Partial Differential Equations — due on May 7 (in class)
MATH 612 — Spring 2013
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Take home part of the FINAL

Exercise 1 **60%**

Let $\Omega \subset \mathbb{R}^3$ be open, bounded, connected and with smooth boundary. Define

$$V := \left\{ v \in H_0^1(\Omega)^3 \ : \ \mathrm{div}(v) = 0 \right\} \subset H_0^1(\Omega)^3,$$

where we equip $H_0^1(\Omega)^3$ with the standard semi-norm. Consider the following formulation of the steady Navier-Stokes system : Find $u \in V$ such that

$$\nu \int_\Omega \nabla u \cdot \nabla v + \int_\Omega (u \cdot \nabla) u \cdot v = \langle f, v \rangle, \qquad \forall v \in V. \tag{1}$$

Here $\nu > 0$ is the fluid viscosity, $f \in H^{-1}(\Omega)^3$ is a given body force and

$$\int_\Omega (u \cdot \nabla) u \cdot v = \sum_{i,j=1}^3 \int_\Omega u_j \partial_j u_i v_i$$

with $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$.
We propose to use the following fixed point theorem to show existence of a weak solution : *Let H be an Hilbert space of finite dimension with norm $|.|_H$ and scalar product $(.,.)_H$. Let F be a continuous mapping from H into H satisfying : There exists $\alpha > 0$ such that for all $f \in H$ with $|f|_H = \alpha$ we have*

$$(F(f), f)_H \geqslant 0,$$

Then F has at least one zero in the ball centered at 0 and of radius α.

1. Show that $(z \cdot \nabla) u \in L^{3/2}(\Omega)^3$ whenever $z, u \in H_0^1(\Omega)^3$ and deduce that for $u, v \in H_0^1(\Omega)^3$
$$(u \cdot \nabla) u \cdot v \in L^1(\Omega)$$
which ensures that the nonlinear term makes sense.
2. Show that V is separable by showing that V is a closed subset of $H_0^1(\Omega)$.
3. Since V is separable, it has a countable basis $\{w_i\}_{i=1}^\infty$ such that for all $v \in V$ and all $\epsilon > 0$, there exists $N \in \mathbb{N}$ and $v_N \in \mathrm{span}(w_1, ..., w_N)$ such that
$$|v - v_N|_{H^1} \leqslant \epsilon.$$
Set
$$V_m = \mathrm{span}(w_1, ..., w_m)$$
and define the mapping Φ_m
$$\begin{cases} V_m \to V_m \\ v \mapsto \Phi_m(v) = \phi_m \end{cases}$$
where $\phi_m \in V_m$ satisfies
$$\int_\Omega \nabla \phi_m \cdot \nabla w_k = \nu \int_\Omega \nabla v \cdot \nabla w_k + \int_\Omega (v \cdot \nabla) v \cdot w_k - \langle f, w_k \rangle$$
for all $1 \leqslant k \leqslant m$. Show that Φ_m is well defined, i.e. for each $v \in V_m$ there exists a unique $\phi_m \in V_m$.

4. Apply the given fixed point theorem to deduce the existence of at least one $v \in V_m$ with
$$|v|_{H^1(\Omega)} \leqslant \frac{1}{\nu} \|f\|_{H^{-1}}$$
and $\Phi_m(v) = 0$.
5. For each m define $u_m \in V_m$ to be one of the v's and deduce that there is a subsequence (still indexed by m) and $\tilde{u} \in H_0^1(\Omega)^3$ such that
$$u_m \rightharpoonup \tilde{u}.$$
Show that $\tilde{u} \in V$.
6. Fix $k_0 \in \mathbb{N}$ and consider $m \geqslant k_0$. Show that
$$\nu \int_\Omega \nabla u_m \cdot \nabla w_{k_0} \to \nu \int_\Omega \nabla \tilde{u} \cdot \nabla w_{k_0}.$$
7. Prove using a density argument that for all $u, v, w \in H^1(\Omega)^3$ with at least one in $H_0^1(\Omega)^3$ there holds
$$\int_\Omega (w \cdot \nabla) u \cdot v = -\int_\Omega (w \cdot \nabla) v \cdot u - \int_\Omega \operatorname{div}(w) u \cdot v.$$
Deduce then that
$$\int_\Omega (u_m \cdot \nabla) u_m \cdot w_{k_0} \to \int_\Omega (\tilde{u} \cdot \nabla) \tilde{u} \cdot w_{k_0}.$$
8. Conclude that $\tilde{u}$ is a weak solution.