Numerical Methods
MATH 417 - 502
A. Bonito

March 10
Spring 2016

**First Name:** ____________________ **Last Name:** ____________________

# Exam 1

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- 75 minute individual exam;
- Answer the questions in the space provided. If you run out of space, continue onto the back of the page. Additional space is provided at the end;
- **Show and explain all work**;
- **Underline** the answer of each steps;
- The use of books, personal notes, **calculator**, cellphone, laptop, and communication with others is forbidden;
- By taking this exam, you agree to follow the university's code of academic integrity.

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| Ex 1 | Ex 2 | Ex 3 | Ex 4 | Total |
|---|---|---|---|---|
| | | | | |

## Exercise 1 **10%**

Compute the first 3 steps of the newton method to find the approximate $2^{1/4}$ starting with the value 1.

## Exercise 2 **30%**

Consider the second difference approximation

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

of $f''(x)$. Recall that there exists $\xi \in [x-h, x+h]$ such that

$$f''(x) - \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = -f^{(4)}(\xi)\frac{h^2}{12}$$

- Assume that every evaluation $f(y)$ is perturbed by the roundoff error

  $$\bar{f}(y) = f(y) + e(y)$$

  where $|e(y)| \leqslant \epsilon$. Determine an error bound for

  $$\left| f''(x) - \frac{\bar{f}(x+h) - 2\bar{f}(x) + \bar{f}(x-h)}{h^2} \right|$$

  for any $x \in [a+h, b-h]$ provided $\max_{s\in[a,b]} |f^{(4)}(s)| \leqslant M$.

- Determine the optimal value of $h$ (as a function of $\epsilon$) which minimizes the error and deduce the smallest error achievable.

## Exercise 3 **30%**

Given $-1 < \alpha < 0$ and $\omega_0, \omega_1 \in \mathbb{R}$. Consider the quadrature

$$J(g) = \omega_0 g(\alpha) + \omega_1 g(1)$$

to approximate

$$\int_{-1}^{1} g(t)dt.$$

1. Find $\omega_0$ and $\omega_1$ such that the quadrature is exact for polynomials of degree $\leqslant 1$.
2. Find $\alpha$ such that the quadrature is exact for polynomials of degree 2.
3. Is the quadrature exact for polynomial of degree 3?
4. Approximate the integral
$$\int_{-1}^{3} (t^2 - 1)dt$$
using the quadrature you discovered in the previous item.
5. How close is your approximate value from the exact integral?

## Exercise 4 **30%**

Replace the symbols $W, X, Y, Z$ in the matlab code below to solve the linear system

$$Ax = b$$

when $A$ is tri-diagonal, i.e. the coefficients satisfy $a_{ij} = 0$ whenever $|i - j| > 1$.

```
%%%% GaussTri %%%%
%% Input:  tridiagonal square matrix A (not checked)
%%         vector b of corresponding size (not checked)
%% Output: solution to Ax=b

function  x=GaussTri(A,b)

N=size(A,1);

for i=1:N-1
        %multiply the ith row by pivot
        p=1/A(i,i);
        A(i,W) = p*A(i,X);
        b(i) = p*b(i);

        % eliminate the ith column
        A(i+1,i+1)=A(i+1,i+1)-A(Y,i)*A(i,Y);
        b(i+1) = b(i+1)-A(Z,i)*b(i);
end

%last step
p = 1/A(N,N);
b(N)=p*b(N);

%once the matrix is upper triangular (with one on the diagonal)
%solve (the solution is stored in b)
for i=N-1:-1:1
         b(i)=b(i)-A(i,i+1)*b(i+1);
end

x=b;
%%%% END %%%%
```

Apply the above algorithm to find the solution of

$$Ax = b,$$

where

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Specify ALL steps within the algorithm.

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# Exam 1: solutions

## Exercise 1 **10%**

The newton iterates are given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

Here $f(x) = x^4 - 2$ and $f'(x) = 4x^3$. Starting with $x_0 = 1$, this yields $x_1 = \frac{5}{4}$, $x_2 = \frac{5}{4} - \frac{(5/4)^4-2}{4(5/4)^3}$.

## Exercise 2 **30%**

Using a triangle inequality, we find that

$$\left| f''(x) - \frac{\bar{f}(x+h) - 2\bar{f}(x) + \bar{f}(x-h)}{h^2} \right| \leqslant \frac{1}{12}Mh^2 + \frac{4\epsilon}{h^2} =: g(h).$$

The parameter $h$ leading leading to the minimal value is characterized by $g'(\overline{h}) = 0$, i.e.

$$\overline{h} = \left(\frac{48\epsilon}{M}\right)^{1/4}.$$

This corresponds to the value

$$g(\overline{h}) = \frac{1}{6}M^{1/2}(48\epsilon)^{1/2}.$$

## Exercise 3 **30%**

1. The weights are given by the integral of the Lagrange basis, i.e.

$$\omega_0 = \int_{-1}^{1} \frac{t-1}{\alpha-1}dt = \frac{2}{1-\alpha}$$

and

$$\omega_1 = \int_{-1}^{1} \frac{t-\alpha}{1-\alpha}dt = -\frac{2\alpha}{1-\alpha}.$$

This leads to a quadrature exact for polynomial of degree $\leqslant 1$.

2. To guarantee that the quadrature is exact for polynomial of degree $\leqslant 2$, we find $\alpha$ such that

$$\frac{2}{3} = \int_{-1}^{1} t^2 dt = \frac{2}{1-\alpha}(\alpha)^2 - \frac{2\alpha}{1-\alpha},$$

i.e.

$$\alpha = -\frac{1}{3}$$

and therefore

$$\omega_0 = \frac{3}{2}, \qquad \omega_1 = \frac{1}{2}.$$

3. This quadrature is not exact for polynomial of degree $\leqslant 3$ since

$$0 = \int_{-1}^{1} t^3 dt \neq \frac{3}{2}(-1/3)^3 + \frac{1}{2}.$$

4. To apply the above quadrature formula, we first set $t = 2s + 1$ so that

$$\int_{-1} 3(t^2 - 1)dt = 2\int_{-1}^{1} ((2s+1)^2 - 1)ds.$$

Applying the above quadrature, we find

$$\int_{-1} 3(t^2 - 1)dt = 2\left(\frac{3}{2}\left(\left(\frac{1}{3}\right)^2 - 1\right) + \frac{1}{2}(9-1)\right) = \frac{16}{3}.$$

5. The numerical approximation is exact since applied to a polynomial of degree 2.

## Exercise 4 **30%**

```
%%%% GaussTri %%%%
%% Input:  tridiagonal square matrix A (not checked)
%%         vector b of corresponding size (not checked)
%% Output: solution to Ax=b

function  x=GaussTri(A,b)

N=size(A,1);

for i=1:N-1
        %multiply the ith row by pivot
        p=1/A(i,i);
        A(i,i+1) = p*A(i,i+1);
        b(i) = p*b(i);

        % eliminate the ith column
        A(i+1,i+1)=A(i+1,i+1)-A(i+1,i)*A(i,i+1);
        b(i+1) = b(i+1)-A(i+1,i)*b(i);
end

%last step
p = 1/A(N,N);
b(N)=p*b(N);

%once the matrix is upper triangular (with one on the diagonal)
%solve (the solution is stored in b)
for i=N-1:-1:1
         b(i)=b(i)-A(i,i+1)*b(i+1);
end

x=b;
%%%% END %%%%
```

We now apply the above algorithm to solve the system

$$Ax = b,$$

where

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The successive matrices and righthand sides are

$$(1) \quad \left(\begin{array}{cc|c} 2 & -1 & 1 \\ -1 & 2 & 1 \end{array}\right); \qquad (2) \quad \left(\begin{array}{cc|c} 1 & -1/2 & 1/2 \\ -1 & 2 & 1 \end{array}\right);$$

$$(3)\quad \left(\begin{array}{cc|c} 1 & -1/2 & 1/2 \\ \boxed{-1} & 3/2 & 3/2 \end{array}\right); \qquad (4)\quad \left(\begin{array}{cc|c} 1 & -1/2 & 1/2 \\ \boxed{-1} & 1 & 1 \end{array}\right);$$

and

$$(5)\quad \left(\begin{array}{cc|c} 1 & \boxed{-1/2} & 1 \\ \boxed{-1} & 1 & 1 \end{array}\right),$$

where the boxed values should be interpreted as 0. The solution is then $x = (1,1)^t$.