1 Model Setup

1.1 Bicartesian Doubly Closed Category

Given a category $\mathcal{C}$, its presheaf category ($\widehat{\mathcal{C}} := [\mathcal{C}^{op}, Set]$) is bicartesian closed. Given a monoidal category $(\mathcal{C}, \otimes_C, I_C)$, its presheaf category is bicartesian closed and monoidal closed via the Day convolution. The monoidal product is given by:

$$(P \otimes^{Day} Q)(x) = \int^{y,z} \mathcal{C}[x, y \otimes_C z] \times P(y) \times Q(z)$$

With a useful fact:

$$^{inc}[h, A(f)(a'), B(g)(b')] =^{inc} [h; (f \otimes g), a', b']$$

where $h : C[x, y \otimes z]$, $f : C[y, y']$, $g : C[z, z']$, $a' : A(y')$, and $b' : B(z')$. The Day monoidal product has the universal property that any maps out of it are in bijective correspondence with a family of maps natural in x and y (Agda): [1]

$$\widehat{\mathcal{C}}[P \otimes^{Day} Q, R] \;\cong\; \widehat{\mathcal{C} \times \mathcal{C}}[P \overline{\times} Q, R \circ \otimes_C] \;\cong\; \Pi_{x,y:\, ob\ C}\ Set[P(x) \times Q(y), R(x \otimes_C y)]$$

The monoidal closed structure is given by:

$$(P -\!\!* Q)(X) = \widehat{\mathcal{C}}[P, Q(X, -)]$$

With the universal property that the closed structure is right adjoint to the tensor (Agda):

$$\widehat{\mathcal{C}}[A \otimes_C B, C] \cong \widehat{\mathcal{C}}[A, B -\!\!* C] \tag{1}$$

1.2 Towards Bunched CBPV with Dynamic Store

Let $(C, \otimes_C, I_C)$ be a monoidal category, the value category be $\mathcal{V} := [C^{op}, Set]$, computation category $\mathcal{C} := [C, Set]$, and use the *standard* monad for ground dynamic store with $F : \mathcal{V} \to \mathcal{C}$ as:

$$F(A)(x) := \Sigma_{y:ob\ C} \Sigma_{f:C^{op}[x,y]} A(y)$$

and $U : \mathcal{C} \to \mathcal{V}$ as :

$$U(\underline{B})(x) := \Pi_{y:ob\ C} \Pi_{f:C^{op}[x,y]} \underline{B}(y)$$

The oblique morphisms in this model are given by families of maps:

$$\mathcal{O}[A, \underline{B}] := \Pi_{x:ob\ C} Set[A(x), \underline{B}(x)]$$

we have the following isomorphims:

$$\mathcal{V}[A, U(\underline{B})] \cong \mathcal{O}[A, \underline{B}] \cong \mathcal{C}[F(A), \underline{B}]$$

and a monad on $\mathcal{V}$ via the adjunction between F and U.

$$T(A)(x) := \Pi_{y:ob\ C} \Pi_{f:C^{op}[x,y]} \Sigma_{z:ob\ C} \Sigma_{g:C^{op}[y,z]} A(z)$$

The monadic Unit

$$\eta_A(x)(a : A(x)) = \lambda x'.\lambda f : FS[x, x'].(x', id_{x'}, A(f)(a))$$

[1] here $\overline{\times}$ is the *external* product

2 Issue with Abstract Model

It seems we cannot define the strength map:

$$str\otimes_{P,Q} : \mathcal{V}[P \otimes T(Q), T(P \otimes Q)]$$

via the universal property of tensor, it suffices to construct components of the form:

$$Set[P(x) \times T(Q)(y), T(P \otimes Q)(x \otimes_C y)]$$

introducing terms and unfolding definitions, we have:

$$\begin{aligned}
&p : P(x) \\
&q : T(Q)(y) = \Pi_{u:ob\ C}\Pi_{f:C^{op}[y,u]}\Sigma_{v:ob\ C}\Sigma_{g:C^{op}[u,v]}Q(v) \\
&h : C^{op}[x \otimes_C y, z]
\end{aligned}$$

we get to choose some future world w of z, $C^{op}[z,w]$, at which we must provide

$$? : (P \otimes Q)(w) = \int^{r,s} \mathcal{C}[w, r \otimes_C s] \times P(r) \times Q(s)$$

How to proceed? Notice that in order to *extract* a value $Q(\_)$ from q, we need to provide a morphism $C^{op}[y,u]$ for some u. Working abstractly with no additional assumptions about C, all we can provide is id_y which yields:

$$q(y)(id_y) : \Sigma_{v:ob\ C}\Sigma_{g:C^{op}[y,v]}Q(v)$$

Then our context becomes:

$$\begin{aligned}
&p : P(x) \\
&g : C^{op}[y,v] \\
&q' : Q(v) \\
&h : C^{op}[x \otimes_C y, z]
\end{aligned}$$

We still need to choose some future world w of z. From the data available to us, it seems we are stuck using identity, id_z, once again. This leaves us with the obligation:

$$? : (P \otimes Q)(z) = \int^{r,s} \mathcal{C}[z, r \otimes_C s] \times P(r) \times Q(s)$$

The choice of r seems forced to be x for which we have $p : P(x)$. The natural choices for s would be v or y, but it seems we've hit a variance issue yet again. Observe that if $g : C^{op}[y,v]$ was instead $g : C^{op}[v,y]$ we'd have

$$((id \otimes_C g); h, p, q') = (h, p, Q(g)(q'))$$

which are equal relative to the coend quotient.

3 Concrete Model

Now we consider substituting the monoidal category $(C, \otimes_C, I_C)$ with $(FinSet^{op}_{mono}, \uplus, \emptyset)$ where $\uplus$ is disjoint union of sets. Take $\mathcal{FS}$ as shorthand for $FinSet_{mono}$.

3.1 Components

Define as a bilinear map.

$$\begin{aligned}
&str_{A,B} : \Pi_{x,y:ob\mathcal{FS}}\ Set[A(x) \times T(B)(y), T(A \otimes B)(x \uplus y)] \\
&str_{A,B}(x,y)(a : A(x), tb : T(B)(y)) = \lambda z.\lambda f : \mathcal{FS}[x \uplus y, z].\ ??
\end{aligned}$$

where

$$?? : \Sigma_{w:ob\mathcal{FS}}\Sigma_{g:\mathcal{FS}[z,w]}(A \otimes B)(w)$$

We recognize that since f is injective and the domain is a disjoint union, z is partitioned into three parts

$$\begin{aligned} z_x &: \text{the range of } f_x\text{, f restricted to x} \\ z_y &: \text{the range of } f_y\text{, f restricted to y} \\ z_m &: z - (z_x \uplus z_y) \\ \text{where } z &\cong z_x \uplus z_y \uplus z_m \end{aligned}$$

Using $f_y : y \to z_y$

$$(v, g : FS[z_y, v], b : B(v)) = tb(zy)(f_y)$$

We can construct future world $w := z_x \uplus v \uplus z_m$, and map $z \xrightarrow{z_x \uplus g \uplus z_m} w$. Now we must provide

$$(A \otimes B)(w) = \int^{r,s} \mathcal{FS}[r \uplus s, w] \times A(r) \times B(s)$$

have $r := z_x$ and $s := v \uplus z_m$. Then the element at the future world is

$$^{inc}[id_w, A(f_x)(a), B(inl)(b)]$$

3.2 Naturality

Given $f : FS[x, x']$ and $g : FS[y, y']$.

$$\begin{tikzcd} A(x) \times T(B)(y) \arrow[r, "A(f)\times T(B)(g)"] \arrow[d, "Str_{A,B}(x,y)"] & A(x') \times T(B)(y') \arrow[d, "Str_{A,B}(x',y')"] \\ T(A \otimes B)(x \otimes y) \arrow[r, "T(A\otimes B)(f\otimes g)"] & T(A \otimes B)(x' \otimes y') \end{tikzcd}$$

3.2.1 RHS

Starting in the upper left at (a, tb), we get $(a' = A(f)(a), tb')$ via the top map where

$$tb' = T(B)(g)(tb) = \lambda z.\lambda h : FS[y', z].tb(z)(g; h)$$

Then, computing the strength map, we have for z, $h : FS[x' \otimes y', z]$ an $h_{y'} : FS[y', z_{y'}]$ with

$$tb'(z_{y'})(h_{y'}) = (\lambda z.\lambda h : FS[y', z].tb(z)(g; h))(z_{y'})(h_{y'}) = tb(z_{y'})(g; h_{y'}) = (v, j : FS[z_y, v], b : B(v))$$

resulting in

$$(z_{x'} \uplus v \uplus z_{m'}, id \uplus j \uplus id, ^{inc}[id, A(f; h_{x'})(a), B(inl)(b)])$$

3.2.2 LHS

Starting in the upper left at (a, tb), we also take in z and $h : FS[x' \otimes y', z]$. The LHS calculates the strength given $m = (f \otimes g); h$. Note that we have:

$$tb(z_y)(m_y) = tb(z_{y'})(g; h_{y'}) = (v, j : FS[z_y, v], b : B(v))$$

which is the same future data as the RHS.

$$
\begin{tikzcd}
 & A \otimes B \arrow[dl, "id_A \otimes \eta_B"'] \arrow[dr, "\eta_{A\otimes B}"] & \\
A \otimes T(B) \arrow[rr, "str_{A,B}"'] & & T(A \otimes B)
\end{tikzcd}
$$

3.3 Laws

3.3.1 Strength with I is Irrelevant

3.3.2 Strength Respects Associators

3.3.3 Strength Commutes with Monad Unit

$\eta_{A\otimes B}$ as a bilinear map computes to:

$$
\begin{aligned}
&\eta' : \Pi_{x,y:ob\mathcal{FS}}\ Set[A(x) \times B(y), T(A \otimes B)(x \uplus y)] \\
&\eta'(x,y)(a : A(x), b : B(y)) = \lambda z.\lambda f : \mathcal{FS}[x \uplus y, z].^{inc}[f, a, b]
\end{aligned}
$$

The strength map precomposed with $id_A \otimes \eta_B$ yields:

$$(z_y, id_{z_y}, b' = B(f_y)(b)) = tb(z_y)(f_y)$$

thus, the future worlds are equal $z \cong z_x \uplus z_y \uplus z_m \cong w$ and the maps $z \to w$ are equal. It remains to show that

$$^{inc}[id, A(f_x)(a), B(inl)(b')] =^{inc} [f, a, b]$$

Recall $^{inc}[h, A(f)(a), B(g)(b)] =^{inc} [(f \uplus g); h, a, b]$

$$
\begin{aligned}
{}^{inc}[id_{z_x \uplus (z_y \uplus z_m)}, A(f_x)(a), B(inl)(b')] &= \\
{}^{inc}[id_{z_x \uplus (z_y \uplus z_m)}, A(f_x; id_{z_x})(a), B(f_y; inl)(b)] &= \\
{}^{inc}[(f_x \uplus f_y); (id_{z_x} \uplus inl), a, b] &= \\
{}^{inc}[f, a, b]&
\end{aligned}
$$

$\square$

3.3.4 Strength Commutes with Monad Multiplication