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Syntactic characterizations of completeness using duals and operators

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Abstract

This article extends the work laid down by Medina and Immerman for the syntactic characterization of complete problems via first-order projections. Our first contribution is a general canonical form for the sentences that characterize complete problems. This canonical form works for a wide collection of complexity classes, including L, NL, P, NP and PSPACE, which can be given in terms of any complete problem in the class and generalizes the canonical form for NP given by Medina and Immerman. Our second contribution is the definition of a new class of syntactic operators that can be used to show the completeness of a problem by purely syntactic means. We prove basic properties of the operators including the fact that any complete problem can be shown to be complete using such operators. The practical relevance of the operators is illustrated in a number of applications which includes new completeness results, and also the application of operators at problems at the second level of the polynomial-time hierarchy. In both contributions, duals of first-order projections play a major role. Thus, our results show that such duals are in fact very powerful syntactic tools in the field.

Keywords: Descriptive complexity, completeness, canonical forms, syntactic operators, dual operator.

1 Introduction

In descriptive complexity [3, 4], problems are understood as sets of finite models described by logical formulae over given vocabularies, and reductions between problems correspond to logical relations between the sets of models that characterize them. As important as polynomial many-one reductions in structural complexity, there is the notion of first-order reductions in descriptive complexity, and among such, first-order projections (fops). A fop is a weak type of many-one reduction whose study have provided interesting results such as that common NP-complete problems remain complete with respect to fop reductions, and that NP-complete problems can be described by logical sentences in a canonical form [2, 4].

A large collection of NP-complete problems via many-one polynomial-time reductions had been shown to be also complete via fops. The way for establishing completeness can be either by the standard method of first showing that the problem belongs to the class and then giving a fop-reduction from a known complete problem in the class to it, or, as often

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done in descriptive complexity, using syntactic methods that establish that a given logical formula indeed defines a complete problem.

In the latter direction, Medina and Immerman [5, 6] gave a canonical form for NP which is based on the INDEPENDENT SET problem. This canonical form establishes that every sentence defining a complete problem via fops is equivalent to a sentence of the form $(\beta_\rho \wedge \Upsilon_{IS}) \vee (\neg\beta_\rho \wedge \Gamma)$, where β_ρ is a FO-formula, Γ is a SO$\exists$-formula, and Υ_{IS} is a *generalized IS-form*; a sentence of a format akin to the one that defines INDEPENDENT SET. The canonical form can be understood as that every sentence defining an NP-complete problem P can be decomposed in two disjuncts where the first characterizes a fragment 'as hard' as INDEPENDENT SET and the second the 'rest' of P. The canonical form provides a syntactic tool for showing completeness since if the sentence that defines P has the required format, then P is guaranteed to be NP-complete via fops. Unfortunately, Medina and Immerman's canonical form has limitations such as that it is restricted to the class NP, and only works with respect to INDEPENDENT SET.

Our first result is to give a novel canonical form that generalizes Medina and Immerman's form in several directions. First, the canonical form works for a large collection of complexity classes that includes L, NL, P, NP, PSPACE and others. Second, the form can be given in terms of any complete problem in the class; e.g. not just INDEPENDENT SET in the case of NP. The generalized form does not need to be provided a priori since it is the result of applying a syntactic operator (the dual operator of a fop) to the sentence that defines the complete problem that guides the decomposition.

Although this generalization provides a more powerful tool, it is often difficult to use it directly for showing the completeness of a given problem. This difficulty, also observed by Medina and Immerman, naturally leads to the idea of considering syntactic operators that transform logical sentences while preserving some of their properties. For instance, Medina and Immerman studied operators T that preserve NP-completeness in the following sense: if Φ defines an NP-complete problem, so does $T\Phi$. Provided with a number of such operators, they were able to show the NP-completeness via fops for a large number of problems.

Our second contribution is to continue the approach based on syntactic operators. However, by slightly changing the invariant property, we are able to generalize Medina and Immerman's results over complexity classes other than NP. Indeed, instead of requiring that operators preserve completeness, we focus on operators that *sustain completeness*: if $T\Phi$ is complete then so is Φ. This simple, but powerful, idea permits us to lay down basic principles for a general theory of syntactic operators that work for a large collection of complexity classes, to relate both types of operators to each other, and to show how the duals of fops play a major role in the theory of syntactic operators.

As for applications, using these operators we present novel completeness results for several important decision problems, including the so-called numerical problems. Up to our knowledge, these are the first NP-completeness via fops for this class of problems. On the other hand, we apply syntactic operators to problems at the second level of the polynomial-time hierarchy in order to illustrate the generality of our approach.

The article is organized as follows. Section 2 delineates the framework and provides the notation. Section 3 gives the general theorem for canonical forms. (A first version of this result appeared at the 9th International Workshop on Logic and Computational Complexity in Wroclaw, Poland, 2007.) Section 4 and 5 develops the theory of syntactic operators and illustrates some applications. Section 6 concludes with a summary and discussion.

2 Preliminaries

2.1 Logics, finite models and decision problems

We consider logical vocabularies without functional symbols of the form $\sigma = \langle R_1^{a_1}, \ldots, R_r^{a_r}, c_1, \ldots, c_s \rangle$, where the R_j's are relational symbols of arity a_j and the c_i's are constant symbols. A structure for σ, also called a σ-structure or just structure, is a tuple $\mathcal{A} = \langle |\mathcal{A}|, R_1^{\mathcal{A}}, \ldots, R_r^{\mathcal{A}}, c_1^{\mathcal{A}}, \ldots, c_s^{\mathcal{A}} \rangle$ where $|\mathcal{A}|$ is the universe (or domain) of $\mathcal{A}$, each $R_j^{\mathcal{A}} \subseteq |\mathcal{A}|^{a_j}$ is a a_j-ary relation over $|\mathcal{A}|$, and each $c_j \in |\mathcal{A}|$ is an element of $|\mathcal{A}|$. For vocabulary σ, STRUC[σ] denotes the class of all finite structures with size $\|\mathcal{A}\| \geq 2$, i.e. structures whose universe is an initial segment $\{0, 1, \ldots, n-1\}$ for $n \geq 2$. Furthermore, we assume that every vocabulary has the *numerical relational* symbols '=', '$\leq$', 'PLUS', 'TIMES', 'BIT' and 'SUC', the *numerical constant* symbols '0', '1' and 'max', and that all these symbols have the *standard interpretations* on every structure $\mathcal{A}$.

If $\mathcal{L}$ denotes a logic, the language $\mathcal{L}(\sigma)$ is the set of all well-formed formulae from $\mathcal{L}$ over the vocabulary σ. A numerical formula in $\mathcal{L}(\sigma)$ is a formula with only numerical symbols. For example, SO$\exists(\sigma)$ is the set of all second-order formulae of the form $\exists Q_1 \cdots \exists Q_n \varphi$, where the Q_i's are relational variables and φ is a first-order formula over vocabulary $\langle \sigma, Q_1, \ldots, Q_n \rangle$. As usual, FO denotes first-order logic, SO denotes second-order logic, SO$\exists$ denotes the existential fragment of second-order logic, and so on.

A decision problem Π is characterized as a subset of STRUC[σ] for some fixed σ. For example, the problem CLIQUE can be characterized by structures $\mathcal{A} = \langle |\mathcal{A}|, E^{\mathcal{A}}, K^{\mathcal{A}} \rangle$ over the vocabulary $\sigma = \langle E^2, K \rangle$, where E is a binary relational symbol and K is a constant, such that $G = (|\mathcal{A}|, E^{\mathcal{A}})$ makes up an undirected graph and $K^{\mathcal{A}} \in \{0, \ldots, |\mathcal{A}| - 1\}$ denotes the size of a clique in G. Such models are typically characterized by a sentence Ψ over some fragment $\mathcal{L}$. CLIQUE, for example, can be characterized with the sentence:

$$\Psi_{CLIQUE} = (\exists f \in \mathrm{Inj})(\forall xy)[x \neq y \wedge f(x) \leq K \wedge f(y) \leq K \rightarrow E(x, y)]$$

where $(\exists f \in \mathrm{Inj})\varphi$ is an abbreviation for the second-order formula

$$\begin{aligned}&(\exists F)[\varphi \wedge (\forall x)(\exists z) F(x, z) \wedge \\ &\qquad (\forall xyz)[(F(x, y) \wedge F(x, z)) \vee (F(y, x) \wedge F(z, x)) \rightarrow y = z],\end{aligned}$$

which asserts that f is a total injective function. In this case, each $f(x)$ in φ must be replaced by a new variable z and the formula $\exists z F(x, z)$ must be added as well. Therefore, the problem CLIQUE that contains all pairs $\langle G, K \rangle$ such that G is an undirected graph with a complete subgraph of size K corresponds to the class MOD[Ψ_{CLIQUE}] of structures $\mathcal{A}$ such that $\mathcal{A} \vDash \Psi_{CLIQUE}$.

In the rest of the article, we shall stick to the following standards on notation:

- Lowercase (resp. uppercase) Greek letters are used to represent first-order (resp. second-order) formulae.
- If φ is a first-order formula whose free variables are among $x_1, \ldots, x_m$, this is denoted as $\varphi(x_1, \ldots, x_m)$ and similarly for second-order formulae.
- A tuple of variables $\langle x_1, \ldots, x_m \rangle$ is often denoted as $\bar{x}$; e.g. $\varphi(\bar{x})$ denotes a first-order formula φ whose free variables are among those in $\bar{x}$.

- If $\varphi(x_1,\dots,x_m)$ is a formula over vocabulary σ, $\mathcal{A}\in\mathrm{STRUC}[\sigma]$, and $a_1,\dots,a_m$ is a tuple over $|\mathcal{A}|$, then $\mathcal{A}\vDash\varphi(a_1,\dots,a_m)$ means that $\langle\mathcal{A},i\rangle\vDash\varphi$ for every interpretation i that maps x_j into a_j for $1\leq j\leq m$.
- The formulae $(\forall\bar{x}\theta(\bar{x}))\varphi(\bar{x})$ and $(\exists\bar{x}\theta(\bar{x}))\varphi(\bar{x})$ are abbreviation of $(\forall\bar{x})(\theta\rightarrow\varphi)$ and $(\exists\bar{x})(\theta\wedge\varphi)$, respectively.
- Finally, if φ and $\psi(\bar{x})$ are two formulae, then $\varphi(\psi(\bar{x}))$ is used to denote the fact that $\psi(\bar{x})$ appears as subformula of φ. Furthermore, $\varphi[\psi/\xi]$ denotes the formula obtained by replacing each occurrence of ψ by ξ in φ.

2.2 First-order queries and projections

Let σ and $\tau=\langle R_1^{a_1},\dots,R_r^{a_r},c_1,\dots,c_s\rangle$ be two vocabularies. Let $k\geq 1$ and consider the tuple $I=\langle\varphi_1,\dots,\varphi_r,\psi_1,\dots,\psi_s\rangle$ of $r+s$ first-order formulae in $\mathrm{FO}(\sigma)$ of the form $\varphi_i(x_1,\dots,x_{ka_i})$ for $1\leq i\leq r$, and $\psi_j(x_1,\dots,x_k)$ for $1\leq j\leq s$. That is, φ_i has at most ka_i free variables among $x_1,\dots,x_{ka_i}$, and ψ_j has at most k free variables among $x_1,\dots,x_k$.

The tuple I defines a mapping $\mathcal{A}\mapsto I(\mathcal{A})$, called a *first-order query* of arity k, from σ-structures into τ-structures, $I:\mathrm{STRUC}[\sigma]\rightarrow\mathrm{STRUC}[\tau]$, given by:

- Universe $|I(\mathcal{A})|\doteq|\mathcal{A}|^k$,
- Relations $R_i^{I(\mathcal{A})}\doteq\{(\bar{u}_1,\dots,\bar{u}_{a_i})\in|\mathcal{A}|^{ka_i}:\mathcal{A}\vDash\varphi_i(\bar{u}_1,\dots,\bar{u}_{a_i})\}$, and
- Constants $c_j^{I(\mathcal{A})}\doteq\bar{u}$ for the *unique* $\bar{u}$ with $\mathcal{A}\vDash\psi_j(\bar{u})$.

The numerical relations in $I(\mathcal{A})$ are defined in the standard way such that the k-tuples in $|I(\mathcal{A})|$ become ordered lexicographically, and the symbols PLUS, TIMES, BIT and SUC obtain the intended interpretations. It is not difficult to show that the formulae defining the numerical predicates are all first-order formulae [4]. Some authors consider mappings I extended with a formula φ_0 used to define the universe as $|I(\mathcal{A})|=\{\bar{u}\in|A|^k:\varphi_0(\bar{u})\}$. However, this often causes difficulties when defining the interpretation of the numerical predicates as, in some cases, the formulae defining them cease to be first-order [4]. For this reason, we shall not consider such formula φ_0.

If $\Pi\subseteq\mathrm{STRUC}[\sigma]$ and $\Omega\subseteq\mathrm{STRUC}[\tau]$ are two problems, and the query I is such that $\mathcal{A}\in\Pi$ iff $I(\mathcal{A})\in\Omega$, then I is called a *first-order reduction* from Π to Ω. A first-order query is called a *first-order projection* (fop) if each φ_i, and each ψ_j, has the form $\alpha_0(\bar{x})\vee(\alpha_1(\bar{x})\wedge\lambda_1(\bar{x}))\vee\cdots\vee(\alpha_e(\bar{x})\wedge\lambda_e(\bar{x}))$ where the α_i's are numerical and mutually exclusive, and each λ_i is a σ-literal. Projections are typically denoted by the letter ρ. If ρ is a reduction from Π to Ω, we write $\rho:\Pi\leq_{\mathrm{fop}}\Omega$, and if Π is complete for the class $\mathbf{C}$ via $\leq_{\mathrm{fop}}$ reductions, then we say that Π is $\leq_{\mathrm{fop}}$-complete for $\mathbf{C}$, that Π is $\mathbf{C}$-complete via fops, etc. Likewise, we say that Π is $\leq_m^p$-complete for $\mathbf{C}$ if Π is complete for the class $\mathbf{C}$ via many-one polynomial-time reductions.

First-order projections have interesting properties; e.g. for each fop ρ there is a first-order sentence $\beta_\rho\in\mathrm{FO}(\sigma)$ that characterizes the image of ρ [1]; i.e. $\mathcal{B}\vDash\beta_\rho$ iff $\mathcal{B}=\rho(\mathcal{A})$ for some $\mathcal{A}\in\mathrm{STRUC}[\sigma]$. We call the sentence β_ρ the *characteristic sentence* for ρ.

2.3 Dual of a first-order query

There is a syntactic operator associated to each first-order query that plays a fundamental role in our results. Let σ and $\tau=\langle R_1^{a_1},\dots,R_r^{a_r},c_1,\dots,c_s\rangle$ be two vocabularies and

$I=\{\varphi_1,\ldots,\varphi_r,\psi_1,\ldots,\psi_s\}$ a k-ary first-order query that maps STRUC$[\sigma]$ in STRUC$[\tau]$. The *dual operator* $\hat{I}$ maps formulae in $\mathcal{L}(\tau)$ to formulae in $\mathcal{L}(\sigma)$ in the following way (cf. [4]). Let us define the mapping f_I:

- $f_I(x)\doteq\bar{x}$ where $\bar{x}=\langle x^1,\ldots,x^k\rangle$ for variable x,
- $f_I(R_i(v_1,\ldots,v_{a_i}))\doteq\varphi_i(f_I(v_1),\ldots,f_I(v_{a_i}))$ for relation R_i,
- $f_I(c_j)\doteq\bar{z}_j$ where $\bar{z}_j=\langle z_j^1,\ldots,z_j^k\rangle$ for constant c_j,
- $f_I(Qx)\doteq(Qf_I(x))$ for $Q\in\{\exists,\forall\}$,
- $f_I(QR^a)\doteq(QR^{ka})$ for $Q\in\{\exists,\forall\}$ where R^{ka} is a ka-arity relation,
- f_I is identity for the boolean connectives.

The dual operator is $\hat{I}(\varphi)\doteq(\exists\bar{z}_1)\cdots(\exists\bar{z}_s)[f_I(\varphi)\wedge\psi_1(\bar{z}_1)\wedge\cdots\wedge\psi_s(\bar{z}_s)]$. The *fundamental property of duals* is $\mathcal{A}\vDash\hat{I}(\theta)$ iff $I(\mathcal{A})\vDash\theta$ for every $\theta\in\mathcal{L}(\tau)$ and $\mathcal{A}\in$STRUC$[\sigma]$

As we shall see, this property is very powerful and can be fruitfully exploited in a number of results. However, up to our knowledge, duals had been mainly used only to show that the major complexity classes are closed under first-order reductions. Thus, one of our general contribution is to show that duals have an important role in descriptive complexity.

3 Canonical forms of complete problems

Medina and Immerman gave a syntactic characterization of NP-completeness via fops with respect to the so-called generalized IS-forms [5, 6]. An IS-form is a formula that resembles the formula defining INDEPENDENT SET:

$$\Psi_{IS} \;=\; (\exists f\in\mathrm{Inj})(\forall x,y)\big[x\neq y\wedge f(x)\leq K\wedge f(y)\leq K\rightarrow\neg E(x,y)\big].$$

Theorem 1 ([6])
Let $\sigma=\langle Q^1\rangle$ be the vocabulary for binary strings, and $L\subseteq$STRUC$[\sigma]$ a NP problem characterized by $\Psi\in\mathrm{SO}\exists(\sigma)$. Then, L is NP-complete via fops iff there is an *injective* fop ρ:STRUC$[\langle E^2,K\rangle]\rightarrow$STRUC$[\sigma]$ such that

$$\Psi \;\equiv\; (\beta_\rho\wedge\Upsilon_{IS})\vee(\neg\beta_\rho\wedge\Lambda)$$

where $\beta_\rho\in\mathrm{FO}(\sigma)$ is the characteristic sentence of ρ, $\Upsilon_{IS}\in\mathrm{SO}\exists(\sigma)$ is a *generalized IS-form*, and Λ is a $\mathrm{SO}\exists(\sigma)$ sentence.

This result can be generalized over a large collection of complexity classes, that includes L, P, NP, PSPACE and for arbitrary vocabularies. Furthermore, the decomposition can be obtained modulo any $\leq_{\mathrm{fop}}$-complete problem for the given class, and the fop ρ does not need to be injective. The main obstacle for obtaining the generalization is to take care of the sentence Υ_{IS} for classes other than NP, and for problems other than INDEPENDENT SET. As it will be shown, we do not have to do a case-by-case analysis for obtaining the generalized form. Instead, it can be obtained from the sentence that defines the complete problem using the dual operator.

For a problem $\Pi\subseteq$STRUC$[\tau]$, define the relation $\cong_\Pi$ over STRUC$[\tau]$ as $\mathcal{A}\cong_\Pi\mathcal{B}$ iff $\mathcal{A}\in\Pi\Leftrightarrow\mathcal{B}\in\Pi$. Clearly, $\cong_\Pi$ is an equivalence relation.

Theorem 2
Let σ and τ be two vocabularies, and **C** a complexity class captured by a fragment $\mathcal{L}$ that is closed under disjunctions and closed under conjunctions with FO. Let $\Pi\subseteq$STRUC$[\tau]$ be a

$\leq_{\text{fop}}$-complete problem for $\mathbf{C}$ characterized by the sentence $\Psi \in \mathcal{L}(\tau)$, and B a problem over vocabulary σ. Then, B is $\leq_{\text{fop}}$-complete for $\mathbf{C}$ iff there is a fop $\rho : \text{STRUC}[\tau] \rightarrow \text{STRUC}[\sigma]$ such that for all $\mathcal{B} \in \text{STRUC}[\sigma]$:

$$\mathcal{B} \in B \quad \text{iff} \quad \mathcal{B} \models (\beta_\rho \wedge \hat{I}(\Psi)) \vee (\neg\beta_\rho \wedge \Lambda) \tag{1}$$

where (i) $\beta_\rho \in \text{FO}(\sigma)$ is the characteristic sentence of ρ, (ii) $\Lambda \in \mathcal{L}(\sigma)$, and (iii) $I : \text{STRUC}[\sigma] \rightarrow \text{STRUC}[\tau]$ is a first-order query such that $I(\rho(\mathcal{A})) \cong_\Pi \mathcal{A}$ for all $\mathcal{A} \in \text{STRUC}[\tau]$.

Proof. For the necessity, assume that B is $\leq_{\text{fop}}$-complete for $\mathbf{C}$ characterized by $\Lambda \in \mathcal{L}(\sigma)$. Then, there is $\rho : \Pi \leq_{\text{fop}} B$. Let $\mathcal{B} \in B$ and consider whether $\mathcal{B} \notin \rho(\text{STRUC}[\tau])$ or not. In the first case, $\mathcal{B} \models \neg\beta_\rho \wedge \Lambda$ and we are done. In the second case, $\mathcal{B} \models \beta_\rho$ and $\mathcal{B} = \rho(\mathcal{A})$ for some $\mathcal{A} \in \text{STRUC}[\tau]$. Then,

$$\mathcal{B} = \rho(\mathcal{A}) \Rightarrow \mathcal{A} \in \Pi \Rightarrow \mathcal{A} \models \Psi \Rightarrow I(\rho(\mathcal{A})) \models \Psi \Rightarrow \rho(\mathcal{A}) \models \hat{I}(\Psi).$$

The first implication because ρ is a reduction and $\mathcal{B} \in B$, the second because Ψ characterizes Π, the third because condition (*iii*), and the last by the fundamental property of duals. Therefore, $\mathcal{B} \in B \Longrightarrow \mathcal{B} \models (\beta_\rho \wedge \hat{I}(\Psi)) \vee (\neg\beta_\rho \wedge \Lambda)$. Now, let $\mathcal{B} \in \text{STRUC}[\sigma]$ be such that $\mathcal{B} \models (\beta_\rho \wedge \hat{I}(\Psi)) \vee (\neg\beta_\rho \wedge \Lambda)$. If $\mathcal{B} \models \Lambda$, then $\mathcal{B} \in B$. Otherwise,

$$\begin{aligned}
\mathcal{B} \models \beta_\rho \wedge \hat{I}(\Psi) \Rightarrow\ & \mathcal{B} = \rho(\mathcal{A}) \text{ and } \rho(\mathcal{A}) \models \hat{I}(\Psi) && (\text{for some } \mathcal{A} \in \text{STRUC}[\tau]) \\
\Rightarrow\ & I(\rho(\mathcal{A})) \models \Psi && (\text{property of duals}) \\
\Rightarrow\ & \mathcal{A} \models \Psi && (\text{condition } (iii)) \\
\Rightarrow\ & \mathcal{A} \in \Pi && (\Psi \text{ characterizes } \Pi) \\
\Rightarrow\ & \mathcal{B} \in B && (\rho \text{ is reduction}).
\end{aligned}$$

It remains to show that there are first-order queries satisfying condition (*iii*). Since Π is complete, there is a fop $I : \text{STRUC}[\sigma] \rightarrow \text{STRUC}[\tau]$ that reduces $\rho(\Pi)$ to Π. Note that $\rho(\Pi) \subseteq B$ since ρ is also a reduction. For $\mathcal{A} \in \text{STRUC}[\tau]$,

$$\begin{gathered}
\mathcal{A} \in \Pi \ \Rightarrow\ \rho(\mathcal{A}) \in \rho(\Pi) \ \Rightarrow\ I(\rho(\mathcal{A})) \in \Pi, \\
I(\rho(\mathcal{A})) \in \Pi \ \Rightarrow\ \rho(\mathcal{A}) \in \rho(\Pi) \ \Rightarrow\ \rho(\mathcal{A}) \in B \Rightarrow \mathcal{A} \in \Pi.
\end{gathered}$$

Thus, $I : \rho(\Pi) \leq_{\text{fop}} \Pi$ satisfies $\mathcal{A} \in \Pi$ iff $I(\rho(\mathcal{A})) \in \Pi$.

We now show sufficiency. Assume there is a fop $\rho : \text{STRUC}[\tau] \rightarrow \text{STRUC}[\sigma]$ such that (1) holds for all $\mathcal{B} \in \text{STRUC}[\sigma]$. We need to show that B is complete for $\mathbf{C}$. The inclusion $B \in \mathbf{C}$ is direct from the closure properties on $\mathcal{L}$. For the hardness, we show that ρ is indeed a reduction from Π to B. For $\mathcal{A} \in \text{STRUC}[\tau]$, we have $\rho(\mathcal{A}) \models \beta_\rho$. If $\mathcal{A} \in \Pi$, then

$$\mathcal{A} \models \Psi \Rightarrow I(\rho(\mathcal{A})) \models \Psi \Rightarrow \rho(\mathcal{A}) \models \hat{I}(\Psi) \Rightarrow \rho(\mathcal{A}) \in B.$$

The first implication because condition (*iii*), the second by property of duals, and the last one by supposition. On the other hand, if $\rho(\mathcal{A}) \in B$, then

$$\rho(\mathcal{A}) \models \beta_\rho \Rightarrow \rho(\mathcal{A}) \models \hat{I}(\Psi) \Rightarrow I(\rho(\mathcal{A})) \models \Psi \Rightarrow \mathcal{A} \models \Psi \Rightarrow \mathcal{A} \in \Pi.$$

The first implication by supposition, the second by property of duals, the third by condition (*iii*), and the last one since Ψ characterizes Π. Thus, $\mathcal{A} \in \Pi$ iff $\rho(\mathcal{A}) \in B$, ρ is a reduction, and B is complete. ∎

As shown in the proof, we can replace the query I satisfying (*iii*) by the $\leq_{\text{fop}}$-reduction $I: \rho(\Pi) \leq_{\text{fop}} \Pi$ that exists since Π is complete. Furthermore, any first-order query J that satisfies (*iii*) is essentially equivalent (with respect to Ψ) to the reduction I; i.e. for $\mathcal{B} = \rho(\mathcal{A})$, $\mathcal{B} \vDash \hat{J}(\Psi)$ iff $\mathcal{B} \vDash \hat{I}(\Psi)$. Finally, if we consider nice complexity classes, the fop ρ can be assumed to be injective [1], and therefore Theorem 1 is a special case of Theorem 2 when $\mathbf{C} = \mathrm{NP}$ since $\Upsilon_{IS} \equiv \hat{I}(\Psi_{IS})$ on $\rho(\mathrm{STRUC}[\tau])$.

4 Syntactic operators

Immerman and Medina used syntactic operators that preserve NP-completeness to show the completeness of several problems [5, 6]. In this section, we generalize the idea of completeness-preserving operators, and consider another class of operators called operators that sustain completeness.

From now on $\mathcal{L}$ denotes a logic, τ and σ two vocabularies, and $\mathbf{C}$ is a complexity class. A *syntactic operator*, or just operator, is an application $T: \mathrm{Dom}\ T \subseteq \mathcal{L}(\tau) \to \mathcal{L}(\sigma)$ that maps formulae in $\mathrm{Dom}\ T$ into formulae in $\mathcal{L}(\sigma)$. The application and its domain are both equally important in this definition. The image of $\Psi \in \mathrm{Dom}\ T$ is denoted by $T\Psi$ and the range of T by $\mathrm{Ran}\ T = \{T\Psi : \Psi \in \mathrm{Dom}\ T\}$. If $\mathrm{Dom}\ T = \mathcal{L}(\tau)$, we simply write $T: \mathcal{L}(\tau) \to \mathcal{L}(\sigma)$. If $T: \mathrm{Dom}\ T \to \mathcal{L}(\sigma)$ and $T': \mathrm{Dom}\ T' \to \mathcal{L}(\sigma)$ are two operators such that $\mathrm{Dom}\ T \subseteq \mathrm{Dom}\ T'$ and $T'\Psi = T\Psi$ for each $\Psi \in \mathrm{Dom}\ T$, then T is called a *restriction* of T' and T' is called an *extension* of T.

Definition 1
Let $\mathcal{L}$ capture $\mathbf{C}$, and $T: \mathrm{Dom}\ T \subseteq \mathcal{L}(\tau) \to \mathcal{L}(\sigma)$ be an operator:

(1) T is *bounded* if it takes *sentences* in $\mathrm{Dom}\ T$ into *sentences* in $\mathcal{L}(\sigma)$.
(2) T *preserves* $\mathbf{C}$*-completeness* if T is bounded, and for every sentence $\Psi \in \mathrm{Dom}\ T$, if $\mathrm{MOD}[\Psi]$ is $\mathbf{C}$-complete, then $\mathrm{MOD}[T\Psi]$ is $\mathbf{C}$-complete.
(3) T *sustains* $\mathbf{C}$*-completeness* if T is bounded, and for every sentence $\Psi \in \mathrm{Dom}\ T$, if $\mathrm{MOD}[T\Psi]$ is $\mathbf{C}$-complete, then $\mathrm{MOD}[\Psi]$ is $\mathbf{C}$-complete.

Theorem 3
Let $\mathcal{L}$ be a logic that captures complexity class $\mathbf{C}$, and $T: \mathcal{L}(\tau) \to \mathcal{L}(\sigma)$ an operator. If $T = \hat{\rho}$ for some fop $\rho: \mathrm{STRUC}[\sigma] \to \mathrm{STRUC}[\tau]$, then T sustains $\mathbf{C}$-completeness.

Proof. Let $\rho: \mathrm{STRUC}[\sigma] \to \mathrm{STRUC}[\tau]$ be a fop such that $T = \hat{\rho}$, and $\Psi \in \mathcal{L}(\tau)$ a sentence such that $\mathrm{MOD}[T\Psi]$ is $\leq_{\text{fop}}$-complete for $\mathbf{C}$. We will show that $\mathrm{MOD}[\Psi]$ is $\leq_{\text{fop}}$-complete for $\mathbf{C}$. We know that $\mathrm{MOD}[\Psi] \in \mathbf{C}$ because $\Psi \in \mathcal{L}(\tau)$ and $\mathcal{L}$ captures $\mathbf{C}$. It remains to show the hardness of $\mathrm{MOD}[\Psi]$. Given $\mathcal{A} \in \mathrm{STRUC}[\sigma]$ then $\mathcal{A} \vDash T\Psi \Leftrightarrow \mathcal{A} \vDash \hat{\rho}(\Psi) \Leftrightarrow \rho(\mathcal{A}) \vDash \Psi$. Hence, ρ reduces $\mathrm{MOD}[T\Psi]$ to $\mathrm{MOD}[\Psi]$ and thus $\mathrm{MOD}[\Psi]$ is $\leq_{\text{fop}}$-complete for $\mathbf{C}$. ∎

We now show that it is always possible to show the completeness of a problem using an operator that sustains completeness. This result establishes the generality provided by the new class of operators.

COROLLARY 4
Let $\mathcal{L}$ be a logic that captures complexity class $\mathbf{C}$, and $\Omega = \text{MOD}[\Psi]$ a $\mathbf{C}$-complete problem. Then, for each $\mathbf{C}$-complete problem $\Pi = \text{MOD}[\Phi]$ there is an operator T that sustains completeness such that $T\Psi \equiv \Phi$.

PROOF. Due to the completeness of Ω, there is a fop $\rho : \Pi \leq_{\text{fop}} \Omega$. By the theorem, the operator $T = \hat{\rho} : \mathcal{L}(\tau) \to \mathcal{L}(\sigma)$ sustains completeness. For $\mathcal{A} \in \text{STRUC}[\sigma]$, $\mathcal{A} \vDash T\Psi \Leftrightarrow \mathcal{A} \vDash \hat{\rho}(\Psi) \Leftrightarrow \rho(\mathcal{A}) \vDash \Psi \Leftrightarrow \mathcal{A} \vDash \Phi$ where the third equivalence is because ρ is a reduction. Hence, $T\Psi \equiv \Phi$. ∎

DEFINITION 2
The operator $T : \mathcal{L}(\tau) \to \mathcal{L}(\sigma)$ is a *predicate substitution* [5] if

$$T\Psi \doteq \Psi[P_1(\bar{x}_1)/\varphi_1(\bar{x}_1), \ldots, P_r(\bar{x}_r)/\varphi_r(\bar{x}_r), c_1/t_1, \ldots, c_s/t_s]$$

where each $P_i^{a_i} \in \tau$ is a relational symbol of arity a_i, each $\varphi_i(\bar{x}_i) \in \text{FO}(\sigma)$, each $c_j \in \tau$ is a constant symbol, and each t_j is a bounded (i.e. with no free variables) σ-term.

COROLLARY 5
If T substitutes predicates such that each φ_i is a first-order projective formula, then T sustains $\mathbf{C}$-completeness for every class $\mathbf{C}$.

PROOF. Direct since T is the dual of a unary fop. ∎

EXAMPLE
If $\tau = \langle R_1^a, \ldots, R_m^a \rangle$ and σ is a vocabulary containing the relational symbol P of arity a, then the operator $T_u : \mathcal{L}(\tau) \to \mathcal{L}(\sigma)$ defined as

$$T_u\Psi \doteq \Psi[R_1(\bar{x})/P(\bar{x}), \ldots, R_m/P(\bar{x})]$$

is a bounded operator. Furthermore, by previous result, T_u sustains completeness regardless of the complexity class.

It is possible to build operators from others by composition while maintaining their fundamental properties. The proof is straightforward.

PROPOSITION 6
Let $T_1 : \text{Dom } T_1 \subseteq \mathcal{L}(\tau) \to \mathcal{L}(\sigma)$ and $T_2 : \text{Dom } T_2 \supseteq \text{Ran } T_1 \to \mathcal{L}(\nu)$ be two operators. If T_1 and T_2 preserve completeness, then T_1T_2, defined as $(T_1T_2)\Psi = T_2(T_1\Psi)$, preserves completeness. If T_1 and T_2 sustain completeness, then T_1T_2 sustains completeness as well.

Furthermore, completeness preserving and sustaining operators are closely related when the operators are invertible. Let us develop this idea.

DEFINITION 3
An operator $T : \text{Dom } T \subseteq \mathcal{L}(\tau) \to \mathcal{L}(\sigma)$ is *invertible* if there is an operator $H : \text{Dom } H \subseteq \mathcal{L}(\sigma) \to \mathcal{L}(\tau)$ such that

- Dom $H = \text{Ran } T$ and Dom $T = \text{Ran } H$,
- $H(T\Psi) \equiv \Psi$ for each $\Psi \in \text{Dom } T$, and
- $T(H\Psi) \equiv \Psi$ for each $\Psi \in \text{Dom } H$.

Note that the last conditions involve equivalence and not equality. If such H exists, it is called *an inverse* of T.

PROPOSITION 7
If $T:\mathrm{Dom}\ T\subseteq\mathcal{L}(\tau)\rightarrow\mathcal{L}(\sigma)$ is an invertible operator, and H_1, H_2 are two inverses, then H_1 and H_2 are equivalent on the image of T; i.e., $H_1\Phi\equiv H_2\Phi$ for $\Phi\in\mathrm{Ran}\ T$. Therefore, we talk about the inverse of T.

PROPOSITION 8
Let $T:\mathrm{Dom}\ T\subseteq\mathcal{L}(\tau)\rightarrow\mathcal{L}(\sigma)$ be an invertible operator and H its inverse. If T preserves **C**-completeness, then H sustains **C**-completeness. If T sustains **C**-completeness, then H preserves **C**-completeness.

PROOF. We have $\mathrm{Dom}\ H=\mathrm{Ran}\ T$. Therefore, if $\Phi\in\mathrm{Dom}\ H$, there is $\Psi\in\mathrm{Dom}\ T$ with $\Phi= T\Psi$. For the first claim,

$$\begin{aligned}\mathrm{MOD}[H\Phi] \text{ is } \mathbf{C}\text{-complete} &\Longrightarrow \mathrm{MOD}[HT\Psi] \text{ is } \mathbf{C}\text{-complete}\\ &\Longrightarrow \mathrm{MOD}[\Psi] \text{ is } \mathbf{C}\text{-complete}\\ &\Longrightarrow \mathrm{MOD}[T\Psi] \text{ is } \mathbf{C}\text{-complete}\\ &\Longrightarrow \mathrm{MOD}[\Phi] \text{ is } \mathbf{C}\text{-complete},\end{aligned}$$

where the second implication holds since H is an inverse of T, and the third since T preserves **C**-completeness. The second claim is similarly shown. ∎

PROPOSITION 9
Let $\tau=\langle P_1,\ldots,P_r,c_1,\ldots,c_s\rangle$. If $T=\hat{\rho}$ where $\rho:\mathrm{STRUC}[\sigma]\rightarrow\mathrm{STRUC}[\tau]$ is a unary fop of the form $\rho=\langle\varphi_1,\ldots,\varphi_{r+s}\rangle$ such that φ_i and φ_j share no common subformula for $i\neq j$, and $\varphi_i\notin\mathcal{L}(\tau)$, then T is invertible.

PROOF. If $\Psi\in\mathcal{L}(\tau)$, its dual $\hat{\rho}(\Psi)$ is $\Psi[P_1/\varphi_1,\ldots,c_s/\varphi_{r+s}]$. Since the φ_i's share no common subformula and do not belong to $\mathcal{L}(\tau)$, each $\varphi_i\in\hat{\rho}(\Psi)$ can be substituted again; i.e., $H\Phi\doteq \Phi[\varphi_1/P_1,\ldots,\varphi_{r+s}/c_s]$ is the inverse of T. ∎

5 Applications

In this section, we present interesting application of syntactic operators. First, we show that Medina's operators are special cases of more general operators that can be obtained from our results. Second, we give a logical characterization for the restriction of a problem, and show that the unrestricted problem is complete if the restricted problem is complete. Using this result, we give new NP-completeness results for a number of problems. Finally, we apply operators to problems at the second level of the polynomial-time hierarchy showing the generality of our approach.

5.1 Medina's operators

The fundamental tool used by Medina to show the completeness of a large number of problems is a collection of six operators that preserve completeness [5]. The first three operators

- $T_1\Psi\doteq\Psi[P(x_1,x_2)/Q(x_2,x_1)]$,
- $T_2\Psi\doteq\Psi[P(x_1,x_2)/\neg Q(x_1,x_2)]$, and
- $T_3\Psi\doteq\Psi[P(x_1,x_2)/R(x_1,x_2,c)]$

are *invertible* predicate-substitution operators and thus are the inverses of completeness-sustaining operators. In all cases, $T_i:\mathcal{L}(\langle\sigma, P\rangle)\rightarrow\mathcal{L}(\langle\sigma, Q^2, R^3, c\rangle)$ where $Q, R, c\notin\sigma$. By Corollary 5 and Propositions 9 and 8, the operators T_1, T_2 and T_3 preserve completeness for every complexity class **C**, not only for NP.

The other three fundamental operators are named Edge Creation 0, I and II in Medina's thesis. Although not explicitly mentioned in the thesis, these operators sustain completeness for every complexity class. For example, Medina's formulation of Edge Creation 0 (EC0) is as follows.

EC0
Let $\Phi_1=(\exists f\in\mathrm{Inj})(\forall x, y)[x\neq y\rightarrow\varphi(P(x,y))]$ be a sentence that defines an NP-complete property. Let $\beta(x,y)=(\exists u)(U_2(u)\wedge Q(x,u)\wedge Q(u,y))$ where Q is a new binary relation, and U_1 and U_2 are new unary relations. Then,

$$\Phi_2=(\exists f\in\mathrm{Inj})(\forall x, y)[U_1(x)\wedge U_2(y)\wedge x\neq y\rightarrow\varphi(\beta(x,y))]$$

defines an NP-complete property.

The claim in EC0, i.e. if Φ_1 defines an NP-complete problem, then Φ_2 also does, is shown by Medina with a 3-ary reduction $\rho:\mathrm{MOD}[\Phi_1]\leq_{\mathrm{fop}}\mathrm{MOD}[\Phi_2]$. This reduction can be used to define a syntactic operator $T_{EC0}=\hat{\rho}:\mathcal{L}(\langle U_1^1, U_2^1, Q^2\rangle)\rightarrow\mathcal{L}(P^2)$. Using structural induction on the formula φ, it is not hard to show that $T_{EC0}\Phi_2\equiv\Phi_1$. By Theorem 3, T_{EC0} sustains completeness regardless of the complexity class. Hence, if Φ_1 defines a complete problem, so does Φ_2.

By similar arguments, it can be shown that the operators ECI and ECII also preserve completeness regardless of the complexity class since they are duals of first-order projections.

5.2 *Restrictions*

Let σ and τ be two vocabularies such that $\tau=\langle\sigma, R_1^{a_1},\ldots, R_r^{a_r}, c_1,\ldots, c_s\rangle$. We say that τ extends σ with the relational symbols $\{R_i\}_{i=1}^r$ and constants $\{c_j\}_{j=1}^s$. If $\mathcal{A}\in\mathrm{STRUC}[\tau]$, $\mathcal{A}|_\sigma$ denotes the σ-structure obtained from $\mathcal{A}$ by dropping the relations $\{R_i^{\mathcal{A}}\}_i$ and the constants $\{c_j^{\mathcal{A}}\}_j$ from $\mathcal{A}$. If Π is a class of τ-structures, $\Pi|_\sigma\doteq\{\mathcal{A}|_\sigma:\mathcal{A}\in\Pi\}$.

Definition 4
Let σ and τ be vocabularies such that τ extends σ with relational symbols $\{R_i^{a_i}\}_i$ and constants $\{c_j\}_j$. Problem $\Pi\subseteq\mathrm{STRUC}[\sigma]$ is a *restriction* of problem $\Omega\subseteq\mathrm{STRUC}[\tau]$ iff there are formulae $\varphi_1,\ldots,\varphi_r\in\mathrm{FO}(\sigma)$ and *numerical* formulae $\psi_1(x),\ldots,\psi_s(x)$ (each satisfied by a unique element in each τ-structure) such that $\Pi=(\Omega\cap\Delta)|_\sigma$ where

$$\begin{aligned}\Delta=\{\mathcal{A}\in\mathrm{STRUC}[\tau]:\ &\mathcal{A}\models R_i(\bar{x}_i)\leftrightarrow\varphi(\bar{x}_i)\text{ for }1\leq i\leq r,\\ &\mathcal{A}\models c_j=x\leftrightarrow\psi_j(x)\text{ for }1\leq j\leq s\}.\end{aligned}$$

Theorem 10
Let σ and τ be vocabularies, and Π a restriction of $\Omega=\mathrm{MOD}[\Psi]$, $\Psi\in\mathcal{L}(\tau)$, as in previous definition. If Π is $\leq_{\mathrm{fop}}$-complete for class **C**, then Ω is $\leq_{\mathrm{fop}}$-complete for class **C**.

Proof. Let $\rho:\mathrm{STRUC}[\sigma]\rightarrow\mathrm{STRUC}[\tau]$ be the 1-ary fop $\langle\varphi_1,\ldots,\varphi_r,\psi_1,\ldots,\psi_s\rangle$ for the relational and constant symbols $\{R_i\}_{i=1}^r$ and $\{c_j\}_{j=1}^s$, and the identity over the symbols in

σ. Note that $\rho(\mathcal{A}|_\sigma)=\mathcal{A}$ iff $\mathcal{A}\in\Delta$. Since, by Theorem 3, $T=\hat{\rho}$ sustains **C**-completeness, we only need to show that $\Pi=\mathrm{MOD}[T\Psi]$. Let $\mathcal{A}|_\sigma\in(\Omega\cap\Delta)|_\sigma$. Then, $\rho(\mathcal{A}|_\sigma)\vDash\Psi$ and thus $\mathcal{A}|_\sigma\vDash\hat{\rho}(\Psi)$. Hence, $\mathcal{A}|_\sigma\in\mathrm{MOD}[T\Psi]$. Let $\mathcal{B}\in\mathrm{MOD}[T\Psi]$. Then, $\rho(\mathcal{B})\vDash\Psi$ and $\rho(\mathcal{B})\in\Delta$. Hence, $\rho(\mathcal{B})\in\Omega\cap\Delta$ and $\rho(\mathcal{B})|_\sigma\in\Pi$. Finish by observing that $\mathcal{B}=\rho(\mathcal{B})|_\sigma$. ∎

Corollary 11
The following problems are $\leq_{\text{fop}}$-complete for NP:

(1) MAX 3SAT
(2) MAX SAT
(3) CHROMATIC NUMBER
(4) PARTITION INTO ISOMORPHIC SUBGRAPHS
(5) PARTITION INTO HAMILTONIAN SUBGRAPHS (w/ param. K)
(6) HAMILTONIAN PATH BETWEEN TWO POINTS
(7) HAMILTONIAN PATH
(8) HAMILTONIAN COMPLETION
(9) HAMILTONIAN PATH COMPLETION
(10) SUBGRAPH ISOMORPHISM
(11) LARGEST COMMON SUBGRAPH
(12) DEGREE CONSTRAINED SPANNING TREE
(13) ISOMORPHIC SPANNING TREE

Proof. In the following, for problem Ω, 'Δ yields Π' means $\Pi=(\Omega\cap\Delta)|_\sigma$:

(1) $\Delta=\{K=\max\}$ yields 3SAT.
(2) $\Delta=\{K=\max\}$ yields SAT.
(3) $\Delta=\{K=3\}$ yields 3-COLORABILITY.
(4) $\Delta=\{K=3, E'(x,y)\equiv\text{true}\}$ yields PARTITION INTO TRIANGLES.
(5) $\Delta=\{K=1\}$ yields HAMILTONIAN CIRCUIT.
(6) $\Delta=\{S=0, T=0\}$ yields HAMILTONIAN CIRCUIT.
(7) See Appendix A.
(8) $\Delta=\{K=0\}$ yields HAMILTONIAN CIRCUIT.
(9) $\Delta=\{K=0\}$ yields HAMILTONIAN PATH.
(10) $\Delta=\{E_2(x,y)\equiv\text{true}\}$ yields CLIQUE.
(11) $\Delta=\{K_1=K_2\}$ yields SUBGRAPH ISOMORPHISM.
(12) $\Delta=\{K=2\}$ yields HAMILTONIAN PATH.
(13) $\Delta=\{E_T(x,y)\equiv\mathrm{SUC}(x,y)\}$ yields HAMILTONIAN PATH.

Use that 3SAT, SAT, 3-COLORABILITY, PARTITION INTO TRIANGLES, HAMILTONIAN CIRCUIT and CLIQUE are NP-complete via fops [5]. ∎

From this list, only results 3, 4 and 10 are already known [5]. On the other hand, MAX 2-SAT was known to be NP-complete via first-order queries (not first-order projections). The other results are new. It should be clear that this list is not exhaustive and thus that there are other problems that can be shown to be complete via fops using the restriction technique.

5.3 Numerical problems

We say that a decision problem is numerical if it involves weights, sizes and, in general, quantities that are expressed in binary notation. Typically, a numerical problem is only NP-complete in the strong sense meaning that if such quantities are expressed in unary, the problem becomes solvable in polynomial time; e.g. using dynamic programming.

Very little is known about the $\leq_{\text{fop}}$-completeness for NP of numerical problems. By directly using the syntactic technique of restrictions, we can directly show the following results. Up to our knowledge, these are novel results for NP via fops.

Corollary 12
The following problems are $\leq_{\text{fop}}$-complete for NP:

(1) LONGEST CIRCUIT
(2) KNAPSACK
(3) PARTIALLY ORDERED KNAPSACK
(4) NUMERICAL 3DM
(5) WEIGHTED HAMILTONIAN PATH COMPLETION

Proof. In the following, for problem Ω, 'Δ yields Π' means $\Pi = (\Omega \cap \Delta)|_{\sigma}$:

(1) $\Delta = \{K = \max, L(e,i) \equiv [i=0]\}$ yields HAMILTONIAN CIRCUIT.
(2) $\Delta = \{K = B, S(x,i) = V(x,i)\}$ yields SUBSET SUM.
(3) $\Delta = \{x \lessdot y \equiv \text{false}\}$ yields KNAPSACK.
(4) $\Delta = \{S(x,i) \equiv [i=0], B(i) \equiv [i=0] \vee [i=1]\}$ yields 3DM.
(5) $\Delta = \{W(e,i) \equiv [i=0]\}$ yields HAMILTONIAN PATH COMPLETION.

Use now that HAMILTONIAN CIRCUIT, 3DM and HAMILTONIAN PATH COMPLETION are NP-complete, and that SUBSET SUM is NP-complete (Appendix B). ∎

As before, it should be clear that this list is not exhaustive. Indeed, any numerical problem whose restriction to constant weights is known to be NP-complete via fops, it is also complete.

5.4 Reductions at the second level of the P-T hierarchy

We now consider two problems about graph colouring that are at the second level of the polynomial-time hierarchy. Let F, G and H be three (undirected) graphs. A (G,H)-*colouring* of F is a 2-colouring of the edges of F (e.g. with red and blue) such that there is no red (isomorphic) copy of G in F, and there is no blue copy of H in F. The existence of such colouring is written as $F \not\to (G,H)$, while $F \to (G,H)$ denotes that there is no such colouring; i.e. that for every 2-colouring of the edges of F there is either a red copy of G or a blue copy of H. Consider the two colouring problems:

GENERALIZED GRAPH COLOURING (GGC)
Instance: Two graphs F and G.
Question: $F \to (G,G)$?

NON-ARROWING [8]
Instance: Three graphs F, G and H.

Question: $F \not\to (G,H)$?

GGC is $\leq_m^p$-complete for Π_2^p [7]. Its complement $\overline{\text{GGC}}$ is $\leq_m^p$-complete for Σ_2^p and consists of all pairs of graphs (F,G) such that $F \not\to (G,G)$. NON ARROWING, on the other hand, is $\leq_m^p$-complete for Σ_2^p [8]. NON ARROWING can be defined over vocabulary $\tau = \langle F^2, G^2, H^2 \rangle$ with the sentence

$$(\forall R \in \text{2-Col})(\exists f \in \text{Inj})[(\forall xy)(G(x,y) \to R(f(x),f(y))) \vee \\ (\forall xy)(H(x,y) \to F(f(x),f(y)) \wedge \neg R(f(x),f(y)))]$$

where $R \in \text{2-Col}$ abbreviates the formula $(\forall xy)(R(x,y) \to F(x,y))$ meaning that R is a 2-colouring of the edges of F. If all instances of H are replaced by G, then the result is the sentence

$$(\forall R \in \text{2-Col})(\exists f \in \text{Inj})[(\forall xy)(G(x,y) \to R(f(x),f(y))) \vee \\ (\forall xy)(G(x,y) \to F(f(x),f(y)) \wedge \neg R(f(x),f(y)))]$$

that defines $\overline{\text{GGC}}$. Hence, for $T_u \Psi \doteq \Psi[H/G]$, $T_u \Psi_{AR} = \Phi_{\overline{\text{GGC}}}$. Therefore, if $\overline{\text{GGC}}$ is $\leq_{\text{fop}}$-complete for Σ_2^p, then NON ARROWING is $\leq_{\text{fop}}$-complete for Σ_2^p as well. Finally, observe that T_u is not invertible so it is not the inverse of a completeness-preserving operator.

6 Discussion

We have extended the canonical form proposed by Medina and Immerman to all complexity classes characterized by fragments $\mathcal{L}$ closed under disjunctions, and under conjunctions with FO. Although Medina and Immerman's method can be generalized to other nice classes beyond NP, it requires the formulation of 'generalized' sentences. Our method circumvent this requirement by considering the dual operator. Additionally, it is not clear how Medina and Immerman's method could be used to find canonical forms with respect to problems that are not 'graph' problems, or on classes that do not have complete problems based on explicit graphs.

We defined and studied a new class of syntactic operators called sustaining operators. These operators satisfy the converse property of preserving operators, yet they seem to be more general and well-behaved than the latter. Indeed, we showed that some preserving operators are inverses of sustaining operators, and that sustaining operators suffice to demonstrate the completeness of every $\leq_{\text{fop}}$-complete problem regardless of the complexity class (although, of course, we may need an infinite number of such operators). These results demonstrate the generality and potential of the new class of operators.

We gave sufficient conditions for an operator to be preserving and to be sustaining, and demonstrated a close connection between duals of fops and sustaining operators. Also, note the resemblance between sustaining operators and the inverse image of functions. The latter play a fundamental role in Mathematics; e.g. analysis and topology.

All the theory of syntactic operators is developed independently of the complexity class and thus applies to other classes as well. Indeed, we showed that if the complement of GENERALIZED GRAPH COLORING is $\leq_{\text{fop}}$-complete for Π_2^p then so is ARROWING. Such results cannot be achieved with other syntactic operators as they only apply to the class NP.

In the future, we are interested in studying a conjecture posed by Medina [5] that says that if $\Phi \wedge \varphi$ defines a NP-complete problem via fops and φ is a first-order formula, then it must be the case that Φ is NP-complete. This claim looks very plausible and intuitive because first-order logic captures AC^0 (i.e. languages recognized by circuits of constant depth and unbounded fan-in) which is known to be strictly contained in P. However, Medina's conjecture is still open and, up to our knowledge, no real advances had been made.

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Appendix A

HP is $\leq_{\text{fop}}$-Complete for NP

We show that HAMILTONIAN PATH (HP) is NP-Complete via fops in two steps. We first reduce HAMILTONIAN PATH BETWEEN TWO POINTS (ST-HP) to HAMILTONIAN PATH BETWEEN 0-1 (01-HP) showing the completeness of the latter since ST-HP is complete (Corollary 11). Then, we reduce 01-HP to HP. The vocabularies for HP, ST-HP and

01-HP are $\langle E^2\rangle$ for HP and 01-HP, and $\langle E^2, S, T\rangle$ for ST-HP. It should be clear that these problems can be expressed with SO∃-sentences.

For the reduction ST-HP $\leq_{\text{fop}}$ 01-HP, consider one that interchanges S with 0, T with 1, and leaves everything else equal. This reduction is a fop and thus 01-HP is $\leq_{\text{fop}}$-complete.

For the reduction 01-HP $\leq_{\text{fop}}$ HP, consider the 2-arity fop ρ given by:

$$E(x_1 x_2, y_1 y_2) \equiv [(x1 = y1) \wedge E(x_2, y_2)] \vee [\mathrm{SUC}(x_1, y_1) \wedge (x_2 = 1) \wedge (y_2 = 0)]$$

If $\mathcal{A}$ denotes a graph G with n vertices, the reduction creates a graph $G' = \rho(\mathcal{A})$ with n^2 vertices that contains n copies $G_1, \ldots, G_n$ of G. Furthermore, the vertex 1 of the i-th copy is connected to the vertex 0 of the $(i+1)$-th copy. Therefore, G contains a Hamiltonian path between 0 and 1 iff G' contains a Hamiltonian path. Clearly, ρ is a fop and thus HP is $\leq_{\text{fop}}$-complete for NP.

Appendix B

SUBSET SUM is $\leq_{\text{fop}}$-Complete for NP

SUBSET SUM
Instance: Set U, weights $w(u) \in \mathbb{Z}^+$, and integer $Z \in \mathbb{Z}^+$.
Is there a subset $U' \subseteq U$ such that $\sum_{u \in U'} w(u) = Z$?

We show that SUBSET SUM is $\leq_{\text{fop}}$-complete for NP with a reduction from

3-DIMENSIONAL MATCHING (3DM)
Instance: Set S with n elements and subset M of triplets over S.
Question: Does M contain a 3-dimensional matching; i.e. a subset $M' \subseteq M$ such that for each $x \in S$ there are triplets $\langle x, a_0, b_0\rangle$, $\langle a_1, x, b_1\rangle$ and $\langle a_2, b_2, x\rangle$ in M', and that for every pair of distinct $\langle x, y, z\rangle, \langle x', y', z'\rangle \in M'$ is true that $x \neq x'$, $y \neq y'$ and $z \neq z'$?

3DM is known to be $\leq_{\text{fop}}$-complete for NP [5] and can be defined with the following sentence over $\sigma = \langle M^3\rangle$:

$$\begin{aligned}\Phi_{3\mathrm{DM}} = (\exists P^3)\big[&(\forall xyz)[P(x,y,z) \rightarrow M(x,y,z)] \wedge \\ &(\forall x \exists u_1 v_1 u_2 v_2 u_3 v_3)[P(x,u_1,v_1) \wedge P(u_2,x,v_2) \wedge P(u_3,v_3,x)] \wedge \\ &(\forall xyz \forall uvw)[(x \neq u \vee y \neq v \vee z \neq w) \wedge \\ &P(x,y,z) \wedge P(u,v,w) \rightarrow (x \neq u \wedge y \neq v \wedge z \neq w)]\big].\end{aligned}$$

SUBSET SUM is defined with the vocabulary $\tau = \langle W^2, Z^1\rangle$ where the symbols W and Z are interpreted as:

- $(u, i) \in W^{\mathcal{A}}$ iff the i-th bit in the binary representation of $w(u)$ is 1, and
- $i \in Z^{\mathcal{A}}$ iff the i-th bit in the binary representation of Z is 1.

THEOREM 13
SUBSET SUM is $\leq_{\text{fop}}$-complete for NP.

Proof. Consider the fop $\rho:\mathrm{STRUC}[\sigma]\to\mathrm{STRUC}[\tau]$ of arity 3 that defines the weights and the bound as follows:

$$\begin{aligned}
W(\bar{x},\bar{y}) \;&=\; [y_1=0\wedge y_2=x_1\wedge y_3=0\wedge M(\bar{x})]\;\vee\\
&\quad\;\; [y_1=1\wedge y_2=x_2\wedge y_3=0\wedge M(\bar{x})]\;\vee\\
&\quad\;\; [y_1=2\wedge y_2=x_3\wedge y_3=0\wedge M(\bar{x})],\\
B(\bar{x}) \;&=\; [(x_1=0\vee x_1=1\vee x_1=2)\wedge x_3=0].
\end{aligned}$$

If $\|\mathcal{A}\|=n$, then $\|\rho(\mathcal{A})\|=n^3$. As usual, the triplets $\bar{x}\in|\rho(\mathcal{A})|$ are ordered lexicographically. In the following, we use interchangeably the triplet $\bar{x}=(x_1,x_2,x_3)$ and the integer $x_1n^2+x_2n+x_3$ that ranks $\bar{x}$ in the lexicographical order $<$; e.g. $(0,1,1)<(1,0,0)$.

Let $\mathrm{bits}(m)$ denote the position of the bits with value 1 in the binary representation of m; e.g. $\mathrm{bits}(0)=\emptyset$, $\mathrm{bits}(8)=\{3\}$ and $\mathrm{bits}(12)=\{0,2,3\}$. First, observe that the fop is such that

$$\begin{aligned}
\mathrm{bits}(w(\bar{x})) \;&=\; \begin{cases}\emptyset & \text{if } \neg M(\bar{x}),\\ \{\langle 0,x_1,0\rangle,\langle 1,x_2,0\rangle,\langle 2,x_3,0\rangle\} & \text{if } M(\bar{x}),\end{cases}\\
\mathrm{bits}(B) \;&=\; \{\langle i,j,0\rangle : 0\le i<3, 0\le j<n\}.
\end{aligned}$$

We need to show that $\mathcal{A}\in 3\mathrm{DM}$ if and only if $\rho(\mathcal{A})\in\mathrm{SUBSET\ SUM}$.

Let $\mathcal{A}\in 3\mathrm{DM}$. Then, there is 3DM $M'\subseteq|\mathcal{A}|^3$. We need to show that $\sum_{\bar{x}\in M'}w(\bar{x})=B$ for $\rho(\mathcal{A})\in\mathrm{SUBSET\ SUM}$. Let $\bar{x},\bar{y}\in M'$ with $\mathrm{bits}(w(\bar{x}))\cap\mathrm{bits}(w(\bar{y}))\neq\emptyset$. Then, there is $\langle i,v,0\rangle$ in the intersection with $i=0$, 1 or 2. Without loss of generality assume that $i=0$. Then, $\bar{x}=\langle v,x_2,y_2\rangle$ and $\bar{y}=\langle v,y_2,y_3\rangle$ and, by the definition of matching, it must be $\bar{x}=\bar{y}$. Therefore, $\mathrm{bits}(w(\bar{x}))\cap\mathrm{bits}(w(\bar{y}))=\emptyset$ for distinct $\bar{x},\bar{y}\in M'$, and thus

$$\mathrm{bits}\Big(\sum_{\bar{x}\in M'}w(\bar{x})\Big) \;=\; \bigcup_{\bar{x}\in M'}\mathrm{bits}(w(\bar{x})). \tag{1}$$

On the other hand, every 3DM M' can be thought as the set $M'=\{\langle f_i,g_i,h_i\rangle : 0\le i<n\}$ for three injective functions $f,g,h:|\mathcal{A}|\to|\mathcal{A}|$. This implies that $\cup_{\bar{x}\in M'}\mathrm{bits}(w(\bar{x}))=\{\langle i,j,0\rangle : 0\le i<3, 0\le j<n\}$ and $\sum_{\bar{x}\in M'}w(\bar{x})=B$.

It remains to show that if $\rho(\mathcal{A})\in\mathrm{SUBSET\ SUM}$ then $\mathcal{A}\in 3\mathrm{DM}$. Suppose $\rho(\mathcal{A})\in\mathrm{SUBSET\ SUM}$ and let U' be such that $\sum_{\bar{x}\in U'}w(\bar{x})=B$ and that there is no $\bar{x}\in U'$ with $w(\bar{x})=0$.

First note that the equality (1) holds if there is no carry at the bit-level in the sum $\sum_{\bar{x}\in A'}w(\bar{x})$. If this happens, U' is a 3DM. We will show that no carry is generated.

For a carry to appear it is needed that there are two different $\bar{x},\bar{x}'\in U'$ such that $\mathrm{bits}(w(\bar{x}))\cap\mathrm{bits}(w(\bar{x}'))\neq\emptyset$. Suppose that the bit $\langle 0,0,0\rangle$ appears at least twice in U'. Their sum generates the bit $\langle 0,0,1\rangle$ which is not in B. It is not hard to see that in order to get rid of $\langle 0,0,1\rangle$ and to have the bit $\langle 0,0,0\rangle$ in the final sum, it is necessary that $\langle 0,0,0\rangle$ appears at least 2^n+1 times. Since each bit appears at most n^2 times, e.g. $\langle 0,0,0\rangle$ appears at most once for each $\langle 0,y,z\rangle\in|\rho(\mathcal{A})|$, then it must be $n^2\ge 2^n+1$ but this is only solvable for $n=3$. Therefore, if $n\neq 3$, it is impossible to generate a carry when summing the bits $\langle 0,0,0\rangle$ and thus this bit appears exactly once. This argument is repeated for the other bits to show that no carry is generated in the sum. Therefore, if $\|\mathcal{A}\|\neq 3$, then $\rho(\mathcal{A})\in\mathrm{SUBSET\ SUM}$ implies $\mathcal{A}\in 3\mathrm{DM}$.

We address the case $n = 3$. Suppose there are multiple instances of $\langle 0,0,0\rangle$. Then, there must be exactly 9 instances that comes from the tuples $\langle 0,y,z\rangle \in U'$ that also satisfy M. In particular, $\{\langle 0,0,2\rangle, \langle 0,1,2\rangle, \langle 0,2,2\rangle\} \subseteq U'$. For each of these, the bit $\langle 2,2,0\rangle$ is one and thus this bit is added at last three times. But this generates a carry beyond the bit $\langle 2,2,0\rangle$ that is the most significant bit in B. Therefore, $\sum_{\bar{x}\in A'} w(\bar{x}) \neq B$ that contradicts the fact that $\mathcal{A} \in$ SUBSET SUM. ∎