7. Lecture 7

In the previous lecture, we discussed piecewise interpolation with $\mathbb{P}^k$, $k \geq 1$, polynomial. These approximations are globally continuous if the endpoints of each interval are interpolation points.

We are now contemplating the possibility of constructing globally C^1 interpolant.

Example 7.1 (Globally $\mathbb{P}^2$ and C^1 interpolant)**.** *Find $p \in \mathbb{P}^2$ satisfying*

$$p(-1) = a, \quad p'(0) = b, \quad p(1) = c \tag{4}$$

for given $a, b, c \in \mathbb{R}$.

Let $p(x) = \alpha + \beta x + \gamma x^2$ for α , β and $\gamma \in \mathbb{R}$. The above 3 equations lead to the linear system

$$\underbrace{\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{b}.$$

Note that $\det(A) = 1 \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$ and so A is singular. This means that depending on b, there are either infinitely many solutions or no solutions. To give an example, if $p(x)$ solves (4), so does $p(x) + \eta(x^2 - 1)$ for any $\eta \in \mathbb{R}$. Also there are no solutions to (4) when $a = 0$, $b = 1$ and $c = 0$ because otherwise both $(x - 1)$ and $(x + 1)$ would have to divide $p(x)$, i.e. $p(x)$ would have to be a multiple of $(x^2 - 1)$ and so $p'(0) = 0$!

Exercise 7.1. *Consider the interpolation problem, find $p \in \mathbb{P}^3$ satisfying*

$$p(-1) = y_0, \quad p(0) = y_1, \quad p'(0) = y_2, \qquad p(1) = y_3.$$

Show that the above problem always as a unique solution.
Hint: Look for a solution $p(x) = \alpha + \beta x + \gamma x^2 + \delta x^3$, derive the matrix system for the unknowns α, β, γ and δ and show that its determinant is non-zero.

A general uniquely solvable interpolation problem. Assume $\{x_0, x_1, ..., x_n\}$ are distinct. Find $p \in \mathbb{P}^N$ satisfying for $j = 0, 1, ..., m_i$ and $i = 0, 1, ..., n$

$$p^{(j)}(x_i) = y_{i,j},$$

where $y_{i,j}$ are given.

The number of equations is

$$\sum_{i=0}^{n} (m_i + 1)$$

so we must take $N = \sum_{i=0}^{n} (m_i + 1) - 1$.

Theorem 7.1 (Globally Smooth Interpolation)**.** *The above interpolation problem has a unique solution given any set $\{y_{i,j}\}_{i=0;j=0}^{n;m_j}$*

Notice that if you include a derivative of order j at x_i, you must also include $p^{(l)}(x_i)$, $l = 0, 1, ..., j!$.

The general Hermite interpolation problem. Given $\{x_0, ..., x_n\}$ distinct and $f \in C^1[x_0, x_n]$. Find $p \in \mathbb{P}^{2n+1}$ satisfying

$$p(x_i) = f(x_i), \qquad p'(x_i) = f'(x_i).$$

According to the previous theorem, this interpolation problem has a unique solution.

Theorem 7.2 (Error with Hermite Inteprolation)**.** *Suppose that* $f \in C^{2n+2}[a, b]$ *and that* $\{x_i\} \subset [a, b]$*. Let* p *be the Hermite interpolant. Then for* $x \in [a, b]$*, there is a* $\xi_x \in (a, b)$ *satisfying*

$$f(x) - p(x) = \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!} \prod_{i=0}^{n} (x - x_i)^2.$$

Proof. The formula is valid when $x = x_i$, $i = 0, ..., n$. Therefore, we assume that $x \notin \{x_0, ..., x_n\}$.

Set $w(t) = \prod_{i=0}^{m} (t - x_i)^2$ and

$$\phi(t) = f(t) - p(t) - \lambda w(t), \tag{5}$$

where

$$\lambda = \frac{f(x) - p(x)}{w(x)}.$$

Note that $\phi'(x_i) = 0$ since $f'(x_i) = p'(x_i)$ and $w'(x_i) = 0$. Now $\phi(x_i) = 0$ for $i = 0, 1, ..., n$ and $\phi(x) = 0$ from the definition of λ. Rolles theorem implies that ϕ' has at least $n+1$ additional zeros which are not in $\{x_0, ..., x_n\}$. Therefore, ϕ' has at least $2n+2$ distinct zero. Applying Rolles theorem again but to these zeros implies that ϕ'' has at least $2n+1$ zero. Repeating this process we find that $\phi^{(2n+2)}$ has at least 1 zero, i.e. $\phi^{(2n+2)}(\xi_x) = 0$ for some $\xi_x \in (a, b)$.

Differentiating (5) $2n+1$ times and evaluating at ξ_x

$$0 = \phi^{(2n+2)}(\xi_x) = f^{(2n+2)}(\xi_x) - \lambda(2n+2)!$$

and the claim follows by simple algebra. □

Example 7.2 (Hermite $n = 0$)**.** *Find* $p \in \mathbb{P}^1$ *satisfying*

$$p(x_0) = f(x_0), \qquad p'(x_0) = f'(x_0).$$

The solution is

$$p(x) = f(x_0) + f'(x_0)(x - x_0),$$

i.e. the 2 term Taylor polynomial.

Example 7.3 (Cubic Hermite $n = 1$)**.** *Find* $p \in \mathbb{P}^3$ *satisfying*

$$p(x_i) = f(x_i), \qquad p'(x_i) = f'(x_i), \qquad i = 0, 1.$$

We use the Newton form solution. From the first example

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0),$$

solves $p_1(x_0) = f(x_0)$ *and* $p_1'(x_0) = f'(x_0)$*. Look for*

$$p_2(x) = p_1(x) + c_2(x - x_0)^2$$

satisfying $p_2(x_1) = f(x_1)$*, i.e.*

$$c_2 = \frac{f(x_1) - p_1(x_1)}{(x_1 - x_0)^2} = \frac{f(x_1) - f(x_0) - f'(x_0)(x_1 - x_0)}{(x_1 - x_0)^2}.$$

Then we look for

$$p_3(x) = p_2(x) + c_3(x - x_0)^2(x - x_1)$$

satisfying $p_3'(x_1) = f'(x_1)$.

Exercise 7.2. *Derive an expression for* $p_3(x)$ *in term of*

$$x_0, x_1, f(x_0), f'(x_0), f(x_1), f'(x_1).$$

Piecewise Hermite interpolation. Let $a = x_0 < x_1 < ... < x_m = b$ and consider piecewise Hermite cubic approximation f_h, i.e.

$$f_h(x) = p_i(x) \qquad \text{on } [x_{i-1}, x_i]$$

with $p_i \in \mathbb{P}^3$ solving

$$\begin{aligned} p_i(x_{i-1}) = f(x_{i-1}), \qquad p_i'(x_{i-1}) = f'(x_{i-1}) \\ p_i(x_i) = f(x_i), \qquad p_i'(x_i) = f'(x_i), \end{aligned}$$

where $h = \max_{i=1,..,N}(x_i - x_{i-1})$. Note that $f_h \in C^1[a, b]$.

Exercise 7.3. *Assume that* $f \in C^4[a, b]$. *Use the previous theorem to prove a* 4*th order (in* h*) error bound for* $|f_h(x) - f(x)|$.