product is semicartesian

$$\frac{\Gamma \vdash_v M : A_1 \quad \Delta \vdash_v N : A_2}{\Gamma \circ \Delta \vdash_v (M * N) : A_1 * A_2} \text{Intro}$$

$$\frac{\Gamma \circ \Delta \vdash_v P : A_1 * A_2}{\Gamma \vdash_v \rho_1 P : A_1} \text{Elim}_1$$

$$\frac{\Gamma \circ \Delta \vdash_v P : A_1 * A_2}{\Delta \vdash_v \rho_2 P : A_2} \text{Elim}_2$$

2.2.7 Thunk

$$\frac{\Gamma \mid \cdot \vdash_c M : B}{\Gamma \vdash_v \text{thunk } M : UB} \text{Intro}$$

$$\frac{\Gamma \vdash_v V : UB}{\Gamma \mid \cdot \vdash_c \text{force } V : B} \text{Elim}$$

2.3 Computation Typing

2.3.1 Function

$$\frac{\Gamma; (x : A) \mid \cdot \vdash_c V : B}{\Gamma \mid \cdot \vdash_c (\lambda x : A. V) : A \rightarrow B} \text{Intro}$$

$$\frac{\Gamma \mid \Theta \vdash_c M : A \rightarrow B \quad \Delta \vdash_v N : A}{\Gamma; \Delta \mid \Theta \vdash_c MN : B} \text{Elim}$$

2.3.2 Separating Function

$$\frac{\Gamma \circ (x : A) \mid \cdot \vdash_c V : B}{\Gamma \mid \cdot \vdash_c (\alpha x : A. V) : A \multimap B} \text{Intro}$$

$$\frac{\Gamma \mid \Theta \vdash_c M : A \multimap B \quad \Delta \vdash_v N : A}{\Gamma \circ \Delta \mid \Theta \vdash_c MN : B} \text{Elim}$$

2.3.3 Return

$$\frac{\Gamma \vdash_v V : A}{\Gamma \mid \cdot \vdash_c \text{ret } V : FA} \text{Intro}$$

$$\frac{\Gamma \mid \Theta \vdash_c M : FA \quad \Gamma; (x : A) \mid \cdot \vdash_c N : B}{\Gamma \mid \Theta \vdash_c x \leftarrow M; N : B} \text{Elim}$$

2.3.4 Error

$$\frac{}{\Gamma \mid \cdot \vdash_c \text{O} : B} \text{Intro}$$

2.4 Equations

for effects (but also computation rules)

2.4.1 Allocation

The following are from Ian Stark's Categorical Models for Local Names [?]

1. swap - allocation swap
2. drop - (our monad might not support this)
3. Fresh - (a kind of congruence)

2.4.2 Dynamic Type

the prism laws here

- 1.

3 Semantics

3.1 Worlds

Normally, the value types in a CBPV would be interpreted as Sets. However, in our case this is insufficient as the interpretation of the extensible sum type, OSum, is dependent on what Cases have been "allocated" via newcase. Thus we interpret value types in a presheaf category on a category of worlds where objects are a map from a finite set to syntactic types. Let SynTy be a set of syntactic types that can be used in the OSum Case store.

$$\begin{array}{lcl} \text{SynTy} & ::= & \text{Unit} \\ & | & \text{Bool} \\ & | & \text{SynTy} \times \text{SynTy} \end{array}$$

The denotation of these syntactic types is independent of what Cases have been allocated. Limiting the extensible sum type to contain only base types and products of base types allows us to temporarily avoid two separate issues.

- The first issue has to do with circularity of definitions. If we say a world is a mapping from a finite set to semantic types and that semantic types are presheaves on the category of worlds, then we have tied ourselves in a knot. This issue is pointed out in, among other places, [?].

$$\text{world} \cong X \rightarrow \text{SemTy} \quad \text{SemTy} \cong [\mathbf{World}, \mathbf{Set}]$$

- The second issue is well-foundedness of the interpretation of OSum. Consider if we allow the extensible sum type to store the type (OSum \rightarrow OSum) and try to recursively interpret OSum.

Here we define the category **World**^{1 2}. A world $X \rightarrow \text{SynTy}$ is a map from a finite set into the set of syntactic types. Morphisms $f : X \rightarrow Y$ in **World** are injective functions $i : Y \hookrightarrow X$ such that the following triangle commutes. The finite sets here represent a finite collection of case symbols and the injective map i represents *renaming*. We may denote a world $X \xrightarrow{w} \text{SynTy}$ by w .

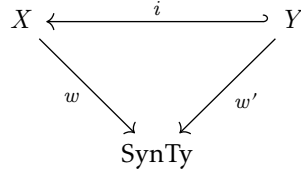


Figure 1: Morphisms in **World**

3.1.1 Monoidal Structure

We need to introduce additional structure on **World** to interpret the separating connectives $A * B$ and $A \multimap B$. For this, we will pull from existing literature on categorical semantics of bunched typing, specifically [?] and section 3.1 of [?]. We use the Day convolution [?] to get a monoidal structure on **World**³ from a monoidal structure on **World**. First, we define an operation $w \otimes_w w'$ on *worlds* by taking the set coproduct of their domains⁴ and using the recursor $X + Y \xrightarrow{\text{rec } w \ w'} \text{SynTy}$ to define a map into syntactic types. The action on

¹Similar categories: *HeapV* [?], *IFam_X* [?], polymorphic algebraic theories [?], [?]

²Formalized here as $\mathbf{World} := (\text{Inc} \downarrow \text{SynTy})^{op}$, where *Inc* is the inclusion functor from the category of finite sets and injections into the category **Set** and *SynTy* is a constant functor.

³ $\mathbf{Set}(\mathbf{World}^{op})$; contravariant presheaves on **World**

⁴This is not the coproduct in *FinSet_{mono}* [?]

morphisms is given by defining the top injection via $map : (A \rightarrow C) \rightarrow (B \rightarrow D) \rightarrow A + B \rightarrow C + D$ and the commuting triangles still hold. The unit for this monoid structure is the empty map $\lambda()$ ⁵.

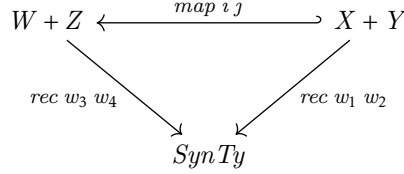


Figure 2: \otimes_w action on morphisms

Using this operation, we define the monoidal product for $\widehat{\mathbf{World}}$. The action on objects being:

$$(A \otimes B)X = \int^{Y,Z} A(Y) \times B(Z) \times \mathbf{World}[X, Y \otimes_w Z]$$

The monoidal unit is given by:

$$I(X) = \mathbf{World}[X, \lambda()]$$

The separating function is defined by:

$$(A \star B)X = \int_Z \mathbf{Set}[A(Z), B(X \otimes_w Z)] \cong \widehat{\mathbf{World}}[A, B(X \otimes_w -)]$$

Finally, we have that:

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, B \star C)$$

These definitions are standard in the literature on bunched implication with the exception that our *resources* are finite maps specifying typing of allocated case symbols. The bicartesian stucture on \mathbf{World} along with the monoidal structure (\otimes) and its right adjoint (\star) bundled together gives us the structure of a bicartesian doubly closed category or BiDCC.

3.1.2 Affine BiDCC

The concrete model that we are working with has the property that $\mathbf{1} \cong I$ ⁶. That is, the cartesian unit is isomorphic to the monoidal unit. This property validates weakening for \circ . Note that:

$$\begin{aligned} \mathbf{1}(X) &= \{\ast\} \\ I(X) &= \mathbf{World}[X, \lambda()] \end{aligned}$$

$\lambda()$ is terminal in \mathbf{World} and the cardinality of $|\mathbf{World}[-, \lambda()]| = 1$.

⁵The empty map is also the terminal object in \mathbf{Worlds}

⁶See section 3.3 of [?]

3.1.3 Concrete Separating Connectives

It may be possible for us to have an alternative interpretation of the separating product if we work in a full subcategory of $\widehat{\mathbf{World}}$. O'Hearn has affine models [?] [?] for his $\alpha\lambda$ -calculus⁷ using a presheaf category on $\mathbf{FinSet}_{\text{mono}}$. He shows that (section 4, [?]) the full subcategory of $\mathbf{Set}^{\mathbf{FinSet}_{\text{mono}}^{op}}$ which consists of pullback preserving functors lets you ask for the *smallest* world that any $a \in A(X)$ comes from, aka $\text{support}(a)$ (Section 9.2, [?]). With this notion of support, you can define non-interference as:

$$a \# b \Leftrightarrow (\text{support}(a) \cap \text{support}(b) = \emptyset)$$

With non-interference, $A * B$ can be directly interpreted as:

$$\begin{aligned} (A * B)X &= \{(a, b) \in A(X) \times B(X) \mid a \# b\} \\ (A * B)f(a, b) &= (A(f)(a), B(f)(b)) \end{aligned}$$

It should be noted that the full subcategory mentioned above is equivalent to the Shanel topos (Remark 6.9 [?]). It seems this would extend to our *multi-sorted* scenario⁸ [?] [?].

3.2 Allocation Monad

To model allocation of case symbols, we can use a modified version of the indexed state monad [?] [?]. This monad is a special case of a general construction⁹. Given a functor $F : C \rightarrow D$ between two categories, it induces a functor on presheaf categories $F^* : [D^{op}, \mathbf{Set}] \rightarrow [C^{op}, \mathbf{Set}]$ via precomposition of F which has both a left and right adjoint given by Kan extensions.

$$\begin{array}{ccc} & \begin{array}{c} \xleftarrow{\Sigma} \\ \perp \\ \xrightarrow{\quad} \end{array} & \\ [D^{op}, \mathbf{Set}] & \xrightarrow{F^*} & [C^{op}, \mathbf{Set}] \\ & \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\vee} \end{array} & \end{array}$$

Figure 3: essential geometric morphism between presheaf topoi

We instantiate this construction twice¹⁰, using the inclusion functor from the discrete category $\text{Inc} : |\mathbf{World}| \rightarrow \mathbf{World}$ and its opposite Inc^{op} . This yields a monad on our value category, contravariant presheaves on \mathbf{World} , by composing the top maps with the bottom maps¹¹.

⁷And an extension with commands, memory cells, and assignment called he calls SCI+ which is based on Reynold's Syntactic Control of Interference

⁸The Schanuel topos models classical HOL, is this problematic?

⁹<https://ncatlab.org/nlab/show/geometric+morphismBetweenPresheafToposes>

¹⁰code here

¹¹The usual presentation of the indexed state monad has covariant presheaves for the value category. ours is flipped to accomodate the definition of \mathbf{World}

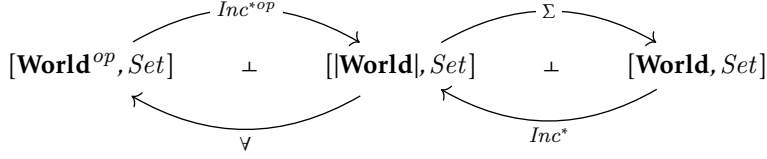


Figure 4: our allocation monad

The concrete interpretation for this monad's action on some object X at some world w_1 is: forall future worlds w_2 reachable by w_1 , there exists some yet future world w_3 reachable from w_2 with a value $X(w_3)$.

$$T(X)(w_1) = \forall_{w_2} \forall_{w_1 \rightarrow w_2} \sum_{w_3} \sum_{w_2 \rightarrow w_3} X(w_3)$$

3.2.1 Limitations

There are plenty of limitations with our current approach to modeling the dynamic type. Leveraging the analogy with state¹², our model supports *dynamic* allocation of cases but only for *ground* types and allocation is *globally* observable. We can adapt Sterling's model [?] to store semantic types instead of syntactic types. Kammar has a candidate monad for local store [?]. It is unclear to me if local store with rich semantic types exists yet.

3.3 Interpreting Types

Taking contravariant presheaves on **World** as our value category and covariant presheaves on **World** as our computation category, we can give a denotation of the syntax. Interpretation of syntactic types $\llbracket - \rrbracket_{el} : \text{SynTy} \rightarrow \widehat{\mathbf{World}}$ is not dependent on what cases have been allocated. **1** is the terminal presheaf in $\widehat{\mathbf{World}}$ and $+, \times$ come from the bicartesian structure of $\widehat{\mathbf{World}}$.

$$\begin{aligned} \llbracket Unit \rrbracket_{el} &= \mathbf{1} \\ \llbracket Bool \rrbracket_{el} &= \mathbf{1} + \mathbf{1} \\ \llbracket A \times B \rrbracket_{el} &= \llbracket A \rrbracket_{el} \times \llbracket B \rrbracket_{el} \end{aligned}$$

Next, we interpret the remaining value types in $\widehat{\mathbf{World}}$. The main definitions of interest are the interpretation of OSum and Case. OSum is an extensible sum type where the current world determines the number and type of cases in the extensible sum. Case A is interpreted to be the set of allocated case symbols for type A in the current world. Note that syntactic type equality¹³ is used in the set comprehension.

¹²Using Jon's breakdown [?]

¹³Although, I currently use the usual path type in the Agda development. We might want Sterling's univalent reference types [?]

$$\begin{aligned}
\llbracket Unit \rrbracket_v &= \llbracket Unit \rrbracket_{el} \\
\llbracket Bool \rrbracket_v &= \llbracket Bool \rrbracket_{el} \\
\llbracket A \times B \rrbracket_v &= \llbracket A \times B \rrbracket_{el} \\
\llbracket OSum \rrbracket_v(w) &= \sum_{\sigma \in dom(w)} \llbracket w(\sigma) \rrbracket_{el}(w) \\
\llbracket Case A \rrbracket_v(w) &= \{\sigma \mid \sigma \in dom(w) \wedge w(\sigma) = A\} \\
\llbracket A * B \rrbracket_v &= \llbracket A \rrbracket_v \otimes \llbracket B \rrbracket_v \\
\llbracket U \underline{B} \rrbracket_v &= U \llbracket B \rrbracket_c
\end{aligned}$$

Lastly, we interpret the computation types

I just noticed this (6-11), I naively thought the function type would map to an exponential but no. The denotation for the regular function type is somewhat surprising. See section 7.1 of Levy's thesis (or 6.7.3 of the CBPV book). Levy's model for cell generation has:

$$\llbracket A \rightarrow B \rrbracket_c(w) = \mathbf{Set}[\llbracket A \rrbracket_v(w), \llbracket B \rrbracket_c(w)]$$

That is, the denotation of the computational function type is given pointwise where A is a covariant presheaf, and B is a contravariant presheaf and the result should be a contravariant presheaf. TODO our situation is oped, check this. From O'Hearn's affine $\alpha\lambda$ -calculus, we would expect

$$\llbracket A \rightarrow B \rrbracket_v(w) = \mathbf{Set}^{\mathbf{World}}[\llbracket A \rrbracket_v(w \otimes_w -), \llbracket B \rrbracket_c(w \otimes_w -)]$$

however, the situation is now more complicated because those monoidal operations exist in two different categories! (need to check this!) For the separating function, we should expect

$$\llbracket A \rightarrow B \rrbracket_v(w) = \mathbf{Set}^{\mathbf{World}}[\llbracket A \rrbracket_v, \llbracket B \rrbracket_c(w \otimes_w -)]$$

$$\begin{aligned}
\llbracket A \rightarrow B \rrbracket_c &= \\
\llbracket A * B \rrbracket_c &= \\
\llbracket FA \rrbracket_c &= F \llbracket A \rrbracket_v
\end{aligned}$$

3.4 Interpreting Contexts

In *regular* type theory, contexts have the structure of a list and are interpreted as an n-ary product in the category used to interpret types. In the bunched type theory we are working with here, there are two connectives for constructing contexts (";" and "⌣") and the structure of contexts are *tree-like*. We use

; to denote the usual context construction operation that is interpreted as a cartesian product and $\mathbin{\text{\tiny ;}}$ to denote the new context construction operation that is interpreted as the tensor product.

$$\begin{aligned}\llbracket x : A \rrbracket_{ctx} &= \llbracket A \rrbracket_v \\ \llbracket \emptyset \rrbracket_{ctx} &= \mathbf{1} \\ \llbracket \Gamma ; \Gamma \rrbracket_{ctx} &= \llbracket \Gamma \rrbracket_{ctx} \times \llbracket \Gamma \rrbracket_{ctx} \\ \llbracket \Gamma \mathbin{\text{\tiny ;}} \Gamma \rrbracket_{ctx} &= \llbracket \Gamma \rrbracket_{ctx} \otimes \llbracket \Gamma \rrbracket_{ctx}\end{aligned}$$

The cartesian fragment of the context supports weakening and contraction. The monoidal fragment supports weakening¹⁴.

3.5 Interpreting Terms

Value terms $\Gamma \vdash M : A$ are interpreted as morphisms $\llbracket \Gamma \rrbracket_{ctx} \xrightarrow{M} \llbracket A \rrbracket_v$ in $\widehat{\mathbf{World}}$ which are natural transformations from $\mathbf{World}^{\mathbf{op}}$ to \mathbf{Set} . So for any term $\Gamma \vdash M : A$, we need to provide a family of maps $\forall w, \alpha_w : \llbracket \Gamma \rrbracket(w) \rightarrow \llbracket A \rrbracket(w)$ such that given any morphism $f : w \rightarrow w'$ in $\mathbf{World}^{\mathbf{op}}$ we have:

$$\begin{array}{ccc}\llbracket \Gamma \rrbracket(w) & \xrightarrow{\llbracket \Gamma \rrbracket(f)} & \llbracket \Gamma \rrbracket(w') \\ \downarrow \alpha_w & & \downarrow \alpha_{w'} \\ \llbracket A \rrbracket(w) & \xrightarrow{\llbracket A \rrbracket(f)} & \llbracket A \rrbracket(w')\end{array}$$

3.5.1 Value Terms

Here we define the terms as natural transformations by giving the components. w is the current world and $\gamma : \llbracket \Gamma \rrbracket(w)$.

$$\begin{aligned}\llbracket \text{tt} \rrbracket_{vtn}(w, \gamma) &= * \\ \llbracket \text{true} \rrbracket_{vtn}(w, \gamma) &= \text{inl} * \\ \llbracket \text{false} \rrbracket_{vtn}(w, \gamma) &= \text{inr} *\end{aligned}$$

Extracting a variable from the context is a matter of using the correct cartesian and monoidal projections¹⁵. We can use this process (call it *lookup*) to construct the proper denotation for variables.

¹⁴consequence of $\mathbf{1} \cong I$ in our model

¹⁵monoidal projections given by a bilinear map $! \otimes id$ composed with the monoidal identity laws [?](Definition 2.1.ii)

$$\llbracket x \rrbracket_{vtm} = \text{lookup}(x)$$

For injection of values into the open sum type, we can assume we already have $\llbracket \Gamma \rrbracket \xrightarrow{\sigma} \llbracket \text{Case } A \rrbracket$ and $\llbracket \Gamma \rrbracket \xrightarrow{V} \llbracket A \rrbracket$. We then want to construct $\llbracket \text{inj}_\sigma V \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \text{OSum} \rrbracket$. It suffices to construct $\llbracket \text{Case } A \times A \rrbracket \rightarrow \llbracket \text{OSum} \rrbracket$ ¹⁶.

$$\llbracket \text{inj}_\sigma V \rrbracket_{vtm}(w, ((x, p : w(x) = A), a)) = (x, a)$$

The denotation of product and separating product are both akin to constructing bilinear maps.

$$\begin{aligned} \llbracket (V_1, V_2) \rrbracket_{vtm} &= \llbracket V_1 \rrbracket, \llbracket V_2 \rrbracket \\ \llbracket (V_1 * V_2) \rrbracket_{vtm} &= \llbracket V_1 \rrbracket * \llbracket V_2 \rrbracket \end{aligned}$$

TODO projection maps

And finally, the interpretation of thunk M .

$$\llbracket \text{thunk } M \rrbracket_{vtm} =$$

3.5.2 Computation Terms

the computation judgments in Levy's model are not required to satisfy naturality..? It seems the problem is the store type .. which we don't have (section 6.7.2 paragraph 2 [?])

$$\begin{aligned} \llbracket \text{U} \rrbracket_{ctm} &= \text{add error to alloc monad} \\ \llbracket \text{force } V \rrbracket_{ctm} &= \\ \llbracket \text{ret } V \rrbracket_{ctm} &= \\ \llbracket x \leftarrow M; N \rrbracket_{ctm} &= \\ \llbracket M N \rrbracket_{ctm} &= \\ \llbracket M @ N \rrbracket_{ctm} &= \\ \llbracket \lambda x : A. M \rrbracket_{ctm} &= \\ \llbracket \alpha x : A. M \rrbracket_{ctm} &= \\ \llbracket \text{newcase}_A \sigma; M \rrbracket_{ctm} &= \\ \llbracket \text{match } V_1 \text{ with } V_2 \{x. M \mid N\} \rrbracket_{ctm} &= \\ \llbracket \text{if } V \text{ then } M \text{ else } N \rrbracket_{ctm} &= \end{aligned}$$

¹⁶It is morally transport p a in the formalization