1. Lecture 1: Preliminaries

Definition 1.1 (Limits)**.** *Let $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. We say that the limit of f when x tends to c exists and is equal to L if*

(1) *f is well defined in a neighborhood of c (but not necessarily at c);*
(2) *given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|f(x) - L| < \varepsilon \qquad \textit{when} \qquad x \in (c - \delta, c + \delta) \setminus \{c\}.$$

In this case, we write

$$\lim_{x \to c} f(x) = L.$$

Definition 1.2 (Continuity)**.** *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$ is the limit of f at c exists and*

$$\lim_{x \to c} f(x) = f(c).$$

Definition 1.3 ($C(a, b)$)**.** *We say that $f \in C(a, b)$ if $f : \mathbb{R} \to \mathbb{R}$ is continuous for each $x \in (a, b)$.*

Definition 1.4 ($C[a, b]$)**.** *We say that $f \in C[a, b]$ if $f \in C(a, b)$ and*

$$\lim_{x \to a^+} f(x) = f(a) \qquad \textit{and} \qquad \lim_{x \to b^-} f(x) = f(b).$$

Definition 1.5 (Differentiability)**.** *$f : \mathbb{R} \to \mathbb{R}$ is said to be differentiable at $c \in \mathbb{R}$ if f is defined in a neighborhood of c and*

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. In that case, we write

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Result 1.1 (Differentiability and Continuity)**.** *If f' exists at c, then f is continuous at c.*

Definition 1.6 ($C^1(a, b)$)**.** *We say that $f \in C^1(a, b)$ if $f'(x)$ exists for each $x \in (a, b)$.*

Definition 1.7 ($C^1[a, b]$)**.** *We say that $f \in C^1[a, b]$ if $f \in C^1(a, b)$, $f \in C[a, b]$ and $f' \in C[a, b]$.*

Example 1.1 (Continuity and Differentiability)**.** *Two examples:*

(1) *$f(x) = 1/x$ is in $C(0, 1)$ but not in $C[0, 1]$;*
(2) *$f(x) = x^{1/2}$ is in $C[0, 1]$ not in $C^1[0, 1]$.*

Theorem 1.1 (Intermediate Values)**.** *If $f \in C[a, b]$, then f takes all the calues between $f(a)$ and $f(b)$.*

Theorem 1.2 (Mean Value)**.** *If $f \in C^1[a, b]$, then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 1.3 (Rolles)**.** *If $f \in C^1[a, b]$ and $f(a) = f(b)$, then there exists $c \in (a, b)$ with $f'(c) = 0$.*

The Rolles theorem is illustrated in Figure 1.

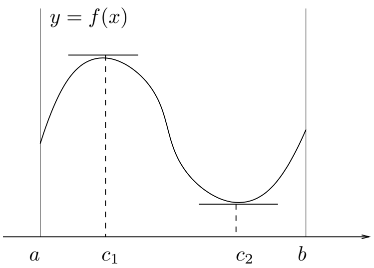

FIGURE 1. Illustration of Rolles theorem. In this case, there are two possible c in the interval (a, b).

Definition 1.8 ($C^m[a, b]$)**.** *We say that* $f \in C^m[a, b]$ *if* $f, f', ..., f^{(m)} \in C[a, b]$.

Theorem 1.4 (Taylor)**.** *Assume* $f \in C^n[a, b]$ *and* $f^{(n+1)}$ *exists on* (a, b)*. Then for* $c \in (a, b)$ *and* $x \in [a, b]$

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(c)}{j!}(x - c)^j + E_{n+1}(x).$$

The error term $E_{n+1}(x)$ *is given by*

$$E_{n+1}(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x - c)^{n+1} \tag{1}$$

for some ξ *between* x *and* c*; or*

$$E_{n+1}(x) = \frac{1}{n!} \int_c^x f^{(n+1)}(t)(x - t)^n dt. \tag{2}$$

Taylors Theorem provides a *numerical approximation*

$$\sum_{j=0}^{n} \frac{f^{(j)}(c)}{j!}(x - c)^j$$

of the function f together with an error bound $E_{n+1}(x)$.

Example 1.2 (Approximation using Taylors Theorem)**.** *We use the* 3 *term Maclaurin series (Taylor series with* $c = 0$*) to approximate* $\cosh(x)$ *for* $x \in [-1, 1]$ *and bound the error. To do this, we compute* $(\cosh(x)' = \sinh(x)$*,* $(\cosh(x)'' = \cosh(x)$ *and* $(\cosh(x)''' = \sinh(x)$*. This leads to the approximation*

$$\cosh(x) \approx \cosh(0) + \sinh(0)x + \frac{1}{2}\cosh(0)x^2 = 1 + \frac{x^2}{2}.$$

To bound the error $|cosh(x) - (1 + \frac{x^2}{2})| = |E_3(x)|$ *we resort the expression of* E_3 *given by* (1)*, which reads in this case*

$$E_3(x) = \frac{1}{3!}\sinh(\xi)x^3$$

for some $\xi \in (-1, 1)$*. Since* $|\sinh(\xi)| \leq \sinh(|\xi|)$ *and* $\sinh(t)$ *is increasing for positive* t*, we deduce that*

$$|\sinh(\xi)| \leq \sinh(1) = \frac{e - e^{-1}}{2}.$$

As a consequence, we obtain the error bound

$$|E_3(x)| \leq \frac{e - e^{-1}}{12}.$$

Example 1.3 (Effect of the Length of the Approximation Interval)**.** *Let us consider the quadratic (3 term) Maclaurin series approximating* $\cos(x)$ *on* $[-\pi, \pi]$*:*

$$\cos(x) \approx 1 - \frac{1}{2}x^2.$$

The corresponding error term reads

$$E_3(x) = \frac{1}{6}\sin(\xi)x^3$$

for some $\xi \in (-\pi, \pi)$*. As a consequence, we get*

$$|E_3(x)| \leq \frac{\pi^3}{6} \approx 5.16.$$

This indicates that Polynomial approximations may not be very good over large intervals.

Exercise 1.1 (Effect of the Interval)**.** *Consider the same problem as in Example 1.3 but on (i)* $[-1/2, 1/2]$ *and (ii)* $[-1/10, 1/10]$.