Homework 1

Exercise 1 35%

Recall that for sufficiently smooth functions $u : \mathbb{R} \to \mathbb{R}$, the following relation holds

$$u(x) = u(y) + \int_y^x u'(s)ds.$$

Let $C^1([0,1])$ be the set of function $u : [0,1] \to \mathbb{R}$ that are continuous and have continuous first derivative on $[0,1]$. Consider the following subsets of $C^1([0,1])$:

$$\mathcal{A} = \{u \in C^1([0,1]) : \ u(0) = u(1) = 0\}, \quad \mathcal{B} = \{u \in C^1([0,1]) : \ u(0) = 0\}, \quad \mathcal{C} = C^1([0,1]).$$

1. Derive the following inequalities:

$$\int_0^1 u^2 dx \leqslant \frac{1}{2}\int_0^1 u'^2 dx, \quad \forall u \in \mathcal{B},$$

$$\int_0^1 u^2 dx \leqslant \frac{1}{4}\int_0^1 u'^2 dx, \quad \forall u \in \mathcal{A}.$$

2. Show that for $u \in \mathcal{C}$ the following inequalities are valid:

$$\int_0^1 u^2 dx \leqslant \frac{1}{6}\int_0^1 u'^2 dx + \Big(\int_0^1 u dx\Big)^2;$$

$$\max_{x\in[0,1]} |u(x)|^2 \leqslant 2u^2(1) + 2\int_0^1 u'^2 dx;$$

$$\int_0^1 u^2(x) dx \leqslant 2u^2(1) + 2\int_0^1 u'^2(x) dx;$$

$$\max_{x\in[0,1]} |u(x)|^2 \leqslant 2\int_0^1 (u^2 + u'^2) dx.$$

3. show that $H^1(0,1) \subset C^0([0,1])$, i.e. for $u \in H^1(0,1)$, there exists $\tilde{u} \in C^0([0,1])$ such that $\tilde{u} = u$ a.e. and $\|\tilde{u}\|_{L^\infty(0,1)} \leqslant C\|u\|_{H^1(0,1)}$ for some absolute constant C.

Exercise 2 35%

We say that $u \in L^2(a,b)$, $a, b \in \mathbb{R}$, has a weak derivative in $L^2(a,b)$ if there exists $v \in L^2(a,b)$ satisfying

$$\int_a^b u(x)\phi'(x)dx = -\int_a^b v(x)\phi(x)dx, \quad \forall \phi \in C_0^\infty([a,b]).$$

The function v is called the weak derivative of u and the space of functions in $L^2(a,b)$ with weak derivative in $L^2(a,b)$ is denoted by $H^1(a,b)$.

1. Consider the function $u(x) = |x|^{\frac{2}{3}} - 1$ defined on the interval $(-1,1)$. Determine whether $u \in H^1(-1,1)$ and if yes, explicit its weak derivative.

2. For wich value of α the function $|x|^\alpha$ belongs to $H^1(-1,1)$.
3. Let $u_1 \in H^1(-1,0)$ and $u_2 \in H^1(0,1)$. Show that if $u_1(0) = u_2(0)$ then the function

$$u(x) = \begin{cases} u_1 & -1 < x < 0; \\ u_2 & 0 < x < 1; \end{cases}$$

belongs to $H^1(-1,1)$. Explain using your own words why if $u_1(0) \neq u_2(0)$ then $u \notin H^1(-1,1)$.

Exercise 3 **30%**

Consider the following second order boundary value problem: Seek $u : (0,1) \to \mathbb{R}$ such that

$$(-k(x)u')' + b(x)u' + q(x)u = f, \qquad x \in (0,1)$$

supplemented with the boundary conditions $u(0) = u(1) = 0$. Here $f \in L^2(0,1)$ and $k, q, b, b' \in L^\infty(0,1)$ satisfy $k(x) > 0$ and $q(x) - \frac{1}{2}b'(x) \geqslant 0$ for almost every $x \in (0,1)$.

1. Determine a weak formulation of the above strong problem.
2. Show that there exists a unique solution $u \in H_0^1(0,1)$.