Numerical Methods for PDEs
MATH 610 - 600
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Due April 19
Spring 2018

Homework 6

Exercise 1 **100%**

Let Ω be a bounded domain with suitably smooth boundary $\partial\Omega$ and exterior unit normal $\mathbf{n}$. Let $T > 0$ be a given final time, f be a given real valued function in $C^0(\overline{\Omega}\times[0,T])$, and let u_0 be a given real valued function in $H^1(\Omega)$. Consider the problem: Find $u \in L^\infty(0,T;L^2(\Omega)\cap L^2(0,T;H^1(\Omega))$ such that

$$\begin{aligned} \frac{\partial u}{\partial t}(\mathbf{x},t) - \Delta u(\mathbf{x},t) = f(\mathbf{x},t) \qquad &\text{in} \qquad \Omega\times(0,T), \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x},t) = 0 \qquad &\text{on} \qquad \partial\Omega\times(0,T), \\ u(\mathbf{x},0) = u_0(\mathbf{x}) \qquad &\text{in} \qquad \Omega. \end{aligned} \tag{1}$$

We focus on the semi-discretization in time.

1. (10%) Show that if u is a solution of problem (1), then there is no constant c (other than zero) for which $u + c$ is a solution of (1).

   From now on, accept as a fact that (1) has one and only one solution u that is sufficiently smooth, and satisfies for all $v \in H^1(\Omega)$

   $$\int_\Omega \frac{\partial u}{\partial t}(\mathbf{x},t)v(\mathbf{x})\ d\mathbf{x} + \int_\Omega \nabla u(\mathbf{x},t)\cdot\nabla v(\mathbf{x})\ d\mathbf{x} = \int_\Omega f(\mathbf{x},t)v(\mathbf{x})\ d\mathbf{x} \tag{2}$$

   and $u(0,\mathbf{x}) = u_0(\mathbf{x})$ a.e. in Ω.

2. (10%) Let $N \geqslant 2$ be an integer, set $\tau = T/N$, define $t_n = n\ \tau$ for $0 \leqslant n \leqslant N$ and set

   $$f^n(\mathbf{x}) := \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(\mathbf{x},s)ds.$$

   Then, starting from $u^0 = u_0$, consider the problem: For each $1 \leqslant n \leqslant N$, knowing u^{n-1} find $u^n \in H^1(\Omega)$ solving

   $$\forall v \in H^1(\Omega),\ \frac{1}{\tau}\int_\Omega (u^n - u^{n-1})v + \int_\Omega \nabla u^n\cdot\nabla v = \int_\Omega f^n\ v. \tag{3}$$

   Prove that (3) has one and only one solution $u^n \in H^1(\Omega)$.

3. (10%) Prove the following bound: For any $\alpha > 0$, for $1 \leqslant n \leqslant N$,

   $$\begin{aligned} &\frac{1}{2\tau}\left(\|u^n\|^2_{L^2(\Omega)} - \|u^{n-1}\|^2_{L^2(\Omega)} + \|u^n - u^{n-1}\|^2_{L^2(\Omega)}\right) + \|\nabla u^n\|^2_{L^2(\Omega)} \\ &\quad\leqslant \frac{1}{2}\left(\alpha\|u^n - u^{n-1}\|^2_{L^2(\Omega)} + \frac{1}{\alpha}\|f^n\|^2_{L^2(\Omega)}\right) + \frac{1}{2}\left(\alpha\|u^{n-1}\|^2_{L^2(\Omega)} + \frac{1}{\alpha}\|f^n\|^2_{L^2(\Omega)}\right). \end{aligned}$$

4. (10%) By choosing α show that

   $$\begin{aligned} &\|u^n\|^2_{L^2(\Omega)} + (1-\tau)\sum_{i=1}^n \|u^i - u^{i-1}\|^2_{L^2(\Omega)} + 2\tau\sum_{i=1}^n \|\nabla u^i\|^2_{L^2(\Omega)} \\ &\qquad\qquad \leqslant \|u_0\|^2_{L^2(\Omega)} + 2\tau\sum_{i=1}^n \|f^i\|^2_{L^2(\Omega)} + \tau\sum_{i=0}^{n-1}\|u^i\|^2_{L^2(\Omega)}. \end{aligned}$$

5. (10%) Recall the discrete Growall Lemma (a proof can be obtained by induction but is not required): Let $\lambda > 0$ be a real number and $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$ be two sequences of non-negative numbers such that $(b_n)_{n\geqslant 0}$ is non-decreasing,

$$a_0 \leqslant b_0 \qquad \text{and} \qquad \forall n \geqslant 1,\ a_n \leqslant b_n + \lambda \sum_{m=0}^{n-1} a_m.$$

Then for all $n \geqslant 0$

$$a_n \leqslant b_n(1+\lambda)^n.$$

Assume from now on that $\tau \leqslant \tau_0 < 1$ for some $\tau_0 > 0$ and prove that

$$\sup_{1\leqslant n\leqslant N} \|u^n\|^2_{L^2(\Omega)} \leqslant \left(\|u_0\|^2_{L^2(\Omega)} + 2\tau \sum_{i=1}^{N} \|f^i\|^2_{L^2(\Omega)} \right) \exp(T).$$

6. (10%) Derive an upper bound uniform in n and τ for

$$\sum_{n=1}^{N} \tau \|\nabla u^n\|^2_{L^2(\Omega)} \qquad \text{and} \qquad (1-\tau)\sum_{n=1}^{N} \|u^n - u^{n-1}\|^2_{L^2(\Omega)}.$$

7. (10%) Show that the solution u of (2) satisfies for all $v \in H^1(\Omega)$

$$\frac{1}{\tau}\int_\Omega (u(\mathbf{x},t_n) - u(\mathbf{x},t_{n-1}))v(\mathbf{x})\, d\mathbf{x} + \int_\Omega \nabla u(\mathbf{x},t_n)\cdot\nabla v(\mathbf{x})\, d\mathbf{x} = \int_\Omega f^n(\mathbf{x})v(\mathbf{x})\, d\mathbf{x} + \int_\Omega E^n(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x},$$

where

$$E^n(\mathbf{x}) := \frac{1}{\tau}(u(\mathbf{x},t_n) - u(\mathbf{x},t_{n-1})) - \frac{\partial u}{\partial t}(\mathbf{x},t_n)) - f^n(\mathbf{x}) + f(\mathbf{x},t_n).$$

8. (20%) Prove that if f and u are sufficiently smooth, there exists a constant C such that for all $1 \leqslant n \leqslant N$,

$$\|E^n\|_{L^2(\Omega)} \leqslant C\tau^{1/2} \left(\|\frac{\partial^2}{\partial t^2} u\|_{L^2(t_{n-1},t_n;L^2(\Omega))} + \|\frac{\partial f}{\partial t}\|_{L^2(t_{n-1},t_n;L^2(\Omega))} \right).$$

Hint: Use the Taylor formula with integral reminder.

9. (10%) Deduce an estimate for the error

$$\left(\sup_{1\leqslant n\leqslant N} \|u(.,t_n) - u^n(.)\|^2_{L^2(\Omega)} + \sum_{n=1}^{N} \tau \|\nabla(u(.,t_n) - u^n(.))\|^2_{L^2(\Omega)} \right)^{1/2}.$$