Homework 4

Exercise 1 **40%**

Let $\widehat{K}$ be the reference triangle in $\mathbb{R}^2$ and K be a triangle in $\mathbb{R}^d$, $d \geqslant 1$. Let $h = \mathrm{diam}(K)$ and $F : \widehat{K} \to K$ be an affine tranformation.
Show that there exists a constant c such that for any $\hat{v} \in H^1(\widehat{K})$ there holds

$$\|\hat{v}\|_{L^2(\partial\widehat{K})} \leqslant c\left(\|\hat{v}\|_{L^2(\widehat{K})} + \|\widehat{\nabla}\hat{v}\|_{L^2(\widehat{K})}\right).$$

Deduce that there exists a constant c independent of h such that for any $v \in H^1(K)$,

$$\|v\|_{L^2(\partial K)} \leqslant c\left(h^{-1/2}\|v\|_{L^2(K)} + h^{1/2}\|\nabla v\|_{L^2(K)}\right).$$

Exercise 2 **40%**

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, $\beta \in C^0(\overline{\Omega})^2$, $\mu \in C^0(\overline{\Omega})$ and $f \in L^2(\Omega)$. Assume that for some positive constants μ_0, μ_1, β_1, there holds $0 < \mu_0 \leqslant \mu(x) \leqslant \mu_1$ and $|\beta(x)| \leqslant \beta_1$ for all $x \in \Omega$. The outward pointing unit normal to Ω is denoted by $\boldsymbol{\nu}$ (defined a.e.) and we consider a decomposition of the domain boundary $\Gamma := \partial\Omega$ in two disjoint pieces $\Gamma = \Gamma_1 \cup \Gamma_2$. In addition, we suppose that $\mathrm{div}(\beta) = 0$ and $\beta \cdot \boldsymbol{\nu} \geqslant 0$ on Γ_2. We propose to analyze a *stabilized* numerical method for the following convection-diffusion problem: Seek $u : \Omega \to \mathbb{R}$ such that

$$-\mathrm{div}(\mu\nabla u) + \beta \cdot \nabla u = f, \qquad \text{in } \Omega$$

together with the boundary conditions

$$u = 0 \qquad \text{on } \Gamma_1 \qquad \text{and} \qquad \mu\frac{\partial}{\partial\nu}u = 0 \qquad \text{on} \Gamma_2.$$

1. Derive an adequate weak formulation of the above problem: Seek $u \in X$ such that

$$a(u, v) = F(v), \qquad \forall v \in X.$$

2. Show that for all $v, w \in X$, there exist a constant $M > 0$ such that

$$a(v, v) \geqslant \mu_0\|\nabla v\|^2_{L^2(\Omega)} \qquad |a(w, v)| \leqslant M\|\nabla w\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}$$

   and deduce the existence and uniqueness of a weak solution.

3. Given a shape-regular, quasi-uniform triangulation $\mathcal{T}_h$ of Ω (h denotes the largest outer circle diameter), we set for $k \geqslant 1$

$$\mathbb{V}_h := \left\{v_h \in C^0(\overline{\Omega}) \mid v_h|_T \in \mathbb{P}^k(T), \quad \forall T \in \mathcal{T}_h\right\} \cap X$$

   and define the approximate bilinear form on $\mathbb{V}_h \times \mathbb{V}_h$

$$a_h(v_h, w_h) := a(v_h, w_h) + \alpha h \int_\Omega \nabla w_h \cdot \nabla v_h, \qquad \forall w_h, v_h \in \mathbb{V}_h.$$

Consider the finite element approximation $u_h \in \mathbb{V}_h$ of $u \in X$ defined as satisfying

$$a_h(u_h, v_h) = F(v_h) \qquad \forall v_h \in \mathbb{V}_h.$$

Show that $a_h(v_h, v_h) \geqslant \mu_h \|\nabla v_h\|^2_{L^2(\Omega)}$ with $\mu_h := \mu_0 + \alpha h$. Deduce the existence and uniqueness of $u_h \in \mathbb{V}_h$.

4. Show that for all $v_h \in \mathbb{V}_h$

$$\mu_h \|\nabla(v_h - u_h)\|^2_{L^2(\Omega)} \leqslant a_h(v_h, v_h - u_h) - F(v_h - u_h),$$

deduce that

$$\|\nabla(v_h - u_h)\|_{L^2(\Omega)} \leqslant \frac{1}{\mu_h} \sup_{w_h \in \mathbb{V}_h} \frac{|a_h(v_h, w_h) - a(v_h, w_h)|}{\|\nabla w_h\|_{L^2(\Omega)}} + \frac{M}{\mu_h} \|\nabla(v_h - u)\|_{L^2(\Omega)}$$

and therefore

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leqslant \inf_{v_h \in \mathbb{V}_h} \left\{ \frac{1}{\mu_h} \sup_{w_h \in \mathbb{V}_h} \frac{|a_h(v_h, w_h) - a(v_h, w_h)|}{\|\nabla w_h\|_{L^2(\Omega)}} + \left(1 + \frac{M}{\mu_h}\right) \|\nabla(v_h - u)\|_{L^2(\Omega)} \right\}.$$

(Strang's lemma).

5. Prove that for all $v_h \in \mathbb{V}_h$, there holds

$$\sup_{w_h \in \mathbb{V}_h} \frac{|a_h(v_h, w_h) - a(v_h, w_h)|}{\|\nabla w_h\|_{L^2(\Omega)}} \leqslant \alpha h \|\nabla v_h\|_{L^2(\Omega)}.$$

6. Recall that there exists a constant C such that (interpolation estimates)

$$\|\nabla(I_{\mathcal{T}_h} u - u)\|_{L^2(\Omega)} \leqslant C_1 h^k |u|_{H^{k+1}(\Omega)}$$

provided $u \in H^{k+1}(\Omega)$ (which is assumed from now on). (OPTIONAL) Deduce that there exist a constant C_2 such that

$$\|\nabla(I_{\mathcal{T}_h} u)\|_{L^2(\Omega)} \leqslant \|\nabla u\|_{L^2(\Omega)} + C_2 h^k |u|_{H^{k+1}(\Omega)}.$$

7. Show the existence of a constant C_3 such that

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leqslant C_3 \left\{ \left(1 + \frac{M}{\mu_h}\right) |u|_{H^{k+1}(\Omega)} h^k + \frac{\alpha}{\mu_h} \|\nabla u\|_{L^2(\Omega)} h \right\}.$$

Exercise 3 **20%**

Let $K = [0,1]^2$ be the unit square and denote by q_i, $i = 1, ..., 4$, its vertices and by a_i, $i = 1, .., 4$, the midpoints of its sides. Set $\mathcal{P} = \mathbb{Q}^1 := \{p(x,y) = (ax+b)(cy+d) \ : \ a,b,c,d \in \mathbb{R}\}$ be the space of polynomial of degree at most 1 in each direction.

1. For $\mathcal{N} := \{N_1, N_2, N_2, N_4\}$, where $N_i(p) = p(q_i)$, $i = 1, ..., 4$, show that the finite element triplet $(K, \mathcal{P}, \mathcal{N})$ is unisolvent.

2. For $\widetilde{\mathcal{N}} := \left\{\widetilde{N}_1, \widetilde{N}_2, \widetilde{N}_3, \widetilde{N}_4\right\}$, where $\widetilde{N}_i(p) = p(a_i)$, $i = 1, ..., 4$, show that the finite element triplet $(K, \mathcal{P}, \widetilde{\mathcal{N}})$ is **not** unisolvent.