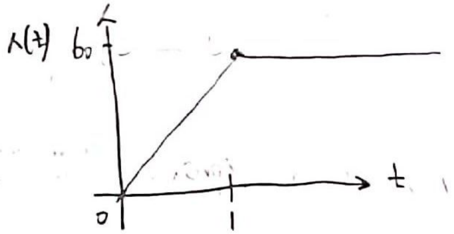
## Formulation of time varying function

**Exercise 15**

*During the first hour* $(0 \le t \le 1)$, $\lambda(t)$ *increases linearly from* $0$ *to* $60$. *After the first hour,* $\lambda(t)$ *is constant at* $60$. *Draw plot for* $\lambda(t)$ *and express the function in math form.*
















"Man can learn nothing unless he proceeds from the known to the unknown. - Claude Bernard"

## IV. Series and Others





**Exercise 13 (Finite geometric series)**

*Simplify the following.* When $r \neq 1$, $S = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

In Exercise 12, already solved problem.

$(r \neq 1)$

$$S = \frac{a(1-r^n)}{1-r}$$



**Exercise 12 (Infinite geometric series)**

*Simplify the following.* When $|r| < 1$, $S = a + ar + ar^2 + ar^3 + \ldots$

$$S = a + ar + ar^2 + ar^3 \cdots ar^{n-2} + ar^{n-1}$$

$$rS = ar + ar^2 + ar^3 + \cdots ar^{n-1} + ar^n$$

$$S - rS = (1-r)S = a - ar^n$$

$$S = \frac{a - ar^n}{1-r} = \frac{a(1-r^n)}{1-r}$$

$(r \neq 1)$

If $|r| < 1$ $\lim_{n\to\infty} r^n = 0$

$$S = \frac{a}{1-r}$$





멱급수

**Exercise 14 (Power series)**

*Simplify the following.* When $|r| < 1$, $S = r + 2r^2 + 3r^3 + 4r^4 + \ldots$

$$S = r + 2r^2 + 3r^3 + 4r^4$$

$$rS = r^2 + 2r^3 + 3r^4 \cdots$$

$$S(1-r) = r + r^2 + r^3 + r^4 \cdots$$

$$S = \frac{r + r^2 + r^3 + r^4 \cdots}{1-r} = \frac{r(1 + r + r^2 \cdots)}{1-r} = \frac{r \times \frac{1}{1-r}}{1-r}$$

$$= \frac{r}{(1-r)^2}$$

### Exercise 9

?

*Solve the following system of equations.*

$$(\pi_0 \quad \pi_1 \quad \pi_2)\begin{pmatrix} -2 & 2 & \\ 3 & -5 & 2 \\ & 3 & -3 \end{pmatrix} = (0 \quad 0 \quad 0)$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$-2\pi_0 + 3\pi_1 = 0$

$2\pi_0 - 5\pi_1 + 3\pi_2 = 0$

$2\pi_1 - 3\pi_2 = 0$

$\pi_0 + \pi_1 + \pi_2 = 1$

$$(\pi_0 \ \pi_1 \ \pi_2)\begin{pmatrix} -2 & 2 & \\ 3 & -5 & 2 \\ & 3 & -3 \end{pmatrix} = (0, 0, 0)$$

$$\left(\begin{array}{ccc|c} -2 & 3 & 0 & 0 \\ 2 & -5 & 3 & 0 \\ 0 & 2 & -3 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \xrightarrow{\text{Gaussian method}} \left(\begin{array}{ccc|c} 1 & -\frac{3}{2} & 0 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 2 & -3 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{9}{19} \\ 0 & 1 & 0 & \frac{6}{19} \\ 0 & 0 & 1 & \frac{4}{19} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\pi_0 = \frac{9}{19}, \pi_1 = \frac{6}{19}, \pi_2 = \frac{4}{19}$





### Exercise 10

?

*Solve the following system of equations.*

$$(\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4)\begin{pmatrix} .7 & .3 & & \\ .5 & .5 & & \\ & & .6 & .4 \\ & & .3 & .7 \end{pmatrix} = (\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4)$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

$0.7\pi_1 + 0.5\pi_2 = \pi_1$

$0.3\pi_1 + 0.5\pi_2 = \pi_2$

$0.6\pi_3 + 0.3\pi_4 = \pi_3$

$0.4\pi_3 + 0.7\pi_4 = \pi_4$

$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

Gauss-Jordan method

$$\left(\begin{array}{cccc|c} -\frac{3}{10} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{3}{10} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{5} & \frac{3}{10} & 0 \\ 0 & 0 & \frac{2}{5} & -\frac{3}{10} & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{35}{32} & \frac{5}{8} \\ 0 & 1 & 0 & \frac{21}{32} & \frac{3}{8} \\ 0 & 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$





### Exercise 11

*Solve following and express $\pi_i$ for $i = 0, 1, 2, \ldots$*

$$\begin{aligned} \pi_0 + \pi_1 + \pi_2 + \ldots &= 1 \\ 0.02\pi_0 + 0.02\pi_1 + 0.02\pi_2 + \ldots &= \pi_0 \\ 0.98\pi_0 &= \pi_1 \\ 0.98\pi_1 &= \pi_2 \\ 0.98\pi_2 &= \pi_3 \\ \ldots &= \ldots \end{aligned}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & \cdots & & 1 \\ -0.98 & 0.02 & 0.02 & \cdots & & 0 \\ 0.98 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0.98 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0.98 & -1 & & \\ \cdots & & & & & \end{array}\right)$$





$\pi_1 = \frac{5}{8} - \frac{35}{32}\pi_4$

$\pi_2 = \frac{3}{8} - \frac{21}{32}\pi_4$

$\pi_3 = \frac{3}{4}\pi_4$

$\pi_4 = $ any value.



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## Matrix multiplication

Exercise 5

*Solve the followings.*

$.6 = 0.6$ ?

$$(.6 \quad .4)\begin{pmatrix}.7 & .3\\ .5 & .5\end{pmatrix} =$$

$(0.6, 0.4)\begin{pmatrix}0.7 & 0.3\\ 0.5 & 0.5\end{pmatrix} = (0.6\times0.7 + 0.4\times0.5,\ 0.6\times0.3 + 0.4\times0.5)$

$(1\times2)(2\times2) = (6.2,\ 3.8)$

$= (1\times2)$



Exercise 6

*What is $P^2$?*

$$P = \begin{pmatrix}.7 & .3\\ .5 & .5\end{pmatrix}$$

$\begin{pmatrix}0.7 & 0.3\\ 0.5 & 0.5\end{pmatrix}\begin{pmatrix}0.7 & 0.3\\ 0.5 & 0.5\end{pmatrix} =$

$= \begin{pmatrix}0.7\times0.7 + 0.3\times0.5, & 0.7\times0.3 + 0.3\times0.5\\ 0.7\times0.5 + 0.5\times0.5, & 0.5\times0.3 + 0.5\times0.5\end{pmatrix}$

$= \begin{pmatrix}6.4, & 3.6\\ 6, & 4\end{pmatrix}$







## Solution to system of linear equations

Exercise 7

*Solve the followings.*

$$(\pi_1 \quad \pi_2)\begin{pmatrix}.7 & .3\\ .5 & .5\end{pmatrix} = (\pi_1 \quad \pi_2)$$

$$\pi_1 + \pi_2 = 1$$

$(0.7\pi_1 + 0.5\pi_2,\ 0.3\pi_1 + 0.5\pi_2) = (\pi_1, \pi_2)$

1) $0.7\pi_1 + 0.5\pi_2 = \pi_1 \rightarrow \pi_2 = \frac{+3}{5}\pi_1$

2) $0.3\pi_1 + 0.5\pi_2 = \pi_2$

3) $\pi_1 + \pi_2 = 1 \leftarrow \pi_1 = \frac{5}{8},\ \pi_2 = \frac{3}{8}$



Exercise 8

*Solve the following system of equations.*

$$x = y$$
$$y = 0.5z$$
$$z = 0.6 + 0.4x$$
$$x + y + z = 1$$

$x - y = 0$

$y - 0.5z = 0$

$z - 0.4x - 0.6 = 0$

$x - y - z - 1 = 0$

$\rightarrow \begin{pmatrix}1 & -1 & 0 & 0\\ 0 & 1 & -0.5 & 0\\ -0.4 & 0 & 1 & -0.6\\ 1 & -1 & -1 & -1\end{pmatrix}$ by Gauss Jordan Elimination

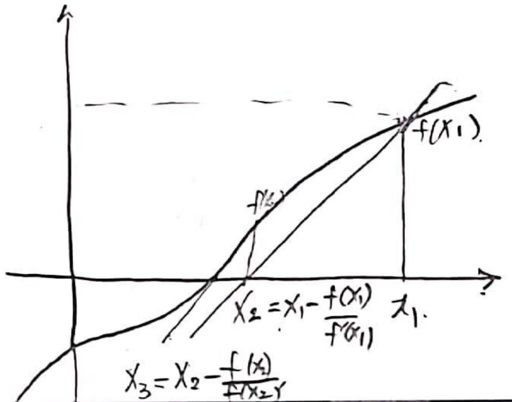
$\rightarrow \begin{pmatrix}1 & -1 & 0 & 0\\ 0 & 1 & -0.5 & 0\\ 0 & -0.4 & 1 & 0.6\\ 0 & 0 & 1 & 1\end{pmatrix} \rightarrow \begin{pmatrix}1 & -1 & 0 & 0\\ 0 & 1 & -\frac{1}{2} & 0\\ 0 & 0 & 1 & \frac{3}{4}\\ \boxed{0 \quad 0 \quad 0} & 1\end{pmatrix}$

It doesn't have solution

- Newton method approximates the function $f$ near $x_k$ by the tangent line at $f(x_k)$.

접선

$x_0$ = initial guess
for $k$=0,1,2,...
    $x_{k+1} = x_k - f(x_k)/f'(x_k)$
    break if $|x_{k+1} - x_k| < tol$
end








- Root-finding numerical methods such as bisection method and newton method has a few common properties.
  1. It is characterized as a *iterative process* (such as $x_0 \to x_1 \to x_2 \to \cdots$).
  2. In each *iteration*, the current candidate *gets closer* to the true value.
  3. It converges. That is, it is theoretically reach the *exact value* up to tolerance.
- Many iterative numerical methods share the properties above.
- The famous back propagation in deep neural network is also motivated by Newton method.
- Major algorithms for dynamic programming are called *policy iteration* and *value iteration* that also share the properties above.









## III. Matrix Algebra

## Solving an equation

- For the rest of this section, we consider a nonlinear and differentiable (thus, continuous) function $f : \mathbb{R} \to \mathbb{R}$, we aim to find a point $x^* \in \mathbb{R}$ such that $f(x^*) = 0$. We call such $x^*$ as a *solution* or a *root*.





## Bisection Method

- The *bisection* method aims to find a very short interval $[a, b]$ in which $f$ changes a sign.
- Why? Changing a sign from $a$ to $b$ means the function crosses the $\{y = 0\}$-axis, (a.k.a. $x$-axis), at least once. It means $x^*$ such that $f(x^*) = 0$ is in this interval. Since $[a, b]$ is a very short interval, We may simply say $x^* = \frac{a+b}{2}$.

**Definition 7 (sign function)**

$sgn(\cdot)$ is called a *sign function* that returns 1 if the input is positive, -1 if negative, and 0 if zero.

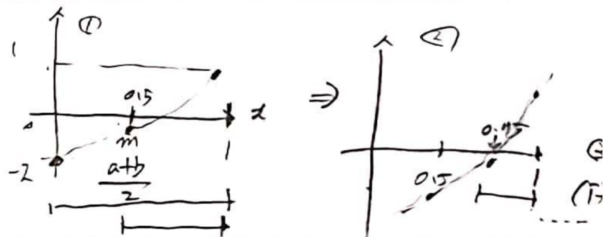
ex) let $f(x) = x^3+2x-2$

$f(0) = -2$

$f(1) = 1$

$f(0.5) = -0.875$

$f(0.75) = -0.078$





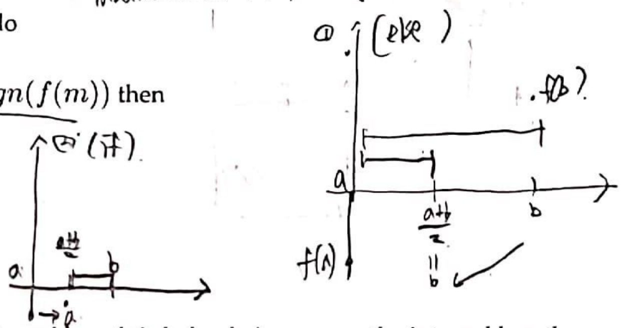


## Bisection algorithm → slow but sure Convergence

- Let $tol$ be the maximum allowable length of the *short interval* and an initial interval $[a, b]$ be such that $sgn(f(a)) \neq sgn(f(b))$.
- The *bisection algorithm* is the following.

maximum allowable length

while $((b - a) > tol)$ do
  $m = \frac{a+b}{2}$
  if $sgn(f(a)) = sgn(f(m))$ then
    $a = m$
  else
    $b = m$
  end
end



- At each *iteration*, the interval length is halved. As soon as the interval length becomes smaller than $tol$, then the algorithm stops.





## Newton Method

- The bisection technique makes no used of the function values other than their signs, resulting in slow but sure convergence.
- More rapid convergence can be achieved by using the function values to obtain a more accurate approximation to the solution *at each iteration*.
- Newton method is a method that use both the function value and derivative value.

**Remark 3**

The followings are popular antiderivatives.

- For $p \neq 1$, $f(x) = x^p \Rightarrow \int f(x)dx = \frac{1}{p+1}x^{p+1} + C$ (polyomial)
- $f(x) = \frac{1}{x} \Rightarrow \int f(x)dx = log(x) + C$ (fraction)
- $f(x) = e^x \Rightarrow \int f(x)dx = e^x + C$ (exponential)
- $f(x) = \frac{g'(x)}{g(x)} \Rightarrow \int f(x)dx = log(g(x)) + C$ (See Theorem 4 above)

**Exercise 3**

*Derive $\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g(x)'\,dx$. (Hint: Use Theorem 2 above.)*

by theorem 2. $h(x)=f(x)g(x) \xrightarrow{\text{then}} h'(x) = f'(x)g(x) + f(x)g'(x) = (f(x)g(x))'$

$\Rightarrow (f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$f'(x)g(x) = (f(x)g(x))' - f(x)g'(x)$

$\int f'(x)g(x) = f(x)g(x) - \int f(x)g'(x)$





**Exercise 4**

*Find $\int xe^x\,dx$, and evaluate $\int_0^1 xe^x\,dx$. (Hint: Use Exercise 3 above.)*

1) $(\int e^x \cdot x)$ $f(x) = e^x$ $f'(x) = e^x$

$g(x) = x$ $g'(x) = 1$

$\int f'(x)g(x) = e^x x = f(x)g(x) - \int f(x)g'(x) = xe^x - \int e^x \cdot 1$

$= x \cdot e^x - e^x = (x-1)e^x + C$

2) $\int_0^1 xe^x$

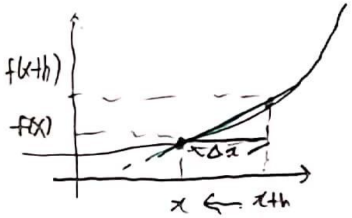
$(x-1)e^x \Big]_0^1 = (1-1)e^1 - \{(0-1)e^0\}$

$= -1$

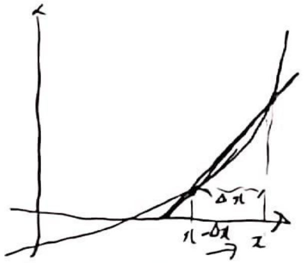




(forward difference formula)

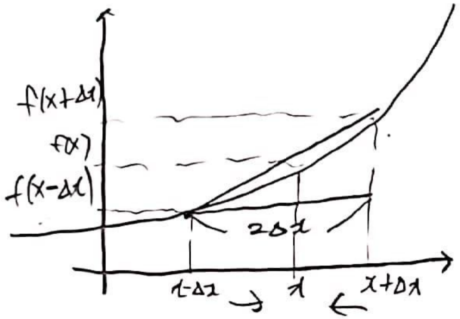



(backward difference formula)




## II. Numerical Methods

(centered difference formula)








## Differentiation

- Oftentimes, finding analytic derivative is hard, but finding numerical derivative is often possible.

↳ ex) $x^3 + 2x - 2 = 0$ ↳ Approximate

**Definition 6**

For a function $f$ and small constant $h$,

- $f'(x) \approx \frac{f(x+h)-f(x)}{h}$ *(forward difference formula)*
- $f'(x) \approx \frac{f(x)-f(x-h)}{h}$ *(backward difference formula)*
- $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$ *(centered difference formula)*

**Definition 3 (differentiable)**

If $\lim_{h\to 0} \frac{f(x+h/2)-f(x-h/2)}{h}$ exists for a function $f$ at $x$, we say the function $f$ is *differentiable at* $x$. That is, $f'(x) = \lim_{h\to 0} \frac{f(x+h/2)-f(x-h/2)}{h}$. If $f$ is differentiable for all $x$, then we say $f$ is *differentiable (everywhere)*.

**Remark 2**

The followings are popular derivatives.

- $f(x) = x^p \Rightarrow f'(x) = px^{p-1}$ (polyomial)
- $f(x) = e^x \Rightarrow f'(x) = e^x$ (exponential)
- $f(x) = log(x) \Rightarrow f'(x) = 1/x$ (log function; not differentiable at $x = 0$)

**Theorem 1**

*Differentiation is linear. That is,* $h(x) = f(x) + g(x)$ *implies* $h'(x) = f'(x) + g'(x)$.



**Theorem 2 (differentiation of product)**

*If* $h(x) = f(x)g(x)$, *then* $h'(x) = f'(x)g(x) + f(x)g'(x)$.

**Exercise 1**

*Suppose* $f(x) = xe^x$, *find* $f'(x)$.

$h(x) = xe^x = f(x)g(x)$, $\Rightarrow$ $h'(x) = f'(x)g(x) + f(x)g'(x)$

$h'(x) = 1\cdot e^x + xe^x$

$f'(x) = 1 \leftarrow f(x) = x$

$g(x)' = e^x \leftarrow g(x) = e^x$

**Theorem 3 (differentiation of fraction)**

*If* $h(x) = \frac{f(x)}{g(x)}$, *then* $h'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$.



**Theorem 4 (composite function)**

*If* $h(x) = f(g(x))$, *then* $h'(x) = f'(g(x)) \cdot g'(x)$.

**Exercise 2**

*Suppose* $f(x) = e^{2x}$, *find* $f'(x)$.

$h(x) = f(g(x))$, $h'(x) = f'(g(x))g'(x)$

$f(x) = e^x$. $h(x) = f(g(x)) = e^{g(x)}$

$g(x) = 2x$ $h'(x) = e^{g(x)} \cdot g'(x) = e^{2x} \cdot 2$

$\therefore 2e^{2x}$



## Integration

**Definition 4 (integration)**

Integration is the computation of an integral, which is a reverse operation of differentiation up to an additive constant.

**Definition 5 (antiderivative)**

Let's say a function $f$ is a derivative of $g$, or $g'(x) = f(x)$, then we say $g$ is an *antiderivative* of $f$, written as $g(x) = \int f(x)dx + C$, where $C$ is a integration constant.

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# Lecture A1. Math Review

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1. I. Differentiation and Integration
2. II. Numerical Methods
3. III. Matrix Algebra
4. IV. Series and Others





## I. Differentiation and Integration





## Differentiation

**Definition 1 (differentiation)**

Differentiation is the action of computing a derivative.

**Definition 2 (derivative)**

The derivative of a function $y = f(x)$ of a variable $x$ is a measure of the rate at which the value $y$ of the function changes with respect to (wrt., hereafter) the change of the variable $x$. It is notated as $f'(x)$ and called derivative of $f$ wrt. $x$.

**Remark 1**

If $x$ and $y$ are real numbers, and if the graph of $f$ is plotted against $x$, the derivative is the slope of this graph at each point.