NUMERICAL APPROXIMATION OF FRACTIONAL POWERS OF REGULARLY ACCRETIVE OPERATORS

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ABSTRACT. We study the numerical approximation of fractional powers of accretive operators in this paper. Namely, if A is the accretive operator associated with a regular sesquilinear form $A(\cdot, \cdot)$ defined on a Hilbert space $\mathbb{V}$ contained in $L^2(\Omega)$, we approximate $A^{-\beta}$ for $\beta \in (0, 1)$. The fractional powers are defined in terms of the so-called Balakrishnan integral formula.

Given a finite element approximation space $\mathbb{V}_h \subset \mathbb{V}$, $A^{-\beta}$ is approximated by $A_h^{-\beta} \pi_h$ where A_h is the operator associated with the form $A(\cdot, \cdot)$ restricted to $\mathbb{V}_h$ and π_h is the $L^2(\Omega)$-projection onto $\mathbb{V}_h$. We first provide error estimates for $(A^{-\beta} - A_h^{-\beta} \pi_h)f$ in Sobolev norms with index in [0,1] for appropriate f. These results depend on elliptic regularity properties of variational solutions involving the form $A(\cdot, \cdot)$ and are valid for the case of less than full elliptic regularity. We also construct and analyze an exponentially convergent sinc quadrature approximation to the Balakrishnan integral defining $A_h^{-\beta} \pi_h f$. Finally, the results of numerical computations illustrating the proposed method are given.

1. INTRODUCTION.

The mathematical study of integral or nonlocal operators has received much attention due to their wide range of applications, see for instance [16, 4, 21, 9, 24, 22, 38, 44, 18].

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with a Lipschitz continuous boundary Γ which is the disjoint union of an open set Γ_N and its complement $\Gamma_D = \Gamma \setminus \Gamma_N$. We define $\mathbb{V}$ to be the functions in $H^1(\Omega)$ vanishing on Γ_D. We then consider a sesquilinear form $A(\cdot, \cdot)$ defined for $u, v \in \mathbb{V}$ given by

$$
(1) \quad A(u, v) := \int_\Omega \left(\sum_{i,j=1}^d a_{i,j}(x) u_i(x) \overline{v_j(x)} + \sum_{i=1}^d (a_{i,0}(x) u_i(x) \overline{v(x)} \right.
$$
$$
\left. + a_{0,i}(x) u(x) \overline{v_i(x)}) + a_{0,0}(x) u(x) \overline{v(x)} \right).
$$

Here the subscript on u and v denotes the partial derivative with respect to x_i, $i = 1, .., d$, and $\overline{v}$ denotes the complex conjugate of v. We further assume that $A(\cdot, \cdot)$ is coercive and bounded (see, (7) and (8) below). Such a sesquilinear form is called *regular* [31].

There is an unbounded operator A on $L^2(\Omega)$ with domain of definition $D(A)$ associated with a regular sesquilinear form (see [31] and Section 2 below). The unbounded operator associated with such a form is called a *regularly accretive* operator [31]. For such operators, the fractional powers are well defined, typically,

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in terms of Dunford-Taylor integrals. When $0 < \beta < 1$, one can also use the Balakrishnan formula [5, 30, 31]:

$$(2) \qquad A^{-\beta} = \frac{\sin(\beta\pi)}{\pi} \int_0^\infty \mu^{-\beta}(\mu I + A)^{-1}\, d\mu.$$

In this paper, we propose a numerical method for the approximation $A^{-\beta}f$ based on the finite element method with an approximation space $\mathbb{V}_h \subset \mathbb{V}$. It is worth mentioning that a similar formula can be used to represent the solution of a Cauchy problem. This is the focus of [10].

Several techniques for approximating $S^{-\beta}f$ are available when S is the operator associated with a Hermitian form (or symmetric and real valued on a real-valued functional space). The most natural technique involves approximating $S^{-\beta}$ by $S_h^{-\beta}$ where S_h is a discretization of S_h, e.g., via the finite element method using $\mathbb{V}_h$. In this case, $S_h^{-\beta}f$ for $f \in \mathbb{V}_h$ can be expressed in terms of the discrete eigenvector expansion [29, 28, 46]:

$$S_h^{-\beta}f = \sum_j c_j \lambda_{j,h}^{-\beta} \psi_{j,h} \quad \text{where} \quad f = \sum_j c_j \psi_{j,h}.$$

Here $(\lambda_{j,h}, \psi_{j,h})$ denote the eigenpairs of S_h. An alternative approach is based on a representation of $S^{-\beta}f$ via a "Neumann to Dirichlet" map [15]. The numerical algorithm proposed and analyzed in [42, 17] consists of a finite element method in one higher dimension. It takes advantage of the rapid decay of the solution in the additional direction enabling truncation to a bounded domain of modest size. A third approach which is valid for more general A, is based on finite element approximation with an analysis employing the Dunford-Taylor characterization of $A^{-\beta}f$ [23] (see, also, [37, 45]) and is most closely related to the approach which we will take in this paper. However, the analysis of [23] only provides errors in $L^2(\Omega)$, requires full elliptic regularity and fails to elucidate the relation between the convergence rate and the smoothness of f. For example, the result in [23] does not hold for problems on non-convex domains or problems with jumps in the higher order coefficients.

The approach that we shall take in this paper is based on (2). The introduction of finite elements on a subspace $\mathbb{V}_h \subset \mathbb{V}$ leads to a discrete approximation A_h to A. The finite element approximation to (2) is then given by

$$(3) \qquad A_h^{-\beta}\pi_h := \frac{\sin(\beta\pi)}{\pi} \int_0^\infty \mu^{-\beta}(\mu I + A_h)^{-1}\pi_h\, d\mu,$$

where π_h is the $L^2(\Omega)$-projection onto $\mathbb{V}_h$. In [11], we proved the convergence in $L^2(\Omega)$ of an equivalent version of this method when $A = S$ is real, symmetric and positive definite. We also showed the exponential convergence of a sinc quadrature approximation.

The current paper extends the approach of [11] to the case when A is a regularly accretive operator. The proof provided in [11] is based on the fact that, in the Hermitian case, the domain of S^γ, for $\gamma \in \mathbb{R}$, is naturally characterized in terms of the decay of the coefficients in expansions involving the eigenvectors of S. Assuming elliptic regularity, it is then possible to show that $D(S^{s/2})$ for $0 \le s \le 1+\alpha$ coincides with standard Hilbert spaces. Here α is the regularity parameter (see below). Thus, norms of the operator $(\mu I + S)^{-1}$ acting between the standard Sobolev spaces can be bounded using their series expansions, the norms in $D(S^{s/2})$, and Young's

inequalities. In contrast, the spaces $D(A^\gamma)$ cannot be characterized in such a simple way when A is not Hermitian.

The main result of this paper is the following error estimate (Theorem 6.2 and Remark 6.1): for any $0 \leq r \leq 2\beta$ there exists $\delta \geq 0$ and a constant C independent of h such that for $f \in D(A^\delta)$

$$(4) \qquad \|(A^{-\beta} - A_h^{-\beta}\pi_h)f\|_{H^r(\Omega)} \leq Ch^{\alpha + \min(1-r,\alpha)}$$

with C being replaced by $C\log(h^{-1})$ for certain combinations of r, α, δ and β. Here $\alpha > 0$ is the so-called elliptic regularity pick-up which is the regularity above $H^1(\Omega)$ expected for $A^{-1}f$ for appropriate f, see Assumption 1. Even in the case of Hermitian A, the above result extends those in [11] to $r > 0$.

This paper shows that the general approach for proving (4) in [11] can be extended to the case of regularly accretive A with additional technical machinery. Some of the most challenging issues involve the relationship between $D(A^{s/2})$, for $s \in [0, 1 + \alpha]$ and fractional Sobolev spaces. The case of $s \in [0, 1)$ is contained in the acclaimed paper by Kato [31] showing that for regularly accretive operators, $D(A^{s/2})$ coincides with the interpolation space between $L^2(\Omega)$ and $\mathbb{V}$ defined using the real method. The case of $s = 1$ is the celebrated Kato Square Root Problem. This is a deep result which has been intensively studied (see, [1, 2, 39, 37] and the references in [39]). The results in those papers give conditions when one can conclude that $D(A^{1/2}) = D((A^*)^{1/2}) = \mathbb{V}$, where A^* stands for the adjoint of A. Motivated by the approach of Agranovich and Selitskii [1] for proving the Kato Square Root Problem, we show in this paper that under elliptic regularity assumptions, $H^s(\Omega) \cap \mathbb{V} \subset D(A^{s/2})$ and $H^s(\Omega) \cap \mathbb{V} \subset D((A^*)^{s/2})$ for $s \in [0, 1 + \alpha]$ with equality when additional injectivity assumptions on A^{-1} and $(A^*)^{-1}$ hold. With this information, the norms of $(\mu I + A)^{-1}$ acting between Sobolev spaces can be bounded in terms of the $L^2(\Omega)$ operator norm of $A^t(\mu I + A)^{-1}$ with $t \in [0, 1]$. This, in turn, (see Lemma 6.3) can be bounded by interpolation using the fact that $D(A^t)$ coincides with the interpolation scale between $L^2(\Omega)$ and $D(A)$ (using the complex method).

Similar results for $0 \leq s \leq 1$ are also required for the finite element approximation A_h. In a discrete setting, the question is to guarantee the existence of a constant C independent of h such that for all $v_h \in \mathbb{V}_h$

$$C^{-1}\|v_h\|_{H^s(\Omega)} \leq \|A_h^{s/2}v_h\|_{L^2(\Omega)} \leq C\|v_h\|_{H^s(\Omega)}.$$

This is provided by Lemma 5.2 for $0 \leq s < 1$ and a solution to the discrete Kato problem ($s = 1$) is given in Theorem 6.5.

We also study a sinc quadrature approximation to $A_h^{-\beta}f$ for $f \in \mathbb{V}_h$. A change of integration variable shows that

$$(5) \qquad A_h^{-\beta} = \frac{\sin(\beta\pi)}{\pi} \int_{-\infty}^{\infty} e^{(1-\beta)y}(e^y I + A_h)^{-1}\, dy.$$

Motivated by [36], the sinc quadrature approximation to $A_h^{-\beta}$ is given by

$$(6) \qquad \mathcal{Q}_k^{-\beta}(A_h) := \frac{k\sin(\pi\beta)}{\pi} \sum_{\ell=-N}^{N} e^{(1-\beta)y_\ell}(e^{y_\ell} I + A_h)^{-1}.$$

Here $k := 1/\sqrt{N}$ is the quadrature step size, N is a positive integer and $y_\ell := \ell k$ for $\ell = -N, ..., N$. The standard tools related to the sinc quadrature together with the

characterization of $D(A^{s/2})$ for $s \in [0, 1/2]$ mentioned above yield the quadrature error estimate (Remark 7.2)

$$\|(A_h^{-\beta} - \mathcal{Q}_k^{-\beta}(A_h))\pi_h\|_{\dot{H}^{2s} \to \dot{H}^{2s}} \leq C_Q e^{-\pi^2/(2k)},$$

where C_Q is a constant independent of k and h.

The outline of this paper is as follows. In Section 2, we introduce the notations and properties related to operator calculus with non-Hermitian operators. Section 3 is devoted to the study of the Hermitian part of A and the related dotted spaces. The latter is instrumental for the characterization of $D(A^{\frac{s}{2}})$ discussed in Section 4. The finite element approximations are then introduced in Section 5, which also contains the proof of the error estimate (4). This coupled with the exponentially convergent sinc quadrature studied in Section 7.2, yields the final error estimate for the fully discrete and implementable approximation. We end this work with Section 8 providing a numerical illustration of the approximation of fractional convection-diffusion problems.

2. Fractional Powers of non-Hermitian Operators

We recall that Ω is a bounded domain in $\mathbb{R}^d$ with a Lipschitz continuous boundary Γ which is the disjoint union of an open set Γ_N and its complement $\Gamma_D = \Gamma \setminus \Gamma_N$. Let $L^2(\Omega)$ be the space of complex valued functions on Ω with square integrable absolute value and denote by $\|.\|$ and $(.,.)$ the corresponding norm and Hermitian inner product. Let $H^1(\Omega)$ be the Sobolev space of complex valued functions on Ω and set

$$\mathbb{V} := \{v \in H^1(\Omega), \ v = 0 \text{ on } \Gamma_D\},$$

the restriction of $H^1(\Omega)$ to functions with vanishing traces on Γ_D. We implicitly assume that Γ_D is such that the trace operator from $H^1(\Omega)$ is bounded into $L^2(\Gamma_D)$, e.g., Γ_D does not contain any isolated sets of zero $d-2$ dimensional measure. We denote $\|.\|_1$ and $\|.\|_{\mathbb{V}}$ to be the norms on $H^1(\Omega)$ and $\mathbb{V}$ defined respectively by

$$\|v\|_1 := \left(\int_\Omega |v|^2 + \int_\Omega |\nabla v|^2\right)^{1/2}, \qquad \|v\|_{\mathbb{V}} := \left(\int_\Omega |\nabla v|^2\right)^{1/2}.$$

For convenience, we avoid the situation where the variational space requires $L^2(\Omega)$-orthogonalization, i.e., the Neumann problem without a zeroth order term.

For a bounded operator $G : X(\Omega) \to Y(\Omega)$ between two Banach spaces $(X(\Omega), \|.\|_X)$ and $(Y(\Omega), \|.\|_Y)$ we write

$$\|G\|_{X \to Y} := \sup_{u \in X, \|u\|_X = 1} \|Gu\|_Y$$

and, in short, $\|G\| := \|G\|_{L^2 \to L^2}$.

Throughout this paper we use the notation $A \preceq B$ to denote $A \leq CB$ with a constant C independent of A, B and the discretization mesh parameter h (defined later). When appropriate, we shall be more explicit on the dependence of C.

Consider the sesquilinear form (1) for u, v in $\mathbb{V}$. We assume that $A(\cdot, \cdot)$ is strongly elliptic and bounded. The assumption of strong ellipticity is the existence of a positive constant c_0 satisfying

$$(7) \qquad\qquad \mathfrak{Re}(A(v, v)) \geq c_0 \|v\|_{\mathbb{V}}^2, \qquad \text{for all } v \in \mathbb{V}.$$

The boundedness of $A(\cdot, \cdot)$ on $\mathbb{V}$ implies the existence of a positive constant c_1 satisfying

$$(8) \qquad |A(u, v)| \leq c_1 \|u\|_{\mathbb{V}} \|v\|_{\mathbb{V}}, \qquad \text{for all } u, v \in \mathbb{V}.$$

The conditions (7) and (8) imply that the sesquilinear form $A(\cdot, \cdot)$ is regular on $\mathbb{V}$ (see, Section 2 of [31]).

Following [31], we define the Hermitian forms

$$\mathfrak{Re}\,A(u, v) := \frac{A(u, v) + \overline{A(v, u)}}{2} \quad \text{and} \quad \mathfrak{Im}\,A(u, v) := \frac{A(u, v) - \overline{A(v, u)}}{2i}.$$

Note that (7) implies that $\mathfrak{Re}\,A(u, u)$ is equivalent to $\|u\|_{\mathbb{V}}^2$, for all $v \in \mathbb{V}$ and (8) is equivalent to

$$(9) \qquad |\mathfrak{Im}\,A(u, u)| \leq \eta\,\mathfrak{Re}\,A(u, u), \qquad \text{for all } u \in \mathbb{V},$$

for some $\eta > 0$. The smallest constant η above is called the index of $A(\cdot, \cdot)$.

We now define operators associated with regular sesquilinear forms. Let A be the unique closed m(aximal)-accretive operator of Theorem 2.1 of [31], which is defined as follows. We set $\widetilde{T} : L^2(\Omega) \to \mathbb{V}$ by

$$A(\widetilde{T}u, \phi) = (u, \phi), \qquad \text{for all } \phi \in \mathbb{V}$$

(which is uniquely defined by the Lax-Milgram Theorem) and set

$$D(A) := \text{Range}(\widetilde{T}) \subset \mathbb{V}.$$

As $\widetilde{T}$ is one to one, we define $Aw := \widetilde{T}^{-1}w$, for $w \in D(A)$. The operator A associated with a regular sesquilinear form is said to be *regularly accretive*.

We denote by $\mathbb{V}_a^*$ denotes the set of bounded antilinear functionals on $\mathbb{V}$. It will be useful to consider also a related bounded operator $T_a : \mathbb{V}_a^* \to \mathbb{V}$ defined for $F \in \mathbb{V}_a^*$ by the unique solution (Lax-Milgram again) to

$$A(T_a F, \phi) = \langle F, \phi \rangle, \qquad \text{for all } \phi \in \mathbb{V}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the antilinear functional/function pairing. The operator T_a is a bijection of $\mathbb{V}_a^*$ onto $\mathbb{V}$ and we denote its inverse by A_a. ¿From the definition of T_a, we readily deduce that for $u \in \mathbb{V}$, $A_a u$ satisfies

$$(10) \qquad \langle A_a u, v \rangle = A(u, v), \qquad \text{for all } v \in \mathbb{V}.$$

It is also clear from the definition of A that $u \in D(A)$ if and only if $A_a u$ extends to a bounded antilinear functional on $L^2(\Omega)$ and, then,

$$(Au, v) = \langle A_a u, v \rangle = A(u, v), \qquad \text{for all } u \in D(A), v \in \mathbb{V}.$$

The above constructions can be repeated for adjoints defining $\mathbb{V}_l^*$ the set of bounded linear functionals on $\mathbb{V}$, $\widetilde{T}^* : L^2(\Omega) \to \mathbb{V}$, $D(A^*) := \text{Range}(\widetilde{T}^*) \subset \mathbb{V}$, $A^* := (\widetilde{T}^*)^{-1} : D(A^*) \to L^2(\Omega)$, $T_l^* : \mathbb{V}_l^* \to \mathbb{V}$ and $A_l^* := (T_l^*)^{-1}$. In this case, for $v \in \mathbb{V}$, $A_l v$ satisfies

$$(11) \qquad \langle u, A_l v \rangle = A(u, v), \qquad \text{for all } u \in \mathbb{V}.$$

with $\langle \cdot, \cdot \rangle$ also denoting the function/linear functional pairing. We also have that $v \in D(A^*)$ if and only if $A_l v$ extends to a bounded linear functional on $L^2(\Omega)$ and, then,

$$(u, A^* v) = \langle u, A_l^* v \rangle = A(u, v), \qquad \text{for all } u \in \mathbb{V}, v \in D(A^*).$$

Of course, these definitions imply that for $u \in D(A)$ and $v \in D(A^*)$, $(Au, v) = (u, A^*v)$.

By construction, the operators A and A^* defined from regular sesquilinear forms are regularly accretive (cf. [31]). They satisfy the following theorem (see, also [23]):

Theorem 2.1 (Theorem 2.2 of [31]). *Let A be the unique regularly accretive operator defined from a regular sesquilinear form $A(\cdot, \cdot)$ with index η. Set $\omega := \arctan(\eta)$. Then the numerical range and the spectrum of A are subsets of the sector $S_\omega := \{z \in \mathbb{C} : |\arg z| \le \omega\}$. Further, the resolvent set $\rho(A)$ of A contains $S_\omega^c := \mathbb{C} \setminus S_\omega$ and on this set the resolvent $R_z(A) := (A - zI)^{-1}$ satisfies*

$$\|R_z(A)\| \le \begin{cases} [|z| \sin(\arg(z) - \omega)]^{-1} & \text{for} \quad \omega < \arg(z) \le \frac{\pi}{2} + \omega, \\ |z|^{-1} & \text{for} \quad \arg(z) > \frac{\pi}{2} + \omega. \end{cases}$$

The result also holds for A replaced by A^*.

Remark 2.1. *It easily follows from* (7) *that for $\mathfrak{Re}(z) \le c_0/2$,*

$$\frac{c_0}{2} \|u\|_1^2 \le |A(u, u) - z(u, u)|$$

which implies that

$$\|R_z(A)f\|_1 \le \frac{2}{c_0} \|f\|.$$

It follows from Theorem 2.1 and Remark 2.1 that the Bochner integral appearing in (2), for $\beta \in (0, 1)$, is well defined and gives a bounded operator $A^{-\beta}$ on $L^2(\Omega)$. Fractional powers for positive indices can be defined from those with negative indices. For $\beta \in (0, 1)$,

$$D(A^\beta) = \{u \in L^2(\Omega) : A^{\beta-1}u \in D(A)\}$$

and $A^\beta u := A(A^{\beta-1})u$ for $u \in D(A^\beta)$.

An alternative but equivalent definition of fractional powers of positive operators (for $\beta \in (-1, 1)$) is given in, e.g., [35]. We shall recall some additional properties provided there (for $\beta \in (0, 1)$). Theorem 4.6 of [35] implies that $D(A) \subset D(A^\beta)$ and for $v \in D(A)$, $A^\beta v = A^{\beta-1}Av$ and $Av = A^\beta A^{1-\beta}v = A^{1-\beta}A^\beta v$. Also, for any $\beta > 0$, $D(A^\beta) = \{A^{-\beta}v : v \in L^2(\Omega)\}$ and $A^\beta v = (A^{-\beta})^{-1}v$ for $v \in D(A^\beta)$.

Set $w := A^\beta v$. The last statement in the previous paragraph implies that $w = (A^{-\beta})^{-1}v$. Now as $(A^{-\beta})w = v$,

$$\|v\| = \|A^{-\beta}w\| \le \|A^{-\beta}\|\|w\| = \|A^{-\beta}\|\|A^\beta v\|.$$

This implies that we can take

$$(12) \qquad \qquad \|v\|_{D(A^s)} := \|A^s v\|$$

as our norm on $D(A^s)$ for $s \in [0, 1]$.

Using the above and techniques from functional calculus [26], we can conclude similar facts concerning products of fractional powers and the resolvent, namely,

$$(13) \qquad A^{-\beta}R_z(A)u = R_z(A)A^{-\beta}u, \qquad \text{for all } u \in L^2(\Omega), \beta \ge 0,$$

and

$$(14) \qquad A^\beta R_z(A)u = R_z(A)A^\beta u, \qquad \text{for all } u \in D(A^\beta), \beta \in [0, 1].$$

We shall also connect fractional powers of operators with their adjoints in the $L^2(\Omega)$-inner product. We have already noted that for $u \in D(A)$ and $v \in D(A^*)$,

$(Au, v) = (u, A^*v)$. This holds for fractional powers as well, i.e., $(A^\beta u, v) = (u, (A^*)^\beta v)$ provided that $u \in D(A)$ and $v \in D(A^*)$.

3. The Hermitian Operator and the Dotted Spaces.

For notational simplicity, we set $S(u, v) := \Re A(u, v)$. As already noted, $S(u, u)^{1/2}$ provides an equivalent norm on $\mathbb{V}$ and we redefine $\|u\|_{\mathbb{V}} := S(u, u)^{1/2}$. As $S(\cdot, \cdot)$ is regular (i.e. satisfies (7) and (8) with $A(.,.)$ replaced by $S(.,.)$), there is an associated (m-accretive) unbounded operator S. The latter is defined similarly as A from $A(.,.)$ in Section 2 (see also [31]). This is, upon first defining $T_S : L^2(\Omega) \to \mathbb{V}$ by $T_S f := w$ where $w \in \mathbb{V}$ is the unique solution of

$$S(w, \phi) = (f, \phi) \qquad \text{for all } \phi \in \mathbb{V}$$

and then setting $D(S) := \text{Range}(T_S)$, $S := T_S^{-1}$.

In addition, as $S(\cdot, \cdot)$ is symmetric and coercive, S is self-adjoint and satisfies

$$S(u, v) = (S^{1/2}u, S^{1/2}v).$$

We consider the Hilbert scale of spaces defined by $\dot{H}^s := D(S^s)$ for $s \geq 0$. The above discussion implies

(15) $$\dot{H}^1 = \mathbb{V} \qquad \text{and} \qquad \dot{H}^0 = L^2(\Omega).$$

Moreover, the operator T_S is a compact Hermitian operator on $L^2(\Omega)$ and so there is a countable $L^2(\Omega)$-orthonormal basis $\{\psi_i, \ i = 1, \ldots, \infty\}$ of eigenfunctions for T_S. The corresponding eigenvalues $\{\mu_i\}$ can be ordered so that they are non-increasing with limit 0 and we set $\lambda_i = \mu_i^{-1}$. This leads to a realization of $\dot{H}^s$ in terms of eigenfunction expansions, namely, for $s \in (0, 1)$

(16) $$D(S^{s/2}) := \dot{H}^s = \left\{ w = \sum_{j=1}^{\infty} (w, \psi_j)\psi_j \in L^2(\Omega) \ : \ \sum_{j=1}^{\infty} |(w, \psi_j)|^2 \lambda_j^s < \infty \right\}.$$

The spaces $\dot{H}^s$ are Hilbert spaces with inner product

$$(u, v)_s := \sum_{j=1}^{\infty} \lambda_j^s (u, \psi_j)\overline{(v, \psi_j)}.$$

Moreover, they are a Hilbert scale of spaces and are also connected by the real interpolation method.

As already mentioned $\dot{H}^1 = \mathbb{V}$ so that the set of antilinear functionals on $\mathbb{V}$, denoted $\mathbb{V}_a^*$, can be characterized by

$$\mathbb{V}_a^* = \dot{H}_a^{-1} := \left\{ \left\langle \sum_{j=1}^{\infty} c_j \psi_j, \cdot \right\rangle \ : \ \sum_{j=1}^{\infty} |c_j|^2 \lambda_j^{-1} < \infty \right\},$$

where $\left\langle \sum_{j=1}^{\infty} c_j \psi_j, \sum_{i=1}^{\infty} d_j \psi_j \right\rangle := \sum_{j=1}^{\infty} c_j \overline{d}_j$.

In addition, the set of antilinear functionals on $L^2(\Omega)$, denoted by $L^2(\Omega)_a^*$, is given by

$$L^2(\Omega)_a^* = \dot{H}_a^0 := \left\{ \left\langle \sum_{j=1}^{\infty} c_j \psi_j, \cdot \right\rangle \ : \ \sum_{j=1}^{\infty} |c_j|^2 < \infty \right\}.$$

Hence, the intermediate spaces are defined by

$$\dot{H}_a^{-s} := \left\{ \left\langle \sum_{j=1}^{\infty} c_j \psi_j, \cdot \right\rangle \; : \; \sum_{j=1}^{\infty} |c_j|^2 \lambda_j^{-s} < \infty \right\}$$

and are Hilbert spaces with the obvious inner product. These also are a Hilbert scale of interpolation spaces for $s \in [-1, 0]$. In addition, these spaces are dual to $\dot{H}^s$, i.e., if $s \in [0, 1]$ and

$$\langle w, \cdot \rangle = \left\langle \sum_{j=1}^{\infty} c_j \psi_j, \cdot \right\rangle \in \dot{H}_a^{-s}$$

then

$$(17) \qquad \|w\|_{\dot{H}_a^{-s}} = \left(\sum_{j=1}^{\infty} \lambda_j^{-s} |c_j|^2 \right)^{1/2} = \sup_{\theta \in \dot{H}^s} \frac{\langle w, \theta \rangle}{\|\theta\|_{\dot{H}^s}}$$

and if $\theta \in \dot{H}^s$,

$$(18) \qquad \|\theta\|_{\dot{H}^s} = \sup_{w \in \dot{H}_a^{-s}} \frac{\langle w, \theta \rangle}{\|w\|_{\dot{H}_a^{-s}}}.$$

Considering linear functionals instead of antilinear functionals and replacing $\mathbb{V}_a^*$ and $L^2(\Omega)_a^*$ by spaces of linear functionals, $\mathbb{V}_l^*$ and $L^2(\Omega)_l^*$ give rise to the analogous Hilbert scale with $\dot{H}_l^{-1} = \mathbb{V}_l^*$ and $\dot{H}_l^0 = L^2(\Omega)_l^*$ as endpoints with equalities similar to (17) and (18) holding for these as well.

As we shall see in Section 4, $D(A^{s/2})$ relates either to $\dot{H}^s$ or $H^s(\Omega) \cap \mathbb{V}$ depending on whether $s > 0$ is smaller or greater that 1. In order to unify the presentation, we introduce the following spaces equipped with their natural norms:

$$\widetilde{H}^s := \begin{cases} \dot{H}^s & \text{for } s \in [0, 1], \\ H^s(\Omega) \cap \mathbb{V} & \text{for } s \geq 1. \end{cases}$$

4. CHARACTERIZATION OF $D(A^{\frac{s}{2}})$

In this section, we first observe that the dotted spaces for $s \in [0, 1)$ coincide with the domains of fractional powers of A and A^* (c.f., [31]). In addition, we note that the dotted spaces $\dot{H}_a^{-s}$ and $\dot{H}_l^{-s}$ can be identified with the dual space of $\dot{H}^s$, for $s \in [0, 1]$. The case of $\widetilde{H}^s := \dot{H}^s$, $s \in (0, 1)$ is addressed in the following theorem which is an immediate consequence of Theorem 3.1 of [31].

Theorem 4.1 (Characterization of $D(A^{\frac{s}{2}})$ for $0 \leq s < 1$). *Assume that* (7) *and* (8) *hold. Then for $s \in [0, 1)$,*

$$D(A^{s/2}) = D((A^*)^{s/2}) = D(S^{s/2}) = \widetilde{H}^s,$$

with equivalent norms.

The identification of the negative dotted spaces with the duals is given in the following remark.

Remark 4.1 (Characterization of Negative Spaces). *We identify $f \in L^2(\Omega)$ with the functional $F_a^f \in L^2(\Omega)_a^*$ defined by*

$$\langle F_a^f, \theta \rangle = (f, \theta), \qquad \text{for all } \theta \in L^2(\Omega).$$

It follows from Theorem 4.1 and (17) that the norms $\|F_a^f\|_{\dot{H}_a^{-2s}}$ and $\|A^{-s}f\|$ are equivalent (for $s \in [0, 1/2)$). Indeed,

$$\|F_a^f\|_{\dot{H}_a^{-2s}} = \sup_{\phi \in \dot{H}^{2s}} \frac{\langle F_a^f, \phi \rangle}{\|\phi\|_{\dot{H}^{2s}}} \approx \sup_{\phi \in \dot{H}^{2s}} \frac{(f, \phi)}{\|(A^*)^s \phi\|}$$

$$= \sup_{\theta \in L^2(\Omega)} \frac{(f, (A^*)^{-s}\theta)}{\|\theta\|_{\dot{H}^{2s}}} = \|A^{-s}f\|.$$

Here $\approx$ denotes comparability with constants independent of f. For simplicity, we shall write $\|f\|_{\dot{H}_a^{-2s}}$ instead of $\|F_a^f\|_{\dot{H}_a^{-2s}}$. We can identify $L^2(\Omega)$ with $L^2(\Omega)_l^*$ in an analogous way and similar norm equivalences hold.

Elliptic regularity is required to obtain convergence rates for finite element approximation. Such results for boundary value problems have been studied by many authors [3, 19, 20, 25, 32, 34, 33, 40, 41]. The follow assumption illustrates the type of elliptic regularity results available.

Assumption 1 (Elliptic Regularity). *We shall assume elliptic regularity for the form $A(\cdot, \cdot)$ with indices $\alpha \in (0, 1]$. Specifically, we assume that for $s \in (0, \alpha]$, T_a is a bounded map of $\dot{H}_a^{-1+s}$ into $\widetilde{H}^{1+s}(\Omega)$ and T_l^* is a bounded map of $\dot{H}_l^{-1+s}$ into $\widetilde{H}^{1+s}$.*

The above assumptions imply the following theorem.

Theorem 4.2 (Property of $D(A^{\frac{s}{2}})$ for $s > 1$). *Assume that (7), (8) and the elliptic regularity assumptions (Assumption 1) hold. Then for $s \in (1, 1 + \alpha]$,*

$$D(A^{s/2}) \subset \widetilde{H}^s \qquad \text{and} \qquad D((A^*)^{s/2}) \subset \widetilde{H}^s,$$

with continuous embeddings.

Remark 4.2 (Kato Square Root Problem). *The case of $s = 1$, i.e., $D(A^{1/2}) \subset \mathbb{V} =: \widetilde{H}^1$ with continuous imbedding is contained in the Kato Square Root Theorem. This is a deep theorem which has been intensively studied, see [1, 2, 39, 37] and the references in [39]. The Kato Square Root Theorem holds for our problem under fairly weak regularity assumptions on the coefficients defining our bilinear form [1]. In fact, it requires the existence of $\epsilon > 0$ such that A_a and A_l^* are bounded operators from $\widetilde{H}^{1+\gamma}$ to $\dot{H}_a^{-1+\gamma}$ and $\dot{H}_l^{-1+\gamma}$, respectively, for $|\gamma| \leq \epsilon$.*

Proof of Theorem 4.2. We consider the case of A as the case of A^* is similar. Suppose that u is in $D(A)$ and v is in $D(A^*)$. Then, for $t \in [0, \alpha]$,

$$A(u, v) = (Au, v) = (A^{(1-t)/2}A^{(1+t)/2}u, v)$$

$$= (A^{(1+t)/2}u, (A^*)^{(1-t)/2}v) := F(v).$$

Thus, Theorem 4.1 gives

$$|F(v)| \leq \|A^{(1+t)/2}u\| \|(A^*)^{(1-t)/2}v\|$$

$$\preceq \|A^{(1+t)/2}u\| \|v\|_{\dot{H}^{1-t}}.$$

This implies that $F \in \dot{H}_l^{-1+t}$. The elliptic regularity Assumption 1 implies that $u = T_l^* F$ is in $\widetilde{H}^{1+t}$ and satisfies

$$\|u\|_{H^{1+t}} \preceq \|A^{(1+t)/2}u\|.$$

As $D(A)$ is dense in $D(A^{(1+t)/2})$, $D(A^{(1+t)/2}) \subset \widetilde{H}^{1+t}$ follows. $\qquad \square$

5. Finite element approximation

In this section, we define finite element approximations to the operator $A^{-\beta}$ for $\beta \in (0, 1)$. For simplicity, we assume that the domain Ω is polyhedral so that it can be partitioned into a conforming subdivision made of simplices. Further, we assume that we are given a finite dimensional subspace $\mathbb{V}_h \subset \mathbb{V}$ consisting of continuous complex valued functions, vanishing on Γ_D, which are piecewise linear with respect to a conforming subdivision of simplices of maximal size diameter $h \le 1$. Notice that when the form $A(v, w)$ is real for real v, w, so is the finite element space (see Remark 5.1). We also need to assume that the triangulation matches the partitioning $\Gamma = \Gamma_D \cup \Gamma_N$. This means that any mesh simplex of dimension less than d which lies on Γ is contained in either $\bar{\Gamma}_N$ or Γ_D. Given a universal constant $\rho > 0$, we restrict further our considerations to quasi-uniform partitions $\mathcal{T}$, i.e. satisfying

$$(19) \qquad \frac{\max_{T \in \mathcal{T}} \operatorname{diam}(T)}{\min_{T \in \mathcal{T}} \operatorname{diam}(T)} \le \rho.$$

Let π_h denote the $L^2(\Omega)$-orthogonal projector onto $\mathbb{V}_h$. Given a sequence of conforming subdivisions $\{\mathcal{T}\}_h$ satisfying (19), there holds

$$(20) \qquad \|\pi_h v\|_1 \le C\|v\|_1, \qquad \text{for all } v \in \mathbb{V},$$

where the constant C is independent of h; see [14]. Obviously, π_h is a bounded operator on $L^2(\Omega)$ and, by interpolation, is a bounded operator on $\dot{H}^s$ for $s \in [0, 1]$ with bounds independent of h.

We shall need the following lemma providing approximation properties for π_h. These results are a consequence of the Scott-Zhang approximation operator [43] and operator interpolation. As the specific form of these results are needed for the analysis in the remainder of this paper, we include a proof for completeness.

Lemma 5.1. *Let s be in $[0, 1]$ and $\sigma > 0$ be such that $s + \sigma \le 2$. Then there is a constant $C = C(s, \sigma)$ not depending on h and satisfying*

$$\|(I - \pi_h)u\|_{\widetilde{H}^s} \le Ch^\sigma \|u\|_{\widetilde{H}^{s+\sigma}}, \qquad \text{for all } u \in \widetilde{H}^{s+\sigma}.$$

Proof. Let $\widetilde{\pi}_h$ denote the Scott-Zhang approximation operator [43] mapping onto the set of piecewise linear polynomials with respect to the above triangulation (without any imposed boundary conditions). This operator satisfies, for $\ell = 0, 1$ and $k = 1, 2$,

$$(21) \qquad \|(I - \widetilde{\pi}_h)u\|_{H^\ell} \preceq h^{k-\ell}\|u\|_{H^k}, \qquad \text{for all } u \in H^k(\Omega).$$

In addition, $\widetilde{\pi}_h u \in \mathbb{V}_h$ for $u \in \mathbb{V}$.

We first verify the lemma when $s + \sigma \le 1$. We clearly have

$$(22) \qquad \|(I - \pi_h)u\| \le \|(I - \widetilde{\pi}_h)u\| \preceq h\|u\|_{\mathbb{V}}, \qquad \text{for all } u \in \mathbb{V}.$$

It immediately follows from (20) that

$$(23) \qquad \|(I - \pi_h)u\|_{\mathbb{V}} \preceq \|u\|_{\mathbb{V}}, \qquad \text{for all } u \in \mathbb{V}.$$

Interpolating this and (22) gives

$$(24) \qquad \|(I - \pi_h)u\|_{\widetilde{H}^s} \preceq h^{1-s}\|u\|_{\mathbb{V}}, \qquad \text{for all } u \in \mathbb{V},$$

for $s \in [0, 1]$. As π_h is stable on $L^2(\Omega)$ and $\mathbb{V}$, interpolation implies that it is stable on $\widetilde{H}^s$ and hence

(25) $$\|(I - \pi_h)u\|_{\widetilde{H}^s} \preceq \|u\|_{\widetilde{H}^s}, \qquad \text{for all } u \in \widetilde{H}^s,$$

for $s \in [0, 1]$. Interpolating (24) and (25) and applying the reiteration theorem gives for $\sigma > 0$ and $s + \sigma \leq 1$,

(26) $$\|(I - \pi_h)u\|_{\widetilde{H}^s} \preceq h^\sigma \|u\|_{\widetilde{H}^{s+\sigma}}, \qquad \text{for all } u \in \widetilde{H}^{s+\sigma}.$$

We next consider the case when $s + \sigma \in (1, 2]$. Taking $\ell = 1$ in (21) and interpolating between the $k = 1$ and $k = 2$ gives for $\zeta \in [1, 2]$,

$$\|(I - \widetilde{\pi}_h)u\|_{H^1} \preceq h^{\zeta-1}\|u\|_{H^\zeta}, \qquad \text{for all } u \in H^\zeta(\Omega).$$

Thus for $u \in \widetilde{H}^{s+\sigma}$, by (26),

$$\|(I - \pi_h)u\|_{\widetilde{H}^s} \preceq h^{1-s}\|(I - \pi_h)u\|_{\mathbb{V}} \preceq h^{1-s}(\|(I - \widetilde{\pi}_h)u\|_{\mathbb{V}} + \|(\widetilde{\pi}_h - \pi_h)u\|_{\mathbb{V}})$$
$$\preceq h^\sigma \|u\|_{\widetilde{H}^{s+\sigma}} + h^{-s}\|(\widetilde{\pi}_h - \pi_h)u\| \preceq h^\sigma \|u\|_{\widetilde{H}^{s+\sigma}}$$

where the last inequality followed from (21) and obvious manipulations. $\qquad \square$

We define $A_h : \mathbb{V}_h \to \mathbb{V}_h$ by

(27) $$(A_h v_h, \varphi_h) = A(v_h, \varphi_h), \qquad \text{for all } \varphi_h \in \mathbb{V}_h.$$

The operator A_h is the discrete analogue of A and we analogously define A_h^*, the discrete analogue of A^*. The fractional powers $A_h^{-\beta}$ for $\beta > 0$ are again given by (2) but with A replaced by A_h, i.e., for $\beta \in (0, 1)$, $A_h^{-\beta} : \mathbb{V}_h \to \mathbb{V}_h$ is given by

(28) $$A_h^{-\beta} := \frac{\sin(\beta\pi)}{\pi} \int_0^\infty \mu^{-\beta}(\mu I + A_h)^{-1}\, d\mu.$$

The goal of this paper is to analyze the error between $A^{-\beta}f$ and $A_h^{-\beta}\pi_h f$.

Remark 5.1 (Real Valued Bilinear Forms and Finite Element Spaces). *When $A(v, w)$ is real for real v, w, the above operators restricted to real valued functions are real valued and hence we may use Sobolev spaces and approximation spaces $\mathbb{V}_h$ of real valued functions.*

Similarly, let $S_h : \mathbb{V}_h \to \mathbb{V}_h$ be defined by

$$(S_h v_h, w_h) = S(v_h, w_h), \qquad \text{for all } v_h, w_h \in \mathbb{V}_h.$$

Theorem 3.1 of [31] applied to the discrete operators A_h and S_h shows that for $s \in [0, 1/2)$,

$$\left(1 - \tan\frac{\pi s}{2}\right)\|S_h^s v_h\| \leq \|A_h^s v_h\| \leq \left[1 + \left(\frac{s}{\pi}\tan \pi s\right)^{1/2}(\eta + \eta^2)\right]\|S_h^s v_h\|,$$

for all $v_h \in \mathbb{V}_h$. Here η is the index of $A(\cdot, \cdot)$ (see, (9)). This also holds for A_h^*.

The bound (20) implies (see, e.g., [7]) that there are positive constants c and C, not depending on h such that for $s \in [0, 1]$,

(29) $$c\|S_h^{s/2} v_h\| \leq \|v_h\|_{\dot{H}^s} \leq C\|S_h^{s/2} v_h\|, \qquad \text{for all } v_h \in \mathbb{V}_h.$$

Combining the preceding two sets of inequalities proves the following lemma.

Lemma 5.2 (Discrete Characterization of $\dot{H}^s$ for $s \in [0,1)$). *There exist positive constants c, C independent of h such that for all $v_h \in \mathbb{V}_h$ and $s \in [0,1)$,*

$$c\|v_h\|_{\dot{H}^s} \leq \|A_h^{s/2}v_h\| \leq C\|v_h\|_{\dot{H}^s}.$$

This result holds with A_h replaced by A_h^*.

Remark 5.2 (Quasi-uniformity Assumption). *The above lemma still holds as long as* (20) *holds. This is for instance the case for certain mesh refinement strategies* [8, 12].

6. Error Estimates

In this section, we study numerical approximation to the operators $A^{-\beta}$ for $\beta \in (0,1)$. Specifically, this involves bounding the errors $(A^{-\beta} - A_h^{-\beta}\pi_h)v$ for v having appropriate smoothness.

We shall use our finite element spaces to approximate $A_a^{-1} := T_a$. Specifically, $T_{h,a} : \mathbb{V}_a^* \to \mathbb{V}_h$ is defined for $F \in \mathbb{V}_a^*$ by

$$A(T_{h,a}F, \phi_h) = \langle F, \phi_h \rangle, \qquad \text{for all } \phi_h \in \mathbb{V}_h.$$

We define $T_{h,l}^*$ corresponding to $T_l^* := (A_l^*)^{-1}$ analogously.

The following lemma provides approximation error bounds in terms of the norms needed for our subsequent analysis. Although the techniques in the proof (Galerkin orthogonality and Nitsche finite element duality) are completely classical, the results are not quotable (as far as we know). We include a proof for completeness.

Lemma 6.1 (Finite Element Error). *Assume that* (7), (8) *and the elliptic regularity Assumption 1 hold. Let $s \in [0, \frac{1}{2}]$ and set $\alpha_* := \frac{1}{2}(\alpha + \min(1 - 2s, \alpha))$. There is a positive constant c not depending on h satisfying*

$$(30) \qquad \|T_a - T_{a,h}\|_{\dot{H}_a^{\alpha-1} \to \dot{H}^{2s}} \leq ch^{2\alpha_*}.$$

The above immediately implies

$$\|T_a - T_{a,h}\| \leq ch^{2\alpha}.$$

Proof. The proof of this lemma is classical and we only include details for completeness. We distinguish two cases.

$\boxed{1}$ When $2s \leq 1 - \alpha$ then we can fully take advantage of the elliptic regularity assumption. For $F \in \dot{H}_a^{\alpha-1}$, we set $e = (T_a - T_{a,h})F$. By (18) and the elliptic regularity Assumption 1,

$$\|e\|_{\dot{H}^{2s}} \leq \|e\|_{\dot{H}^{1-\alpha}} \preceq \sup_{G \in \dot{H}_l^{\alpha-1}} \frac{\langle e, G \rangle}{\|G\|_{\dot{H}_l^{\alpha-1}}} \preceq \sup_{G \in \dot{H}_l^{\alpha-1}} \frac{A(e, T_l^*G)}{\|T_l^*G\|_{H^{1+\alpha}}}$$

$$= \sup_{w \in \widetilde{H}^{1+\alpha}} \frac{A(e, w)}{\|w\|_{H^{1+\alpha}}} = \sup_{w \in \widetilde{H}^{1+\alpha}} \frac{A(e, w - w_h)}{\|w\|_{H^{1+\alpha}}}.$$

We used Galerkin orthogonality for the last equality (which holds for any $w_h \in \mathbb{V}_h$). Using the approximation property

$$\inf_{w_h \in \mathbb{V}_h} \|w - w_h\|_1 \preceq h^\alpha \|w\|_{H^{1+\alpha}}, \qquad \text{for all } w \in \widetilde{H}^{1+\alpha},$$

(8) and the above inequalities give

$$\|e\|_{\dot{H}^{1-\alpha}} \preceq h^\alpha \|e\|_1.$$

$\boxed{2}$ This duality argument yields a reduced order of convergence when $2s > 1 - \alpha$. Indeed, proceeding similarly

$$\|e\|_{\dot{H}^{2s}} \preceq \sup_{G \in \dot{H}_l^{-2s}} \frac{\langle e, G \rangle}{\|G\|_{\dot{H}_l^{-2s}}} \preceq \sup_{G \in \dot{H}_l^{-2s}} \frac{A(e, T_l^* G)}{\|T_l^* G\|_{H^{2-2s}}}$$

$$= \sup_{w \in \widetilde{H}^{2-2s}} \frac{A(e, w)}{\|w\|_{H^{2-2s}}} = \sup_{w \in \widetilde{H}^{2-2s}} \frac{A(e, w - w_h)}{\|w\|_{H^{2-2s}}},$$

so that together with the approximation property

$$\inf_{w_h \in \mathbb{V}_h} \|w - w_h\|_1 \preceq h^{1-2s} \|w\|_{H^{2-2s}}, \qquad \text{for all } w \in \widetilde{H}^{2-2s},$$

we deduce that

$$\|e\|_{\dot{H}^{2s}} \preceq h^{1-2s} \|e\|_1.$$

Gathering the two cases $2s > 1 + \alpha$ and $2s \leq 1 + \alpha$, we get

$$\|e\|_{\dot{H}^{2s}} \preceq h^{\min(1-2s, \alpha)} \|e\|_1.$$

Whence, together with the estimate

$$\|e\|_1 \preceq h^\alpha \|T_a F\|_{H^{1+\alpha}} \preceq h^\alpha \|F\|_{\dot{H}_a^{\alpha-1}}$$

guaranteed by Cea's Lemma and elliptic regularity Assumption 1, we obtain (30) as desired. □

We can now state and prove our main convergence results. It requires data in the abstract space $D(A^\delta)$ for some $\delta \geq 0$. A characterization of $D(A^\delta)$ is provided in Theorem 6.4 below.

Theorem 6.2 (Convergence). *Suppose that (7) and (8) as well as the elliptic regularity Assumption 1 hold. Given $s \in [0, \frac{1}{2})$ with $s \leq \beta$, set $\alpha_* := \frac{1}{2}(\alpha + \min(1 - 2s, \alpha))$ and $\gamma := \max(s + \alpha^* - \beta, 0)$ and let $\delta \geq \gamma$. Assume finally, that $s + \alpha_* \neq 1/2$ when $\delta = \gamma$ and $s + \alpha_* \geq \beta$. Then there exists a constant C independent of h and δ such that*

$$\|(A^{-\beta} - A_h^{-\beta} \pi_h) f\|_{\dot{H}^{2s}} \leq C_{\delta,h} h^{2\alpha_*} \|A^\delta f\|, \qquad \text{for all } f \in D(A^\delta),$$

where

(31) $$C_{\delta,h} = \begin{cases} C \ln(2/h) : & \text{when } \delta = \gamma \text{ and } s + \alpha_* \geq \beta, \ s + \alpha_* \neq \dfrac{1}{2} \\ C : & \text{when } \delta > \gamma \text{ and } s + \alpha_* \geq \beta, \\ C : & \text{when } \delta = 0 \text{ and } \beta > s + \alpha_*. \end{cases}$$

Remark 6.1 (Critical Case $s + \alpha_* = \frac{1}{2}$). *The condition $s + \alpha_* \neq \frac{1}{2}$ in the above theorem can be removed provided that the Kato Square Root Theorem holds as well (see, Remark 4.2).*

Remark 6.2 (Critical Case $2s = 1$). *The above results also hold when $2s = 1$ provided that the continuous and discrete Kato Square Root Theorem hold. As already mentioned above, the former relies on the additional assumption requiring the existence of $\epsilon > 0$ such that A_a and A_l^* are bounded from $\widetilde{H}^{1+\gamma}$ to $\dot{H}_a^{\gamma-1}$ and $\dot{H}_l^{\gamma-1}$, respectively, for $|\gamma| \leq \epsilon$. For the discrete Kato Theorem, we will need to assume similar conditions for operators based on the S form, see Theorem 6.4 below.*

The above theorem depends on an auxiliary lemma.

Lemma 6.3. *For $s \in [0,1]$, there is a constant C not depending on h such that for any $\mu \in (0, \infty)$*

$$\|A^s(\mu + A)^{-1}v\| \le C\mu^{s-1}\|v\|, \qquad \text{for all } v \in L^2(\Omega)$$

and

$$\|A_h^s(\mu + A_h)^{-1}v_h\| \le c\mu^{s-1}\|v_h\|, \qquad \text{for all } v_h \in \mathbb{V}_h.$$

Proof. The claim relies on interpolation estimates. As the same argument is used for both estimates, we only prove the first.

Theorem 4.29 of [35] implies A^{it} is a bounded operator satisfying

(32) $$\|A^{it}\| \le e^{\pi|t|/2}, \qquad \text{for all } t \in \mathbb{R}.$$

This, in turn, implies that (e.g., Lemma 4.31 of [35]) for $s \in [0, \frac{1}{2}]$,

(33) $$[L^2(\Omega), D(A^{1/2})]_{2s} = D(A^s).$$

Here $[X, Y]_{2s}$ denotes the intermediate space between X and Y obtained by the complex interpolation method. Thus, for $w \in D(A)$, Corollary 2.8 of [35] gives

$$\|A^s w\| \preceq \|w\|_{[L^2(\Omega), D(A)]_s} \le \|Aw\|^s\|w\|^{1-s}.$$

Now if $w = (\mu + A)^{-1}v$ with $v \in L^2(\Omega)$, then

$$\mu\|w\|^2 + \mathfrak{Re}A(w, w) = \mathfrak{Re}(v, w)$$

and hence (7) immediately implies $\|(\mu + A)^{-1}v\| \le \mu^{-1}\|v\|$. In addition,

$$\|Aw\| = \|v - \mu(\mu + A)^{-1}v\| \le 2\|v\|.$$

The lemma follows combining the above estimates. $\qquad \square$

Proof of Theorem 6.2. It suffices to prove the theorem for $\delta \le \gamma < (1 + \alpha)/2$.
$\boxed{1}$ We first show that

(34) $$\|(I - \pi_h)A^{-\beta}f\|_{\dot{H}^{2s}} \preceq h^{2\alpha_*}\|A^\delta f\|.$$

Theorem 4.6 of [35] implies that $A^{-\beta}f$ is in $D(A^{\alpha_*})$ when f is in $D(A^{\alpha_* - \beta})$ and we now discuss separately the cases $s + \alpha_* \in (0, \frac{1}{2})$, $s + \alpha_* = \frac{1}{2}$ and $s + \alpha_* \in (\frac{1}{2}, \frac{1}{2}(1+\alpha)]$. When $s + \alpha_* \in (0, \frac{1}{2})$, we apply Theorem 4.1 and obtain

(35) $$\begin{aligned} \|(I - \pi_h)A^{-\beta}f\|_{\dot{H}^{2s}} &\preceq h^{2\alpha_*}\|A^{-\beta}f\|_{\dot{H}^{2s+2\alpha_*}} \preceq h^{2\alpha_*}\|A^{s+\alpha_* - \beta}f\| \\ &\preceq h^{2\alpha_*}\|A^\delta f\|, \end{aligned}$$

recalling that $\delta \ge \gamma \ge s + \alpha_* - \beta$. For $s + \alpha_* \in (\frac{1}{2}, \frac{1}{2}(1+\alpha)]$, we apply Theorem 4.2 to conclude that $A^{-\beta}f$ is in $\widetilde{H}^{2s+2\alpha_*}$. And again,

(36) $$\begin{aligned} \|(I - \pi_h)A^{-\beta}f\|_{\dot{H}^{2s}} &\preceq h^{2\alpha_*}\|A^{-\beta}f\|_{H^{2s+2\alpha_*}} \preceq h^{2\alpha_*}\|A^{s+\alpha_* - \beta}f\| \\ &\preceq h^{2\alpha_*}\|A^\delta f\|. \end{aligned}$$

Finally, we consider the case $s + \alpha_* = \frac{1}{2}$, which entails $\alpha \le 1 - s$ so that $\alpha_* = \alpha$ and $2s + 2\alpha = 1$. We choose $0 < \epsilon < \alpha$ (further restricted below) so that as above

$$\|(I - \pi_h)A^{-\beta}f\|_{\dot{H}^{2s}} \preceq h^{2\alpha+\epsilon}\|A^{-\beta}f\|_{H^{1+\epsilon}} \preceq h^{2\alpha+\epsilon}\|A^{\frac{1+\epsilon}{2} - \beta}f\|.$$

In addition the assumption $\delta > \gamma := \max(\frac{1}{2} - \beta, 0)$ yields $\frac{1}{2} - \beta + \frac{\epsilon}{2} < \delta$ upon choosing a sufficiently small ϵ. Hence, we deduce

$$\|(I - \pi_h)A^{-\beta}f\|_{\dot{H}^{2s}} \preceq h^{2\alpha}\|A^\delta f\|.$$

This, (35) and (36) yield (34).

$\boxed{2}$ By the triangle inequality, it suffices now to bound

(37)
$$\|(\pi_h A^{-\beta} - A_h^{-\beta} \pi_h)\|_{D(A^\delta) \to \dot{H}^{2s}}$$
$$\leq \frac{\sin(\beta\pi)}{\pi} \int_0^\infty \mu^{-\beta} \left\|\pi_h(\mu + A)^{-1} - (\mu + A_h)^{-1} \pi_h\right\|_{D(A^\delta) \to \dot{H}^{2s}} d\mu.$$

Assuming without loss of generality that $h \leq 1$, we shall break the above integral into integrals on three subintervals, namely, $(0,1)$, $(1, h^{-2\alpha_*/\beta})$ and $(h^{-2\alpha_*/\beta}, \infty)$.

$\boxed{3}$ We start with $(h^{-2\alpha_*/\beta}, \infty)$. Recalling the definition of the operator norm (12) as well as the characterizations of the dotted space provided by Theorem 4.1 , we get

$$\|\pi_h(\mu + A)^{-1}\|_{D(A^\delta) \to \dot{H}^{2s}} \preceq \|A^{\max(s-\delta,0)}(\mu + A)^{-1}\|,$$

where we used in addition the stability of π_h in $D(A^\delta)$ and the boundedness of A^{-r} from $L^2(\Omega)$ to $L^2(\Omega)$, for $r \geq 0$ (see discussion below Remark 2.1). Hence, applying Lemma 5.2 yields

$$\|\pi_h(\mu + A)^{-1}\|_{D(A^\delta) \to \dot{H}^{2s}} \preceq \mu^{\max(s-\delta,0)-1}.$$

Similarly, but using the discrete characterization provided by Lemma 5.2, we obtain

$$\|(\mu + A_h)^{-1} \pi_h\|_{D(A^\delta) \to \dot{H}^{2s}} \preceq \mu^{\max(s-\delta,0)-1}.$$

Thus, invoking Lemma 6.3, we deduce that

$$\int_{h^{-2\alpha_*/\beta}}^\infty \mu^{-\beta} \|\pi_h(\mu + A)^{-1} - (\mu + A_h)^{-1} \pi_h\|_{D(A^\delta) \to \dot{H}^{2s}} d\mu$$
$$\preceq \int_{h^{-2\alpha_*/\beta}}^\infty \mu^{-\beta+\max(s-\delta,0)-1} d\mu \preceq \int_{h^{-2\alpha_*/\beta}}^\infty \mu^{\max(-\alpha_*,-\beta)-1} d\mu \preceq h^{2\alpha_*},$$

because $\delta \geq s + \alpha_* - \beta$.

$\boxed{4}$ For the remaining two cases, we use the identity

$$\pi_h(\mu + A)^{-1} - (\mu + A_h)^{-1} \pi_h = (\mu + A_h)^{-1} A_h \pi_h (T_a - T_{h,a}) A(\mu + A)^{-1}.$$

The latter, follows from the identification of Remark 4.1 and that the observation that for $u \in D(A)$ and $v \in \mathbb{V}$,

$$(T_a Au, v) = A(T_a Au, T_l^* v) = (Au, T_l^* v) = A(u, T_l^* v) = (u, v),$$

i.e., $T_a Au = u$. Also, it is easy to see that $A_h \pi_h T_{h,a} = \pi_h$. Thus, for $u \in D(A)$,

$$A_h \pi_h (T_a - T_{h,a}) Au = (\mu + A_h)\pi_h u - \pi_h(\mu + A)u,$$

which leads to the desired identity.

$\boxed{5}$ For $\mu \in (1, h^{-2\alpha_*/\beta})$, we set $\theta = \min(1 - 2s, \alpha)$, note that $(1+\theta)/2 + s \leq 1$ and write

$$\int_1^{h^{-2\alpha_*/\beta}} \mu^{-\beta} \|A_h(\mu + A_h)^{-1} \pi_h (T_a - T_{h,a}) A(\mu + A)^{-1}\|_{D(A^\delta) \to \dot{H}^{2s}} d\mu$$
$$\leq \int_1^{h^{-2\alpha_*/\beta}} \mu^{-\beta} \|A_h(\mu + A_h)^{-1} \pi_h\|_{\dot{H}^{1-\theta} \to \dot{H}^{2s}} \|(T_a - T_{h,a})\|_{\dot{H}_a^{-1+\alpha} \to \dot{H}^{1-\theta}}$$
$$\|A(\mu + A)^{-1}\|_{D(A^\delta) \to \dot{H}_a^{-1+\alpha}} d\mu$$

Now, the definition (12) of $\|.\|_{D(A^\delta)}$ together with the characterization of the negative spaces provided in Remark 4.1, imply that

$$\|A(\mu + A)^{-1}\|_{D(A^\delta)\to \dot{H}_a^{\alpha-1}} \preceq \|A^{(1+\alpha)/2-\delta}(\mu + A)^{-1}\| \preceq \mu^{(\alpha-1)/2-\delta}.$$

We use the condition $\delta < (1+\alpha)/2$ and Lemma 6.3 for the last inequality above.

In addition, using Lemma 5.2, we obtain

$$\|A_h(\mu + A_h)^{-1}\pi_h\|_{\dot{H}^{1-\theta}\to\dot{H}^{2s}} = \sup_{f\in\dot{H}^{1-\theta}} \frac{\|A_h(\mu + A_h)^{-1}\pi_h f\|_{\dot{H}^{2s}}}{\|\pi_h f\|_{\dot{H}^{1-\theta}}}$$

$$\preceq \sup_{f\in\dot{H}^{1-\theta}} \frac{\|A_h(\mu + A_h)^{-1}\pi_h f\|_{\dot{H}^{2s}}}{\|A_h^{(1-\theta)/2}\pi_h f\|}$$

$$\preceq \sup_{g\in\mathbb{V}_h} \frac{\|A_h^{\frac{1+\theta}{2}+s}(\mu + A_h)^{-1}g\|}{\|g\|}$$

$$\preceq \mu^{\frac{-1+\theta}{2}+s}.$$

The last inequality followed from Lemma 6.3 and the fact that $(1+\theta)/2 + s \leq 1$.

The above three estimates together with Lemma 6.1 yield

$$\int_1^{h^{-2\alpha_*/\beta}} \mu^{-\beta}\|A_h(\mu + A_h)^{-1}\pi_h(T_a - T_{h,a})A(\mu + A)^{-1}\|_{D(A^\delta)\to\dot{H}^{2s}}\, d\mu$$

$$\preceq h^{2\alpha_*}\int_1^{h^{-2\alpha_*/\beta}} \mu^{-1+s+\alpha_*-\beta-\delta} \preceq \begin{cases} h^{2\alpha_*}\ln(2/h) & : & \text{if } \delta = s + \alpha_* - \beta, \\ h^{2\alpha_*} & : & \text{otherwise.} \end{cases}$$

6 Finally for $\mu \in (0,1)$, we write

$$\int_0^1 \mu^{-\beta}\|A_h(\mu + A_h)^{-1}\pi_h(T_a - T_{h,a})A(\mu + A)^{-1}\|_{D(A^\delta)\to\dot{H}^{2s}}\, d\mu$$

$$\preceq \int_0^1 \mu^{-\beta}\|A_h(\mu + A_h)^{-1}\pi_h\|_{\dot{H}^{2s}\to\dot{H}^{2s}}\|T_a - T_{h,a}\|_{L^2\to\dot{H}^{2s}}\|A(\mu + A)^{-1}\|_{L^2\to L^2}\, d\mu$$

$$\preceq h^{2\alpha_*}\int_0^1 \mu^{-\beta}\, d\mu \preceq h^{2\alpha_*}.$$

7 The proof of the theorem is complete upon collecting the estimates obtained in Steps 3, 5 and 6.   □

Theorem 4.1 and the Kato Square Root Theorem characterize $D(A^s)$ for $s \in [0, 1/2]$. The characterization can be extended to $s \in (1/2, (1+\alpha)/2]$ when A_a maps $\widetilde{H}^{2s}$ into $\dot{H}_a^{2s-2}$. This is of particular importance to characterize the regularity assumption $f \in D(A^\delta)$ in Theorem 6.2.

Theorem 6.4 (Characterization of $D(A^{(1+s)/2})$ for $s \in (0, \alpha]$). *Suppose that (7) and (8) hold. Assume furthermore that for $s \in (0, \alpha]$,*

(38) T_a *is an isomorphism from $\dot{H}_a^{-1+s}$ into $\widetilde{H}^{1+s}$.*

Then,

$$D(A^{(1+s)/2}) = \widetilde{H}^{1+s}$$

with equivalent norms.

Proof. By Theorem 4.2, we need only to prove that $\widetilde{H}^{1+s} \subset D(A^{(1+s)/2})$. We first observe that $D(A) \cap \widetilde{H}^{1+s}$ is dense in $\widetilde{H}^{1+s}$. Indeed, if w is in $\widetilde{H}^{1+s}$ then (38) implies that $A_a w$ is in $\dot{H}_a^{s-1}$. As $\dot{H}_a^0$ is dense in $\dot{H}_a^{s-1}$, there is a sequence $F_n \in \dot{H}_a^0$ converging to $A_a w$ in $\dot{H}_a^{s-1}$. Setting $u_n := (A_a)^{-1} F_n$, elliptic regularity implies that u_n converges to w in $\widetilde{H}^{1+s}$. Clearly u_n is in $D(A)$, i.e., $D(A) \cap \widetilde{H}^{1+s}$ is dense in $\widetilde{H}^{1+s}$ as claimed.

Suppose that $u \in D(A) \cap \widetilde{H}^{1+s}$. We first show that

$$(39) \qquad \|A^{(1+s)/2} u\| \le C \|u\|_{H^{1+s}}.$$

For $v \in D(A^*)$ and $\delta := (1-s)/2 \in [0, 1/2)$,

$$(A^{(1+s)/2} u, v) = (A^{-\delta} A u, v) = (Au, (A^*)^{-\delta} v).$$

Since $u \in D(A)$ and $(A^*)^{-\delta} v \in D(A^*) \subset \mathbb{V}$,

$$|(Au, (A^*)^{-\delta} v)| = |\langle A_a u, (A^*)^{-\delta} v \rangle| \le \|A_a u\|_{\dot{H}_a^{-1+s}} \|(A^*)^{-\delta} v\|_{\dot{H}^{1-s}}$$
$$\preceq \|u\|_{H^{1+s}} \|(A^*)^{-\delta} v\|_{\dot{H}^{1-s}}$$

where we also used (17) and (38). Now, Theorem 4.1 ensures that

$$\|(A^*)^{-\delta} v\|_{\dot{H}^{1-s}} \preceq \|(A^*)^{\delta} (A^*)^{-\delta} v\| = \|v\|.$$

Combining the above inequalities shows that (39) holds for $u \in D(A) \cap \widetilde{H}^{1+s}$. The inclusion $\widetilde{H}^{1+s} \subset D(A^{(1+s)/2})$ and (39) for $u \in \widetilde{H}^{1+s}$ hold by density. $\qquad \square$

Let $T_{S,a}$ be defined similarly to T_a but using the form $S(\cdot, \cdot)$. The final result in this section shows that under suitable assumptions, Lemma 5.2 holds for $s = 1$. This is a discrete Kato Square Root Theorem. Its proof was motivated by the proof of the Kato Square Root Theorem given in [1].

Theorem 6.5 (Discrete Kato Square Root Theorem). *Suppose that* (7) *and* (8) *hold. Assume furthermore that for some $s \in (0, 1/2)$, T_a, T_a^* and $T_{S,a}$ are isomorphisms from $\dot{H}_a^{-1+s}$ into $\widetilde{H}^{1+s}$. Then, there are positive constants c, C independent of h such that*

$$c\|v_h\|_{\mathbb{V}} \le \|A_h^{1/2} v_h\| \le C\|v_h\|_{\mathbb{V}}, \qquad \textit{for all } v_h \in \mathbb{V}_h.$$

The analogous inequalities hold with A_h^* replacing A_h above.

Proof. In this proof C denotes a generic constant independent of h. The assumption on $T_{S,a}$ implies that for $t \in (0, s]$, $\widetilde{H}^{1+t} = \dot{H}^{1+t}$, with equivalent norms [11].

We first observe that it suffices to show that

$$(40) \qquad \|A_h^{1/2} u_h\| \le C\|u_h\|_{\mathbb{V}}, \qquad \textit{for all } u_h \in \mathbb{V}_h$$

along with the analogous inequality involving A_h^*. Indeed, if (40) holds then by (7),

$$c_0 \|u_h\|_{\mathbb{V}}^2 \le |(A_h u_h, u_h)| \le \|A_h^{1/2} u_h\| \|(A_h^*)^{1/2} u_h\|$$
$$\le C\|(A_h^*)^{1/2} u_h\| \|u_h\|_{\mathbb{V}}$$

and hence

$$c_0 \|u_h\|_{\mathbb{V}} \le C\|(A_h^*)^{1/2} u_h\|.$$

The proof of the lower bound involving $A_h^{1/2}$ is similar.

Applying Lemma 5.2 gives, for $u_h \in \mathbb{V}_h$,

$$(41) \quad \|A_h^{(1+s)/2} u_h\| = \sup_{\theta \in \mathbb{V}_h} \frac{(A_h^{(1+s)/2} u_h, (A_h^*)^{(1-s)/2} \theta)}{\|(A_h^*)^{(1-s)/2} \theta\|} \leq C \sup_{\theta \in \mathbb{V}_h} \frac{(A u_h, \theta)}{\|\theta\|_{\dot{H}^{1-s}}}$$

$$\leq C\|A_a u_h\|_{\dot{H}^{s-1}} \leq C\|u_h\|_{\dot{H}^{1+s}}$$

where we used the assumption on $T_a = (A_a)^{-1}$.

Let $\widetilde{\pi}_h$ denote the S-elliptic projection onto $\mathbb{V}_h$, i.e., for $v \in \mathbb{V}$, $w_h = \widetilde{\pi}_h v \in \mathbb{V}_h$ solves

$$S(w_h, \theta_h) = S(v, \theta_h), \qquad \text{for all } \theta_h \in \mathbb{V}_h.$$

It is a consequence of the isomorphism assumption on $T_{S,a}$ that $\widetilde{\pi}_h$ is a uniformly (independent of h) bounded operator on $\widetilde{H}^{1+s}$ (see, e.g., [13]). Thus, recalling the eigenvalue decomposition (16), it holds for $u_h \in \mathbb{V}_h$,

$$\|u_h\|_{\dot{H}^{1-s}} = \sup_{\phi \in \dot{H}^{1+s}} \frac{S(u_h, \phi)}{\|\phi\|_{\dot{H}^{1+s}}} = \sup_{\phi \in \dot{H}^{1+s}} \frac{S(u_h, \widetilde{\pi}_h \phi)}{\|\widetilde{\pi}_h \phi\|_{\dot{H}^{1+s}}} \frac{\|\widetilde{\pi}_h \phi\|_{\dot{H}^{1+s}}}{\|\phi\|_{\dot{H}^{1+s}}}$$

$$\leq C \sup_{\phi_h \in \mathbb{V}_h} \frac{S(u_h, \phi_h)}{\|\phi_h\|_{\dot{H}^{1+s}}}.$$

Similarly, for all $u_h \in \mathbb{V}_h$,

$$\|u_h\|_{\dot{H}^{1+s}} = \sup_{\phi_h \in \mathbb{V}_h} \frac{S(u_h, \phi_h)}{\|\phi_h\|_{\dot{H}^{1-s}}}.$$

Thus, the characterization (29) gives

$$\|u_h\|_{\dot{H}^{1+s}} \leq C \sup_{\phi_h \in \mathbb{V}_h} \frac{S(u_h, \phi_h)}{\|S_h^{(1-s)/2} \phi_h\|} = C\|S_h^{(1+s)/2} u_h\|.$$

Combining this with (41) gives

$$\|A_h^{(1+s)/2} u_h\| \leq C\|S_h^{(1+s)/2} u_h\|, \qquad \text{for all } u_h \in \mathbb{V}_h.$$

Interpolating this result with the trivial inequality

$$\|A_h^0 u_h\| \leq \|S_h^0 u_h\|, \qquad \text{for all } u_h \in \mathbb{V}_h,$$

gives

$$\|A_h^{1/2} u_h\| \leq C\|S_h^{1/2} u_h\| = C\|u_h\|_{\mathbb{V}}, \qquad \text{for all } u_h \in \mathbb{V}_h$$

and verifies (40). The proof for the analogous inequality involving A_h^* is similar. $\square$

7. EXPONENTIALLY CONVERGENT SINC QUADRATURE.

Theorem 6.2 and Remark 6.1 provide estimates for the errors $\|(A^{-\beta} - A_h^{-\beta} \pi_h) f\|_{\dot{H}^{2s}}$. We now apply an exponentially convergent sinc quadrature (see, for example, [36]) to approximate the integral in (3). Since the argument below does not require the operator $A_h^{-\beta} \pi_h$ to be discrete, the analysis includes and focuses on the case of $A^{-\beta}$.

The change of variable and sinc quadrature approximations for A are thus

$$A^{-\beta} = \frac{\sin(\pi \beta)}{\pi} \int_{-\infty}^{\infty} e^{(1-\beta)y} (e^y I + A)^{-1} \, dy$$

and

$$\mathcal{Q}_k^{-\beta}(A) := \frac{k\sin(\pi\beta)}{\pi} \sum_{\ell=-N}^{N} e^{(1-\beta)y_\ell}(e^{y_\ell}I + A)^{-1}.$$

Here, for any positive integer N, $y_\ell := \ell k$ and $k := 1/\sqrt{N}$.

To estimate the quadrature error, we start by defining $D_{\pi/2} := \{z \in \mathbb{C}, |\mathfrak{Im}(z)| < \pi/2\}$ and denote by $\overline{D}_{\pi/2}$ its closure. Here $\omega = \arctan(\eta)$ and η is the index of the sesquilinear form $A(.,.)$, see (9). For any functions $u, v \in L^2(\Omega)$ and $z \in \mathbb{C}$ with $-e^z$ not in the spectrum of A, we define

$$f(z; u, v) := e^{(1-\beta)z}\left((e^z I + A)^{-1}u, v\right).$$

Note that $\mathfrak{Re}(e^z)$ is non negative for $z \in \overline{D}_{\pi/2}$ and hence Theorem 2.1 and Remark 2.1 imply that $f(z, u, v)$ is well defined for $z \in \overline{D}_{\pi/2}$.

We apply the classical analysis for these types of quadrature approximations given in [36] with a particular attention on deriving estimates uniform in $u, v \in L^2(\Omega)$. Using the resolvent estimate (Theorem 2.1) when $\mathfrak{Re}(z) > 0$ and Remark 2.1 when $\mathfrak{Re}(z) \leq 0$ gives

$$\|(e^z I + A)^{-1}\| \leq \begin{cases} |e^{-z}|/\sin(\pi/2 - \omega) : & \text{for } z \in \overline{D}_{\pi/2},\ \mathfrak{Re}(z) > 0, \\ \dfrac{2}{c_0} : & \text{for } z \in \overline{D}_{\pi/2},\ \mathfrak{Re}(z) \leq 0 \end{cases}$$

It is a consequence of the above inequality that $f(t, u, v)$ is an analytic function of t for $t \in D_{\pi/2}$. Moreover, for $z \in \overline{D}_{\pi/2}$:

$$(42) \quad \begin{aligned} &|e^{(1-\beta)z}\left((e^z I + A)^{-1}u, v\right)| \\ &\quad \leq \|u\|\|v\| \begin{cases} e^{-\beta\mathfrak{Re}(z)}/\sin(\pi/2 - \omega) & : & \text{for } \mathfrak{Re}(z) > 0, \\ \frac{2}{c_0}e^{(1-\beta)\mathfrak{Re}(z)} & : & \text{for } \mathfrak{Re}(z) \leq 0. \end{cases} \end{aligned}$$

This implies that

$$(43) \quad \begin{aligned} &\int_{-\infty}^{\infty} \left(|f(y - i\pi/2; u, v)| + |f(y + i\pi/2; u, v)|\right) dy \\ &\quad \leq \left(2(\beta\sin(\pi/2 - \omega))^{-1} + 4((1-\beta)c_0)^{-1}\right)\|u\|\|v\| := N(D_{\pi/2})\|u\|\|v\|. \end{aligned}$$

In addition, applying (42) gives

$$\int_{-\pi/2}^{\pi/2} |f(t + iy, u, v)| dy \leq C, \qquad \text{for all } t \in \mathbb{R}.$$

Hence, we can apply Theorem 2.20 of [36] to conclude that for $k > 0$,

$$(44) \quad \left|\int_{-\infty}^{\infty} f(y; u, v)\, dy - k \sum_{\ell=-\infty}^{\infty} f(\ell k; u, v)\right| \leq \frac{N(D_{\pi/2}; u, v)}{2\sinh(\pi^2/(4k))}e^{-\pi^2/(4k)}.$$

This leads to the following result for the sinc quadrature error.

Theorem 7.1 (sinc Quadrature Error). *For $N > 0$ and $k := 1/\sqrt{N}$, let $\mathcal{Q}_k^{-\beta}(\cdot)$ be defined by (6). Then for $s \in [0, 1)$, there exists a constant $C(\beta)$ independent of k*

such that

$$(45) \quad \|A^{-\beta} - \mathcal{Q}_k^{-\beta}(A)\|_{\dot{H}^s \to \dot{H}^s} \leq C(\beta) \left[\frac{N(D_{\pi/2})}{2\sinh(\pi^2/(4k))} e^{-\pi^2/(4k)} \right.$$
$$\left. + \frac{1}{\beta} e^{-\beta/k} + \frac{2}{(1-\beta)c_0} e^{-(1-\beta)/k} \right].$$

Similarly, there exists a constant $C(\beta)$ independent of k and h such that

$$(46) \quad \|(A_h^{-\beta} - \mathcal{Q}_k^{-\beta}(A_h))\pi_h\|_{\dot{H}^s \to \dot{H}^s} \leq C(\beta) \left[\frac{N(D_{\pi/2})}{2\sinh(\pi^2/(4k))} e^{-\pi^2/(4k)} \right.$$
$$\left. + \frac{1}{\beta} e^{-\beta/k} + \frac{2}{(1-\beta)c_0} e^{-(1-\beta)/k} \right].$$

Proof. Both estimates when $s = 0$ directly follow from (44) and the estimates

$$k \sum_{\ell=-N-1}^{-\infty} |f(\ell k; u, v)| \leq \frac{2}{(1-\beta)c_0} e^{-(1-\beta)/k} \|u\| \|v\|,$$

$$k \sum_{\ell=N+1}^{\infty} |f(\ell k; u, v)| \leq \frac{1}{\beta} e^{-\beta/k} \|u\| \|v\|,$$

(which are direct consequences of (42)).

For the case $s \in (0, 1)$, we first consider (45). Using Theorem 4.1 and the commutativity of A and $(e^y I + A)^{-1}$, we have

$$\|A^{-\beta} - \mathcal{Q}_k^{-\beta}(A)\|_{\dot{H}^s \to \dot{H}^s} \preceq \|A^{s/2}(A^{-\beta} - \mathcal{Q}_k^{-\beta}(A))A^{-s/2}\|.$$

Applying (13) shows that the right hand side above equals

$$\|A^{-\beta} - \mathcal{Q}_k^{-\beta}(A)\|$$

and so the desired estimate follows again from the $s = 0$ case.

For (46) when $s > 0$, we apply Lemma 5.2 to see that

$$\|(A_h^{-\beta} - \mathcal{Q}_k^{-\beta}(A_h))\pi_h\|_{\dot{H}^s \to \dot{H}^s} \leq \|A_h^{s/2}(A_h^{-\beta} - \mathcal{Q}_k^{-\beta}(A_h))A_h^{-s/2}\| \|\pi_h\|_{\dot{H}^s \to \dot{H}^s}$$
$$\preceq \|(A_h^{-\beta} - \mathcal{Q}_k^{-\beta}(A_h))\pi_h\|.$$

The last inequality followed from commutativity and the fact that π_h is a bounded operator on $\dot{H}^s$. The result now follows from the $s = 0$ case. $\qquad \square$

Remark 7.1 (Critical Case $s = 1$). *The quadrature error estimate still holds when $s = 1$ provided that the discrete and continuous Kato Square Theorems hold. These results are given by Remark 4.2 for the continuous operator A and Theorem 6.5 for the discrete operator A_h.*

Remark 7.2 (Exponential Decay). *The error from the three exponentials above can essentially be equalized by setting*

$$(47) \quad \mathcal{Q}_k^{-\beta}(A_h) := \frac{k \sin(\pi\beta)}{\pi} \sum_{\ell=-M}^{N} e^{(1-\beta)y_\ell} (e^{y_\ell} I + A_h)^{-1}$$

with

$$\pi^2/(2k) \approx 2(1-\beta)kM \approx 2\beta kN.$$

Thus, given $k > 0$, we set

$$M = \left\lceil \frac{\pi^2}{4(1-\beta)k^2} \right\rceil \quad and \quad N = \left\lceil \frac{\pi^2}{4\beta k^2} \right\rceil$$

and get the estimate

$$\|(A_h^{-\beta} - \mathcal{Q}_k^{-\beta}(A_h))\pi_h\|_{\dot{H}^s \to \dot{H}^s} \le C(\beta)\left[\frac{1}{2\beta} + \frac{1}{2(1-\beta)\lambda_0}\right]\left[\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)}\right].$$

We note that the right hand side above asymptotically behaves like

$$C(\beta)\left[\frac{1}{2\beta} + \frac{1}{2(1-\beta)\lambda_0}\right]e^{-\pi^2/(2k)}$$

as $k \to 0$.

This, together with the finite element approximation estimates provided in Theorem 6.2 and Remark 6.2, yields the fully discrete convergence estimate stated below.

Corollary 7.2 (Fully Discrete Convergence Estimate). *Suppose that (7) and (8) as well as the elliptic regularity Assumption 1 hold. Given $s \in [0, \frac{1}{2})$, set $\alpha_* := \frac{1}{2}(\alpha + \min(1 - 2s, \alpha))$ and $\gamma := \max(s + \alpha^* - \beta, s)$. For $\delta \ge \gamma$, then there exists a constant C independent of $k < 1$ and h such that*

$$\|(A^{-\beta} - \mathcal{Q}_k^{-\beta}(A_h)\pi_h)f\|_{\dot{H}^{2s}} \le C_{\delta,h}h^{2\alpha_*}\|A^{\delta}f\| + Ce^{-\pi^2/2k}\|f\|_{\dot{H}^{2s}},$$

$$\forall f \in D(A^{\delta}) \cap \dot{H}^{2s},$$

where $C_{\delta,h}$ is given by (31) and $\mathcal{Q}_h^{-\beta}$ is defined by (6).

In addition, if the continuous and discrete Kato Square Root Theorems hold (see Remark 4.2 and Theorem 6.5) then the above estimate also holds for $s = \frac{1}{2}$.

8. Numerical Illustrations for the Convection-Diffusion Problem

In order to illustrate the performance of the proposed algorithm, we set $\mathbb{V} = H_0^1(\Omega)$ with $\Omega = (0,1)^2$ and for $b \in \mathbb{R}$, consider the sesquilinear form

$$A(u,v) := \int_\Omega (\nabla u \cdot \nabla \bar{v} + b(u_x + u_y)\bar{v}), \quad \text{for all } u, v \in \mathbb{V}.$$

This form is regular and the corresponding regularly accretive operator A has domain $H^2(\Omega) \cap \mathbb{V}$.

In general, it is difficult to analytically compute solutions to $u = A^{-\beta}f$ although it is possible in this case. Indeed, we consider the Hermitian form:

$$\widetilde{S}(u,v) = \int_\Omega \left(\nabla u \cdot \nabla \bar{v} + \frac{b^2}{2}u\bar{v}\right), \quad \text{for all } u, v \in \mathbb{V}.$$

Fix $f \in L^2(\Omega)$ and for $\mu \ge 0$, let $w \in \mathbb{V}$ solve

$$\mu(w,\phi) + A(w,\phi) = (f,\phi), \quad \text{for all } \phi \in \mathbb{V}.$$

Putting $v = e^{-b(x+y)/2}w$ and $\phi = e^{-b(x+y)/2}\theta$ (for $\theta \in \mathbb{V}$) in the above equation and integrating by parts when appropriate, we see that $v \in \mathbb{V}$ is the solution of

$$\mu(v,\theta) + \widetilde{S}(v,\theta) = (e^{-b(x+y)/2}f,\theta), \quad \text{for all } \theta \in \mathbb{V},$$

i.e.,

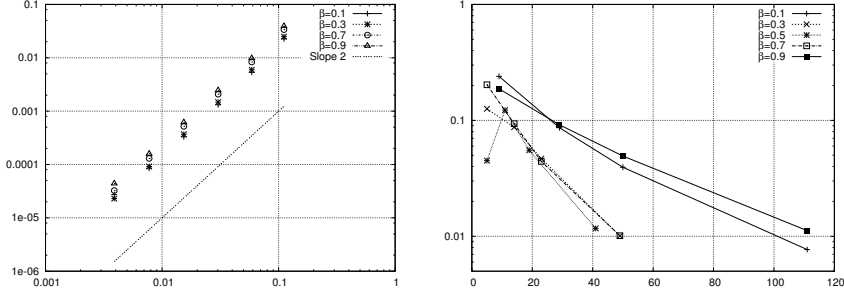$$e^{-b(x+y)/2}(\mu + A)^{-1}f = (\mu + \widetilde{S})^{-1}e^{-b(x+y)/2}f.$$

FIGURE 1. (Left) Decay of e_h^k versus the uniform mesh size h in a log-log plot. Second order rate of convergence is observed for all values of β. The number of quadrature points is taken large enough not to interfere with the spatial discretization error. (Right) Exponential decay of e_h^k as a function of the square root of number of points $M + N + 1$ in a semi-log plot (see Remark 7.2 for the definition of k). The spatial discretization is fixed and consists in 10 uniform refinements of Ω.

Substituting this into (2) shows that

$$A^{-\beta}f = e^{b(x+y)/2}\widetilde{S}^{-\beta}(e^{-b(x+y)/2}f).$$

Since, for example, $\sin(\pi x)\sin(2\pi y)$ is an eigenvector of $\widetilde{S}$ with eigenvalue $5\pi^2 + b^2/2$,

$$u := A^{-\beta}f = e^{b(x+y)/2}(5\pi^2 + b^2/2)^{-\beta}\sin(\pi x)\sin(2\pi y)$$

when

(48) $$f = e^{b(x+y)/2}\sin(\pi x)\sin(2\pi y).$$

The space discretization consists of continuous piecewise bi-linear finite element subordinate to successive quadrilateral refinements of Ω. Figure 1 provides the behavior of the errors $e_h^k := \|(A^{-\beta} - Q_k^{-\beta}(A_h)\pi_h)f\|$ when the advection coefficient is given by $b = 1$ and f is given by (48) . For a fixed number of quadrature points $e_h^k \sim h^2$ while for a fixed spatial resolution e_h^k is exponential decaying. This is in agreement with the estimate provided in Corollary 7.2.

We now set $b = 10$, $f \equiv 1$ and study the boundary layer inherent to convection-diffusion problems. For this, we consider 8 successive quadrilateral refinements of Ω for the space discretization. In particular, the corresponding mesh size $h := 2^{-8}$ is fine enough for the Galerkin representation not to require any stabilization. The value of the approximations $A_h^{-\beta}1$ for $\beta = 0.1, 0.3, 0.5, 0.7, 0.9$ over the segment joining the points $(0,0)$ and $(1,1)$ are plotted in Figure 8 together with the graphs of $5A_h^{-\beta}1$ for $\beta = 0.1$ and $\beta = 0.9$. The results indicate that the width of the boundary layer for convection-diffusion problems remains proportional to the ratio diffusion / convection and is therefore independent of β. However, its intensity decreases with increasing β.
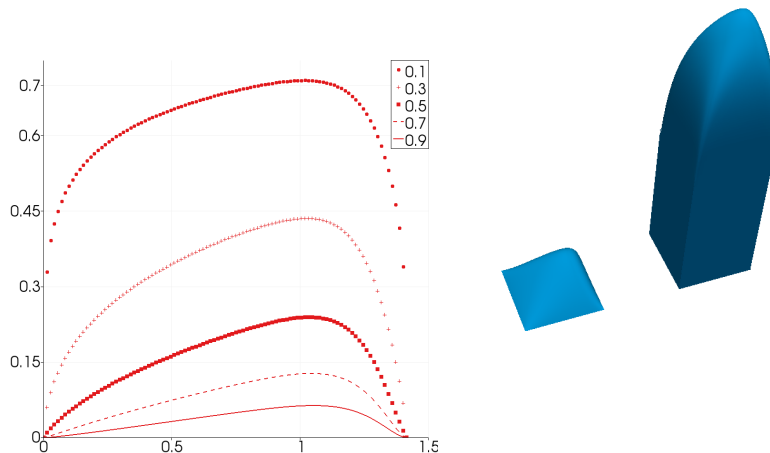
FIGURE 2. Approximations of $A^{-\beta}1$ on a subdivision of the unit square using 4^8 quadrilaterals and 401 quadrature points. The width of the boundary layer for convection-diffusion problems appears independent of β while its intensity decreases with increasing β. (Left) Plots over the segment joining the $(0,0)$ and $(1,1)$ for $\beta = 0.1, 0.3, 0.5, 0.7, 0.9$. (Right) Approximations scaled by a factor 5 for $\beta = 0.9$ and $\beta = 0.1$.

Acknowledgment

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