Honey Bunches of OSum

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1 Untyped Syntax

value types	A	$::=$	$\mathbf{1}$	
		$\mid$	$A \times A$	
		$\mid$	$A * A$	
		$\mid$	$U\underline{B}$	
computation types	B	$::=$	$A \to \underline{B}$	
		$\mid$	$A \mathrel{-\!\!*} \underline{B}$	
		$\mid$	$F\ A$	
value typing context	Γ	$::=$	$x : A$	
		$\mid$	ϕ	*additive unit*
		$\mid$	$\Gamma; \Gamma$	*additive combination (product)*
		$\mid$	φ	*multiplicative unit*
		$\mid$	$\Gamma \fatsemi \Gamma$	*multiplicative combination (tensor)*
stoup	Δ	$::=$	$\cdot$	
		$\mid$	$\bullet$	

2 Typed Syntax

2.1 Var

$$\frac{}{x : A \vdash x : A}\ Id_v$$

$$\frac{}{\Gamma \mid \bullet : \underline{B} \vdash \bullet : \underline{B}}\ Id_c$$

2.2 One

$$\frac{}{\Gamma \vdash^v () : \mathbf{1}}\ \mathbf{1}Intro$$

$$\frac{\Gamma \vdash^v M : \mathbf{1}}{\Gamma \vdash^v () = M : \mathbf{1}}\ \mathbf{1}\eta$$

2.3 Product

$$\frac{\Gamma_1 \vdash^v V_1 : A_1 \qquad \Gamma_2 \vdash^v V_2 : A_2}{\Gamma_1 ; \Gamma_2 \vdash^v (V_1, V_2) : A_1 \times A_2} \times Intro$$

$$\frac{\Gamma \vdash^v M : A_1 \times A_2}{\Gamma \vdash^v \pi_i M : A_i} \times Elim_i$$

$$\frac{\Gamma \vdash^v M_1 : A_1 \qquad \Gamma \vdash^v M_2 : A_2}{\Gamma \vdash^v \pi_i(M_1, M_2) = M_i : A_i} \times \beta_i$$

$$\frac{\Gamma \vdash^v M : A_1 \times A_2}{\Gamma \vdash^v M = (\pi_1 M, \pi_2 M) : A_1 \times A_2} \times \eta$$

2.4 Sep Product

$$\frac{\Gamma_1 \vdash^v V_1 : A_1 \qquad \Gamma_2 \vdash^v V_2 : A_2}{\Gamma_1 \fatsemi \Gamma_2 \vdash^v V_1 * V_2 : A_1 * A_2} *Intro$$

$$\frac{\Gamma_1(x : A \fatsemi y : B) \vdash^v N : C \qquad \Gamma_2 \vdash^v M : A * B}{\Gamma_1(\Gamma_2) \vdash^v \mathbf{let}(x, y) = M \ \mathbf{in}\ N : C} *Elim$$

[1] $$\frac{\Gamma_1 \vdash^v M_1 : A \qquad \Gamma_2 \vdash^v M_2 : B \qquad \Gamma_3(x : A \fatsemi y : B) \vdash^v N : C}{\Gamma? \vdash^v (\mathbf{let}(x, y) = (M_1 * M_2)\ \mathbf{in}\ N) = N[M_1/x, M_2/y] : C} *\beta$$

$$\frac{\Gamma \vdash^v M : A_1 * A_2}{\Gamma \vdash^v (\mathbf{let}(x, y) = M \ \mathbf{in}\ x * y) = M : A_1 * A_2} *\eta$$

2.5 U

$$\frac{\Gamma \mid \cdot \vdash^c M : \underline{B}}{\Gamma \vdash^v \mathbf{thunk} M : U\underline{B}} \ \mathbf{tf}\ Intro$$

$$\frac{\Gamma \vdash^v V : U\underline{B}}{\Gamma \mid \cdot \vdash^c \mathbf{force} V : \underline{B}} \ \mathbf{tf}\ Elim$$

$$\frac{\Gamma \mid \cdot \vdash^c M : \underline{B}}{\Gamma \mid \cdot \vdash^c \mathbf{force}\ (\mathbf{thunk}\ M) = M : \underline{B}} \ \mathbf{tf}\ \beta$$

$$\frac{\Gamma \vdash^v V : U\underline{B}}{\Gamma \vdash^v V = \mathbf{thunk}\ (\mathbf{force}\ V) : U\underline{B}} \ \mathbf{tf}\ \eta$$

[1] beta eta from page 21 of [2]

2.6 ”Normal” functions

This is where things get tricky..

$$\frac{\Gamma;\,(x:A)\,|\,\underline{\Delta}\vdash^{c} M:\underline{B}}{\Gamma\,|\,\underline{\Delta}\vdash^{c}(\lambda x:A.\,M):A\to\underline{B}}\to Intro$$

$$\frac{\Gamma_1\,|\,\underline{\Delta}\vdash^{c} M:A\to\underline{B}\qquad \Gamma_2\vdash^{v} V:A}{\Gamma_1;\,\Gamma_2\,|\,\underline{\Delta}\vdash^{c} MV:\underline{B}}\to Elim$$

$$\frac{\Gamma_1;\,(x:A)\,|\,\underline{\Delta}\vdash^{c} M:\underline{B}\qquad \Gamma_2\vdash^{v} N:A}{\Gamma_1;\,\Gamma_2\,|\,\underline{\Delta}\vdash^{c}(\lambda x:A.\,M)N=M[N/x]:\underline{B}}\to\beta$$

$$\frac{\Gamma;\,(x:A)\,|\,\underline{\Delta}\vdash^{c} M:\underline{B}\qquad x\notin FV(M)}{\Gamma\,|\,\underline{\Delta}\vdash^{c}(\lambda x:A.\,Mx)=M:A\to\underline{B}}\to\eta$$

2.7 Wand

This is the same as $\to$, just ”alpha renamed” symbols ($\fatsemi, @, \alpha, -\!\!*$).

$$\frac{\Gamma\fatsemi(x:A)\,|\,\underline{\Delta}\vdash^{c} M:\underline{B}}{\Gamma\,|\,\underline{\Delta}\vdash^{c}(\alpha x:A.\,M):A-\!\!*\,\underline{B}}-\!\!*\,Intro$$

$$\frac{\Gamma_1\,|\,\underline{\Delta}\vdash^{c} M:A-\!\!*\,\underline{B}\qquad \Gamma_2\vdash^{v} V:A}{\Gamma_1\fatsemi\Gamma_2\,|\,\underline{\Delta}\vdash^{c} M@V:\underline{B}}-\!\!*\,Elim$$

$$\frac{\Gamma_1\fatsemi(x:A)\,|\,\underline{\Delta}\vdash^{c} M:\underline{B}\qquad \Gamma_2\vdash^{v} N:A}{\Gamma_1\fatsemi\Gamma_2\,|\,\underline{\Delta}\vdash^{c}(\alpha x:A.\,M)@N=M[N/x]:\underline{B}}-\!\!*\,\beta$$

$$\frac{\Gamma\fatsemi(x:A)\,|\,\underline{\Delta}\vdash^{c} M:\underline{B}\qquad x\notin FV(M)}{\Gamma\,|\,\underline{\Delta}\vdash^{c}(\alpha x:A.\,M@\,x)=M:A-\!\!*\,\underline{B}}-\!\!*\,\eta$$

2.8 F

[2]

$$\frac{\Gamma\vdash^{v} M:A}{\Gamma\,|\,\cdot\vdash^{c}:\mathbf{ret}M:F\underline{A}}\,\mathbf{ret}Intro$$

$$\frac{\Gamma_1\,|\,\underline{\Delta}\vdash^{c} M:FA\qquad \Gamma_2;\,x:A\,|\,\cdot\vdash^{c} N:\underline{B}}{\Gamma_1;\,\Gamma_2\,|\,\underline{\Delta}\vdash^{c} x\leftarrow M;\,N:\underline{B}}\,\mathbf{ret}Elim(;\,)$$

$$\frac{\Gamma_1\,|\,\underline{\Delta}\vdash^{c} M:FA\qquad \Gamma_2\fatsemi x:A\,|\,\cdot\vdash^{c} N:\underline{B}}{\Gamma_1\fatsemi\Gamma_2\,|\,\underline{\Delta}\vdash^{c} x\leftarrow M;\,N:\underline{B}}\,\mathbf{ret}Elim(\fatsemi)$$

[2]following [1]

$$\frac{\Gamma_1 \vdash^v V : A \qquad \Gamma_2;\, x : A \mid \cdot \vdash^c M : \underline{B}}{\Gamma_1 \,;\, \Gamma_2 \mid \cdot \vdash^c (x \leftarrow \mathbf{ret}\, V;\ M) = M[V/x] : \underline{B}}\ \mathbf{ret}\ \beta(;\ )$$

$$\frac{\Gamma_1 \vdash^v V : A \qquad \Gamma_2 \fatsemi x : A \mid \cdot \vdash^c M : \underline{B}}{\Gamma_1 \fatsemi \Gamma_2 \mid \cdot \vdash^c (x \leftarrow \mathbf{ret}\, V;\ M) = M[V/x] : \underline{B}}\ \mathbf{ret}\ \beta(\fatsemi)$$

$$\frac{\Gamma \mid \underline{\Delta} \vdash^c N : FA \qquad \Gamma \mid \bullet : FA \vdash^c M : \underline{B}}{\Gamma \mid \underline{\Delta} \vdash^c M[N] = (x \leftarrow N;\ M[\mathbf{ret} x]) : \underline{B}}\ \mathbf{ret}\ \eta$$

3 Structural rules

see page 20 of https://link.springer.com/book/10.1007/978-94-017-0091-7

$$\frac{\Gamma \vdash^v M : A \qquad \Upsilon(x : A) \vdash^v N : B}{\Upsilon(\Gamma) \vdash^v N[M/x] : B}\ Cut$$

is this ^ needed?

$$\frac{\Gamma(\Upsilon) \vdash^v M : A}{\Gamma(\Upsilon \,;\, \Upsilon') \vdash^v M : A}\ \mathit{Weakening}\ \text{(for ;)}$$

$$\frac{\Gamma(\Upsilon \,;\, \Upsilon') \vdash^v M : A}{\Gamma(\Upsilon) \vdash^v M[i(\Upsilon)/i(\Upsilon')] : A}\ (\Upsilon' \cong \Upsilon)\mathit{Contraction}\ \text{(for ;)}$$

$\cong$ is isomorphism of bunches
$i(\Upsilon)$ denote an in order traversal of the identifiers in Υ

$$\frac{\Gamma \vdash^v M : A}{\Upsilon \vdash^v M : A}\ \text{(where } \Gamma \equiv \Upsilon)\mathit{Exchange}$$

$\equiv$ is a coherence equivalece defined by

- Commutative monoid equations for ;
- Commutative monoid equations for $\fatsemi$
- Congruence: $\Upsilon \equiv \Upsilon' \Rightarrow \Gamma(\Upsilon) \equiv \Gamma(\Upsilon')$

4 Semantics

Preliminary definitions and constructions

4.1 Presheaves

Redo using the presentation of Presheaves as "Predicators"

Let C be a small category. A **presheaf** on C is a functor $F : C^{op} \to Set$. $Psh(C)$ denotes the functor category where objects are presheaves on C. This category will be used to denote value types. We will want a doubly cartesian closed structure on this category.

4.1.1 Terminal Object

The terminal object $\top$ is a functor $\top : C^{op} \to Set$. Have $\top_0(X) := \{*\}$ where $\top$ maps all objects of C^{op} to a singleton set. Have $\top_1(f : X \to Y) : * \to *$ so $\top$ maps morphisms in C^{op} to the identity function on the singleton set $*$. Say $F : C^{op} \to Set$ is an object in $Psh(C)$ and consider the natural transformation $F \Rightarrow \top$.

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"] & F(Y) \arrow[d, "\alpha_Y"] \\ \perp(X) \arrow[r, "id_{\{*\}}"] & \perp(Y) \end{tikzcd} .$$

Any natural transformation from F to $\perp$ is determined by the components, of which there is no choice but the terminal map in Set. Thus, $Psh(C)$ has a terminal object.

4.1.2 Products

Given F, G presheaves on C, construct their product object in $[C^{op}, Set]$.

On objects:

$$(F \times_{Psh(C)} G)(X) := F_0(X) \times_{Set} G_0(X)$$

On morphisms: given $(f : X \to Y)$[3]

$$\begin{aligned} &(F \times_{Psh(C)} G)(f) : (F_0(X) \times_{Set} G_0(X)) \to (F_0(Y) \times_{Set} G_0(Y)) \\ &(F \times_{Psh(C)} G)(f)(Fx, Gy) := (F_1(f)(Fx), G_1(f)(Gy)) \end{aligned}$$

Preserves identity (holds pairwise)[4]

$$\begin{aligned} (F \times_{Psh(C)} G)(id_X) &= F_1(id_X), G_1(id_X) \\ &= id_{F_0(X)}, id_{G_0(X)} \\ &= id_{(F \times_{Psh(C)} G)(X)} \end{aligned}$$

Preserves composition (holds pairwise)

$$\begin{aligned} (F \times_{Psh(C)} G)(g \circ f) &= F_1(g \circ f), G_1(g \circ f) \\ &= F_1(g) \circ F_1(f), G_1(g) \circ G_1(f) \end{aligned}$$

[3] $(f^{op} : Y \to X)$. Alternatively, we could write something like $\langle F_1(f) \circ \pi_1, G_1(f) \circ \pi_2 \rangle$. Being in set, we are abusing notation in the function definition by implicitly abstracting over $(F_0(X) \times_{Set} G_0(X))$ and unpacking/repacking products

[4] Again I am abusing notation by performing implicit "computation" in Set

This describes an object $F \times_{Psh(C)} G \in Psh(C)$. The projection maps π_1, π_2 are natural transformations. Consider $\pi_1 : (F \times G) \Rightarrow F$.

$$\begin{tikzcd}
F(X) \times G(X) \arrow[r, "F(f)\times G(f)"] \arrow[d, "\pi_{1-Set}"] & F(Y) \times G(Y) \arrow[d, "\pi_{1-Set}"] \\
F(X) \arrow[r, "F(f)"] & F(Y)
\end{tikzcd}$$

The commuting diagram lives in Set and $(F \times G)(X) := (F(X), G(X))$. So the components of π_1 are the projections maps of products in Set. From what we know about Set, we know this diagram commutes. What remains is to demonstrate the universal properties for products.

$$\begin{tikzcd}
& C \arrow[dl, Rightarrow] \arrow[d, dotted] \arrow[dr, Rightarrow] & \\
F & F \times G \arrow[l, Rightarrow, "\pi_1"'] \arrow[r, Rightarrow, "\pi_2"] & G
\end{tikzcd}$$

This looks like the usual definition of product except that all the morphism are natural transformations. We can work with functions and sets if consider the underlying components and we fix an arbitrary object $X \in Ob\ C$. We know this commutes because Set has products.

$$\begin{tikzcd}
& C(X) \arrow[dl, "i"] \arrow[d, dotted, "h"] \arrow[dr, "j"] & \\
F(X) & F(X) \times G(X) \arrow[l, "\pi_{1-Set}"'] \arrow[r, "\pi_{2-Set}"] & G
\end{tikzcd}$$

4.1.3 Exponentials

To construct the exponential object for two presheaves F, G, we first need to describe the Yoneda embedding.

4.1.4 Yonedda Embedding

The Yonedda Embedding is a functor $よ : C \to Psh(C)$ which maps an object in C to its contravariant hom functor.

$$よ_0(X) := Hom(\_, X)$$

Where Any Y is mapped to the set of maps from Y to X.

$$Hom(\_, X)_1(f : B \to A) : Hom(A, X) \to Hom(B, X)$$
$$Hom(\_, X)_1(f) := \_ \circ f$$

And the action on morphisms $f^{op} : A \to B$ is precomposition.

$$
\begin{aligned}
Hom(\_, X)(id_A)(g : A \to X) &= g \circ id_A \\
&= g
\end{aligned}
$$

When written pointfree, $Hom(\_, X)(id) = id_{Hom(A,X)}$.
Finally, given $f : B \to A$, $g : C \to B$

$$
\begin{aligned}
Hom(\_, X)(g \circ^{op} f) &: Hom(A, X) \to Hom(C, X) \\
Hom(\_, X)(g \circ^{op} f)(h) &= h \circ (f \circ g) \\
&= (h \circ f) \circ g \\
&= (Hom(\_, X)(g) \circ Hom(\_, X)(f))(h)
\end{aligned}
$$

$よ_1$ maps morphisms in C to a natural transformation between hom functors.

$$よ_1(f : X \to Y) : Hom(\_, X) \Rightarrow Hom(\_, Y)$$

With components

$$
\begin{aligned}
&\eta(Z) : Hom(Z, X) \to Hom(Z, Y) \\
&\eta(Z) = f \circ \_
\end{aligned}
$$

such that for any morphism $g : W \to V$ in C,

$$
\begin{tikzcd}
Hom(V, X) \arrow[r, "Hom(g)=\_\circ g"] \arrow[d, "\eta=f\circ\_"] & Hom(W, X) \arrow[d, "\eta=f\circ\_"] \\
Hom(V, Y) \arrow[r, "Hom(g)=\_\circ g"] & Hom(W, Y)
\end{tikzcd}
$$

Pointwise, the naturality condition is $(f \circ h) \circ g = f \circ (h \circ g)$ which holds by associativity of functions in Set. What remains is the preservation of identity and composition.

$$よ_1(id_X) : Hom(\_, X) \Rightarrow Hom(\_, X)$$

Where, for a morphism $g : W \to V$,

$$
\begin{tikzcd}
Hom(V, X) \arrow[r, "Hom(g)=\_\circ g"] \arrow[d, "\eta=id_X\circ\_"'] & Hom(W, X) \arrow[d, "\eta=id_X\circ\_"] \\
Hom(V, X) \arrow[r, "Hom(g)=\_\circ g"] & Hom(W, X)
\end{tikzcd}
$$

The identity natural transformation where components are the identity function. Both commuting squares boil down to the equation $\_ \circ g = \_ \circ g$.

$$\begin{tikzcd}
Hom(V,X) \arrow[r, "Hom(g)=_\circ g"] \arrow[d, "id"'] & Hom(W,X) \arrow[d, "id"] \\
Hom(V,X) \arrow[r, "Hom(g)=_\circ g"] & Hom(W,X)
\end{tikzcd}$$

For morphisms $f : A \to B$, $g : B \to C$,

$$\text{よ}_1(g \circ f) = \text{よ}_1(g) \circ \text{よ}_1(f)$$

$$W \xrightarrow{\quad h \quad} V$$

$$\begin{tikzcd}
Hom(V,A) \arrow[r, "Hom(h)=_\circ h"] \arrow[d, "(f\circ g)\circ_"] & Hom(W,A) \arrow[d, "(f\circ g)\circ_"] \\
Hom(V,C) \arrow[r, "Hom(h)=_\circ h"] & Hom(W,C)
\end{tikzcd}$$

$$\begin{tikzcd}
Hom(V,A) \arrow[r, "_\circ h"] \arrow[d, "f\circ_"] & Hom(W,A) \arrow[d, "f\circ_"] \\
Hom(V,B) \arrow[r, "_\circ h"] \arrow[d, "g\circ_"] & Hom(W,B) \arrow[d, "g\circ_"] \\
Hom(V,C) \arrow[r, "_\circ h"] & Hom(W,C)
\end{tikzcd}$$

The components of each natural transformations are equal.

4.1.5 Exponential Object

Given presheaves F, G, the exponential object G^F is also a presheaf. For an object X of C, we get a set of natural transformations.

$$G_0^F(X) := Hom_{[C^{op},Set]}(\text{よ}_0(X) \times_{psh} F, G)$$

For a morphism $f : X \to Y$ of C, we have a function between sets of natural transformations

$$G_1^F(f) : Hom_{[C^{op},Set]}(\text{よ}_0(Y) \times_{psh} F, G) \to Hom_{[C^{op},Set]}(\text{よ}_0(X) \times_{psh} F, G)$$

Say we are given $nt : Hom_{[C^{op},Set]}(\text{よ}_0(Y) \times_{psh} F, G)$ which has components

$$\alpha : (Z : ObC) \to (Hom(Z,Y), F(Z)) \to G(Z)$$

We map nt to a natural transformation in $Hom_{[C^{op},Set]}(よ_0(X) \times_{psh} F, G)$, where the components are defined to be:

$$\eta : (Z : ObC) \to (Hom(Z, X), F(Z)) \to G(Z)$$
$$\eta(Z)(g : Z \to X, fz : F(Z)) := \alpha(Z)(f \circ g, fz)$$

Given $g : W \to V$, a morphism in C, the naturality square is of the form:[5]

$$F_1(g) : F(V) \to F(W) \qquad G_1(g) : G(V) \to G(W)$$

$$\begin{tikzcd}
Hom(V, X) \times F(W) \arrow[r, "(_\circ g) \times F_1(g)"] \arrow[d, "\eta_V = \alpha_V(f \circ _ \times id_{F(X)})"'] & Hom(W, X) \times F(W) \arrow[d, "\eta_W = \alpha_W(f \circ _ \times id_{F(X)})"] \\
G(V) \arrow[r, "G(g)"] & G(W)
\end{tikzcd}$$

From our original natural transformation nt, we have the naturality square:

4.2 Day Convolution

The Day convolution is used to model our separating connectives $\_ * \_$ and $\_ \mathrel{-\!\!*} \_$. First, we define the category of **Worlds** and a partial monoidal structure on **Worlds**. Let N be a finite set. Take the objects of **Worlds** to be subsets of N and the morphisms to be set inclusion. Define separation on the subsets of N as:

$$X * Y := \begin{cases} X \cup Y & X \cap Y = \varnothing \\ undefined & otherwise \end{cases}$$

This ensures that $X * Y$ is only defined when X and Y are disjoint subsets of N. $(N, \varnothing, *)$ is a partial commutative monoid. When both sides of the equations are defined, we get the usual commutative monoid structure on $(N, \varnothing, \cup)$.

To "lift" this to a bifunctor, $\_ * \_$ is the action on objects.(but the operation is only partial... make N a pointed set?). Given $(X * Y)$, $f : X \subseteq X'$, and $g : Y \subseteq Y'$, produce $(X' * Y')$ not target may not exist.. something seems to be missing?

The Day convolution take the monoidal structure on $World$ to a monoidal structure on $\widehat{World}$.

$$(A * B)X = \int^{Y,Z} A(Y) \times B(Z) \times World^{op}[X, Y * Z]$$

The concrete interpretation for this model is

$$(A * B)X = \{[(Y, Z, a \in A(Y), b \in B(Z))] \mid Y * Z \text{ is defined and } Y * Z \subseteq X\}$$

[5] note the type of F_1 and G_1 since the source category is opposite

where $[\cdot]$ denotes an equivalence class. $(Y, Z, a \in A(Y), b \in B(Z))$ and $(Y', Z', a' \in A(Y'), b' \in B(Z'))$ are equivalent if they have the same parent in the order determined by

$$(Y, Z, a \in A(Y), b \in B(Z)) \leq (Y', Z', a' \in A(Y'), b' \in B(Z'))$$
$$\text{if}$$
$$f : Y \subseteq Y', g : Z \subseteq Z', a' = A(f)a, b' = B(g)b$$

And for the "magic wand"

$$(A \twoheadrightarrow B)X = \int_Z Set^{World^{op}}[E(Z), F(X * Z)] \cong Set^{World^{op}}[E, F(X * \_)]$$

References

[1] https://leccap.engin.umich.edu/leccap/site/z02eb2esrpaddy7cnwz.

[2] https://link.springer.com/content/pdf/10.1007/978-94-017-0091-7.pdf.