1 Object Language

1.1 Raw Terms

The untyped syntax for CBPV OSum, add separating connectives, remove error. Add $\forall \underline{X}.\underline{B}$?

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Value Types
                             A
                                           X
                                           Unit
                                           Case A
                                           OSum
                                           A \times A
                                           A*A
                                           \exists X.A
                                           U\underline{B}
Computation Types \underline{B}
                                           A \to \underline{B}
                                           A \twoheadrightarrow \underline{B}
                                           \forall X.\underline{B}
                                           FA
Values
                             V
                                           \boldsymbol{x}
                                           tt
                                           \mathrm{inj}_V V
                                           (V, V)
                                           (V * V)
                                           pack (A, V) as \exists X.A
                                           thunk M
                                           \lambda x : A.M
Computations
                             M
                                  ::=
                                           MV
                                           \alpha x : A.M
                                           M@V
                                           \Lambda X.M
                                           M[A]
                                           ret V
                                           x \leftarrow M; N
                                           force V
                                           \mathrm{newcase}_A x; M
                                           match V with V { inj x.M||N| }
                                           let (x, x) = V; M
                                           let (x*x) = V; M
                                           unpack (X, x) = V; M
                             Γ
Value Context
                                    ::=
                                           \Gamma, x \colon A
                                           \Gamma * x : A
Type Context
                                           \Xi, X
```

1.2 Typed Terms

$$\overline{\Xi; \Gamma, x \colon A \vdash_{v} x \colon A}$$

$$\overline{\Xi; \Gamma \vdash_{v} \text{tt} \colon \text{Unit}}$$

$$\overline{\Xi; \Gamma \vdash_{v} \text{oscae} A} \qquad \Xi; \Gamma \vdash_{v} V \colon A$$

$$\overline{\Xi; \Gamma \vdash_{v} \text{inj}_{\sigma} V \colon \text{OSum}}$$

$$\underline{\Xi; \Gamma \vdash_{v} V_{1} \colon A_{1}} \qquad \Xi; \Gamma \vdash_{v} V_{2} \colon A_{2}$$

$$\overline{\Xi; \Gamma \vdash_{v} V_{1} \colon A_{1}} \qquad \Xi; \Gamma \vdash_{v} V_{2} \colon A_{2}$$

$$\overline{\Xi; \Gamma \vdash_{v} V_{1} \colon A_{1}} \qquad \Xi; \Gamma_{2} \vdash_{v} V_{2} \colon A_{2}$$

$$\overline{\Xi; \Gamma \vdash_{v} V_{1} \colon A_{1}} \qquad \Xi; \Gamma_{2} \vdash_{v} V_{2} \colon A_{2}$$

$$\overline{\Xi; \Gamma \vdash_{v} V_{1} \colon A_{1}} \qquad \Xi; \Gamma_{2} \vdash_{v} V_{2} \colon A_{2}$$

$$\overline{\Xi; \Gamma \vdash_{v} V_{1} \colon A_{1}} \qquad \Xi; \Gamma \vdash_{v} V_{2} \colon A_{1} \ast A_{2}$$

$$\Xi; \Gamma \vdash_{v} V \colon A[A'/X]$$

$$\overline{\Xi; \Gamma \vdash_{v} V} \Rightarrow A[A \mapsto B]$$

$$\overline{\Xi; \Gamma \vdash_{v} Ax \colon A \vdash_{v} M \colon B}$$

$$\overline{\Xi; \Gamma \vdash_{v} Ax \colon A \vdash_{v} M \colon B}$$

$$\overline{\Xi; \Gamma \vdash_{v} Ax \colon A \vdash_{v} M \colon B}$$

$$\overline{\Xi; \Gamma \vdash_{v} Ax \colon A \vdash_{v} M \colon B}$$

$$\overline{\Xi; \Gamma \vdash_{v} Ax \colon A \vdash_{v} M \colon B}$$

$$\overline{\Xi; \Gamma \vdash_{v} Ax \colon A \vdash_{v} B} \Rightarrow \Xi \vdash A$$

$$\overline{\Xi; \Gamma \vdash_{v} V} \Rightarrow A$$

$$\overline{\Xi; \Gamma \vdash_{v} Ax \colon A \vdash_{v} V \colon A}$$

$$\overline{\Xi; \Gamma \vdash_{v} \nabla \vdash_{v} A}$$

$$\begin{array}{c} \Xi;\Gamma\vdash_{c}M:FA \qquad \Xi;\Gamma,x:A\vdash_{c}N:\underline{B} \\ \hline \Xi;\Gamma\vdash_{c}x\leftarrow M;N:\underline{B} \\ \hline \Xi;\Gamma\vdash_{v}V:U\underline{B} \\ \hline \Xi;\Gamma\vdash_{v}V:U\underline{B} \\ \hline \Xi;\Gamma\vdash_{c}:\mathrm{force}V:\underline{B} \\ \hline \end{array}$$

$$\begin{array}{c} \Xi;\Gamma\vdash_{v}V:U\underline{B} \\ \hline \Xi;\Gamma\vdash_{c}:\mathrm{force}V:\underline{B} \\ \hline \end{array}$$

$$\begin{array}{c} \Xi;\Gamma\vdash_{v}V:\mathrm{Case}A)\vdash_{c}M:\underline{B} \qquad \Xi\vdash A \\ \hline \Xi;\Gamma\vdash_{c}\mathrm{newcase}_{A}x;M:\underline{B} \\ \hline \end{array}$$

$$\begin{array}{c} \Xi;\Gamma\vdash_{v}v:\mathrm{Case}A \qquad \Xi;\Gamma,x:A\vdash M:\underline{B} \qquad \Xi;\Gamma\vdash_{c}N:\underline{B} \\ \hline \Xi;\Gamma\vdash_{c}\mathrm{match}V \text{ with } \sigma\{\text{ inj }x.M\parallel N\}:\underline{B} \\ \hline \Xi;\Gamma\vdash_{c}\mathrm{match}V \text{ with } \sigma\{\text{ inj }x.M\parallel N\}:\underline{B} \\ \hline \Xi;\Gamma\vdash_{c}\mathrm{let}(x,y)=V;M:\underline{B} \\ \hline \Xi;\Gamma\vdash_{c}\mathrm{let}(x,y)=V;M:\underline{B} \\ \hline \Xi;\Gamma\vdash_{c}\mathrm{let}(x*y)=V;M:\underline{B} \\ \hline \Xi\vdash\underline{B} \qquad \Xi;\Gamma\vdash_{c}\mathrm{let}(x*y)=V;M:\underline{B} \\ \hline \Xi;\Gamma\vdash_{c}\mathrm{let}(x*y)=V;M:\underline{B} \\ \hline \Xi\vdash_{c}\mathrm{let}(x*y)=V;M:\underline{B} \\ \hline \Xi\vdash_{c}\mathrm{let}(x*y)=V;M:\underline{B} \\ \hline \Xi;\Gamma\vdash_{c}\mathrm{let}(x*y)=V;M:\underline{B} \\ \hline \end{array}$$

2 Logic

2.1 Judgments

The relation environment, Θ in PE logic contains both value and computation relations. How does this work in the semantics when value relations are denoted as objects of $Sub_{\mathcal{V}}(A \times B)$ for $Rel_{\mathcal{V}}[A,B]$ and computation relations are denoted as objects of $Sub_{\mathcal{C}}(\underline{A} \times \underline{B})$ for $Rel_{\mathcal{V}}[\underline{A},\underline{B}]$? Maybe we have separate relation environments?

$$\begin{split} &\Xi; \Gamma; \Theta \vdash \phi \text{ VProp} \\ &\Xi; \Gamma; \Theta \vdash (x:A,y:B).\phi : Rel_{\mathcal{V}}[A,B] \\ &\Xi; \Gamma; \Theta | \Phi \vdash \phi \\ &\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\phi} \text{ CProp} \\ &\Xi; \Gamma; \Delta; \Theta; \Omega \vdash (x:\underline{A},y:\underline{B}).\underline{\phi} : Rel_{\mathcal{C}}[\underline{A},\underline{B}] \\ &\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \underline{\Psi} \vdash \phi \end{split}$$

for type environment Ξ , term environment Γ , stoup Δ , value relation environment Θ , computation relation environment Ω , value proposition environment Φ , and computation proposition environment $\underline{\Psi}$.

2.2 Formation Rules

2.2.1 Value Propositions

$$\phi := \top |\phi \wedge \phi| t =_{A} u | R(t, u)$$

$$\overline{\Xi; \Gamma; \Theta \vdash \top \text{VProp}}$$

$$\underline{\Xi; \Gamma; \Theta \vdash \phi \text{ VProp}} \quad \Xi; \Gamma; \Theta \vdash \psi \text{ VProp}$$

$$\underline{\Xi; \Gamma; \Theta \vdash \phi \wedge \psi \text{ VProp}}$$

$$\underline{\Xi; \Gamma \vdash_{v} t : A} \quad \Xi; \Gamma \vdash_{v} u : A$$

$$\overline{\Xi; \Gamma; \Theta \vdash t =_{A} u \text{ VProp}}$$

$$\underline{\Xi; \Gamma \vdash_{v} t : A} \quad \Xi; \Gamma \vdash_{v} u : B \quad R : Rel_{\mathcal{V}}[A, B] \in \Theta$$

$$\Xi; \Gamma; \Theta \vdash R(t, u) \text{ VProp}$$

2.2.2 Computation Propositions

$$\underline{\psi} := \underline{\top} |\underline{\psi} \wedge \underline{\psi}| t =_{\underline{B}} u | \phi \implies \underline{\psi} |\underline{R}(t,u)| \forall (x:A).\underline{\psi}| \forall X.\underline{\psi}| \forall X.\underline{\psi}| \forall (R:Rel_{\mathcal{V}}[A,B]).\underline{\psi}| \forall (R:Rel_{\mathcal{C}}[\underline{A},\underline{B}]).\underline{\psi}$$

$$\underline{\Xi}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \text{ CProp}$$

$$\underline{\Xi}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \text{ CProp}$$

$$\underline{\Xi}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \wedge \underline{\psi} \text{ CProp}$$

$$\underline{\Xi}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \wedge \underline{\psi} \text{ CProp}$$

$$\underline{\Xi}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \wedge \underline{\psi} \text{ CProp}$$

$$\underline{\Xi}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \wedge \underline{\psi} \text{ CProp}$$

$$\underline{\Xi}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \wedge \underline{\psi} \text{ CProp}$$

$$\underline{\Xi}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \wedge \underline{\psi} \text{ CProp}$$

$$\underline{\Xi}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\mu} \wedge \underline{\psi} \wedge \underline{\psi} \text{ CProp}$$

$$\underline{\Xi}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \wedge \underline{\psi} \wedge \underline{\psi} \text{ CProp}$$

$$\underline{\Xi}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \wedge \underline{\psi} \wedge \underline{\psi} \wedge \underline{\psi} \text{ CProp}$$

$$\underline{\Xi}; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\psi} \wedge \underline{\psi} \wedge$$

- 2.2.3 Value Relations
- 2.2.4 Computation Relations
- 2.3 Derivation Rules
- **2.3.1** Values

Rel

$$\frac{\Xi;\Gamma;\Theta|\Phi\vdash}{\Xi;\Gamma;\Theta|\Phi\vdash}$$

2.3.2 Computation

Rel?

 $\frac{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \forall (R : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]).\underline{\phi} \qquad \Xi; \Gamma; \Delta; \Theta; \Omega, \vdash (x : \underline{A}, y : \underline{B}).\underline{\psi} : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash \forall (R : Rel_{\mathcal{C}}[\underline{A}, \underline{B}]).\phi [\psi[t/x, u/y]/R(t, u)]} \to \text{E-\forall crel }$

2.3.3 Congruences

$$\frac{\Xi; \Gamma \vdash_{c} t : \underline{B}}{\Xi; \Gamma; \Delta; \Theta; \Omega | \Phi; \Psi \vdash t = u} \xrightarrow{\Xi; \Gamma, x : A; \Delta; \Theta; \Omega | \Phi; \Psi \vdash t = u} \operatorname{cong-}\lambda$$

2.4 Axioms

Beta/Eta/(parametricity schema?)

2.5 Logical Interpretation of Types

Let X and \underline{X} be vectors of value type and computation type variables of length n. Let $\boldsymbol{\rho}$ be a vector of value relations $\Xi; \Gamma; \Theta \vdash \rho_i : Rel_{\mathcal{V}}[C_i, C_i']$ for all $i \in 1..n$. Let $\underline{\boldsymbol{\rho}}$ be a vector of computation relations $\Xi; \Gamma; \Delta; \Theta; \Omega \vdash \underline{\rho_i} : Rel_{\mathcal{C}}[\underline{C_i}, \underline{C_i'}]$ for all $i \in 1..n$.

Let A be a value type with $FTV(A) \in \{X, \underline{X}\}$. Define:

$$A[\rho/X, \rho/\underline{X}] : Rel_{\mathcal{V}}[A[C/X, \underline{C}/\underline{X}], A[C'/X, \underline{C'}/\underline{X}]]$$

by induction on A.

$$\begin{split} X_i[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \rho_i \\ \operatorname{Unit}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= (x:Unit, y:Unit).x =_{Unit} y \\ \operatorname{Case}A[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= (x:\operatorname{Case}\ (A[\boldsymbol{C}/\boldsymbol{X},\underline{\boldsymbol{C}}/\underline{\boldsymbol{X}}]), y:\operatorname{Case}\ (A[\boldsymbol{C}'/\boldsymbol{X},\underline{\boldsymbol{C}'}/\underline{\boldsymbol{X}}])). \\ &\qquad \qquad \text{think exists?} \\ \operatorname{OSum}[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= \text{think exists?} \\ A \times A'[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}] &= ((x,y):A \times A'[\boldsymbol{C}/\boldsymbol{X},\underline{\boldsymbol{C}}/\underline{\boldsymbol{X}}], (x',y'):A \times A'[\boldsymbol{C}'/\boldsymbol{X},\underline{\boldsymbol{C}'}/\underline{\boldsymbol{X}}]). \\ &\qquad \qquad A[\boldsymbol{\rho}, \underline{\boldsymbol{\rho}}](x,x') \wedge A'[\boldsymbol{\rho},\underline{\boldsymbol{\rho}}](y,y') \\ A * A'[\boldsymbol{\rho},\underline{\boldsymbol{\rho}}] &= \text{similar to product?} \\ \exists X.A[\boldsymbol{\rho},\underline{\boldsymbol{\rho}}] &= \text{standard} \\ U\underline{B}[\boldsymbol{\rho},\boldsymbol{\rho}] &= \text{related thunks?} \end{split}$$

Let \underline{B} be a **computation type** with $FTV(\underline{B}) \in \{X, \underline{X}\}$. Define:

$$\underline{B}[\rho/X, \rho/\underline{X}] : Rel_{\mathcal{C}}[\underline{B}[C/X, \underline{C}/\underline{X}], \underline{B}[C'/X, \underline{C'}/\underline{X}]]$$

by induction on \underline{B} .

3 Goal

Try to prove

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\begin{split} &\vdash_{d} \forall A \ \underline{B}. \\ &\forall (y: \mathrm{OSum}). \\ &\forall (f: U(\forall X. \mathrm{Case} \ X \twoheadrightarrow \mathrm{OSum} \to FX)). \\ &\forall (k \ k': U(A \to \underline{B})). \\ &\mathrm{newcase}_{A} \sigma; x \leftarrow (!f)[A] \sigma y; (!k) x \\ &= \\ &\mathrm{newcase}_{A} \sigma; x \leftarrow (!f)[A] \sigma y; (!k') x \end{split}
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