Numerical Methods
MATH 417 - 501
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First Name: ____________________ **Last Name:** ____________________

Homework 2

Exercise 1 **45%**

The goal of this exercise is to write a matlab code based on the Vandermonde approach to determine the polynomial p_n of degre n such that

$$p_n(x_i) = y_i, \qquad i = 0, ..., n$$

for given distinct x_i's and y_i's. Recall that is p_n is given by $p_n(x) = a_0 + a_1 x + ... + a_n x^n$, the coefficients a_i solve the following system

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Before proceeding further, we need a routine able to efficiently evaluate a polynomial $p_n(x) = a_0 + a_1 x + ... + a_n x^n$ for different values of x. This is the purpose of *POLYEVAL* below which takes advantage of the Horner representation

$$p_n(x) = (...((a_n x + a_{n-1})x + a_{n-2})...)x - a_0$$

to limit the amount of multiplications performed
which can be downloaded here

`http://www.math.tamu.edu/~bonito/Teaching/MTH417/POLYEVAL.m`

1. Familiar yourself with the Horner representation of polynomials and understand the routine *POLYEVAL*.

2. Write a matlab m-file which solve the above system. Its first line should read

   ```
   function [a] = VANDERMONDE(x,y)
   ```

   where x and y are columns arrays of the same dimension. Use the matlab *backslash* solver

   ```
   a = V\y
   ```

   to solve the linear system

   ```
   Va = y
   ```

   where $V \in \mathbb{R}^{n \times n}$ and $a, y \in \mathbb{R}^n$.

3. Let $x_0 = 0$, $x_1 = 0.6$ and $x_2 = 0.9$. Use your algorithm to construct interpolation polynomial p_2 to approximate $f(0.45)$ for (a) $f(x) = \cos(x)$, (b) $f(x) = \sqrt{1 + x}$, $f(x) = \tan(x)$. For each case, compute

   $$|p_2(0.45) - f(0.45)|.$$

Exercise 2 **45%**

This exercise explores the Runge interpolation paradox.
Consider the function

$$f(x) = \frac{1}{1 + 25x^2}$$

on the interval $[-1, 1]$. Use the following *matlab* commands to see how the function looks like

```
z = linspace(-1,1,200);
yexact = 1./(1+25.*z.^2);
plot(z,yexact)
```

We propose to construct interpolations of this function using equally spaced interpolation points on $[-1, 1]$. Given $N + 1$ interpolation points, define the interpolations nodes to be equally distributed in the interval $[-1, 1]$

```
x = linspace(-1,1,N+1);
```

Use the *matlab* commands *polyfit* to construct the associated interpolation polynomial of degree N (your *VANDERMONDE* implementation does the same job except that *polyfit* returns $a = (a_n, a_{n-1}, ..., a_1, a_0)^t$)

```
a = polyfit(x,1./(1+25.*x.^2),N);
```

In order to compare the interpolation polynomial with the function f, we also need to compute the value of the interpolation polynomial at the points z defined above. To do this, you can use the matlab command *polyval*

```
yinter = polyval(a,z);
```

Again, notice that *polyval* does the same as *POLYEVAL* in the previous exercise except that the coefficients a are flipped, i.e. *polyval* expect the coefficient in the following order $a = (a_n, a_{n-1}, ..., a_0)^t$. This matches the output from *polyfit*. Finally, the commands

```
plot(z,yexact,z,yinter);
legend('exact','interpolation');
```

graph the exact function and its interpolation on the same plot.

1. Chose $N = 2, 5, 10, 15, 20, 30$ and graph for each case the function f and interpolation polynomial on the same plot.

2. What do you conclude? explain why using the error bound on polynomial approximations discussed in class.

3. Instead of equi-distributed interpolation points x, define

$$x_i = -1 + \left(1 + \cos\left(\frac{(2i+1)\pi}{2(N+1)}\right)\right), \qquad \text{for } i = 0, ..., N.$$

   Perform the same computations with these points and return the corresponding graphs. Notice where the interpolation points lie in the interval $[-1, 1]$. What do you conclude?

4. Instead of global interpolation, define the piecewise linear interpolant. For given N set $x_i := -1 + 2i/N$ and defined $P_1^N(x)$ on each interval $[x_i, x_{i+1}]$ by

$$P_1^N(x)|_{[x_i,x_{i+1}]} := f(x_i) + \frac{x - x_i}{x_{i+1} - x_i}(f(x_{i+1}) - f(x_i)).$$

   Chose $N = 2, 5, 10, 15, 20, 30$ and return the corresponding graphs. What do you conclude?

Exercise 3 **10%**

Pen and paper homework from book:
Section 2.1: 1,14.
Section 2.3: 1.