PREFACE VOLUME 21

ANDREA BONITO AND RICARDO H. NOCHETTO

This and the following volumes are devoted to the numerical approximation of *geometric partial differential equations* (GPDEs). Before describing the objectives and contents of this project, it is appropriate to explain the meaning of GPDEs and the reasons why they deserve two volumes of the Handbook of Numerical Analysis. GPDEs are governing equations of natural, social, and economic phenomena where geometry play a prominent to dominant role. Examples abound from interfaces and free boundaries in fluids and solids — such as modeling of surface tension and bending effects involving second fundamental forms — to the development of defects in director and line fields in liquid crystal modeling, motion, merging and splitting of droplets within incompressible fluids, total variation minimization in imaging science, and shape morphing and extrapolation, just to name a few. These problems possess an intrinsic mathematical beauty and pose a formidable challenge both in analysis and computation. Besides their overwhelming mathematical richness, GPDEs are ubiquitous in many scientific, engineering and industrial applications, such as fluid and solid mechanics, materials science, biology, chemistry, astrophysics, plasma physics, imaging, and computer animation.

The last three decades have witnessed the development of powerful algorithms and corresponding numerical analysis for the description and computation of interfaces. The level set and phase field methods have joined the more traditional front tracking techniques and, together with the advent of ever more powerful and versatile computers, have allowed for the simulation and understanding of rather complex phenomena involving interfaces. It is fair to assert that, besides theory and experimentation, mathematical modeling and computation have established themselves as the third pillar of scientific inquiry. A well designed computational model can replace a very expensive or even unrealizable experimental setting, and give new insight into the theoretical developments of a specific discipline. Because of their technical complexity and practical relevance, GPDEs are a chief example.

The purpose of this two-volume contribution is to provide a missing reference book that portrays the state-of-the-art on basic algorithms for GPDEs and their analysis, along with their impacts in a wide variety of areas of science and engineering. Since this field has grown tremendously over the last few years, the selection of topics and authors is a formidable task. Our intention is to present different and complementary approaches ranging from fundamental numerical analysis of basic algorithms to scientific computation and exciting applications. The choice of authors reflects their expertise on various aspects of this ambitious and multifaceted project.

This first volume consists of eight chapters which encompass numerical analysis of GPDEs, algorithm design, analysis and simulation of interfaces, and shape morphing. A brief description follows.

- *Numerical analysis of GPDEs.* Chapter 1, by Bonito, Demlow and Nochetto, reviews and extends three popular finite element methods to approximate the Laplace-Beltrami operator on a co-dimension one surface; they are the parametric, trace and narrow band methods. The discussion centers around the relationship between the (minimal) regularity of the surface and the manner it is represented and approximated. Chapter 2, by Neilan, Salgado and Zhang, describes and analyzes several approximation techniques for the fully nonlinear Monge-Ampère equation. Key notions such as stability, consistency and continuous dependence on data are developed in the max norm and used to derive rates of convergence to the viscosity solution.
- *Geometrically nonlinear plates and rods.* Chapter 3, by Bartels, discusses geometrically nonlinear bending models for plates and rods that allow large deformations. Plate deformations are constrained to be isometries whereas rods are elastic and self-avoiding. Recent finite element algorithms are reviewed along with their numerical analysis.
- *Interfaces and free boundaries.* This is the focus of the next four chapters. Chapter 4, by Barrett, Garcke and Nürnberg, collects their work on the parametric finite element approximation of curvature driven interface evolutions. Finite element methods for surface geometric equations, coupling of surface geometric equations with bulk equations, and two phase flows are presented and analyzed. Chapter 5, by Du and Feng, overviews the phase-field modeling and corresponding diffuse interface approximation. The main numerical analysis techniques are described and applied. Chapter 6, by Saye and Sethian, reviews the level-set method starting from basic ideas and techniques and culminating with complex and intricate interface dynamics. A description of the inherent challenges to multi-phase and sharp-interface physics is provided along with numerical algorithms to overcome them. These three chapters present variational front tracking, phase field and level set methods, three competing techniques for interface evolution. Chapter 7, by Bänsch and Schmidt, discusses different aspects of phase transitions in materials science and fluids along with efficient algorithms for their approximation.
- *Riemmanian calculus and applications.* Chapter 8, by Heeren, Rumpf, Wardetzky and Wirth, develops a Riemmanian calculus on the space of discrete triangular shells along with its discretization. This leads to adequate algorithms for shape morphing, shape extrapolation, parallel transport, smooth interpolation, and other geometric processes of interest. Computer animated video sequences and movies are presented.

Acknowledgments

A.B is partially supported by NSF grant DMS-1817691. RHN is partially supported by NSF grants DMS–1411808 and DMS–1908267.