Partial Differential Equations
MATH 612
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Due April 9
Spring 2013

Assignement 3

Exercise 1   30%

We saw in class that for two Hilbert spaces X and Y and a continuous bilinear form $b : X \times Y \to \mathbb{R}$ the inf sup condition : there exists $\beta > 0$ such that

$$\inf_{q\in Y,\ \|q\|_Y=1} \sup_{v\in X,\ \|v\|_X=1} b(v,q) \geqslant \beta$$

implies that the operatore $B : X_0^\perp \to Y^*$ is an isomorphism. Here

$$X_0 := \{v \in X \ : \ b(v,q) = 0, \qquad \forall q \in Y\}.$$

Prove that if B is an isomorphism such that there exists a constant C satisfying

$$\|B\varphi\|_{Y^*} \leqslant C\|\varphi\|_X$$

for all $\varphi \in X_0^\perp$, then the inf-sup condition holds.

Exercise 2   35%

Let $\Omega \subset \mathbb{R}^2$ be open, bounded and with smooth boundary. Given $f \in L_2(\Omega)^2$, we consider the stationary Stokes system

$$-\Delta u + \nabla p = f, \qquad \text{and} \qquad \mathrm{div} u = 0, \qquad \text{in} \quad \Omega$$

supplemented by the boundary condition $u = 0$ on $\partial\Omega$.
Show that there exists a unique solution $(u,p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ where

$$L_0^2(\Omega) := \left\{ q \in L^2(\Omega) \ : \ \int_\Omega q = 0 \right\}.$$

Hint : You may assume that there exists a constant C such that for every $q \in L_0^2(\Omega)$, there exists a $v \in H_0^1(\Omega)^2$ such that

$$\mathrm{div} v = q$$

with

$$\|v\|_{H^1(\Omega)^2} \leqslant C\|q\|_{L^2(\Omega)}.$$

Exercise 3   35%

Let $\Omega \subset \mathbb{R}^2$ be open, bounded and with smooth boundary. Given $f \in L_2(\Omega)$, we consider the mixed formulation of the Poisson equation

$$u = \nabla p, \qquad \text{and} \qquad -\mathrm{div} u = f, \qquad \text{in} \quad \Omega$$

supplemented by the boundary condition $p = 0$ on $\partial\Omega$.
Find the adequate spaces X and Y such that the above system has a unique solution $(u,p) \in X \times Y$. Justify your answer by proving that with the proposed choice, there exists, indeed, a unique solution.