Numerical Methods | March 6
MATH 417 - 501 | Spring 2018
A. Bonito

First Name: ____________ **Last Name:** ____________

Exam 1

- 75 minute individual exam;
- Answer the questions in the space provided. If you run out of space, continue onto the back of the page. Additional space is provided at the end;
- **Show and explain all work**;
- **Underline** the answer of each steps;
- The use of books, personal notes, **calculator**, cellphone, laptop, and communication with others is forbidden;
- By taking this exam, you agree to follow the university's code of academic integrity.

Ex 1	Ex 2	Ex 3	Ex 4	Total

Exercise 1 **10%**

Compute the first 3 steps of the newton method to find a rational approximation of $\sqrt{2}$ starting with the value 1.

Exercise 2 **30%**

Let $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $y_0 = 0$, $y_1 = 1$ and $y_2 = \sqrt{2}$.

1. Compute the Lagrange basis associated with the points x_0, x_1, x_2.
2. Write an expression of the polynomial of degree 2 passing through the points (x_i, y_i), $i = 0, 1, 2$ using the above Lagrange basis.
3. Compute the piecewise polynomial of degree 1 associated with the points (x_i, y_i), $i = 0, 1, 2$. It suffices to give its expression on $(0, 1)$ and on $(1, 2)$ separately.

Exercise 3 **30%**

Consider the second difference approximation

$$\frac{f(x+h)-2f(x)+f(x-h)}{h^2}$$

of $f''(x)$. Recall that there exists $\xi \in [x-h, x+h]$ such that

$$f''(x) - \frac{f(x+h)-2f(x)+f(x-h)}{h^2} = -f^{(4)}(\xi)\frac{h^2}{12}$$

- Assume that every evaluation $f(y)$ is perturbed by the roundoff error

  $$\bar{f}(y) = f(y) + e(y)$$

  where $|e(y)| \leqslant \epsilon$. Determine an error bound for

  $$\left| f''(x) - \frac{\bar{f}(x+h)-2\bar{f}(x)+\bar{f}(x-h)}{h^2} \right|$$

  for any $x \in [a+h, b-h]$ provided $\max_{s\in[a,b]} |f^{(4)}(s)| \leqslant M$.

- Determine the optimal value of h (as a function of ϵ) which minimizes the error and deduce the smallest error achievable.

Exercise 4 **30%**

Given $0 < \alpha < 1$ and $\omega_0, \omega_1 \in \mathbb{R}$. Consider the quadrature

$$J(g) = \omega_0 g(-1/2) + \omega_1 g(\alpha)$$

to approximate

$$\int_{-1}^{1} g(t)dt.$$

1. Find ω_0 and ω_1 such that the quadrature is exact for polynomials of degree $\leqslant 1$.
2. Find α such that the quadrature is exact for polynomials of degree 2.
3. Is the quadrature exact for polynomial of degree 3?
4. Approximate the integral

   $$\int_{-1}^{3} (t^2 - 1)dt$$

   using the quadrature you discovered in the previous item.
5. How close is your approximate value from the exact integral?

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Exam 1: solutions

Exercise 1 **10%**

The newton iterates are given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

Here $f(x) = x^2 - 2$ and $f'(x) = 2x$. Starting with $x_0 = 1$, this yields $x_1 = \frac{3}{2}$, $x_2 = \frac{17}{12}$ and $x_3 = \frac{577}{408}$.

Exercise 2 **30%**

1. The Lagrange basis associated with the points 0, 1, and 2 are

$$L_{2,0}(x) = \frac{(x-1)(x-2)}{2}, \quad L_{2,1}(x) = x(2-x), \quad L_{2,2}(x) = x(x-1).$$

2. An expression of desired polynomial is

$$p_2(x) = x(2-x) + \sqrt{2}x(x-1).$$

3. On $(0,1)$ we have

$$L_{1,0}(x) = 1 - x, \qquad L_{1,1}(x) = x$$

and so the desired polynomial on $(0,1)$ is

$$p_1(x) = x.$$

On $(1,2)$ we have

$$L_{1,0}(x) = 2 - x, \qquad L_{1,1}(x) = x - 1$$

and so the desired polynomial on $(1,2)$ is

$$p_1(x) = (2-x) + \sqrt{2}(x-1).$$

Exercise 3 **30%**

Using a triangle inequality, we find that

$$\left| f''(x) - \frac{\bar{f}(x+h) - 2\bar{f}(x) + \bar{f}(x-h)}{h^2} \right| \leqslant \frac{1}{12}Mh^2 + \frac{4\epsilon}{h^2} =: g(h).$$

The parameter h leading leading to the minimal value is characterized by $g'(\overline{h}) = 0$, i.e.

$$\overline{h} = \left(\frac{48\epsilon}{M}\right)^{1/4}.$$

This corresponds to the value

$$g(\overline{h}) = \frac{1}{6}M^{1/2}(48\epsilon)^{1/2}.$$

Exercise 4 **30%**

1. The weights are given by the integral of the Lagrange basis, i.e.

$$\omega_0 = \int_{-1}^{1} \frac{t-\alpha}{-1/2-\alpha} dt = \frac{2\alpha}{1/2+\alpha}$$

and

$$\omega_1 = \int_{-1}^{1} \frac{t+1/2}{\alpha+1/2} dt = \frac{1}{1/2+\alpha}.$$

This leads to a quadrature exact for polynomial of degree $\leqslant 1$.

2. To guarantee that the quadrature is exact for polynomial of degree $\leqslant 2$, we find α such that

$$\frac{2}{3} = \int_{-1}^{1} t^2 dt = \frac{2\alpha}{1/2+\alpha}(-1/2)^2 + \frac{1}{1/2+\alpha}\alpha^2,$$

i.e.

$$\alpha = \frac{2}{3}$$

and therefore

$$\omega_0 = \frac{8}{7}, \qquad \omega_1 = \frac{6}{7}.$$

3. This quadrature is not exact for polynomial of degree $\leqslant 3$ since

$$0 = \int_{-1}^{1} t^3 dt \neq \frac{8}{7}(-1/2)^3 + \frac{6}{7}(2/3)^3.$$

4. To apply the above quadrature formula, we first set $t = 2s+1$ so that

$$\int_{-1}^{3} (t^2-1) dt = 2\int_{-1}^{1} ((2s+1)^2-1) ds.$$

Applying the above quadrature, we find

$$\int_{-1}^{3} (t^2-1) dt = 2\left(-\frac{8}{7} + \frac{6}{7}\left(\left(\frac{7}{3}\right)^2 - 1\right)\right) = \frac{16}{3}.$$

5. The numerical approximation is exact since applied to a polynomial of degree 2.