First Name: _____   **Last Name:** _____

Homework 5

Exercise 1   60% (MATLAB)

Create two matlab routines, which do composite Simpson and Gauss-Legendre approximation to an integral using equidistant points.

For the Gauss-Legendre approximation use the points $t_0 = -\frac{\sqrt{3}}{3}$, $t_1 = \frac{\sqrt{3}}{3}$ and corresponding weights $w_0 = w_1 = 1$. The interfaces should look like

```
function r=Simpsons(a,b,N,f)
function r=GaussLegendre(a,b,N,f)
```

Here a, b are the limits of integration, N is the number of subintervals and f is the integrand.

1. Apply your composite Simpson and Gauss-Legendre code to approximate

$$\int_0^1 \frac{dx}{(1+x)^\alpha}.$$

   Run your code for $N = 10, 20, 40, 80, 160, 320$ and $\alpha = 1/4, 1/2, 3/4$. Report the error obtained in each case (compared with the answer obtained using calculus applied to the integral over $(0, 1)$). If your code is working correctly, you will be able to see the 4th order global convergence, i.e., when you multiply N by two, the error should decrease roughly by a factor of 16. To see this more clearly, for each α plot the error versus $1/N$ in log-log scales.

2. Apply your composite Gauss-Legendre code to approximate the improper integral

$$\int_0^1 \frac{dx}{x^\alpha}.$$

   The simplest way to do this is to skip the first interval (the one containing 0), i.e., add the contributions from $[h, 2h], [2h, 3h], ..., [(N-1)h, 1]$. Here $h = 1/N$. Report the errors (compared with the answer obtained using calculus applied to the integral over $(0, 1)$) for the same values of N and α as above.

3. The uniform mesh on the improper integral does not do a very good job. The convergence can be improved by using a non-uniform quadrature node spacing (discussed in recitation). We shall take a different approach, namely, we first apply the change of variable $x = e^{-y}$ to obtain

$$\int_0^1 \frac{dx}{x^\alpha} = \int_0^\infty e^{(\alpha-1)y} dy.$$

   We shall use the equally spaced composite approximation to the integral on the right hand side. The first step is to truncate the interval,

$$\int_0^\infty e^{(\alpha-1)y} dy \approx \int_0^M e^{(\alpha-1)y} dy$$

and subsequently apply the equally spaced quadrature scheme on the truncated interval (M defined below). The idea is to try to balance the integral truncation error with the 4th order error of the quadrature scheme. The truncation error behaves like $e^{(\alpha-1)M}$ (check it). To attempt to equalize the number of steps for the same reduction (independent of α), we set $h = ((1-\alpha)N)^{-1}$ with N being our interval parameter. To equalize the truncation and 4th order error, we set M by $e^{(\alpha-1)M} = h^4$. Solving this, we find that we should take L steps (of size h) with L given by

$$4\ln(h^{-1}) = (1-\alpha)M = (1-\alpha)hL = L/N.$$

This formula makes sense only when $h < 1$ (and, of course, L needs to be an integer). Thus, we take

$$L = \left\lceil \max(4N, 4N\ln(h^{-1})) \right\rceil.$$

Run $\alpha = 1/4, 1/2, 3/4$ but use $N = 4, 8, 16, 32, 64$ with L steps of size h. The above scheme increases the number of intervals only logarithmically but asymptotically recovers 4th order convergence. Again report the error (comparing with the calculus result for the integral on $(0,1)$) and L for each combination of N and α.

Exercise 2   40% (BY HAND)

1. Approximate the following integrals using the trapezoidal rule.

   - $\int_{-0.25}^{0.25} \cos(x)^2 dx$.
   - $\int_{-0.5}^{0} x \ln(x+1) dx$.
   - $\int_{0.75}^{1.3} \left(\sin(x)^2 - 2x\sin(x) + 1 \right) dx$.
   - $\int_{e}^{e+1} \frac{dx}{x \ln x}$.

2. Find a bound for the error in all the above integrals using the error formula and compare this with the actual error.