8. Lecture 8

Trigonometric Interpolation. We now recall some results on Fourier in series.

We set $\Omega = [0, 2\pi]$, $\psi_j(x) = e^{ijx}$ for $x \in \Omega$, $i = \sqrt{-1}$ and $j \in \mathbb{Z}$.
We recall the Euler's formula for $\theta \in \mathbb{R}$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

see Figure 2 for an illustration.

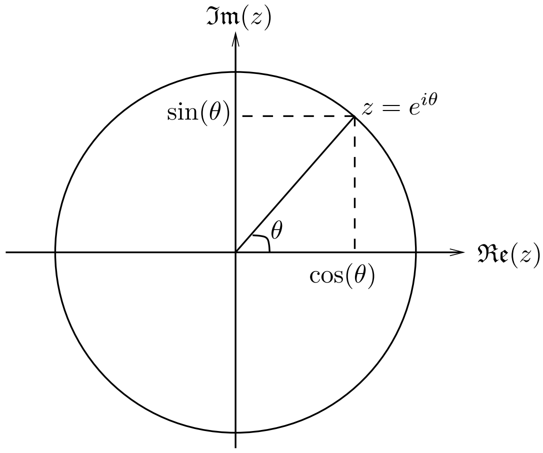

Figure 2. Illustration of the Euler's formula on the Complex plane.

Using the Euler's formula we find that

$$\psi_j(x) = \cos(jx) + i\sin(jx), \qquad j \in \mathbb{Z}.$$

Define the space

$$L^2(\Omega) := \left\{ f : [0, 2\pi] \to \mathbb{C} \ : \ \int_0^{2\pi} |f(x)|^2 dx < \infty \right\}$$

with norm

$$\|f\|_{L^2(\Omega)} := \left(\int_0^{2\pi} |f(x)|^2 dx \right)^{1/2}.$$

Remark 8.1 ($L^2(\Omega)$).
1. $L^2(\Omega)$ is a vector space (infinite dimensional) of functions on $[0, 2\pi]$ with scalar field $\mathbb{C}$.
2. A norm $\|.\|$ on a vector space $\mathbb{V}$ over $\mathbb{C}$ satisfies
   (a) $\|v\| \geq 0$ for all $v \in \mathbb{V}$ and $\|v\| = 0$ only if $v = 0$.
   (b) $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{C}$ and $v \in \mathbb{V}$.
   (c) $\|v + w\| \leq \|v\| + \|w\|$ for $v, w \in \mathbb{V}$ (triangle inequality).
3. Norms on a vector space $\mathbb{V}$ give a notion of distances between elements of $\mathbb{V}$, i.e. the distance between two elements $v, w \in \mathbb{V}$ is $\|v - w\|$.

Definition 8.1 (Convergence of Series in Normed Vector Space)**.** *Let $\mathbb{V}$ be a vector space and $\|.\|$ a norm on $\mathbb{V}$. Assume $\{v_j\} \subset \mathbb{V}$ and $v \in \mathbb{V}$. Then $\sum_{j=1}^{\infty} v_j$ converges* *to vin $\|.\|$ if the sequence of partial sums $S_l := \sum_{j=1}^{l} v_j$ converges to v. This means that given $\varepsilon > 0$, there exists $N = N(\varepsilon)$ satisfying*

$$\|v - S_l\| \leq \varepsilon \qquad \textit{when } l > N.$$

Theorem 8.1 (Fourier Series)**.** *For $f \in L^2(\Omega)$, the series*

$$\sum_{j=-\infty}^{+\infty} c_j \psi_j(x),$$

with

$$c_j = \frac{1}{2\pi} \int_0^{2\pi} f(x) \psi_j(x) dx \in \mathbb{C}$$

converges to f. (In this case we set $S_l(x) := \sum_{j=-l}^{l} c_j \psi_j(x)$.) In addition,

$$\|f\|_{L^2(\Omega)}^2 = 2\pi \sum_{j \in \mathbb{Z}} |c_j|^2.$$

Remark 8.2. Some remarks are in order.

(1) For $f \in L^2(\Omega)$, the series $\sum_{j=-\infty}^{\infty} |c_j|^2$ converges.
(2) Note that the functions $\psi_j(x)$ are periodic, i.e.

$$\lim_{x \to 0^+} \psi_j(x) = \lim_{x \to 2\pi^-} \psi_j(x)$$

and

$$\lim_{x \to 0^+} \psi_j'(x) = \lim_{x \to 2\pi^-} \psi_j'(x)$$

and so on for all derivatives.
(3) Depending on the smoothness of f, i.e. $f \in C^n[0, 2\pi]$ and $f, f', f^(2), ..., f^{(n)}$ are periodic, then the series

$$\sum_{j \in \mathbb{Z}} |c_j|^2 j^{2n} < \infty.$$

The set of such functions is denoted $\dot{H}^n$ and we set

$$\|f\|_{\dot{H}^n} := \left(\sum_{j \in \mathbb{Z}} |c_j|^2 (1 + j)^{2n} \right)^{1/2}.$$

Theorem 8.2 (Spectral approximation)**.** *Suppose that $f \in \dot{H}^n$. Then the truncated series*

$$S_l := \sum_{j=-l}^{l} c_j \psi_j(x)$$

satisfies

$$\|f - S_N\|_{L^2(\Omega)}^2 = 2\pi \sum_{|j| > N} |c_j|^2 \leq \frac{2\pi}{(N+1)^{2n}} \|f\|_{\dot{H}^n}^2.$$

Proof. We have

$$\|f - S_N\|_{L^2(\Omega)}^2 = 2\pi \sum_{|j| > N} |c_j|^2 = 2\pi \sum_{|j| > N} |c_j|^2 \frac{j^{2n}}{j^{2n}}.$$

Now for $|j| > N$, $\frac{1}{j^{2n}} \leq \frac{1}{(N+1)^{2n}}$ and the claims follow. ☐

Some remarks are in order.

Remark 8.3.

(1) The rate of convergence for the truncated series is better for smooth f.
(2) To compute the coefficients c_j, you need compute integrals with complex integrand. We provide an alternative next.

Trigonometric interpolation. Given an integer $N \geq 0$, set

$$h = \frac{2\pi}{2N+1} \qquad \text{and} \qquad x_j = jh, \quad j = 0, ..., 2N.$$

and

$$\mathbb{V}_{2N+1} := \text{span}\{\psi_j \ , \ j = -N, ..., N\} = \left\{ \sum_{|j| \leq N} d_j \psi_j \ , \ \{d_j\} \subset \mathbb{C} \right\}.$$

The *trigonometric interpolation problem* reads: Find $f_{2N+1} \in \mathbb{V}_{2N+1}$ satisfying

$$f_{2N+1}(x_j) = f(x_j), \qquad j = 0, 1, ..., 2N.$$

Note that $\dim(\mathbb{V}_{2N+1}) = 2N+1$ (f_{2N+1} involves $2N+1$ coefficients) and there are $2N+1$ equations. Maybe there is a unique solution!

The DFT - Discrete Fourier Transform. Define $E(j) := e^{\frac{2\pi i j}{2N+1}}$ for $j \in \mathbb{Z}$ and note that

$$E(j + (2N+1)) = e^{\frac{2\pi i j}{2N+1}} e^{\frac{2\pi i(2N+1)}{2N+1}} = E(j) \underbrace{e^{2\pi i}}_{=1} = E(j).$$

This shows that $E(j)$ is periodic with period $2N+1$. Also

$$E(j)^{(2N+1)} = e^{\frac{2\pi i j(2N+1)}{2N+1}} = e^{2\pi i j} = 1.$$

In fact, $E(j)$, $j = 0, 1, ..., 2N$ are the $2N+1$ roots of $x^{2N+1} - 1 = 0$, see Figure 3.

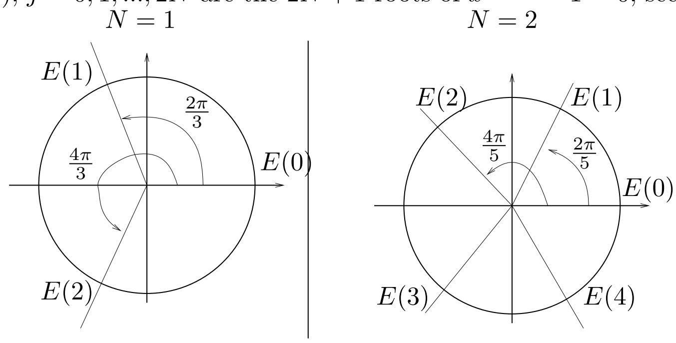

FIGURE 3. Illustration of DFT: (left) $N = 1$ where $E(0) = 1$, $E(1) = e^{\frac{2\pi i}{3}}$, $E(2) = e^{\frac{4\pi i}{3}}$, $E(3) = E(0) = 1$ and (right) $N = 2$ where $E(0) = 1$, $E(1) = e^{\frac{2\pi i}{5}}$, $E(2) = e^{\frac{4\pi i}{5}}$, $E(3) = e^{\frac{6\pi i}{5}}$, $E(4) = e^{\frac{8\pi i}{5}}$, $E(5) = E(0) = 1$.

For $d \in \mathbb{C}^{2N+1}$ we define $DFT_{\pm}(d) \in \mathbb{C}^{2N+1}$ by

$$DFT_{\pm}(d)(j) = \sum_{m=0}^{2N} d_m E(\pm jm).$$

We now return to the interpolation problem. If

$$f_{2N+1} = \sum_{|j|\leq N} c_j \psi_j \in \mathbb{V}_N$$

satisfies

$$f_{2N+1}(x_l) = f(x_l), \qquad l = 0, 1, ..., 2N,$$

then

$$f(x_l) = f_{2N+1}(x_l) = \sum_{j=-N}^{N} c_j e^{ijx_l} = \sum_{j=-N}^{N} c_j e^{\frac{ijl2\pi}{2N+1}} = \sum_{j=-N}^{N} c_j E(jl) = \sum_{j=0}^{2N} d_j E(jl) = DFT_+(d)(l),$$

where

$$d_j = \begin{cases} c_j & \text{if } 0 \leq j \leq N \\ c_{j-(2N+1)} & \text{if } N < j \leq 2N. \end{cases}$$

This means that

$$F := \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{2N}) \end{pmatrix} = DFT_+(d).$$

The next theorem guarantees that the interpolation problem (finding the coefficients c_j or d_j) can be solved for any data and as this is a square linear system, unique solvability follows.

Theorem 8.3 (Inverse DFT)**.**

$$(DFT_+)^{-1} = \frac{1}{2N+1} DFT_-$$

so that

$$d = \frac{1}{2N+1} DFT_-(F).$$

9. Lecture 9

Proof of Theorem 8.3. We begin by proving the identity

$$(DFT)_{+}^{-1} = \frac{1}{2N+1} DFT_{-} \qquad .$$

For $c \in \mathbb{C}^{2N+1}$, we have

$$DFT_{+}(c)(j) = \sum_{l=0}^{2N} c_l E(jl),$$

where $E(jl) = e^{\frac{2\pi i j l}{2N+1}}$. Then,

$$\begin{aligned} DFT_{-}(DFT_{+}(c))(m) &= \sum_{j=0}^{2N} DFT_{+}(c)(j)E(-jm) = \sum_{j=0}^{2N} \left(\sum_{l=0}^{2N} c_l E(lj) \right) E(-jm) \\ &= \sum_{j=0}^{2N} \sum_{l=0}^{2N} c_l E((l-m)j) = \sum_{l=0}^{2N} c_l \left(\sum_{j=0}^{2N} E((l-m)j) \right). \end{aligned}$$

Now, if $l = m$,

$$sum_{j=0}^{2N} E((l-m)j) = \sum_{j=0}^{2N} 1 = 2N+1.$$

If $l \neq m$,

$$E((l-m)j) = e^{\frac{2\pi i (l-m) j}{2N+1}} = \xi^j,$$

with

$$\xi := e^{\frac{2\pi i (l-m)}{2N+1}}.$$

Therefore, we have

$$\sum_{j=0}^{2N} E((l-m)j) = 1 + \xi + ... + \xi^{2N} = \frac{1 - \xi^{2N+1}}{1-\xi} = \frac{1 - e^{\frac{2\pi i (l-m) j \cancel{(2N+1)}}{\cancel{2N+1}}}}{1-\xi} = \frac{1-1}{1-\xi} = 0$$

so that

$$DFT_{-}(DFT_{+}(c))(m) = (2N+1)c_m,$$

and the desired relation follows. $\square$

Remark 9.1 (Trigonometric interpolation using $2N$ points). Recall the interpolation problem: find $f_{2N+1}(x) = \sum_{j=-N}^{N} c_j \psi_j(x)$ such that

$$f_{2N+1}(x_i) = f(x_j), \qquad j = 0, 1, ..., 2N,$$

where $\psi_j(x) = e^{ijx}$.

The discrete Fourier transforms using $2N$ points reads

$$DFT_{\pm} : (c_0, ..., c_{2N-1}) \subset \mathbb{C}^{2N} \to \mathbb{C}^{2N},$$

where

$$DFT_{\pm}(c)(j) = \sum_{l=0}^{2N-1} c_l E(\pm l j), \qquad j = 0, ..., 2N-1,$$

and

$$E(m) = e^{\frac{2\pi i m}{2N}}.$$

Hence, if we set $x_j = jh$, with $h = \frac{2\pi}{2N} = \frac{\pi}{N}$, the the trigonometric interpolation problem has solution

$$f_{2N}(x) = \sum_{j=-N+1}^{N} c_j \psi_j(x),$$

where

$$c_j = \begin{cases} d_j & \text{for } j = 0, ..., N \\ d_{j+2N} & \text{for } j = -N+1, ..., -1, \end{cases}$$

and

$$d = \frac{1}{2N} DFT_-(F), \qquad \text{with } F_j = f(x_j), \quad j = 0, 1, ..., 2N-1.$$

Remark 9.2 (Computational Cost using DFT). We recall that

$$DFT_\pm(c)(j) = \sum_{l=0}^{2N-1} c_l E(lj).$$

Each value j requires $2N$ multiplications (complex) and $2N-1$ additions or a total of $O(N)$ flops (floating point operations). This entails an overall cost of $O(N^2)$ to compute all the output values.

Remark 9.3 (The Fast "Discrete" Fourier Transform - FFT). If $2N$ is highly factorable, the FFT algorithm will compute the DFT in $O(Nlog(N))$ operations.

The discussion in the audio file now regards *Homework 5* (available from ecampus).