Introduction to Automated Planning

(Draft)

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Foreword

These are the lecture notes of the AI planning course at the Albert-Ludwigs-University Freiburg in summer term 2005, based on earlier notes for the course in winter 2002/2003 and in summer 2004.

I would like to thank all the students who have participated in the planning course and given comments, pointed out errors, and suggested other improvements, including Slawomir Grzonka, Bernd Gutmann, Raimund Renner, Richard Schmidt, and Martin Wehrle.

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Chapter 1

Introduction

Planning in Artificial Intelligence is *decision making* about the *actions* to be taken.

Consider an intelligent robot. The robot is a computational mechanism that takes input through its sensors that allow the robot to *observe* its environment and to build *a representation* of its immediate surroundings and parts of the world it has observed earlier. For a robot to be useful it has to be able to *act*. A robot acts through its *effectors* which are devices that allow the robot to move itself and other objects in its immediate surroundings. A robot resembling a human being has hands and feet, or their muscles, as effectors.

At an abstract level, a robot is a mechanism that maps its observations, which are obtained through the sensors, to actions which are performed by means of the effectors. Planning is the decision making needed in producing a sequence of actions given a sequence of observations. The more complicated the environment and the tasks of the robot are, the more intelligent the robot has to be. For genuine intelligence it is important that the robot is able to plan its actions also in challenging situations.

No intelligent robots exist yet. The most intelligent existing robots carry out tasks that do not require genuine intelligence, like transporting objects from one place to another in environments that are predictable and known in advance. For more challenging tasks in which the working environment of the robot is not exactly known in advance, the biggest challenges are currently in interpreting the sensor data reliably and controlling the basic movements of the robot effectively. Before these research problems have been solved adequately, the employment of robots for more intelligent tasks is not feasible. When this stage will be reached some time in the future, powerful techniques for knowledge representation and task planning will be needed to bring the intelligence of the robots to a sufficiently high level.

Impediments for the success of AI in producing genuinely intelligent beings are related to perceiving and representing knowledge concerning the world. The real world is very complicated in all its physical and geometric as well as social aspects, and representing all the knowledge required by an intelligent being may be too inflexible and complicated by the logical and symbolical means almost exclusively used in artificial intelligence and in planning. This has been criticized by many researchers [Brooks, 1991] and it is a topic of continuing scientific debate.

AI planning – like knowledge representation and learning techniques in AI in general – are currently best applicable in restricted domains in which it is easy to identify what the atomic facts are and to exactly describe how the world behaves. These properties are best fulfilled by systems that are completely man-made, or systems in which planning can view the world at a sufficiently abstract level.

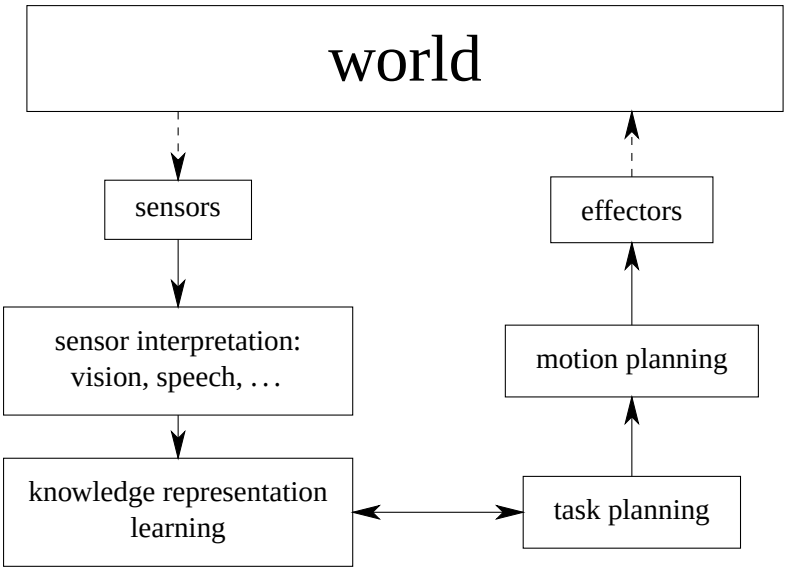


Figure 1.1: Software architecture of an intelligent robot

An example of a completely man-made system to which planning techniques have successfully been applied include the control of autonomous spacecraft [Muscettola *et al.*, 1998]. The vacuum of the outer space is a very simple environment without most of the uncertainties typically present on the surface of the earth. Other current robotic applications like delivering mail in an office or distributing medicine in a hospital, employ only very little from the potential of AI planning.

A simple real-world application in which abstracting away the details of the real world is possible is transportation planning: how to get from Freiburg to London by public transportation, trains, airplanes and buses. If a robot were capable of finding its way between the couple of hundred of meters between the various forms of transportation and recognize the trains and buses to board it could easily travel all over the world. Planning what transportation to use is an easy problem in this case.

1.1 Types of planning problems

The word *planning* is very general and denotes many different things. Even in the AI and robotics context there are many types of planning.

Simply controlling the basic movement of robots is a very challenging problem. *Path planning* is needed in finding a way for one location to another, and *motion planning* is needed in moving the hands and feet of the robot to produce meaningful behavior. and they are not discussed in this lecture as they require specialized representations of the geometric properties of the world and cannot usually be efficiently represented in the general state-based model we are interested in. There is also the well established research area of *scheduling* which is concerned with ordering and choosing a schedule for executing a number of predefined actions.

The topic of this lecture is sometimes called *task planning* in order to distinguish it from the

more concrete geometric and physical forms of planning which are used in controlling the movements of robots and similar systems.

Even within task planning, there are many different types of planning problems, depending on the assumptions concerning the properties of actions and the world that are made. Some of these are the following.

1. Determinism versus nondeterminism.

   In the simplest form of planning the state of the world at any moment is unambiguously determined by the initial state of the world and the sequence of actions that have been taken. Hence the world is completely deterministic.

   The assumption of a deterministic world holds in many simple planning problems. However, when the world is modeled in more detail and more realistically, the assumption does not hold any more: the plans have to take into account events that take place independently of the actions and also the possibility that the effects of an action are not the same every time the action is taken, even when the world appears to be the same.

   Nondeterminism comes from two different sources.

   First, any feasible model of the world is very incomplete, and events that are possible as far as our beliefs are concerned can be viewed as nondeterministic: we do not know whether somebody is going to phone or visit us, and the visit or phone call can be modeled as a nondeterministic event that may or may not take place.

   Second, many actions themselves are by their nature nondeterministic, either intentionally or unintentionally. Throwing two dice and summing the result has 11 possible outcomes that cannot be predicted. Throwing an object to a garbage bin from a distance may or may not succeed.

   Notice that there is still the possibility that the physical universe is completely deterministic, but as long as we do not know the exact causes of events, we might just as well consider them nondeterministic.

2. Observability.

   For deterministic planning problems with one initial state the world is completely predictable. As the state of the world after taking certain actions can be completely predicted, there is no need to use observations. Hence a plan, if one exists, is simply a sequence of actions.

   When the actions or the environment can be nondeterministic, or when the initial state is not exactly known, it is not in general possible to reach the goals by using one fixed sequence of actions. The actions have to depend on the observations.

   There are two possibilities. First, planning could be interleaved with plan execution: only one action is chosen at a time, it is executed, and based on the observations the next action is chosen, and so on. Second, a complete plan is generated, covering all possible events that can happen, and it is executed, without further planning during execution. This kind of plans could be formalized as programs with conditionals (*if-then-else*) and loops.

   These two approaches are computationally very close, but the first approach does not require explicitly representing all the action sequences that might be needed, it only has to find a guarantee that such action sequences exist.

The possible observations have a strong impact on how exactly the actual state of the world can be determined: the more facts can be observed, the more precisely the current state of the world can be determined, and the better the most appropriate action can be chosen. If there is a lot of uncertainty concerning the current state of the world it may be impossible to choose an appropriate action.

If the current state can always be determined uniquely we have *full observability*. If the current state cannot be determined uniquely we have *partial observability*, and planning algorithms are forced to consider sets of possible current states.

3. Time.

   Most work on planning uses discrete (integer) time and actions of unit duration. This means that all changes caused by an action at time point t are visible at time point $t+1$. So changes in the world take only one unit of time, and what happens between two time points is not analyzed further.

   More complicated models of time and change are possible, but in this lecture we consider only discrete time. Most types of problems can be analyzed in terms of discrete time by making the unit duration sufficiently small. Rational and real time cause conceptual difficulties. Effects of actions that are not immediate can be reduced to the basic case by encoding the delayed effects in the state description.

4. Control information and plan structure.

   In the basic planning problem a plan is to be synthesized based on a generic description of how the actions affect the world.

   There may be, however, further control information that may affect the planning process and the plans that are produced. In hierarchical planning, for example, information on the structure of the possible plans is given in the form of a hierarchical task network, and the plans that are produced must conform to this structure. This kind of structural information may substantially improve the efficiency of planning. Another way of restricting the structure of plans, for efficiency or other reasons, is by using temporal logics [Bacchus and Kabanza, 2000].

5. Plan quality.

   The purpose of a plan is often just to reach one of the predefined goal states, and plans are judged only with respect to the satisfaction of this property. However, actions may have differing costs and durations, and plans could be assessed in terms of their time consumption or cost.

   As different executions of a plan in a nondeterministic world produce different sequences of actions, plans can be valued in terms of their expected costs, best-case costs, worst-case costs, and probability of eventually reaching the goals.

   Plans with an infinite execution length can also be considered, and then plans may be valued according to their average cost per unit time, or according to their geometrically discounted costs.

1.2 Related topics

Reasoning about action has emerged as a separate research area with the goal of making inferences about actions and their effects [Ginsberg and Smith, 1988; Shoham, 1988; Sandewall, 1994a; 1994b; Stein and Morgenstern, 1994]. Important research topics include the qualification and the ramification problems, which respectively involve deciding whether a certain action can be performed to have its anticipated effects and what are the indirect effects of an action. These problems are important because of their relation to the reasoning performed by human beings and their importance in representing the world as required by intelligent systems employing planning. In this lecture, however, we assume that a description of some actions is given, with all preconditions and direct and indirect effects fully spelled out, and concentrate on what kind of planning can be performed with these actions. The problems are also fully orthogonal, that is, the planning algorithms do not need to depend on the solution to the ramification and qualification problems that are used.

Markov decision processes [Puterman, 1994] in operations research is essentially a formalization of planning. In contrast to AI planning, work in that area has used explicit enumerative representations of transition systems, like those used in Section 2.1, and as a consequence the algorithms have a different flavor than most planning algorithms do. However, most recent work on probabilistic planning is based on Markov decision processes.

Discrete event systems (DES) in control engineering have been proposed as a model for synthesizing controllers for systems like automated factories [Ramadge and Wonham, 1987; Wonham, 1988], and this topic is closely related to planning. Again, there are differences in the problem formulation, with state spaces being represented enumeratively or more succinctly, for example as Petri nets [Ichikawa and Hiraishi, 1988] or vector additions systems [Li and Wonham, 1993].

Synthesis of programs for reactive systems that work in nondeterministic and partially observable environments is similar to planning under same conditions. Program synthesis has been considered for example from specifications of their input-output behavior in different types of temporal logics [Vardi and Stockmeyer, 1985; Kupferman and Vardi, 1999].

1.3 Early research on AI planning

Research that has lead to current AI planning started in the 1960's in the form of programs that tried to simulate problem solving abilities of human beings. One of the first programs of this kind was the General Problem Solver (GPS) by Newell and Simon [Ernst *et al.*, 1969]. GPS performed state space search guided by estimated differences between the current state and the goal states.

At the end of 1960's Green proposed the use of theorem-provers for constructing plans [Green, 1969]. However, because of the immaturity of theorem-proving techniques at that time, this approach was soon mostly abandoned in favor of specialized planning algorithms. There was theoretically oriented work on deductive planning which used different kinds of modal and dynamic logics [Rosenschein, 1981] but these works had little impact on the development of efficient planning algorithms. Deductive and logic-based approaches to planning gained popularity again only at the end of the 1990's as a consequence of the development of more sophisticated programs for the satisfiability problem of the classical propositional logic [Kautz and Selman, 1996].

One of the most well known early planning systems is the STRIPS planner from the beginning of the 1970's [Fikes and Nilsson, 1971]. The states in STRIPS are sets of formulae, and the operators change these state descriptions by adding and deleting formulae in the sets. Heuristics similar

to the ones used in the GPS system were used in guiding the search. The definition of operators, with a *precondition* as well as *add* and *delete* lists, corresponding to the facts that respectively become true and false, and the associated terminology, is still in common use, although restricted to atomic facts, that is, the add list is simply the set of state variables that the action makes true, and the delete list similarly consists of the state variables that become false.

Starting in the mid 1970's the dominating approach to domain-independent planning was the so-called partial-order, or causal link, or nonlinear planning [Sacerdoti, 1975; McAllester and Rosenblitt, 1991], which remained popular until the mid-1990's and the introduction of the Graphplan planner [Blum and Furst, 1997] which started the shift away from partial-order planning to types of algorithms that had earlier been considered infeasible, even the then-notorious total-order planners. The basic idea of partial-order planning is that a plan is incrementally constructed starting from the initial state and the goals, by either adding an action to the plan so that one of the open goals or operator preconditions is fulfilled, or adding an ordering constraint on operators already in the plan in order to resolve a potential conflict between them. In contrast to the forward or backward search strategies in Chapter 3 partial-order planners tried to avoid unnecessarily imposing an ordering on operators. The main advantages of both partial-order planners and Graphplan are present in the SAT/CSP approach to planning which is discussed in Section 3.6.

In parallel to partial-order planning, the notion of hierarchical planning emerged [Sacerdoti, 1974], and it has been deployed in many real-world applications. The idea in hierarchical planning is that the problem description imposes a structure on solutions and restricts the number of choices the planning algorithm has to make. A hierarchical plan consists of a main task which is decomposed to smaller tasks which are recursively solved. For each task there is a choice between solution methods. The less choice there is, the more efficiently the problem is solved. Furthermore, many hierarchical planners allow the embedding of problem-specific heuristics and problem-solvers to further speed up planning.

A collection of articles on AI planning starting from the late 1960's has been edited by Allen et al. [1990]. Many of the papers are mainly of historical interest, and some of them outline ideas that are still in use.

1.4 This book

My intention in writing these lecture notes was to cover planning problems of different generality and some of the most important approaches to solving each type of problem. Of course, during the last several decades of planning research a lot of work has been done that are not covered in these notes.

Important differences to most textbooks and research papers on planning is that I use a unified and rather expressive syntax for representing operators, including nondeterministic and conditional effects. This has several implications on the material covered in this book. For example, it may surprising that I do not use a concept viewed very central for deterministic planning by some researchers, *the planning graphs* of Blum and Furst [1997]. This is a direct implication of the general syntax for operators I use, as discussed in more detail in Section 3.8. It seems that any useful graph-theoretic properties planning graphs have lose their meaning when a definition of operators more general than STRIPS operators is used.

One of the messages of these notes is the importance of logic (propositional logic in our case) for all forms of planning ranging from the simplest deterministic case to the most general types

of planning with partial observability. As we will see, states, sets of states, belief states and transition relations associated with operators are often most naturally represented as propositional formulae. This representation shows up once and again in connection with different types of planning algorithms, including backward search in classical/deterministic planning, planning as satisfiability, and in implementations of nondeterministic planning algorithms by means of binary decision diagrams and similar data structures.

In addition to generalizing many existing techniques to the more general definition of planning problems, many of the algorithms are either new or have been developed further from earlier algorithms. I cite the original sources in the literature sections in the end of every chapter. Some of my contributions can be singled out rather precisely. They include the following.

1. The definition of regression for conditional and nondeterministic operators in Sections 3.1.2 and 4.1.1.

2. The algorithm for computing invariants in Section 3.5. The computation of mutexes in Blum and Furst's [1997] planning graphs can be viewed as a special case of my algorithm, restricted to unconditional operators only.

3. The algorithm for planning with full observability in Section 4.4.2. This algorithm is based on a similar but more complicated algorithm by Cimatti et al. [2003].

4. The representation of planning without observability as quantified Boolean formulae in Section 4.6.

5. The framework for non-probabilistic planning with partial observability in Section 4.7.

6. The complexity results in Section 4.8.3, most importantly the 2-EXP-completeness result for conditional planning with partial observability.

Chapter 2

Background

In this chapter we will define the formal machinery which is needed for describing different planning problems and algorithms. We will give the basic definitions related to the classical propositional logic and the transition system model which is the basis of most work on planning and which is closely related to finite automata and transition systems in other areas of computer science.

2.1 Transition systems

We define transition systems in which states are atomic objects and actions are represented as binary relations on the set of states.

Definition 2.1 *A* transition system *is a 5-tuple* $\Pi = \langle S, I, O, G, P \rangle$ *where*

1. S *is a finite set of states,*
2. $I \subseteq S$ *is the set of initial states,*
3. O *is a finite set of actions* $o \subseteq S \times S$*,*
4. $G \subseteq S$ *is the set of goal states, and*
5. $P = (C_1, \ldots, C_n)$ *is a partition of* S *to non-empty classes of observationally indistinguishable states satisfying* $\bigcup\{C_1, \ldots, C_n\} = S$ *and* $C_i \cap C_j = \emptyset$ *for all* i, j *such that* $1 \leq i < j \leq n$*.*

Making an observation tells which set C_i the current state belongs to. Distinguishing states within a given C_i is not possible by observations. If two states are observationally distinguishable then plan execution can proceed differently for them.

The number n of components in the partition P determines different classes of planning problems with respect to observability restrictions. If $n = |S|$ then every state is observationally distinguishable from every other state. This is called *full observability*. If $n = 1$ then no observations are possible and the transition system is *unobservable*. The general case $n \in \{1, \ldots, |S|\}$ is called *partial observability*.

An action o is *applicable* in states for which it associates at least one successor state. We define *images* of states as $\mathit{img}_o(s) = \{s' \in S | sos'\}$ and (weak) *preimages* of states as $\mathit{preimg}_o(s') = \{s \in S | sos'\}$. Generalization to sets of states is $\mathit{img}_o(T) = \bigcup_{s \in T} \mathit{img}_o(s)$ and $\mathit{preimg}_o(T) =$

$\bigcup_{s \in T} \mathit{preimg}_o(s)$. For sequences $o_1, \ldots, o_n$ of actions we define $\mathit{img}_{o_1;\ldots;o_n}(T) = \mathit{img}_{o_n}(\cdots \mathit{img}_{o_1}(T) \cdots)$ and $\mathit{preimg}_{o_1;\ldots;o_n}(T) = \mathit{preimg}_{o_1}(\cdots \mathit{preimg}_{o_n}(T) \cdots)$. The *strong preimage* of a set T of states is the set of states for which all successor states are in T, defined as $\mathit{spreimg}_o(T) = \{s \in S | s' \in T, sos', \mathit{img}_o(s) \subseteq T\}$.

Lemma 2.2 *Images, strong preimages and weak preimages of sets of states are related to each other as follows. Let o be any action and S and S' any sets of states.*

1. $\mathit{spreimg}_o(T) \subseteq \mathit{preimg}_o(T)$
2. $\mathit{img}_o(\mathit{spreimg}_o(T)) \subseteq T$
3. *If* $T \subseteq T'$ *then* $\mathit{img}_o(T) \subseteq \mathit{img}_o(T')$.
4. $\mathit{preimg}_o(T \cup T') = \mathit{preimg}_o(T) \cup \mathit{preimg}_o(T')$.
5. $s' \in \mathit{img}_o(s)$ *if and only if* $s \in \mathit{preimg}_o(s)$.

Proof:

1. $\mathit{spreimg}_o(T) = \{s \in S | s' \in T, sos', \mathit{img}_o(s) \subseteq T\} \subseteq \{s \in S | s' \in T, sos'\} = \bigcup_{s' \in T} \{s \in S | sos'\} = \bigcup_{s' \in T} \mathit{preimg}_o(s') = \mathit{preimg}_o(T)$.
2. Take any $s' \in \mathit{img}_o(\mathit{spreimg}_o(T))$. Hence there is $s \in \mathit{spreimg}_o(T)$ so that sos'. As $s \in \mathit{spreimg}_o(T)$, $\mathit{img}_o(s) \subseteq T$. Since $s' \in \mathit{img}_o(s)$, $s' \in T$.
3. Assume $T \subseteq T'$ and $s' \in \mathit{img}_o(T)$. Hence sos' for some $s \in T$ by definition of images. Hence sos' for some $s \in T'$ because $T \subseteq T'$. Hence $s' \in \mathit{img}_o(T')$ by definition of images.
4. $\mathit{preimg}_o(T \cup T') = \bigcup_{s' \in T \cup T'} \{s \in S | sos'\} = \bigcup_{s' \in T} \{s \in S | sos'\} \cup \bigcup_{s' \in T'} \{s \in S | sos'\} = \mathit{preimg}_o(T) \cup \mathit{preimg}_o(T')$
5. $s' \in \mathit{img}_o(s)$ iff sos' iff $s \in \mathit{preimg}_o(s)$.

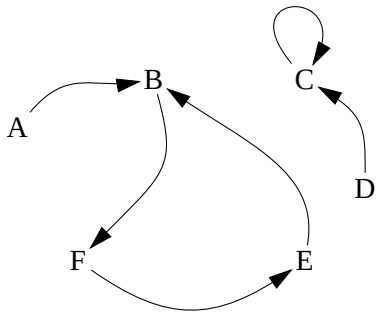
□

2.1.1 Deterministic transition systems

Transition systems which we use in Chapter 3 have only one initial state and deterministic actions. For this subclass observability is irrelevant because the state of the transition system after a given sequence of actions can be predicted exactly. We use a simpler formalization of them.

Definition 2.3 *A* deterministic transition system *is a 4-tuple* $\Pi = \langle S, I, O, G \rangle$ *where*

1. S *is a finite set of states,*
2. $I \in S$ *is the initial state,*
3. O *is a finite set of actions* $o \subseteq S \times S$ *that are partial functions, and*
4. $G \subseteq S$ *is the set of goal states.*



	A	B	C	D	E	F
A	0	1	0	0	0	0
B	0	0	0	0	0	1
C	0	0	1	0	0	0
D	0	0	1	0	0	0
E	0	1	0	0	0	0
F	0	0	0	0	1	0

Figure 2.1: The transition graph and the incidence matrix of a deterministic action

That the actions are partial functions means that for any $s \in S$ and $o \in O$ there is at most one state s' such that sos'. We denote the unique successor state s' of a state s in which operator o is applicable by $s' = app_o(s)$. For sequences $o_1; \ldots; o_n$ of operators we define $app_{o_1;\ldots;o_n}(s)$ as $app_{o_n}(\cdots app_{o_1}(s)\cdots)$.

2.1.2 Incidence matrices

Actions and other binary relations can be represented in terms of incidence matrices M (adjacency matrices) in which the element in row i and column j indicates whether a transition from state i to j is possible.

Figure 2.1 depicts the transition graph of an action and the corresponding incidence matrix. The action can be seen to be deterministic because for every state there is at most one arrow going out of it, and each row of the matrix contains at most one non-zero element.

For matrices $M_1, \ldots, M_n$ which represent the transition relations of actions $a_1, \ldots, a_n$ the combined transition relation is $M = M_1 + M_2 + \cdots + M_n$. The matrix M now tells whether a state can be reached from another state by at least one of the actions.

Here $+$ is the usual matrix addition that uses the Boolean addition for integers 0 and 1, which is defined as $0 + 0 = 0$, and $b + b' = 1$ if $b = 1$ or $b' = 1$. Boolean addition is used because in the presence of nondeterminism we could have 1 for both of two transitions from A to B and from A to C. For probabilistic planning problems normal addition is used and matrix elements are interpreted as probabilities of nondeterministic transitions.

The incidence matrix corresponding to first taking action a_1 and then a_2 is $M_1 M_2$. This is illustrated by Figure 2.2 The inner product of two vectors in the definition of matrix product corresponds to the reachability of a state from another state through all possible intermediate states.

Now we can compute for all pairs s, s' of states whether s' is reachable from s by a sequence of actions. Let M be the matrix that is the (Boolean) sum of the matrices of the individual actions. Then define

$$\begin{aligned} R_0 &= I_{n\times n} \\ R_i &= R_{i-1} + MR_{i-1} \text{ for } i \geq 1. \end{aligned}$$

Here n is the number of states and $I_{n\times n}$ is the unit matrix of size n. By Tarski's fixpoint theorem $R_i = R_j$ for some $i \geq 0$ and all $j \geq i$ because of the monotonicity property that every element that is 1 for some i is 1 also for all $j > i$. Matrix $R_i = M^0 \cup M^1 \cup \cdots \cup M^i$ represents reachability

	A	B	C	D	E	F
A	0	1	0	0	0	0
B	0	0	0	0	0	1
C	0	0	1	0	0	0
D	0	0	1	0	0	0
E	0	1	0	0	0	0
F	0	0	0	0	1	0

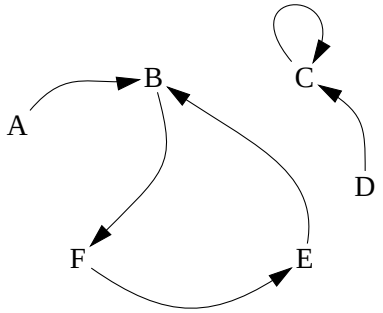
$\times$

	A	B	C	D	E	F
A	0	1	0	0	0	0
B	0	0	0	0	0	1
C	1	0	0	0	0	0
D	0	0	0	1	0	0
E	0	0	0	0	1	0
F	0	0	0	1	0	0

$=$

	A	B	C	D	E	F
A	0	0	0	0	0	1
B	0	0	0	**1**	0	0
C	1	0	0	0	0	0
D	1	0	0	0	0	0
E	0	0	0	0	0	1
F	0	0	0	0	1	0

Figure 2.2: Matrix product corresponds to sequential composition.



	A	B	C	D	E	F
A	0	1	0	0	0	0
B	0	0	0	0	0	1
C	0	0	1	0	0	0
D	0	0	1	0	0	0
E	0	1	0	0	0	0
F	0	0	0	0	1	0

Figure 2.3: A transition graph and the corresponding matrix M

by i actions or less.

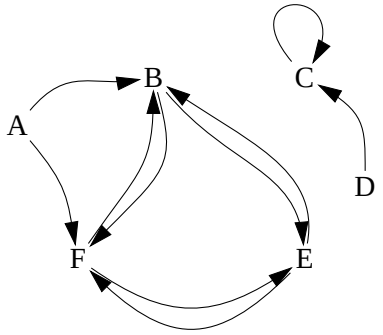
2.2 Classical propositional logic

Let A be a set of propositional variables (atomic propositions). We define the set of propositional formulae inductively as follows.

1. For all $a \in A$, a is a propositional formula.
2. If ϕ is a propositional formula, then so is $\neg\phi$.
3. If ϕ and ϕ' are propositional formulae, then so is $\phi \vee \phi'$.
4. If ϕ and ϕ' are propositional formulae, then so is $\phi \wedge \phi'$.
5. The symbols $\bot$ and $\top$, respectively denoting truth-values false and true, are propositional formulae.

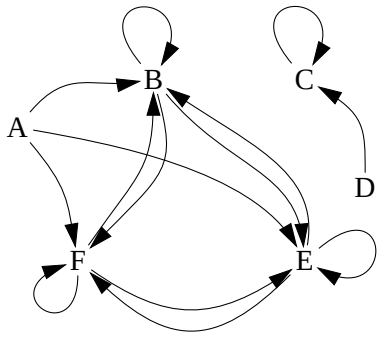
The symbols $\wedge$, $\vee$ and $\neg$ are *connectives* respectively denoting the *conjunction*, *disjunction* and *negation*. We define the implication $\phi \to \phi'$ as an abbreviation for $\neg\phi \vee \phi'$, and the equivalence $\phi \leftrightarrow \phi'$ as an abbreviation for $(\phi \to \phi') \wedge (\phi' \to \phi)$.

A valuation of A is a function $v : A \to \{0, 1\}$ where 0 denotes false and 1 denotes true. Valuations are also known as *assignments* or *models*. For propositional variables $a \in A$ we define



	A	B	C	D	E	F
A	0	1	0	0	0	1
B	0	0	0	0	1	1
C	0	0	1	0	0	0
D	0	0	1	0	0	0
E	0	1	0	0	0	1
F	0	1	0	0	1	0

Figure 2.4: A transition graph extended with composed paths of length 2 and the corresponding matrix $M + M^2$



	A	B	C	D	E	F
A	0	1	0	0	1	1
B	0	1	0	0	1	1
C	0	0	1	0	0	0
D	0	0	1	0	0	0
E	0	1	0	0	1	1
F	0	1	0	0	1	1

Figure 2.5: A transition graph extended with composed paths of length 3 and the corresponding matrix $M + M^2 + M^3$

$v \models a$ if and only if $v(a) = 1$. A valuation of the propositional variables in A can be extended to a valuation of all propositional formulae over A as follows.

1. $v \models \neg\phi$ if and only if $v \not\models \phi$
2. $v \models \phi \vee \phi'$ if and only if $v \models \phi$ or $v \models \phi'$
3. $v \models \phi \wedge \phi'$ if and only if $v \models \phi$ and $v \models \phi'$
4. $v \models \top$
5. $v \not\models \bot$

Computing the truth-value of a formula under a given valuation of propositional variables is polynomial time in the size of the formula by the obvious recursive procedure.

A propositional formula ϕ is *satisfiable* (*consistent*) if there is at least one valuation v so that $v \models \phi$. Otherwise it is *unsatisfiable* (*inconsistent*). A finite set F of formulae is satisfiable if $\bigwedge_{\phi \in F} \phi$ is. A propositional formula ϕ is *valid* or a *tautology* if $v \models \phi$ for all valuations v. We denote this by $\models \phi$. A propositional formula ϕ is *a logical consequence* of a propositional formula ϕ', written $\phi' \models \phi$, if $v \models \phi$ for all valuations v such that $v \models \phi'$. A propositional formula that

is a proposition variable a or a negated propositional variable $\neg a$ for some $a \in A$ is *a literal*. A formula that is a disjunction of literals is *a clause*.

A formula ϕ is in *negation normal form* (NNF) if all occurrences of negations are directly in front of propositional variables. Any formula can be transformed to negation normal form by applications of the De Morgan rules $\neg(\phi \vee \phi') \equiv \neg\phi \wedge \neg\phi'$ and $\neg(\phi \wedge \phi') \equiv \neg\phi \vee \neg\phi'$, the double negation rule $\neg\neg\phi \equiv \phi$. A formula ϕ is in *conjunctive normal form* (CNF) if it is a conjunction of disjunctions of literals. A formula ϕ is in *disjunctive normal form* (DNF) if it is a disjunction of conjunctions of literals. Any formula in CNF or in DNF is also in NNF.

2.2.1 Quantified Boolean formulae

There is an extension of the satisfiability and validity problems of the classical propositional logic with quantification over the truth-values of propositional variables. *Quantified Boolean formulae* (QBF) are like propositional formulae but there are two new syntactic rules for the quantifiers.

6. If ϕ is a formula and $a \in A$, then $\forall a\phi$ is a formula.

7. If ϕ is a formula and $a \in A$, then $\exists a\phi$ is a formula.

Further, there is the requirement that every variable is quantified, that is, every occurrence of $a \in A$ in a QBF is in the scope of either $\exists a$ or $\forall a$.

Define $\phi[\psi/x]$ as the formula obtained from ϕ by replacing occurrences of the propositional variable x by ψ.

We define the truth-value of QBF by reducing them to ordinary propositional formulae without occurrences of propositional variables. The atomic formulae in these formulae are the constants $\top$ and $\bot$. The truth-value of these formulae is independent of the valuation, and is recursively computed by the Boolean functions associated with the connectives $\vee$, $\wedge$ and $\neg$.

Definition 2.4 (Truth of QBF) *A formula $\exists x\phi$ is true if and only if $\phi[\top/x] \vee \phi[\bot/x]$ is true. (Equivalently, if $\phi[\top/x]$ is true or $\phi[\bot/x]$ is true.)*

A formula $\forall x\phi$ is true if and only if $\phi[\top/x] \wedge \phi[\bot/x]$ is true. (Equivalently, if $\phi[\top/x]$ is true and $\phi[\bot/x]$ is true.)

A formula ϕ with an empty prefix (and consequently without occurrences of propositional variables) is true if and only if ϕ is satisfiable (equivalently, valid: for formulae without propositional variables validity coincides with satisfiability.)

Example 2.5 The formulae $\forall x\exists y(x \leftrightarrow y)$ and $\exists x\exists y(x \wedge y)$ are true.

The formulae $\exists x\forall y(x \leftrightarrow y)$ and $\forall x\forall y(x \vee y)$ are false. ■

Notice that a QBF with only existential quantifiers is true if and only if the formula without the quantifiers is satisfiable. Similarly, truth of QBF with only universal quantifiers coincides with the validity of the corresponding formulae without quantifiers.

Changing the order of two consecutive variables quantified by the same quantifier does not affect the truth-value of the formula. It is often useful to ignore the ordering in these cases and to view each quantifier as quantifying a set of formulae, for example $\exists x_1 x_2 \forall y_1 y_2 \phi$.

Quantified Boolean formulae are interesting because evaluating their truth-value is PSPACE-complete [Meyer and Stockmeyer, 1972], and many computational problems that presumably cannot be translated into the satisfiability problem of the propositional logic in polynomial time (assuming that NP$\neq$PSPACE) can be efficiently translated into QBF.

2.2.2 Binary decision diagrams

Propositional formulae can be transformed to different normal forms. The most well-known normal forms are the conjunctive normal form (CNF) and the disjunctive normal form (DNF). Formulae in conjunctive normal form are conjunctions of disjunctions of literals, and in disjunctive normal form they are disjunctions of conjunctions of literals. For every propositional formula there is a logically equivalent one in both of these normal forms. However, the formula in normal form may be exponentially bigger.

Normal forms are useful for at least two reasons. First, certain types of algorithms are easier to describe when assumptions of the syntactic form of the formulae can be made. For example, the resolution rule which is the basis of many theorem-proving algorithms, is defined for formulae in the conjunctive normal form only (the clausal form). Defining resolution for non-clausal formulae is more difficult.

The second reason is that certain computational problems can be solved more efficiently for formulae in normal form. For example, testing the validity of propositional formulae is in general co-NP-hard, but if the formulae are in CNF then it is polynomial time: just check whether every conjunct contains both p and $\neg p$ for some proposition p.

Transformation into a normal form in general is not a good solution to any computationally intractable problem like validity testing, because for example in the case of CNF, polynomial-time validity testing became possible only by allowing a potentially exponential increase in the size of the formula.

However, there are certain normal forms for propositional formulae that have proved very useful in various types of reasoning needed in planning and other related areas, like model-checking in computer-aided verification.

In this section we discuss (ordered) binary decision diagrams (BDDs) [Bryant, 1992]. Other normal forms of propositional formulae that have found use in AI and could be applied to planning include the decomposable negation normal form [Darwiche, 2001] which is less restricted than binary decision diagrams (formulae in DNNF can be viewed as a superclass of BDDs) and are sometimes much smaller. However, smaller size means that some of the logical operations that can be performed in polynomial time for BDDs, like equivalence testing, are NP-hard for formulae in DNNF.

The main reason for using BDDs is that the logical equivalence of BDDs coincides with syntactic equivalence: two BDDs are logically equivalent if and only if they are the same BDD. Propositional formulae in general, or formulae in CNF or in DNF do not have this property. Furthermore, computing a BDD that represents the conjunction or disjunction of two BDDs or the negation of a BDDs also takes only polynomial time.

However, like with other normal forms, a BDD can be exponentially bigger than a corresponding unrestricted propositional formula. One example of such a propositional formulae is the binary multiplier: Any BDD representation of n-bit multipliers has a size exponential in n. Also, even though many of the basic operations on BDDs can be computed in polynomial time in the size of the component BDDs, iterating these operations may increase the size exponentially: some of these operator may double the size of the BDD, and doubling n times is exponential in n and in the size of the original BDD.

A main application of BDDs has been model-checking in computer-aided verification [Burch *et al.*, 1994; Clarke *et al.*, 1994], and in recent years these same techniques have been applied to AI planning as well. We will discuss BDD-based planning algorithms in Chapter 4.

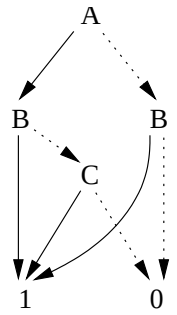


Figure 2.6: A BDD

BDDs are expressed in terms of the ternary Boolean operator if-then-else $ite(p, \phi_1, \phi_2)$ defined as $(p \wedge \phi_1) \vee (\neg p \wedge \phi_2)$, where p is a proposition. Any Boolean formula can be represented by using this operator together with propositions and the constants $\top$ and $\bot$. Figure 2.6 depicts a BDD for the formula $(A \vee B) \wedge (B \vee C)$. The normal arrow coming from a node for P corresponds to the case in which P is true, and the dotted arrow to the case in which P is false. Note that BDDs are graphs, not trees like formulae, and this provides a further reduction in the BDD size as a subformula never occurs more than once.

There is an ordering condition on BDDs: the occurrences of propositions on any path from the root to a leaf node must obey a fixed ordering of the propositions. This ordering condition together with the graph representation is required for the good computational properties of BDDs, like the polynomial time equivalence test.

A BDD corresponding to a propositional formula can be obtained by repeated application of an equivalence called the Shannon expansion.

$$\phi \equiv (p \wedge \phi[\top/p]) \vee (\neg p \wedge \phi[\bot/p]) \equiv ite(p, \phi[\top/p], \phi[\bot/p])$$

Example 2.6 We show how the BDD for $(A \vee B) \wedge (B \vee C)$ is produced by repeated application of the Shannon expansion. We use the variable ordering A, B, C and use the Shannon expansion to eliminate the variables in this order.

$$\begin{aligned}
& (A \vee B) \wedge (B \vee C) \\
\equiv\ & ite(A, (\top \vee B) \wedge (B \vee C), (\bot \vee B) \wedge (B \vee C)) \\
\equiv\ & ite(A, B \vee C, B) \\
\equiv\ & ite(A, ite(B, \top \vee C, \bot \vee C), ite(B, \top, \bot)) \\
\equiv\ & ite(A, ite(B, \top, C), ite(B, \top, \bot)) \\
\equiv\ & ite(A, ite(B, \top, ite(C, \top, \bot)), ite(B, \top, \bot))
\end{aligned}$$

The simplifications in the intermediate steps are by the equivalences $\top \vee \phi \equiv \top$ and $\bot \vee \phi \equiv \phi$ and $\top \wedge \phi \equiv \phi$ and $\bot \wedge \phi \equiv \bot$. When

$$ite(A, ite(B, \top, ite(C, \top, \bot)), ite(B, \top, \bot))$$

is first turned into a tree and then equivalent subtrees are identified, we get the BDD in Figure 2.6. The terminal node 1 corresponds to $\top$ and the terminal node 0 to $\bot$. ■

There are many operations on BDDs that are computable in polynomial time. These include forming the conjunction $\wedge$ and the disjunction $\vee$ of two BDDs, and forming the negation $\neg$ of a

BDD. However, conjunction and disjunction of n BDDs may have a size that is exponential in n, as adding a new disjunct or conjunct may double the size of the BDD.

An important operation in many applications of BDDs is the existential abstraction operation $\exists p.\phi$, which is defined by

$$\exists p.\phi = \phi[\top/p] \vee \phi[\bot/p]$$

where $\phi[\psi/p]$ means replacing all occurrences of p in ϕ by ψ. Also this is computable in polynomial time, but existentially abstracting n variables may result in a BDD that has size exponential in n, and hence may take exponential time. Existential abstraction can of course be used for any propositional formulae, not only for BDDs.

The formula ϕ' obtained from ϕ by existentially abstracting p is in general not equivalent to ϕ, but has many properties that make the abstraction operation useful.

Lemma 2.7 *Let ϕ be a formula and p a proposition. Let $\phi' = \exists p.\phi = \phi[\top/p] \vee \phi[\bot/p]$. Now the following hold.*

1. *ϕ is satisfiable if and only if ϕ' is.*
2. *ϕ is valid if and only if ϕ' is.*
3. *If χ is a formula without occurrences of p, then $\phi \models \chi$ if and only if $\phi' \models \chi$.*

Example 2.8

$$\begin{aligned}
&\exists B.((A \rightarrow B) \wedge (B \rightarrow C)) \\
&= ((A \rightarrow \top) \wedge (\top \rightarrow C)) \vee ((A \rightarrow \bot) \wedge (\bot \rightarrow C)) \\
&\equiv C \vee \neg A \equiv A \rightarrow C
\end{aligned}$$

$$\exists AB.(A \vee B) = \exists B.(\top \vee B) \vee (\bot \vee B) = ((\top \vee \top) \vee (\bot \vee \top)) \vee ((\top \vee \bot) \vee (\bot \vee \bot))$$

■

2.2.3 Algebraic decision diagrams

Algebraic decision diagrams (ADDs) [Fujita *et al.*, 1997; Bahar *et al.*, 1997] are a generalization of binary decision diagrams that has been applied to many kinds of probabilistic extensions of problems solved by BDDs. BDDs have only two terminal nodes, 1 and 0, and ADDs generalize this to a finite number of real numbers.

While BDDs represent Boolean functions, ADDs represent mapping from valuations to real numbers. The Boolean operations on BDDs, like taking the disjunction or conjunction of two BDDs, generalize to the arithmetic operations to take the arithmetic sum or the arithmetic product of two functions. There are further operations on ADDs that have no counterpart for BDDs, like constructing a function that on any valuation equals the maximum of two functions.

Figure 2.7 depicts three ADDs, the first of which is also a BDD. The product of ADDs is a generalization of conjunction of BDDs: if for some valuation/state ADD A assigns the value r_1 and ADD B assigns the value r_2, then the product ADD $A \cdot B$ assigns the value $r_1 \cdot r_2$ to the valuation.

The following are some of the operations typically available in implementations of ADDs. Here we denote ADDs by f and g and view them as functions from valuations x to real numbers.

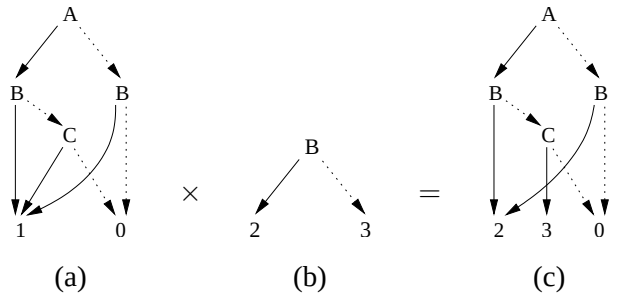


Figure 2.7: Three ADDs, the first of which is also a BDD.

operation	notation	meaning
sum	$f + g$	$(f + g)(x) = f(x) + g(x)$
product	$f \cdot g$	$(f \cdot g)(x) = f(x) \cdot g(x)$
maximization	$\max(f, g)$	$(\max(f, g))(x) = max(f(x), g(x))$

There is an operation for ADDs that corresponds to the existential abstraction operation on BDDs, and that is used in multiplication of matrices represented as ADDs, just like existential abstraction is used in multiplication of Boolean matrices represented as BDDs.

Let f be an ADD and p a proposition. Then *arithmetic existential abstraction* of f, written $\exists p.f$, is an ADD that satisfies the following.

$$(\exists p.f)(x) = (f[\top/p])(x) + (f[\bot/p])(x)$$

2.3 Succinct transition systems

It is often more natural to represent the states of a transition system as valuations of state variables instead of enumeratively as in Section 2.1. The binary relations that correspond to actions can often be represented compactly in terms of the changes the actions cause to the values of state variables.

We represent states in terms of a set A of Boolean state variables which take the values *true* or *false*. Each *state* is a valuation of A, that is, a function $s : A \to \{0, 1\}$.

Since we identify states with valuations of state variables, we can now identify sets of states with propositional formulae over the state variables. This allows us to perform set-theoretic operations on sets as logical operations and test relations between sets by inference in the propositional logic as summarized in Table 2.1

The actions of a succinct transition system are described by operators. An operator has two components. The precondition describes the set of states in which the action can be taken. The effect describes the successor states of each state in terms of the changes made to the values of the state variables.

Definition 2.9 *Let A be a set of state variables. An* operator *is a pair $\langle c, e\rangle$ where c is a propositional formula over A (the* precondition*), and e is an* effect *over A. Effects over A are recursively defined as follows.*

set	formula
$T \cup U$	$T \vee U$
$T \cap U$	$T \wedge U$
$\overline{T}$	$\neg T$
$T \backslash U$	$T \wedge \neg U$
$\emptyset$	$\bot$
the universal set	$\top$

question about sets	question about formulae
$T \subseteq U$?	$\models T \rightarrow U$?
$T \subset U$?	$\models T \rightarrow U$ and $\not\models U \rightarrow T$?
$T = U$?	$\models T \leftrightarrow U$?

Table 2.1: Correspondence between set-theoretical and logical operations

1. *a and $\neg a$ for state variables $a \in A$ are effects over A.*
2. *$e_1 \wedge \cdots \wedge e_n$ is an effect over A if $e_1, \ldots, e_n$ are effects over A (the special case with $n = 0$ is the empty effect $\top$.)*
3. *$c \rhd e$ is an effect over A if c is a formula over A and e is an effect over A.*
4. *$e_1 | \cdots | e_n$ is an effect over A if $e_1, \ldots, e_n$ for $n \geq 2$ are effects over A.*

The compound effects $e_1 \wedge \cdots \wedge e_n$ denote executing all the effects $e_1, \ldots, e_n$ simultaneously. In conditional effects $c \rhd e$ the effect e is executed if c is true in the current state. The effects $e_1 | \cdots | e_n$ denote nondeterministic choice between the effects $e_1, \ldots, e_n$. Exactly one of these effects is chosen randomly.

Operators describe a binary relation on the set of states as follows.

Definition 2.10 (Operator application) *Let $\langle c, e \rangle$ be an operator over A. Let s be a state (a valuation of A). The operator is* applicable in *s if $s \models c$ and every set $E \in [e]_s$ is consistent. The set $[e]_s$ is recursively defined as follows.*

1. *$[a]_s = \{\{a\}\}$ and $[\neg a]_s = \{\{\neg a\}\}$ for $a \in A$.*
2. *$[e_1 \wedge \cdots \wedge e_n]_s = \{\bigcup_{i=1}^{n} E_i | E_1 \in [e_1]_s, \ldots, E_n \in [e_n]_s\}$.*
3. *$[c' \rhd e]_s = [e]_s$ if $s \models c'$ and $[c' \rhd e]_s = \{\emptyset\}$ otherwise.*
4. *$[e_1 | \cdots | e_n]_s = [e_1]_s \cup \cdots \cup [e_n]_s$.*

An operator $\langle c, e \rangle$ induces a binary relation $R\langle c, e \rangle$ on states as follows: states s and s' are related by $R\langle c, e \rangle$ if $s \models c$ and s' is obtained from s by making the literals in some $E \in [e]_s$ true and retaining the values of state variables not occurring in E.

We define images and preimages for operators o in terms of $R(o)$, for instance by *preimg*$_o(s) =$ *preimg*$_{R(o)}(s)$.

Definition 2.11 *A* succinct transition system *is a 5-tuple $\Pi = \langle A, I, O, G, V \rangle$ where*

1. *A is a finite set of state variables,*
2. *I is a formula over A describing the initial states,*
3. *O is a finite set of operators over A,*
4. *G is a formula over A describing the goal states, and*
5. *$V \subseteq A$ is the set of observable state variables.*

Succinct transition systems with $V = A$ are *fully observable*, and succinct transition systems with $V = \emptyset$ are *unobservable*. Without restrictions on V the succinct transition systems are *partially observable*.

We can associate a transition system with every succinct transition system.

Definition 2.12 *Given a succinct transition system $\Pi = \langle A, I, O, G, V\rangle$, construct the transition system $F(\Pi) = \langle S, I', O', G', P\rangle$ where*

1. *S is the set of all Boolean valuations of A,*
2. *$I' = \{s \in S | s \models I\}$,*
3. *$O' = \{R(o) | o \in O\}$,*
4. *$G' = \{s \in S | s \models G\}$, and*
5. *$P = (C_1, \ldots, C_n)$ where $v_1, \ldots, v_n$ for $n = 2^{|V|}$ are all the Boolean valuations of V and $C_i = \{s \in S | s(a) = v_i(a)$ for all $a \in V\}$ for all $i \in \{1, \ldots, n\}$.*

The transition system may have a size that is exponential in the size of the succinct transition system. However, the construction takes only polynomial time in the size of the transition system.

2.3.1 Deterministic succinct transition systems

A deterministic operator has no occurrences of | in the effect. Further, in this special case the definition of operator application is slightly simpler.

Definition 2.13 (Operator application) *Let $\langle c, e\rangle$ be a deterministic operator over A. Let s be a state (a valuation of A). The operator is* applicable in *s if $s \models c$ and the set $[e]_s^{det}$ is consistent. The set $[e]_s^{det}$ is recursively defined as follows.*

1. *$[a]_s^{det} = \{a\}$ and $[\neg a]_s^{det} = \{\neg a\}$ for $a \in A$.*
2. *$[e_1 \wedge \cdots \wedge e_n]_s^{det} = \bigcup_{i=1}^{n} [e_i]_s^{det}$.*
3. *$[c' \rhd e]_s^{det} = [e]_s^{det}$ if $s \models c'$ and $[c' \rhd e]_s^{det} = \emptyset$ otherwise.*

A deterministic operator $\langle c, e\rangle$ induces a partial function $R\langle c, e\rangle$ on states as follows: two states s and s' are related by $R\langle c, e\rangle$ if $s \models c$ and s' is obtained from s by making the literals in $[e]_s^{det}$ true and retaining the truth-values of state variables not occurring in $[e]_s^{det}$.

We define $app_o(s) = s'$ by $sR(o)s'$ and $app_{o_1;\ldots;o_n}(s) = s'$ by $app_{o_n}(\ldots app_{o_1}(s)\ldots)$, just like for non-succinct transition systems.

We formally define deterministic succinct transition systems.

Definition 2.14 *A* deterministic succinct transition system *is a 4-tuple* $\Pi = \langle A, I, O, G\rangle$ *where*

1. A *is a finite set of state variables,*
2. I *is an initial state,*
3. O *is a finite set of operators over* A*, and*
4. G *is a formula over* A *describing the goal states.*

We can associate a deterministic transition system with every deterministic succinct transition system.

Definition 2.15 *Given a deterministic succinct transition system* $\Pi = \langle A, I, O, G\rangle$*, define the deterministic transition system* $F(\Pi) = \langle S, I, O', G'\rangle$ *where*

1. S *is the set of all Boolean valuations of* A*,*
2. $O' = \{R(o)|o \in O\}$*, and*
3. $G' = \{s \in S|s \models G\}$.

A subclass of operators considered in many early and recent works restrict to *STRIPS* operators. An operator $\langle c, e\rangle$ is a STRIPS operator if c is a conjunction of state variables and e is a conjunction of literals. STRIPS operators do not allow disjunctivity in formulae nor conditional effects. This class of operators is sufficient in the sense that any transition system can be expressed in terms of STRIPS operators only if the identities of operators are not important: when expressing a transition system in terms of STRIPS operators only some operators correspond to an exponential number of STRIPS operators.

Example 2.16 Let $A = \{a_1, \ldots, a_n\}$ be the set of state variables. Let $o = \langle \top, e\rangle$ where

$$e = (a_1 \rhd \neg a_1) \wedge (\neg a_1 \rhd a_1) \wedge \cdots \wedge (a_n \rhd \neg a_n) \wedge (\neg a_n \rhd a_n)\rangle.$$

This operator reverses the values of all state variables. As its set of active effects $[e]_s^{det}$ is different in every one of 2^n states, this operator corresponds to 2^n STRIPS operators.

$$\begin{aligned}
o_0 &= \langle \neg a_1 \wedge \neg a_2 \wedge \cdots \wedge \neg a_n, a_1 \wedge a_2 \wedge \cdots \wedge a_n\rangle \\
o_1 &= \langle a_1 \wedge \neg a_2 \wedge \cdots \wedge \neg a_n, \neg a_1 \wedge a_2 \wedge \cdots \wedge a_n\rangle \\
o_2 &= \langle \neg a_1 \wedge a_2 \wedge \cdots \wedge \neg a_n, a_1 \wedge \neg a_2 \wedge \cdots \wedge a_n\rangle \\
o_3 &= \langle a_1 \wedge a_2 \wedge \cdots \wedge \neg a_n, \neg a_1 \wedge \neg a_2 \wedge \cdots \wedge a_n\rangle \\
&\vdots \\
o_{2^n-1} &= \langle a_1 \wedge a_2 \wedge \cdots \wedge a_n, \neg a_1 \wedge \neg a_2 \cdots \wedge \neg a_n\rangle
\end{aligned}$$

■

$$c \rhd (e_1 \wedge \cdots \wedge e_n) \equiv (c \rhd e_1) \wedge \cdots \wedge (c \rhd e_n) \quad (2.1)$$
$$c_1 \rhd (c_2 \rhd e) \equiv (c_1 \wedge c_2) \rhd e \quad (2.2)$$
$$(c_1 \rhd e) \wedge (c_2 \rhd e) \equiv (c_1 \vee c_2) \rhd e \quad (2.3)$$
$$e \wedge (c \rhd e) \equiv e \quad (2.4)$$
$$e \equiv \top \rhd e \quad (2.5)$$
$$e_1 \wedge (e_2 \wedge e_3) \equiv (e_1 \wedge e_2) \wedge e_3 \quad (2.6)$$
$$e_1 \wedge e_2 \equiv e_2 \wedge e_1 \quad (2.7)$$
$$c \rhd \top \equiv \top \quad (2.8)$$
$$e \wedge \top \equiv e \quad (2.9)$$

Table 2.2: Equivalences on effects

2.3.2 Extensions

The basic language for effects could be extended with further constructs. A natural construct is *sequential composition* of effects. If e and e' are effects, then also $e; e'$ is an effect that corresponds to first executing e and then e'. Definition 3.11 and Theorem 3.12 show how effects with sequential composition can be reduced to effects without sequential composition.

2.3.3 Normal form for deterministic operators

Deterministic operators can be transformed to a particularly simple form without nesting of conditionality $\rhd$ and with only atomic effects e as antecedents of conditionals $\phi \rhd e$. Normal forms are useful as they allow concentrating on a particularly simple form of effects.

Table 2.2 lists a number of equivalences on effects. Their proofs of correctness with Definition 2.13 are straightforward. An effect e is equivalent to $\top \wedge e$, and conjunctions of effects can be arbitrarily reordered without affecting the meaning of the operator. These trivial equivalences will later be used without explicitly mentioning them, for example in the definitions of the normal forms and when applying equivalences.

The normal form corresponds to moving all occurrences of $\rhd$ inside $\wedge$ so that the consequents of $\rhd$ are atomic effects.

Definition 2.17 *A deterministic effect e is in* normal form *if it is $\top$ or a conjunction of one or more effects $c \rhd a$ and $c \rhd \neg a$ with at most one occurrence of atomic effect a and $\neg a$ for any $a \in A$. An operator $\langle c, e\rangle$ is in normal form if e is in normal form.*

Theorem 2.18 *For every deterministic operator there is an equivalent one in normal form. There is one that has a size that is polynomial in the size of the operator.*

Proof: We can transform any deterministic operator into normal form by using the equivalences in Table 2.2. The proof is by structural induction on the effect e of the operator $\langle c, e\rangle$.

Induction hypothesis: the effect e can be transformed to normal form.

Base case 1, $e = \top$: This is already in normal form.

Base case 2, $e = a$ or $e = \neg a$: An equivalent effect in normal form is $\top \rhd e$ by Equivalence 2.5.

Inductive case 1, $e = e_1 \wedge e_2$: By the induction hypothesis e_1 and e_2 can be transformed into normal form, so assume that they already are. If one of e_1 and e_2 is $\top$, by Equivalence 2.9 we can eliminate it.

Assume e_1 contains $c_1 \rhd l$ for some literal l and e_2 contains $c_2 \rhd l$. We can reorder $e_1 \wedge e_2$ with Equivalences 2.6 and 2.7 so that one of the conjuncts is $(c_1 \rhd l) \wedge (c_2 \rhd l)$. Then by Equivalence 2.3 it can be replaced by $(c_1 \vee c_2) \rhd l$. Since this can be done repeatedly for every literal l, we can transform $e_1 \wedge e_2$ into normal form.

Inductive case 2, $e = z \rhd e_1$: By the induction hypothesis e_1 can be transformed to normal form, so assume that it already is.

If e_1 is $\top$, e can be replaced by $\top$ which is in normal form.

If $e_1 = z' \rhd e_2$ for some z' and e_2, then e can be replaced by the equivalent (by Equivalence 2.2) effect $(z \wedge z') \rhd e_2$ in normal form.

Otherwise, e_1 is a conjunction of effects $z \rhd l$. By Equivalence 2.1 we can move z inside the conjunction. Applications of Equivalences 2.2 transform the effect into normal form.

In this transformation the conditions c in $c \rhd e$ are copied into front of the atomic effects. Let m be the sum of the sizes of all the conditions c, and let n be the number of occurrences of atomic effects a and $\neg a$ in the effect. An upper bound on size of the new effect is $\mathcal{O}(nm)$ which is polynomial in the size of the original effect. □

2.3.4 Normal forms for nondeterministic operators

We can generalize the normal form defined in Section 2.3.3 to nondeterministic effects and operators. In the normal form nondeterministic choices and conjunctions are the outermost constructs, and consequents e of conditional effects $c \rhd e$ are atomic effects.

Definition 2.19 (Normal form for nondeterministic operators) *A deterministic effect is in normal form if it is $\top$ or a conjunction of one or more effects $c \rhd a$ and $c \rhd \neg a$ with at most one occurrence of a and $\neg a$ for any $a \in A$.*

A nondeterministic effect is in normal form if it is $e_1 | \cdots | e_n$ or $e_1 \wedge \cdots \wedge e_n$ for effects e_i that are in normal form.

A nondeterministic operator $\langle c, e \rangle$ is in normal form if e is in normal form.

For showing that every nondeterministic effect can be transformed into normal form we use further equivalences that are given in Table 2.3.

Theorem 2.20 *For every operator there is an equivalent one in normal form. There is one that has a size that is polynomial in the size of the former.*

Proof: Transformation to normal form is like in the proof of Theorem 2.18. Additional equivalences needed for nondeterministic choices are 2.10 and 2.11. □

Example 2.21 The effect

$$a \rhd (b | (c \wedge f)) \wedge ((d \wedge e) | (b \rhd e))$$

$$c \rhd (e_1|\cdots|e_n) \equiv (c \rhd e_1)|\cdots|(c \rhd e_n) \quad (2.10)$$
$$e \wedge (e_1|\cdots|e_n) \equiv (e \wedge e_1)|\cdots|(e \wedge e_n) \quad (2.11)$$
$$(e'_1|\cdots|e'_{n'})|e_2|\cdots|e_n \equiv e'_1|\cdots|e'_{n'}|e_2|\cdots|e_n \quad (2.12)$$
$$(e' \wedge (c \rhd e_1))|e_2|\cdots|e_n \equiv (c \rhd ((e' \wedge e_1)|e_2|\cdots|e_n)) \wedge (\neg c \rhd (e'|e_2|\cdots|e_n)) \quad (2.13)$$

Table 2.3: Equivalences on nondeterministic effects

in normal form is

$$((a \rhd b)|((a \rhd c) \wedge (a \rhd f))) \wedge (((\top \rhd d) \wedge (\top \rhd e))|(b \rhd e)).$$

■

For some applications a still simpler form of operators is useful. In the second normal form for nondeterministic operators nondeterminism may appear only at the outermost structure in the effect.

Definition 2.22 (Normal form II for nondeterministic operators) *A deterministic effect is in normal formal II if it is $\top$ or a conjunction of one or more effects $c \rhd a$ and $c \rhd \neg a$ with at most one occurrence of a and $\neg a$ for any $a \in A$.*

A nondeterministic effect is in normal form II if it is of form $e_1|\cdots|e_n$ where e_i are deterministic effects in normal form II.

A nondeterministic operator $\langle c, e\rangle$ is in normal form II if e is in normal form II.

Theorem 2.23 *For every operator there is an equivalent one in normal form II.*

Proof: By Theorem 2.20 there is an equivalent operator in normal form. The transformation further into normal form II requires equivalences 2.11 and 2.12. □

2.4 Computational complexity

In this section we discuss deterministic, nondeterministic and alternating Turing machines (DTMs, NDTMs and ATMs) and define several complexity classes in terms of them. For a detailed introduction to computational complexity see any of the standard textbooks [Balcázar *et al.*, 1988; 1990; Papadimitriou, 1994].

The definition of ATMs we use is like that of Balcázar et al. [1990] but without a separate input tape. Deterministic and nondeterministic Turing machines (DTMs, NDTMs) are a special case of alternating Turing machines.

Definition 2.24 *An* alternating Turing machine *is a tuple $\langle \Sigma, Q, \delta, q_0, g\rangle$ where*

- *Σ is a finite alphabet (the contents of tape cells),*
- *Q is a finite set of states (the internal states of the ATM),*

- δ *is a transition function* $\delta : Q \times (\Sigma \cup \{|, \Box\}) \to 2^{(\Sigma \cup \{|\}) \times Q \times \{L,N,R\}}$,
- q_0 *is the initial state, and*
- $g : Q \to \{\forall, \exists, \textit{accept}, \textit{reject}\}$ *is a labeling of the states.*

The symbols | and $\Box$, the end-of-tape symbol and the blank symbol, in the definition of δ respectively refer to the beginning of the tape and to the end of the tape. It is required that $s = |$ and $m = R$ for all $\langle s, q', m\rangle \in \delta(q, |)$ for any $q \in Q$, that is, at the left end of the tape the movement is always to the right and the end-of-tape symbol | may not be changed. For $s \in \Sigma$ we restrict s' in $\langle s', q', m\rangle \in \delta(q, s)$ to $s' \in \Sigma$, that is, | gets written onto the tape only in the special case when the R/W head is on the end-of-tape symbol. Notice that the transition function is a total function, and the ATM computation terminated upon reaching an accepting or a rejecting state.

A configuration of an ATM is $\langle q, \sigma, \sigma'\rangle$ where q is the current state, σ is the tape contents left of the R/W head with the rightmost symbol under the R/W head, and σ' is the tape contents strictly right of the R/W head. This is a finite representation of the finite non-blank segment of the tape of the ATM. The configuration is universal ($\forall$) if $g(q) = \forall$, and existential ($\exists$) if $g(q) = \exists$.

The computation of an ATM starts from the initial configuration $\langle q_0, |a, \sigma\rangle$ where $a\sigma$ is the input string of the Turing machine. Below ϵ denotes the empty string.

Successor configurations are defined as follows.

1. A successor of $\langle q, \sigma a, \sigma'\rangle$ is $\langle q', \sigma, a'\sigma'\rangle$ if $\langle a', q', L\rangle \in \delta(q, a)$.
2. A successor of $\langle q, \sigma a, \sigma'\rangle$ is $\langle q', \sigma a', \sigma'\rangle$ if $\langle a', q', N\rangle \in \delta(q, a)$.
3. A successor of $\langle q, \sigma a, b\sigma'\rangle$ is $\langle q', \sigma a' b, \sigma'\rangle$ if $\langle a', q', R\rangle \in \delta(q, a)$.
4. A successor of $\langle q, \sigma a, \epsilon\rangle$ is $\langle q', \sigma a'\Box, \epsilon\rangle$ if $\langle a', q', R\rangle \in \delta(q, a)$.

We write $\langle q, \sigma\rangle \vdash \langle q', \sigma'\rangle$ if the latter is a successor configuration of the former. A configuration $\langle q, \sigma, \sigma'\rangle$ of an ATM is *final* if $g(q) =$ accept or $g(q) =$ reject.

The acceptance of an input string by an ATM is defined recursively starting from final configurations. A final configuration is 0-accepting if $g(q) =$ accept. A non-final universal configuration is n-accepting for $n \geq 1$ if its every successor configuration is m-accepting for some $m < n$ and one of its successor configurations is $n-1$-accepting. A non-final existential configuration is n-accepting for $n \geq 1$ if at least one of its successor configurations is $n-1$-accepting and it has no m-accepting successor configurations for any $m < n-1$. Finally, an ATM accepts a given input string if its initial configuration is n-accepting for some $n \geq 0$. A configuration is *accepting* if it is n-accepting for some $n \geq 0$.

If an ATM accepts a given input string, then we can define *an accepting computation subtree* of the ATM and the input string as a set T of accepting configurations such that

1. the initial configuration is in T,
2. if $c \in T$ is a $\forall$-configuration then $c' \in T$ for all configurations c' such that $c \vdash c'$,
3. if $c \in T$ is an n-accepting $\exists$-configuration then $c' \in T$ for at least one c' such that $c \vdash c'$ and c' is m-accepting for some $m < n$.

A nondeterministic Turing machine is an ATM without universal states. A deterministic Turing machine is an ATM with $|\delta(q, s)| = 1$ for all $q \in Q$ and $s \in \Sigma$.

The complexity classes used in this lecture are the following. PSPACE is the class of decision problems solvable by deterministic Turing machines that use a number of tape cells bounded by a polynomial on the input length n. Formally,

$$\text{PSPACE} = \bigcup_{k \geq 0} \text{DSPACE}(n^k).$$

Other complexity classes are similarly defined in terms of the time consumption on a deterministic Turing machine (DTIME($f(n)$), time consumption on a nondeterministic Turing machine (NTIME($f(n)$), or time or space consumption on alternating Turing machines (ATIME($f(n)$) or ASPACE($f(n)$)) [Balcázar *et al.*, 1988; 1990].

$$\begin{aligned}
\text{P} &= \textstyle\bigcup_{k \geq 0} \text{DTIME}(n^k) \\
\text{NP} &= \textstyle\bigcup_{k \geq 0} \text{NTIME}(n^k) \\
\text{EXP} &= \textstyle\bigcup_{k \geq 0} \text{DTIME}(2^{n^k}) \\
\text{NEXP} &= \textstyle\bigcup_{k \geq 0} \text{NTIME}(2^{n^k}) \\
\text{EXPSPACE} &= \textstyle\bigcup_{k \geq 0} \text{DSPACE}(2^{n^k}) \\
\text{2-EXP} &= \textstyle\bigcup_{k \geq 0} \text{DTIME}(2^{2^{n^k}}) \\
\text{2-NEXP} &= \textstyle\bigcup_{k \geq 0} \text{NTIME}(2^{2^{n^k}}) \\
\\
\text{APSPACE} &= \textstyle\bigcup_{k \geq 0} \text{ASPACE}(n^k) \\
\text{AEXPSPACE} &= \textstyle\bigcup_{k \geq 0} \text{ASPACE}(2^{n^k})
\end{aligned}$$

There are many useful connections between complexity classes defined in terms of deterministic and alternating Turing machines [Chandra *et al.*, 1981], for example

$$\begin{aligned}
\text{EXP} &= \text{APSPACE} \\
\text{2-EXP} &= \text{AEXPSPACE}.
\end{aligned}$$

Roughly, an exponential deterministic time bound corresponds to a polynomial alternating space bound.

We have defined all the complexity classes in terms of Turing machines. However, for all purposes of this lecture, we can equivalently use conventional programming languages (like C or Java) or simplified variants of them for describing computation. The main difference between conventional programming languages and Turing machines is that the former use random-access memory whereas memory access in Turing machines is local and only the current tape cell can be directly accessed. However, these two computational models can be simulated with each other with a polynomial overhead and are therefore for our purposes equivalent. The differences show up in complexity classes with very strict (subpolynomial) restrictions on time and space consumption.

Later in this lecture, the proofs of membership of a given computational problem in a certain complexity class are usually given in terms of a program in a simple programming language comparable to a small subset of C or Java, instead of giving a formal description of a Turing machine because the latter would usually be very complicated and difficult to understand.

A problem L is *C-hard* (where C is any of the complexity classes) if all problems in the class C are polynomial time *many-one reducible* to it; that is, for all problems $L' \in C$ there is a function

$f_{L'}$ that can be computed in polynomial time on the size of its input and $f_{L'}(x) \in L$ if and only if $x \in L'$ for all inputs x. We say that the function $f_{L'}$ is a translation from L' to L. A problem is *C-complete* if it belongs to the class C and is C-hard.

In complexity theory the most important distinction between computational problems is that between *tractable* and *intractable* problems. A problem is considered to be tractable, efficiently solvable, if it can be solved in polynomial time. Otherwise it is intractable. Most planning problems are highly intractable, but for many algorithmic approaches to planning it is important that certain basic steps in these algorithms can be guaranteed to be tractable.

In this lecture we analyze the complexity of many computational problems, showing them to be complete problems for some of the classes mentioned above. The proofs consist of two parts. We show that the problem belongs to the class. This is typically by giving an algorithm for the problem, possibly a nondeterministic one, and then showing that the algorithm obeys the resource bounds on time or memory consumption as required by the complexity class. Then we show the hardness of the problem for the class, that is, we can reduce any problem in the class to the problem in polynomial time. This can be either by simulating all Turing machines that represent computation in the class, or by reducing a complete problem in the class to the problem in question in polynomial time (a many-one reduction).

For almost all commonly used complexity classes there are more or less natural complete problems that often have a central role in proving the completeness of other problems for the class in question. Some complete problems for the complexity classes mentioned above are the following.[1]

class	complete problem
P	truth-value of formulae in the propositional logic in a given valuation
NP	satisfiability of formulae in the propositional logic (SAT)
PSPACE	truth-value of quantified Boolean formulae

Complete problems for classes like EXP and NEXP can be obtained from the P-complete and NP-problems by representing propositional formulae succinctly in terms of other propositional formulae [Papadimitriou and Yannakakis, 1986].

2.5 Exercises

2.1 Show that any transition system in which the states are valuations of a set A of propositional variables can be translated into an equivalent succinct transition system.

2.2 Show that conditional effects with $\rhd$ are necessary, that is, find a transition system where states are valuations of a set of state variables and the actions cannot be represented as operators without conditional effects with $\rhd$. *Hint*: There is an example with two states and one state variable.

[1] For definition of P-hard problems we have to use more restricted many-one reductions that use only logarithmic space instead of polynomial time. Otherwise all non-trivial problems in P would be P-hard and P-complete.

Chapter 3

Deterministic planning

The simplest planning problems involves finding a sequence of actions that lead from a given initial state to a goal state. Only deterministic actions are considered. Determinism and the uniqueness of the initial state mean that the state of the transition system after any sequence of actions is exactly predictable. The problem instances in this chapter are deterministic succinct transition systems as defined in Section 2.3.1.

3.1 State-space search

The simplest possible planning algorithm generates all states (valuations of the state variables), constructs the transition graph, and then finds a path from the initial state I to a goal state $g \in G$ for example by a shortest-path algorithm. The plan is then simply the sequence of operators corresponding to the edges on the shortest path from the initial state to a goal state. However, this algorithm is not feasible when the number of state variables is higher than 20 or 30 because the number of valuations is very high: $2^{20} = 1048576 \sim 10^6$ for 20 Boolean state variables and $2^{30} = 1073741824 \sim 10^9$ for 30.

Instead, it will often be much more efficient to avoid generating most of the state space explicitly and to produce only the successor or predecessor states of the states currently under consideration. This form of plan search can be easiest viewed as the application of general-purpose search algorithms that can be employed in solving a wide range of search problems. The best known *heuristic search algorithms* are A$*$, IDA$*$ and their variants [Hart *et al.*, 1968; Pearl, 1984; Korf, 1985] which can be used in finding shortest plans or plans that are guaranteed to be close to the shortest ones.

There are two main possibilities to find a path from the initial state to a goal state: traverse the transition graph forwards starting from the initial state, or traverse it backwards starting from the goal states. The main difference between these possibilities is that there may be several goal states (and one state may have several predecessor states with respect to one operator) but only one initial state: in forward traversal we repeatedly compute the unique successor state of the current state, whereas with backward traversal we are forced to keep track of a possibly very high number of possible predecessor states of the goal states. Backward search is slightly more complicated to implement but it allows to simultaneously consider several paths leading to a goal state.

3.1.1 Progression and forward search

We have already defined *progression* for single states s as $app_o(s)$. The simplest algorithm for the deterministic planning problem does not require the explicit representation of the whole transition graph. The search starts in the initial state. New states are generated by progression. As soon as a state s such that $s \models G$ is found a plan is guaranteed to exist: it is the sequence of operators with which the state s is reached from the initial state.

A planner can use progression in connection with any of the standard search algorithms. Later in this chapter we will discuss how heuristic search algorithms together with heuristics yield an efficient planning method.

3.1.2 Regression and backward search

With backward search the starting point is a propositional formula G that describes the set of goal states. An operator is selected, the set of possible predecessor states is computed, and this set is again described by a propositional formula. A plan has been found when a formula that is true in the initial state is reached. The computation of a formula representing the predecessor states of the states represented by another formula is called *regression*. Regression is more powerful than progression because it allows handling potentially very big sets of states, but it is also more expensive.

Definition 3.1 *We define the condition $EPC_l(e)$ of literal l made true when an operator with the effect e is applied recursively as follows.*

$$\begin{aligned} EPC_l(\top) &= \bot \\ EPC_l(l) &= \top \\ EPC_l(l') &= \bot \text{ when } l \neq l' \text{ (for literals } l') \\ EPC_l(e_1 \wedge \cdots \wedge e_n) &= EPC_l(e_1) \vee \cdots \vee EPC_l(e_n) \\ EPC_l(c \rhd e) &= c \wedge EPC_l(e) \end{aligned}$$

The case $EPC_l(e_1 \wedge \cdots \wedge e_n) = EPC_l(e_1) \vee \cdots \vee EPC_l(e_n)$ is defined as a disjunction because it is sufficient that at least one of the effects makes l true.

Definition 3.2 *Let A be the set of state variables. We define the condition $EPC_l(o)$ of operator $o = \langle c, e \rangle$ being applicable so that literal l is made true as $c \wedge EPC_l(e) \wedge \bigwedge_{a \in A} \neg(EPC_a(e) \wedge EPC_{\neg a}(e))$.*

For effects e the truth-value of the formula $EPC_l(e)$ indicates in which states l is a literal to which the effect e assigns the value true. The connection to the earlier definition of $[e]_s^{det}$ is stated in the following lemma.

Lemma 3.3 *Let A be the set of state variables, s a state on A, l a literal on A, and o and operator with effect e. Then*

1. *$l \in [e]_s^{det}$ if and only if $s \models EPC_l(e)$, and*
2. *$app_o(s)$ is defined and $l \in [e]_s^{det}$ if and only if $s \models EPC_l(o)$.*

Proof: We first prove (1) by induction on the structure of the effect e.

Base case 1, $e = \top$: By definition of $[\top]_s^{det}$ we have $l \notin [\top]_s^{det} = \emptyset$, and by definition of $EPC_l(\top)$ we have $s \not\models EPC_l(\top) = \bot$, so the equivalence holds.

Base case 2, $e = l$: $l \in [l]_s^{det} = \{l\}$ by definition, and $s \models EPC_l(l) = \top$ by definition.

Base case 3, $e = l'$ for some literal $l' \neq l$: $l \notin [l']_s^{det} = \{l'\}$ by definition, and $s \not\models EPC_l(l') = \bot$ by definition.

Inductive case 1, $e = e_1 \wedge \cdots \wedge e_n$:

$l \in [e]_s^{det}$ if and only if $l \in [e']_s^{det}$ for some $e' \in \{e_1, \ldots, e_n\}$
if and only if $s \models EPC_l(e')$ for some $e' \in \{e_1, \ldots, e_n\}$
if and only if $s \models EPC_l(e_1) \vee \cdots \vee EPC_l(e_n)$
if and only if $s \models EPC_l(e_1 \wedge \cdots \wedge e_n)$.

The second equivalence is by the induction hypothesis, the other equivalences are by the definitions of $EPC_l(e)$ and $[e]_s^{det}$ as well as elementary facts about propositional formulae.

Inductive case 2, $e = c \rhd e'$:

$l \in [c \rhd e']_s^{det}$ if and only if $l \in [e']_s^{det}$ and $s \models c$
if and only if $s \models EPC_l(e')$ and $s \models c$
if and only if $s \models EPC_l(c \rhd e')$.

The second equivalence is by the induction hypothesis. This completes the proof of (1).

(2) follows from the fact that the conjuncts c and $\bigwedge_{a \in A} \neg(EPC_a(e) \wedge EPC_{\neg a}(e))$ in $EPC_l(o)$ exactly state the applicability conditions of o. □

Notice that any operator $\langle c, e \rangle$ can be expressed in normal form in terms of $EPC_a(e)$ as

$$\left\langle c, \bigwedge_{a \in A} (EPC_a(e) \rhd a) \wedge (EPC_{\neg a}(e) \rhd \neg a) \right\rangle.$$

The formula $EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$ expresses the condition for the truth $a \in A$ after the effect e is executed in terms of truth-values of state variables before: either a becomes true, or a is true before and does not become false.

Lemma 3.4 *Let $a \in A$ be a state variable, $o = \langle c, e \rangle \in O$ an operator, and s and $s' = app_o(s)$ states. Then $s \models EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$ if and only if $s' \models a$.*

Proof: Assume that $s \models EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$. We perform a case analysis and show that $s' \models a$ holds in both cases.

Case 1: Assume that $s \models EPC_a(e)$. By Lemma 3.3 $a \in [e]_s^{det}$, and hence $s' \models a$.

Case 2: Assume that $s \models a \wedge \neg EPC_{\neg a}(e)$. By Lemma 3.3 $\neg a \notin [e]_s^{det}$. Hence a is true in s'.

For the other half of the equivalence, assume that $s \not\models EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$. Hence $s \models \neg EPC_a(e) \wedge (\neg a \vee EPC_{\neg a}(e))$.

Case 1: Assume that $s \models a$. Now $s \models EPC_{\neg a}(e)$ because $s \models \neg a \vee EPC_{\neg a}(e)$, and hence by Lemma 3.3 $\neg a \in [e]_s^{det}$ and hence $s' \not\models a$.

Case 2: Assume that $s \not\models a$. Since $s \models \neg EPC_a(e)$, by Lemma 3.3 $a \notin [e]_s^{det}$ and hence $s' \not\models a$.

Therefore $s' \not\models a$ in all cases. □

The formulae $EPC_l(e)$ can be used in defining regression.

Definition 3.5 (Regression) *Let ϕ be a propositional formula and $o = \langle c, e\rangle$ an operator. The* regression *of ϕ with respect to o is $regr_o(\phi) = \phi_r \wedge c \wedge \chi$ where $\chi = \bigwedge_{a\in A} \neg(EPC_a(e) \wedge EPC_{\neg a}(e))$ and ϕ_r is obtained from ϕ by replacing every $a \in A$ by $EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$. Define $regr_e(\phi) = \phi_r \wedge \chi$ and use the notation $regr_{o_1;\ldots;o_n}(\phi) = regr_{o_1}(\cdots regr_{o_n}(\phi)\cdots)$.*

The conjuncts of χ say that none of the state variables may simultaneously become true and false. The operator is not applicable in states in which χ is false.

Remark 3.6 *Regression can be equivalently defined in terms of the conditions the state variables stay or become false, that is, we could use the formula $EPC_{\neg a}(e) \vee (\neg a \wedge \neg EPC_a(e))$ which tells when a is false. The negation of this formula, which can be written as $(EPC_a(e) \wedge \neg EPC_{\neg a}(e)) \vee (a \wedge \neg EPC_{\neg a}(e))$, is not equivalent to $EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$. However, if $EPC_a(e)$ and $EPC_{\neg a}(e)$ are not simultaneously true, we do get equivalence, that is,*

$$\begin{aligned}\neg(EPC_a(e) \wedge EPC_{\neg a}(e)) \models\ & ((EPC_a(e) \wedge \neg EPC_{\neg a}(e)) \vee (a \wedge \neg EPC_{\neg a}(e))) \\ & \leftrightarrow (EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e)))\end{aligned}$$

because $\neg(EPC_a(e) \wedge EPC_{\neg a}(e)) \models (EPC_a(e) \wedge \neg EPC_{\neg a}(e)) \leftrightarrow EPC_a(e)$.

An upper bound on the size of the formula obtained by regression with operators $o_1, \ldots, o_n$ starting from ϕ is the product of the sizes of $\phi, o_1, \ldots, o_n$, which is exponential in n. However, the formulae can often be simplified because there are many occurrences of $\top$ and $\bot$, for example by using the equivalences $\top \wedge \phi \equiv \phi$, $\bot \wedge \phi \equiv \bot$, $\top \vee \phi \equiv \top$, $\bot \vee \phi \equiv \phi$, $\neg\bot \equiv \top$, and $\neg\top \equiv \bot$. For unconditional operators $o_1, \ldots, o_n$ (with no occurrences of $\rhd$), an upper bound on the size of the formula (after eliminating $\top$ and $\bot$) is the sum of the sizes of $o_1, \ldots, o_n$ and ϕ.

The reason why regression is useful for planning is that it allows to compute the predecessor states by simple formula manipulation. The same does not seem to be possible for progression because there is no known simple definition of successor states of a *set* of states expressed in terms of a formula: simple syntactic progression is restricted to individual states only (see Section 4.2 for a general but expensive definition of progression for arbitrary formulae.)

The important property of regression is formalized in the following lemma.

Theorem 3.7 *Let ϕ be a formula over A, o an operator over A, and S the set of all states i.e. valuations of A. Then $\{s \in S | s \models regr_o(\phi)\} = \{s \in S | app_o(s) \models \phi\}$.*

Proof: We show that for any state s, $s \models regr_o(\phi)$ if and only if $app_o(s)$ is defined and $app_o(s) \models \phi$. By definition $regr_o(\phi) = \phi_r \wedge c \wedge \chi$ for $o = \langle c, e\rangle$ where ϕ_r is obtained from ϕ by replacing every state variable $a \in A$ by $EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$ and $\chi = \bigwedge_{a\in A} \neg(EPC_a(e) \wedge EPC_{\neg a}(e))$.

First we show that $s \models c \wedge \chi$ if and only if $app_o(s)$ is defined.

$s \models c \wedge \chi$	iff	$s \models c$ and $\{a, \neg a\} \not\subseteq [e]^{det}_s$ for all $a \in A$	by Lemma 3.3
	iff	$app_o(s)$ is defined	by Definition 2.13.

Then we show that $s \models \phi_r$ if and only if $app_o(s) \models \phi$. This is by structural induction over subformulae ϕ' of ϕ and formulae ϕ'_r obtained from ϕ' by replacing $a \in A$ by $EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$

Induction hypothesis: $s \models \phi'_r$ if and only if $app_o(s) \models \phi'$.

Base case 1, $\phi' = \top$: Now $\phi'_r = \top$ and both are true in the respective states.

Base case 2, $\phi' = \bot$: Now $\phi'_r = \bot$ and both are false in the respective states.

Base case 3, $\phi' = a$ for some $a \in A$: Now $\phi'_r = EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$. By Lemma 3.4 $s \models \phi'_r$ if and only if $app_o(s) \models \phi'$.

Inductive case 1, $\phi' = \neg\theta$: By the induction hypothesis $s \models \theta_r$ iff $app_o(s) \models \theta$. Hence $s \models \phi'_r$ iff $app_o(s) \models \phi'$ by the truth-definition of $\neg$.

Inductive case 2, $\phi' = \theta \vee \theta'$: By the induction hypothesis $s \models \theta_r$ iff $app_o(s) \models \theta$, and $s \models \theta'_r$ iff $app_o(s) \models \theta'$. Hence $s \models \phi'_r$ iff $app_o(s) \models \phi'$ by the truth-definition of $\vee$.

Inductive case 3, $\phi' = \theta \wedge \theta'$: By the induction hypothesis $s \models \theta_r$ iff $app_o(s) \models \theta$, and $s \models \theta'_r$ iff $app_o(s) \models \theta'$. Hence $s \models \phi'_r$ iff $app_o(s) \models \phi'$ by the truth-definition of $\wedge$. □

Regression can be performed with any operator but not all applications of regression are useful. First, regressing for example the formula a with the effect $\neg a$ is not useful because the new unsatisfiable formula describes the empty set of states. Hence the sequence of operators of the previous regressions steps do not lead to a goal from any state. Second, regressing a with the operator $\langle b, c\rangle$ yields $regr_{\langle b,c\rangle}(a) = a \wedge b$. Finding a plan for reaching a state satisfying a is easier than finding a plan for reaching a state satisfying $a \wedge b$. Hence the regression step produced a subproblem that is more difficult than the original problem, and it would therefore be better not to take this regression step.

Lemma 3.8 *Let there be a plan $o_1, \ldots, o_n$ for $\langle A, I, O, G\rangle$. If $regr_{o_k;\ldots;o_n}(G) \models regr_{o_{k+1};\ldots;o_n}(G)$ for some $k \in \{1, \ldots, n-1\}$, then also $o_1, \ldots, o_{k-1}, o_{k+1}, \ldots, o_n$ is a plan for $\langle A, I, O, G\rangle$.*

Proof: By Theorem 3.7 $app_{o_{k+1};\ldots;o_n}(s) \models G$ for any s such that $s \models regr_{o_{k+1};\ldots;o_n}(G)$. Since $app_{o_1;\ldots;o_{k-1}}(I) \models regr_{o_k;\ldots;o_n}(G)$ and $regr_{o_k;\ldots;o_n}(G) \models regr_{o_{k+1};\ldots;o_n}(G)$ also $app_{o_1;\ldots;o_{k-1}}(I) \models regr_{o_{k+1};\ldots;o_n}(G)$. Hence $app_{o_1;\ldots;o_{k-1};o_{k+1};\ldots;o_n}(I) \models G$ and $o_1; \ldots; o_{k-1}; o_{k+1}; \ldots; o_n$ is a plan for $\langle A, I, O, G\rangle$. □

Therefore any regression step that makes the set of states smaller in the set-inclusion sense is unnecessary. However, testing whether this is the case may be computationally expensive. Although the following two problems are closely related to SAT, it could be possible that the formulae obtained by reduction to SAT would fall in some polynomial-time subclass. We show that this is not the case.

Lemma 3.9 *The problem of testing whether $regr_o(\phi) \not\models \phi$ is NP-hard.*

Proof: We give a reduction from SAT to the problem. Let ϕ be any formula. Let a be a state variable not occurring in ϕ. Now $regr_{\langle\neg\phi\rightarrow a,a\rangle}(a) \not\models a$ if and only if $(\neg\phi \rightarrow a) \not\models a$, because $regr_{\langle\neg\phi\rightarrow a,a\rangle}(a) = \neg\phi \rightarrow a$. $(\neg\phi \rightarrow a) \not\models a$ is equivalent to $\not\models (\neg\phi \rightarrow a) \rightarrow a$ that is equivalent to the satisfiability of $\neg((\neg\phi \rightarrow a) \rightarrow a)$. Further, $\neg((\neg\phi \rightarrow a) \rightarrow a)$ is logically equivalent to $\neg(\neg(\phi \vee a) \vee a)$ and further to $\neg(\neg\phi \vee a)$ and $\phi \wedge \neg a$.

Satisfiability of $\phi \wedge \neg a$ is equivalent to the satisfiability of ϕ as a does not occur in ϕ: if ϕ is satisfiable, there is a valuation v such that $v \models \phi$, we can set a false in v to obtain v', and as a does not occur in ϕ, we still have $v' \models \phi$, and further $v' \models \phi \wedge \neg a$. Clearly, if ϕ is unsatisfiable also $\phi \wedge \neg a$ is.

Hence $regr_{\langle\neg\phi\rightarrow a,a\rangle}(a) \not\models a$ if and only if ϕ is satisfiable. □

Also the problem of testing whether a regression step leads to an empty set of states is difficult.

Lemma 3.10 *The problem of testing that $regr_o(\phi)$ is satisfiable is NP-hard.*

Proof: Proof is a reduction from SAT. Let ϕ be a formula. $\mathit{regr}_{\langle\phi,a\rangle}(a)$ is satisfiable if and only if ϕ is satisfiable because $\mathit{regr}_{\langle\phi,a\rangle}(a) \equiv \phi$.

The problem is NP-hard even if we restrict to operators that have a satisfiable precondition: ϕ is satisfiable if and only if $(\phi \vee \neg a) \wedge a$ is satisfiable if and only if $\mathit{regr}_{\langle\phi\vee\neg a,b\rangle}(a \wedge b)$ is satisfiable. Here a is a state variable that does not occur in ϕ. Clearly, $\phi \vee \neg a$ is true when a is false, and hence $\phi \vee \neg a$ is satisfiable. □

Of course, testing that $\mathit{regr}_o(\phi) \not\models \phi$ or that $\mathit{regr}_o(\phi)$ is satisfiable is not necessary for the correctness of backward search, but avoiding useless steps improves efficiency.

Early work on planning restricted to goals and operator preconditions that are conjunctions of state variables and to unconditional effects (STRIPS operators with only positive literals in preconditions.) In this special case both goals G and operator effects e can be viewed as sets of literals, and the definition of regression is particularly simple: regressing G with respect to $\langle c, e\rangle$ is $(G \backslash e) \cup c$. If there is $a \in A$ such that $a \in G$ and $\neg a \in e$, then the result of regression is $\bot$, that is, the empty set of states. We do not use this restricted type of regression in this lecture.

Some planners that use backward search and have operators with disjunctive preconditions and conditional effects eliminate all disjunctivity by branching. For example, the backward step from g with operator $\langle a \vee b, g\rangle$ yields $a \vee b$. This formula corresponds to two non-disjunctive goals, a and b. For each of these new goals a separate subtree is produced. Disjunctivity caused by conditional effects can similarly be handled by branching. However, this branching may lead to a very high branching factor and thus to poor performance.

In addition to being the basis of backward search, regression has many other applications in reasoning about actions. One of them is the composition of operators. The composition $o_1 \circ o_2$ of operators $o_1 = \langle c_1, e_1\rangle$ and $o_2 = \langle c_2, e_2\rangle$ is an operator that behaves like applying o_1 followed by o_2. For a to be true after o_2 we can regress a with respect to o_2, obtaining $\mathit{EPC}_a(e_2) \vee (a \wedge \neg \mathit{EPC}_{\neg a}(e_2))$. Condition for this formula to be true after o_1 is obtained by regressing with e_1, leading to

$$\begin{aligned}
&\mathit{regr}_{e_1}(\mathit{EPC}_a(e_2) \vee (a \wedge \neg \mathit{EPC}_{\neg a}(e_2)))\\
&= \mathit{regr}_{e_1}(\mathit{EPC}_a(e_2)) \vee (\mathit{regr}_{e_1}(a) \wedge \neg \mathit{regr}_{e_1}(\mathit{EPC}_{\neg a}(e_2)))\\
&= \mathit{regr}_{e_1}(\mathit{EPC}_a(e_2)) \vee ((\mathit{EPC}_a(e_1) \vee (a \wedge \neg \mathit{EPC}_{\neg a}(e_2))) \wedge \neg \mathit{regr}_{e_1}(\mathit{EPC}_{\neg a}(e_2))).
\end{aligned}$$

Since we want to define an effect $\phi \rhd a$ of $o_1 \circ o_2$ so that a becomes true whenever o_1 followed by o_2 would make it true, the formula ϕ does not have to represent the case in which a is true already before the application of $o_1 \circ o_2$. Hence we can simplify the above formula to

$$\mathit{regr}_{e_1}(\mathit{EPC}_a(e_2)) \vee (\mathit{EPC}_a(e_1) \wedge \neg \mathit{regr}_{e_1}(\mathit{EPC}_{\neg a}(e_2))).$$

An analogous formula is needed for making $\neg a$ false. This leads to the following definition.

Definition 3.11 (Composition of operators) *Let $o_1 = \langle c_1, e_1\rangle$ and $o_2 = \langle c_2, e_2\rangle$ be two operators on A. Then their* composition $o_1 \circ o_2$ *is defined as*

$$\left\langle c, \bigwedge_{a\in A} \begin{pmatrix} ((\mathit{regr}_{e_1}(\mathit{EPC}_a(e_2)) \vee (\mathit{EPC}_a(e_1) \wedge \neg \mathit{regr}_{e_1}(\mathit{EPC}_{\neg a}(e_2)))) \rhd a) \wedge \\ ((\mathit{regr}_{e_1}(\mathit{EPC}_{\neg a}(e_2)) \vee (\mathit{EPC}_{\neg a}(e_1) \wedge \neg \mathit{regr}_{e_1}(\mathit{EPC}_a(e_2)))) \rhd \neg a) \end{pmatrix} \right\rangle$$

where $c = c_1 \wedge \mathit{regr}_{e_1}(c_2) \wedge \bigwedge_{a\in A} \neg(\mathit{EPC}_a(e_1) \wedge \mathit{EPC}_{\neg a}(e_1))$.

Notice that in $o_1 \circ o_2$ first o_1 is applied and then o_2, so the ordering is opposite to the usual notation for the composition of functions.

Theorem 3.12 *Let o_1 and o_2 be operators and s a state. Then $app_{o_1 \circ o_2}(s)$ is defined if and only if $app_{o_1;o_2}(s)$ is defined, and $app_{o_1 \circ o_2}(s) = app_{o_1;o_2}(s)$.*

Proof: Let $o_1 = \langle c_1, e_1 \rangle$ and $o_2 = \langle c_2, e_2 \rangle$. Assume $app_{o_1 \circ o_2}(s)$ is defined. Hence $s \models c_1 \wedge regr_{e_1}(c_2) \wedge \bigwedge_{a \in A} \neg (EPC_a(e_1) \wedge EPC_{\neg a}(e_1))$, that is, the precondition of $o_1 \circ o_2$ is true, and $s \not\models (regr_{e_1}(EPC_a(e_2)) \vee (EPC_a(e_1) \wedge \neg regr_{e_1}(EPC_{\neg a}(e_2)))) \wedge (((regr_{e_1}(EPC_{\neg a}(e_2)) \vee (EPC_{\neg a}(e_1) \wedge \neg regr_{e_1}(EPC_a(e_2))))))$ for all $a \in A$, that is, the effects do not contradict each other.

Now $app_{o_1}(s)$ in $app_{o_1;o_2}(s) = app_{o_2}(app_{o_1}(s))$ defined because $s \models c_1 \wedge \bigwedge_{a \in A} \neg (EPC_a(e_1) \wedge EPC_{\neg a}(e_1))$. Further $app_{o_1}(s) \models c_2$ by Theorem 3.7 because $s \models regr_{e_1}(c_2)$. From $s \not\models (regr_{e_1}(EPC_a(e_2)) \vee (EPC_a(e_1) \wedge \neg regr_{e_1}(EPC_{\neg a}(e_2)))) \wedge (((regr_{e_1}(EPC_{\neg a}(e_2)) \vee (EPC_{\neg a}(e_1) \wedge \neg regr_{e_1}(EPC_a(e_2))))))$ for all $a \in A$ logically follows $s \not\models regr_{e_1}(EPC_a(e_2)) \wedge regr_{e_1}(EPC_{\neg a}(e_2))$ for all $a \in A$. Hence by Theorem 3.7 $app_{o_1}(s) \not\models EPC_a(e_2) \wedge EPC_{\neg a}(e_2)$ for all $a \in A$, and by Lemma 3.3 $app_{o_2}(app_{o_1}(s))$ is defined.

For the other direction, since $app_{o_1}(s)$ is defined, $s \models c_1 \wedge \bigwedge_{a \in A} \neg (EPC_a(e_1) \wedge EPC_{\neg a}(e_1))$. Since $app_{o_2}(app_{o_1}(s))$ is defined, $s \models regr_{e_1}(c_2)$ by Theorem 3.7.

It remains to show that the effects of $o_1 \circ o_2$ do not contradict. Since $app_{o_2}(app_{o_1}(s))$ is defined $app_{o_1}(s) \not\models EPC_a(e_2) \wedge EPC_{\neg a}(e_2)$ and $s \not\models EPC_a(e_1) \wedge EPC_{\neg a}(e_1)$ for all $a \in A$. Hence by Theorem 3.7 $s \not\models regr_{e_1}(EPC_a(e_2)) \wedge regr_{e_1}(EPC_{\neg a}(e_2))$ for all $a \in A$. Assume that for some $a \in A$ $s \models regr_{e_1}(EPC_a(e_2)) \vee (EPC_a(e_1) \wedge \neg regr_{e_1}(EPC_{\neg a}(e_2)))$, that is, $a \in [o_1 \circ o_2]_s^{det}$. If $s \models regr_{e_1}(EPC_a(e_2))$ then $s \not\models regr_{e_1}(EPC_{\neg a}(e_2)) \vee \neg regr_{e_1}(EPC_a(e_2))$. Otherwise $s \models EPC_a(e_1) \wedge \neg regr_{e_1}(EPC_{\neg a}(e_2))$ and hence $s \not\models EPC_{\neg a}(e_1)$. Hence in both cases $s \not\models regr_{e_1}(EPC_{\neg a}(e_2)) \vee (EPC_{\neg a}(e_1) \wedge \neg regr_{e_1}(EPC_a(e_2)))$, that is, $\neg a \notin [o_1 \circ o_2]_s^{det}$. Therefore $app_{o_1 \circ o_2}(s)$ is defined.

We show that for any $a \in A$, $app_{o_1 \circ o_2}(s) \models a$ if and only if $app_{o_1}(app_{o_2}(s)) \models a$. Assume $app_{o_1 \circ o_2}(s) \models a$. Hence one of two cases hold.

1. Assume $s \models regr_{e_1}(EPC_a(e_2)) \vee (EPC_a(e_1) \wedge \neg regr_{e_1}(EPC_{\neg a}(e_2)))$.

   If $s \models regr_{e_1}(EPC_a(e_2))$ then by Theorem 3.7 and Lemma 3.3 $a \in [e_1]^{det}_{app_{o_1}(s)}$. Hence $app_{o_1;o_2}(s) \models a$.

   Assume $s \models EPC_a(e_1) \wedge \neg regr_{e_1}(EPC_{\neg a}(e_2))$. Hence by Lemma 3.3 $a \in [e_1]^{det}_s$ and $app_{o_1}(s) \models a$, and $app_{o_1}(s) \not\models EPC_{\neg a}(e_2)$ and $\neg a \notin [e_2]^{det}_{app_{o_1}(s)}$. Hence $app_{o_1;o_2}(s) \models a$.

2. Assume $s \models a$ and $s \not\models regr_{e_1}(EPC_{\neg a}(e_2)) \vee (EPC_{\neg a}(e_1) \wedge \neg regr_{e_1}(EPC_a(e_2)))$.

   Since $s \not\models regr_{e_1}(EPC_{\neg a}(e_2))$ by Theorem 3.7 $app_{o_1}(s) \not\models EPC_{\neg a}(e_2)$ and hence $\neg a \notin [e_2]^{det}_{app_{o_1}(s)}$.

   Since $s \not\models EPC_{\neg a}(e_1) \wedge \neg regr_{e_1}(EPC_a(e_2))$ by Lemma 3.3 $\neg a \notin [e_1]^{det}_s$ or $app_{e_1}(s) \models EPC_a(e_2)$ and hence by Theorem 3.7 $a \in [e_2]^{det}_{app_{o_1}(s)}$.

Hence either o_1 does not make a false, or if it makes, makes o_2 it true again so that $app_{o_1;o_2}(s) \models a$ in all cases.

Assume $app_{o_1;o_2}(s) \models a$. Hence one of the following three cases must hold.

1. If $a \in [e_2]^{det}_{app_{o_1}(s)}$ then by Lemma 3.3 $app_{o_1}(s) \models EPC_a(e_2)$. By Theorem 3.7 $s \models regr_{e_1}(EPC_a(e_2))$.

2. If $a \in [e_1]_s^{det}$ and $\neg a \notin [e_2]_{app_{o_1}(s)}^{det}$ then by Lemma 3.3 $app_{o_1}(s) \not\models EPC_{\neg a}(e_2)$. By Theorem 3.7 $s \models EPC_a(e_1) \wedge \neg regr_{e_1}(EPC_{\neg a}(e_2))$.
3. If $s \models a$ and $\neg a \notin [e_2]_{app_{o_1}(s)}^{det}$ and $\neg a \notin [e_1]_s^{det}$ then by Lemma 3.3 $app_{o_1}(s) \not\models EPC_{\neg a}(e_2)$. By Theorem 3.7 $s \not\models regr_{e_1}(EPC_{\neg a}(e_2))$.

   By Lemma 3.3 $s \not\models EPC_{\neg a}(e_1)$.

In the first two cases the antecedent of the first conditional in the definition of $o_1 \circ o_2$ is true, meaning that $app_{o_1 \circ o_2}(s) \models a$, and in the third case $s \models a$ and the antecedent of the second conditional effect is false, also meaning that $app_{o_1 \circ o_2}(s) \models a$. □

The above construction can be used to eliminate *sequential composition* from operator effects (Section 2.3.2).

3.2 Planning by heuristic search algorithms

Search for plans can be performed forwards or backwards respectively with progression or regression as described in Sections 3.1.1 and 3.1.2. There are several algorithms that can be used for the purpose, including depth-first search, breadth-first search, and iterative deepening, but without informed selection of operators these algorithms perform poorly.

The use of additional information for guiding search is essential for achieving efficient planning with general-purpose search algorithms. Algorithms that use heuristic estimates on the values of the nodes in the search space for guiding the search have been applied to planning very successfully. Some of the more sophisticated search algorithms that can be used are A$*$ [Hart *et al.*, 1968], WA$*$ [Pearl, 1984], IDA$*$ [Korf, 1985], and simulated annealing [Kirkpatrick *et al.*, 1983].

The effectiveness of these algorithms is dependent on good heuristics for guiding the search. The most important heuristics are estimates of distances between states. The distance is the minimum number of operators needed for reaching a state from another state. In Section 3.4 we will present techniques for estimating the distances between states and sets of states. In this section we will discuss how heuristic search algorithms are applied in planning.

When search proceeds forwards by progression starting from the initial state, we estimate the distance between the current state and the set of goal states. When search proceeds backwards by regression starting from the goal states, we estimate the distance between the initial state and the current set of goal states as computed by regression.

All the systematic heuristic search algorithms can easily be implemented to keep track of the search history which for planning equals the sequence of operators in the incomplete plan under consideration. Therefore the algorithms are started from the initial state I (forward search) or from the goal formula G (backward search) and then proceed forwards with progression or backwards with regression. Whenever the search successfully finishes, the plan can be recovered from the data structures maintained by the algorithm.

Local search algorithms do not keep track of the search history, and we have to define the elements of the search space as prefixes or suffixes of plans. For forward search we use sequences of operators (prefixes of plans)

$$o_1; o_2; \ldots; o_n.$$

The search starts from the empty sequence. The neighbors of an incomplete plan are obtained by adding an operator to the end of the plan or by deleting some of the last operators.

Definition 3.13 (Neighbors for local search with progression) *Let $\langle A, I, O, G\rangle$ be a succinct transition system. For forward search, the neighbors of an incomplete plan $o_1; o_2; \ldots; o_n$ are the following.*

1. $o_1; o_2; \ldots; o_n; o$ *for any* $o \in O$ *such that* $app_{o_1;\ldots;o_n;o}(I)$ *is defined*
2. $o_1; o_2; \ldots; o_i$ *for any* $i < n$

When $app_{o_1;o_2;\ldots;o_n}(I) \models G$ then $o_1; \ldots; o_n$ is a plan.

Also for backward search the incomplete plans are sequence of operators (suffixes of plans)

$$o_n; \ldots; o_1.$$

The search starts from the empty sequence. The neighbors of an incomplete plan are obtained by adding an operator to the beginning of the plan or by deleting some of the first operators.

Definition 3.14 (Neighbors for local search with regression) *Let $\langle A, I, O, G\rangle$ be a succinct transition system. For backward search, the children of an incomplete plan $o_n; \ldots; o_1$ are the following.*

1. $o; o_n; \ldots; o_1$ *for any* $o \in O$ *such that* $regr_{o;o_n;\ldots;o_1}(G)$ *is defined*

2. $o_i; \ldots; o_1$ *for any* $i < n$

When $I \models regr_{o_n;\ldots;o_1}(G)$ then $o_n; \ldots; o_1$ is a plan.

Backward search and forward search are not the only possibilities to define planning as a search problem. In partial-order planning [McAllester and Rosenblitt, 1991] the search space consists of incomplete plans which are partially ordered multisets of operators. The neighbors of an incomplete plan are those obtained by adding an operator or an ordering constraint. Incomplete plans can also be formalized as fixed length sequences of operators in which zero or more of the operators are missing. This leads to the constraint-based approaches to planning, including the planning as satisfiability approach that is presented in Section 3.6.

3.3 Reachability

The notion of reachability is important in defining whether a planning problem is solvable and in deriving techniques that speed up search for plans.

3.3.1 Distances

First we define the distances between states in a transition system in which all operators are deterministic. Heuristics in Section 3.4 are approximations of distances.

Definition 3.15 *Let I be an initial state and O a set of operators. Define the* forward distance sets D_i^{fwd} *for I, O that consist of those states that are reachable from I by at most i operator applications as follows.*

$$\begin{aligned} D_0^{fwd} &= \{I\} \\ D_i^{fwd} &= D_{i-1}^{fwd} \cup \{s | o \in O, s \in img_o(D_{i-1}^{fwd})\} \text{ for all } i \geq 1 \end{aligned}$$

Definition 3.16 *Let I be a state, O a set of operators, and $D_0^{fwd}, D_1^{fwd}, \ldots$ the forward distance sets for I, O. Then* the forward distance *of a state s from I is*

$$\delta_I^{fwd}(s) = \begin{cases} 0 \text{ if } s = I \\ i \text{ if } s \in D_i^{fwd} \backslash D_{i-1}^{fwd}. \end{cases}$$

If $s \notin D_i^{fwd}$ for all $i \geq 0$ then $\delta_I^{fwd}(s) = \infty$. States that have a finite forward distance are reachable *(from I with O).*

Distances can also be defined for formulae.

Definition 3.17 *Let ϕ be a formula. Then the* forward distance $\delta_I^{fwd}(\phi)$ *of ϕ is i if there is state s such that $s \models \phi$ and $\delta_I^{fwd}(s) = i$ and there is no state s' such that $s' \models \phi$ and $\delta_I^{fwd}(s) < i$. If $I \models \phi$ then $\delta_I^{fwd}(\phi) = 0$.*

A formula ϕ has a finite distance $< \infty$ if and only if $\langle A, I, O, \phi \rangle$ has a plan.

Reachability and distances are useful for implementing efficient planning systems. We mention two applications.

First, if we know that no state satisfying a formula ϕ is reachable from the initial states, then we know that no operator $\langle \phi, e \rangle$ can be a part of a plan, and we can ignore any such operator.

Second, distances help in finding a plan. Consider a deterministic planning problem with goal state G. We can now produce a shortest plan by finding an operator o so that $\delta_I^{fwd}(regr_o(G)) < \delta_I^{fwd}(G)$, using $regr_o(G)$ as the new goal state and repeating the process until the initial state I is reached.

Of course, since computing distances is in the worst case just as difficult as planning (PSPACE-complete) it is in general not useful to use subprocedures based on exact distances in a planning algorithm. Instead, different kinds of *approximations* of distances and reachability have to be used. The most important approximations allow the computation of useful reachability and distance information in polynomial time in the size of the succinct transition system. In Section 3.4 we will consider some of them.

3.3.2 Invariants

An *invariant* is a formula that is true in the initial state and in every state that is reached by applying an operator in a state in which it holds. Invariants are closely connected to reachability and distances: a formula ϕ is an invariant if and only if the distance of $\neg\phi$ from the initial state is ∞. Invariants can be used for example to speed up algorithms based on regression.

Definition 3.18 *Let I be a set of initial states and O a set of operators. An formula ϕ is* an invariant *of I, O if $s \models \phi$ for all states s that are reachable from I by a sequence of 0 or more operators in O.*

An invariant ϕ is *the strongest invariant* if $\phi \models \psi$ for any invariant ψ. The strongest invariant exactly characterizes the set of all states that are reachable from the initial state: for every state s, $s \models \phi$ if and only if s is reachable from the initial state. We say "the strongest invariant" even though there are actually several strongest invariants: if ϕ satisfies the properties of the strongest invariant, any other formula that is logically equivalent to ϕ, for example $\phi \vee \phi$, also does. Hence the uniqueness of the strongest invariant has to be understood up to logical equivalence.

Example 3.19 Consider a set of blocks that can be on the table or stacked on top of other blocks. Every block can be on at most one block and on every block there can be one block at most. The actions for moving the blocks can be described by the following schematic operators.

$$\langle \text{ontable}(x) \wedge \text{clear}(x) \wedge \text{clear}(y), \text{on}(x,y) \wedge \neg\text{clear}(y) \wedge \neg\text{ontable}(x) \rangle$$
$$\langle \text{clear}(x) \wedge \text{on}(x,y), \text{ontable}(x) \wedge \text{clear}(y) \wedge \neg\text{on}(x,y) \rangle$$
$$\langle \text{clear}(x) \wedge \text{on}(x,y) \wedge \text{clear}(z), \text{on}(x,z) \wedge \text{clear}(y) \wedge \neg\text{clear}(z) \wedge \neg\text{on}(x,y) \rangle$$

We consider the operators obtained by instantiating the schemata with the objects A, B and C. Let all the blocks be initially on the table. Hence the initial state satisfies the formula

$$\text{clear}(A) \wedge \text{clear}(B) \wedge \text{clear}(C) \wedge \text{ontable}(A) \wedge \text{ontable}(B) \wedge \text{ontable}(C) \wedge$$
$$\neg\text{on}(A,B) \wedge \neg\text{on}(A,C) \wedge \neg\text{on}(B,A) \wedge \neg\text{on}(B,C) \wedge \neg\text{on}(C,A) \wedge \neg\text{on}(C,B)$$

that determines the truth-values of all state variables uniquely. The strongest invariant of this problem is the conjunction of the following formulae.

$$\text{clear}(A) \leftrightarrow (\neg\text{on}(B,A) \wedge \neg\text{on}(C,A)) \qquad \text{clear}(B) \leftrightarrow (\neg\text{on}(A,B) \wedge \neg\text{on}(C,B))$$
$$\text{clear}(C) \leftrightarrow (\neg\text{on}(A,C) \wedge \neg\text{on}(B,C)) \qquad \text{ontable}(A) \leftrightarrow (\neg\text{on}(A,B) \wedge \neg\text{on}(A,C))$$
$$\text{ontable}(B) \leftrightarrow (\neg\text{on}(B,A) \wedge \neg\text{on}(B,C)) \qquad \text{ontable}(C) \leftrightarrow (\neg\text{on}(C,A) \wedge \neg\text{on}(C,B))$$
$$\neg\text{on}(A,B) \vee \neg\text{on}(A,C) \qquad \neg\text{on}(B,A) \vee \neg\text{on}(B,C)$$
$$\neg\text{on}(C,A) \vee \neg\text{on}(C,B)$$
$$\neg\text{on}(B,A) \vee \neg\text{on}(C,A) \qquad \neg\text{on}(A,B) \vee \neg\text{on}(C,B)$$
$$\neg\text{on}(A,C) \vee \neg\text{on}(B,C)$$
$$\neg(\text{on}(A,B) \wedge \text{on}(B,C) \wedge \text{on}(C,A)) \qquad \neg(\text{on}(A,C) \wedge \text{on}(C,B) \wedge \text{on}(B,A))$$

We can schematically give the invariants for any set X of blocks as follows.

$\text{clear}(x) \leftrightarrow \forall y \in X \backslash \{x\}.\neg\text{on}(y,x)$
$\text{ontable}(x) \leftrightarrow \forall y \in X \backslash \{x\}.\neg\text{on}(x,y)$
$\neg\text{on}(x,y) \vee \neg\text{on}(x,z)$ when $y \neq z$
$\neg\text{on}(y,x) \vee \neg\text{on}(z,x)$ when $y \neq z$
$\neg(\text{on}(x_1,x_2) \wedge \text{on}(x_2,x_3) \wedge \cdots \wedge \text{on}(x_{n-1},x_n) \wedge \text{on}(x_n,x_1))$ for all $n \geq 1, \{x_1,\ldots,x_n\} \subseteq X$

The last formula says that the *on* relation is acyclic. ■

3.4 Approximations of distances

The approximations of distances are based on the following idea. Instead of considering the number of operators required to reach individual states, we approximately compute the number of operators to reach a state in which a certain state variable has a certain value. So instead of using distances of states, we use distances of literals.

The estimates are not accurate for two reasons. First, and more importantly, distance estimation is done one state variable at a time and dependencies between state variables are ignored. Second, to achieve polynomial-time computation, satisfiability tests for a formula and a set of literals to test the applicability of an operator and to compute the distance estimate of a formula, have to be performed by an inaccurate polynomial-time algorithm that approximates NP-hard satisfiability testing. As we are interested in computing distance estimates efficiently the inaccuracy is a necessary and acceptable compromise.

3.4.1 Admissible max heuristic

We give a recursive procedure that computes a lower bound on the number of operator applications that are needed for reaching from a state I a state in which state variables $a \in A$ have certain values. This is by computing a sequence of sets D_i^{max} of literals. The set D_i^{max} consists literals that are true in all states that have distance $\leq i$ from the state I.

Recall Definition 3.2 of $EPC_l(o)$ for literals l and operators $o = \langle c, e \rangle$:

$$EPC_l(o) = c \wedge EPC_l(e) \wedge \bigwedge_{a \in A} \neg(EPC_a(e) \wedge EPC_{\neg a}(e)).$$

Definition 3.20 *Let $L = A \cup \{\neg a | a \in A\}$ be the set of literals on A and I a state. Define the sets D_i^{max} for $i \geq 0$ as follows.*

$$\begin{aligned} D_0^{max} &= \{l \in L | I \models l\} \\ D_i^{max} &= D_{i-1}^{max} \backslash \{l \in L | o \in O, D_{i-1}^{max} \cup \{EPC_{\bar{l}}(o)\} \text{ is satisfiable}\}, \text{ for } i \geq 1 \end{aligned}$$

Since we consider only finite sets A of state variables and $|D_0^{max}| = |A|$ and $D_{i+1}^{max} \subseteq D_i^{max}$ for all $i \geq 0$, necessarily $D_i^{max} = D_j^{max}$ for some $i \leq |A|$ and all $j > i$.

The above computation starts from the set D_0^{max} of all literals that are true in the initial state I. This set of literals characterizes those states that have distance 0 from the initial state. The initial state is the only such state.

Then we repeatedly compute sets of literals characterizing sets of states that are reachable with 1, 2 and more operators. Each set D_i^{max} is computed from the preceding set D_{i-1}^{max} as follows. For each operator o it is tested whether it is applicable in one of the distance $i-1$ states and whether it could make a literal l false. This is by testing whether $EPC_{\bar{l}}(o)$ is true in one of the distance $i-1$ states. If this is the case, the literal l could be false, and it will not be included in D_i^{max}.

The sets of states in which the literals D_i^{max} are true are an upper bound (set-inclusion) on the set of states that have forward distance i.

Theorem 3.21 *Let $D_i^{fwd}, i \geq 0$ be the forward distance sets and D_i^{max} the max-distance sets for I and O. Then for all $i \geq 0$, $D_i^{fwd} \subseteq \{s \in S | s \models D_i^{max}\}$ where S is the set of all states.*

Proof: By induction on i.

Base case $i = 0$: D_0^{fwd} consists of the unique initial state I and D_0^{max} consists of exactly those literals that are true in I, identifying it uniquely. Hence $D_i^{fwd} = \{s \in S | s \models D_i^{max}\}$.

Inductive case $i \geq 1$: Let s be any state in D_i^{fwd}. We show that $s \models D_i^{max}$. Let l be any literal in D_i^{max}.

Assume $s \in D_{i-1}^{fwd}$. As $D_i^{max} \subseteq D_{i-1}^{max}$ also $l \in D_{i-1}^{max}$. By the induction hypothesis $s \models l$.

Otherwise $s \in D_i^{fwd} \backslash D_{i-1}^{fwd}$. Hence there is $o \in O$ and $s_0 \in D_{i-1}^{fwd}$ with $s = app_o(s_0)$. By $D_i^{max} \subseteq D_{i-1}^{max}$ and the induction hypothesis $s_0 \models l$. As $l \in D_i^{max}$, by definition of D_i^{max} the set $D_{i-1}^{max} \cup \{EPC_{\bar{l}}(o)\}$ is not satisfiable. By $s_0 \in D_{i-1}^{fwd}$ and the induction hypothesis $s_0 \models D_{i-1}^{max}$. Hence $s_0 \not\models EPC_{\bar{l}}(o)$. By Lemma 3.3 applying o in s_0 does not make l false. Hence $s \models l$. □

The sets D_i^{max} can be used for estimating the distances of formulae. The distance of a formula is the minimum of the distances of states that satisfy the formula.

Definition 3.22 *Let ϕ be a formula. Define*

$$\delta_I^{max}(\phi) = \begin{cases} 0 \text{ iff } D_0^{max} \cup \{\phi\} \text{ is satisfiable} \\ d \text{ iff } D_d^{max} \cup \{\phi\} \text{ is satisfiable and } D_{d-1}^{max} \cup \{\phi\} \text{ is not satisfiable, for } d \geq 1. \end{cases}$$

Lemma 3.23 *Let I be a state, O a set of operators, and $D_0^{max}, D_1^{max}, \ldots$ the sets given in Definition 3.20 for I and O. Then $app_{o_1;\ldots;o_n}(I) \models D_n^{max}$ for any operators $\{o_1, \ldots, o_n\} \subseteq O$.*

Proof: By induction on n.

Base case $n = 0$: The length of the operator sequence is zero, and hence $app_\epsilon(I) = I$. The set D_0^{max} consists exactly of those literals that are true in s, and hence $I \models D_0^{max}$.

Inductive case $n \geq 1$: By the induction hypothesis $app_{o_1;\ldots;o_{n-1}}(I) \models D_{n-1}^{max}$.

Let l be any literal in D_n^{max}. We show it is true in $app_{o_1;\ldots;o_n}(I)$. Since $l \in D_n^{max}$ and $D_n^{max} \subseteq D_{n-1}^{max}$, also $l \in D_{n-1}^{max}$, and hence by the induction hypothesis $app_{o_1;\ldots;o_{n-1}}(I) \models l$. Since $l \in D_n^{max}$ it must be that $D_{n-1}^{max} \cup \{EPC_{\bar{l}}(o_n)\}$ is not satisfiable (definition of D_n^{max}) and further that $app_{o_1;\ldots;o_{n-1}}(I) \not\models EPC_{\bar{l}}(o_n)$. Hence applying o_n in $app_{o_1;\ldots;o_{n-1}}(I)$ does not make l false, and consequently $app_{o_1;\ldots;o_n}(I) \models l$.

□

The next theorem shows that the distance estimates given for formulae yield a lower bound on the number of actions needed to reach a state satisfying the formula.

Theorem 3.24 *Let I be a state, O a set of operators, ϕ a formula, and $D_0^{max}, D_1^{max}, \ldots$ the sets given in Definition 3.20 for I and O. If $app_{o_1;\ldots;o_n}(I) \models \phi$, then $D_n^{max} \cup \{\phi\}$ is satisfiable.*

Proof: By Lemma 3.23 $app_{o_1;\ldots;o_n}(I) \models D_n^{max}$. By assumption $app_{o_1;\ldots;o_n}(I) \models \phi$. Hence $D_n^{max} \cup \{\phi\}$ is satisfiable. □

Corollary 3.25 *Let I be a state and ϕ a formula. Then for any sequence $o_1, \ldots, o_n$ of operators such that $app_{o_1;\ldots;o_n}(I) \models \phi$, $n \geq \delta_I^{max}(\phi)$.*

The estimate $\delta_s^{max}(\phi)$ never overestimates the distance from s to ϕ and it is therefore an admissible heuristic. It may severely underestimate the distance, as discussed in the end of this section.

Distance estimation in polynomial time

The algorithm for computing the sets D_i^{max} runs in polynomial time except that the satisfiability tests for $D \cup \{\phi\}$ are instances of the NP-complete SAT problem. For polynomial time computation we perform these tests by a polynomial-time approximation that has the property that if $D \cup \{\phi\}$ is satisfiable then asat(D, ϕ) returns true, but not necessarily vice versa. A counterpart of Theorem 3.21 can be established when the satisfiability tests $D \cup \{\phi\}$ are replaced by tests asat(D, ϕ).

The function asat(D, ϕ) tests whether there is a state in which ϕ and the literals D are true, or equivalently, whether $D \cup \{\phi\}$ is satisfiable. This algorithm does not accurately test satisfiability, and may claim that $D \cup \{\phi\}$ is satisfiable even when it is not. This, however, never leads to

overestimating the distances, only underestimating. The algorithm runs in polynomial time and is defined as follows.

$$\begin{aligned}
\text{asat}(D, \bot) &= \text{false} \\
\text{asat}(D, \top) &= \text{true} \\
\text{asat}(D, a) &= \text{true iff } \neg a \notin D \text{ (for state variables } a \in A) \\
\text{asat}(D, \neg a) &= \text{true iff } a \notin D \text{ (for state variables } a \in A) \\
\text{asat}(D, \neg\neg\phi) &= \text{asat}(D, \phi) \\
\text{asat}(D, \phi_1 \vee \phi_2) &= \text{asat}(D, \phi_1) \text{ or asat}(D, \phi_2) \\
\text{asat}(D, \phi_1 \wedge \phi_2) &= \text{asat}(D, \phi_1) \text{ and asat}(D, \phi_2) \\
\text{asat}(D, \neg(\phi_1 \vee \phi_2)) &= \text{asat}(D, \neg\phi_1) \text{ and asat}(D, \neg\phi_2) \\
\text{asat}(D, \neg(\phi_1 \wedge \phi_2)) &= \text{asat}(D, \neg\phi_1) \text{ or asat}(D, \neg\phi_2)
\end{aligned}$$

In this and other recursive definitions about formulae the cases for $\neg(\phi_1 \wedge \phi_2)$ and $\neg(\phi_1 \vee \phi_2)$ are obtained respectively from the cases for $\phi_1 \vee \phi_2$ and $\phi_1 \wedge \phi_2$ by the De Morgan laws.

The reason why the satisfiability test is not accurate is that for formulae $\phi \wedge \psi$ (respectively $\neg(\phi \vee \psi)$) we make recursively two satisfiability tests that do not require that the subformulae ϕ and ψ (respectively $\neg\phi$ and $\neg\psi$) are *simultaneously* satisfiable.

We give a lemma that states the connection between $\text{asat}(D, \phi)$ and the satisfiability of $D \cup \{\phi\}$.

Lemma 3.26 *Let ϕ be a formula and D a consistent set of literals (it contains at most one of a and $\neg a$ for every $a \in A$.) If $D \cup \{\phi\}$ is satisfiable, then asat(D, ϕ) returns true.*

Proof: The proof is by induction on the structure of ϕ.

Base case 1, $\phi = \bot$: The set $D \cup \{\bot\}$ is not satisfiable, and hence the implication trivially holds.

Base case 2, $\phi = \top$: $\text{asat}(D, \top)$ always returns true, and hence the implication trivially holds.

Base case 3, $\phi = a$ for some $a \in A$: If $D \cup \{a\}$ is satisfiable, then $\neg a \notin D$, and hence $\text{asat}(D, a)$ returns true.

Base case 4, $\phi = \neg a$ for some $a \in A$: If $D \cup \{\neg a\}$ is satisfiable, then $a \notin D$, and hence $\text{asat}(D, \neg a)$ returns true.

Inductive case 1, $\phi = \neg\neg\phi'$ for some ϕ': The formulae are logically equivalent, and by the induction hypothesis we directly establish the claim.

Inductive case 2, $\phi = \phi_1 \vee \phi_2$: If $D \cup \{\phi_1 \vee \phi_2\}$ is satisfiable, then either $D \cup \{\phi_1\}$ or $D \cup \{\phi_2\}$ is satisfiable and by the induction hypothesis at least one of $\text{asat}(D, \phi_1)$ and $\text{asat}(D, \phi_2)$ returns true. Hence $\text{asat}(D, \phi_1 \vee \phi_2)$ returns true.

Inductive case 3, $\phi = \phi_1 \wedge \phi_2$: If $D \cup \{\phi_1 \wedge \phi_2\}$ is satisfiable, then both $D \cup \{\phi_1\}$ and $D \cup \{\phi_2\}$ are satisfiable and by the induction hypothesis both $\text{asat}(D, \phi_1)$ and $\text{asat}(D, \phi_2)$ return true. Hence $\text{asat}(D, \phi_1 \wedge \phi_2)$ returns true.

Inductive cases 4 and 5, $\phi = \neg(\phi_1 \vee \phi_2)$ and $\phi = \neg(\phi_1 \wedge \phi_2)$: Like cases 2 and 3 by logical equivalence. □

The other direction of the implication does not hold because for example $\text{asat}(\emptyset, a \wedge \neg a)$ returns true even though the formula is not satisfiable. The procedure is a polynomial-time approximation of the logical consequence test from a set of literals: $\text{asat}(D, \phi)$ always returns true if $D \cup \{\phi\}$ is satisfiable, but it may return true also when the set is not satisfiable.

Informativeness of the max heuristic

The max heuristic often underestimates distances. Consider an initial state in which all n state variables are false and a goal state in which all state variables are true and a set of n operators each of which is always applicable and makes one of the state variables true. The max heuristic assigns the distance 1 to the goal state although the distance is n.

The problem is that assigning every state variable the desired value requires a different operator, and taking the maximum number of operators for each state variable ignores this fact. In this case the actual distance is obtained as the *sum* of the distances suggested by each of the n state variables. In other cases the max heuristic works well when the desired state variable values can be reached with the same operators.

Next we will consider heuristics that are not admissible like the max heuristic but in many cases provide a much better estimate of the distances.

3.4.2 Inadmissible additive heuristic

The max heuristic is very optimistic about the distances, and in many cases very seriously underestimates them. If two goal literals have to be made true, the maximum of the goal costs (distances) is assumed to be the combined cost. This however is only accurate when the easier goal is achieved for free while achieving the more difficult goal. Often the goals are independent and then a more accurate estimate would be the sum of the individual costs. This suggests another heuristic, first considered by Bonet and Geffner [2001] as a more practical variant of the max heuristic in the previous section. Our formalization differs from the one given by Bonet and Geffner.

Definition 3.27 *Let I be a state and $L = A \cup \{\neg a | a \in A\}$ the set of literals. Define the sets D_i^+ for $i \geq 0$ as follows.*

$$\begin{aligned} D_0^+ &= \{l \in L | I \models l\} \\ D_i^+ &= D_{i-1}^+ \backslash \{l \in L | o \in O, \mathit{cost}(\mathit{EPC}_{\bar{l}}(o), i) < i\} \text{ for all } i \geq 1 \end{aligned}$$

We define $\mathit{cost}(\phi, i)$ by the following recursive definition.

$$\begin{aligned} \mathit{cost}(\bot, i) &= \infty \\ \mathit{cost}(\top, i) &= 0 \\ \mathit{cost}(a, i) &= 0 \text{ if } \neg a \notin D_0^+, \text{ for } a \in A \\ \mathit{cost}(\neg a, i) &= 0 \text{ if } a \notin D_0^+, \text{ for } a \in A \\ \mathit{cost}(a, i) &= j \text{ if } \neg a \in D_{j-1}^+ \backslash D_j^+ \text{ for some } j < i \\ \mathit{cost}(\neg a, i) &= j \text{ if } a \in D_{j-1}^+ \backslash D_j^+ \text{ for some } j < i \\ \mathit{cost}(a, i) &= \infty \text{ if } \neg a \in D_j^+ \text{ for all } j < i \\ \mathit{cost}(\neg a, i) &= \infty \text{ if } a \in D_j^+ \text{ for all } j < i \\ \mathit{cost}(\phi_1 \vee \phi_2, i) &= \min(\mathit{cost}(\phi_1, i), \mathit{cost}(\phi_2, i)) \\ \mathit{cost}(\phi_1 \wedge \phi_2, i) &= \mathit{cost}(\phi_1, i) + \mathit{cost}(\phi_2, i) \\ \mathit{cost}(\neg\neg\phi, i) &= \mathit{cost}(\phi, i) \\ \mathit{cost}(\neg(\phi_1 \wedge \phi_2), i) &= \min(\mathit{cost}(\neg\phi_1, i), \mathit{cost}(\neg\phi_2, i)) \\ \mathit{cost}(\neg(\phi_1 \vee \phi_2), i) &= \mathit{cost}(\neg\phi_1, i) + \mathit{cost}(\neg\phi_2, i) \end{aligned}$$

Notice that a variant of the definition of the max heuristic could be obtained by replacing the sum $+$ in the definition of costs of conjunctions by $\max$. The definition of $\mathrm{cost}(\phi, i)$ approximates

satisfiability tests similarly to the definition of asat(D, ϕ) by ignoring the dependencies between propositions.

Similarly to max distances we can define distances of formulae.

Definition 3.28 *Let ϕ be a formula. Define*

$$\delta_I^+(\phi) = \mathit{cost}(\phi, n)$$

where n is the smallest i such that $D_i^+ = D_{i-1}^+$.

The following theorem shows that the distance estimates given by the sum heuristic for literals are at least as high as those given by the max heuristic.

Theorem 3.29 *Let $D_i^{max}, i \geq 0$ be the sets defined in terms of the approximate satisfiability tests asat(D, ϕ). Then $D_i^{max} \subseteq D_i^+$ for all $i \geq 0$.*

Proof: The proof is by induction on i.

Base case $i = 0$: By definition $D_0^+ = D_0^{max}$.

Inductive case $i \geq 1$: We have to show that $D_{i-1}^{max} \backslash \{l \in L | o \in O, \text{asat}(D_{i-1}^{max}, \mathit{EPC}_{\bar{l}}(o))\} \subseteq D_{i-1}^+ \backslash \{l \in L | o \in O, \text{cost}(\mathit{EPC}_{\bar{l}}(o), i) < i\}$. By the induction hypothesis $D_{i-1}^{max} \subseteq D_{i-1}^+$. It is sufficient to show that cost$(\mathit{EPC}_{\bar{l}}(o), i) < i$ implies asat$(D_{i-1}^{max}, \mathit{EPC}_{\bar{l}}(o))$.

We show this by induction on the structure of $\phi = \mathit{EPC}_{\bar{l}}(o)$.

Induction hypothesis: cost$(\phi, i) < i$ implies asat(D_{i-1}^{max}, ϕ)=true.

Base case 1, $\phi = \bot$: cost$(\bot, i) = \infty$ and asat$(D_i^{max}, \bot)$=false.

Base case 2, $\phi = \top$: cost$(\bot, i) = 0$ and asat$(D_i^{max}, \bot)$=true.

Base case 3, $\phi = a$: If cost$(a, i) < i$ then $\neg a \notin D_j^+$ for some $j < i$ or $\neg a \notin D_0^+$. Hence $\neg a \notin D_{i-1}^+$. By the outer induction hypothesis $\neg a \notin D_{i-1}^{max}$ and consequently $\neg a \notin D_i^{max}$. Hence asat$(D_i^{max}, \bot)$=true.

Base case 4, $\phi = \neg a$: Analogous to the case $\phi = a$.

Inductive case 5, $\phi = \phi_1 \vee \phi_2$: Assume cost$(\phi_1 \vee \phi_2, i) < i$. Since cost$(\phi_1 \vee \phi_2, i) = \min(\text{cost}(\phi_1, i), \text{cost}(\phi_2, i))$, either cost$(\phi_1, i) < i$ or cost$(\phi_2, i) < i$. By the induction hypothesis cost$(\phi_1, i) < i$ implies asat(D_{i-1}^{max}, ϕ_1), and cost$(\phi_2, i) < i$ implies asat(D_{i-1}^{max}, ϕ_2). Hence either asat(D_{i-1}^{max}, ϕ_1) or asat(D_{i-1}^{max}, ϕ_2). Therefore by definition asat$(D_{i-1}^{max}, \phi_1 \vee \phi_2)$.

Inductive case 6, $\phi = \phi_1 \wedge \phi_2$: Assume cost$(\phi_1 \wedge \phi_2, i) < i$. Since $i \geq 1$ and cost$(\phi_1 \vee \phi_2, i) =$ cost(ϕ_1, i) + cost(ϕ_2, i), both cost$(\phi_1, i) < i$ and cost$(\phi_2, i) < i$. By the induction hypothesis cost$(\phi_1, i) < i$ implies asat(D_{i-1}^{max}, ϕ_1), and cost$(\phi_2, i) < i$ implies asat(D_{i-1}^{max}, ϕ_2). Hence both asat(D_{i-1}^{max}, ϕ_1) an asat(D_{i-1}^{max}, ϕ_2). Therefore by definition asat$(D_{i-1}^{max}, \phi_1 \wedge \phi_2)$.

Inductive case 7, $\phi = \neg\neg\phi_1$: By the induction hypothesis cost$(\phi_1, i) < i$ implies asat(D_{i-1}^{max}, ϕ_1). By definition cost$(\neg\neg\phi_1, i) =$ cost(ϕ_1, i) and asat$(D, \neg\neg\phi) =$ asat(D, ϕ). By the induction hypothesis cost$(\neg\neg\phi_1, i) < i$ implies asat$(D_{i-1}^{max}, \neg\neg\phi_1)$.

Inductive case 8, $\phi = \neg(\phi_1 \vee \phi_2)$: Analogous to the case $\phi = \phi_1 \wedge \phi_2$.

Inductive case 9, $\phi = \neg(\phi_1 \wedge \phi_2)$: Analogous to the case $\phi = \phi_1 \vee \phi_2$. □

That the sum heuristic gives higher estimates than the max heuristic could in many cases be viewed as an advantage because the estimates would be more accurate. However, in some cases this leads to overestimating the actual distance, and therefore the sum distances are not an admissible heuristic.

Example 3.30 Consider an initial state such that $I \models \neg a \wedge \neg b \wedge \neg c$ and the operator $\langle \top, a \wedge b \wedge c \rangle$. A state satisfying $a \wedge b \wedge c$ is reached by this operator in one step but $\delta_I^+(a \wedge b \wedge c) = 3$. ■

3.4.3 Relaxed plan heuristic

The max heuristic and the additive heuristic represent two extremes. The first assumes that sets of operators required for reaching the individual goal literals maximally overlap in the sense that the operators needed for the most difficult goal literal include the operators needed for all the remaining ones. The second assumes that the required operators are completely disjoint.

Usually, of course, the reality is somewhere in between and which notion is better depends on the properties of the operators. This suggests yet another heuristic: we attempt to find a set of operators that approximates, in a sense that will become clear later, the smallest set of operators that are needed to reach a state from another state. This idea has been considered by Hoffman and Nebel [2001]. If the approximation is exact, the cardinality of this set equals the actual distance between the states. The approximation may both overestimate and underestimate the actual distance, and hence it is does not yield an admissible heuristic.

The idea of the heuristic is the following. We first choose a set of goal literals the truth of which is sufficient for the truth of G. These literals must be reachable in the sense of the sets D_i^{max} which we defined earlier. Then we identify those goal literals that were the last to become reachable and a set of operators making them true. A new goal formula represents the conditions under which these operator can make the literals true, and a new set of goal literals is produced by a simplified form of regression from the new goal formula. The computation is repeated until we have a set of goal literals that are true in the initial state.

The function goals(D, ϕ) recursively finds a set M of literals such that $M \models \phi$ and each literal in M is consistent with D. Notice that M itself is not necessarily consistent, for example for $D = \emptyset$ and $\phi = a \wedge \neg a$ we get $M = \{a, \neg a\}$. If a set M is found goals$(D, \phi) = \{M\}$ and otherwise goals$(D, \phi) = \emptyset$.

Definition 3.31 *Let D be a set of literals.*

$$
\begin{aligned}
\mathit{goals}(D, \bot) &= \emptyset \\
\mathit{goals}(D, \top) &= \{\emptyset\} \\
\mathit{goals}(D, a) &= \{\{a\}\} \text{ if } \neg a \notin D \\
\mathit{goals}(D, a) &= \emptyset \text{ if } \neg a \in D \\
\mathit{goals}(D, \neg a) &= \{\{\neg a\}\} \text{ if } a \notin D \\
\mathit{goals}(D, \neg a) &= \emptyset \text{ if } a \in D \\
\mathit{goals}(D, \neg\neg\phi) &= \mathit{goals}(D, \phi) \\
\mathit{goals}(D, \phi_1 \vee \phi_2) &= \begin{cases} \mathit{goals}(D, \phi_1) \text{ if } \mathit{goals}(D, \phi_1) \neq \emptyset \\ \mathit{goals}(D, \phi_2) \text{ otherwise} \end{cases} \\
\mathit{goals}(D, \phi_1 \wedge \phi_2) &= \begin{cases} \{L_1 \cup L_2\} & \text{if } \mathit{goals}(D, \phi_1) = \{L_1\} \text{ and } \mathit{goals}(D, \phi_2) = \{L_2\} \\ \emptyset & \text{otherwise} \end{cases} \\
\mathit{goals}(D, \neg(\phi_1 \wedge \phi_2)) &= \begin{cases} \mathit{goals}(D, \neg\phi_1) \text{ if } \mathit{goals}(D, \neg\phi_1) \neq \emptyset \\ \mathit{goals}(D, \neg\phi_2) \text{ otherwise} \end{cases} \\
\mathit{goals}(D, \neg(\phi_1 \vee \phi_2)) &= \begin{cases} \{L_1 \cup L_2\} & \text{if } \mathit{goals}(D, \neg\phi_1) = \{L_1\} \text{ and } \mathit{goals}(D, \neg\phi_2) = \{L_2\} \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}
$$

Above in the case for $\phi_1 \vee \phi_2$ if both ϕ_1 and ϕ_2 yield a set of goal literals the set for ϕ_1 is always chosen. A practically better implementation is to choose the smaller of the two sets.

Lemma 3.32 *Let D be a set of literals and ϕ a formula.*

1. *goals$(D, \phi) \neq \emptyset$ if and only if asat$(D, \phi) =$ true.*
2. *If goals$(D, \phi) = \{M\}$ then $\{\bar{l} | l \in M\} \cap D = \emptyset$ and asat$(D, \bigwedge_{l \in M} l) =$ true.*

Proof:

1. This is by an easy induction proof on the structure of ϕ based on the definitions of asat(D, ϕ) and goals(D, ϕ).
2. This is because $\bar{l} \notin D$ for all $l \in M$. This can be shown by a simple induction proof.

□

Lemma 3.33 *Let D and $D' \subseteq D$ be sets of literals. If goals$(D, \phi) = \emptyset$ and goals$(D', \phi) = \{M\}$ for some M, then there is $l \in M$ such that $\bar{l} \in D \backslash D'$.*

Proof: Proof is by induction in the structure of formulae ϕ.

Induction hypothesis: If goals$(D, \phi) = \emptyset$ and goals$(D', \phi) = \{M\}$ for some M, then there is $l \in M$ such that $\bar{l} \in D \backslash D'$.

Base cases 1 & 2, $\phi = \top$ and 2 $\phi = \bot$: Trivial as the condition cannot hold.

Base case 3, $\phi = a$: If goals$(D, a) = \emptyset$ and goals$(D', a) = M = \{\{a\}\}$, then respectively $\neg a \in D$ and $\neg a \notin D'$. Hence there is $a \in M$ such that $\overline{a} \in D \backslash D'$.

Inductive case 1, $\phi = \neg\neg\phi'$: By the induction hypothesis as goals$(D, \neg\neg\phi') =$ goals(D, ϕ').

Inductive case 2, $\phi = \phi_1 \vee \phi_2$: Assume goals$(D, \phi_1 \vee \phi_2) = \emptyset$ and goals$(D', \phi_1 \vee \phi_2) = \{M\}$ for some M. Hence goals$(D, \phi_1) = \emptyset$ and goals$(D, \phi_2) = \emptyset$, and goals$(D', \phi_1) = \{M\}$ or goals$(D', \phi_2) = \{M\}$. Hence by the induction hypothesis with ϕ_1 or ϕ_2 there is $l \in M$ such that $\bar{l} \in D \backslash D'$.

Inductive case 3, $\phi = \phi_1 \wedge \phi_2$: Assume goals$(D, \phi_1 \wedge \phi_2) = \emptyset$ and goals$(D', \phi_1 \wedge \phi_2) = \{M\}$ for some M. Hence goals$(D, \phi_1) = \emptyset$ or goals$(D, \phi_2) = \emptyset$, and goals$(D', \phi_1) = \{L_1\}$ and goals$(D', \phi_2) = \{L_2\}$ for some L_1 and L_2 such that $M = L_1 \cup L_2$. Hence by the induction hypothesis with ϕ_1 or ϕ_2 there is either $l \in L_1$ or $l \in L_2$ such that $\bar{l} \in D \backslash D'$.

Inductive cases $\phi = \neg(\phi_1 \wedge \phi_2)$ and $\phi = \neg(\phi_1 \vee \phi_2)$ are analogous to cases 2 and 3. □

Definition 3.34 *Define $\delta_I^{rlx}(\phi) =$ relaxedplan(A, I, O, ϕ).*

Like the sum heuristic, the relaxed plan heuristic gives higher distance estimates than the max heuristic.

Theorem 3.35 *Let ϕ be a formula and $\delta_I^{max}(\phi)$ the max-distance defined in terms of asat(D, ϕ). Then $\delta_I^{rlx}(\phi) \geq \delta_I^{max}(\phi)$.*

Proof: We have to show that for any formula G the procedure call *relaxedplan*(A,I,O,G) returns a number $\geq \delta_I^{max}(G)$.

First, the procedure returns ∞ if and only if asat$(D_i^{max}, G) =$ false for all $i \geq 0$. In this case by definition $\delta_I^{max}(G) = \infty$.

1: **procedure** relaxedplan(A,I,O,G);
2: $L := A \cup \{\neg a | a \in A\}$; *(* All literals *)*
3: compute sets D_i^{max} as in Definition 3.20;
4: **if** asat$(D_i^{max}, G) =$ false for all $i \geq 0$ **then return** ∞; *(* Goal not reachable *)*
5: $t := \delta_I^{max}(G)$;
6: $L_{t+1}^G := \emptyset$;
7: $N_{t+1} := \emptyset$;
8: $G_t := G$;
9: **for** $i := t$ **downto** 1 **do**
10: **begin**
11: $L_i^G := (L_{i+1}^G \backslash N_{i+1}) \cup \{l \in M | M \in \text{goals}(D_i^{max}, G_i)\}$; *(* The goal literals *)*
12: $N_i := \{l \in L_i^G | \bar{l} \in D_{i-1}^{max}\}$; *(* Goal literals that become true between $i-1$ and i *)*
13: $T_i :=$ a minimal subset of O so that $N_i \subseteq \{l \in L | o \in T_i, \text{asat}(D_{i-1}^{max}, EPC_l(o))\}$;
14: $G_{i-1} := \bigwedge_{l \in N_i} \bigvee \{EPC_l(o) | o \in T_i\}$; *(* New goal formula *)*
15: **end**
16: **return** $|T_1| + |T_2| + \cdots + |T_t|$;

Figure 3.1: Algorithm for finding a relaxed plan

Otherwise $t = \delta_I^{max}(G)$. Now $t = 0$ if and only if asat$(D_0^{max}, G) =$ true. In this case the procedure returns 0 without iterating the loop starting on line 9.

We show that if $t \geq 1$ then for every $i \in \{1, \ldots, t\}$ the set T_i is non-empty, entailing $|T_1| + \cdots + |T_t| \geq t = \delta_I^{max}(G)$. This is by an induction proof from t to 1.

We use the following auxiliary result. If asat$(D_{i-1}^{max}, G_i) =$ false and asat$(D_i^{max}, G_i) =$ true and $\bar{l} \notin D_i^{max}$ for all $l \in L_i^G$ then T_i is well-defined and $T_i \neq \emptyset$. The proof is as follows.

> By Lemma 3.32 goals$(D_{i-1}^{max}, G_i) = \emptyset$ and goals$(D_i^{max}, G_i) = \{M\}$ for some M. By Lemma 3.33 there is $l \in M$ such that $\bar{l} \in D_{i-1}^{max}$ and hence $N_i \neq \emptyset$. By definition $\bar{l} \in D_{i-1}^{max}$ for all $l \in N_i$. By $N_i \subseteq L_i^G$ and the assumption about L_i^G $\bar{l} \notin D_i^{max}$ for all $l \in N_i$. Hence $\bar{l} \in D_{i-1}^{max} \backslash D_i^{max}$ for all $l \in N_i$. Hence by definition of D_i^{max} for every $l \in N_i$ there is $o \in O$ such that asat$(D_{i-1}^{max}, EPC_l(o))$. Hence there is $T_i \subseteq O$ so that $N_i \subseteq \{l \in L | o \in T_i, \text{asat}(D_{i-1}^{max}, EPC_l(o))\}$ and the value of T_i is defined. As $N_i \neq \emptyset$ also $T_i \neq \emptyset$.

In the induction proof we establish the assumptions of the auxiliary result and then invoke the auxiliary result itself.

Induction hypothesis: For all $j \in \{i, \ldots, t\}$

1. $\bar{l} \notin D_j^{max}$ for all $l \in L_j^G$,
2. asat$(D_j^{max}, G_j) =$ true and asat$(D_{j-1}^{max}, G_j) =$ false, and
3. $T_j \neq \emptyset$.

Base case $i = t$:

1. $\bar{l} \notin D_t^{max}$ for all $l \in L_t^G$ by (2) of Lemma 3.32 because $L_t^G = \{l \in \text{goals}(D_t^{max}, G_t)\}$.
2. As $t = \delta_I^{max}(G_t)$ by definition asat$(D_{t-1}^{max}, G_t) =$ false and asat$(D_t^{max}, G_t) =$ true.

3. By the auxiliary result from the preceding case.

Inductive case $i < t$:

1. We have $\bar{l} \notin D_i^{max}$ for all $l \in L_i^G$ because $L_i^G = (L_{i+1}^G \backslash N_{i+1}) \cup \{l \in \text{goals}(D_i^{max}, G_i)\}$ and by the induction hypothesis $\bar{l} \notin D_{i+1}^{max}$ for all $l \in L_{i+1}^G$ and by (2) of Lemma 3.32 $\bar{l} \notin D_i^{max}$ for all $l \in M$ for $M \in \text{goals}(D_i^{max}, G_i)$.

2. By definition $G_i = \bigwedge_{l \in N_{i+1}} \bigvee \{EPC_l(o) | o \in T_{i+1}\}$. By definition of T_{i+1} for every $l \in N_{i+1}$ there is $o \in T_{i+1}$ such that $\text{asat}(D_i^{max}, EPC_l(o)) = \text{true}$. By definition of $\text{asat}(D_i^{max}, \phi_1 \vee \phi_2)$ and $\text{asat}(D_i^{max}, \phi_1 \wedge \phi_2)$ for ϕ_1 and ϕ_2 also $\text{asat}(D_i^{max}, G_i) = \text{true}$.

   Then we show that $\text{asat}(D_{i-1}^{max}, G_i) = \text{false}$. By definition of D_i^{max}, $\text{asat}(D_{i-1}^{max}, EPC_{\bar{l}}(o)) = \text{false}$ for all $l \in D_i^{max}$ and $o \in O$. Hence $\text{asat}(D_{i-1}^{max}, EPC_l(o)) = \text{false}$ for all $l \in N_{i+1}$ and $o \in O$ because $\bar{l} \in D_i^{max}$. Hence $\text{asat}(D_{i-1}^{max}, EPC_l(o)) = \text{false}$ for all $l \in N_{i+1}$ and $o \in T_{i+1}$ because $T_{i+1} \subseteq O$. By definition $G_i = \bigwedge_{l \in N_{i+1}} \bigvee \{EPC_l(o) | o \in T_{i+1}\}$. Hence by definition of $\text{asat}(D, \phi)$ also $\text{asat}(D_{i-1}^{max}, G_i) = \text{false}$.

3. By the auxiliary result from the preceding case.

□

3.5 Algorithm for computing invariants

Planning with backward search and regression suffers from the following problem. Often only a fraction of all valuations of state variables represent states that are reachable from the initial state and represent possible world states. The goal formula and many of the formulae produced by regression often represent many unreachable states. If the formulae represent only unreachable states a planning algorithm may waste a lot of effort determining that a certain sequence of actions is not the suffix of any plan[1]. Also planning with propositional logic (Section 3.6) suffers from the same problem.

Planning can be made more efficient by restricting search to states that are reachable from the initial state. However, determining whether a given state is reachable from the initial state is PSPACE-complete. Consequently, exact information on the reachability of states could not be used for speeding up the basic forward and backward search algorithms: solving the subproblem would be just as complex as solving the problem itself.

In this section we will present a polynomial time algorithm for computing a class of invariants that approximately characterize the set of reachable states. These invariants help in improving the efficiency of planning algorithms based on backward search and on satisfiability testing in the propositional logic (Section 3.6).

Our algorithm computes invariants that are clauses with at most n literals, for some fixed n. For representing the strongest invariant arbitrarily high n may be needed. Although the runtime is polynomial for any fixed n, the runtimes grow quickly as n increases. However, for many applications short invariants of length $n = 2$ are sufficient, and longer invariants are less important.

[1] A symmetric problem arises with forward search because with progression one may reach states from which goal states are unreachable.

```
 1: procedure preserved(φ,C,o);
 2: φ = l_1 ∨ ··· ∨ l_n for some l_1, ..., l_n and o = ⟨c, e⟩ for some c and e;
 3: for each l ∈ {l_1, ..., l_n} do
 4:   if C ∪ {EPC_l̄(o)} is unsatisfiable then goto OK;        (* l cannot become false. *)
 5:   for each l' ∈ {l_1, ..., l_n}\{l} do   (* Otherwise another literal in φ must be true. *)
 6:     if C ∪ {EPC_l̄(o)} ⊨ EPC_l'(o) then goto OK;          (* l' becomes true. *)
 7:     if C ∪ {EPC_l̄(o)} ⊨ l' ∧ ¬EPC_l̄'(o) then goto OK;    (* l' was and stays true. *)
 8:   end do
 9:   return false;                  (* Truth of the clause could not be guaranteed. *)
10:   OK:
11: end do
12: return true;
```

Figure 3.2: Algorithm that tests whether o may falsify $l_1 \vee \cdots \vee l_n$ in a state satisfying C

The algorithm first computes the set of all 1-literal clauses that are true in the initial state. This set exactly characterizes the set of distance 0 states consisting of the initial state only. Then the algorithm considers the application of every operator. If an operator is applicable it may make some of the clauses false. These clauses are removed and replaced by weaker clauses which are also tested against every operator. When no further clauses are falsified, we have a set of clauses that are guaranteed to be true in all distance 1 states. This computation is repeated for distances 2, 3, and so on, until the clause set does not change. The resulting clauses are invariants because they are true after any number of operator applications.

The flavor of the algorithm is similar to the distance estimation in Section 3.4: starting from a description of what is possible in the initial state, inductively determine what is possible after i operator applications. In contrast to the distance estimation method in Section 3.4 the state sets are characterized by sets of clauses instead of sets of literals.

Let C_i be a set of clauses that characterizes those states that are reachable by i operator applications. Similarly to distance computation, we consider for each operator and for each clause in C_i whether applying the operator may make the clause false. If it can, the clause could be false after i operator applications and therefore will not be in the set C_{i+1}.

Figure 3.2 gives an algorithm that tests whether applying an operator $o \in O$ in some state s may make a formula $l_1 \vee \cdots \vee l_n$ false assuming that $s \models C \cup \{l_1 \vee \cdots \vee l_n\}$.

The algorithm performs a case analysis for every literal in the clause, testing in each case whether the clause remains true: if a literal becomes false, either another literal becomes true simultaneously or another literal was true before and does not become false.

Lemma 3.36 *Let C be a set of clauses, $\phi = l_1 \vee \cdots \vee l_n$ a clause, and o an operator. If preserved(ϕ,C,o) returns* true*, then $app_o(s) \models \phi$ for any state s such that $s \models C \cup \{\phi\}$ and o is applicable in s. (It may under these conditions also return* false*).*

Proof: Assume s is a state such that $s \models C \wedge \phi$, $app_o(s)$ is defined and $app_o(s) \not\models \phi$. We show that the procedure returns *false*.

Since $s \models \phi$ and $app_o(s) \not\models \phi$ at least one literal in ϕ is made false by o. Let $\{l_1^\perp, \ldots, l_m^\perp\} \subseteq \{l_1, \ldots, l_n\}$ be the set of all such literals. Hence $s \models l_1^\perp \wedge \cdots \wedge l_m^\perp$ and $\{\overline{l_1^\perp}, \ldots, \overline{l_m^\perp}\} \subseteq [e]_s^{det}$. The literals in $\{l_1, \ldots, l_n\}\backslash\{l_1^\perp, \ldots, l_m^\perp\}$ are false in s and o does not make them true.

```
 1: procedure invariants(A, I, O, n);
 2: C := {a ∈ A | I ⊨ a} ∪ {¬a | a ∈ A, I ⊭ a};          (* Clauses true in the initial state *)
 3: repeat
 4:   C' := C;
 5:   for each o ∈ O and l_1 ∨ ··· ∨ l_m ∈ C such that not preserved(l_1 ∨ ··· ∨ l_m,C',o) do
 6:     C := C\{l_1 ∨ ··· ∨ l_m};
 7:     if m < n then                                    (* Clause length within pre-defined limit. *)
 8:       begin                                          (* Add weaker clauses. *)
 9:         C := C ∪ {l_1 ∨ ··· ∨ l_m ∨ a | a ∈ A, {a, ¬a} ∩ {l_1, ..., l_m} = ∅};
10:         C := C ∪ {l_1 ∨ ··· ∨ l_m ∨ ¬a | a ∈ A, {a, ¬a} ∩ {l_1, ..., l_m} = ∅};
11:       end
12:   end do
13: until C = C';
14: return C;
```

Figure 3.3: Algorithm for computing a set of invariant clauses

Choose any $l \in \{l_1^\perp, \ldots, l_m^\perp\}$. We show that when the outermost *for each* loop starting on line 3 considers l the procedure will return *false*.

Since $\overline{l} \in [e]_s^{det}$ and o is applicable in s by Lemma 3.3 $s \models EPC_{\overline{l}}(o)$. Since by assumption $s \models C$, the condition of the *if* statement on line 4 is not satisfied and the execution proceeds by iteration of the inner *for each* loop.

Let l' be any of the literals in ϕ except l. Since $app_o(s) \not\models \phi$, $l' \notin [e]_s^{det}$. Hence by Lemma 3.3 $s \not\models EPC_{l'}(o)$, and as $s \models C \cup \{EPC_{\overline{l}}(o)\}$ the condition of the *if* statement on line 6 is not satisfied and the execution continues from line 7. Analyze two cases.

1. If $l' \in \{l_1^\perp, \ldots, l_m^\perp\}$ then by assumption $\overline{l'} \in [e]_s^{det}$ and by Lemma 3.3 $s \models EPC_{\overline{l'}}(o)$. Hence $C \cup \{EPC_{\overline{l}}(o)\} \not\models \neg EPC_{\overline{l'}}(o)$ and the condition of the *if* statement on line 7 is not satisfied.

2. If $l' \notin \{l_1^\perp, \ldots, l_m^\perp\}$ then $s \not\models l'$. Hence $C \cup \{EPC_{\overline{l}}(o)\} \not\models l'$ and the condition of the *if* statement on line 7 is not satisfied.

Hence on none of the iterations of the inner *for each* loop is a *goto OK* executed, and as the loop exits, the procedure returns *false*. □

Figure 3.3 gives the algorithm for computing invariants consisting of at most n literals. The loop on line 5 is repeated until there are no $o \in O$ and clauses ϕ in C such that preserved(ϕ,C',o) returns false. This exit condition for the loop is critical for the correctness proof.

Theorem 3.37 *Let A be a set of state variables, I a state, O a set of operators, and $n \geq 1$ an integer. Then the procedure call invariants(A, I, O, n) returns a set C of clauses with at most n literals so that for any sequence $o_1; \ldots; o_m$ of operators from O $app_{o_1;\ldots;o_m}(I) \models C$.*

Proof: Let C_0 be the value first assigned to the variable C in the procedure *invariants*, and $C_1, C_2, \ldots$ be the values of the variable in the end of each iteration of the outermost *repeat* loop.

Induction hypothesis: for every $\{o_1, \ldots, o_i\} \subseteq O$ and $\phi \in C_i$, $app_{o_1;\ldots;o_i}(I) \models \phi$.

Base case $i = 0$: $app_\epsilon(I)$ for the empty sequence is by definition I itself, and by construction C_0 consists of only formulae that are true in the initial state.

Inductive case $i \geq 1$: Take any $\{o_1, \ldots, o_i\} \subseteq O$ and $\phi \in C_i$. First notice that preserved(ϕ,C_i,o) returns *true* because otherwise ϕ could not be in C_i. Analyze two cases.

1. If $\phi \in C_{i-1}$, then by the induction hypothesis $app_{o_1;\ldots;o_{i-1}}(I) \models \phi$. Since $\phi \in C_i$ preserved(ϕ,C_{i-1},o) returns *true*. Hence by Lemma 3.36 $app_{o_1;\ldots;o_i}(I) \models \phi$.

2. If $\phi \notin C_{i-1}$, it must be because preserved(ϕ',C_{i-1},o') returns *false* for some $o' \in O$ and $\phi' \in C_{i-1}$ such that ϕ is obtained from ϕ' by conjoining some literals to it. Hence $\phi' \models \phi$.

   Since $\phi' \in C_{i-1}$ by the induction hypothesis $app_{o_1;\ldots;o_{i-1}}(I) \models \phi'$. Since $\phi' \models \phi$ also $app_{o_1;\ldots;o_{i-1}}(I) \models \phi$. Since the function call preserved(ϕ,C_i,o) returns *true* by Lemma 3.36 $app_{o_1;\ldots;o_i}(I) \models \phi$.

This finishes the induction proof. The iteration of the procedure stops when $C_i = C_{i-1}$, meaning that the claim of the theorem holds for arbitrarily long sequences $o_1; \ldots; o_m$ of operators. □

The algorithm does not find the strongest invariant for two reasons. First, only clauses until some fixed length are considered. Expressing the strongest invariant may require clauses that are longer. Second, the test performed by *preserved* tries to prove for one of the literals in the clause that it is true after an operator application. Consider the clause $a \vee b \vee c$ and the operator $\langle b \vee c, \neg a \rangle$. We cannot show for any literal that it is true after applying the operator but we know that either b or c is true. The test performed by *preserved* could be strengthened to handle cases like these, for example by using the techniques discussed in Section 4.2, but this would make the computation more expensive and eventually lead to intractability.

To make the algorithm run in polynomial time the satisfiability and logical consequence tests should be performed by algorithms that approximate these tests in polynomial time. The procedure asat(D, ϕ) is not suitable because it assumes that D is a set of literals, whereas for *preserved* the set C usually contain clauses with 2 or more literals. There are generalizations of the ideas behind asat(D, ϕ) to this more general case but we do not discuss the topic further.

3.5.1 Applications of invariants in planning by regression and satisfiability

Invariants can be used to speed up backward search with regression. Consider the blocks world with the goal *AonB*∧*BonC*. Regression with the operator that moves B onto C from the table yields *AonB* ∧ *Bclear* ∧ *Cclear* ∧ *BonT*. This formula does not correspond to an intended blocks world state because *AonB* is incompatible with *Bclear*, and indeed, ¬*AonB* ∨ ¬*Bclear* is an invariant for the blocks world. Any regression step that leads to a formula that is incompatible with the invariants can be ignored because that formula does not represent any state that is reachable from the initial state, and hence no plan extending the current incomplete plan can reach the goals.

Another application of invariants and the intermediate sets C_i produced by our invariant algorithm is improving the heuristics in Section 3.4. Using D_i^{max} for testing whether an operator precondition, for example $a \wedge b$, has distance i from the initial state, the distances of a and b are used separately. But even when it is possible to reach both a and b with i operator applications, it might still not be possible to reach them both simultaneously with i operator applications. For example, for $i = 1$ and an initial state in which both a and b are false, there might be no single operator that makes them both true, but two operators, each of which makes only one of them true. If $\neg a \vee \neg b \in C_i$, we know that after i operator applications one of a or b must still be false, and then we know that the operator in question is not applicable at time point i. Therefore the invariants and the sets C_i produced during the invariant computation can improve distance estimates.

3.6 Planning as satisfiability in the propositional logic

A very powerful approach to deterministic planning was introduced in 1992 by Kautz and Selman [1992; 1996]. In this approach the problem of reachability of a goal state from a given initial state is translated into propositional formulae $\phi_0, \phi_1, \phi_2, \ldots$ so that every valuation that satisfies formula ϕ_i corresponds to a plan of length i. Planning proceeds by first testing the satisfiability of ϕ_0. If ϕ_0 is unsatisfiable, continue with ϕ_1, ϕ_2, and so on, until a satisfiable formula ϕ_n is found. From a valuation that satisfies ϕ_n a plan of length n can be constructed.

3.6.1 Actions as propositional formulae

First we need a representation of actions in the propositional logic. We can view arbitrary propositional formulae as actions, or we can translate operators into formulae in the propositional logic. We discuss both of these possibilities.

Given a set of state variables $A = \{a_1, \ldots, a_n\}$, one could describe an action directly as a propositional formula ϕ over propositional variables $A \cup A'$ where $A' = \{a'_1, \ldots, a'_n\}$. Here the variables A represent the values of state variables in the state s in which an action is taken, and variables A' the values of state variables in a successor state s'.

A pair of valuations s and s' can be understood as a valuation of $A \cup A'$ (the state s assigns a value to variables A and s' to variables A'), and a transition from s to s' is possible if and only if $s, s' \models \phi$.

Example 3.38 The action that reverses the values of state variables a_1 and a_2 is described by $\phi = (a_1 \leftrightarrow \neg a'_1) \wedge (a_2 \leftrightarrow \neg a'_2)$. The following 4×4 incidence matrix represents this action.

	$a'_1a'_2$	$a'_1a'_2$	$a'_1a'_2$	$a'_1a'_2$
a_1a_2	00	01	10	11
00	0	0	0	1
01	0	0	1	0
10	0	1	0	0
11	1	0	0	0

The matrix can be equivalently represented as the following truth-table.

a_1	a_2	a'_1	a'_2	ϕ
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	1
0	1	1	1	0
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

■

Example 3.39 Let the set of state variables be $A = \{a_1, a_2, a_3\}$. The formula $(a_1 \leftrightarrow a'_2) \wedge (a_2 \leftrightarrow a'_3) \wedge (a_3 \leftrightarrow a'_1)$ represents the action that rotates the values of the state variables a_1, a_2 and a_3 one position right. The formula can be represented as the following adjacency matrix. The rows correspond to valuations of A and the columns to valuations of $A' = \{a'_1, a'_2, a'_3\}$.

	000	001	010	011	100	101	110	111
000	1	0	0	0	0	0	0	0
001	0	0	0	0	1	0	0	0
010	0	1	0	0	0	0	0	0
011	0	0	0	0	0	1	0	0
100	0	0	1	0	0	0	0	0
101	0	0	0	0	0	0	1	0
110	0	0	0	1	0	0	0	0
111	0	0	0	0	0	0	0	1

A more conventional way of depicting the valuations of this formula would be as a truth-table with one row for every valuation of $A \cup A'$, a total of 64 rows. ■

The action in Example 3.39 is deterministic. Not all actions represented by propositional formulae are deterministic. A sufficient (but not necessary) condition for determinism is that the formula is of the form $(\phi_1 \leftrightarrow a'_1) \wedge \cdots \wedge (\phi_n \leftrightarrow a'_n) \wedge \psi$ where $A = \{a_1, \ldots, a_n\}$ is the set of all state variables, ϕ_i are formulae over A (without occurrences of $A' = \{a'_1, \ldots, a'_n\}$). There are no restrictions on ψ. Formulae of this form uniquely determine the value of every state variable in the successor state in terms of the values in the predecessor state. Therefore they represent deterministic actions.

3.6.2 Translation of operators into propositional logic

We first give the simplest possible translation of deterministic planning into the propositional logic. In this translation every operator is separately translated into a formula, and the choice between the operators is represented as disjunction.

Definition 3.40 *The formula $\tau_A(o)$ that represents operator $o = \langle c, e\rangle$ is defined by*

$$\tau_A(e) = \bigwedge_{a\in A}((EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))) \leftrightarrow a') \wedge \bigwedge_{a\in A} \neg(EPC_a(e) \wedge EPC_{\neg a}(e))$$
$$\tau_A(o) = c \wedge \tau_A(e).$$

The formula $\tau_A(e)$ expresses the value of a in the successor state in terms of the values of the state variables in the predecessor state and requires that executing e may not make any state variable simultaneously true and false. This is like in the definition of regression in Section 3.1.2. The formula $\tau_A(o)$ additionally requires that the operator's precondition is true.

Example 3.41 Consider operator $\langle a \vee b, (b \rhd a) \wedge (c \rhd \neg a) \wedge (a \rhd b)\rangle$. The corresponding propositional formula is

$$\begin{aligned}
& (a \vee b) \wedge((b \vee (a \wedge \neg c)) \leftrightarrow a') \\
& \qquad \wedge((a \vee (b \wedge \neg\bot)) \leftrightarrow b') \\
& \qquad \wedge((\bot \vee (c \wedge \neg\bot)) \leftrightarrow c') \\
& \qquad \wedge\neg(b \wedge c) \wedge \neg(a \wedge \bot) \wedge \neg(\bot \wedge \bot) \\
\equiv\ & (a \vee b) \wedge((b \vee (a \wedge \neg c)) \leftrightarrow a') \\
& \qquad \wedge((a \vee b) \leftrightarrow b') \\
& \qquad \wedge(c \leftrightarrow c') \\
& \qquad \wedge\neg(b \wedge c).
\end{aligned}$$

■

Lemma 3.42 *Let s and s' be states and o an operator. Let $v : A \cup A' \to \{0, 1\}$ be a valuation such that*

1. *for all $a \in A$, $v(a) = s(a)$, and*
2. *for all $a \in A$, $v(a') = s'(a)$.*

Then $v \models \tau_A(o)$ if and only if $s' = app_o(s)$.

Proof: Assume $v \models \tau_A(o)$. Hence $s \models c$ and $s \models \bigwedge_{a\in A} \neg(EPC_a(e) \wedge EPC_{\neg a}(e))$, and therefore $app_o(s)$ is defined. Consider any state variable $a \in A$. By Lemma 3.4 and the assumption $v \models (EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))) \leftrightarrow a'$, the value of every state variable in s' matches the definition of $app_o(s)$. Hence $s' = app_o(s)$.

Assume $s' = app_o(s)$. Since s' is defined, $v \models \tau_A(o)$ and $v \models \bigwedge_{a\in A} \neg(EPC_a(e) \wedge EPC_{\neg a}(e))$. By Lemma 3.4 $v \models EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$ if and only if $s' \models a$. □

Definition 3.43 *Define $\mathcal{R}_1(A, A') = \tau_A(o_1) \vee \cdots \vee \tau_A(o_n)$.*

The valuations that satisfy this formula do not uniquely determine which operator was applied because for a given state more than one operator may produce the same successor state. However, in such cases it does not matter which operator is applied, and when constructing a plan from the valuation any of the operators may be chosen arbitrarily.

It has been noticed that extending $\mathcal{R}_1(A, A')$ by 2-literal invariants (see Section 3.5) reduces runtimes of algorithms that test satisfiability. Notice that invariants do not affect the set of models of a formula representing planning: any satisfying valuation of the original formula also satisfies the invariants because the values of variables describing the values of state variables at any time point corresponds to a state that is reachable from the initial state, and hence this valuation also satisfies any invariant.

3.6.3 Finding plans by satisfiability algorithms

We show how plans can be found by first translating succinct transition systems $\langle A, I, O, G\rangle$ into propositional formulae, and then finding satisfying valuations by a satisfiability algorithm.

In Section 3.6.1 we showed how operators can be described by propositional formulae over sets A and A' of propositional variables, the set A describing the values of the state variables in the state in which the operator is applied, and the set A' describing the values of the state variables in the successor state of that state.

For a fixed plan length n, we use sets $A^0, \ldots, A^n$ of variables to represent the values of state variables at different time points, with variables A^i representing the values at time i. In other words, a valuation of these propositional variables represents a sequence $s_0, \ldots, s_n$ of states. If $a \in A$ is a state variable, then we use the propositional variable a^i for representing the value of a at time point i.

Then we construct a formula so that the state s_0 is determined by I, the state s_n is determined by G, and the changes of state variables between any two consecutive states corresponds to the application of an operator.

Definition 3.44 *Let $\langle A, I, O, G\rangle$ be a deterministic transition system. Define $\iota^0 = \bigwedge\{a^0 | a \in A, I(a) = 1\} \cup \{\neg a^0 | a \in A, I(a) = 0\}$ for the initial state and G^n as the formula G with every variable $a \in A$ replaced by a^n. Define*

$$\Phi_n^{seq} = \iota^0 \wedge \mathcal{R}_1(A^0, A^1) \wedge \mathcal{R}_1(A^1, A^2) \wedge \cdots \wedge \mathcal{R}_1(A^{n-1}, A^n) \wedge G^n$$

where $A^i = \{a^i | a \in A\}$ for all $i \in \{0, \ldots, n\}$.

A plan can be found by using the formulae Φ_i^{seq} as follows. We start with plan length $i = 0$, test the satisfiability of Φ_i^{seq}, and depending on the result, either construct a plan (if Φ_i^{seq} is satisfiable), or increase i by one and repeat the previous steps, until a plan is found.

If there are no plans, it has to be somehow decided when to stop increasing i. An upper bound on plan length is $2^{|A|} - 1$ where A is the set of state variables but this upper bound does not provide a practical termination condition for this procedure. Some work on more practical termination conditions are cited in Section 3.8.

The construction of a plan from a valuation v that satisfies Φ_i^{seq} is straightforward. The plan has exactly i operators, and this plan is known to be the shortest one because the formula Φ_{i-1}^{seq} had already been determined to be unsatisfiable. First construct the execution $s_0, \ldots, s_i$ of the plan from v as follows. For all $j \in \{0, \ldots, i\}$ and $a \in A$, $s_j(a) = v(a_j)$. The plan has the

form $o_1, \ldots, o_i$. Operator o_j for $j \in \{1, \ldots, i\}$ is identified by testing for all $o \in O$ whether $app_o(s_{j-1}) = s_j$. There may be several operators satisfying this condition, and any of them can be chosen.

Example 3.45 Let $A = \{a, b\}$. Let the state I satisfy $I \models a \wedge b$. Let $G = (a \wedge \neg b) \vee (\neg a \wedge b)$ and $o_1 = \langle \top, (a \rhd \neg a) \wedge (\neg a \rhd a) \rangle$ and $o_2 = \langle \top, (b \rhd \neg b) \wedge (\neg b \rhd b) \rangle$. The following formula is satisfiable if and only if $\langle A, I, \{o_1, o_2\}, G \rangle$ has a plan of length 3.

$$
\begin{array}{l}
(a^0 \wedge b^0) \\
\wedge(((a^0 \leftrightarrow a^1) \wedge (b^0 \leftrightarrow \neg b^1)) \vee ((a^0 \leftrightarrow \neg a^1) \wedge (b^0 \leftrightarrow b^1))) \\
\wedge(((a^1 \leftrightarrow a^2) \wedge (b^1 \leftrightarrow \neg b^2)) \vee ((a^1 \leftrightarrow \neg a^2) \wedge (b^1 \leftrightarrow b^2))) \\
\wedge(((a^2 \leftrightarrow a^3) \wedge (b^2 \leftrightarrow \neg b^3)) \vee ((a^2 \leftrightarrow \neg a^3) \wedge (b^2 \leftrightarrow b^3))) \\
\wedge((a^3 \wedge \neg b^3) \vee (\neg a^3 \wedge b^3))
\end{array}
$$

One of the valuations that satisfy the formula is the following.

	time i			
	0	1	2	3
a^i	1	0	0	0
b^i	1	1	0	1

This valuation corresponds to the plan that applies operator o_1 at time point 0, o_2 at time point 1, and o_2 at time point 2. There are also other satisfying valuations. The shortest plans have length 1 and respectively consist of the operators o_1 and o_2. ■

Example 3.46 Consider the following problem. There are two operators, one for rotating the values of bits abc one step right, and the other for inverting the values of all the bits. Consider reaching from the initial state 100 the goal state 001 with two actions. This is represented as the following formula.

$$
\begin{array}{l}
(a^0 \wedge \neg b^0 \wedge \neg c^0) \\
\wedge(((a^0 \leftrightarrow b^1) \wedge (b^0 \leftrightarrow c^1) \wedge (c^0 \leftrightarrow a^1)) \vee ((\neg a^0 \leftrightarrow a_1) \wedge (\neg b^0 \leftrightarrow b^1) \wedge (\neg c^0 \leftrightarrow c^1))) \\
\wedge(((a^1 \leftrightarrow b^2) \wedge (b^1 \leftrightarrow c^2) \wedge (c^1 \leftrightarrow a^2)) \vee ((\neg a^1 \leftrightarrow a^2) \wedge (\neg b^1 \leftrightarrow b^2) \wedge (\neg c^1 \leftrightarrow c^2))) \\
\wedge(\neg a^2 \wedge \neg b^2 \wedge c^2)
\end{array}
$$

Since the literals describing the initial and the goal state must be true, we can replace occurrences of these state variables in the subformulae for operators by $\top$ and $\bot$.

$$
\begin{array}{l}
(a^0 \wedge \neg b^0 \wedge \neg c^0) \\
\wedge(((\top \leftrightarrow b^1) \wedge (\bot \leftrightarrow c^1) \wedge (\bot \leftrightarrow a^1)) \vee ((\neg \top \leftrightarrow a_1) \wedge (\neg \bot \leftrightarrow b^1) \wedge (\neg \bot \leftrightarrow c^1))) \\
\wedge(((a^1 \leftrightarrow \bot) \wedge (b^1 \leftrightarrow \top) \wedge (c^1 \leftrightarrow \bot)) \vee ((\neg a^1 \leftrightarrow \bot) \wedge (\neg b^1 \leftrightarrow \bot) \wedge (\neg c^1 \leftrightarrow \top))) \\
\wedge(\neg a^2 \wedge \neg b^2 \wedge c^2)
\end{array}
$$

After simplifying we have the following.

$$
\begin{array}{l}
(a^0 \wedge \neg b^0 \wedge \neg c^0) \\
\wedge((b^1 \wedge \neg c^1 \wedge \neg a^1) \vee (\neg a_1 \wedge b^1 \wedge c^1) \\
\wedge((\neg a^1 \wedge b^1 \wedge \neg c^1) \vee (a^1 \wedge b^1 \wedge \neg c^1)) \\
\wedge(\neg a^2 \wedge \neg b^2 \wedge c^2)
\end{array}
$$

The only way of satisfying this formula is to make the first disjuncts of both disjunctions true, that is, b^1 must be true and a^1 and c^1 must be false. The resulting valuation corresponds to taking the rotation action twice.

Consider the same problem but now with the goal state 101.

$$
\begin{array}{l}
(a^0 \wedge \neg b^0 \wedge \neg c^0) \\
\wedge(((a^0 \leftrightarrow b^1) \wedge (b^0 \leftrightarrow c^1) \wedge (c^0 \leftrightarrow a^1)) \vee ((\neg a^0 \leftrightarrow a_1) \wedge (\neg b^0 \leftrightarrow b^1) \wedge (\neg c^0 \leftrightarrow c^1))) \\
\wedge(((a^1 \leftrightarrow b^2) \wedge (b^1 \leftrightarrow c^2) \wedge (c^1 \leftrightarrow a^2)) \vee ((\neg a^1 \leftrightarrow a^2) \wedge (\neg b^1 \leftrightarrow b^2) \wedge (\neg c^1 \leftrightarrow c^2))) \\
\wedge(a^2 \wedge \neg b^2 \wedge c^2)
\end{array}
$$

We simplify again and get the following formula.

$$
\begin{array}{l}
(a^0 \wedge \neg b^0 \wedge \neg c^0) \\
\wedge((b^1 \wedge \neg c^1 \wedge \neg a^1) \vee (\neg a_1 \wedge b^1 \wedge c^1)) \\
\wedge((\neg a^1 \wedge b^1 \wedge c^1) \vee (\neg a^1 \wedge b^1 \wedge \neg c^1)) \\
\wedge(a^2 \wedge \neg b^2 \wedge c^2)
\end{array}
$$

Now there are two possible plans, to rotate first and then invert the values, or first invert and then rotate. These respectively correspond to making the first disjunct of the first disjunction and the second disjunct of the second disjunction true, or the second and the first disjunct. ■

3.6.4 Parallel application of operators

For states s and sets T of operators we define $app_T(s)$ as the result of simultaneously applying all operators $o \in T$: the preconditions of all operators in T must be true in s and the state $app_T(s)$ is obtained from s by making the literals in $\bigcup_{\langle p,e\rangle \in T} [e]_s^{det}$ true. Analogously to sequential plans we can define $app_{T_1;T_2;\ldots;T_n}(s)$ as $app_{T_n}(\cdots app_{T_2}(app_{T_1}(s))\cdots)$.

Next we show how the translation of deterministic operators into the propositional logic in Section 3.6.2 can be extended to the simultaneous application of operators as in $app_T(s)$.

Consider the formula $\tau_A(o)$ representing one operator $o = \langle c, e\rangle$.

$$
c \wedge \bigwedge_{a \in A} ((EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))) \leftrightarrow a') \wedge \bigwedge_{a \in A} \neg(EPC_a(e) \wedge EPC_{\neg a}(e)).
$$

This formula can be rewritten to the following logically equivalent formula that separately says which state variables are changed by the operator and which state variables retain their values.

$$
\begin{array}{l}
c\wedge \\
\bigwedge_{a \in A}(EPC_a(e) \rightarrow a')\wedge \\
\bigwedge_{a \in A}(EPC_{\neg a}(e) \rightarrow \neg a')\wedge \\
\bigwedge_{a \in A}((a \wedge \neg a') \rightarrow EPC_{\neg a}(e))\wedge \\
\bigwedge_{a \in A}((\neg a \wedge a') \rightarrow EPC_a(e))
\end{array}
$$

We use this formulation of $\tau_A(o)$ as basis of obtaining encodings of planning that allow *several operators in parallel*. Every operator applied at a given time point causes its effects to be true and requires its precondition to be true. This is expressed by the first three conjuncts. The last two conjuncts say that, assuming the operator that is applied is the only one, certain state variables retain their value. These formulae have to be modified to accommodate the possibility of executing several operators in parallel.

We introduce propositional variables o for denoting the execution of operators $o \in O$.

Definition 3.47 *Let A be the set of state variables and O a set of operators. Let the formula $\tau_A(O)$ denote the conjunction of formulae*

$$(o \rightarrow c) \wedge$$
$$\bigwedge_{a \in A} (o \wedge \mathit{EPC}_a(e) \rightarrow a') \wedge$$
$$\bigwedge_{a \in A} (o \wedge \mathit{EPC}_{\neg a}(e) \rightarrow \neg a')$$

for all $\langle c, e \rangle \in O$ and

$$\bigwedge_{a \in A} ((a \wedge \neg a') \rightarrow ((o_1 \wedge \mathit{EPC}_{\neg a}(e_1)) \vee \cdots \vee (o_n \wedge \mathit{EPC}_{\neg a}(e_n))) \wedge$$
$$\bigwedge_{a \in A} ((\neg a \wedge a') \rightarrow ((o_1 \wedge \mathit{EPC}_a(e_1)) \vee \cdots \vee (o_n \wedge \mathit{EPC}_a(e_n))))$$

where $O = \{o_1, \ldots, o_n\}$ and $e_1, \ldots, e_n$ are the respective effects.

The difference to the definition of $\tau_A(o)$ in Section 3.6.2 is that above the formulae do not assume that there is only one operator explaining the changes that take place.

The formula $\tau_A(O)$ matches the definition of $\mathit{app}_T(s)$.

Lemma 3.48 *Let s and s' be states and O and $T \subseteq O$ sets of operators. Let $v : A \cup A' \cup O \rightarrow \{0, 1\}$ be a valuation such that*

1. *for all $o \in O$, $v(o) = 1$ iff $o \in T$,*
2. *for all $a \in A$, $v(a) = s(a)$, and*
3. *for all $a \in A$, $v(a') = s'(a)$.*

Then $v \models \tau_A(O)$ if and only if $s' = \mathit{app}_T(s)$.

Proof: For the proof from right to left we assume that $s' = \mathit{app}_T(s)$ and show that $v \models \tau_A(O)$.

For the formulae $o \rightarrow c$ consider any $o = \langle c, e \rangle \in O$. If $o \notin T$ then $v \not\models o$ and $v \models o \rightarrow c$. So assume $o \in T$. By assumption s is a state such that $\mathit{app}_T(s)$ is defined. Hence $s \models c$. Hence $v \models o \rightarrow c$.

For the formulae $o \wedge \mathit{EPC}_a(e) \rightarrow a'$ consider any $o = \langle c, e \rangle \in O$. If $o \notin T$ then $v \not\models o$ and $v \models o \wedge \mathit{EPC}_l(e) \rightarrow l$ for all literals l. So assume $o \in T$. Now $v \models o \wedge \mathit{EPC}_l(e) \rightarrow l$ because if $s \models \mathit{EPC}_l(e)$ then $l \in [e]_s^{det}$ by Lemma 3.3 and $s' \models l$. Proof for $o \wedge \mathit{EPC}_{\neg a}(e) \rightarrow \neg a'$ is analogous.

For the formulae $((a \wedge \neg a') \rightarrow ((o_1 \wedge \mathit{EPC}_{\neg a}(e_1)) \vee \cdots \vee (o_n \wedge \mathit{EPC}_{\neg a}(e_n))$ consider any $a \in A$. According to the definition of $s' = \mathit{app}_T(s)$, a can be true in s and false in s' only if $\neg a \in [o]_s^{det}$ for some $o \in T$. By Lemma 3.3 $\neg a \in [o]_s^{det}$ if and only if $s \models \mathit{EPC}_{\neg a}(o)$. So if the antecedent of $(a \wedge \neg a') \rightarrow ((o_1 \wedge \mathit{EPC}_{\neg a}(o_1)) \vee \cdots \vee (o_m \wedge \mathit{EPC}_{\neg a}(o_m)))$ is true, then one of the disjuncts of the consequent is true, where $O = \{o_1, \ldots, o_m\}$. The proof for the change from false to true is analogous.

For the proof from left to right we assume $v \models \tau_A(O)$ and show that $s' = \mathit{app}_T(s)$.

The precondition c of every $o \in T$ is true in s because $v \models o$ and $v \models o \rightarrow c$, and $s' \models [e]_s^{det}$ for every $o = \langle c, e \rangle \in T$ because $v \models o$ and $v \models o \wedge \mathit{EPC}_l(e) \rightarrow l$ for every literal l. This also means that $[T]_s^{det}$ is consistent and $\mathit{app}_T(s)$ is defined.

For state variables a not occurring in $[T]_s^{det}$ we have to show that $s(a) = s'(a)$. Since a does not occur in $[T]_s^{det}$, for every $o \in \{o_1, \ldots, o_m\} = O = \{\langle c_1, e_1 \rangle, \ldots, \langle c_m, e_m \rangle\}$ either $o \notin T$ or both

$a \notin [e]_s^{det}$ and $\neg a \notin [e]_s^{det}$. Hence either $v \not\models o$ or (by Lemma 3.3) $v \models \neg(EPC_a(e)) \wedge \neg EPC_{\neg a}(e)$. This together with the assumptions that $v \models (a \wedge \neg a') \rightarrow ((o_1 \wedge EPC_{\neg a}(e_1)) \vee \cdots \vee (o_m \wedge EPC_{\neg a}(e_m)))$ and $v \models (\neg a \wedge a') \rightarrow ((o_1 \wedge EPC_a(o_1)) \vee \cdots \vee (o_m \wedge EPC_a(e_m)))$ implies $v \models (a \rightarrow a') \wedge (\neg a \rightarrow \neg a')$. Therefore every $a \in A$ not occurring in $[T]_s^{det}$ remains unchanged. Hence $s' = app_T(s)$. □

Example 3.49 Let $o_1 = \langle \neg LAMP1, LAMP1 \rangle$ and $o_2 = \langle \neg LAMP2, LAMP2 \rangle$. The application of none, one or both of these operators is described by the following formula.

$$
\begin{array}{l}
(\neg LAMP1 \wedge LAMP1') \rightarrow ((o_1 \wedge \top) \vee (o_2 \wedge \bot)) \\
(LAMP1 \wedge \neg LAMP1') \rightarrow ((o_1 \wedge \bot) \vee (o_2 \wedge \bot)) \\
(\neg LAMP2 \wedge LAMP2') \rightarrow ((o_1 \wedge \bot) \vee (o_2 \wedge \top)) \\
(LAMP2 \wedge \neg LAMP2') \rightarrow ((o_1 \wedge \bot) \vee (o_2 \wedge \bot)) \\
o_1 \rightarrow LAMP1' \\
o_1 \rightarrow \neg LAMP1 \\
o_2 \rightarrow LAMP2' \\
o_2 \rightarrow \neg LAMP2
\end{array}
$$

■

3.6.5 Partially-ordered plans

In this section we consider a more general notion of plans in which several operators can be applied simultaneously. This kind of plans are formalized as sequences of sets of operators. In such a plan the operators are partially ordered because there is no ordering on the operators taking place at the same time point. This notion of plans is useful for two reasons.

First, consider a number of operators that affect and depend on disjoint state variables so that they can be applied in any order. If there are n such operators, there are $n!$ plans that are equivalent in the sense that each leads to the same state. When a satisfiability algorithm shows that there is no plan of length n consisting of these operators, it has to show that none of the $n!$ plans reaches the goals. This may be combinatorially very difficult if n is high.

Second, when several operators can be applied simultaneously, it is not necessary to represent all intermediate states of the corresponding sequential plans: partially-ordered plans require less time points than the corresponding sequential plans. This reduces the number of propositional variables that are needed for representing the planning problem, which may make testing the satisfiability of these formulae much more efficient.

In Section 3.6.4 we have shown how to represent the parallel application of operators in the propositional logic. However, this definition is too loose because it allows plans that cannot be executed.

Example 3.50 The operators $\langle a, \neg b \rangle$ and $\langle b, \neg a \rangle$ may be executed simultaneously resulting in a state satisfying $\neg a \wedge \neg b$, although this state is not reachable by the two operators sequentially. ■

A realistic way of interpreting parallelism in partially ordered plans is that any total ordering of the simultaneous operators is executable and results in the same state in all cases. This is the definition used in planning research so far.

Definition 3.51 (Step plans) *For a set of operators O and an initial state I,* a step plan for O and I *is a sequence $T = \langle T_0, \ldots, T_{l-1} \rangle$ of sets of operators for some $l \geq 0$ such that there is a sequence of states $s_0, \ldots, s_l$ (the execution of T) such that*

1. *$s_0 = I$,*
2. *for all $i \in \{0, \ldots, l-1\}$ and every total ordering $o_1, \ldots, o_n$ of T_i, $app_{o_1;\ldots;o_n}(s_i)$ is defined and equals s_{i+1}.*

Theorem 3.52 *Testing whether a sequence of sets of operators is a step plan is co-NP-hard.*

Proof: The proof is by reduction from the co-NP-complete validity problem TAUT. Let ϕ be any propositional formula. Let $A = \{a_1, \ldots, a_n\}$ be the set of propositional variables occurring in ϕ. Our set of state variables is A. Let $o_z = \langle \phi, \top \rangle$ and $O = \{\langle \top, a_1 \rangle, \ldots, \langle \top, a_n \rangle, o_z\}$. Let s and s' be states such that $s \not\models a$ and $s' \models a$ for all $a \in A$. We show that ϕ is a tautology if and only if $T = \langle O \rangle$ is a step plan for O and s.

Assume ϕ is a tautology. Now for any total ordering $o_0, \ldots, o_n$ of O the state $app_{o_0;\ldots;o_n}(s)$ is defined and equals s' because all preconditions are true in all states and the set of effects of all operators is A (the set is consistent and making the effects true in s yields s'.) Hence T is a step plan.

Assume T is a step plan. Let v be any valuation. We show that $v \models \phi$. Let $O_v = \{\langle \top, a \rangle | a \in A, v \models a\}$. The operators O can be ordered to $o_0, \ldots, o_n$ so that the operators $O_v = \{o_0, \ldots, o_k\}$ precede o_z and $O \backslash (O_v \cup \{o_z\})$ follow o_z. Since T is a step plan, $app_{o_0;\ldots;o_n}(s)$ is defined. Since also $app_{o_0;\ldots;o_k;o_z}(s)$ is defined, the precondition ϕ of o_z is true in $v = app_{o_0;\ldots;o_k}(s)$. Hence $v \models \phi$. Since this holds for any valuation v, ϕ is a tautology. □

To avoid intractability it is better to restrict to a class of step plans that are easy to recognize. One such class is based on the notion of *interference*.

Definition 3.53 (Affect) *Let A be a set of state variables and $o = \langle c, e \rangle$ and $o' = \langle c', e' \rangle$ operators over A. Then o* affects *o' if there is $a \in A$ such that*

1. *a is an atomic effect in e and a occurs in a formula in e' or it occurs negatively in c', or*
2. *$\neg a$ is an atomic effect in e and a occurs in a formula in e' or it occurs positively in c'.*

Definition 3.54 (Interference) *Operators o and o'* interfere *if o affects o' or o' affects o.*

Testing for interference of two operators is easy polynomial time computation. Non-interference not only guarantees that a set of operators is executable in any order, but it also guarantees that the result equals to applying all the operators simultaneously.

Lemma 3.55 *Let s be a state and T a set of operators so that $app_T(s)$ is defined and no two operators interfere. Then $app_T(s) = app_{o_1;\ldots;o_n}(s)$ for any total ordering $o_1, \ldots, o_n$ of T.*

Proof: Let $o_1, \ldots, o_n$ be any total ordering of T. We prove by induction on the length of a prefix of $o_1, \ldots, o_n$ the following statement for all $i \in \{0, \ldots, n-1\}$ by induction on i: $s \models a$ if and only if $app_{o_1;\ldots;o_i}(s) \models a$ for all state variables a occurring in an antecedent of a conditional effect or a precondition of operators $o_{i+1}, \ldots, o_n$.

Base case $i = 0$: Trivial.

Inductive case $i \geq 1$: By the induction hypothesis the antecedents of conditional effects of o_i have the same value in s and in $app_{o_1;\ldots;o_{i-1}}(s)$, from which follows $[o_i]^{det}_s = [o_i]^{det}_{app_{o_1;\ldots;o_{i-1}}(s)}$. Since o_i does not interfere with operators $o_{i+1}, \ldots, o_n$, no state variable occurring in $[o_i]^{det}_s$ occurs in an antecedent of a conditional effect or in the precondition of $o_{i+1}, \ldots, o_n$, that is, these state variables do not change. Since $[o_i]^{det}_s = [o_i]^{det}_{app_{o_1;\ldots;o_{i-1}}(s)}$ this also holds when o_i is applied in $app_{o_1;\ldots;o_{i-1}}(s)$. This completes the induction proof.

Since $app_T(s)$ is defined, the precondition of every $o \in T$ is true in s and $[o]^{det}_s$ is consistent. By the fact we established above, the precondition of every $o \in T$ is true also in $app_{o_1;\ldots;o_k}(s)$ and $[o]^{det}_{app_{o_1;\ldots;o_k}(s)}$ is consistent for any $\{o_1, \ldots, o_k\} \subseteq T \backslash \{o\}$. Hence any total ordering of the operators is executable. By the fact we established above, $[o]^{det}_s = [o]^{det}_{app_{o_1;\ldots;o_k}(s)}$ for every $\{o_1, \ldots, o_k\} \subseteq T \backslash \{o\}$. Hence every operator causes the same changes no matter what the total ordering is. Since $app_T(s)$ is defined, no operator in T undoes the effects of another operator. Hence the same state $s' = app_T(s)$ is reached in every case. □

For finding plans by using the translation of parallel actions from Section 3.6.4 it remains to encode the condition that no two parallel actions are allowed to interfere.

Definition 3.56 *Define*

$$\mathcal{R}_2(A, A', O) = \tau_A(O) \wedge \bigwedge \{\neg(o \wedge o') | \{o, o'\} \subseteq O, o \neq o', o \text{ and } o' \text{interfere}\}$$

Definition 3.57 *Let $\langle A, I, O, G\rangle$ be a deterministic succinct transition system. Define*

$$\Phi^{par}_n = \iota^0 \wedge \mathcal{R}_2(A^0, A^1, O^0) \wedge \mathcal{R}_2(A^1, A^2, O^1) \wedge \cdots \wedge \mathcal{R}_2(A^{n-1}, A^n, O^{n-1}) \wedge G^n$$

where $A^i = \{a^i | a \in A\}$ for all $i \in \{0, \ldots, n\}$ and $O^i = \{o^i | o \in O\}$ for all $i \in \{1, \ldots, n\}$ and $\iota^0 = \bigwedge\{a^0 | a \in A, I(a) = 1\} \cup \{\neg a^0 | a \in A, I(a) = 0\}$ and G^n is G with every $a \in A$ replaced by a^n.

If Φ^{par}_n is satisfiable and v is a valuation such that $v \models \Phi^{par}_n$, then define $T_i = \{o \in O | v \models o^i\}$ for every $i \in \{1, \ldots, n\}$. Then $\langle T_1, \ldots, T_n\rangle$ is a plan for the transition system, that is, $app_{T_1;\ldots;T_n}(I) \models G$.

It may be tempting to think that non-interference implies that the actions occurring in parallel in a plan could always be executed simultaneously in the real world. This however is not the case. For genuine temporal parallelism the formalization of problems as operators has to fulfill much stronger criteria than when sequential execution is assumed.

Example 3.58 Consider the operators

$$\begin{aligned}\text{transport-A-with-truck-1} &= \langle \text{AinFreiburg}, \text{AinStuttgart} \wedge \neg\text{AinFreiburg}\rangle \\ \text{transport-B-with-truck-1} &= \langle \text{BinFreiburg}, \text{BinKarlsruhe} \wedge \neg\text{BinFreiburg}\rangle\end{aligned}$$

which formalize the transportation of two objects with one vehicle. The operators do not interfere, and our notion of plans allows the simultaneous execution of these operators. However, these actions cannot really be simultaneous because the corresponding real world actions involve the same vehicle going to different destinations. ■

3.7 Computational complexity

In this section we discuss the computational complexity of the main decision problems related to deterministic planning.

The plan existence problem of deterministic planning is PSPACE-complete. The result was proved by Bylander [1994]. He proved the hardness part by giving a simulation of deterministic polynomial-space Turing machines, and the membership part by giving an algorithm that solves the problem in polynomial space. We later generalize his Turing machine simulation to alternating Turing machines to obtain an EXP-hardness proof for nondeterministic planning with full observability in Theorem 4.53.

Theorem 3.59 *The problem of testing the existence of a plan is PSPACE-hard.*

Proof: Let $\langle \Sigma, Q, \delta, q_0, g \rangle$ be any deterministic Turing machine with a polynomial space bound $p(x)$. Let σ be an input string of length n.

We construct a deterministic succinct transition system for simulating the Turing machine. The succinct transition system has a size that is polynomial in the size of the description of the Turing machine and the input string.

The set A of state variables in the succinct transition system consists of

1. $q \in Q$ for denoting the internal states of the TM,
2. s_i for every symbol $s \in \Sigma \cup \{|, \square\}$ and tape cell $i \in \{0, \ldots, p(n)\}$, and
3. h_i for the positions of the R/W head $i \in \{0, \ldots, p(n)+1\}$.

The initial state of the succinct transition system represents the initial configuration of the TM. The initial state I is as follows.

1. $I(q_0) = 1$
2. $I(q) = 0$ for all $q \in Q \backslash \{q_0\}$.
3. $I(s_i) = 1$ if and only if ith input symbol is $s \in \Sigma$, for all $i \in \{1, \ldots, n\}$.
4. $I(s_i) = 0$ for all $s \in \Sigma$ and $i \in \{0, n+1, n+2, \ldots, p(n)\}$.
5. $I(\square_i) = 1$ for all $i \in \{n+1, \ldots, p(n)\}$.
6. $I(\square_i) = 0$ for all $i \in \{0, \ldots, n\}$.
7. $I(|_0) = 1$
8. $I(|_i) = 0$ for all $n \in \{1, \ldots, p(n)\}$
9. $I(h_1) = 1$
10. $I(h_i) = 0$ for all $i \in \{0, 2, 3, 4, \ldots, p(n)+1\}$

The goal is the following formula.

$$G = \bigvee \{q \in Q | g(q) = \text{accept}\}$$

To define the operators, we first define effects corresponding to all possible transitions.

For all $\langle s, q\rangle \in (\Sigma \cup \{|, \square\}) \times Q$, $i \in \{0, \ldots, p(n)\}$ and $\langle s', q', m\rangle \in (\Sigma \cup \{|\}) \times Q \times \{L, N, R\}$ define the effect $\tau_{s,q,i}(s', q', m)$ as $\alpha \wedge \kappa \wedge \theta$ where the effects α, κ and θ are defined as follows.

The effect α describes what happens to the tape symbol under the R/W head. If $s = s'$ then $\alpha = \top$ as nothing on the tape changes. Otherwise, $\alpha = \neg s_i \wedge s'_i$ to denote that the new symbol in the ith tape cell is s' and not s.

The effect κ describes the change to the internal state of the TM. Again, either the state changes or does not, so $\kappa = \neg q \wedge q'$ if $q \neq q'$ and $\top$ otherwise. We define $\kappa = \neg q$ when $i = p(n)$ and $m = R$ so that when the space bound gets violated, no accepting state can be reached.

The effect θ describes the movement of the R/W head. Either there is movement to the left, no movement, or movement to the right. Hence

$$\theta = \begin{cases} \neg h_i \wedge h_{i-1} & \text{if } m = L \\ \top & \text{if } m = N \\ \neg h_i \wedge h_{i+1} & \text{if } m = R \end{cases}$$

By definition of TMs, movement at the left end of the tape is always to the right. Similarly, we have state variable for R/W head position $p(n) + 1$ and moving to that position is possible, but no transitions from that position are possible, as the space bound has been violated.

Now, these effects that represent possible transitions are used in the operators that simulate the Turing machine. Let $\langle s, q\rangle \in (\Sigma \cup \{|, \square\}) \times Q$, $i \in \{0, \ldots, p(n)\}$ and $\delta(s, q) = \{\langle s', q', m\rangle\}$. If $g(q) = \exists$, then define the operator

$$o_{s,q,i} = \langle h_i \wedge s_i \wedge q, \tau_{s,q,i}(s', q', m)\rangle.$$

We claim that the succinct transition system has a plan if and only if the Turing machine accepts without violating the space bound.

If the Turing machine violates the space bound, the state variable $h_{p(n)+1}$ becomes true and an accepting state cannot be reached because no further operator will be applicable.

So, because all deterministic Turing machines with a polynomial space bound can be in polynomial time translated into a planning problem, all decision problems in PSPACE are polynomial time many-one reducible to deterministic planning, and the plan existence problem is PSPACE-hard. □

Theorem 3.60 *The problem of testing the existence of a plan is in PSPACE.*

Proof: A recursive algorithm for testing m-step reachability between two states with $\log m$ memory consumption is given in Figure 3.4. The parameters of the algorithm are the set O of operators, the starting state s, the terminal state s', and m characterizing the maximum number 2^m of operators needed for reaching s' from s.

We show that when the algorithm is called with the number $n = |A|$ of state variables as the last argument, it consumes a polynomial amount of memory in n. The recursion depth is n. At the recursive calls memory is needed for storing the intermediate states s''. The memory needed for this is polynomial in n. Hence at any point of time the space consumption is $\mathcal{O}(m^2)$.

A succinct transition system $\langle A, I, O, G\rangle$ with $n = |A|$ state variables has a plan if and only if reach(O,I,s',n) returns *true* for some s' such that $s' \models G$. Iteration over all states s' can be performed in polynomial space and testing $s' \models G$ can be performed in polynomial time in the

```
 1:  procedure reach(O,s,s',m)
 2:  if m = 0 then                                   (* Plans of length 0 and 1 *)
 3:    if s = s' or there is o ∈ O such that s' = app_o(s) then return true
 4:    else return false
 5:  else
 6:    begin                                         (* Longer plans *)
 7:      for all states s'' do                       (* Iteration over intermediate states *)
 8:        if reach(O,s,s'',m − 1) and reach(O,s'',s',m − 1) then return true
;  9:      end
10:      return false;
11:    end
```

Figure 3.4: Algorithm for testing plan existence in polynomial space

size of G. Hence the whole memory consumption is polynomial. □

Part of the high complexity of planning is due to the fact that plans can be exponentially long. If a polynomial upper bound for plan length exists, testing the existence of plans is still intractable but much easier.

Theorem 3.61 *The problem of whether a plan having a length bounded by a given polynomial exists is NP-hard.*

Proof: We reduce the satisfiability problem of the classical propositional logic to the plan existence problem. The length of the plans, whenever they exist, is bounded by the number of propositional variables and hence is polynomial.

Let ϕ be a formula over the propositional variables in A. Let $N = \langle A, \{(a,0)|a \in A\}, O, \phi\rangle$ where $O = \{\langle \top, a\rangle | a \in A\}$ We show that N has a plan if and only if the formula ϕ is satisfiable.

Assume $\phi \in SAT$, that is, there is a valuation $v : A \to \{0,1\}$ such that $v \models \phi$. Now take the operators $\{\langle \top, a\rangle | v \models a, a \in A\}$ in any order: these operators form a plan that reach the state v that satisfies ϕ.

Assume N has a plan $o_1, \ldots, o_m$. The valuation $v = \{(a,1)|(\top, a) \in \{o_1, \ldots, o_m\}\} \cup \{(a,0)|a \in A, (\top, a) \notin \{o_1, \ldots, o_m\}\}$ of A is the terminal state of the plan and satisfies ϕ. □

Theorem 3.62 *The problem of whether a plan having a length bounded by a given polynomial exists is in NP.*

Proof: Let $p(m)$ be a polynomial. We give a nondeterministic algorithm that runs in polynomial time and determines whether a plan of length $p(m)$ exists.

Let $N = \langle A, I, O, G\rangle$ be a deterministic succinct transition system.

1. Nondeterministically guess a sequence of $l \leq p(m)$ operators $o_1, \ldots, o_l$ from the set O. Since l is bounded by the polynomial $p(m)$, the time consumption $\mathcal{O}(p(m))$ is polynomial in the size of N.

2. Compute $s = app_{o_l}(app_{o_{l-1}}(\cdots app_{o_2}(app_{o_1}(I))\cdots))$. This takes polynomial time in the size of the operators and the number of state variables.

3. Test $s \models G$. This takes polynomial time in the size of the operators and the number of state variables.

This nondeterministic algorithm correctly determines whether a plan of length at most $p(m)$ exists and it runs in nondeterministic polynomial time. Hence the problem is in NP. □

These theorems show the NP-completeness of the plan existence problem for polynomial-length plans.

3.8 Literature

Progression and regression were used early in planning research [Rosenschein, 1981]. Our definition of regression in Section 3.1.2 is related to the weakest precondition predicates for program synthesis [de Bakker and de Roever, 1972; Dijkstra, 1976]. Instead of using the general definition of regression we presented, earlier work on planning with regression and a definition of operators that includes disjunctive preconditions and conditional effects has avoided all disjunctivity by producing only goal formulae that are conjunctions of literals [Anderson *et al.*, 1998]. Essentially, these formulae are the disjuncts of $\mathit{regr}_o(\phi)$ in DNF, although the formulae $\mathit{regr}_o(\phi)$ are not generated. The search algorithm then produces a search tree with one branch for every disjunct of the DNF formula. In comparison to the general definition, this approach often leads to a much higher branching factor and an exponentially bigger search tree.

The use of algorithms for the satisfiability problem of the classical propositional logic in planning was pioneered by Kautz and Selman, originally as a way of testing satisfiability algorithms, and later shown to be more efficient than other planning algorithms at the time [Kautz and Selman, 1992; 1996]. In addition to Kautz and Selman [1996], parallel plans were used by Blum and Furst in their Graphplan planner [Blum and Furst, 1997]. Parallelism in this context serves the same purpose as partial-order reduction [Godefroid, 1991; Valmari, 1991], reducing the number of orderings of independent actions to consider. There are also other notions of parallel plans that may lead to much more efficient planning [Rintanen *et al.*, 2005]. Ernst et al. [1997] have considered translations of planning into the propositional that utilize the regular structure of sets of operators obtained from schematic operators. Planning by satisfiability has been extended to model-checking for testing whether a finite or infinite execution satisfying a given Linear Temporal Logic (LTL) formula exists [Biere *et al.*, 1999]. This approach to model-checking is called *bounded model-checking*.

It is trickier to use a satisfiability algorithm for showing that no plans of any length exist than for finding a plan of a given length. To show that no plans exist all plan lengths up to $2^n - 1$ have to be considered when there are n state variables. In typical planning applications n is often some hundreds or thousands, and generating and testing the satisfiability of all the required formulae is practically impossible. That no plans of a given length $n < 2^{|A|}$ do not exist does not directly imply anything about the existence of longer plans. Some other approaches for solving this problem based on satisfiability algorithms have been recently proposed [McMillan, 2003; Mneimneh and Sakallah, 2003].

The use of general-purpose heuristic search algorithms has recently got a lot of attention. The class of heuristics currently in the focus of interest was first proposed by McDermott [1999] and Bonet and Geffner [2001]. The distance estimates $\delta_I^{max}(\phi)$ and $\delta_I^{+}(\phi)$ in Section 3.4 are based on the ones proposed by Bonet and Geffner [2001]. Many other distance estimates similar to Bonet

and Geffner's exist [Haslum and Geffner, 2000; Hoffmann and Nebel, 2001; Nguyen *et al.*, 2002]. The $\delta_I^{rlx}(\phi)$ estimate generalizes ideas proposed by Hoffmann and Nebel [2001].

Other techniques for speeding up planning with heuristic state-space search include symmetry reduction [Starke, 1991; Emerson and Sistla, 1996] and partial-order reduction [Godefroid, 1991; Valmari, 1991; Alur *et al.*, 1997], both originally introduced outside planning in the context of reachability analysis and model-checking in computer-aided verification. Both of these techniques address the main problem in heuristic state-space search, high branching factor (number of applicable operators) and high number of states.

The algorithm for invariant computation was originally presented for simple operators without conditional effects [Rintanen, 1998]. The computation parallels the construction of planning graphs in the Graphplan algorithm [Blum and Furst, 1997], and it would seem to us that the notion of planning graph emerged when Blum and Furst noticed that the intermediate stages of invariant computation are useful for backward search algorithms: if a depth-bound of n is imposed on the search tree, then formulae obtained by m regression steps (suffixes of possible plans of length m) that do not satisfy clauses C_{n-m} cannot lead to a plan, and the search tree can be pruned. A different approach to find invariants has been proposed by Gerevini and Schubert [1998].

Some researchers extensively use Graphplan's planning graphs [Blum and Furst, 1997] for various purposes but we do not and have not discussed them in more detail for certain reasons. First, the graph character of planning graphs becomes inconvenient when preconditions of operators are arbitrary formulae and effects are conditional. As a result, the basic construction steps of planning graphs become unintuitive. Second, even when the operators have the simple form, the practically and theoretically important properties of planning graphs are not graph-theoretic. We can equivalently represent the contents of planning graphs as sequences of sets of literals and 2-literal clauses, as we have done in Section 3.5. In general it seems that the graph representation does not provide advantages over more conventional logic-based and set-based representations and is primarily useful for visualization purposes.

The algorithms presented in this section cannot in general be ordered in terms of efficiency. The general-purpose search algorithms with distance heuristics are often very effective in solving big problem instances with a sufficiently simple structure. This often entails better runtimes than in the SAT/CSP approach because of the high overheads with handling big formulae or constraint nets in the latter. Similarly, there are problems that are quickly solved by the SAT/CSP approach but on which heuristic state-space search fails.

There are few empirical studies on the behavior of different algorithms on planning problems in general or average. Bylander [1996] gives empirical results suggesting the existence of hard-easy pattern and a phase transition behavior similar to those found in other NP-hard problems like propositional satisfiability [Selman *et al.*, 1996]. Bylander also demonstrates that outside the phase transition region plans can be found by a simple hill-climbing algorithm or the inexistence of plans can be determined by using a simple syntactic test. Rintanen [2004b] complemented Bylander's work by analyzing the behavior of different types of planning algorithms on difficult problems inside the phase transition region, suggesting that current planners based on heuristic state space search are outperformed by satisfiability algorithms on difficult problems.

The PSPACE-completeness of the plan existence problem for deterministic planning is due to Bylander [1994]. The same result for another succinct representation of graphs had been established earlier by Lozano and Balcazar [1990].

Any computational problem that is NP-hard – not to mention PSPACE-hard – is considered too difficult to be solved in general. As planning even in the deterministic case is PSPACE-hard there

has been interest in finding restricted special cases in which efficient (polynomial-time) planning is always guaranteed. Syntactic restrictions have been investigated by several researchers [Bylander, 1994; Bäckström and Nebel, 1995] but the restrictions are so strict that very few interesting problems can be represented.

The computational complexity of planning with schematic operators has also been analyzed. Schematic operators increase the conciseness of the representations of some problem instances exponentially and lift the worst-case complexity accordingly. For example, deterministic planning with schematic operators is EXPSPACE-complete [Erol *et al.*, 1995]. If function symbols are allowed, encoding arbitrary Turing machines becomes possible and the plan existence problem is undecidable [Erol *et al.*, 1995].

3.9 Exercises

3.1 Show that regression for goals G that are sets (conjunctions) of state variables and operators with preconditions p that are sets (conjunctions) of state variables and effects that consist of an add list a (a set of state variables that become true) and a delete list d (a set of state variables that become false) can equivalently be defined as $(G \backslash a) \cup p$ when $d \cap G = \emptyset$.

3.2 Show that the problem in Lemma 3.9 is in NP and therefore NP-complete.

3.3 Satisfiability testing in the propositional logic is tractable in some special cases, like for sets of clauses with at most 2 literals in each, and for Horn clauses, that is sets of clauses with at most one positive literal in each clause.

Can you identify special cases in which existence of an n-step plan can be determined in polynomial time (in n and the size of the problem instance), because the corresponding formula transformed to CNF is a set of 2-literal clauses or a set of Horn clauses?

Chapter 4

Nondeterministic planning

4.1 Nondeterministic operators

In this section we will present a basic translation of nondeterministic operators into the propositional logic and a regression operation for nondeterministic operators. In the next sections we will discuss a general framework for computing with nondeterministic operators and their transition relations which are represented as propositional formulae. This framework provides techniques for computing both regression and progression for sets of states that are represented as formulae.

4.1.1 Regression for nondeterministic operators

Regression for deterministic operators is given in Definition 3.5. It can be easily generalized to a subclass of nondeterministic operators.

Definition 4.1 (Regression for nondeterministic operators) *Let ϕ be a propositional formula and $o = \langle c, e_1 | \cdots | e_n \rangle$ an operator where $e_1, \ldots, e_n$ are deterministic. Define*

$$\mathit{regr}_o^{nd}(\phi) = \mathit{regr}_{\langle c,e_1\rangle}(\phi) \wedge \cdots \wedge \mathit{regr}_{\langle c,e_n\rangle}(\phi).$$

Theorem 4.2 *Let ϕ be a formula over A, o an operator over A, and S the set of all states over A. Then $\{s \in S | s \models \mathit{regr}_o^{nd}(\phi)\} = \mathit{spreimg}_o(\{s \in S | s \models \phi\})$.*

Proof: Let $o = \langle c, (e_1 | \cdots | e_n) \rangle$.

$\{s \in S | s \models \mathit{regr}_o^{nd}(\phi)\}$
$= \{s \in S | s \models \mathit{regr}_{\langle c,e_1\rangle}(\phi) \wedge \cdots \wedge \mathit{regr}_{\langle c,e_n\rangle}(\phi)\}$
$= \{s \in S | s \models \mathit{regr}_{\langle c,e_1\rangle}(\phi), \ldots, s \models \mathit{regr}_{\langle c,e_n\rangle}(\phi)\}$
$= \{s \in S | \mathit{app}_{\langle c,e_1\rangle}(s) \models \phi, \ldots, \mathit{app}_{\langle c,e_n\rangle}(s) \models \phi\}$ T3.7
$= \{s \in S | s' \models \phi$ for all $s' \in \mathit{img}_o(s)$ and there is $s' \models \phi$ with $sos'\}$
$= \mathit{spreimg}_o(\{s \in S | s \models \phi\})$

The second last equality is because $\mathit{img}_o(s) = \{\mathit{app}_{\langle c,e_1\rangle}(s), \ldots, \mathit{app}_{\langle c,e_n\rangle}(s)\}$. □

Example 4.3 Let $o = \langle d, (b | \neg c) \rangle$. Then

$$\begin{aligned} \mathit{regr}_o^{nd}(b \leftrightarrow c) &= \mathit{regr}_{\langle d,b\rangle}(b \leftrightarrow c) \wedge \mathit{regr}_{\langle d,\neg c\rangle}(b \leftrightarrow c) \\ &= (d \wedge (\top \leftrightarrow c)) \wedge (d \wedge (b \leftrightarrow \bot)) \\ &\equiv d \wedge c \wedge \neg b. \end{aligned}$$

■

4.1.2 Translation of nondeterministic operators into propositional logic

In Section 3.6.2 we gave a translation of deterministic operators into the propositional logic. In this section we extend this translation to nondeterministic operators.

We define for effects e the sets *changes*(e) of state variables that are possibly changed by e, or in other words, the set of state variables occurring in an atomic effect in e.

$$\begin{aligned}
\textit{changes}(a) &= \{a\} \\
\textit{changes}(\neg a) &= \{a\} \\
\textit{changes}(c \rhd e) &= \textit{changes}(e) \\
\textit{changes}(e_1 \wedge \cdots \wedge e_n) &= \textit{changes}(e_1) \cup \cdots \cup \textit{changes}(e_n) \\
\textit{changes}(e_1 | \cdots | e_n) &= \textit{changes}(e_1) \cup \cdots \cup \textit{changes}(e_n)
\end{aligned}$$

We make the following assumption to simplify the translation.

Assumption 4.4 *Let $a \in A$ be a state variable. Let $e_1 \wedge \cdots \wedge e_n$ occur in the effect of an operator. If $e_1, \ldots, e_n$ are not all deterministic, then a or $\neg a$ may occur as an atomic effect in at most one of $e_1, \ldots, e_n$.*

This assumption rules out effects like $(a|b) \wedge (\neg a|c)$ that may make a simultaneously true and false. It also rules out effects like $((d \rhd a)|b) \wedge ((\neg d \rhd \neg a)|c)$ that are well-defined and could be translated into the propositional logic. However, the additional complexity outweighs the benefit of allowing them. Effects can often easily be transformed by the equivalences in Table 2.3 to satisfy Assumption 4.4: $((d \rhd a)|b) \wedge ((\neg d \rhd \neg a)|c)$ is equivalent to $((d \rhd a) \wedge (\neg d \rhd \neg a))|((d \rhd a) \wedge c)|(b \wedge (\neg d \rhd \neg a))|(b \wedge c)$.

The problem in the translation that does not show up with deterministic operators is that for nondeterministic choices $e_1|\cdots|e_n$ the formula for each e_i has to express the changes for exactly the same set of state variables. This set B is given as a parameter to the translation function. The set B has to include all state variables possibly changed by the effect.

$$\begin{aligned}
\tau_B^{nd}(e) &= \tau_B(e) \text{ when } e \text{ is deterministic} \\
\tau_B^{nd}(e_1|\cdots|e_n) &= \tau_B^{nd}(e_1) \vee \cdots \vee \tau_B^{nd}(e_n) \\
\tau_B^{nd}(e_1 \wedge \cdots \wedge e_n) &= \tau_{B \setminus (B_2 \cup \cdots \cup B_n)}^{nd}(e_1) \wedge \tau_{B_2}^{nd}(e_2) \wedge \cdots \wedge \tau_{B_n}^{nd}(e_n) \\
&\quad \text{where } B_i = \textit{changes}(e_i) \text{ for all } i \in \{2, \ldots, n\}
\end{aligned}$$

The first part of the translation $\tau_B^{nd}(e)$ for deterministic e is the translation of deterministic effects we presented in Section 3.6.2 restricted to state variables in B. The other two parts cover all nondeterministic effects in normal form. In the translation of $e_1 \wedge \cdots \wedge e_n$ all state variables that are not changed are handled in the translation of e_1. Assumption 4.4 guarantees that for each $\tau_B^{nd}(e)$ all state variables changed by e are in B.

Example 4.5 We translate the effect

$$e = (a|(d \rhd a)) \wedge (c|d)$$

into a propositional formula. The set of state variables is $A = \{a, b, c, d\}$.

$$\begin{aligned}\tau^{nd}_{\{a,b,c,d\}}(e) &= \tau^{nd}_{\{a,b\}}(a|(d \rhd a)) \wedge \tau^{nd}_{\{c,d\}}(c|d)\\ &= (\tau^{nd}_{\{a,b\}}(a) \vee \tau^{nd}_{\{a,b\}}(d \rhd a)) \wedge (\tau^{nd}_{\{c,d\}}(c) \vee \tau^{nd}_{\{c,d\}}(d))\\ &= ((a' \wedge (b \leftrightarrow b')) \vee (((a \vee d) \leftrightarrow a') \wedge (b \leftrightarrow b'))) \wedge\\ &\quad ((c' \wedge (d \leftrightarrow d')) \vee ((c \leftrightarrow c') \wedge d'))\end{aligned}$$

■

For expressing a state in terms of A' instead of A, or vice versa, we need to map a valuation of A to a corresponding valuation of A', or vice versa. for this purpose we define $s[A'/A] = \{\langle a', s(a)\rangle | a \in A\}$.

Definition 4.6 *Let A be a set of state variables. Let $o = \langle c, e\rangle$ be an operator over A in normal form. Define $\tau^{nd}_A(o) = c \wedge \tau^{nd}_A(e)$.*

Lemma 4.7 *Let o be an operator over a set A of state variables. Then*

$$\{v | v \textit{ is a valuation of } A \cup A', v \models \tau^{nd}_A(o)\} = \{s \cup s'[A'/A] | s, s' \in S, s' \in \textit{img}_o(s)\}.$$

Proof: We show that there is a one-to-one match between valuations satisfying $\tau^{nd}_A(o)$ and pairs of states and their successor states.

For the proof from right to left assume that s and s' are states such that $s' \in \textit{img}_o(s)$. Hence there is $E \in [e]_s$ such that s' is obtained from s by making literals in E true. Let $v = s \cup s'[A'/A]$. We show that $v \models \tau^{nd}_A(o)$. Let $o = \langle c, e\rangle$. Since $\textit{img}_o(s)$ is non-empty, $s \models c$. It remains to show that $v \models \tau^{nd}_A(e)$.

Induction hypothesis: Let e be any effect over a set B of state variables, and s and s' states such for some $E \in [e]_s$ $s' \models E$ and $s(a) = s'(a)$ for every $a \in B$ such that $\{a, \neg a\} \cap E = \emptyset$. Then $s \cup s'[A'/A] \models \tau^{nd}_B(e)$.

Base case: e is a deterministic effect. There is only one $E \in [e]_s$. A proof similar to that of Lemma 3.42 shows that $s \cup s'[A'/A] \models \tau^{nd}_B(e)$.

Inductive case 1, $e = e_1 \wedge \cdots \wedge e_n$: By definition $\tau^{nd}_B(e_1 \wedge \cdots \wedge e_n) = \tau^{nd}_{B\setminus(B_2\cup\cdots\cup B_n)}(e_1) \wedge \tau^{nd}_{B_2}(e_2) \wedge \cdots \wedge \tau^{nd}_{B_n}(e_n)$ for $B_i = \textit{changes}(e_i), i \in \{2, \ldots, n\}$. Let E be any member of $[e]_s$ and s' a state such that $s' \models E$ and $s(a) = s'(a)$ for every $a \in B$ such that $\{a, \neg a\} \cap E = \emptyset$. By definition of $[e]_s$ we have $E = E_1 \cup \cdots \cup E_n$ for some $E_i \in [e_i]_s$ for every $i \in \{1, \ldots, n\}$. The assumptions of the induction hypothesis hold for every e_i and $B_i, i \in \{2, \ldots, n\}$:

1. $s' \models E_i$ because $E_i \subseteq E$.
2. By Assumption 4.4 $s(a) = s'(a)$ for every $a \in B_i$ such that $\{a, \neg a\} \cap E_i = \emptyset$.

Similarly for e_1 and $B\setminus(B_2 \cup \cdots \cup B_n)$. Hence $s \cup s'[A'/A] \models \tau^{nd}_{B_i}(e_i)$ for all $i \in \{2, \ldots, n\}$ and $s \cup s'[A'/A] \models \tau^{nd}_{B\setminus(B_2\cup\cdots\cup B_n)}(e_i)$, and therefore $s \cup s'[A'/A] \models \tau^{nd}_B(e)$.

Inductive case 2, $e = e_1 | \cdots | e_n$: By definition $\tau^{nd}_B(e_1 | \cdots | e_n) = \tau^{nd}_B(e_1) \vee \cdots \vee \tau^{nd}_B(e_n)$. By definition $[e_1 | \cdots | e_n]_s = [e_1]_s \cup \cdots \cup [e_n]_s$. Hence $E \in [e_i]_s$ for some $i \in \{1, \ldots, n\}$. Hence the assumptions of the induction hypothesis hold for at least one $e_i, i \in \{1, \ldots, n\}$ and we get $s \cup s'[A'/A] \models \tau^{nd}_B(e_i)$. As $\tau^{nd}_B(e_i)$ is one of the disjuncts of $\tau^{nd}_B(e)$ finally $s \cup s'[A'/A] \models \tau^{nd}_B(e)$.

For the proof from left to right assume that $v \models \tau^{nd}_B(e)$ for $v = s \cup s'[A'/A]$. We prove by structural induction that the changes from s to s' correspond to $[e]_s$.

Induction hypothesis: Let e be any effect, B a set of state variables that includes those occurring in e, and s and s' states such that $v \models \tau^{nd}_B(e)$ where $v = s \cup s'[A'/A]$. Then there is $E \in [e]_s$ such that $s \models E$ and $s(a) = s'(a)$ for all $a \in B$ such that $\{a, \neg a\} \cap E = \emptyset$.

Base case: e is a deterministic effect. There is only one $E \in [e]_s$. A proof similar to that of Lemma 3.42 shows that the changes between s and s' for $a \in B$ correspond to E.

Inductive case 1, $e = e_1 \wedge \cdots \wedge e_n$: By definition $[e]_s = \{E_1 \cup \cdots \cup E_n | E_1 \in [e_1]_s, \ldots, E_n \in [e_n]_s\}$, and by Assumption 4.4 sets of the state variables occurring in $e_1, \ldots, e_n$ are disjoint. By definition $\tau^{nd}_B(e_1 \wedge \cdots \wedge e_n) = \tau^{nd}_{B\setminus(B_2\cup\cdots\cup B_n)}(e_1) \wedge \tau^{nd}_{B_2}(e_2) \wedge \cdots \wedge \tau^{nd}_{B_n}(e_n)$ for $B_i = \mathit{changes}(e_i), i \in \{2, \ldots, n\}$. The induction hypothesis for e and all $a \in B$ is directly by the induction hypothesis for all $a \in B = (B\setminus(B_2 \cup \cdots \cup B_n)) \cup B_2 \cup \cdots \cup B_n$ because $v \models \tau^{nd}_{B\setminus(B_2\cup\cdots\cup B_n)}(e_1) \wedge \tau^{nd}_{B_2}(e_2) \wedge \cdots \wedge \tau^{nd}_{B_n}(e_n)$.

Inductive case 2, $e = e_1 | \cdots | e_n$: By definition $[e_1 | \cdots | e_n]_s = [e_1]_s \cup \cdots \cup [e_n]_s$. By definition $\tau^{nd}_B(e_1 | \cdots | e_n) = \tau^{nd}_B(e_1) \vee \cdots \vee \tau^{nd}_B(e_n)$. Because $v \models \tau^{nd}_B(e_1 | \cdots | e_n)$, $v \models \tau^{nd}_B(e_i)$ for some $i \in \{1, \ldots, n\}$. By the induction hypothesis there is $E \in [e_i]_s$ with the given property. We get the induction hypothesis for e because $[e_i]_s \subseteq [e]_s$ and hence also $E \in [e]_s$.

Therefore s' is obtained from s by making some literals in $E \in [e]_s$ true and retaining the values of state variables not mentioned in E, and $s' \in \mathit{img}_o(s)$. □

4.2 Computing with transition relations as formulae

As discussed in Section 2.3, formulae are a representation of sets of states. In this section we show how operations on transition relations have a counterpart as operations on formulae that represent transition relations.

Most implementations of the techniques in this section are based on binary decision diagrams (BDDs) [Bryant, 1992], a representation (essentially a normal form) of propositional formulae with useful computational properties, but the techniques are applicable to other representations of propositional formulae as well.

4.2.1 Existential and universal abstraction

The most important operations performed on transition relations represented as propositional formulae are based on *existential abstraction* and *universal abstraction*.

Definition 4.8 Existential abstraction *of a formula ϕ with respect to an atomic proposition a is the formula*

$$\exists a.\phi = \phi[\top/a] \vee \phi[\bot/a].$$

Universal abstraction is defined analogously by using conjunction instead of disjunction.

Definition 4.9 Universal abstraction *of a formula ϕ with respect to an atomic proposition a is the formula*

$$\forall a.\phi = \phi[\top/a] \wedge \phi[\bot/a].$$

Existential and universal abstraction of ϕ with respect to a *set of atomic propositions* is defined in the obvious way: for $B = \{b_1, \ldots, b_n\}$ such that B is a subset of the propositional variables

occurring in ϕ define

$$\exists B.\phi = \exists b_1.(\exists b_2.(\ldots \exists b_n.\phi \ldots))$$
$$\forall B.\phi = \forall b_1.(\forall b_2.(\ldots \forall b_n.\phi \ldots)).$$

In the resulting formulae there are no occurrences of variables in B.

Let ϕ be a formula over A. Then $\exists A.\phi$ is a formula that consists of the constants $\top$ and $\bot$ and the logical connectives only. The truth-value of this formula is independent of the valuation of A, that is, its value is the same for all valuations.

The following lemma expresses the important properties of existential and universal abstraction. When we write $v \cup v'$ for a pair of valuations we view valuations v as binary relations, that is, sets of pairs such that $\{(a, b), (a, c)\} \notin v$ for any a, b and c such that $b \neq c$.

Lemma 4.10 *Let ϕ be a formula over $A \cup A'$ and v' a valuation of A'. Then*

1. *$v' \models \exists A.\phi$ if and only if $(v \cup v') \models \phi$ for at least one valuation v of A, and*
2. *$v' \models \forall A.\phi$ if and only if $(v \cup v') \models \phi$ for all valuations v of A.*

Proof: We prove the statements by induction on the cardinality of A. We only give the proof for $\exists$. The proof for $\forall$ is analogous to that for $\exists$.

Base case $|A| = 0$: There is only one valuation $v = \emptyset$ of the empty set $A = \emptyset$. When there is nothing to abstract we have $\exists\emptyset.\phi = \phi$. Hence trivially $v' \models \exists\emptyset.\phi$ if and only if $(v \cup \emptyset) \models \phi$.

Inductive case $|A| \geq 1$: Take any $a \in A$. $v' \models \exists A.\phi$ if and only if $v' \models \exists A\backslash\{a\}.(\phi[\top/a] \vee \phi[\bot/a])$ by the definition of $\exists a.\phi$. By the induction hypothesis $v' \models \exists A\backslash\{a\}.(\phi[\top/a] \vee \phi[\bot/a])$ if and only if $(v_0 \cup v') \models \phi[\top/a] \vee \phi[\bot/a]$ for at least one valuation v_0 of $A\backslash\{a\}$. Since the formula $\phi[\top/a] \vee \phi[\bot/a]$ represents both possible valuations of a in ϕ, the last statement is equivalent to $(v \cup v') \models \phi$ for at least one valuation v of A. □

4.2.2 Images and preimages as formula manipulation

Let $A = \{a_1, \ldots, a_n\}$, $A' = \{a'_1, \ldots, a'_n\}$ and $A'' = \{a''_1, \ldots, a''_n\}$. Let ϕ_1 be a formula over $A \cup A'$ and ϕ_2 be a formula over $A' \cup A''$. The formulae can be viewed as representations of $2^n \times 2^n$ matrices or as transition relations over a state space of size 2^n.

The product matrix of ϕ_1 and ϕ_2 is represented by a the following formula over $A \cup A''$.

$$\exists A'.\phi_1 \wedge \phi_2$$

Example 4.11 Let $\phi_1 = a \leftrightarrow \neg a'$ and $\phi_2 = a' \leftrightarrow a''$ represent two actions, reversing the truth-value of a and doing nothing. The sequential composition of these actions is

$$\begin{aligned}\exists a'.\phi_1 \wedge \phi_2 &= ((a \leftrightarrow \neg\top) \wedge (\top \leftrightarrow a'')) \vee ((a \leftrightarrow \neg\bot) \wedge (\bot \leftrightarrow a'')) \\ &\equiv ((a \leftrightarrow \bot) \wedge (\top \leftrightarrow a'')) \vee ((a \leftrightarrow \top) \wedge (\bot \leftrightarrow a'')) \\ &\equiv a \leftrightarrow \neg a''.\end{aligned}$$

■

This idea can be used for computing the images, preimages and strong preimages of operators and sets of states in terms of formula manipulation by existential and universal abstraction. Table

matrices	formulas	sets of states
vector $V_{1\times n}$	formula over A	set of states
matrix $M_{n\times n}$	formula over $A \cup A'$	transition relation
$V_{1\times n} + V'_{1\times n}$	$\phi_1 \vee \phi_2$	set union
	$\phi_1 \wedge \phi_2$	set intersection
$M_{n\times n} \times N_{n\times n}$	$\exists A'.(\tau_A^{nd}(o) \wedge \tau_A^{nd}(o')[A''/A', A'/A])[A'/A'']$	sequential composition $o \circ o'$
$V_{1\times n} \times M_{n\times n}$	$(\exists A.(\phi \wedge \tau_A^{nd}(o)))[A/A']$	$\mathit{img}_o(T)$
$M_{n\times n} \times V_{n\times 1}$	$\exists A'.(\tau_A^{nd}(o) \wedge \phi[A'/A])$	$\mathit{preimg}_o(T)$
	$\forall A'.(\tau_A^{nd}(o) \rightarrow \phi[A'/A]) \wedge \exists A'.\tau_A^{nd}(o)$	$\mathit{spreimg}_o(T)$

Table 4.1: Correspondence between matrix operations, Boolean operations and set-theoretic/relational operations. Above $T = \{s \in S | s \models \phi\}$, M is the matrix corresponding to $\tau_A^{nd}(o)$ and N is the matrix corresponding to o'.

4.1 outlines a number of connections between operations on vectors and matrices, on propositional formulae, and on sets and relations. For transition relations we use valuations of $A \cup A'$ for representing pairs for states and for states we use valuations of A.

Lemma 4.12 *Let ϕ be a formula over A and v a valuation of A. Then $v \models \phi$ if and only if $v[A'/A] \models \phi[A'/A]$, and $(\phi[A'/A])[A/A'] = \phi$.*

Definition 4.13 *Let o be an operator and ϕ a formula. Define*

$$\begin{aligned} \mathit{img}_o(\phi) &= (\exists A.(\phi \wedge \tau_A^{nd}(o)))[A/A'] \\ \mathit{preimg}_o(\phi) &= \exists A'.(\tau_A^{nd}(o) \wedge \phi[A'/A]) \\ \mathit{spreimg}_o(\phi) &= \forall A'.(\tau_A^{nd}(o) \rightarrow \phi[A'/A]) \wedge \exists A'.\tau_A^{nd}(o). \end{aligned}$$

Theorem 4.14 *Let $T = \{s \in S | s \models \phi\}$. Then $\{s \in S | s \models \mathit{img}_o(\phi)\} = \{s \in S | s \models (\exists A.(\phi \wedge \tau_A^{nd}(o)))[A/A']\} = \mathit{img}_o(T)$.*

Proof: $s' \models (\exists A.(\phi \wedge \tau_A^{nd}(o)))[A/A']$ □
iff $s'[A'/A] \models \exists A.(\phi \wedge \tau_A^{nd}(o))$ L4.12
iff there is valuation s of A such that $(s \cup s'[A'/A]) \models \phi \wedge \tau_A^{nd}(o)$ L4.10
iff there is valuation s of A such that $s \models \phi$ and $(s \cup s'[A'/A]) \models \tau_A^{nd}(o)$
iff there is $s \in T$ such that $(s \cup s'[A'/A]) \models \tau_A^{nd}(o)$
iff there is $s \in T$ such that $s' \in \mathit{img}_o(s)$ L4.7
iff $s' \in \mathit{img}_o(T)$.

Theorem 4.15 *Let $T = \{s \in S | s \models \phi\}$. Then $\{s \in S | s \models \mathit{preimg}_o(\phi)\} = \{s \in S | s \models \exists A'.(\tau_A^{nd}(o) \wedge \phi[A'/A])\} = \mathit{preimg}_o(T)$.*

Proof: $s \models \exists A'.(\tau_A^{nd}(o) \wedge \phi[A'/A])$
iff there is $s'_0 : A' \to \{0,1\}$ such that $(s \cup s'_0) \models \tau_A^{nd}(o) \wedge \phi[A'/A]$
iff there is $s'_0 : A' \to \{0,1\}$ such that $s'_0 \models \phi[A'/A]$ and $(s \cup s'_0) \models \tau_A^{nd}(o)$ L4.10
iff there is $s' : A \to \{0,1\}$ such that $s' \models \phi$ and $(s \cup s'_0) \models \tau_A^{nd}(o)$ L4.12
iff there is $s' \in T$ such that $(s \cup s'[A'/A]) \models \tau_A^{nd}(o)$
iff there is $s' \in T$ such that $s' \in \mathit{img}_o(s)$ L4.7
iff there is $s' \in T$ such that $s \in \mathit{preimg}_o(s')$ (5) of L2.2
iff $s \in \mathit{preimg}_o(T)$.
Above we define $s' = s'_0[A/A']$ (and hence $s'_0 = s'[A'/A]$.) □

Theorem 4.16 *Let* $T = \{s \in S | s \models \phi\}$. *Then* $\{s \in S | s \models \mathit{spreimg}_o(\phi)\} = \{s \in S | s \models \forall A'.(\tau_A^{nd}(o) \to \phi[A'/A]) \wedge \exists A'.\tau_A^{nd}(o)\} = \mathit{spreimg}_o(T)$.

Proof:
$s \models \forall A'.(\tau_A^{nd}(o) \to \phi[A'/A]) \wedge \exists A'.\tau_A^{nd}(o)$
iff $s \models \forall A'.(\tau_A^{nd}(o) \to \phi[A'/A])$ and $s \models \exists A'.\tau_A^{nd}(o)$
iff $(s \cup s'_0) \models \tau_A^{nd}(o) \to \phi[A'/A]$ for all $s'_0 : A' \to \{0,1\}$ and $s \models \exists A'.\tau_A^{nd}(o)$ L4.10
iff $(s \cup s'_0) \not\models \tau_A^{nd}(o)$ or $s'_0 \models \phi[A'/A]$ for all $s'_0 : A' \to \{0,1\}$ and $s \models \exists A'.\tau_A^{nd}(o)$
iff $(s \cup s'[A'/A]) \not\models \tau_A^{nd}(o)$ or $s' \models \phi$ for all $s' : A \to \{0,1\}$ and $s \models \exists A'.\tau_A^{nd}(o)$ L4.12
iff $s' \notin \mathit{img}_o(s)$ or $s' \models \phi$ for all $s' : A \to \{0,1\}$ and $s \models \exists A'.\tau_A^{nd}(o)$ L4.7
iff $s' \in \mathit{img}_o(s)$ implies $s' \models \phi$ for all $s' : A \to \{0,1\}$ and $s \models \exists A'.\tau_A^{nd}(o)$
iff $\mathit{img}_o(s) \subseteq T$ and $s \models \exists A'.\tau_A^{nd}(o)$
iff $\mathit{img}_o(s) \subseteq T$ and there is $s' : A \to \{0,1\}$ with $(s \cup s'[A'/A]) \models \tau_A^{nd}(o)$ L4.10
iff $\mathit{img}_o(s) \subseteq T$ and there is $s' : A \to \{0,1\}$ with $s' \in \mathit{img}_o(s)$ L4.7
iff $\mathit{img}_o(s) \subseteq T$ and there is $s' \in T$ with $s' \in \mathit{img}_o(s)$
iff $\mathit{img}_o(s) \subseteq T$ and there is $s' \in T$ with sos'
iff $s \in \mathit{spreimg}_o(T)$.
Above we define $s' = s'_0[A/A']$ (and hence $s'_0 = s'[A'/A]$.) □

Corollary 4.17 *Let* $o = \langle c, (e_1 | \cdots | e_n) \rangle$ *be an operator such that all* e_i *are deterministic. The formula* $\mathit{spreimg}_o(\phi)$ *is logically equivalent to* $\mathit{regr}_o^{nd}(\phi)$ *as given in Definition 4.1.*

Proof: By Theorems 4.2 and 4.16 $\{s \in S | s \models \mathit{regr}_o(\phi)\} = \mathit{spreimg}_o(\{s \in S | s \models \phi\}) = \{s \in S | s \models \mathit{spreimg}_o(\phi)\}$. □

Example 4.18 Let $o = \langle c, a \wedge (a \rhd b) \rangle$. Then

$$\mathit{regr}_o^{nd}(a \wedge b) = c \wedge (\top \wedge (b \vee a)) \equiv c \wedge (b \vee a).$$

The transition relation of o is represented by

$$\tau_A^{nd}(o) = c \wedge a' \wedge ((b \vee a) \leftrightarrow b') \wedge (c \leftrightarrow c').$$

The preimage of $a \wedge b$ with respect to o is represented by

$$\begin{aligned}\exists a'b'c'.((a' \wedge b') \wedge \tau_A^{nd}(o)) &\equiv \exists a'b'c'.((a' \wedge b') \wedge c \wedge a' \wedge ((b \vee a) \leftrightarrow b') \wedge (c \leftrightarrow c')) \\ &\equiv \exists a'b'c'.(a' \wedge b' \wedge c \wedge (b \vee a) \wedge c') \\ &\equiv \exists b'c'.(b' \wedge c \wedge (b \vee a) \wedge c') \\ &\equiv \exists c'.(c \wedge (b \vee a) \wedge c') \\ &\equiv c \wedge (b \vee a)\end{aligned}$$

■

Hence regression for nondeterministic operators (Definition 4.1) can be viewed as a specialized method for computing preimages of sets of states represented as formulae.

Many algorithms include the computation of the union of images or preimages with respect to all operators, for example $\bigcup_{o \in O} \mathit{img}_o(T)$, or in terms of formulae, $\bigvee_{o \in O} \mathit{img}_o(\phi)$ where $T = \{s \in S | s \models \phi\}$. A technique used by many implementations of such algorithms is the following. Instead of computing the images or preimages one operator at a time, construct a combined transition relation for all operators. For an illustration of the technique, consider $\mathit{img}_{o_1}(\phi) \vee \mathit{img}_{o_2}(\phi)$ that represents the union of state sets represented by $\mathit{img}_{o_1}(\phi)$ and $\mathit{img}_{o_2}(\phi)$. By definition

$$\mathit{img}_{o_1}(\phi) \vee \mathit{img}_{o_2}(\phi) = (\exists A.(\phi \wedge \tau_A^{nd}(o_1)))[A/A'] \vee (\exists A.(\phi \wedge \tau_A^{nd}(o_2)))[A/A'].$$

Since substitution commutes with disjunction we have

$$\mathit{img}_{o_1}(\phi) \vee \mathit{img}_{o_2}(\phi) = (\exists A.(\phi \wedge \tau_A^{nd}(o_1))) \vee (\exists A.(\phi \wedge \tau_A^{nd}(o_2)))[A/A'].$$

Since existential abstraction commutes with disjunction we have

$$\mathit{img}_{o_1}(\phi) \vee \mathit{img}_{o_2}(\phi) = (\exists A.((\phi \wedge \tau_A^{nd}(o_1)) \vee (\phi \wedge \tau_A^{nd}(o_2))))[A/A'].$$

By logical equivalence finally

$$\mathit{img}_{o_1}(\phi) \vee \mathit{img}_{o_2}(\phi) = (\exists A.(\phi \wedge (\tau_A^{nd}(o_1) \vee \tau_A^{nd}(o_2))))[A/A'].$$

Hence an alternative way of computing the union of images $\bigvee_{o \in O} \mathit{img}_o(\phi)$ is to first form the disjunction $\bigvee_{o \in O} \tau_A^{nd}(o)$ and then conjoin the formula with ϕ and only once existentially abstract the propositional variables in A. This may reduce the amount of computation because existential abstraction is in general expensive and it may be possible to simplify the formulae $\bigvee_{o \in O} \tau_A^{nd}(o)$ before existential abstraction.

The definitions of $\mathit{preimg}_o(\phi)$ and $\mathit{spreimg}_o(\phi)$ allow using $\bigvee_{o \in O} \tau_A^{nd}(o)$ in the same way.

Notice that defining progression for arbitrary formulae (sets of states) seems to require the explicit use of existential abstraction with potential exponential increase in formula size. A simple syntactic definition of progression similar to that of regression does not seem to be possible because the value of a state variable in a given state cannot be stated in terms of the values of the state variables in the successor state. This is because of the asymmetry of deterministic actions: the current state and an operator determine the successor state uniquely but the successor state and the operator do not determine the current state uniquely. In other words, the changes that take place are a function of the current state, but not a function of the successor state. Taking an action erases the information that determines which changes take place between two states. This information is visible in the predecessor state but not in the successor state.

4.3 Problem definition

We state the conditional planning problem in the general form. Because the number of observations that are possible has a very strong effect on the type of solution techniques that are applicable, we will discuss algorithms for three classes of planning problems that are defined in terms of restrictions on the set B of observable state variables.

The set B did not appear in the definition of deterministic planning. This is the set of *observable state variables*. The idea is that plans can make decisions about what operations to apply and how the execution proceeds based on the values of the observable state variables. Restrictions on observability and sensing emerge because of various restrictions on the sensors human beings and robots have: typically only a small part of the world can be observed.

However, because of nondeterminism and the possibility of more than one initial state, it is in general not possible to use the same sequence of operators for reaching the goals from all the initial states, and a more general notion of plans has to be used.

Nondeterministic planning problems under certain restrictions have very different properties than the problem in its full generality. In Chapter 3 we had the restriction to one initial state (I was defined as a valuation) and deterministic operators. We relax these two restrictions in this chapter, but still consider two special cases obtained by restrictions on the set B of observable state variables.

1. Full observability.

   This is the most direct extension of the deterministic planning problem of the previous chapter. The difference is that we have to use a more general notion of plans with branches (and with loops, if there is no upper bound on the number of actions that might be needed to reach the goals.)

2. No observability.

   Planning without observability can be considered more difficult than planning with full observability, although they are in many respects not directly comparable.

   The main difference to deterministic planning as discussed in Chapter 3 and to planning with full observability is that during plan execution it is not known what the actual current state is, and there are several possible current states. This complication means that planning takes place in *the belief space*: the role of individual states in deterministic planning is taken by sets of states, called *belief states*.

   Because no observations can be made, branching is not possible, and plans are still just sequences of actions, just like in deterministic planning with one initial state.

The type of observability we consider in this lecture is very restricted as only values of individual state variables can be observed (as opposed to arbitrary formulae) and observations are independent of what operators have been executed before. Hence we cannot for example directly express special sensing actions. However, extensions to the above definition like sensing actions can be relatively easily reduced to the basic definition but we will not discuss this topic further.

4.3.1 Memoryless plans

We use two definitions of plans. The simpler definition, formalized as mappings from states to operators, is applicable to fully observable planning problems only. The general definition

has a sufficient generality for all kinds of planning problems, and includes the sequential plans considered for deterministic planning as a special case.

Definition 4.19 *Let $\langle A, I, O, G, V\rangle$ be a succinct transition system. Let S be the set of states (all Boolean valuations of A). Then a* memoryless plan *is a partial function $\pi : S \to O$.*

To be able to execute a memoryless plan the current state must always be known, otherwise the correct operator in general cannot be correctly chosen. Hence we always assume full observability when using a memoryless plan. In the context of Markov decision processes (see Section 5.5) memoryless plans are also known as *policies* or *history dependent policies*.

We define the satisfaction of plan objectives in terms of the transition system that is obtained when the original transition system is being controlled by a plan, that is, the plan chooses which of the transitions possible in a state is taken.

Definition 4.20 (Execution graph of a memoryless plan) *Let π be a memoryless plan for a succinct transition system $\langle A, I, O, G, V\rangle$. Then the* execution graph *of π and the transition system is the graph $\langle S, E\rangle$ where*

1. *$E \subseteq S \times S$ and*
2. *$(s, s') \in E$ if $s' \in img_o(s)$.*

The states s such that $s \models I$ are *initial nodes* of the execution graph, and the states s such that $s \models G$ are *goal nodes* of the execution graph. We have introduced the nodes of an execution graph as a notion that is separate from the states in the transition system because for the more general notion of plans we define next these two notions do not coincide.

4.3.2 Conditional plans

Plans determine what actions are executed. We formalize plans as a form of directed graphs. Each node is assigned an operator and information on zero or more successor nodes.

Definition 4.21 *Let $\Pi = \langle A, I, O, G, V\rangle$ be a succinct transition system. A* plan *for Π is a triple $\langle N, b, l\rangle$ where*

1. *N is a finite set of nodes,*
2. *$b \subseteq \mathcal{L} \times N$ maps initial states to starting nodes, and*
3. *$l : N \to O \times 2^{\mathcal{L}\times N}$ is a function that assigns each node n an operator and a set of pairs $\langle \phi, n'\rangle$ where ϕ is a formula over the observable state variables V and $n' \in N$ is a successor node.*

   Nodes n with $l(n) = \langle o, \emptyset\rangle$ for some $o \in O$ are terminal nodes.

Ignoring the operators and branch formulae in a plan π we can construct a graph $G(\pi) = \langle N, E\rangle$ with $E \subseteq N \times N$ such that $\langle n, n'\rangle \in E$ iff $\langle \phi, n'\rangle \in B$ for $l(n) = \langle o, B\rangle$ and some ϕ. A plan π is acyclic if there is no non-trivial path starting and ending at the same node in $G(\pi)$.

Plan execution starts from a node $n \in N$ and state s such that $\langle \phi, n\rangle \in b$ and $s \models I \wedge \phi$. Execution in node n with $l(n) = \langle o, B\rangle$ proceeds by executing the operator o and then testing for

each $\langle \phi, n' \rangle \in l(n)$ whether ϕ is true in all possible current states, and if it is, continuing execution from plan node n'. At most one ϕ may be true for this to be well-defined. Plan execution ends when none of the branch labels matches the current state. In a terminal node plan execution necessarily ends.

Definition 4.22 (Execution graph of a conditional plan) *Let $\langle A, I, O, G, V \rangle$ be a succinct transition system and $\pi = \langle N, b, l \rangle$ be a plan. Define* the execution graph *of π as a pair $\langle M, E \rangle$ where*

1. *$M = S \times (N \cup \{\bot\})$, where S is the set of Boolean valuations of A,*
2. *$E \subseteq M \times M$ has an edge from $\langle s, n \rangle \in S \times N$ to $\langle s', n' \rangle \in S \times N$ if and only if $l(n) = \langle o, B \rangle$ and for some $\langle \phi, n' \rangle \in B$*
   (a) $s' \in img_o(s)$ and
   (b) $s' \models \phi$.

   and an edge from $\langle s, n \rangle \in S \times N$ to $\langle s', \bot \rangle$ if and only if
   (a) $l(n) = \langle o, B \rangle$,
   (b) $s' \in img_o(s)$, and
   (c) there is no $\langle \phi, n' \rangle \in B$ such that $s' \models \phi$.

The *initial nodes* of these execution graphs are nodes $\langle s, n \rangle$ such that $s \models I$ and $s \models \phi$ for some $\langle \phi, n \rangle \in b$.

The *goal nodes* of these execution graphs are nodes $\langle s, n \rangle$ such that $s \models G$.

We can translate every memoryless plan to a conditional plan.

Definition 4.23 *Let $\langle A, I, O, G, V \rangle$ be a succinct transition system. Let S be the set of all states on A. Let $\pi : S \to O$ be a memoryless plan. Define $C(\pi) = \langle N, b, l \rangle$ where*

1. *$N = O$,*
2. *$b = \{\langle FMA(\{s \in S | \pi(s) = o\}), o \rangle | o \in O\}$, and*
3. *$l(o) = (o, \{\langle FMA(\{s \in S | \pi(s) = o\}), o' \rangle | o' \in O\})$ for all $o \in O$.*

Above $FMA(T)$ is a formula ϕ such that $T = \{s \in S | s \models \phi\}$.

The memoryless plan π corresponds the conditional plan $C(\pi)$ in the sense that the subgraphs induced by the initial nodes are isomorphic, and this isomorphism preserves both initial and goal nodes.

4.3.3 Decision problems

There are different types of objectives the plans may have to fulfill. The most basic one, considered in much of AI planning research, is the reachability of a goal state. In this case every plan execution has a finite length. Also problems with infinite plan executions can be considered. A plan does not reach a goal and terminate, but is a continuing process that has to repeatedly reach goal states or avoid visiting bad states. Examples of these are different kinds of maintenance tasks: keep a building clean and transport mail from location A to location B.

We consider two objectives defined in terms of finite plan executions. The first objective requires, just like in deterministic planning, that a goal state is reached after a given number of operator executions.

Not all planning problems that have an intuitively plausible solution are solvable under this objective. A problem that is intuitively solvable is tossing a coin until it yields heads. This problem can be in practice always solved after a small number of tosses but because there is no guaranteed upper bound on the number of tosses that are needed, under the first objective it is not solvable. Hence we also consider another objective.

The second objective requires that from every state that can be reached by executing the plan, the plan is either a goal state or a goal state is reachable by executing the plan further.

The third objective we consider is defined in terms of infinite plan executions. The objective requires that all executions of a plan are infinite, and on every execution all states that are visited are goal states. This objective is known as *maintenance* because the transition system has to be kept in one of the goal states.

Other infinite horizon objectives that are defined in terms of expected costs/rewards are used in connection with probabilistic planning, see Section 5.3.

Definition 4.24 (Bounded reachability) *A plan π for $\langle A, I, O, G, V\rangle$ under the* Bounded Reachability *criterion fulfills the following.*

For all initial nodes x in the execution graph, all paths starting from x have a finite length and maximal paths end in a goal node.

Definition 4.25 (Unbounded reachability) *A plan π for $\langle A, I, O, G, V\rangle$ under the* Unbounded Reachability *criterion fulfills the following.*

For all initial nodes x in the execution graph, for every x' to which there is a path from x there is a path from x' of length ≥ 0 to some x'' such that x'' is a goal node without successor nodes.

This plan objective with unbounded looping can be interpreted probabilistically. For every nondeterministic choice in an operator we have to assume that each of the alternatives has a nonzero probability. Then for goal reachability, a plan with unbounded looping is simply a plan that has no finite upper bound on the length of its executions, but that with probability 1 eventually reaches a goal state. A non-looping plan also reaches a goal state with probability 1, but there is a finite upper bound on the execution length.

Definition 4.26 (Maintenance) *A plan π for $\langle A, I, O, G, V\rangle$ under the* Maintenance *criterion fulfills the following.*

All nodes x in the execution graph to which there is a path of length ≥ 0 from an initial node of the execution graph are goal nodes and have a successor node.

Example 4.27 Consider the plan $\langle N, b, l\rangle$ for a problem instance with the operators $O = \{o_1, o_2, o_3\}$, where

$$\begin{aligned} N &= \{1, 2\} \\ b &= \{\langle \top, 1\rangle\} \\ l(1) &= \langle o_3, \{\langle \phi_1, 1\rangle, \langle \phi_2, 2\rangle, \langle \phi_3, 3\rangle\}\rangle \\ l(2) &= \langle o_2, \{\langle \phi_4, 1\rangle, \langle \phi_5, 3\rangle\}\rangle \end{aligned}$$

This could be visualized as the program.

1: o_3
CASE
ϕ_1: GOTO 1
ϕ_2: GOTO 2
$\neg(\phi_1 \vee \phi_2)$: GOTO 3
2: o_2
CASE
ϕ_4: GOTO 1
$\neg\phi_4$: GOTO 3
3:

Every plan $\langle N, b, l\rangle$ can be written as such a program. ■

A plan is *acyclic* if it is a directed acyclic graph in the usual graph theoretic sense.

4.4 Planning with full observability

Nondeterminism causes several The differences to algorithms for deterministic planning The main difference is that successor states are not uniquely determined by the current state and the action, and different action may be needed for each successor state. Further, nondeterminism may require loops. Consider tossing a die until it yields 6. Plan for this task involves tossing the die over and over, and there is no upper bound on the number of tosses that might be needed.[1] Hence we need plans with loops for representing the sequences of actions of unbounded length required for solving the problem.

Below in Section 4.4.1 we first discuss the simplest algorithm for planning with nondeterminism and full observability. The plans this algorithm produces are acyclic, and the algorithm does not find plans for problem instances that only have plans with loops. Then in Section 4.4.2 we present an algorithm that also produces plans with loops. The structure of the algorithm is more complicated. The algorithms can be implemented by using data structures like binary decision diagrams which makes it possible to utilize the regularities in the state space and to solve very big planning problems. Representation of planning problems with these logic-based data structures is explained in Section 4.2.

4.4.1 An algorithm for constructing acyclic plans

Next we present an algorithm for constructing acyclic plans for nondeterministic problem with full observability. Acyclicity means that during any execution of the plan no state is visited more than once. Not all nondeterministic planning problems that have an intuitively acceptable solution have a solution as an acyclic plan. For a more detailed discussion of this topic and related algorithms see [Cimatti *et al.*, 2003].

The basic algorithm is for transition systems as in Definition 2.1 but the techniques in Section 4.2 can be directly applied to obtain a logic-based algorithm for succinct transition systems (Definition 2.11 in Section 2.3) that can be implemented easily by using any publicly available BDD package.

In the first phase the algorithm computes distances of the states. In the second phase the algorithm constructs a plan based on the distances.

[1] However, for every $p > 0$ there is a finite plan that reaches the goal with probability p or higher.

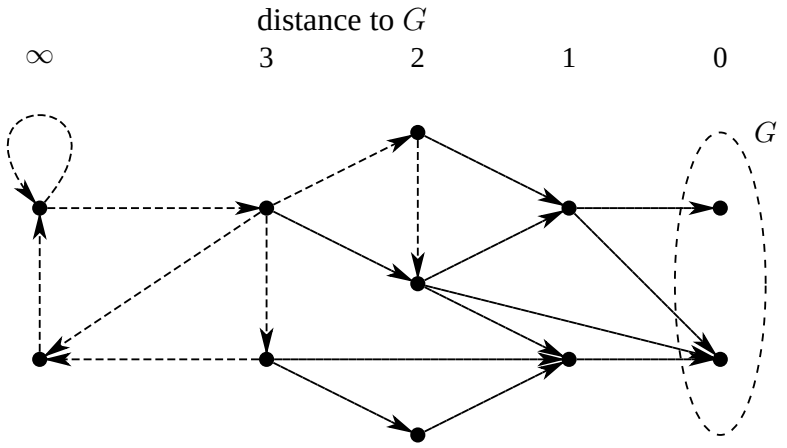


Figure 4.1: Goal distances in a nondeterministic transition system

Let G be a set of states and O a set of operators. Then we define the *backward distance sets* D_i^{bwd} for G, O that consist of those states for which there is a guarantee of reaching a state in G with at most i operator applications.

$$\begin{aligned} D_0^{bwd} &= G \\ D_i^{bwd} &= D_{i-1}^{bwd} \cup \bigcup_{o \in O} \mathit{spreimg}_o(D_{i-1}^{bwd}) \text{ for all } i \geq 1 \end{aligned}$$

Definition 4.28 *Let G be as set of states and O a set of operators, and let $D_0^{bwd}, D_1^{bwd}, \ldots$ be the backward distance sets for G and O. Then* the backward distance *of a state s to G is*

$$\delta_G^{bwd}(s) = \begin{cases} 0 \text{ if } s \in G \\ i \text{ if } s \in D_i^{bwd} \backslash D_{i-1}^{bwd} \end{cases}$$

If $s \notin D_i^{bwd}$ for all $i \geq 0$ then $\delta_G^{bwd}(s) = \infty$.

Example 4.29 We illustrate the distance computation by the diagram in Figure 4.1. The set of states with distance 0 is the set of goal states G. States with distance i are those for which there is an action that always leads to states with distance $i-1$ or smaller. In this example the action depicted by the solid arrow has this property for every state. The dashed arrows depict the second action which for no state is guaranteed to get closer to the goal states. States for which there is no finite upper bound on the number of actions for reaching a goal state have distance ∞. ■

Given the backward distance sets we can construct a plan covering all states having a finite backward distance. Let $S' \subseteq S$ be those states having a finite backward distance. The plan π is defined by assigning for every $s \in S$ such that $\delta_G^{bwd}(s) \geq 1$ $\pi(s)$ any operator $o \in O$ such that $\mathit{img}_o(s) \subseteq D_{i-1}^{bwd}$ where $i = \delta_G^{bwd}(s)$.

The plan execution starts from one of the initial states. As we have full observability, we may observe the current state s and then execute the action corresponding to the operator $\pi(s)$, reaching one of the successor states $s' \in \mathit{img}_o(s)$. The plan execution proceeds by repeatedly observing the new current state s' and executing the associated action $\pi(s')$ until the current state is a goal state.

Lemma 4.30 *Let a state s be in D_j. Then there is a plan that reaches a goal state from s by at most j operator applications.*

The algorithm can be implemented by using logic-based data structures and operations defined in Section 4.2 by representing the set of goal states as a formula, using the logic-based operation $spreimg_o(\phi)$ instead of the set-based operation $spreimg_o(T)$ for computing the sets D_i^{bwd} that are also represented as formulae, and replacing all set-theoretic operations like $\cup$ and $\cap$ by the respective logical operations $\vee$ and $\wedge$.

4.4.2 An algorithm for constructing plans with loops

There are many nondeterministic planning problems that require plans with loops because there is no finite upper bound on the number of actions that might be needed for reaching the goals. These plan executions with an unbounded length cannot be handled in acyclic plans of a finite size. For unbounded execution lengths we have to allow loops (cycles) in the plans.

Example 4.31 ■

The problem is those states that do not have a finite strong distance as defined Section 4.4.1. Reaching the goals from these states is either impossible or there is no finite upper bound on the number of actions that might be needed. For the former states nothing can be done, but the latter states can be handled by plans with loops.

We present an algorithm based on a generalized notion of distances that does not require reachability by a finitely bounded number of actions. The algorithm is based on the procedure *prune* that identifies a set of states for which reaching a goal state eventually is guaranteed. The procedure *prune* is given in Figure 4.2.

We introduce some terminology. Let S be a set of states, O a set of operators, and $x : S \to O$ a mapping from states to operators. A sequence $s_0, \ldots, s_n$ of states is an *execution* if for all $i \in \{1, \ldots, n\}$ there is $o \in O$ such that $s_i \in img_o(s_{i-1})$, and it is an *execution of* x if $s_i \in img_{x(s_{i-1})}(s_{i-1})$ for all $i \in \{1, \ldots, n\}$.

Lemma 4.32 (Procedure prune) *Let S be a set of states, O a set of operators and $G \subseteq S$ a set of states. Then the procedure call prune(S,O,G) will terminate after a finite number of steps returning a set of states $W \subseteq S$ such that there is function $x : W \to O$ such that*

1. *for every $s \in W$ there is an execution $s_0, s_1, \ldots, s_n$ of x with $n \geq 1$ such that $s = s_0$ and $s_n \in G$,*
2. *$img_{x(s)}(\{s\}) \subseteq W \cup G$ for every $s \in W$, and*
3. *There is no function x satisfying the above properties for states not in W: for every $s \in S \backslash W$ and function $x' : S \to O$ there is an execution $s_0, \ldots, s_n$ of x' such that $s = s_0$ and there is no $m \geq n$ and execution $s_n, s_{n+1}, \ldots, s_m$ such that $s_m \in G$.*

Proof: The proof is by two nested induction proofs that respectively correspond to the repeat-until loops on lines 9 and 13 in the procedure *prune*. If there is no plan that is guaranteed to reach a goal state from a state s, then this is because for any plan after some number of executions steps i it is possible to reach a state from which no sequence actions can reach a goal state. A plan covering

```
1:  procedure prune(S,O,G);
2:  W_{-1} := S;
3:  W_0 := ∅;
4:  repeat
5:    W'_0 := W_0;
6:    W_0 := (W'_0 ∪ ⋃_{o∈O} preimg_o(W'_0 ∪ G)) ∩ S;
7:  until W_0 = W'_0;      (* States from which a goal state can be reached by ≥ 1 operators *)
8:  i := 0;
9:  repeat
10:   i := i + 1;
11:   k := 0;
12:   S_0 := ∅;
13:   repeat
14:     k := k + 1;          (* States from which a state in G is reachable with ≤ k steps. *)
15:     S_k := S_{k-1} ∪ ⋃_{o∈O}(S ∩ preimg_o(S_{k-1} ∪ G) ∩ spreimg_o(W_{i-1} ∪ G));
16:   until S_k = S_{k-1};          (* States that stay within W_{i-1} before reaching G. *)
17:   W_i := S_k;
18: until W_i = W_{i-1};           (* States in W_i stay within W_i before reaching G. *)
19: return W_i;
```

Figure 4.2: Algorithm for detecting a loop that eventually makes progress

all other states exists with an execution reaching a goal state in some h steps. The outer loop and induction go through $i = 0, 1, 2, \ldots$ and the inner loop and induction through $h = 0, 1, 2, \ldots$.

Induction hypothesis: There is function $x : W_i \to O$ such that

1. for every $s \in W_i$ there is an execution $s_0, \ldots, s_n$ of x such that $n \geq 1$, $s = s_0$ and $s_n \in G$,
2. $\mathit{img}_{x(s)}(\{s\}) \subseteq W_{i-1} \cup G$ for every $s \in W_i$, and
3. for all functions $x' : S \to O$ and states $s \in S \backslash W_i$ there is $i' \in \{0, \ldots, i\}$ and an execution $s_0, \ldots, s_{i'}$ of x' such that $s_0 = s$ and there is no $h \geq i'$ and execution $s_{i'}, s_{i'+1}, \ldots, s_h$ such that $s_h \in G$.

Base case $i = 0$:

1. W_0 has been computed to fulfill exactly this property. We denote the value of the variables W_0 in the end of iteration i of the first repeat-until loop by $W_{0,i}$.

   Induction hypothesis:

   (a) There is a function $x : W_{0,j} \to O$ such that there is an execution of x for every $s \in W_{0,j}$ of length $j \geq 1$ reaching a state in G.

   (b) For states not in $W_{0,j}$ there is no x with this property.

   Base case $j = 1$: After the first iteration $W_{0,1} = \bigcup_{o \in O} \mathit{preimg}_o(G)$. Hence for every $s \in W_{0,1}$ assign $x(s) = o$ for any o such that $s \in \mathit{preimg}_o(G)$.

   (a) Now there is an execution of length 1 from any $s \in W_{0,1}$ to a state in G.

(b) For states not in $W_{0,1}$ no one operator may reach a state in G.

Inductive case $j \geq 2$: By induction hypothesis there is a function x with execution of length $j-1 \geq 1$ for reaching a state in G for every state for which such an execution exists. We extend this function to cover states $s \in W_{0,j} \backslash W_{0,j-1}$ as follows: $x(s) = o$ for any o such that $s \in \mathit{preimg}_o(W_{0,j-1} \cup G)$.

(a) For any $s \in W_{0,j}$ there is an execution of x reaching a state in G because for states $s \in W_{0,j-1}$ this is by the induction hypothesis, and for states in $W_{0,j} \backslash W_{0,j-1}$ applying the operator $x(s)$ may reach a state in $W_{0,j-1}$ for which an execution reaching G exists by the induction hypothesis.

(b) Let s be a state such that there is a function $x' : S \to O$ with an execution that reaches G from s with j steps. Hence there is a state s' for which an execution with x' reaches G from s' with $j-1$ steps. Hence by the induction hypothesis $s \in W_{0,j-1}$ and consequently $s \in \mathit{preimg}_{x'(s)}(W_{0,j-1})$. Therefore for any state not in $W_{0,j}$ there is no such function x'.

2. Because $W_{-1} = S$ trivially $\mathit{img}_{x(s)}(\{s\}) \subseteq W_{-1} \cup G$.

3. States $s \in W_0 \backslash W_{-1}$ are exactly those states from which no operator sequence leads to G by construction of W_0, as shown above.

Inductive case $i \geq 1$: For the inner *repeat-until* loop we prove inductively the following. Induction hypothesis: There is function $x : S_k \to O$ such that

1. for every $s \in S_k$ there is an execution $s_0, s_1, \ldots, s_n$ of x such that $n \in \{1, \ldots, k\}$, $s = s_0$ and $s_n \in G$,

2. $\mathit{img}_{x(s)}(\{s\}) \subseteq W_{i-1} \cup G$ for every $s \in S_k$, and

3. for all functions $x' : S \to O$ and states $s \in S \backslash S_k$ either

   (a) there is $i' \in \{0, \ldots, i\}$ and an execution $s_0, \ldots, s_{i'}$ of x' such that $s_0 = s$ and there is no $h \geq i'$ and execution $s_{i'}, s_{i'+1}, \ldots, s_h$ such that $s_h \in G$, or

   (b) there is no $k' \in \{1, \ldots, k\}$ and an execution $s_0, \ldots, s_{k'}$ of x' such that $s_0 = s$ and $s_{k'} \in G$.

Base case $k = 0$: Because $S_0 = \emptyset$, cases (1) and (2) trivially hold for every $s \in S_0$. It remains to show the third component of the induction hypothesis.

3. For any $s \in S \backslash S_0 = S$ (3b) is satisfied because it requires executions to be longer than $k = 0$.

Inductive case $k \geq 1$: We extend the function $x : S_{k-1} \to O$ to cover states in $S_k \backslash S_{k-1}$. Let s be any state in S_k. If $s \in S_{k-1}$ then properties (1) and (2) are by the induction hypothesis. Otherwise $s \in S_k \backslash S_{k-1}$. Therefore by definition of S_k, $s \in \mathit{preimg}_o(S_{k-1} \cup G) \cap \mathit{spreimg}_o(W_{i-1} \cup G)$ for some $o \in O$.

1. Because $s \in \mathit{preimg}_o(S_{k-1} \cup G)$ for some $o \in O$, by (4) of Lemma 2.2 either $s \in \mathit{preimg}_o(S_{k-1})$ or $s \in \mathit{preimg}_o(G)$.

   If $s \in \mathit{preimg}_o(G)$ then we set $x(s) = o$. The desired execution just consists of s and a state $s' \in G$.

   If $s \in \mathit{preimg}_o(S_{k-1}) \backslash \mathit{preimg}_o(G)$ then there is a state $s' \in S_{k-1}$ such that $s' \in \mathit{img}_o(\{s\})$. By the induction hypothesis there is an execution of x starting from s' that ends in a goal state. For s such an execution is obtained by prefixing with o, so we define $x(s) = o$.

2. Because $s \in \mathit{spreimg}_o(W_{i-1} \cup G)$, by (2) and (3) of Lemma 2.2 $\mathit{img}_o(\{s\}) \subseteq W_{i-1} \cup G$.

3. Take any $s \in S \backslash S_k$. Now for every operator $o \in O$, either $s \notin \mathit{spreimg}_o(W_{i-1} \cup G)$ or $s \notin \mathit{preimg}_o(S_{k-1} \cup G)$. Consider any function $x' : S \to O$ such that $x'(s) = o$.

   In the first case by the outer induction hypothesis there is $i' \in \{0, \ldots, i-1\}$ and an execution $s_0, \ldots, s_{i'}$ of x' such that $s_0 \in \mathit{img}_o(s)$ and there is no $h \geq i'$ and execution $s_{i'}, s_{i'+1}, \ldots, s_h$ such that $s_h \in G$. Hence executing o first could similarly lead to the state $s_{i'}$ from which no goal could be reached, now requiring i steps.

   In the second case by the inner induction hypothesis for all $s' \in \mathit{img}_o(s)$ there is no execution of length $k-1$ ending in a goal state.

   Because this holds for any $o \in O$, every x' has one of these properties.

This completes the inner induction. To establish the induction step of the outer induction consider the following. The inner repeat-until loops ends when $S_k = S_{k-1}$. This means that $S_z = S_k$ for all $z \geq k$. Hence executions for reaching a goal state for (1) and (3) are allowed to have arbitrarily high length k. The outer induction hypothesis is obtained from the inner induction hypothesis by removing the upper bound and replacing S_k by W_i. By construction $W_i = S_k$.

This finishes the outer induction proof. The claim of the lemma is obtained from the outer induction hypothesis by noticing that the outer loop exits when $W_i = W_{i-1}$ (it will exit after a finite number of iterations because W_0 is finite and its size decreases on every iteration) and then we can replace both W_i and W_{i-1} by W to obtain the claim of the lemma. □

The algorithm in Figure 4.3 uses *prune* to identify states from which a goal state is reachable by some execution and no execution leads to a state outside the set. On line 4 the algorithm tests whether the reachability of a goal state can be guaranteed for the initial states. If not, the algorithm terminates. Starting on line 7 the algorithm computes the *weak backward distances* to G for all states in L. Finally, starting on line 11 the algorithm assigns every state in $L \backslash G$ an operator that may reduce the distance to a goal by one.

4.4.3 An algorithm for constructing plans for maintenance goals

There are many important planning problems in which the objective is not to reach a goal state and then stop execution. A *maintanence goal* is a goal that has to hold in all time points. To achieve a maintenance goals a plan has to keep the state of the system in a goal state indefinitely.

Plans that satisfy a maintenance goal have only infinite executions.

Figure 4.4 gives an algorithm for finding plans for maintenance goals. The algorithm starts with the set G of all states that satisfy the property to be maintained. Then iteratively such states

```
1:  procedure FOplancyclic(I,O,G)
2:  S := the set of all states;
3:  L := G∪ prune(S,O,G);
4:  if I ⊈ L then return false;
5:  D_0 := G;                                   (* States with weak backward distance 0 *)
6:  i := 1;
7:  repeat
8:    D_i := D_{i−1} ∪ ⋃_{o∈O}(preimg_o(D_{i−1}) ∩ spreimg_o(L));
9:    i := i + 1;
10: until D_i = D_{i−1};
11: for each s ∈ D_i\G do
12:   d := number such that s ∈ D_d\D_{d−1};     (* State has weak backward distance d. *)
13:   assign π(s) := o such that img_o(s) ⊆ L and img_o(s) ∩ D_{d−1} ≠ ∅;
14: end do
```

Figure 4.3: Algorithm for nondeterministic planning with full observability

```
1:  procedure FOplanMAINTENANCE(I,O,G)
2:  i := 0;
3:  G_0 := G;
4:  repeat
5:    i := i + 1;          (* The subset of G_{i−1} from which G_{i−1} can be always reached. *)
6:    G_i := ⋃_{o∈O} (spreimg_o(G_{i−1}) ∩ G_{i−1});
7:  until G_i = G_{i−1};
8:  return G_i;
9:  for each s ∈ G_i do
10:   assign π(s) := o such that img_o(s) ⊆ G_i;
11: end do
```

Figure 4.4: Algorithm for nondeterministic planning with full observability and maintenance goals

are removed from G for which the satisfaction of the property cannot be guaranteed in the next time point. More precisely, the sets G_i for $i \geq 0$ consist of all those states in which the goal objective can be maintained for the next i time points. For some i the sets G_i and G_{i-1} coincide, and then $G_j = G_i$ for all $j \geq i$. This means that starting from the states in G_i the goal objective can be maintained indefinitely.

Theorem 4.33 *Let I be a set of initial states, O a set of operator and G a set of goal states. Let G' be the set returned by the procedure FOplanMAINTENANCE in Figure 4.4.*

Then $G' \subseteq G$ and there is a plan π such that $img_{\pi(s)}(s) \subseteq G'$ for every $s \in G'$, and for every $s \in S \backslash G'$ and every plan π' there is $n \geq 1$ and and an execution $s_0, \ldots, s_n$ of π' with $s_0 = s$ such that $s_n \notin G$.

Proof:

Induction hypothesis:

1. $G_i \subseteq G$,
2. there is a plan π such that $img_{\pi(s)}(s) \subseteq G_{i-1}$ for every $s \in G_i$, and
3. for every $s \in S \backslash G_i$ and every plan π' there is $n \in \{1, \ldots, i\}$ and an execution $s_0, \ldots, s_n$ of π' with $s_0 = s$ such that $s_n \notin G$.

Base case $i = 1$:

1. $G_1 \subseteq G_0 = G$ by construction.
2. By construction $G_1 = \bigcup_{o \in O}(spreimg_o(G_0) \cap G_0)$. Hence for every $s \in G_1$ there is $o \in O$ such that $s \in spreimg_o(G_0) \cap G_0 \subseteq spreimg_o(G_0)$. Hence $img_o(s) \subseteq G_0$. Define $\pi(s) = o$. Hence $img_{\pi(s)}(s) \subseteq G_0$ for every $s \in G_1$.
3. Consider any $s \in S \backslash G_1$. For every $o \in O$ $img_o(s) \not\subseteq G_0 = G$ because $s \notin G_1$. Hence for every $s \in S \backslash G_1$ and every plan π' there is an execution s_0, s_1 of π' with $s_0 = s$ such that $s_1 \notin G$.

Inductive case $i \geq 2$:

1. As $G_i \subseteq G_{i-1}$ and by the induction hypothesis $G_{i-1} \subseteq G$, $G_i \subseteq G$.
2. By construction $G_i = \bigcup_{o \in O}(spreimg_o(G_{i-1}) \cap G_{i-1})$. Hence for every $s \in G_i$ there is $o \in O$ such that $s \in spreimg_o(G_{i-1}) \cap G_{i-1} \subseteq spreimg_o(G_{i-1})$. Hence $img_o(s) \subseteq G_{i-1}$. Define $\pi(s) = o$. Hence there is a plan π such that $img_{\pi(s)}(s) \subseteq G_{i-1}$ for every $s \in G_i$.
3. Consider any $s \in S \backslash G_i$. Hence $s \notin spreimg_o(G_{i-1})$ for every $o \in O$. Hence for every $o \in O$ there is $s' \in S \backslash G_{i-1}$ such that $s' \in img_o(s)$. By the induction hypothesis for every plan π' there is $n \in \{1, \ldots, i-1\}$ and an execution $s_0, \ldots, s_n$ of π' with $s_0 = s'$ such that $s_n \notin G$. Hence for every plan π' there is also $n' = n + 1 \in \{1, \ldots, i\}$ and an execution $s, s_0, \ldots, s_n$ of π' with $s_0 = s'$ such that $s_n \notin G$.

Because G_i are finite sets and $G_i \subseteq G_{i-1}$ and every G_{i+1} is a function of G_i, $G_j = G_{j-1}$ for some j and the loop iteration terminates after a finite number of iterations.

Now the claim of the lemma are obtained as follows.

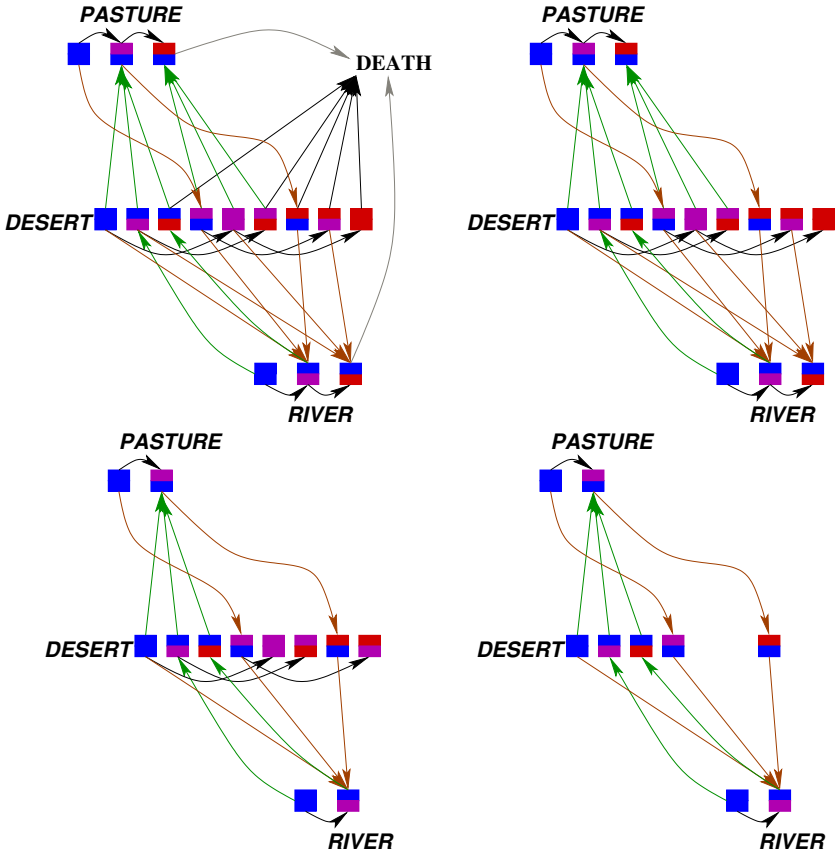


Figure 4.5: Example run of the algorithm for maintenance goals

1. The G' that is returned is $G' = G_j$. By the induction proof $G_j \subseteq G$.

2. By the termination condition of the loop $G_j = G_{j+1} = G'$. Hence by the results of the induction proof there is a plan π such that $img_{\pi(s)}(s) \subseteq G'$ for every $s \in G'$.

3. Because $G' = G_j = G_{j-1}$ and G_j is a function of G_{j-1}, the sets G_k for all $k \geq j$ equal G_j. Hence the constant n for the length of executions leading outside G can be arbitrarily high. By the results of the induction proof for every $s \in S \backslash G'$ and every π' there is $n \geq 0$ and an execution $s_0, \ldots, s_n$ of π' with $s_0 = s$ such that $s_n \notin G$.

□

Example 4.34 Consider the problem depicted in Figure 4.5. An animal may drink at a river and eat at a pasture. To get from the river to the pasture it must go through a desert. Its hunger and thirst increase after every time point after respectively leaving the pasture or the river. If either one reaches level 3 the animal dies. The hunger and thirst levels are indicated by different colors: the upper halves of the rectanges show thirst level and the lower halves the hunger level. Blue means no hunger or thirst, red means much hunger or thirst. The upper left diagram shows all the possible actions the animal can take. The objective of the animal is to stay alive. The three iterations of the

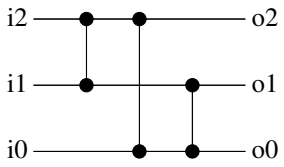


Figure 4.6: A sorting network with three inputs

algorithm for finding a plan that satisfies the goal of staying alive are depicted by the remaining three diagrams. The diagram on upper right depicts all the states that satisfy the goal. The diagram on lower left depicts all the states that satisfy the goal and after which the satisfaction of the goal can be guaranteed for at least one time period. The diagram on lower right depicts all the states that satisfy the goal and after which the satisfaction of the goal can be guaranteed for at least two time periods.

Further iterations of the algorithm do not eliminate further states, and hence the last diagram depicts all those states for which the satisfaction of the goal can be guaranteed indefinitely.

Hence the only plan says that the animal has to go continuously back and forth between the pasture and the river. The only choice the animal has is in the beginning if in the initial state it is not at all hungry or thirsty. For instance, if it is in the desert initially, then it may freely choose whether to first go to the pasture or the river. ■

4.5 Planning without observability

4.5.1 Planning without observability by heuristic search

Planning under unobservability is similar to deterministic planning in the sense that the problem is to find a path from the initial state(s) to the goal states. For unobservable planning, however, the nodes in the graph do not correspond to individual states but to belief states, and the size of the belief space is exponentially higher than the size of the state space. Algorithms for deterministic planning have direct counterparts for unobservable planning, which is not the case for conditional planning with full or partial observability.

Example 4.35 A sorting network [Knuth, 1998, Section 5.3.4 in 2nd edition] consists of a sequence of gates acting on a number of input lines. Each gate combines a comparator and a swapper: if the first value is greater than the second, then swap them. The goal is to sort any given input sequence. The sorting network always has to perform the same operations irrespective of the input, and hence constructing a sorting network corresponds to planning without observability. Figure 4.6 depicts a sorting network with three inputs. An important property of sorting networks is that any network that sorts any sequence of zeros and ones will also sort any sequence of arbitrary numbers. Hence it suffices to consider Boolean 0-1 input values only.

Construction of sorting networks is essentially a planning problem without observability, because there are several initial states and a goal state has to be reached by using the same sequence of actions irrespective of the initial states.

For the 3-input sorting net the initial states are $000, 001, 010, 011, 100, 101, 110, 111$. and the goal states are $000, 001, 011, 111$ Now we can compute the images and strong preimages of the three sorting actions, sort12, sort02 and sort01 respectively starting from the initial or the goal states. These yield the following belief states at different stages of the sorting network.

$000, 001, 010, 011, 100, 101, 110, 111$	initially
$000, 001, 011, 100, 101, 111$	after sort12
$000, 001, 011, 101, 111$	after sort02
$000, 001, 011, 111$	after sort01

■

The most obvious approaches to planning with unobservability is to use regression, strong preimages or images, and to perform backward or forward search in the belief space. The difference to forward search with deterministic operators and one initial state is that belief states are used instead of states. The difference to backward search for deterministic planning is that regression for nondeterministic operators has to be used and testing whether (a subset of) the initial belief state has been reached involves the co-NP-hard inclusion test $\models I \rightarrow \mathit{regr}_o(\phi)$ for the belief states. With one initial state this is an easy polynomial time test $I \models \mathit{regr}_o(\phi)$ of whether $\mathit{regr}_o(\phi)$ is true in the initial state.

Deriving good heuristics for heuristic search in the belief space is more difficult than in deterministic planning. The main approaches have been to use distances in the state space as an estimate for distances in the belief space, and to use the cardinalities of belief spaces as a measure of progress.

Many problems cannot be solved by blindly taking actions that reduce the cardinality of the current belief state: the cardinality of the belief state may stay the same or increase during plan execution, and hence the decrease in cardinality is not characteristic to belief space planning in general, even though in many problems it is a useful measure of progress.

Similarly, distances in the state space ignore the most distinctive aspect of planning with partial observability: the same action must be used in two states if the states are not observationally distinguishable. A given (optimal) plan for an unobservable problem may increase the actual current state-space distance to the goal states (on a given execution) when the distance in the belief-space monotonically decreases, and vice versa. Hence, the state space distances may yield wildly misleading estimates of the distances in the corresponding belief space.

Heuristics based on state-space distances

The most obvious distance heuristics are based on the backward distances in the state space.

$$\begin{aligned} D_0 &= G \\ D_{i+1} &= D_i \cup \textstyle\bigcup_{o \in O} \mathit{spreimg}_o(D_i) \text{ for all } i \geq 1 \end{aligned}$$

A lower bound on plan length for belief state Z is j if $Z \subseteq D_j$ and $Z \not\subseteq D_{j-1}$.

Next we derive distance heuristics for the belief space based on state space distances. Backward distances yield an admissible distance heuristic for belief states.

Definition 4.36 (State space distance) *The* state space distance *of a belief state B is $d \geq 1$ when $B \subseteq D_d$ and $B \not\subseteq D_{d-1}$, and it is 0 when $B \subseteq D_0 = G$.*

Even though computing the exact distances for the operator based representation of state spaces is PSPACE-hard, the much higher complexity of planning problems with partial observability still often justifies it: this computation would in many cases be an inexpensive preprocessing step, preceding the much more expensive solution of the partially observable planning problem. Otherwise cheaper approximations can be used.

Heuristics based on belief state cardinality

The second heuristic that has been used in algorithms for partial observability is simply based on the cardinality of the belief states.

In forward search, prefer operators that maximally decrease the cardinality of the belief state.

In backward search, prefer operators that maximally increase the cardinality of the belief state.

These heuristics are not in general admissible, because there is no direct connection between the distance to a goal belief state and the cardinalities of the current belief state and a goal belief state. The belief state cardinality can decrease or increase arbitrarily much by one step.

4.6 Planning as satisfiability in the propositional logic and QBF

The techniques presented in Sections 3.6 and 3.6.5 can be extended to nondeterministic operators. The notion of parallel application of operators and partially ordered plans can be generalized to nondeterministic operators.

Let T be a set of operators and s a state such that $s \models c$ for every $\langle c, e\rangle \in T$ and $E_1 \cup \cdots \cup E_n$ is consistent for for any $E_i \in [e_i]_s, i \in \{1, \ldots, n\}$ and $T = \{\langle c_1, e_1\rangle, \ldots \langle c_n, e_n\rangle\}$. Then define $\mathit{img}_T(s)$ as the set of states s' that are obtained from s by making $E_1 \cup \cdots \cup E_n$ true in s where $E_i \in [e_i]_s$ for every $i \in \{1, \ldots, n\}$. We also use the notation sTs' for $s' \in \mathit{img}_T(s)$ and $\mathit{img}_T(S) = \bigcup_{s\in S} \mathit{img}_T(s)$.

4.6.1 Advanced translation of nondeterministic operators into propositional logic

In Section 4.1.2 we showed how nondeterministic operators can be translated into formulae in the propositional logic. This translation is not sufficient for reasoning about actions and plans in a setting with more than one agent. This is because the formulae $\tau_A^{nd}(o_1) \vee \cdots \vee \tau_A^{nd}(o_n)$ do not distinguish between the choice of operator in $\{o_1, \ldots, o_n\}$ and the nondeterministic effects (the opponent) of each operator, even though the former is controllable and the latter is not.

In nondeterministic planning in general we have to treat the controllable and uncontrollable choices differently. We cannot do this practically in the propositional logic but by using quantified Boolean formulae (QBF) we can. For the QBF representation of nondeterministic operators we universally quantify over all uncontrollable eventualities (nondeterminism) and existentially quantify over controllable eventualities (the choice of operators).

We need to universally quantify over all the nondeterministic choices because for every choice the remaining operators in the plan must lead to a goal state. This is achieved by associating with every atomic effect a formula that is true if and only if that effect is executed, similarly to functions $\mathit{EPC}_l(e)$ in Definition 3.1, so that for l to become true the universally quantified auxiliary variables that represent nondeterminism have to have values corresponding to an effect that makes l true.

The operators are assumed to be in normal form. For simplicity of presentation we further transform nondeterministic choices $e_1|\cdots|e_n$ so that only binary choices exist. For example $a|b|c|d$ is replaced by $(a|b)|(c|d)$. Each binary choice can be encoded in terms of one auxiliary variable.

The condition for the atomic effect l to be executed when e is executed is $\mathit{EPC}_l^{nd}(e, \sigma)$. The sequence σ of integers is used for deriving unique names for auxiliary variables in $\mathit{EPC}_l^{nd}(e, \sigma)$. The sequences correspond to paths in the tree formed by nested nondeterministic choices and

conjunctions.

$$\begin{array}{ll} EPC_l^{nd}(e,\sigma) & = EPC_l(e) \text{ if } e \text{ is deterministic} \\ EPC_l^{nd}(e_1|e_2,\sigma) & = (x_\sigma \wedge EPC_l^{nd}(e_1,\sigma 1)) \vee (\neg x_\sigma \wedge EPC_l^{nd}(e_2,\sigma 1)) \\ EPC_l^{nd}(e_1 \wedge \cdots \wedge e_n,\sigma) & = EPC_l^{nd}(e_1,\sigma 1) \vee \cdots \vee EPC_l^{nd}(e_n,\sigma n) \end{array}$$

The translation of nondeterministic operators into the propositional logic is similar to the translation for deterministic operators given in Section 3.6.4.

Nondeterminism is encoded by making the effects conditional on the values of the auxiliary variables x_σ. Different valuations of these auxiliary variables correspond to different nondeterministic effects.

The following frame axioms express the conditions under which state variables $a \in A$ may change from true to false and from false to true. Let $e_1, \ldots, e_n$ be the effects of $o_1, \ldots, o_n$ respectively. Each operator $o \in O$ has a unique index $\Omega(o)$.

$$\begin{array}{l} (a \wedge \neg a') \rightarrow ((o_1 \wedge EPC_{\neg a}^{nd}(e_1,\Omega(o_1))) \vee \cdots \vee (o_n \wedge EPC_{\neg a}^{nd}(e_n,\Omega(o_n)))) \\ (\neg a \wedge a') \rightarrow ((o_1 \wedge EPC_{a}^{nd}(e_1,\Omega(o_1))) \vee \cdots \vee (o_n \wedge EPC_{a}^{nd}(e_n,\Omega(o_n)))) \end{array}$$

For $o = \langle c, e\rangle \in O$ there is a formula for describing values of state variables in the predecessor and successor states when the operator is applied.

$$\begin{array}{l} (o \rightarrow c) \wedge \\ \bigwedge_{a\in A}(o \wedge EPC_a^{nd}(e,\Omega(o)) \rightarrow a') \wedge \\ \bigwedge_{a\in A}(o \wedge EPC_{\neg a}^{nd}(e,\Omega(o)) \rightarrow \neg a') \end{array}$$

Example 4.37 Consider $o_1 = \langle \neg a, (b|(c \rhd d)) \wedge (a|c)\rangle$ and $o_2 = \langle \neg b, (((d \rhd b)|c)|a)\rangle$. The application of these operators is described by the following formulae.

$$\begin{array}{ll} \neg(a \wedge \neg a') & (\neg a \wedge a') \rightarrow ((o_1 \wedge x_{12}) \vee (o_2 \wedge \neg x_2)) \\ \neg(b \wedge \neg b') & (\neg b \wedge b') \rightarrow ((o_1 \wedge x_{11}) \vee (o_2 \wedge x_2 \wedge x_{21} \wedge d)) \\ \neg(c \wedge \neg c') & (\neg c \wedge c') \rightarrow ((o_1 \wedge \neg x_{12}) \vee (o_2 \wedge x_2 \wedge \neg x_{21})) \\ \neg(d \wedge \neg d') & (\neg d \wedge d') \rightarrow (o_1 \wedge \neg x_{11} \wedge c) \\ o_1 \rightarrow \neg a & \\ (o_1 \wedge x_{12}) \rightarrow a' & (o_1 \wedge x_{11}) \rightarrow b' \\ (o_1 \wedge \neg x_{12}) \rightarrow c' & (o_1 \wedge \neg x_{11} \wedge c) \rightarrow d' \\ o_2 \rightarrow \neg b & \\ (o_2 \wedge \neg x_2) \rightarrow a' & (o_2 \wedge x_2 \wedge x_{21} \wedge d) \rightarrow b' \\ (o_2 \wedge x_2 \wedge \neg x_{21}) \rightarrow c' & \end{array}$$

■

Two operators o and o' may be applied in parallel only if they do not interfere. Hence we use formulae

$$\neg(o \wedge o')$$

for all operators o and o' that interfere and $o \neq o'$.

Let X be the set of auxiliary variables x_σ in all the above formulae. The conjunction of all the above formulae is denoted by

$$\mathcal{R}_3(A, A', O, X).$$

We need two auxiliary definitions for proving properties about these formulae and the translation of nondeterministic operators into the propositional logic.

Let $\Xi_\sigma(e)$ be the set of propositional variables $x_{\sigma'}$ in the translation of the effect e with a given σ. This is equal to the set of variables $x_{\sigma'}$ in formulae $EPC_a^{nd}(e,\sigma)$ and $EPC_{\neg a}^{nd}(e,\sigma))$ for all $a \in A$.

Definition 4.38 *Define the set of literals $[e]_s^{\sigma,v}$ which are the active effects of e when e is executed in state s and nondeterministic choices are determined by the valuation v of propositional variables $\Xi_\sigma(e)$ as follows.*

$$\begin{aligned}
[e]_s^{\sigma,v} &= [e]_s^{det} \text{ if } e \text{ is deterministic} \\
[e_1|e_2]_s^{\sigma,v} &= \begin{cases} [e_1]_s^{\sigma 1,v} \text{ if } v(x_\sigma) = 1 \\ [e_2]_s^{\sigma 1,v} \text{ if } v(x_\sigma) = 0 \end{cases} \\
[e_1 \wedge \cdots \wedge e_n]_s^{\sigma,v} &= [e_1]_s^{\sigma 1,v} \cup \cdots \cup [e_n]_s^{\sigma n,v}
\end{aligned}$$

Lemma 4.39 *Let s be a state and $\{v_1,\ldots,v_n\}$ all valuations of $X = \Xi_\sigma(e)$. Then $\bigcup_{1\leq i\leq n}[e]_s^{\sigma,v_i} = [e]_s$.*

Lemma 4.40 *Let O and $T \subseteq O$ be sets of operators. Let s and s' be states. Let v_x a valuation of $X = \bigcup_{\langle c,e\rangle\in O}\Xi_{\Omega(\langle c,e\rangle)}(e)$. Let v_o be a valuation of O such that $v_o(o) = 1$ iff $o \in T$.*
Then $s \cup s'[A'/A] \cup v_o \cup v_x \models \mathcal{R}_3(A,A',O,X)$ if and only if

1. *$s \models a$ iff $s' \models a$ for all $a \in A$ such that $\{a,\neg a\} \cap \bigcup_{\langle c,e\rangle\in T}[e]_s^{\Omega(\langle c,e\rangle),v_x} = \emptyset$,*
2. *$s' \models \bigcup_{\langle c,e\rangle\in T}[e]_s^{\Omega(\langle c,e\rangle),v_x}$, and*
3. *$s \models c$ for all $\langle c,e\rangle \in T$.*

The number of auxiliary variables x_σ can be reduced when two operators o and o' interfere. Since they cannot be applied simultaneously the same auxiliary variables can control the nondeterminism in both operators. To share the variables rename the ones occurring in the formulae for one of the operators so that the variables needed for o is a subset of those for o' or vice versa. Having as small a number of auxiliary variables as possible may be important for the efficiency for algorithms evaluating QBF and testing propositional satisfiability.

The formulae $\mathcal{R}_3(A,A',O,X)$ can be used for plan search with algorithms that evaluate QBF (Section 4.6.2) as well as for testing by a satisfiability algorithm whether a conditional plan (with full, partial or no observability) that allows several operators simultaneously indeed is a valid plan.

4.6.2 Finding plans by evaluation of QBF

In deterministic planning in propositional logic (Section 3.6) the problem is to find a sequence of operators so that a goal state is reached when the operators are applied starting in the initial state. When there are several initial states, the operators are nondeterministic and it is not possible to use observations during plan execution for selecting operators, the problem is to find an operator sequence so that a goal state is reached in all possible executions of the operator sequence. There may be several executions because there may be several initial states and the operators may be nondeterministic. Expressing the quantification over all possible executions cannot be concisely

expressed in the propositional logic. This is the reason why quantified Boolean formulae are used instead.

The existence of an n-step partially-ordered plan that reaches a state satisfying G from any state satisfying the formula I can be tested by evaluating the QBF Φ_n^{qpar} defined as

$$\exists V_{plan} \forall V_{nd} \exists V_{exec}$$
$$I^0 \rightarrow (\mathcal{R}_3(A^0, A^1, O^0, X^0) \wedge \mathcal{R}_3(A^1, A^2, O^1, X^1) \wedge \cdots \wedge \mathcal{R}_3(A^{n-1}, A^n, O^{n-1}, X^{n-1}) \wedge G^n).$$

Here $V_{plan} = O^0 \cup \cdots \cup O^{n-1}$, $V_{nd} = A^0 \cup X^0 \cup \cdots \cup X^{n-1}$ and $V_{exec} = A^1 \cup \cdots \cup A^n$. Define $\Phi_n^{qparM} = I^0 \rightarrow (\mathcal{R}_3(A^0, A^1, O^0, X^0) \wedge \mathcal{R}_3(A^1, A^2, O^1, X^1) \wedge \cdots \wedge \mathcal{R}_3(A^{n-1}, A^n, O^{n-1}, X^{n-1}) \wedge G^n)$. The valuation of V_{plan} corresponds to a sequence of sets of operators. For a given valuation of V_{plan} the valuation of V_{nd} determines the execution of these operators. The valuation of V_{exec} is uniquely determined by the valuation of $V_{plan} \cup V_{nd}$.

The algorithms for evaluating QBF that extend the Davis-Putnam procedure traverse an and-or tree in which the and-nodes correspond to universally quantified variables and or-nodes correspond to existentially quantified variables. If the QBF is *true* then these algorithms return a valuation of the outermost existential variables. For a true Φ_n^{qpar} this valuation of V_{plan} corresponds to a plan that can be constructed like the plans in the deterministic case in Section 3.6.5.

Theorem 4.41 *The QBF Φ_n^{qpar} has value* true *if and only if there is a sequence $T_0, \ldots, T_{n-1}$ of sets of operators such that for every $i \in \{0, \ldots, n\}$ and every sequence $s_0, \ldots, s_i$ such that*

1. $s_0 \models I$ *and*
2. $s_0 T_0 s_1 T_1 s_2 \cdots s_{i-1} T_{i-1} s_i$

T_i is applicable in s_i if $i < n$ and $s_i \models G$ if $i = n$.

Proof: We first prove the implication from left to right. Since Φ_n^{qpar} is true there is a valuation v_{plan} of $V_{plan} = O^0 \cup \cdots \cup O^{n-1}$ such that for all valuations v_{nd} of $V_{nd} = A^0 \cup X^0 \cup \cdots \cup X^{n-1}$ there is a valuation v_{exec} of $V_{exec} = A^1 \cup \cdots \cup A^n$ such that $v_{plan} \cup v_{nd} \cup v_{exec} \models I^0 \rightarrow (\mathcal{R}_3(A^0, A^1, O^0, X^0) \wedge \cdots \wedge \mathcal{R}_3(A^{n-1}, A^n, O^{n-1}, X^{n-1}) \wedge G^n)$.

Let $T_0, \ldots, T_{n-1}$ be the sequence of sets of operators such that for all $o \in O$ and $i \in \{0, \ldots, n-1\}$, $o \in T_i$ if and only if $v_{plan}(o^i) = 1$. We prove the right hand side of the theorem by induction on n.

Induction hypothesis: For every $s_0, \ldots, s_i$ such that $s_0 \models I$ and $s_0 T_0 s_1 T_1 s_2 \cdots s_{i-1} T_{i-1} s_i$:

1. T_i is applicable in s_i if $i < n$.
2. $s_i \models G$ if $i = n$.

Base case $i = 0$: Let s_0 be any state sequence such that $s_0 \models I$.

1. If $0 < n$ then we have to show that T_0 is applicable in s_0.

   Let $E = E_1 \cup \cdots \cup E_m$ for all $j \in \{1, \ldots, m\}$ and any $E_j \in [e_j]_{s_0}$, where $e_1, \ldots, e_m$ are respectively the effects of the operators $o_1, \ldots, o_m$ in T_0. Such sets E are the possible active effects of T_0.

   We have to show that E is consistent and the preconditions of operators in T_0 are true in s_0.

   By Lemma 4.39 there is a valuation v of X such that $E = \bigcup_{\langle c,e\rangle \in T_0} [e]_{s_0}^{\Omega(\langle c,e\rangle),v}$.

   Let v_{nd} be any valuation of V_{nd} such that $s_0[A^0/A] \subseteq v_{nd}$ and $v[X^0/X] \subseteq v_{nd}$. Since Φ_n^{qpar} is true there is a valuation of v_{exec} such that $v_{plan} \cup v_{nd} \cup v_{exec} \models \Phi_n^{qparM}$.

   Since $v_{nd} \models I^0$ also $v_{plan} \cup v_{nd} \cup v_{exec} \models \mathcal{R}_3(A^0, A^1, O^0, X^0)$. Hence by Lemma 4.40 the preconditions of operators in T_0 are true in s_0 and $s_1 \models E$ where s_1 is the state such that $s_1(a) = v_{exec}(a^1)$ for all $a \in A$. Since E was chosen arbitrarily from the sets of possible sets of active effects of T_0 and it is consistent, T_0 is applicable in s_0.

2. If $n = 0$ then $V_{plan} = V_{exec} = \emptyset$ and $\forall V_{nd}(I^0 \rightarrow G^0)$ is true, and $v_{nd} \models G^0$ for every valuation v_{nd} of V_{nd} such that $v_{nd} \models I^0$.

Inductive case $i \geq 1$: Let $s_0, \ldots, s_i$ be any sequence such that $s_0 \models I$ and $s_0 T_0 s_1 \ldots s_{i-1} T_{i-1} s_i$.

1. If $i < n$ then we have to show that T_i is applicable in s_i.

   Let $E = E_1 \cup \cdots \cup E_m$ for all $j \in \{1, \ldots, m\}$ and any $E_j \in [e_j]_{s_i}$, where $e_1, \ldots, e_m$ are respectively the effects of the operators $o_1, \ldots, o_m$ in T_i. Such sets E are the possible active effects of T_i.

   We have to show that E is consistent and the preconditions of operators in T_i are true in s_i.

By Lemma 4.39 there is a valuation v of X such that $E = \bigcup_{\langle c,e\rangle \in T_i} [e]_{s_i}^{\Omega(\langle c,e\rangle),v}$.

Since by the induction hypothesis $s_j T_j s_{j+1}$ for all $j \in \{0,\ldots,i-1\}$, by Lemma 4.39 for every $j \in \{0,\ldots,i-1\}$ there is a valuation v_j^x of X such that $s_j[A/A^j] \cup s_{j+1}[A'/A^{j+1}] \cup v_o \cup v_j^x \models \mathcal{R}_3(A, A', O, X)$ where v_o assigns every $o \in O$ value 1 iff $o \in T_j$.

Let v_{nd} be any valuation of V_{nd} such that $s_0[A^0/A] \subseteq v_{nd}$ and $v[X^i/X] \subseteq v_{nd}$ and $v_j^x[X^j/X] \subseteq v_{nd}$ for all $j \in \{0,\ldots,i-1\}$.

Since Φ_n^{qpar} is true there is a valuation of v_{exec} such that $v_{plan} \cup v_{nd} \cup v_{exec} \models \Phi_n^{qparM}$.

Since $v_{nd} \models I^0$ also $v_{plan} \cup v_{nd} \cup v_{exec} \models \mathcal{R}_3(A^i, A^{i+1}, O^i, X^i)$. Hence by Lemma 4.40 the preconditions of operators in T_i are true in s_i and $s_{i+1} \models E$ where s_{i+1} is a state such that $s_{i+1}(a) = v_{exec}(a^{i+1})$ for all $a \in A$. Since any E is consistent, T_i is applicable in s_i.

2. If $i = n$ we have to show that $s_n \models G$. Like in the proof for the previous case we construct valuations v_{nd} and v_{exec} matching the execution $s_0,\ldots,s_n$ and since $v_{plan} \cup v_{nd} \cup v_{exec} \models I^0 \rightarrow G^n$ we have $s_n \models G$.

Then we prove the implication from right to left. So there is sequence $T_0,\ldots,T_{n-1}$ for which all executions are defined and reach G.

We show that Φ_n^{qpar} is true: there is valuation v_{plan} of $V_{plan} = O^0 \cup \cdots \cup O^{n-1}$ such that for every valuation v_{nd} of $V_{nd} = A^0 \cup X^0 \cup \cdots \cup X^{n-1}$ there is a valuation v_{exec} of $V_{exec} = A^1 \cup \cdots \cup A^n$ such that $v_{plan} \cup v_{nd} \cup v_{exec} \models \Phi_n^{qparM}$.

We define the valuation v_{plan} of V_{plan} by $o \in T_i$ iff $v_{plan}(o^i) = 1$ for every $o \in O$ and $i \in \{0,\ldots,n-1\}$.

Take any valuation v_{nd} of V_{nd}. Define the state s_0 by $s_0(a) = 1$ iff $v_{nd}(a^0) = 1$ for every $a \in A$.

If $s_0 \not\models I$ then $v_{nd} \not\models I^0$ and $v_{plan} \cup v_{nd} \cup v_{exec} \models \Phi_n^{qparM}$ for any valuation v_{exec} of V_{exec}.

It remains to consider the case $s_0 \models I$.

Define for every $i \in \{1,\ldots,n\}$ sets E_i and states s_i as follows.

1. Let v_x^i be a valuation of X such that $v_x^i(x) = v_{nd}(x^{i-1})$ for every $x \in X$.

2. Let $E_i = \bigcup_{\langle c,e\rangle \in T_{i-1}} [e]_{s_{i-1}}^{\Omega(\langle c,e\rangle),v_x^i}$.

   We show below that this is the set of literals made true by T_{i-1} in s_{i-1}.

3. Define $s_i(a) = 1$ iff $a \in E_i$ or $s_{i-1}(a) = 1$ and $\neg a \notin E_i$, for every $a \in A$.

Let $v_{exec} = s_1[A^1/A] \cup \cdots \cup s_n[A^n/A]$.

Induction hypothesis: $v_{plan} \cup v_{nd} \cup s_1[A^1/A] \cup \cdots \cup s_i[A^i/A] \models I^0 \wedge \mathcal{R}_3(A^0, A^1, O^0, X^0) \wedge \cdots \wedge \mathcal{R}_3(A^{i-1}, A^i, O^{i-1}, X^{i-1})$ and $s_j T_j s_{j+1}$ for all $j \in \{0,\ldots,i-1\}$.

Base case $i = 0$: Trivial because $v_{nd} \models I^0$.

Inductive case $i \geq 1$: Let $v_x \subseteq v_{nd}$ be the valuation of X^{i-1} determined by v_{nd} and let v_o be the valuation of O^{i-1} such that $v_o(o) = v_{plan}(o^{i-1})$ for every $o \in O$. By Lemma 4.40 $v_{plan} \cup v_{nd} \cup s_{i-1}[A^{i-1}/A] \cup s_i[A^i/A] \models \mathcal{R}_3(A^{i-1}, A^i, O^{i-1}, X^{i-1})$. This together with the claim of the induction hypothesis for $i-1$ establishes the first part of the claim of the hypothesis for i. By Lemma 4.39 the set E_i is one of the possible sets of active effects of T_{i-1} in s_{i-1}. Hence $s_{i-1} T_{i-1} s_i$. This finishes the induction proof.

Hence $v_{plan} \cup v_{nd} \cup v_{exec} \models I^0 \wedge \mathcal{R}_3(A^0, A^1, O^0, X^0) \wedge \cdots \wedge \mathcal{R}_3(A^{n-1}, A^n, O^{n-1}, X^{n-1})$, and $v_{exec} \models G^n$ because $s_n \models G$ by assumption and $s_n[A^n/A] \subseteq v_{exec}$. □

4.7 Planning with partial observability

Planning with partial observability is much more complicated than its two special cases with full and no observability. Like planning without observability, the notion of belief states becomes very important. Like planning with full observability, formalization of plans as sequences of operators is insufficient. However, plans also cannot be formalized as mappings from states to operators because partial observability implies that the current state is not necessarily unambiguously known. Hence we will need the general definition of plans introduced in Section 4.3.1.

When executing operator o in belief state B the set of possible successor states is $img_o(B)$, and based on the observation that are made, this set is restricted to $B' = img_o(B) \cap C$ where C is the equivalence class of observationally indistinguishable states corresponding to the observation.

In planning with unobservability, a backward search algorithm starts from the goal belief state and uses regression or strong preimages for finding predecessor belief states until a belief state covering the initial belief state is found.

With partial observability, plans do not just contain operators but may also branch. With branching the sequence of operators may depend on the observations, and this makes it possible to reach goals also when no fixed sequence of operators reaches the goals. Like strong preimages in backward search correspond to images, the question arises what does branching correspond to in backward search?

Example 4.42 Consider the blocks world with three blocks with the goal state in which all the blocks are on the table. There are three operators, each of which picks up one block (if there is nothing on top of it) and places it on the table. We can only observe which blocks are not below another block. This splits the state space to seven observational classes, corresponding to the valuations of the state variables clear-A, clear-B and clear-C in which at least one block is clear.

The plan construction steps are given in Figure 4.7. Starting from the top left, the first diagram depicts the goal belief state. The second diagram depicts the belief states obtained by computing the strong preimage of the goal belief state with respect to the move-A-onto-table action and splitting the set of states to belief states corresponding to the observational classes. The next two diagrams are similarly for strong preimages of move-B-onto-table and move-C-onto-table.

The fifth diagram depicts the computation of the strong preimage from the union of two existing belief states in which the block A is on the table and C is on B or B is on C. In the resulting belief state A is the topmost block in a stack containing all three blocks. The next two diagrams similarly construct belief states in which respectively B and C are the topmost blocks.

The last three diagrams depict the most interesting cases, constructing belief states that subsume two existing belief states in one observational class. The first diagram depicts the construction of the belief state consisting of both states in which A and B are clear and C is under either A or B. This belief state is obtained as the strong preimage of the union of two existing belief states, the one in which all blocks are on the table and the one in which A is on the table and B is on top of C. The action that moves A onto the table yields the belief state because if A is on C all blocks will be on the table and if A is already on the table nothing will happen. Construction of the belief

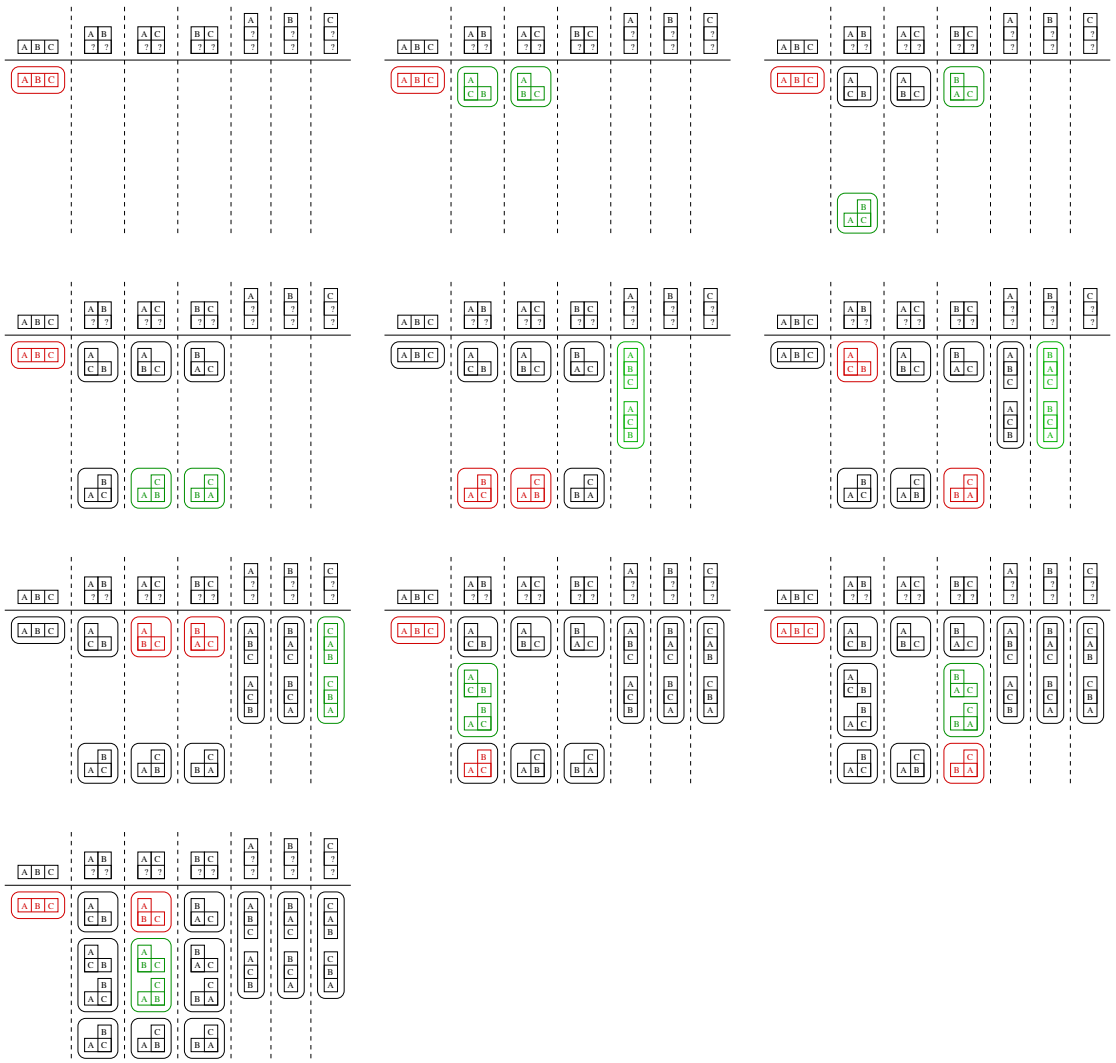


Figure 4.7: Solution of a simple blocks world problem

states in which B and C are clear and A and C are clear is analogous and depicted in the last two diagrams.

The resulting plan reaches the goal state from any state in the blocks world. The plan in the program form is given in Figure 4.8 (order of construction is from the end to the beginning.)

■

We restrict to acyclic plans. Construction of cyclic plans requires looking at more global properties of transition graphs than what is needed for acyclic plans. Taking these local properties into account is difficult because we want to avoid explicit enumeration of the belief states.

4.7.1 Problem representation

Now we introduce the representation for sets of state sets for which a plan for reaching goal states exists.

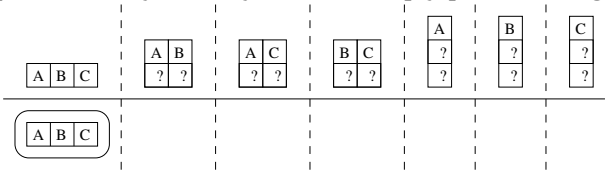
```
16:
  IF clear-A AND clear-B AND clear-C THEN GOTO end
  IF clear-A AND clear-C THEN GOTO 15
  IF clear-B AND clear-C THEN GOTO 13
  IF clear-A AND clear-B THEN GOTO 11
  IF clear-A THEN GOTO 5
  IF clear-B THEN GOTO 7
  IF clear-C THEN GOTO 9
15:
  move-C-onto-table
14:
  IF clear-A AND clear-B AND clear-C THEN GOTO end
  IF clear-A AND clear-C THEN GOTO 1
13:
  move-B-onto-table
12:
  IF clear-A AND clear-B AND clear-C THEN GOTO end
  IF clear-B AND clear-C THEN GOTO 3
11:
  move-A-onto-table
10:
  IF clear-A AND clear-B AND clear-C THEN GOTO end
  IF clear-A AND clear-B THEN GOTO 2
9:
  move-C-onto-table
8:
  IF clear-A AND clear-C THEN GOTO 1
  IF clear-B AND clear-C THEN GOTO 2
7:
  move-B-onto-table
6:
  IF clear-A AND clear-B THEN GOTO 1
  IF clear-B AND clear-C THEN GOTO 3
5:
  move-A-onto-table
4:
  IF clear-A AND clear-B THEN GOTO 2
  IF clear-A AND clear-C THEN GOTO 3
3:
  move-C-onto-table
  GOTO end
2:
  move-B-onto-table
  GOTO end
1:
  move-A-onto-table
end:
```

Figure 4.8: A plan for a partially observable blocks world problem

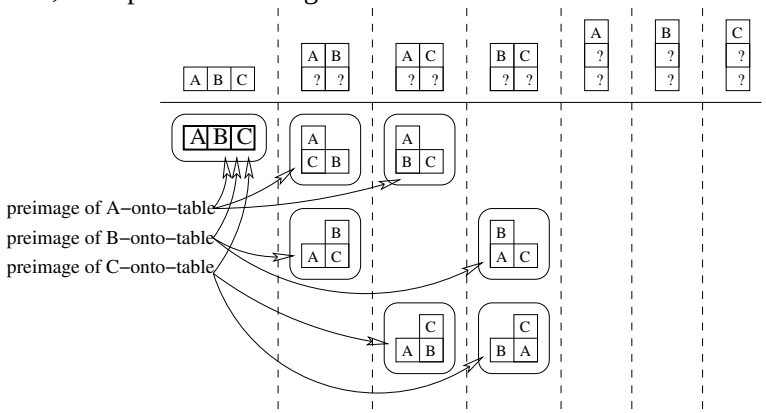
In the following example states are viewed as valuations of state variables, and the observational classes correspond to valuations of those state variables that are observable.

Example 4.43 Consider the blocks world with the state variables *clear(X)* observable, allowing to observe the topmost block of each stack. With three blocks there are 7 observational classes because there are 7 valuations of $\{\text{clear(A)}, \text{clear(B)}, \text{clear(C)}\}$ with at least one block clear.

Consider the problem of trying to reach the state in which all blocks are on the table. For each block there is an action for moving it onto the table from wherever it was before. If a block cannot be moved nothing happens. Initially we only have the empty plan for the goal states.

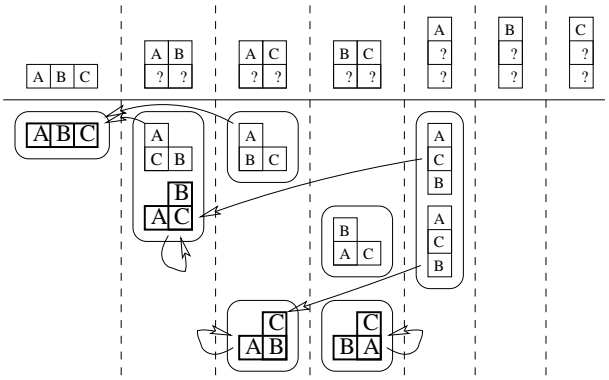


Then we compute the preimages of this set with actions that respectively put the blocks A, B and C onto the table, and split the resulting sets to the different observational classes.



Now for these 7 belief states we have a plan consisting of one or zero actions. But we also have plans for sets of states that are only represented implicitly. These involve branching. For example, we have a plan for the state set consisting of the four states in which respectively all blocks are on the table, A is on C, A is on B, and B is on A. This plan first makes observations and branches, and then executes the plan associated with the belief state obtained in each case. Because 3 observational classes each have 2 belief states, there are 2^3 maximal state sets with a branching plan. From each class only one belief state can be chosen because observations cannot distinguish between belief states in the same class.

We can find more belief states that have plans by computing preimages of existing belief states. Let us choose the belief states in which respectively all blocks are on the table, B is on C, C is on B, and C is on A, and compute their union's preimage with A-onto-table. The preimage intersected with the observational classes yields new belief states: for the class with A and B clear there is a new 2-state belief state covering both previous belief states in the class, and for the class with A clear there is a new 2-state belief state.



Computation of further preimages yields for each observational class a belief state covering all the states in that class, and hence a plan for every belief state. ■

Next we formalize the framework in detail.

Definition 4.44 (Belief space) *Let $P = (C_1, \ldots, C_n)$ be a partition of the set of all states. Then* a belief space *is an n-tuple $\langle G_1, \ldots, G_n \rangle$ where $G_i \subseteq 2^{C_i}$ for all $i \in \{1, \ldots, n\}$ and $B \not\subset B'$ for all $i \in \{1, \ldots, n\}$ and $\{B, B'\} \subseteq G_i$.*

Notice that in each component of a belief space we only have set-inclusion maximal belief states. The simplest belief spaces are obtained from sets B of states as $\mathcal{F}(B) = \langle \{C_1 \cap B\}, \ldots, \{C_n \cap B\} \rangle$. A belief space is extended as follows.

Definition 4.45 (Extension) *Let $P = (C_1, \ldots, C_n)$ be the partition of all states, $G = \langle G_1, \ldots, G_n \rangle$ a belief space, and T a set of states. Define $G \oplus T$ as $\langle G_1 \Cup (T \cap C_1), \ldots, G_n \Cup (T \cap C_n) \rangle$ where the operation $\Cup$ adds the latter set of states to the former set of sets of states and eliminates sets that are not set-inclusion maximal, defined as $U \Cup V = \{R \in U \cup \{V\} | R \not\subset K \text{ for all } K \in U \cup \{V\}\}$.*

A belief space $G = \langle G_1, \ldots, G_n \rangle$ represents the set of sets of states $\text{flat}(G) = \{B_1 \cup \cdots \cup B_n | B_i \in G_i \text{ for all } i \in \{1, \ldots, n\}\}$ and its cardinality is $|G_1| \cdot |G_2| \cdot \ldots \cdot |G_n|$.

4.7.2 Complexity of basic operations

The basic operations on belief spaces needed in planning algorithms are testing the membership of a set of states in a belief space, and finding a set of states whose preimage with respect to an action is not contained in the belief space. Next we analyze the complexity of these operations.

Theorem 4.46 *For belief spaces G and state sets B, testing whether there is $B' \in \text{flat}(G)$ such that $B \subseteq B'$, and computing $G \oplus B$ takes polynomial time.*

Proof: Idea: A linear number of set-inclusion tests suffices. □

Our algorithm for extending belief spaces by computing the preimage of a set of states (Lemma 4.48) uses exhaustive search and runs in worst-case exponential time. This asymptotic worst-case complexity is very likely the best possible because the problem is NP-hard. Our proof for this fact is a reduction from SAT: represent each clause as the set of literals that are not in it, and then a satisfying assignment is a set of literals that is not included in any of the sets, corresponding to the same question about belief spaces.

Theorem 4.47 *Testing if for belief space G there is $R \in$ flat(G) such that preimg$_o(R) \not\subseteq R'$ for all $R' \in$ flat(G) is NP-complete. This holds even for deterministic actions o.*

Proof: Membership is easy: For $G = \langle G_1, \ldots, G_n \rangle$ choose nondeterministically $R_i \in G_i$ for every $i \in \{1, \ldots, n\}$, compute $R =$ preimg$_o(R_1 \cup \cdots \cup R_n)$, and verify that $R \cap C_i \not\subseteq B$ for some $i \in \{1, \ldots, n\}$ and all $B \in G_i$. Each of these steps takes only polynomial time.

Let $T = \{c_1, \ldots, c_m\}$ be a set of clauses over propositions $A = \{a_1, \ldots, a_k\}$. We define a belief space based on states $\{a_1, \ldots, a_k, \hat{a}_1, \ldots, \hat{a}_k, z_1, \ldots, z_k, \hat{z}_1, \ldots, \hat{z}_k\}$. The states $\hat{a}$ represent negative literals. Define

$$\begin{aligned} c'_i &= (A \backslash c_i) \cup \{\hat{a} | a \in A, \neg a \notin c_i\} \text{ for } i \in \{1, \ldots, m\}, \\ G &= \langle \{c'_1, \ldots, c'_m\}, \{\{z_1\}, \{\hat{z}_1\}\}, \ldots, \{\{z_k\}, \{\hat{z}_k\}\} \rangle, \\ o &= \{\langle a_i, z_i \rangle | 1 \leq i \leq k\} \cup \{\langle \hat{a}_i, \hat{z}_i \rangle | 1 \leq i \leq k\}. \end{aligned}$$

We claim that T is satisfiable if and only if there is $B \in$ flat(G) such that preimg$_o(B) \not\subseteq B'$ for all $B' \in$ flat(G).

Assume T is satisfiable, that is, there is M such that $M \models T$. Define $M' = \{z_i | a_i \in A, M \models a_i\} \cup \{\hat{z}_i | a_i \in A, M \not\models a_i\}$. Now $M' \subseteq B$ for some $B \in$ flat(G) because from each class only one of $\{z_i\}$ or $\{\hat{z}_i\}$ is taken. Let $M'' =$ preimg$_o(M') = \{a_i \in A | M \models a_i\} \cup \{\hat{a}_i | a_i \in A, M \not\models a_i\}$. We show that $M'' \not\subseteq B$ for all $B \in$ flat(G). Take any $i \in \{1, \ldots, m\}$. Because $M \models c_i$, there is $a_j \in c_i \cap A$ such that $M \models a_j$ (or $\neg a_j \in c_i$, for which the proof goes similarly.) Now $z_j \in M'$, and therefore $a_j \in M''$. Also, $a_j \notin c'_j$. As there is such an a_j (or $\neg a_j$) for every $i \in \{1, \ldots, m\}$, M'' is not a subset of any c'_i, and hence $M'' \not\subseteq B$ for all $B \in$ flat(G).

Assume there is $B \in$ flat(G) such that $D =$ preimg$_o(B) \not\subseteq B'$ for all $B' \in$ flat(G). Now D is a subset of $A \cup \{\hat{a} | a \in A\}$ with at most one of a_i and $\hat{a}_i$ for any $i \in \{1, \ldots, k\}$. Define a model M such that for all $a \in A$, $M \models a$ if and only if $a \in D$. We show that $M \models T$. Take any $i \in \{1, \ldots, m\}$ (corresponding to a clause.) As $D \not\subseteq B$ for all $B \in$ flat(G), $D \not\subseteq c'_i$. Hence there is a_j or $\hat{a}_j$ in $D \backslash c'_i$. Consider the case with a_j ($\hat{a}_j$ goes similarly.) As $a_j \notin c'_i$, $a_j \in c_i$. By definition of M, $M \models a_j$ and hence $M \models c_i$. As this holds for all $i \in \{1, \ldots, m\}$, $M \models T$. □

4.7.3 Algorithms

Based on the problem representation in the preceding section, we devise a planning algorithm that repeatedly identifies new belief states (and associated plans) until a plan covering the initial states is found. The algorithm in Figure 4.10 tests for plan existence; further book-keeping is needed for outputting a plan. The size of the plan is proportional to the number of iterations the algorithm performs, and outputting the plan takes polynomial time in the size of the plan. The algorithm uses the subprocedure *findnew* (Figure 4.9) for extending the belief space (this is the NP-hard subproblem from Theorem 4.47). Our implementation of the subprocedure orders sets $f_1, \ldots, f_m$ by cardinality in a decreasing order: bigger belief states are tried first. We also use a simple pruning technique for deterministic actions o: If preimg$_o(f_i) \subseteq$ preimg$_o(f_j)$ for some i and j such that $i > j$, then we may ignore f_i.

Lemma 4.48 *Let H be a belief space and o an action. The procedure call findnew(o,$\emptyset$,F, H) returns a set B' of states such that $B' =$ preimg$_o(B)$ for some $B \in$ flat(F) and $B' \not\subseteq B''$ for all $B'' \in$ flat(H), and if no such belief state exists it returns $\emptyset$.*

```
1:  procedure findnew(o,A,F,H);
2:  if F = ⟨⟩ and preimg_o(A) ⊈ B for all B ∈ flat(H)
3:  then return A;
4:  if F = ⟨⟩ then return ∅;
5:  F is ⟨{f_1, ..., f_m}, F_2, ..., F_k⟩ for some k ≥ 1;
6:  for i := 1 to m do
7:    B := findnew(o,A ∪ f_i,⟨F_2, ..., F_k⟩,H);
8:    if B ≠ ∅ then return B;
9:  end;
10: return ∅;
```

Figure 4.9: Algorithm for finding new belief states

```
1:  procedure plan(I,O,G);
2:  H := F(G);
3:  progress := true;
4:  while progress and I ⊈ I' for all I' ∈ flat(H) do
5:    progress := false;
6:    for each o ∈ O do
7:      B := findnew(o,∅,H,H);
8:      if B ≠ ∅ then
9:        begin
10:         H := H ⊕ preimg_o(B);
11:         progress := true;
12:       end;
13:   end;
14: end;
15: if I ⊆ I' for some I' ∈ flat(H) then return true
16: else return false;
```

Figure 4.10: Algorithm for planning with partial observability

Proof: Sketch: The procedure goes through the elements $\langle B_1, \ldots, B_n \rangle$ of $F_1 \times \cdots \times F_n$ and tests whether $\mathit{preimg}_o(B_1 \cup \cdots \cup B_n)$ is in H. The sets $B_1 \cup \cdots \cup B_n$ are the elements of flat(F). The traversal through $F_1 \times \cdots \times F_n$ is by generating a search tree with elements of F_1 as children of the root node, elements of F_2 as children of every child of the root node, and so on, and testing whether the preimage is in H. The second parameter of the procedure represents the state set constructed so far from the belief space, the third parameter is the remaining belief space, and the last parameter is the belief space that is to be extended, that is, the new belief state may not belong to it. □

The correctness proof of the procedure *plan* consists of the following lemma and theorems. The first lemma simply says that extending a belief space H is monotonic in the sense that the members of flat(H) can only become bigger.

Lemma 4.49 *Assume T is any set of states and $B \in$ flat(H). Then there is $B' \in$ flat$(H \oplus T)$ so that $B \subseteq B'$.*

The second lemma says that if we have belief states in different observational classes such that each is included in a belief state of a belief space H, then there is a set in flat(H) that includes all these belief states.

Lemma 4.50 *Let $B_1, \ldots, B_n$ be sets of states so that for every $i \in \{1, \ldots, n\}$ there is $B'_i \in$ flat(H) such that $B_i \subseteq B'_i$, and there is no observational class C such that for some $\{i, j\} \subseteq \{1, \ldots, n\}$ both $i \neq j$ and $B_i \cap C \neq \emptyset$ and $B_j \cap C \neq \emptyset$. Then there is $B' \in$ flat(H) such that $B_1 \cup \cdots \cup B_n \subseteq B'$.*

The proof of the next theorem shows how the algorithm is capable of finding any plan by constructing it bottom up starting from the leaf nodes. The construction is based on first assigning a belief state to each node in the plan, and then showing that the algorithm reaches that belief state from the goal states by repeated computation of preimages.

Theorem 4.51 *Whenever there exists a finite acyclic plan for a problem instance, the algorithm in Figure 4.10 returns* true.

Proof: Assume that there is a plan $\langle N, b, l\rangle$ for a problem instance $\langle S, I, O, G, P\rangle$. We assume that states in S are valuations of a set of state variables. Label all nodes of the plan as follows. Each initial node n_i for $i \in \{1, \ldots, m\}$ with $\{\langle\phi_1, n_1\rangle, \ldots, \langle\phi_m, n_m\rangle\}$ we assign the label $Z(n_i) = \{s \in I | s \models \phi_i\}$.

When all parent nodes $n_1, \ldots, n_m\rangle$ of a node n have a label, we assign a label to n. Let $l(n_i) = \langle o_i, \{\langle\phi_i, n\rangle, \ldots\}\rangle$ for all $i \in \{1, \ldots, m\}$. Then $Z(n) = \bigcup_{i\in\{1,\ldots,n\}}\{s \in \mathit{img}_{o_i}(Z(n_i)) | s \models \phi_i\}$. If the above labeling does not assign anything to a node n, then assign $Z(n) = \emptyset$. Each node is labeled with exactly those states that are possible in that node on some execution.

We show that if plans for $Z(n_1), \ldots, Z(n_k)$ exist, where $n_1, \ldots, n_k$ are children of a node n, then the algorithm determines that a plan for $Z(n)$ exists as well.

Induction hypothesis: for every plan node n such that all paths from it to a terminal node have length i or less, $B = Z(n)$ is a subset of some $B' \in$ flat(H) where H is the value of the program variable H after the *while* loop exits and H could not be extended further.

Base case $i = 0$: Terminal nodes of the plan are labeled with subsets of G. By Lemma 4.49 there is G' such that $G \subseteq G'$ and $G' \in$ flat(H) because initially $H = \mathcal{F}(G)$ and thereafter it was repeatedly extended.

Inductive case $i \geq 1$: Let n be a plan node with $l(n) = (o, \{\langle\phi_1, n_1\rangle, \ldots, \langle\phi_k, n_k\rangle\}$.

We show that $Z(n) \subseteq B$ for some $B \in$ flat(H).

By the induction hypothesis $Z(n_i) \subseteq B$ for some $B \in$ flat(H) for all $i \in \{1, \ldots, k\}$.

For all $i \in \{1, \ldots, k\}$ $\{s \in \mathit{img}_o(Z(n)) | s \models \phi_i\} \subseteq Z(n_i)$.

Hence by Lemma 4.50 $B = \bigcup_{i\in\{1,\ldots,k\}}\{s \in \mathit{img}_o(Z(n)) | s \models \phi_i\} \subseteq B'$ for some $B' \in$ flat(H). Assume that there is no such B''. But now by Lemma 4.48 findnew(o,$\emptyset$,H,H) would return B''' such that $\mathit{preimg}_o(B''') \not\subseteq B$ for all $B \in$ flat(H), and the *while* loop could not have exited with H, contrary to our assumption about H. □

Theorem 4.52 *Let $\Pi = \langle S, I, O, G, P\rangle$ be a problem instance. If plan(I,O,G) returns* true*, then Π has a solution plan.*

Proof: Let $H_0, H_1, \ldots$ be the sequence of belief spaces H produced by the algorithm.

Induction hypothesis: For every $B \in H_{i,j}$ for some $j \in \{1,\ldots,n\}$ and $H_i = \langle H_{i,1},\ldots,H_{i,n}\rangle$ a plan reaching G exists.

Base case $i = 0$: Every component of H_0 consists of a subset of G. The empty plan reaches G.

Inductive case $i \geq 1$: H_{i+1} is obtained as $H_i \oplus \mathit{preimg}_o(B)$ where $B =$ findnew(o,$\emptyset$,H_i,H_i) and o is an operator.

By Lemma 4.48 $B \in$ flat(H_i). By the induction hypothesis there are plans π_i for every $B \cap C_i, i \in \{1,\ldots,n\}$. The plan that executes o followed by π_i on observation C_i reaches G from $\mathit{preimg}_o(B)$.

Let $B' \in H_{i+1,j}$ for $H_{i+1} = \langle H_{i+1,1},\ldots,H_{i+1,n}\rangle$ and some $j \in \{1,\ldots,n\}$. We show that for B' there is a plan for reaching G.

If $B' \in H_{i,j}$ then by the induction hypothesis a plan exists.

Otherwise $B' \subseteq \mathit{preimg}_o(B)$ and we can use the plan for $\mathit{preimg}_o(B)$ that first applies o and then continues with a plan associated with one of the belief states in H_i. □

It would be easy to define an algorithm that systematically generates all belief states (plans) breadth-first and therefore plans with optimal execution lengths, but this algorithm would in practice be much slower and plans would be bigger.

Above we have used only one partition of the state space to observational classes. However, it is straightforward to generalize the above definitions and algorithms to the case in which several partitions are used, each for a different set of actions. This means that the possible observations depend on the action that has last been taken.

4.8 Computational complexity

In this section we analyze the computational complexity of the main decision problems related to nondeterministic planning. The conditional planning problem is a generalization of the deterministic planning problem from Chapter 3, and therefore the plan existence problem is at least PSPACE-hard. In this section we discuss the computational complexity of each of the three planning problems, the fully observable, the unobservable, and the general partially observable planning problem, showing them respectively complete for the complexity classes EXP, EXPSPACE and 2-EXP.

4.8.1 Planning with full observability

We first show that the plan existence problem for nondeterministic planning with full observability is EXP-hard and then that the problem is in EXP.

The EXP-hardness proof in Theorem 4.53 is by simulating polynomial-space alternating Turing machines by nondeterministic planning problems with full observability and the using the fact that the complexity classes EXP and APSPACE are the same (see Section 2.4.) The most interesting thing in the proof is the representation of alternation. Theorem 3.59 already showed how deterministic Turing machines with a polynomial space bound are simulated, and the difference is that we now have nondeterminism, that is, a configuration of the TM may have several successor configurations, and that there are both $\forall$ and $\exists$ states.[2]

[2]Restricting the proof of Theorem 4.53 to $\exists$ states with nondeterministic transitions would yield a proof of the NPSPACE-hardness of deterministic planning, but this is not interesting as PSPACE=NPSPACE.

The $\forall$ states mean that all successor configurations must be accepting (terminal or non-terminal) configurations. The $\exists$ states mean that at least one successor configuration must be an accepting (terminal or non-terminal) configuration. Both of these requirements can be represented in the nondeterministic planning problem.

The transitions from a configuration with a $\forall$ state will correspond to one nondeterministic operator. That all successor configurations must be accepting (terminal or non-terminal) configurations corresponds to requirement in planning that from all successor states of a state a goal state must be reached.

Every transition from a configuration with $\exists$ state will correspond to a deterministic operator, that is, the transition may be chosen, as only one of the successor configurations needs to be accepting.

Theorem 4.53 *The problem of testing the existence of an acyclic plan for problem instances with full observability is EXP-hard.*

Proof: Let $\langle \Sigma, Q, \delta, q_0, g \rangle$ be any alternating Turing machine with a polynomial space bound $p(x)$. Let σ be an input string of length n.

We construct a problem instance in nondeterministic planning with full observability for simulating the Turing machine. The problem instance has a size that is polynomial in the size of the description of the Turing machine and the input string.

The set A of state variables in the problem instance consists of

1. $q \in Q$ for denoting the internal states of the TM,
2. s_i for every symbol $s \in \Sigma \cup \{|, \Box\}$ and tape cell $i \in \{0, \ldots, p(n)\}$, and
3. h_i for the positions of the R/W head $i \in \{0, \ldots, p(n) + 1\}$.

The unique initial state of the problem instance represents the initial configuration of the TM. The corresponding formula is the conjunction of the following literals.

1. q_0
2. $\neg q$ for all $q \in Q \backslash \{q_0\}$.
3. s_i for all $s \in \Sigma$ and $i \in \{1, \ldots, n\}$ such that ith input symbol is s.
4. $\neg s_i$ for all $s \in \Sigma$ and $i \in \{1, \ldots, n\}$ such that ith input symbol is not s.
5. $\neg s_i$ for all $s \in \Sigma$ and $i \in \{0, n+1, n+2, \ldots, p(n)\}$.
6. $\Box_i$ for all $i \in \{n+1, \ldots, p(n)\}$.
7. $\neg \Box_i$ for all $i \in \{0, \ldots, n\}$.
8. $|_0$
9. $\neg |_i$ for all $n \in \{1, \ldots, p(n)\}$
10. h_1
11. $\neg h_i$ for all $i \in \{0, 2, 3, 4, \ldots, p(n) + 1\}$

The goal is the following formula.

$$G = \bigvee \{q \in Q | g(q) = \text{accept}\}$$

Next we define the operators. All the transitions may be nondeterministic, and the important thing is whether the transition is for a $\forall$ state or an $\exists$ state.[3] For a given input symbol and a $\forall$ state, the transition corresponds to one nondeterministic operator, whereas for a given input symbol and an $\exists$ state the transitions corresponds to a set of deterministic operators.

To define the operators, we first define effects corresponding to all possible transitions.

For all $\langle s, q\rangle \in (\Sigma \cup \{|, \square\}) \times Q$, $i \in \{0, \ldots, p(n)\}$ and $\langle s', q', m\rangle \in (\Sigma \cup \{|\}) \times Q \times \{L, N, R\}$ define the effect $\tau_{s,q,i}(s', q', m)$ as $\alpha \wedge \kappa \wedge \theta$ where the effects α, κ and θ are defined as follows.

The effect α describes what happens to the tape symbol under the R/W head. If $s = s'$ then $\alpha = \top$ as nothing on the tape changes. Otherwise, $\alpha = \neg s_i \wedge s'_i$ to denote that the new symbol in the ith tape cell is s' and not s.

The effect κ describes the change to the internal state of the TM. Again, either the state changes or does not, so $\kappa = \neg q \wedge q'$ if $q \neq q'$ and $\top$ otherwise. We define $\kappa = \neg q$ when $i = p(n)$ and $m = R$ so that when the space bound gets violated, no accepting state can be reached.

The effect θ describes the movement of the R/W head. Either there is movement to the left, no movement, or movement to the right. Hence

$$\theta = \begin{cases} \neg h_i \wedge h_{i-1} & \text{if } m = L \\ \top & \text{if } m = N \\ \neg h_i \wedge h_{i+1} & \text{if } m = R \end{cases}$$

By definition of TMs, movement at the left end of the tape is always to the right. Similarly, we have state variable for R/W head position $p(n) + 1$ and moving to that position is possible, but no transitions from that position are possible, as the space bound has been violated.

Now, these effects that represent possible transitions are used in the operators that simulate the ATM. Operators for existential states $q, g(q) = \exists$ and for universal states $q, g(q) = \forall$ differ. Let $\langle s, q\rangle \in (\Sigma \cup \{|, \square\}) \times Q$, $i \in \{0, \ldots, p(n)\}$ and $\delta(s, q) = \{\langle s_1, q_1, m_1\rangle, \ldots, \langle s_k, q_k, m_k\rangle\}$.

If $g(q) = \exists$, then define k deterministic operators

$$\begin{aligned} o_{s,q,i,1} &= \langle h_i \wedge s_i \wedge q, \tau_{s,q,i}(s_1, q_1, m_1)\rangle \\ o_{s,q,i,2} &= \langle h_i \wedge s_i \wedge q, \tau_{s,q,i}(s_2, q_2, m_2)\rangle \\ &\vdots \\ o_{s,q,i,k} &= \langle h_i \wedge s_i \wedge q, \tau_{s,q,i}(s_k, q_k, m_k)\rangle \end{aligned}$$

That is, the plan determines which transition is chosen.

If $g(q) = \forall$, then define one nondeterministic operator

$$\begin{aligned} o_{s,q,i} = \langle h_i \wedge s_i \wedge q,\ & (\tau_{s,q,i}(s_1, q_1, m_1)| \\ & \tau_{s,q,i}(s_2, q_2, m_2)| \\ & \vdots \\ & \tau_{s,q,i}(s_k, q_k, m_k))\rangle. \end{aligned}$$

That is, the transition is chosen nondeterministically.

[3] No operators are needed for accepting or rejecting states.

We claim that the problem instance has a plan if and only if the Turing machine accepts without violating the space bound.

If the Turing machine violates the space bound, the state variable $h_{p(n)+1}$ becomes true and an accepting state cannot be reached because no operator will be applicable.

Otherwise, we show inductively that from a computation tree of an accepting ATM we can extract a conditional plan that always reaches a goal state, and vice versa. For obtaining an correspondence between conditional plans and computation trees it is essential that the plans are acyclic.

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So, because all alternating Turing machines with a polynomial space bound can be in polynomial time translated to a nondeterministic planning problem, all decision problems in APSPACE are polynomial time many-one reducible to nondeterministic planning, and the plan existence problem is APSPACE-hard and consequently EXP-hard. □

We can extend Theorem 4.53 to general plans with loops. The problem looping plans cause in the proofs of this theorem is that a Turing machine computation of infinite length is not accepting but the corresponding infinite length zero-probability plan execution is allowed to be a part of plan and would incorrectly count as an accepting Turing machine computation.

To eliminate infinite plan executions we have to modify the Turing machine simulation. This is by counting the length of the plan executions and failing when at least one state or belief state must have been visited more than once. This modification makes infinite loops ineffective, and any plan containing a loop can be translated to a finite non-looping plan by unfolding the loop. In the absence of loops the simulation of alternating Turing machines is faithful.

Theorem 4.54 *The plan existence problem for problem instances with full observability is EXP-hard.*

Proof: This is an easy extension of the proof of Theorem 4.53. If there are n state variables, an acyclic plan exists if and only if a plan with execution length at most 2^n exists, because visiting any state more than once is unnecessary. Plans that rely on loops can be invalidated by counting the number of actions taken and failing when this exceeds 2^n. This counting can be done by having $n + 1$ auxiliary state variables $c_0, \ldots, c_n$ that are initialized to false. Every operator $\langle p, e \rangle$ is extended to $\langle p, e \wedge t \rangle$ where t is an effect that increments the binary number encoded by $c_0, \ldots, c_n$ by one until the most significant bit c_n becomes one. The goal G is replaced by $G \wedge \neg c_n$.

Then a plan exists if and only if an acyclic plan exists if and only if the alternating Turing machine accepts. □

Theorem 4.55 *The problem of testing the existence of a plan for problem instances with full observability is in EXP.*

Proof: The algorithm in Section 4.4.2 runs in exponential time in the size of the problem instance. □

4.8.2 Planning without observability

The plan existence problem of conditional planning with unobservability is more complex than that of conditional planning with full observability.

To show the unobservable problem EXPSPACE-hard by a direct simulation of exponential space Turing machines, the first problem is how to encode the tape of the TM. With polynomial space, as in the PSPACE-hardness and APSPACE-hardness proofs of deterministic planning and conditional planning with full observability, it was possible to represent all the tape cells as the state variables of the planning problem. With an exponential space bound this is not possible any more, as we would need an exponential number of state variables, and the planning problem could not be constructed in polynomial time.

Hence we have to find a more clever way of encoding the working tape. It turns out that we can use the uncertainty about the initial state for this purpose. When an execution of the plan that simulates the Turing machine is started, we randomly choose one of the tape cells to be the *watched* tape cell. This is the only cell of the tape for which the current symbol is represented in the state variables. On all transitions the plan makes, if the watched tape cell changes, the change is reflected in the state variables.

That the plan corresponds to a simulation of the Turing machine it is tested whether the transition the plan makes when the current tape cell is the watched tape cell is the one that assumes the current symbol to be the one that is stored in the state variables. If it is not, the plan is not a valid plan. Because the watched tape cell could be any of the exponential number of tape cells, all the transitions the plan makes are guaranteed to correspond to the contents of the current tape cell of the Turing machine, so if the plan does not simulate the Turing machine, the plan is not guaranteed to reach the goal states.

The proof requires both several initial states and unobservability. Several initial states are needed for selecting the watched tape cell, and unobservability is needed so that the plan cannot cheat: if the plan can determine what the current tape cell is, it could choose transitions that do not correspond to the Turing machine on all but the watched tape cell. Because of unobservability all the transitions have to correspond to the Turing machine.

Theorem 4.56 *The problem of testing the existence of a plan for problem instances with unobservability is EXPSPACE-hard.*

Proof: Proof is a special case of the proof of Theorem 4.59. We do not have $\forall$ states and restrict to deterministic Turing machines. Nondeterministic Turing machines could be simulated for a NEXPSPACE-hardness proof, but it is already known that EXPSPACE = NEXPSPACE, so this additional generality would not bring anything.

Let $\langle \Sigma, Q, \delta, q_0, g \rangle$ be any deterministic Turing machine with an exponential space bound $e(x)$. Let σ be an input string of length n. We denote the ith symbol of σ by σ_i.

The Turing machine may use space $e(n)$, and for encoding numbers from 0 to $e(n) + 1$ corresponding to the tape cells we need $m = \lceil \log_2(e(n) + 2) \rceil$ Boolean state variables.

We construct a problem instance in nondeterministic planning without observability for simulating the Turing machine. The problem instance has a size that is polynomial in the size of the description of the Turing machine and the input string.

We cannot have a state variable for every tape cell because the reduction from Turing machines to planning would not be polynomial time. It turns out that it is not necessary to encode the whole contents of the tape in the transition system of the planning problem, and that it suffices to keep

track of only one tape cell (which we will call the *watched tape cell*) that is randomly chosen in the beginning of every execution of the plan.

The set A of state variables in the problem instance consists of

1. $q \in Q$ for denoting the internal states of the TM,
2. w_i for $i \in \{0, \ldots, m-1\}$ for the watched tape cell $i \in \{0, \ldots, e(n)\}$,
3. s for every symbol $s \in \Sigma \cup \{|, \square\}$ for the contents of the watched tape cell,
4. h_i for $i \in \{0, \ldots, m-1\}$ for the position of the R/W head $i \in \{0, \ldots, e(n)+1\}$.

The uncertainty in the initial state is about which tape cell is the watched one. Otherwise the formula encodes the initial configuration of the TM, and it is the conjunction of the following formulae.

1. q_0
2. $\neg q$ for all $q \in Q \backslash \{q_0\}$.
3. Formulae for having the contents of the watched tape cell in state variables $\Sigma \cup \{|, \square\}$.

$$\begin{aligned} | &\leftrightarrow (w = 0) \\ \square &\leftrightarrow (w > n) \\ s &\leftrightarrow \textstyle\bigvee_{i \in \{1,\ldots,n\}, \sigma_i = s} (w = i) \text{ for all } s \in \Sigma \end{aligned}$$

4. $h = 1$ for the initial position of the R/W head.

So the initial state formula allows any values for state variables w_i and the values of the state variables $s \in \Sigma$ are determined on the basis of the values of w_i. The expressions $w = i$, $w > i$ denote the obvious formulae for testing integer equality and inequality of the numbers encoded by $w_0, w_1, \ldots$. Later we will also use effects $h := h + 1$ and $h := h - 1$ that represent incrementing and decrementing the number encoded by $h_0, h_1, \ldots$.

The goal is the following formula.

$$G = \bigvee \{q \in Q | g(q) = \text{accept}\}$$

To define the operators, we first define effects corresponding to all possible transitions.

For all $\langle s, q \rangle \in (\Sigma \cup \{|, \square\}) \times Q$ and $\langle s', q', m \rangle \in (\Sigma \cup \{|\}) \times Q \times \{L, N, R\}$ define the effect $\tau_{s,q}(s', q', m)$ as $\alpha \wedge \kappa \wedge \theta$ where the effects α, κ and θ are defined as follows.

The effect α describes what happens to the tape symbol under the R/W head. If $s = s'$ then $\alpha = \top$ as nothing on the tape changes. Otherwise, $\alpha = ((h = w) \rhd (\neg s \wedge s'))$ to denote that the new symbol in the watched tape cell is s' and not s.

The effect κ describes the change to the internal state of the TM. Again, either the state changes or does not, so $\kappa = \neg q \wedge q'$ if $q \neq q'$ and $\top$ otherwise. If R/W head movement is to the right we define $\kappa = \neg q \wedge ((h < e(n)) \rhd q')$ if $q \neq q'$ and $(h = e(n)) \rhd \neg q$ otherwise. This prevents reaching an accepting state if the space bound is violated: no further operator applications are possible.

The effect θ describes the movement of the R/W head. Either there is movement to the left, no movement, or movement to the right. Hence

$$\theta = \begin{cases} h := h - 1 & \text{if } m = L \\ \top & \text{if } m = N \\ h := h + 1 & \text{if } m = R \end{cases}$$

By definition of TMs, movement at the left end of the tape is always to the right.

Now, these effects $\tau_{s,q}(s', q', m)$ which represent possible transitions are used in the operators that simulate the DTM. Let $\langle s, q\rangle \in (\Sigma \cup \{|, \square\}) \times Q$ and $\delta(s, q) = \{\langle s', q', m\rangle\}$.

If $g(q) = \exists$, then define the operator

$$o_{s,q} = \langle((h \neq w) \vee s) \wedge q, \tau_{s,q}(s', q', m)\rangle$$

It is easy to verify that the planning problem simulates the DTM assuming that when operator $o_{s,q}$ is executed the current tape symbol is indeed s. So assume that some $o_{s,q}$ is the first operator that misrepresents the tape contents and that $h = c$ for some tape cell location c. Now there is an execution of the plan so that $w = c$. On this execution the precondition $o_{s,q}$ is not satisfied, and the plan is not executable. Hence a valid plan cannot contain operators that misrepresent the tape contents. □

Theorem 4.57 *The problem of testing the existence of a plan for problem instances with unobservability is in EXPSPACE.*

Proof: Proof is similar to the proof Theorem 3.60 but works at the level of belief states. □

The two theorems together yield the EXPSPACE-completeness of the plan existence problem for conditional planning without observability.

4.8.3 Planning with partial observability

We show that the plan existence problem of the general conditional planning problem with partial observability is 2-EXP-complete. The hardness proof is by a simulation of AEXPSPACE=2-EXP Turing machines. Membership in 2-EXP is obtained directly from the decision procedure discussed earlier: the procedure runs in polynomial time in the size of the enumerated belief space of doubly exponential size.

Showing that the plan existence problem for planning with partial observability is in 2-EXP is straightforward. The easiest way to see this is to view the partially observable planning problem as a nondeterministic fully observable planning problem with belief states viewed as states. An operator maps a belief state to another belief state nondeterministically: compute the image of a belief state with respect to an operator, and choose the subset of its states that correspond to one of the possible observations. Like pointed out in the proof of Theorem 4.55, the algorithms for fully observable planning run in polynomial time in the size of the state space. The state space with the belief states as the states has a doubly exponential size in the size of the problem instance, and hence the algorithm runs in doubly exponential time in the size of the problem instance. This gives us the membership in 2-EXP.

Theorem 4.58 *The plan existence problem for problem instances with partial observability is in 2-EXP.*

The hardness proof is an extension of both the EXP-hardness proof of Theorem 4.53 and of the EXPSPACE-hardness proof of Theorem 4.56. From the first proof we have the simulation of alternating Turing machines, and from the second proof the simulation of Turing machines with an exponentially long tape.

Theorem 4.59 *The problem of testing the existence of an acyclic plan for problem instances with partial observability is 2-EXP-hard.*

Proof: Let $\langle \Sigma, Q, \delta, q_0, g \rangle$ be any alternating Turing machine with an exponential space bound $e(x)$. Let σ be an input string of length n. We denote the ith symbol of σ by σ_i.

The Turing machine may use space $e(n)$, and for encoding numbers from 0 to $e(n)+1$ corresponding to the tape cells we need $m = \lceil \log_2(e(n)+2) \rceil$ Boolean state variables.

We construct a problem instance in nondeterministic planning with full observability for simulating the Turing machine. The problem instance has a size that is polynomial in the size of the description of the Turing machine and the input string.

We cannot have a state variable for every tape cell because the reduction from Turing machines to planning would not be polynomial time. It turns out that it is not necessary to encode the whole contents of the tape in the transition system of the planning problem, and that it suffices to keep track of only one tape cell (which we will call the *watched tape cell*) that is randomly chosen in the beginning of every execution of the plan.

The set A of state variables in the problem instance consists of

1. $q \in Q$ for denoting the internal states of the TM,
2. w_i for $i \in \{0, \ldots, m-1\}$ for the watched tape cell $i \in \{0, \ldots, e(n)\}$,
3. s for every symbol $s \in \Sigma \cup \{|, \Box\}$ for the contents of the watched tape cell,
4. s^* for every $s \in \Sigma \cup \{|\}$ for the symbol last written (important for nondeterministic transitions),
5. L, R and N for the last movement of the R/W head (important for nondeterministic transitions), and
6. h_i for $i \in \{0, \ldots, m-1\}$ for the position of the R/W head $i \in \{0, \ldots, e(n)+1\}$.

The observable state variables are L, N and R, $q \in Q$, and s^* for $s \in \Sigma$. These are needed by the plan to decide how to proceed execution after a nondeterministic transition with a $\forall$ state.

The uncertainty in the initial state is about which tape cell is the watched one. Otherwise the formula encodes the initial configuration of the TM, and it is the conjunction of the following formulae.

1. q_0
2. $\neg q$ for all $q \in Q \backslash \{q_0\}$.
3. $\neg s^*$ for all $s \in \Sigma \cup \{|\}$.

4. Formulae for having the contents of the watched tape cell in state variables $\Sigma \cup \{|, \square\}$.

$$\begin{aligned} | &\leftrightarrow (w = 0) \\ \square &\leftrightarrow (w > n) \\ s &\leftrightarrow \bigvee_{i \in \{1,\ldots,n\}, \sigma_i = s} (w = i) \text{ for all } s \in \Sigma \end{aligned}$$

5. $h = 1$ for the initial position of the R/W head.

So the initial state formula allows any values for state variables w_i and the values of the state variables $s \in \Sigma$ are determined on the basis of the values of w_i. The expressions $w = i$, $w > i$ denote the obvious formulae for testing integer equality and inequality of the numbers encoded by $w_0, w_1, \ldots$. Later we will also use effects $h := h + 1$ and $h := h - 1$ that represent incrementing and decrementing the number encoded by $h_0, h_1, \ldots$.

The goal is the following formula.

$$G = \bigvee \{q \in Q | g(q) = \text{accept}\}$$

Next we define the operators. All the transitions may be nondeterministic, and the important thing is whether the transition is for a $\forall$ state or an $\exists$ state. For a given input symbol and a $\forall$ state, the transition corresponds to one nondeterministic operator, whereas for a given input symbol and an $\exists$ state the transitions corresponds to a set of deterministic operators.

To define the operators, we first define effects corresponding to all possible transitions.

For all $\langle s, q\rangle \in (\Sigma \cup \{|, \square\}) \times Q$ and $\langle s', q', m\rangle \in (\Sigma \cup \{|\}) \times Q \times \{L, N, R\}$ define the effect $\tau_{s,q}(s', q', m)$ as $\alpha \wedge \kappa \wedge \theta$ where the effects α, κ and θ are defined as follows.

The effect α describes what happens to the tape symbol under the R/W head. If $s = s'$ then $\alpha = \top$ as nothing on the tape changes. Otherwise, $\alpha = ((h = w) \rhd (\neg s \wedge s')) \wedge s'^* \wedge \neg s^*$ to denote that the new symbol in the watched tape cell is s' and not s, and to make it possible for the plan to detect which symbol was written to the tape by the possibly nondeterministic transition.

The effect κ describes the change to the internal state of the TM. Again, either the state changes or does not, so $\kappa = \neg q \wedge q'$ if $q \neq q'$ and $\top$ otherwise. If R/W head movement is to the right we define $\kappa = \neg q \wedge ((h < e(n)) \rhd q')$ if $q \neq q'$ and $(h = e(n)) \rhd \neg q$ otherwise. This prevents reaching an accepting state if the space bound is violated: no further operator applications are possible.

The effect θ describes the movement of the R/W head. Either there is movement to the left, no movement, or movement to the right. Hence

$$\theta = \begin{cases} (h := h - 1) \wedge L \wedge \neg N \wedge \neg R & \text{if } m = L \\ N \wedge \neg L \wedge \neg R & \text{if } m = N \\ (h := h + 1) \wedge R \wedge \neg L \wedge \neg N & \text{if } m = R \end{cases}$$

By definition of TMs, movement at the left end of the tape is always to the right.

Now, these effects $\tau_{s,q}(s', q', m)$ which represent possible transitions are used in the operators that simulate the ATM. Operators for existential states $q, g(q) = \exists$ and for universal states $q, g(q) = \forall$ differ. Let $\langle s, q\rangle \in (\Sigma \cup \{|, \square\}) \times Q$ and $\delta(s, q) = \{\langle s_1, q_1, m_1\rangle, \ldots, \langle s_k, q_k, m_k\rangle\}$.

If $g(q) = \exists$, then define k deterministic operators

$$\begin{aligned} o_{s,q,1} &= \langle((h \neq w) \vee s) \wedge q, \tau_{s,q}(s_1, q_1, m_1)\rangle \\ o_{s,q,2} &= \langle((h \neq w) \vee s) \wedge q, \tau_{s,q}(s_2, q_2, m_2)\rangle \\ &\vdots \\ o_{s,q,k} &= \langle((h \neq w) \vee s) \wedge q, \tau_{s,q}(s_k, q_k, m_k)\rangle \end{aligned}$$

That is, the plan determines which transition is chosen.

If $g(q) = \forall$, then define one nondeterministic operator

$$o_{s,q} = \langle(((h \neq w) \vee s) \wedge q,\ (\tau_{s,q}(s_1, q_1, m_1)|\\ \tau_{s,q}(s_2, q_2, m_2)|\\ \vdots\\ \tau_{s,q}(s_k, q_k, m_k)\rangle).$$

That is, the transition is chosen nondeterministically.

We claim that the problem instance has a plan if and only if the Turing machine accepts without violating the space bound. If the Turing machine violates the space bound, then $h > e(n)$ and an accepting state cannot be reached because no further operator will be applicable.

From an accepting computation tree of an ATM we can construct a plan, and vice versa. Accepting final configurations are mapped to terminal nodes of plans, $\exists$-configurations are mapped to operator nodes in which an operator corresponding to the transition to an accepting successor configuration is applied, and $\forall$-configurations are mapped to operator nodes corresponding to the matching nondeterministic operators followed by a branch node that selects the plan nodes corresponding to the successors of the $\forall$ configuration. The successors of $\forall$ and $\exists$ configurations are recursively mapped to plans.

Construction of computation trees from plans is similar, but involves small technicalities. A plan with DAG form can be turned into a tree by having several copies of the shared subplans. Branches not directly following the nondeterministic operator causing the uncertainty can be moved earlier so that every nondeterministic operator is directly followed by a branch that chooses a successor node for every possible new state, written symbol and last tape movement. With these transformations there is an exact match between plans and computation trees of the ATM, and mapping from plans to ATMs is straightforward like in the opposite direction.

Because alternating Turing machines with an exponential space bound are polynomial time reducible to the nondeterministic planning problem with partial observability, the plan existence problem is AEXPSPACE=2-EXP-hard. □

What remains to be done is the extension of the above theorem to the case with arbitrary (possibly cyclic) plans. For the fully observable case counting the execution length does not pose a problem because we only have to count an exponential number of execution steps, which can be represented by a polynomial number of state variables, but in the partially observable case we need to count a doubly exponential number of execution steps, as the number of belief states to be visited may be doubly exponential. A binary representation of these numbers requires an exponential number of bits, and we cannot use an exponential number of state variables for the purpose, because the reduction to planning would not be polynomial time. However, partial observability together with only a polynomial number of auxiliary state variables can be used to force the plans to count doubly exponentially far.

Theorem 4.60 *The plan existence problem for problem instances with partial observability is 2-EXP-hard.*

Proof: We extend the proof of Theorem 4.59 by a counting scheme that makes cyclic plans ineffective. We show how counting the execution length can be achieved within a problem instance obtained from the alternating Turing machine and the input string in polynomial time.

Instead of representing the exponential number of bits explicitly as state variables, we use a randomizing technique for forcing the plans to count the number of Turing machine transitions. The technique has resemblance to the idea in simulating exponentially long tapes in the proofs of Theorems 4.56 and 4.53.

For a problem instance with n state variables (representing the Turing machine configurations) executions that visit each belief state at most once may have length 2^{2^n}. Representing numbers from 0 to $2^{2^n} - 1$ requires 2^n binary digits. We introduce $n + 1$ new unobservable state variables $d_0, \ldots, d_n$ for representing the index of one of the digits and v_d for the value of that digit, and new state variables $c_0, \ldots, c_n$ through which the plan indicates changes in the counter of Turing machine transitions. There is a set of operators by means of which the plan sets the values of these variables before every transition of the Turing machine is made.

The idea of the construction is the following. Whenever the counter of TM transitions is incremented, one of the 2^n digits in the counter changes from 0 to 1 and all of the less significant digits change from 1 to 0. The plan is forced to communicate the index of the digit that changes from 0 to 1 by the state variables $c_0, \ldots, c_n$. The unobservable state variables $d_0, \ldots, d_n, v_d$ store the index and value of one of the digits (chosen randomly in the beginning of the plan execution), that we call *the watched digit*, and they are used for checking that the reporting of $c_0, \ldots, c_n$ by the plan is truthful. The test for truthful reporting is randomized, but this suffices to invalidate plans that incorrectly report the increments, as a valid plan has to reach the goals on every possible execution. The plan is invalid if reporting is false or when the count can exceed 2^{2^n}. For this reason a plan for the problem instance exists if and only if an acyclic plan exists if and only if the Turing machine accepts the input string.

Next we exactly define how the problem instances defined in the proof of Theorem 4.59 are extended with a counter to prevent unbounded looping.

The initial state description is extended with the conjunct $\neg d_v$ to signify that the watched digit is initially 0 (all the digits in the counter implicitly represented in the belief state are 0.) The state variables $d_0, \ldots, d_n$ may have any values which means that the watched digit is chosen randomly. The state variables $d_v, d_0, \ldots, d_n$ are all unobservable so that the plan does not know the watched digit (may not depend on it).

There is also a failure flag f that is initially set to false by having $\neg f$ in the initial states formula.

The goal is extended by $\neg f \wedge ((d_0 \wedge \cdots \wedge d_n) \rightarrow \neg d_v)$ to prevent executions that lead to setting f true or that have length $2^{2^{n+1}-1}$ or more. The conjunct $(d_0 \wedge \cdots \wedge d_n) \rightarrow \neg d_v$ is false if the index of the watched digit is $2^{n+1} - 1$ and the digit is true, indicating an execution of length $\geq 2^{2^{n+1}-1}$.

Then we extend the operators simulating the Turing machine transitions, as well as introduce new operators for indicating which digit changes from 0 to 1.

The operators for indicating the changing digit are

$$\begin{array}{ll} \langle \top, c_i \rangle & \text{for all } i \in \{0, \ldots, n\} \\ \langle \top, \neg c_i \rangle & \text{for all } i \in \{0, \ldots, n\} \end{array}$$

The operators for Turing machine transitions are extended with the randomized test that the digit the plan claims to change from 0 to 1 is indeed the one: every operator $\langle p, e \rangle$ defined in the proof of Theorem 4.59 is replaced by $\langle p, e \wedge t \rangle$ where the test t is the conjunction of the following effects.

$$\begin{array}{rl} ((c = d) \wedge d_v) & \rhd f \\ (c = d) & \rhd d_v \\ ((c > d) \wedge \neg d_v) & \rhd f \\ (c > d) & \rhd \neg d_v \end{array}$$

Here $c = d$ denotes $(c_0 \leftrightarrow d_0) \wedge \cdots \wedge (c_n \leftrightarrow d_n)$ and $c > d$ encodes the greater-than test for the binary numbers encoded by $c_0, \ldots, c_n$ and $d_0, \ldots, d_n$.

The above effects do the following.

1. When the plan claims that the watched digit changes from 0 to 1 and the value of d_v is 1, fail.
2. When the plan claims that the watched digit changes from 0 to 1, change d_v to 1.
3. When the plan claims that a more significant digit changes from 0 to 1 and the value of d_v is 0, fail.
4. When the plan claims that a more significant digit changes from 0 to 1, set the value of d_v to 0.

That these effects guarantee the invalidity of a plan that relies on unbounded looping is because the failure flag f will be set if the plan lies about the count, or the most significant bit with index $2^{n+1} - 1$ will be set if the count reaches $2^{2^{n+1}-1}$. Attempts of unfair counting are recognized and consequently f is set to true because of the following.

Assume that the binary digit at index i changes from 0 to 1 (and therefore all less significant digits change from 1 to 0) and the plan incorrectly claims that it is the digit j that changes, and this is the first time on that execution that the plan lies (hence the value of d_v is the true value of the watched digit.)

If $j > i$, then i could be the watched digit (and hence $c > d$), and for j to change from 0 to 1 the less significant bit i should be 1, but we would know that it is not because d_v is false. Consequently on this plan execution the failure flag f would be set.

If $j < i$, then j could be the watched digit (and hence $c = d$), and the value of d_v would indicate that the current value of digit j is 1, not 0. Consequently on this plan execution the failure flag f would be set.

So, if the plan does not correctly report the digit that changes from 0 to 1, then the plan is not valid. Hence any valid plan correctly counts the execution length which cannot exceed $2^{2^{n+1}-1}$. □

4.8.4 Polynomial size plans

We showed in Section 3.7 that the plan existence problem of deterministic planning is only NP-complete, in contrast to PSPACE-complete, when a restriction to plans of polynomial length is made. Here we investigate the same question for conditional plans.

Theorem 4.61 *The plan existence problem for conditional planning without observability restricted to polynomial length plans is in* Σ_2^p.

Proof: Let $p(n)$ be any polynomial. We give an NP^{NP} algorithm (Turing machine) that solves the problem. Let the problem instance $\langle A, I, O, G, \emptyset \rangle$ have size n.

First guess a sequence of operators $\sigma = o_0, o_1, \ldots, o_k$ for $k < p(n)$. This is nondeterministic polynomial time computation.

Then use an NP-oracle for testing that σ is a solution. The oracle is a nondeterministic polynomial-time Turing machine that accepts if a plan execution does not lead to a goal state

or if the plan is not executable (operator precondition not satisfied). The oracle guesses an initial state and for each nondeterministic operator for each step which nondeterministic choices are made, and then in polynomial time tests whether the execution of the operator sequence leads to a goal state.

1. Guess valuation I' that satisfies I.
2. Guess the results of the nondeterministic choices for every operator in the plan: replace every $p_1e_1|\cdots|p_ne_n$ by a nondeterministically selected e_i.
3. Compute $s_j = \mathit{app}_{o_j}(\mathit{app}_{o_{j-1}}(\cdots \mathit{app}_{o_2}(\mathit{app}_{o_1}(I'))))$ for $j = 0, j = 1, j = 2, \ldots, j = k$.
4. If $s_j \not\models c_j$ for $o_j = \langle c_j, e_j\rangle$, accept.
5. If $s_k \not\models G$, accept.
6. Otherwise reject.

□

Theorem 4.62 *The plan existence problem for conditional planning without observability restricted to polynomial length plans is Σ_2^p-hard.*

Proof: Truth of QBF of the form $\exists x_1 \cdots x_n \forall y_1 \cdots y_m \phi$ is Σ_2^p-complete. We reduce this problem to the plan existence problem of unobservable planning with polynomial length plans.

- $A = \{x_1, \ldots, x_n, y_1, \ldots, y_m, s, g\}$
- $I = \neg x_1 \wedge \cdots \wedge \neg x_n \wedge \neg g \wedge s$
- $O = \{\langle s, x_1\rangle, \langle s, x_2\rangle, \ldots, \langle s, x_n\rangle, \langle s, \neg s \wedge (\phi \rhd g)\rangle\}$
- $G = g$

Out claim is that there is a plan if and only if $\exists x_1 \cdots x_n \forall y_1 \cdots y_m \phi$ is true.

Assume the QBF is true, that is, there is a valuation x for $x_1, \ldots, x_n$ so that $x, y \models \phi$ for any valuation y of $y_1, \ldots, y_m$. Let $X = \{\langle s, x_i\rangle | i \in \{1, \ldots, n\}, x(x_i) = 1\}$. Now the operators X in any order followed by $\langle s, \neg s \wedge (\phi \rhd g)\rangle$ is a plan: whatever values $y_1, \ldots, y_m$ have, ϕ is true after executing the operators X, and hence the last operator makes $G = g$ true.

Assume there is a plan. The plan has one occurrence of $\langle s, \neg s \wedge (\phi \rhd g)\rangle$ and it must be the last operator. Define the valuation x of $x_1, \ldots, x_n$ as follows. Let $x(x_i) = 1$ iff $\langle s, x_i\rangle$ is one of the operators in the plan, for all $i \in \{1, \ldots, n\}$. Because g is reached, $x, y \models \phi$ for any valuation y of $y_1, \ldots, y_m$, and the QBF is therefore true. □

	deterministic context-independent	deterministic context-dependent	non-deterministic context-dependent
full observability	PSPACE	PSPACE	EXPTIME
no observability	PSPACE	EXPSPACE	EXPSPACE
partial observability	PSPACE	EXPSPACE	2-EXPTIME

Table 4.2: Computational complexity of plan existence problems

	deterministic context-independent	deterministic context-dependent	non-deterministic context-dependent
full observability	PSPACE	PSPACE	EXPTIME
no observability	PSPACE	PSPACE	EXPSPACE
partial observability	PSPACE	PSPACE	2-EXPTIME

Table 4.3: Computational complexity of plan existence problems with one initial state

4.8.5 Summary of the results

The complexities of the plan existence problem under different restrictions on operators and observability are summarized in Tables 4.2 (with an arbitrary number of initial states) and 4.3 (with one initial state). The different columns list the complexities with different restrictions on the operators. In the previous sections we have considered the general problems with arbitrary operators containing conditional effects and nondeterministic choice. These results are summarized in the third column. The second column lists the complexities in the case without nondeterminism (choice $|$), and the first column without nondeterminism (choice $|$) and without conditional effects ($\rhd$). These results are not given in this lecture.

4.9 Literature

There is a difficult trade-off between the two extreme approaches, producing a conditional plan covering all situations that might be encountered, and planning only one action ahead. Schoppers [1987] proposed *universal plans* as a solution to the high complexity of planning. Ginsberg [1989] attacked Schopper's idea. Schopper's proposal was to have memoryless plans that map any given observations to an action. He argued that plans have to be memoryless in order to be able to react to all the unforeseeable situations that might be encountered during plan execution. Ginsberg argued that plans that are able to react to all possible situations are necessarily much too big to be practical. It seems to us that Schopper's insistence on using plans without a memory is not realistic nor necessary, and that most of Ginsberg's argumentation on impracticality of universal plans relies on the lack of any memory in the plan execution mechanism. Of course, we agree that a conditional plan that can be executed efficiently can be much bigger than a plan or a planner that has no restrictions on the amount of time consumed in deciding about the action to be taken. Plans without such restrictions could have as high expressivity as Turing machines, for example, and then a conditional plan does not have to be less succinct than the description of a general purpose planning algorithm.

There is some early work on conditional planning that mostly restricts to the fully observable case and is based on partial-order planning [Etzioni *et al.*, 1992; Peot and Smith, 1992; Pryor and

Collins, 1996]. We have not discussed these algorithms because they have only been shown to solve very small problem instances.

A variant of the algorithm for constructing plans for nondeterministic planning with full observability in Section 4.4.1 was first presented by Cimatti et al. [2003]. The algorithms by Cimatti et al. construct mappings of states to actions whereas our presentation in Section 4.4 focuses on the computation of distances of states, and plans are synthesized afterwards on the basis of the distances. We believe that our algorithms are conceptually simpler. Cimatti et al. also presented an algorithm for finding *weak plans* that may reach the goals but are not guaranteed to. However, finding weak plans is polynomially equivalent to the deterministic planning problem of Chapter 3.

The nondeterministic planning problem with unobservability is not very interesting because all robots and intelligent beings can sense their environment to at least some extent. However, there are problems (outside AI) that are equivalent to the unobservable planning problem. Finding homing/reset/synchronization sequences of circuits/automata is an example of such a problem [Pixley *et al.*, 1992]. There are extensions of the distance and cardinality based heuristics for planning without observability not discussed in this lecture [Rintanen, 2004a].

Bertoli et al. have presented a forward search algorithm for finding conditional plans in the general partially observable case [Bertoli *et al.*, 2001].

The computational complexity of conditional planning was first investigated by Littman [1997] and Haslum and Jonsson [2000]. They presented proofs for the EXPTIME-completeness of planning with full observability and the EXPSPACE-completeness of planning without observability. The hardness parts of the proofs were reductions respectively from the existence problem of winning strategies for the game G_4 [Stockmeyer and Chandra, 1979] and from the universality problem of regular expressions with exponentiation [Hopcroft and Ullman, 1979]. In this chapter we gave more direct hardness proofs by direct simulation of alternating polynomial space (exponential time) and exponential space Turing machines.

Chapter 5

Probabilistic planning

Probabilistic planning is an extension of nondeterministic planning with information on the probabilities of nondeterministic events.

Probabilites are important in quantifying the costs and success probabilities of plans when the actions are nondeterministic. In many applications it is not sufficient just to have a plan. It is important to have a plan that is efficient in the sense that the cost of the actions does not outweigh the benefits of reaching the goals. On some other problems there are no plans that are guaranteed to reach the goals. In these cases it is important to maximize the probability of reaching the goals, and hence it is vitally important to use information on the probabilities of different effects of operators.

Probabilities complicate planning, both conceptually and computationally. Whereas in the non-probabilistic of conditional planning with partial observability it is sufficient to work in a finite discrete belief space, probabilities make the belief space continuous and thereby infinite.

In this section a number of algorithms for probabilistic planning are presented. In Sections 5.1 and 5.2 we present the transition system model with probabilities that extend the definitions given in Sections 2.1 and 2.3 for non-succinct and succinct transition systems, respectively. Like in Chapter 4 we start from planning with full observability in Section 5.3. Many probabilistic planning problems with full observability are closely related to Markov decision processes [Puterman, 1994].

5.1 Probabilistic transition systems

In many types of probabilistic planning problems considered in the literature the objective is not to reach one of a set of designated goal states. Instead, the objective is to act in a way that maximizes the *rewards* or minimizes the *costs*. Planning problems with a designated set of goal states can be expressed in terms of rewards, but not vice versa.

Definition 5.1 *A* probabilistic transition system *is a 5-tuple* $\Pi = \langle S, I, O, G, P \rangle$ *where*

1. S *is a finite set of states,*
2. I *is a probability distribution over* S*,*
3. O *is a finite set of actions* o *that are partial functions that map each state to a probability distribution over* S*,*

4. *$G \subseteq S$ is the set of goal states, and*

5. *$P = (C_1, \ldots, C_n)$ is a partition of S to classes of observationally indistinguishable states satisfying $\bigcup\{C_1, \ldots, C_n\} = S$ and $C_i \cap C_j = \emptyset$ for all i, j such that $1 \leq i < j \leq n$.*

An action o is *applicable* in states for which $o(s)$ is defined. These states we denote by $\text{prec}(o) = \{s \in S | o(s) \text{ is defined } \}$. Below, we will denote the set of actions applicable in a state $s \in S$ by $O(S)$. We also require that $O(s)$ is non-empty for every $s \in S$.

A major difference to the definition of Markov decision processes [Puterman, 1994] is that $o \in O$ are partial functions. This means that an action does not associate every state with a probability distribution because an action is not necessarily applicable in all states.

Instead of using a designated set of goal states and have reaching a goal state or staying in the goal states as an objective, in many types of planning problems the objective is to maximize rewards or minimize costs. To formalize this we use instead of a set of goal states a cost function that associates every action and state a numerical cost.

Definition 5.2 *A* probabilistic transition system with rewards *is a 5-tuple $\Pi = \langle S, I, O, C, P \rangle$ where the components S, I, O and P are as in Definition 5.1 and $C : O \times S \to \mathcal{R}$ is a function from actions and states to real numbers, indicating the* cost *associated with an action in a given state.*

5.2 Succinct probabilistic transition systems

Probabilistic transition system can be represented exponentially more succinctly in terms of state variables and operators.

Definition 5.3 *Let A be a set of state variables. An* operator *is a pair $\langle c, e \rangle$ where c is a propositional formula over A (the* precondition*), and e is an* effect *over A. Effects over A are recursively defined as follows.*

1. *a and $\neg a$ for state variables $a \in A$ are effects over A.*

2. *$e_1 \wedge \cdots \wedge e_n$ is an effect over A if $e_1, \ldots, e_n$ are effects over A (the special case with $n = 0$ is the empty effect $\top$.)*

3. *$c \rhd e$ is an effect over A if c is a formula over A and e is an effect over A.*

4. *$p_1 e_1 | \cdots | p_n e_n$ is an effect over A if $n \geq 2$ and $e_1, \ldots, e_n$ for $n \geq 2$ are effects over A and $p_1, \ldots, p_n$ are real numbers such that $p_1 + \cdots + p_n = 1$ and $0 \leq p_i \leq 1$ for all $i \in \{1, \ldots, n\}$.*

Operators map states to probability distributions over their successor states.

Definition 5.4 (Operator application) *Let $\langle c, e \rangle$ be an operator over A. Let s be a state (a valuation of A). The operator is* applicable *in s if $s \models c$ and for every set $E \in [e]_s$ the set $\bigcup\{M | \langle p, M \rangle \in E, p > 0\}$ is consistent.*

Recursively assign each effect e a set $[e]_s$ of pairs $\langle p, M \rangle$ where p is a probability $0 \leq p \leq 1$ and M is a set of literals a and $\neg a$ where $a \in A$.

1. $[a]_s = \{\langle 1, \{a\}\rangle\}$ *and* $[\neg a]_s = \{\langle 1, \{\neg a\}\rangle\}$ *for* $a \in A$.
2. $[e_1 \wedge \cdots \wedge e_n]_s = \{\langle \Pi_{i=1}^n p_i, \bigcup_{i=1}^n M_i\rangle | \langle p_1, M_1\rangle \in [e_1]_s, \ldots, \langle p_n, M_n\rangle \in [e_n]_s\}$.
3. $[c' \rhd e]_s = [e]_s$ *if* $s \models c'$ *and* $[c' \rhd e]_s = \{\langle 1, \emptyset\rangle\}$ *otherwise.*
4. $[p_1 e_1 | \cdots | p_n e_n]_s = \{\langle p_1 \cdot p, e\rangle | \langle p, e\rangle \in [e_1]_s\} \cup \cdots \cup \{\langle p_n \cdot p, e\rangle | \langle p, e\rangle \in [e_n]_s\}$

For handling effects like $(0.2a|0.8b) \wedge (0.2a|0.8b)$*, which produces sets* $\{a\}, \{a, b\}, \{a, b\}, \{b\}$ *respectively with probabilities* 0.04*,* 0.16*,* 0.16 *and* 0.64*, the sets in this definition are understood as multisets so that the probability 0.16 of* $\{a, b\}$ *is counted twice. Alternatively, in (4) the union of sets is defined so that for example* $\{\langle 0.2, \{a\}\rangle\} \cup \{\langle 0.2, \{a\}\rangle\} = \{\langle 0.4, \{a\}\rangle\}$*: same sets of changes are combined by summing their probabilities.*

The successor states of s *under the operator are ones that are obtained from* s *by making the literals in* M *for* $\langle p, M\rangle \in [e]_s$ *true and retaining the truth-values of state variables not occurring in* M*. The probability of a successor state is the sum of the probabilities* p *for* $\langle p, M\rangle \in [e]_s$ *that lead to it.*

Definition 5.5 *A* succinct probabilistic transition system *is a 5-tuple* $\Pi = \langle A, I, O, G, V\rangle$ *where*

1. A *is a finite set of state variables,*
2. I *which describes a probability distribution over the possible initial states is a set of pairs* $\langle p, \phi\rangle$ *where* p *is a number such that* $0 \leq p \leq 1$ *and* ϕ *is a formula over* A *such that* $(\sum_{s \in S, s \models \phi_1} p_1) + \cdots + (\sum_{s \in S, s \models \phi_n} p_n) = 1$ *where* $I = \{\langle p_1, \phi_1\rangle, \ldots, \langle p_n, \phi_n\rangle\}$*,*
3. O *is a finite set of operators over* A*,*
4. G *is a formula over* A *describing the goal states, and*
5. $V \subseteq A$ *is the set of observable state variables.*

Definition 5.6 *A* succinct probabilistic transition system with rewards *is a 5-tuple* $\Pi = \langle A, I, O, C, V\rangle$ *where the components* A, I, O *and* V *are as in Definition 5.5 and* R *is a function from operators to pairs* $\langle \phi, r\rangle$ *where* ϕ *is a formula over* A *and* r *is a real number indicating the* cost *associated with an action in a given state: cost of operator* $o \in O$ *in state* s *is* r *if there is* $\langle \phi, r\rangle \in C(o)$ *such that* $s \models \phi$*. For this to be well defined there may be no* $\{\langle \phi_1, r_1\rangle, \langle \phi_2, r_2\rangle\} \subseteq C(o)$ *such that* $\phi_1 \wedge \phi_2$ *is satisfiable.*

We can associate a probabilistic transition system with every succinct probabilistic transition system.

5.3 Problem definition

A given plan produces infinite sequences of rewards $r_1, r_2, \ldots$. Clearly, if the planning problem has several initial states or if the actions are nondeterministic this sequence of rewards is not unique. In either case, possible plans are assessed in terms of these rewards, and there are several possibilities how good plans are defined. Because the sequences are infinite, we in general cannot simply take their sum and compare them. Instead, several other possibilities have been considered.

1. Expected total rewards over a finite horizon.

   This is a natural alternative that allows using the normal arithmetic sum of the rewards. However, there is typically no natural bound on the horizon length.

2. Expected average rewards over an infinite horizon.

   This is for many applications that involve very long actions sequences the most natural way of assessing plans. However, there are several technical complications that make average rewards difficult to use.

3. Expected discounted rewards over an infinite horizon.

   This is the most often used criterion in connection with Markov decision processes. Discounting means multiplying the ith reward by λ^{i-1} and it means that early rewards are much more important than rewards obtained much later. The discount constant λ has a value strictly between 0.0 and 1.0. The sum of the geometrically discounted rewards is finite. Like with choosing the horizon length when evaluating plans with respect to their behavior within a finite horizon, it is often difficult to say why a certain discount constant λ is used.

For the latter two infinite horizon problems there always is an optimal plan that is a mapping from states to actions. For finite horizon problems the optimal actions in a given state at different time points may be different. The optimal plans are therefore time-dependent.

5.4 Algorithms for finding finite horizon plans

Conceptually the simplest probabilistic planning is when plan executions are restricted to have a finite horizon of length N. We briefly describe this problem to illustrate the techniques that are used in connection with the infinite horizon planning problems.

The optimum values $v_i(s)$ that can be obtained in state $s \in S$ at time point $i \in \{1, \ldots, N\}$ fulfill the following equations.

$$v_N(s) = \max_{a \in O(s)} R(s,a)$$

$$v_i(s) = \max_{a \in O(s)} \left(R(s,a) + \sum_{s' \in S} p(s'|s,a) v_{i+1}(s') \right), \text{ for } i \in \{1, \ldots, N-1\}$$

The value at the last stage N is simply the best immediate reward that can be obtained, and values of states for the other stages are obtained in terms of the values of states for the later stages.

These equations also directly yield an algorithm for computing the optimal values and optimal plans: first compute v_N, then v_{N-1}, v_{N-2} and so on, until v_1 is obtained. The action to be taken in state $s \in S$ at time point i is $\pi(s,i)$ defined by

$$\pi(s,N) = \arg \max_{a \in O(s)} R(s,a)$$

$$\pi(s,i) = \arg \max_{a \in O(s)} \left(R(s,a) + \sum_{s' \in S} p(s'|s,a) v_{i+1}(s') \right), \text{ for } i \in \{1, \ldots, N-1\}$$

5.5 Algorithms for finding plans under discounted rewards

The value $v(s)$ of a state $s \in S$ is the discounted sum of the expected rewards that can be obtained by choosing the best possible action in s and assuming that the best possible actions are also chosen in all the possible successor states. The following equations, one for each state $s \in S$, characterize the relations between the values of states of a stochastic transition system under an optimal plan and geometrically discounted rewards with discount constant λ.

$$v(s) = \max_{a \in O(s)} \left(R(s,a) + \sum_{s' \in S} \lambda p(s'|s,a) v(s') \right) \tag{5.1}$$

These equations are called the optimality equations or the Bellman equations, and they are the basis of the most important algorithms for finding optimal plans for probabilistic planning problems with full observability.

5.5.1 Evaluating the value of a given plan

Given a plan π its value under discounted rewards with discount constant λ satisfies the following equation for every $s \in S$.

$$v(s) = R(s, \pi(s)) + \sum_{s' \in S} \lambda p(s'|s, \pi(s)) v(s') \tag{5.2}$$

This yields a system of linear equation with $|S|$ equations and unknowns. The solution of these equations yields the value of the plan in each state.

5.5.2 Value iteration

The value iteration algorithm finds an approximation of the value of the optimal λ-discounted plan within a constant ϵ, and a plan with at least this value.

1. $n := 0$
2. Assign (arbitrary) initial values to $v^0(s)$ for all $s \in S$.
3. For each $s \in S$, assign

   $$v^{n+1}(s) := \max_{a \in O(s)} \left(R(s,a) + \sum_{s' \in S} \lambda p(s'|s,a) v^n(s') \right)$$

   If $|v^{n+1}(s) - v^n(s)| < \frac{\epsilon(1-\lambda)}{2\lambda}$ for all $s \in S$ then go to step 4.

   Otherwise, set $n := n + 1$ and go to step 3.
4. Assign

   $$\pi(s) := \arg\max_{a \in O(s)} \left(R(s,a) + \sum_{s' \in S} \lambda p(s'|s,a) v^{n+1}(s') \right)$$

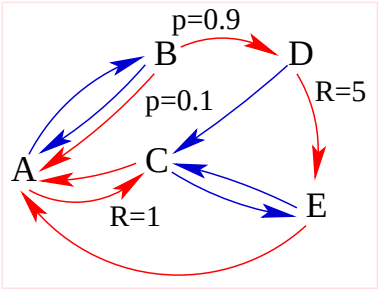

Figure 5.1: A stochastic transition system

Theorem 5.7 *Let v_π be the value function of the plan produced by the value iteration algorithm, and let v^* be the value function of an optimal plan. Then $|v^*(s) - v_\pi(s)| \leq \epsilon$ for all $s \in S$.*

Notice that unlike in partially observable planning problems, under full observability there is never a trade-off between the values of two states: if the optimal value for state s_1 is r_1 and the optimal value for state s_2 is r_2, then there is one plan that achieves these both.

Example 5.8 Consider the stochastic transition system in Figure 5.1. Only one of the actions is nondeterministic and only in state B, and all the other actions and states have zero reward except one of the actions in states A and D, with rewards 1 and 5, respectively. ■

5.5.3 Policy iteration

The second, also rather widely used algorithm for finding plans, is policy iteration[1]. It is slightly more complicated to implement than value iteration, but it typically converges after a smaller number of iterations, and it is guaranteed to produce an optimal plan.

The idea is to start with an arbitrary plan (assignment of actions to states), compute its value, and repeatedly choose for every state an action that is better than its old action.

1. Assign $n := 0$.
2. Let π^0 be any mapping from states to actions.
3. Compute the values $v^n(s)$ of all $s \in S$ under π^n.
4. Let $\pi^{n+1}(s) = \arg\max_{a \in O(s)} \left(R(s,a) + \sum_{s' \in S} \lambda p(s'|s,a) v^n(s') \right)$.
5. Assign $n := n + 1$.
6. If $n = 1$ or $v^n \neq v^{n-1}$ then go to 3.

Theorem 5.9 *The policy iteration algorithm terminates after a finite number of steps and returns an optimal plan.*

Proof: Outline: There is only a finite number of different plans, and at each step the new plan assigns at least as high a value to each state as the old plan. □

[1] In connection with Markov decision processes the word *policy* is typically used instead of the word *plan*.

It can be shown that the convergence rate of policy iteration is always at least as fast as that of value iteration [Puterman, 1994], that is, the number of iterations needed for finding an ϵ-optimal plan for policy iteration is never higher than the number of iterations needed by value iteration.

In practise policy iteration often finds an optimal plan after just a few iterations. However, the amount of computation in one round of policy iteration is substantially higher than in value iteration, and value iteration is often considered more practical.

5.5.4 Implementation of the algorithms with ADDs

Similarly to the techniques in Section 4.2 that allow representing state sets and transition relations as formulae or binary decision diagrams, also probabilistic planning algorithms can be implemented with data structures that allow the compact representation of probability distributions.

A main difference to the non-probabilistic case (Sections 4.4.1 and 4.4.2) is that for probabilistic planning propositional formulae and binary decision diagrams are not suitable for representing the probabilities of nondeterministic operators nor the probabilities of the value functions needed in the value and policy iteration algorithms. *Algebraic decision diagrams* ADDs are a generalization of BDDs can represent probability distributions. (Section 2.2.3).

In Section 4.1.2 we gave a translation from nondeterministic operators to propositional formulae. The definition of nondeterministic operators and the translation does not use probabilities.

Next we define a similar translation from nondeterministic operators to ADDs that represents the probabilities. The translation is based on a function $\tau_B^{prob}(e)$ that translates an effect e with that possibly affects state variables in B to an ADD.

$$\begin{aligned}
\tau_B^{prob}(e) &= \tau_B(e) \text{ when } e \text{ is deterministic} \\
\tau_B^{prob}(p_1 e_1 | \cdots | p_n e_n) &= p_1 \cdot \tau_B^{prob}(e_1) + \cdots + p_n \cdot \tau_B^{prob}(e_n) \\
\tau_B^{prob}(e_1 \wedge \cdots \wedge e_n) &= \tau_{B \setminus (B_2 \cup \cdots \cup B_n)}^{prob}(e_1) \cdot \tau_{B_2}^{prob}(e_2) \cdot \ldots \cdot \tau_{B_n}^{prob}(e_n) \\
&\quad \text{where } B_i = \mathit{changes}(e_i) \text{ for all } i \in \{1, \ldots, n\}
\end{aligned}$$

The first part of the translation $\tau_B^{prob}(e)$ for deterministic e is the translation of deterministic effects we presented in Section 3.6.2, but restricted to state variables in B. The result of this translation is a normal propositional formula, which can be further transformed to a BDD and an ADD with only two terminal nodes 0 and 1. The other two cases cover all nondeterministic effects in normal form.

The translation of an effect e in normal form into an ADD is $\tau_A^{prob}(e)$ where A is the set of all state variables. Translating an operator $o = \langle c, e \rangle$ to an ADD representing its incidence matrix is as $T_o = c \cdot \tau_A^{prob}(e)$, where c is the ADD representing the precondition.

Example 5.10 Consider effect $(0.2\neg A | 0.8A) \wedge (0.5(b \rhd \neg b) | 0.5\top)$. The two conjunct translated to functions

aa'	f_a
00	0.2
01	0.8
10	0.2
11	0.8

bb'	f_b
00	1.0
01	0.0
10	0.5
11	0.5

Notice that the sum of the probabilities of the successor states is 1.0. These functions are below depicted in the same table. Notice that the third column, with the two functions componentwise

multiplied, has the property that the sum of successor states of each state is 1.0.

$aba'b'$	f_a	f_b	$f_a \cdot f_b$
0000	0.2	1.0	0.2
0001	0.2	0.0	0.0
0010	0.8	1.0	0.8
0011	0.8	0.0	0.0
0100	0.2	0.5	0.1
0101	0.2	0.5	0.1
0110	0.8	0.5	0.4
0111	0.8	0.5	0.4
1000	0.2	1.0	0.2
1001	0.2	0.0	0.0
1010	0.8	1.0	0.8
1011	0.8	0.0	0.0
1100	0.2	0.5	0.1
1101	0.2	0.5	0.1
1110	0.8	0.5	0.4
1111	0.8	0.5	0.4

■

We represent the rewards produced by operator $o = \langle c, e\rangle \in O$ in different states compactly as a list $R(o) = \{\langle \phi_1, r_1\rangle, \ldots, \langle \phi_n, r_n\rangle\}$ of pairs $\langle \phi, r\rangle$, meaning that when o is applied in a state satisfying ϕ the reward r is obtained. In any state only one of the formulae ϕ_i may be true, that is $\phi_i \models \neg\phi_j$ for all $\{i, j\} \subseteq \{1, \ldots, n\}$ such that $i \neq j$. If none of the formula is true in a given state, then the reward is zero. Hence R_o is simply a mapping from states to a real numbers.

The reward functions $R(o)$ can be easily translated to ADDs. First construct the BDDs for $\phi_1, \ldots, \phi_n$ and then multiply them with the respective rewards as

$$R_o = r_1 \cdot \phi_1 + \cdots + r_n \cdot \phi_n - \infty \cdot \neg c.$$

The summand $\infty \cdot \neg c$ handles the case in which the precondition of the operator is not satisfied: application yields immediate reward minus infinity. This prevent using the operator in any state.

Similarly, the probability distribution on possible initial states can be represented as $I = \{\langle \phi_1, p_1\rangle, \ldots, \langle \phi_n, p_n\rangle\}$ and translated to an ADD.

Now the value iteration algorithm can be rephrased in terms of ADD operations as follows.

1. Assign $n := 0$ and let v^n be an ADD that is constant 0.

2. $$v^{n+1} := \max_{\langle c,e\rangle = o \in O} \left(R_o + \lambda \cdot \exists A'.(T_o \cdot (v^n[A'/A])\right) \text{ for every } s \in S$$

   If all terminal nodes of ADD $|v^{n+1} - v^n|$ are $< \frac{\epsilon(1-\lambda)}{2\lambda}$ then stop.

   Otherwise, set $n := n + 1$ and repeat step 2.

5.6 Literature

A comprehensive book on (fully observable) Markov decision processes has been written by Puterman [1994], and our presentation of the algorithms in Section 5.5 (5.5.2 and 5.5.3) follows that of Puterman. The book represents the traditional research on MDPs and uses exclusively enumerative representations of state spaces and transition probabilities. The book discusses all the main optimality criteria as well as algorithms for solving MDPs by iterative techniques and linear programming. There are also many other books on solving MDPs.

A planning system that implements the value iteration algorithm with ADDs is described by Hoey et al. [1999] and is shown to be capable of solving problems that could not be efficiently solved by conventional implementations of value iteration.

The best known algorithms for solving partially observable Markov decision processes were presented by Sondik and Smallwood in the early 1970's [Sondik, 1978; Smallwood and Sondik, 1973] and even today most of the work on POMDPs is based on those algorithms [Kaelbling *et al.*, 1998]. In this section we have presented the standard value iteration algorithm with the simplification that there is no sensing uncertainty, that is, for every state the same observation, dependent on the state, is always made.

The most general infinite-horizon planning problems and POMDP solution construction are undecidable [Madani *et al.*, 2003]. The complexity of probabilistic planning has been investigated for example by Mundhenk et al. [2000] and Littman [1997].

Bonet and Geffner [2000] and Hansen and Zilberstein [2001] have presented algorithms for probabilistic planning with Markov decision processes that use heuristic search.

5.7 Exercises

5.1 Prove that on each step of policy iteration the policy improves.

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