Ordinary Differential Equations
MATH 308H - 201
A. Bonito

April 21
Spring 2019

First Name: ____________________ **Last Name:** ____________________

Exam 2

- 75 minute individual exam;
- Answer the questions in the space provided. If you run out of space, continue onto the back of the page. Additional space is provided at the end;
- **Show and explain all work**;
- **Underline** the answer of each steps;
- The use of books, personal notes, **calculator**, cellphone, laptop, and communication with others is forbidden;
- By taking this exam, you agree to follow the university's code of academic integrity.

Ex 1	Ex 2	Ex 3	Ex 4	Total

Some Laplace Transforms

f	$\mathcal{L}(f)$		f	$\mathcal{L}(f)$	
1	$\frac{1}{s}$	$s>0$	$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$	$s>0$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$	$s>-\alpha$	$e^{-\alpha t}\ t^n$	$\frac{n!}{(s+\alpha)^{n+1}}$	$s>-\alpha$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s>0$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s>0$
$e^{\alpha t}\sin(\omega t)$	$\frac{\omega}{(s-\alpha)^2+\omega^2}$	$s>\alpha$	$e^{\alpha t}\cos(\omega t)$	$\frac{s-\alpha}{(s-\alpha)^2+\omega^2}$	$s>\alpha$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$s>\lvert\omega\rvert$	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$s>\lvert\omega\rvert$
$H_\alpha(t)$	$\frac{e^{-\alpha s}}{s}$	$s>0$	$\delta_\alpha(t)$	$e^{-\alpha s}$	$s>-\infty$

Some Properties of the Laplace Transforms

Let $f,g:[0,+\infty)\to\mathbb{R}$ be piecewise continuous functions with piecewise continuous derivatives. Assume there exists $K\geqslant 0$ and $a_1,\ a_2\in\mathbb{R}$ such that

$$|f(t)|\leqslant Ke^{a_1t},\qquad |g(t)|\leqslant Me^{a_2t},\qquad \forall t\in[0,+\infty).$$

Then there holds

$(\imath.)$ $\mathcal{L}\left(\frac{d^n}{dt^n}f(t)\right)(s)=s^n\mathcal{L}\left(f(t)\right)-s^{n-1}f(0)-...-s\frac{d^{n-2}}{dt^{n-2}}f(0)-\frac{d^{n-1}}{dt^{n-1}}f(0),$
$\forall s>a_1,\ (f\in C^{n-1}([0,\infty)),\ \frac{d^n}{dt^n}f$ piecewise continuous)

$(\imath\imath.)$ $\mathcal{L}\left(\int_0^t f(\tau)d\tau\right)(s)=\frac{1}{s}\mathcal{L}\left(f(t)\right)(s),\qquad \forall s>a_1,$

$(\imath\imath\imath.)$ $\mathcal{L}\left((-1)^nt^nf(t)\right)(s)=\frac{d^n}{ds^n}\mathcal{L}\left(f(t)\right)(s),\qquad \forall s>a_1,$

$(\imath v.)$ $\mathcal{L}\left(e^{-\alpha t}f(t)\right)(s)=\mathcal{L}\left(f(t)\right)(s+\alpha),\qquad \forall s>a_1+\alpha,\ \alpha\geqslant 0,$

$(v.)$ $\mathcal{L}\left(H_\alpha(t)f(t-\alpha)\right)(s)=e^{-\alpha s}\mathcal{L}\left(f(t)\right)(s),\qquad \forall s>a_1,\ \alpha\geqslant 0,$

$(v\imath.)$ $\mathcal{L}\left((f*g)(t)\right)(s)=\mathcal{L}(f(t))(s)\cdot\mathcal{L}(g(t))(s),\qquad \forall s>\max(a_1,a_2).$

<u>Exercise 1</u> **25%**

Find the solution of

$$y'' + 2y' + 3y = 5\delta_{3\pi}, \qquad y(0) = y'(0) = 0.$$

Exercise 2 **25%**

- Solve the following system

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

  and draw the phase portrait.

- Among the solution found in the previous step, determine the one satisfying

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

  and plot the evolution of $y_1(t)$ vs t.

Exercise 3 **25%**

Determine the critical point $\overline{\mathbf{y}}$ of the following system. Use the change of variable $\mathbf{y} = \mathbf{z} - \overline{\mathbf{y}}$, find the eigenvalues of the resulting system and deduce whether $\overline{\mathbf{y}}$ is stable or not.

$$\frac{d}{dt}\mathbf{y} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{y} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Exercise 4 **25%**

Find the general solution to the following ODE

$$y'' + y = (1 + x)\sin(x)$$

and graph the evolution of $y(x)$ for large values of $x > 0$.

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Exam 2: solutions

Exercise 1 **25%**

Taking the Laplace transform yields

$$(s^2+2s+3)Y = 5e^{-3\pi s}$$

or

$$Y = \frac{5}{s^2+2s+3}e^{-3\pi s}.$$

We first find the Laplace transform inverse of

$$F(s) := \frac{5}{s^2+2s+3} = \frac{5}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2},$$

which is given by

$$\mathcal{L}^{-1}(F)(t) = \frac{5}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t).$$

Hence, using the formula table we find that the Laplace transform inverse of $F(s)e^{-cs})$ is

$$\mathcal{L}^{-1}(F)(t-c)u_c(t)$$

and we conclude that

$$y(t) = \frac{5}{\sqrt{2}}e^{-(t-3\pi)}\sin(\sqrt{2}(t-3\pi))u_{3\pi}(t).$$

Exercise 2 **25%**

- We find the eigenvalues-eigenvectors of the matrix. The eigenvalues are found solving

  $$\det\begin{pmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix} = 0,$$

  i.e.

  $$\lambda^2+2\lambda+5=0$$

  or $\lambda = -1 \pm 2i$.

  We only consider $\lambda = -1+2i$ and find the associated eigenvector:

  $$\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = (-1+2i)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

  or

  $$2i\xi_1 + 4\xi_2 = 0.$$

  Hence all eigenvectors are given by

  $$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \alpha\begin{pmatrix} 2 \\ -i \end{pmatrix}, \qquad \forall\alpha\neq 0.$$

As a consequence two linearly independent solution are given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \Re\left(e^{(-1+2i)t}\begin{pmatrix} 2 \\ -i \end{pmatrix}\right) = e^{-t}\begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}$$

and

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \Im\left(e^{(-1+2i)t}\begin{pmatrix} 2 \\ -i \end{pmatrix}\right) = e^{-t}\begin{pmatrix} 2\sin(2t) \\ -\cos(2t) \end{pmatrix}.$$

Gathering the above results, we find that the general solution reads

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{-t}\left(C_1\begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix} + C_2\begin{pmatrix} 2\sin(2t) \\ -\cos(2t) \end{pmatrix}\right).$$

The phase portrait is provided in Fig. 1.

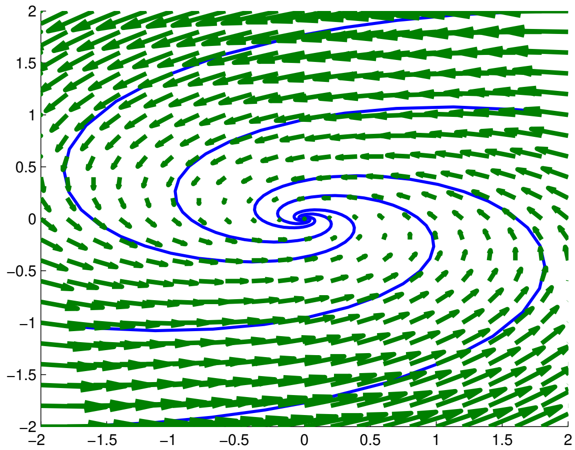

Figure 1: Phase portrait

- We use the provided initial condition to solve for the constants C_1 and C_2:

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2C_1 \\ -C_2 \end{pmatrix}$$

to deduce $C_1 = 1$ and $C_2 = 0$. Hence

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{-t}\begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}.$$

The plot of $y_1(t)$ is provided in Fig. 2.

Exercise 3 **25%**

The critical points are defined as the solution to

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{y} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This means

$$y_1 + y_2 = 2 \qquad \text{and} \qquad y_1 - y_2 = 0$$

or

$$y_1 = y_2 = 1.$$

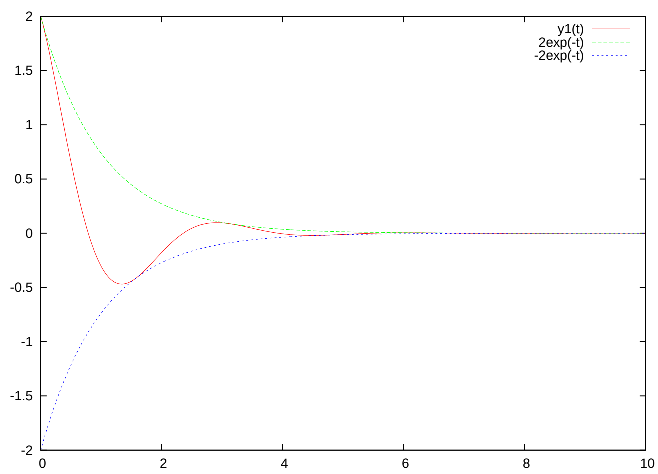

Figure 2: y_1 vs t.

Using the change of variable $\mathbf{y} = \mathbf{z} - (1,1)^t$ one gets

$$\frac{d}{dt}\mathbf{z} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{z}$$

so that the stability of $\overline{\mathbf{y}}$ depends on eigenvalues of the above matrix.
The eigenvalues satisfy

$$\lambda^2 - 2 = 0$$

or

$$\lambda = \pm\sqrt{2}.$$

The eigenvalues having different signes, we conclude that the critical point is unstable.

Exercise 4 **25%**

We first consider the homogeneous equation by solving the characteristic equation

$$\lambda^2 + 1 = 0.$$

This is $\lambda = \pm i$. Therefore two linearly independent solutions of the homogeneous equation are given by

$$y_1(x) = \mathrm{Re}(e^{ix}) = \cos(x), \qquad y_2(x) = \mathrm{Im}(e^{ix}) = \sin(x).$$

We now guess a particular solution of the form

$$y_p(x) = \mathrm{Im}(z_p(x)), \qquad \text{where} \qquad z_p(x) = w_p(x)e^{ix}.$$

Pluging $z_p(x)$ into the ODE we get

$$w_p'' + 2iw_p' = (1+x), \tag{1}$$

leading to the educated guess for $w_p(x)$

$$w_p(x) = Ax^2 + Bx$$

for some constants A and B. Pluging $w_p(x)$ in (1) yields

$$A = -\frac{i}{4}, \qquad B = \frac{1}{4} - \frac{i}{2}.$$

Therefore,

$$y_p(x) = \mathrm{Im}(z_p(x)) = \mathrm{Im}\left(\left(-\frac{i}{4}x^2 + (\frac{1}{4} - \frac{i}{2})x\right)(\cos(x) + i\sin(x))\right)$$
$$= \frac{1}{4}x\sin(x) - \left(\frac{x^2}{4} + \frac{x}{2}\right)\cos(x).$$

and

$$y(x) = C_1\cos(x) + C_2\sin(x) + \frac{1}{4}x\sin(x) - (\frac{x^2}{4} + \frac{x}{2})\cos(x),$$

for some constants C_1 and C_2.
For large values of x the solution $y(x)$ looks like

$$y(x) \approx -\frac{x^2}{4}\cos(x).$$

The graph is provided in Fig. 3.

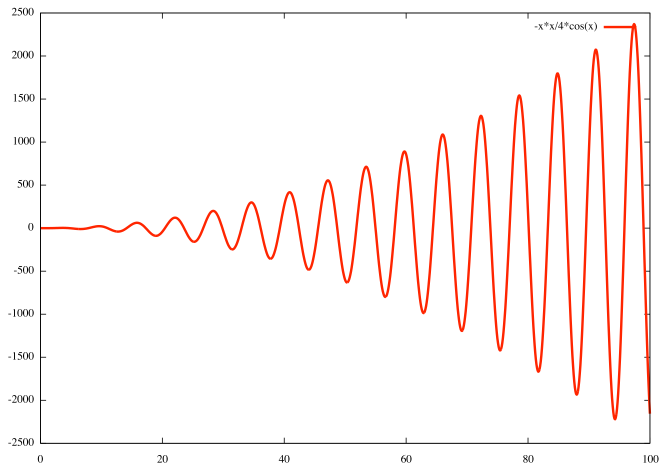

Figure 3: graph of $y(x) = -\frac{x^2}{4}\cos(x)$.