Numerical Methods
MATH 417 - 501
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First Name: ____________________ **Last Name:** ____________________

Exam 1

- 75 minute individual exam;
- Answer the questions in the space provided. If you run out of space, continue onto the back of the page. Additional space is provided at the end;
- **Show and explain all work**;
- **Underline** the answer of each steps;
- The use of books, personal notes, **calculator**, cellphone, laptop, and communication with others is forbidden;
- By taking this exam, you agree to follow the university's code of academic integrity.

Ex 1	Ex 2	Ex 3	Ex 4	Total

<u>Exercise 1</u> **10%**

Compute the first 3 steps of the newton method to find a rational approximation of $\sqrt{2}$ starting with the value 1.

Exercise 2 **30%**

Consider the second difference approximation

$$\frac{f(x+h)-f(x)}{h}$$

of $f'(x)$ and recall that there exists $\xi \in [x, x+h]$ such that

$$f'(x) - \frac{f(x+h)-f(x)}{h} = -\frac{1}{2} f''(\xi) h.$$

- Assume that every evaluation $f(y)$ is perturbed by the roundoff error

  $$\bar{f}(y) = f(y) + e(y)$$

  where $|e(y)| \leqslant \epsilon$. Determine an error bound for

  $$\left| f'(x) - \frac{\bar{f}(x+h)-\bar{f}(x)}{h} \right|$$

  for any $x \in [a+h, b-h]$ provided $\max_{s\in[a,b]} |f^{(2)}(s)| \leqslant M$.

- Determine the optimal value of h (as a function of ϵ) which minimizes the error and deduce the smallest error achievable.

Exercise 3 **30%**

Given $0 < \alpha < 1$ and $\omega_0, \omega_1 \in \mathbb{R}$. Consider the quadrature

$$J(g) = \omega_0 g(-1) + \omega_1 g(\alpha)$$

to approximate

$$\int_{-1}^{1} g(t)dt.$$

1. Find ω_0 and ω_1 such that the quadrature is exact for polynomials of degree $\leqslant 1$.
2. Find α such that the quadrature is exact for polynomials of degree 2.
3. Is the quadrature exact for polynomial of degree 3?
4. Approximate the integral

   $$\int_{-1}^{3} (t^2 - 1)dt$$

   using the quadrature you discovered in the previous item.
5. How close is your approximate value from the exact integral?

Exercise 4 **30%**

1. Replace the symbols W, X, Y and Z in the matlab code below to find L from the LU factorization of the tri-diagonal matrix A:

```
%%%% LUTri %%%%
%% Input:  tridiagonal symetric square matrix A (not checked)
%% Output: A where the lower triangular part is L and the strictly upper triangular is U

function  A=LUTri(A)

N=size(A,1);


% First row of U (the first column of L is the first column of A)
for i=2:N
        A(1,i) = A(1,i)/W;
end

% kth Colomn of  L and kth row of U
for k=2:N-1
    % L(k,k)
    A(k,k) = A(k,k) - A(k,k-1)*A(k-1,k)
    % L(k+1,k)
    A(k+1,k) = A(k+1,k) - A(k+1,k-1)*A(k-1,X)
    % U(k,k+1)
    A(k,k+1) = (A(k,Y) - A(k,k-1)*A(k-1,k+1))/A(k,k)
end

% Construction of L(N,N)
A(N,N) = A(Z,Z) - A(N,N-1)*A(N-1,N);

%%%% END %%%%
```

2. Apply the above algorithm to the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Exam 1: solutions

Exercise 1 **10%**

The newton iterates are given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

Here $f(x) = x^2 - 2$ and $f'(x) = 2x$. Starting with $x_0 = 1$, this yields $x_1 = \frac{3}{2}$, $x_2 = \frac{17}{12}$ and $x_3 = \frac{577}{408}$.

Exercise 2 **30%**

Using a triangular inequality, we have that

$$\left| f'(x) - \frac{\bar{f}(x+h) - \bar{f}(x)}{h} \right| \leqslant \frac{1}{2}Mh + \frac{2\epsilon}{h} =: g(h).$$

The parameter h leading to the minimal value is characterized by $g'(\overline{h}) = 0$, i.e.

$$\overline{h} = \left(\frac{4\epsilon}{M}\right)^{1/2}.$$

The corresponding value is

$$g(\overline{h}) = 2(\epsilon M)^{1/2}.$$

Exercise 3 **30%**

1. The weights are given by the integral of the Lagrange basis, i.e.

   $$\omega_0 = \int_{-1}^{1} \frac{t-\alpha}{-1-\alpha} dt = \frac{2\alpha}{1+\alpha}$$

   and

   $$\omega_1 = \int_{-1}^{1} \frac{t+1}{\alpha+1} dt = \frac{2}{1+\alpha}.$$

   This leads to a quadrature exact for polynomial of degree $\leqslant 1$.

2. To guarantee that the quadrature is exact for polynomial of degree $\leqslant 2$, we find α such that

   $$\frac{2}{3} = \int_{-1}^{1} t^2 dt = \frac{2\alpha}{1+\alpha}(-1)^2 + \frac{2}{1+\alpha}\alpha^2,$$

   i.e.

   $$\alpha = \frac{1}{3}$$

   and therefore

   $$\omega_0 = \frac{1}{2}, \qquad \omega_1 = \frac{3}{2}.$$

3. This quadrature is not exact for polynomial of degree $\leqslant 3$ since

$$0 = \int_{-1}^{1} t^3 dt \neq \frac{1}{2}(-1)^3 + \frac{3}{2}(1/3)^3.$$

4. To apply the above quadrature formula, we first set $t = 2s+1$ so that

$$\int_{-1} 3(t^2-1)dt = 2\int_{-1}^{1} ((2s+1)^2 - 1)ds.$$

   Applying the above quadrature, we find

$$\int_{-1} 3(t^2-1)dt = 2\left(\frac{1}{2}0 + \frac{3}{2}\left(\left(\frac{5}{3}\right)^2 - 1\right)\right) = \frac{16}{3}.$$

5. The numerical approximation is exact since applied to a polynomial of degree 2.

Exercise 4 **30%**

```
%%%% CholeskyTri %%%%
%% Input:  tridiagonal symetric square matrix A (not checked)
%% Output: L

function  A=LUTri(A)

N=size(A,1);


% First row of U (the first column of L is the first column of A)
for i=2:N
        A(1,i) = A(1,i)/A(1,1);
end

% kth Colomn of  L and kth row of U
for k=2:N-1
    % L(k,k)
    A(k,k) = A(k,k) - A(k,k-1)*A(k-1,k)
    % L(k+1,k)
    A(k+1,k) = A(k+1,k) - A(k+1,k-1)*A(k-1,k)
    % U(k,k+1)
    A(k,k+1) = (A(k,k+1) - A(k,k-1)*A(k-1,k+1))/A(k,k)
end

% Construction of L(N,N)
A(N,N) = A(N,N) - A(N,N-1)*A(N-1,N);

%%%% END %%%%
```

The above algorithm applied to the matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

produces the following matrices

$$(1) \quad \begin{pmatrix} l_{11} & u_{12} & u_{13} \\ l_{21} & l_{22} & u_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix}; \qquad (2) \quad \begin{pmatrix} 2 & -1/2 & 0 \\ -1 & l_{22} & u_{23} \\ 0 & l_{32} & l_{33} \end{pmatrix};$$

$$(3) \quad \begin{pmatrix} 2 & -1/2 & 0 \\ -1 & 3/2 & -2/3 \\ 0 & -1 & l_{33} \end{pmatrix}; \qquad (4) \quad \begin{pmatrix} 2 & -1/2 & 0 \\ -1 & 3/2 & -2/3 \\ 0 & -1 & 4/3 \end{pmatrix}.$$