Numerical Approximations of Fractional Operators using Dunford-Taylor Representations

Andrea Bonito

Department of Mathematics
Texas A&M University

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Department of Applied Mathematics • The Hong Kong Polytechnic University

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Examples of Non-local operators

- Nonlocal denoising filters [Gilboa and Osher 2008], [Gatto and Hesthaven 2015];
- Geostrophic model: Temperature toy model - study of 3D blow-up in fluid dynamics [Held, Pierrehumbert, Garner and Swanson 1995], [Constantin and Wu 1999], [Constantin, Majda and Tabak 2001], [Constantin 2002], [Cordoba 2003];
- Porous media without scale separation [Cushman and Glinn 1993];
- Turbulence model [Bakunin 2008];
- Electro-convection [Daya et al. 1998], [Tsai et al. 2007], [Constantin et al. 2016];
- Fractal conservation laws [Biler, Karch and Woyczynsky 2001], [Droniou 2010];
- Peridynamics: elastic theory with long range interaction and fractures [Silling 2000];
- Fractional kinetics and anomalous transport [Zaslavsky 2002];
- Nonlocal Allen-Cahn equation - atoms interacts weakly but at long range [Bales 2006];
- American option pricing [Pham 1997], [Tankov 2003], [Levendorskii 2004].

Prototype Non-local Problem

Domain

$\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3, ...$, open, bounded, Lipschitz boundary.

Data

Right hand side $f : \Omega \to \mathbb{R}$ and fractional power $0 < \beta < 1$.

Boundary value problem

We want to approximate the solution $u : \Omega \to \mathbb{R}$ satisfying

$$(-\Delta)^{\beta} u = f, \quad \text{in} \quad \Omega \qquad + \qquad \text{conditions on } \mathbb{R}^d \setminus \Omega.$$

How to define/approximate $(-\Delta)^{-\beta} f$?

What does it mean?

Spectral Fractional - Spectral Decomposition

Let $\{\psi_i\} \subset H_0^1(\Omega)$ be an orthonormal basis of $L_2(\Omega)$ (orthogonal in H_0^1) made of eigenfunctions of $(-\Delta)$ with corresponding eigenvalues $\{\lambda_i\}$:

$$C_0^\infty(\overline{\Omega}) \ni u = \sum_{i=1}^\infty u_i \psi_i \quad \Longrightarrow \quad (-\Delta)^\beta u(x) := \sum_{i=1}^\infty \lambda_i^\beta u_i \psi_i(x).$$

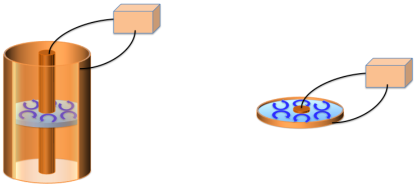
Integral Fractional Laplacian - Fourier Transform

Extend $u \in C_0^\infty(\overline{\Omega})$ by 0 outside Ω

$$G(\xi) := \mathcal{F}((-\Delta)^\beta u)(\xi) := (|\xi|^2)^\beta \mathcal{F}(u)(\xi) \quad \Longrightarrow \quad (-\Delta)^\beta u := \mathcal{F}^{-1}(G)|_\Omega.$$

Other Definitions: [Lischke et al.: What is the Fractional Laplacian?, 2020].

ElectroConvection of Thin Liquid Crystals [with P. Wei 2020]



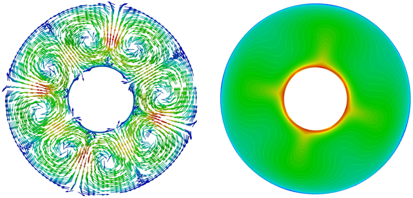
- Thin liquid Crystal fluid encapsulate within two concentric electrodes:
  - Infinite - Electrodes extend to infinity in the vertical direction.
  - Slim - Electrodes are flat.
- Semectic-A phase: Liquid crystals are aligned in the vertical direction
  - no motion along the thickness: planar motion;
- Poor conductivity in the planar direction
  - negligible current and magnetic field;
- Charges aggregate at the boundary of the liquid domain $\rightarrow$ surface charge density.

Electro-Convection on thin fluids (Morris, U. Toronto)

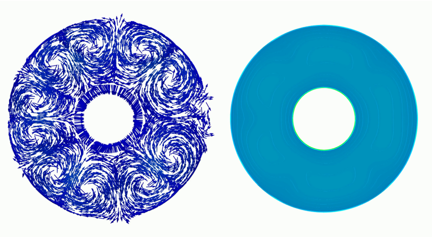
https://www.youtube.com/watch?v=aaKwymX7pqY

Slim VS Infinite Electrodes

Slim $\leftrightarrow$ Integral



Infinite $\leftrightarrow$ Spectral



Electroconvection appears and remains at lower voltages in the slim electrodes case

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Spectral Representation

$$(-\Delta)^\beta u = f, \quad \text{in} \quad \Omega \qquad u = 0 \qquad \text{on } \partial\Omega.$$

Spectral Decomposition

$$u(x) = \sum_{i=1}^{\infty} \lambda_i^{-\beta} f_i \psi_i(x), \qquad \text{for} \quad f(x) = \sum_{i=1}^{\infty} f_i \psi_i(x)$$

and $\{\psi_i\} \subset H_0^1(\Omega)$ is an orthonormal basis of $L_2(\Omega)$ (orthogonal in H_0^1) made of eigenfunctions of $(-\Delta)$ with corresponding eigenvalues $\{\lambda_i\}$.

Numerical Method [Ilic et al. 2005]

- Construct a finite dimensional approximation of $-\Delta$.
- Compute its eigenfunctions (orthonormal in L_2, orthogonal in H_0^1) $\{\psi_{i,N}\}$ and eigenvalues $\{\lambda_{i,N}\}$, $i = 1, ..., N$.
- Compute the approximate Fourier coefficients $\{f_{i,N}\}$ of f and define

$$u_N := \sum_{i=1}^{N} f_{i,N} \lambda_{i,N}^{-\beta} \psi_{i,N}$$

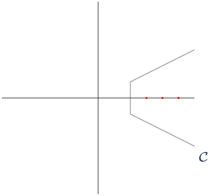
Dunford Taylor Representation of Fractional Powers of Elliptic Operators

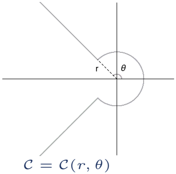
Dunford Integral

For $\beta \in (0,1)$,

$$u = (-\Delta)^{-\beta} f = \frac{1}{2\pi i} \int_{\mathcal{C}} z^{-\beta} (z+\Delta)^{-1} f \, dz$$

with $z^{-\beta} = |z|^{-\beta} e^{-i\beta \arg z}$.





Balakrishnan formula

Let $r \to 0$ and $\theta \to \pi$

$$u = (-\Delta)^{-\beta} f = \frac{2\sin(\pi\beta)}{\pi} \int_0^\infty \mu^{-\beta} (\mu - \Delta)^{-1} f \, d\mu.$$

The Balakrishnan Formula Representation

$$(-\Delta)^{-\beta} f = \frac{2\sin(\pi\beta)}{\pi} \int_0^\infty \mu^{-\beta} (\mu - \Delta)^{-1} f \, d\mu.$$

Sanity Check

If $\psi \in H_0^1(\Omega)$ is an eigenfunction of $(-\Delta)$ with associated eigenvalue $\lambda > 0$ then

$$(-\Delta)^{-\beta}\psi = C(\beta) \int_0^\infty \frac{\mu^{-\beta}}{\mu+\lambda} \psi d\mu \overset{\mu=\lambda t}{=} \lambda^{-\beta}\psi C(\beta) \int_0^\infty \frac{t^{-\beta}}{t+1} dt = \lambda^{-\beta}\psi.$$

Numerical method: Game plan

(Step 1) use quadrature for the μ variable;
(Step 2) use standard finite element methods on the same mesh to approximate

$$u_\mu \in H_0^1(\Omega): \qquad \mu u_\mu - \Delta u_\mu = f \quad \text{in} \quad \Omega.$$

(Step 3) Gather all contributions.

Quadrature for the μ variable: SINC Quadrature

Change of variable: $\mu = e^y$

$$u = (-\Delta)^{-\beta} f = \frac{2\sin(\pi\beta)}{\pi} \int_{-\infty}^{\infty} e^{(1-\beta)y} (e^y - \Delta)^{-1} f \, dy.$$

Given $N \in \mathbb{N}$, define $k = 1/\sqrt{N}$, $y_l = lk$ and the quadrature approximation

$$Q^N f = \frac{2\sin(\pi\beta)k}{\pi} \sum_{l=-N}^{N} e^{(1-\beta)y_l} (e^{y_l} - \Delta)^{-1} f.$$

Theorem [with Pasciak 2015],[with Pasciak and Lei 2019]

For any $s \in [0,1]$, let $f \in H_0^\gamma(\Omega) := [H_0^1(\Omega), L_2(\Omega)]_{1-s}$ with $\gamma > s - 2\beta$

$$\|(-\Delta)^{-\beta} f - Q^N f\|_{H_0^s} \leq C e^{-c\sqrt{N}} \|f\|_{H_0^{\max(\gamma,0)}}.$$

Exponential Convergence!

Finite Element Method

Approximate each sub-problems $\mu u_\mu - \Delta u_\mu = f$ using continuous piecewise linear FEM.

Assumptions

- Pick-up regularity: There is $0 < \alpha \leq 1$ such that

$$(-\Delta)^{-1} : H^{-1+\alpha}(\Omega) \to H^{1+\alpha}(\Omega) \cap H_0^1(\Omega)$$

  is bounded. The solution of the standard laplacian problem $(-\Delta)v = g$ (with vanishing bc) satisfies $v \in H^{1+\alpha}(\Omega) \cap H_0^1(\Omega)$ for $g \in H^{-1+\alpha}(\Omega) \subset L_2(\Omega)$.
- $\{\mathcal{T}_h\}_{h>0}$ is a sequence of quasi-uniform subdivision of Ω.

Theorem [with Pasciak 2015]

$$\|u - Q_h^N f\|_{H^1} \lesssim (h^\alpha + e^{-c\sqrt{N}})\|f\|_{H_0^{\max(1+\alpha-2\beta,0)}},$$

$$\|u - Q_h^N f\|_{L_2} \lesssim (h^{2\alpha} + e^{-c\sqrt{N}})\|f\|_{H_0^{\max(2\alpha-2\beta,0)}}$$

(up to log terms for some combinations of β and α).

Rate observed in practice

Properties of the Proposed Method

Advantages

- The method is easily parallelizable: Each subproblems (sinc quad) are independent. We tried with $15'000$ cores.
- Minimal changes in existing codes: It relies on standard finite element in $\mathbb{R}^d$, i.e. the quadrature component is an additional external loop.
- Preconditionner: Standard preconditionners can be used. Moreover, the iterative solvers at each quadrature points benefits from the previous quadrature point as starting guess.

Extensions

Works Self-Adjoint, Coercive operators

In this case the pick-up regularity depends on the operator as well. E.g. diffusion with discontinuous coefficients, different boundary conditions, Laplace-Beltrami operators.

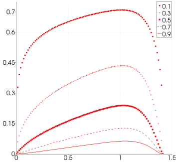
Non-Hermitian problems [with Pasciak 2017]

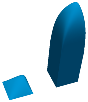
Consider a regularly non-accretive operator A, e.g. $Au = -\Delta u + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \nabla u$.

Then

$$A^{-\beta} f = \frac{2\sin(\pi\beta)}{\pi} \int_0^\infty \mu^{-\beta} (\mu + A)^{-1} f d\mu$$

Case $f = 1$ with $\Omega = (0,1)^2$





$\beta = 0.9$ $\beta = 0.1$

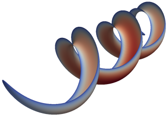
Extensions

Laplace - Beltrami Operator on surfaces [with W. Lei (submitted)]

For a C^3 compact surface γ:

$$(-\Delta_\gamma)^{-\beta} f = \frac{2\sin(\pi\beta)}{\pi} \int_0^\infty \mu^{-\beta}(\mu - \Delta_\gamma)^{-1} f d\mu$$

$$(-\Delta_\gamma)^{1/2} u = 1$$

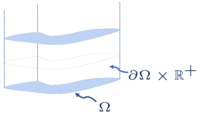


Reduced Order Method [with Guignard and Zhang 2020]

The set of solutions $\mu \mapsto u_\mu$ of the sub problems $\mu u_\mu - \Delta u_\mu = f$ is a one dimensional manifolds independent of the fractional power. This one dimensional manifold can be approximated with exponential accuracy using a greedy method (offline / online paradigm).

Different Representation - Extended Problem [Stinga and Torrea 2009]

Harmonic Problem in $\mathbb{R}^{d+1}$ - Case $\beta = 1/2$



$$-\Delta_{d+1} w = 0 \quad \text{in } \Omega\times\mathbb{R}_+, \qquad \partial_z w = -f \quad \text{on} \quad \Omega, \qquad w = 0 \quad \text{on} \quad \partial\Omega\times\mathbb{R}^+.$$

- Use spectral decomposition $w(x,y,z) = \sum_{i=1}^\infty \psi_i(x,y) Z_i(z)$ for $(\lambda_i, \psi_i)_{i=1}^\infty$ eigenpairs of $(-\Delta)$:

$$0 = -\Delta_{d+1} w = \sum_{i=1}^\infty (\lambda_i Z_i - Z_i'')\psi_i = 0 \quad \Longrightarrow \quad Z_i(z) = \frac{f_i}{\sqrt{\lambda_i}} e^{-\sqrt{\lambda_i} z}.$$

- $v(x,y) := w(x,y,0) := \sum_{i=1}^\infty v_i \psi_i \Longrightarrow v_i = f_i/\sqrt{\lambda_i}$

$$\sum_{i=1}^\infty \lambda_i^{1/2} v_i \psi_i = \sum_{i=1}^\infty f_i \psi_i = f \quad \text{in } \Omega \quad \Longrightarrow \quad (-\Delta)^{1/2} v = f.$$

Case $\beta \neq \frac{1}{2}$: Fractional normal derivative and diffusion problem with weight.
Numerical Method: [Nochetto, Otárola and Salgado 2015].

The Surface Quasi-Geostrophic Flows System [with Nazarov 2021]

Modeling Assumptions

- Incompressible, inviscid, adiabatic and in hydrostatic balance
  $\rightarrow$ 3d Navier-Stokes
- Fluid constrained by environmental rotation (small Rossby no) and stratification (small Froude no)
  $\rightarrow$ 3d Quasi-geostrophic flow
  $\rightarrow$ large scale mid-latitude atmospheric / oceanic motions.
- Uniform potential vorticity ψ $\rightarrow$ SQG

Difficulties

- Transport equation with no or limited smoothing effect from Ekman pumping (not discussed);
- Solutions develop discontinuities - Frontogenesis;
- Solutions exhibit fine structures.

Mathematical Model

System for the velocity $\mathbf{u}$ and temperature potential or buoyancy θ

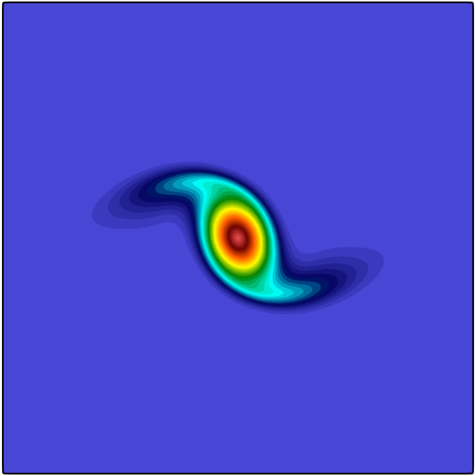
$$\partial_t\theta + \mathbf{u}(\theta)\cdot\nabla\theta = 0, \quad \text{on } \mathbb{T}^2\times(0,\infty);$$
$$\theta(0) = v, \quad \text{on } \mathbb{T}^2;$$

- Torus $\mathbb{T}^2 = \mathbb{R}^2/(\pi\mathbb{Z})^2 \to$ 2 dimensional model;
- $\boxed{\partial_t + \mathbf{u}\cdot\nabla}$ transport;
- $\boxed{(-\Delta)^{\frac{1}{2}}\psi = \theta}$, ψ: potential vorticity $\to$ hydrostatic relation $-\partial_z\psi = \theta$ on $\mathbb{T}^2$ and uniform potential velocity $\Delta_{3d}\psi = 0$ in $\mathbb{T}^2\times\mathbb{R}^+\times\mathbb{R}^+$;
- $\boxed{\mathbf{u}(\theta) = \nabla^\perp\psi}$, motion along iso-bar $\to$ balance between Coriolis and pressure gradient;
- $\boxed{(-\Delta)^{\frac{1}{2}}}$ is the spectral laplacians.

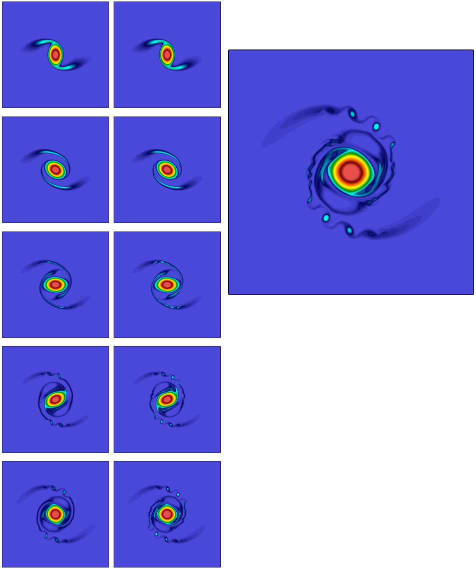
Vortex Rotation

Parameters: space discretization consisting of 512×512 vertices, CFL=0.4

$$\theta_0(x_1, x_2) = e^{-(x_1-\pi)^2 - 16(x_2-\pi)^2}.$$

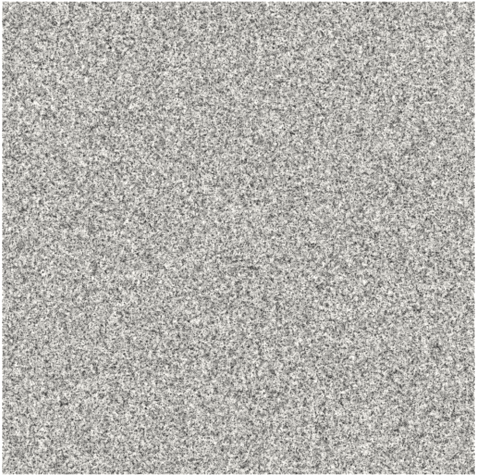


Vortex Rotation

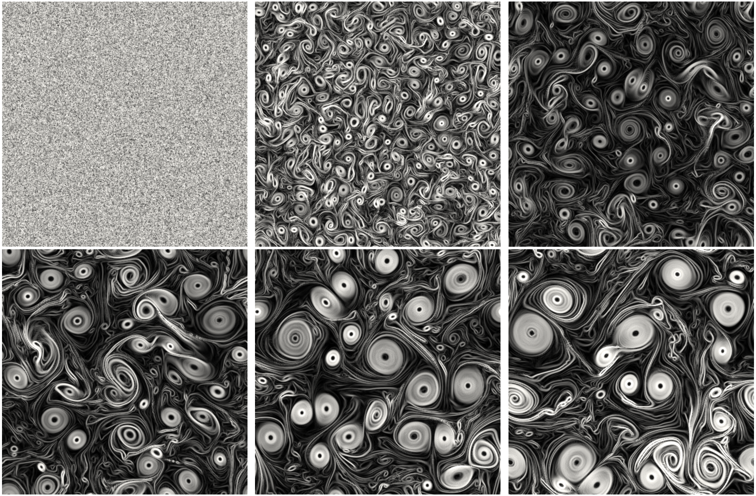


Freely decaying turbulence

Parameters: space discretization consisting of 512×512 vertices, CFL=0.4

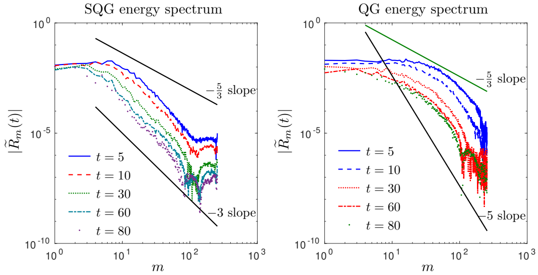


Freely decaying turbulence



Freely decaying turbulence

Spectrum of the kinetic energy $K(t) = \frac{1}{2}\int_{\mathbb{T}^2} \theta^2(t) = \int_{\mathbb{T}^2} \theta^2(0)$



For the quasi-geostrophic flow, $(-\Delta)^{1/2}\psi = \theta$ is replaced by $(-\Delta)\psi = \theta$.

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Integral Fractional Laplacian Problem

$$
\begin{aligned}
u &= 0, && \text{in } \Omega^c := \mathbb{R}^d \setminus \Omega; \\
(-\Delta)^\beta u|_\Omega &= f, && \text{in } \Omega.
\end{aligned}
$$

The fractional Laplacian is defined by the Fourier transform: for $u \in L^2(\mathbb{R}^d)$,

$$\mathcal{F}((-\Delta)^\beta u)(\zeta) = |\zeta|^{2\beta} \mathcal{F}(u)(\zeta).$$

Existing Methods

Finite Difference: [Huang and Oberman 2014];
Finite Element: [D'Elia and Gunzburger 2013] and [Acosta and Borthagaray 2016].

Dunford-Taylor Strategy

Dunford-Taylor representations of $(-\Delta)^{-\beta} f$ or $(-\Delta)^\beta u = (-\Delta)^{\beta-1}(-\Delta)u$ are not well defined.

Approximation of Variational Formulation

Variational Formulation: Find $u \in \widetilde{H}^\beta$ such that

$$a(u,\phi) := ((-\Delta)^{\beta/2}u, (-\Delta)^{\beta/2}\phi)_{\mathbb{R}^d} := \int_{\mathbb{R}^d} |\xi|^{\beta/2}\mathcal{F}(u)|\xi|^{\beta/2}\mathcal{F}(\phi)d\xi = \int_\Omega f\phi$$

for $\phi \in \widetilde{H}^\beta := \widetilde{H}^\beta(\Omega)$, the functions in $H^\beta(\mathbb{R}^d)$ vanishing on Ω^c.

Alternate Representation

$$I(u,v) := C(\beta)\int_0^\infty \mu^{\beta-1}((\mu-\Delta)^{-1}(-\Delta)u, v)\, d\mu = \text{``}((-\Delta)^{\beta-1}(-\Delta)u, v)_{L_2(\Omega)}\text{''}$$

Theorem: evaluation of the bilinear form

The integral defined above converges absolutely for u, $v \in \widetilde{H}^\beta$ and satisfies

$$I(u,v) = a(u,v), \qquad \forall u, v \in \widetilde{H}^\beta,$$

i.e. the targeted solution $u \in \widetilde{H}^\beta$ solves

$$C(\beta)\int_0^\infty \mu^{\beta-1}((\mu-\Delta)^{-1}(-\Delta)u, v)_{L_2(\Omega)}\, d\mu = \int_\Omega fv \qquad \forall v \in \widetilde{H}^\beta.$$

Inverse Laplcians in $\mathbb{R}^d$.

Inner Problems

For $\eta, \theta \in \widetilde{H}^\beta(\Omega)$, define

$$I(\eta,\theta) = C(\beta)\int_0^\infty \mu^{\beta-1}((\mu I-\Delta)^{-1}(-\Delta)\eta,\theta)_{L_2(\Omega)}\,d\mu$$

$$\stackrel{\mu=t^{-2}}{=} C(\beta)\int_0^\infty t^{-2\beta}(\underbrace{(I-t^2\Delta)^{-1}t^2(-\Delta)\eta}_{=:w(t,\eta):=w(t)},\theta)_{L_2(\Omega)}\frac{dt}{t}$$

$$= C(\beta)\int_0^\infty t^{-2\beta-1}(w(t)+\eta,\theta)_{L_2(\Omega)}dt,$$

where $w(t) = w(t,\eta) \in H^1(\mathbb{R}^d)$ solves

$$w - t^2\Delta w = -\eta \qquad \text{in } \mathbb{R}^d$$

- This gives a method to evaluate the bilinear form but do not provide an expression of the solution u directly;
- The sub-problems are in $\mathbb{R}^d$;
- The support of η is Ω and only the values of $w(t)$ in Ω are needed.

Error estimate - Strang's Lemma

Numerical method: Game plan

(Step 1) use a preconditioned conjugate gradient (matrix free);
(Step 2) use quadrature for the t/μ variable;
(Step 3) for each quadrature point, truncate the computational domain to $\Omega(t)$ of diameter $\sim Mt$;
(Step 4) use standard finite element methods to approximate $v \in H_0^1(\Omega(t))$

$$v - t^2 \Delta v = -\eta \qquad \text{in } \Omega(t).$$

Theorem [with Pasciak and Lei 2019]

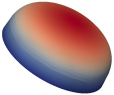
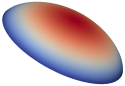
Assume $u \in \widetilde{H}^\delta(\Omega)$ for $\delta \in (\beta, 3/2)$ and that $e^{-\pi^2/(4k)} h \leq C$,

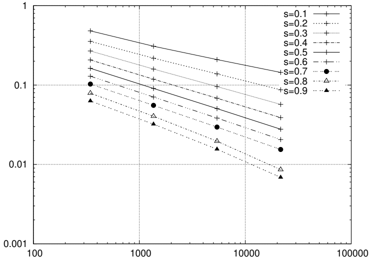
$$\|u - u_h^{N,M}\|_{\widetilde{H}^\beta(\Omega)} \preceq (e^{-c\sqrt{N}} + e^{-cM} + h^{\delta-\beta}) \|u\|_{\widetilde{H}^\delta(\Omega)}.$$

Numerical Illustration

Let the domain Ω be the unit ball. We want to find $u \in \widetilde{H}^{\beta}(\Omega)$ satisfying

$$a(u, \phi) = (1, \phi)_{\Omega}, \forall \phi \in \widetilde{H}^{\beta}(\Omega).$$

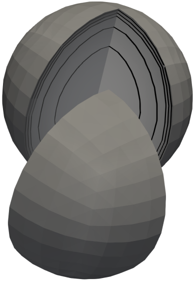
$\beta = 0.3$   $\beta = 0.7$



number of DOFS (extended mesh) - Convergence rate $\min(\beta + 1/2, 1)$ as predicted.

Works in 3D

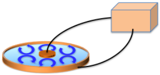
- Consider the unit ball of $\mathbb{R}^3$.
- $\beta = 0.3,\ f = 1$



The approximation of u (darker $=$ 0.0, whiter $=$ 0.7). The lines represent the isosurfaces $\{u(x) = k/10\}$ for $k = 0, ..., 7$. This illustrate the singular behavior of the solution towards the domain boundary

$$u(x) \approx \mathrm{dist}(x, \partial\Omega)^{\beta} + \mathrm{smooth} \qquad\qquad [\text{Grubb 2015}].$$

Back to Electro-Convection [with P. Wei 2020]



- Liquid cristal $\Omega \subset \mathbb{R}^2$, electrodes $E_{i/o}$ of radii $R_i = \alpha < 1 = R_o$
- Electric potential w: No charge in free space $\mathbb{R}^3 \setminus (\Omega \cup E_{i/o})$, no magnetic effect
$$-\Delta_{3d} w = 0 \quad \text{in } \mathbb{R}^3 \setminus (\Omega \cup E_{i/o}).$$
- Boundary condition: $w = V$ on E_i, $w = 0$ on E_o.
- Interface condition: $[\partial_z w] = -2q$ across Ω, where q is the surface charge density in liquid.
- Symmetry: restrict to $\{z \geq 0\}$ and use $\partial_z w = -q$ on Ω.
- Condition at $z \to \infty$: $\lim_{z\to\infty} w(x,y,z) = 0$, $(x,y) \in \mathbb{R}^2$;
- Domain of w is $\mathbb{R}^3$.

Dimension Reduction for Slim Electrodes [Caffarelli-Silvestre 2007]

- $w = w_0 + \overline{w}$.
- $\overline{w}$ harmonic lifting of boundary conditions: $\partial_z \overline{w}|_{z=0} = g(x^2 + y^2)$ can be computed explicitly in term of hyper-geometric functions g.
- PDE for w_0:

$$-\Delta_{3d} w_0 = 0 \qquad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+,$$

with

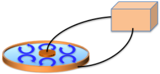
$$w_0(x, y, 0) = 0 \qquad \text{on} \quad E_{i/o} = \mathbb{R}^2 \backslash \Omega, \qquad -\partial_z w_0(x, y, 0) = q + g \quad \text{on} \quad \Omega.$$

- Proceed as for the representation of the spectral laplacian in $\mathbb{R}^3$ but use Fourier transform.

Electric Potential for Slim Electrodes (Integral Laplacian)

$$(-\Delta)^{1/2} w_0 = q + g \ \text{ in } \Omega, \quad w_0 = 0 \text{ in } \mathbb{R}^2 \setminus \Omega, \qquad w = w_0 + \overline{w}.$$

Mathematical Modeling continued



- Fluid Dynamic:

$$(\partial_t + \mathbf{u}\cdot\nabla)\,\mathbf{u} - \mathcal{P}\Delta\mathbf{u} + \nabla p = -\mathcal{R}\mathcal{P}q\nabla w, \qquad \mathrm{div}\mathbf{u} = 0, \qquad \text{in} \quad \Omega$$

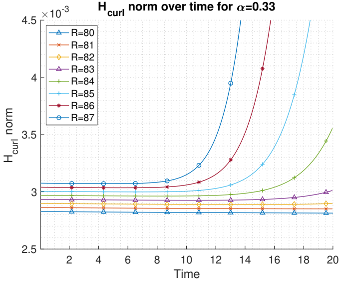
  $\mathbf{u}$ is the 2d fluid velocity satisfying $\mathbf{u} = 0$ on $\partial\Omega$, p is the pressure,
  - $\mathcal{P}$ is the Prandtl number $\rightarrow$ fluid viscous relaxation ability relative to its charge relaxation ability
  - $\mathcal{R}$ is the Rayleigh number $\rightarrow$ ratio between electric and dissipation forces
- Charge Conservation using $\mathrm{div}E = \Delta w$, $E = -\nabla w$ (Ohm's law)

$$\frac{\partial}{\partial t}q + \mathbf{u}\cdot\nabla q - \Delta w = 0 \quad \text{in} \quad \Omega, \quad \int_\Omega q = Cst$$

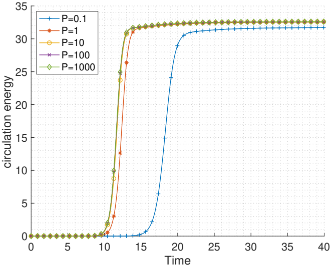
Two dimensional coupled system

Unknown $(\mathbf{u}, p, w, q)$ defined on Ω only $\rightarrow$ dimension reduction.

Effect of $\mathcal{R}$ and $\mathcal{P}$



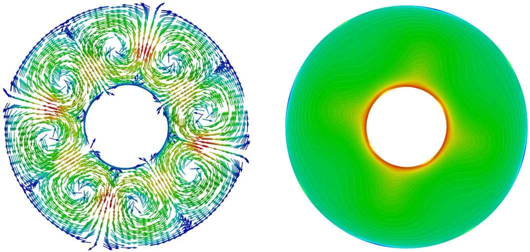
Effect of Rayleigh
ratio between electric
and dissipation forces



Effect Prandtl
Fluid viscous relaxation ability
relative to its charge relaxation ability

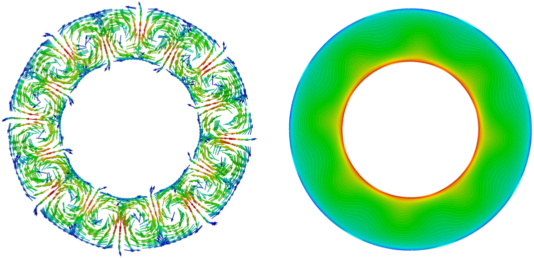
Slim Electrodes: Different Aspect Ratio $\alpha = R_i/R_o$

$\alpha = 0.5$

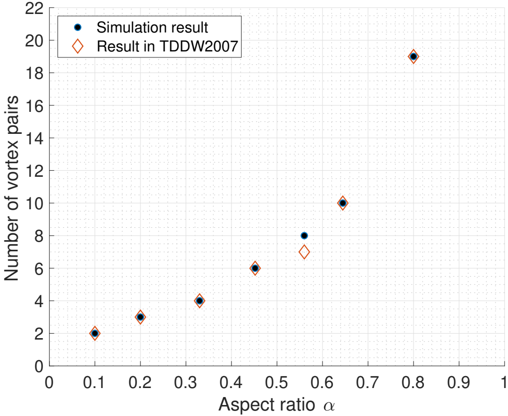


$\mathcal{R} = 100$, $\mathcal{P} = 10$

$\alpha = 0.33$



Slim Electrodes: Different Aspect Ratio $\alpha = R_i/R_o$



Our results match experiments

Conclusion

The Balakrishnan formula provides an adequate formulation for finite element approximation of fractional operators.

- Spectral Fractional Laplacian
  - Representation of the solution $(-\Delta)^{-\beta} f$ using the Balakrishnan formula;
  - Exponential convergent SINC quadrature;
  - Optimal FEM discretizations.
- Integral Fractional Laplacian
  - The Balakrishnan formula cannot be used to represent the solution
  - Action of the bilinear form is approximated instead;
  - Matrix free iterative algorithm;
  - Additional exponentially convergent Domain Truncation.
- Applications Discussed
  - Surface Quasi-Geostrophic Flow;
  - Electro-convection.