Numerical Methods for PDEs
MATH 610 - 600 Spring 2018
A. Bonito

First Name: ____________________ **Last Name:** ____________________

Take Home Exam

Exercise 1 **50%**

Let Ω be a bounded domain with suitably smooth boundary $\partial\Omega$. Let $T > 0$ be a given final time, f be a given real valued function in $C^0(\overline{\Omega} \times [0,T])$, and let u_0 be a given real valued function in $H^1(\Omega)$. Consider the problem: Find $u \in L^\infty(0,T; L^2(\Omega) \cap L^2(0,T; H^1(\Omega))$ such that

$$\begin{aligned} \frac{\partial u}{\partial t}(\mathbf{x},t) - \Delta u(\mathbf{x},t) &= f(\mathbf{x},t) \qquad \text{in} \qquad \Omega \times (0,T), \\ u(\mathbf{x},t) &= 0 \qquad \text{on} \qquad \partial\Omega \times (0,T), \\ u(\mathbf{x},0) &= u_0(\mathbf{x}) \qquad \text{in} \qquad \Omega. \end{aligned} \tag{1}$$

We focus on a *second order* semi-discretization in time.

1. Prove an energy (stability) estimate for the solution of problem (1), i.e. for any $t \in (0,T]$ obtain a bound only depending on the data for the quantity

$$\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u\|^2_{L^2(\Omega)}.$$

Deduce that there exists at most one solution to problem (1).

From now on, accept as a fact that (1) has one and only one solution u that is sufficiently smooth, and satisfies for all $v \in H^1(\Omega)$

$$\int_\Omega \frac{\partial u}{\partial t}(\mathbf{x},t)v(\mathbf{x})\ d\mathbf{x} + \int_\Omega \nabla u(\mathbf{x},t) \cdot \nabla v(\mathbf{x})\ d\mathbf{x} = \int_\Omega f(\mathbf{x},t)v(\mathbf{x})\ d\mathbf{x} \tag{2}$$

and $u(0,\mathbf{x}) = u_0(\mathbf{x})$ a.e. in Ω.

2. (15%) Let $N \geqslant 2$ be an integer, set $\tau = T/N$, define $t_n = n\ \tau$ for $0 \leqslant n \leqslant N$ and set

$$f^{n-1/2}(\mathbf{x}) := \frac{1}{2}\left(f(\mathbf{x},t_{n-1}) + f(\mathbf{x},t_n)\right).$$

Then, starting from $u^0 = u_0$, consider the problem: For each $1 \leqslant n \leqslant N$, knowing u^{n-1} find $u^n \in H^1(\Omega)$ satisfying for any $v \in H^1(\Omega)$,

$$\begin{aligned} \frac{1}{\tau}\int_\Omega (u^n(\mathbf{x}) - u^{n-1}(\mathbf{x}))v(\mathbf{x})d\mathbf{x} + \int_\Omega \nabla\left(\frac{u^n(\mathbf{x}) + u^{n-1}(\mathbf{x})}{2}\right) \cdot \nabla v(\mathbf{x})\ d\mathbf{x} \\ = \int_\Omega f^{n-1/2}(\mathbf{x})\ v(\mathbf{x})\ d\mathbf{x}. \end{aligned} \tag{3}$$

Prove that (3) has one and only one solution $u^n \in H^1(\Omega)$.

3. Show that for any $n = 1, ..., N$ there holds

$$\|u^N\|^2_{L^2(\Omega)} + \sum_{n=1}^N \tau \|\nabla(\frac{u^n + u^{n-1}}{2})\|^2_{L^2(\Omega)} \leqslant \|u^0\|^2_{L^2(\Omega)} + 4C_\Omega^2 \sum_{n=1}^N \tau\|f^{n-1/2}\|^2_{L^2(\Omega)},$$

where C_Ω is the Poincaré constant.

4. Show that the solution u of (2) satisfies for all $v \in H^1(\Omega)$

$$\frac{1}{\tau}\int_\Omega (u(\mathbf{x},t_n) - u(\mathbf{x},t_{n-1}))v(\mathbf{x})\ d\mathbf{x} + \int_\Omega \nabla(\frac{u(\mathbf{x},t_n)+u(\mathbf{x},t_n)}{2})\cdot\nabla v(\mathbf{x})\ d\mathbf{x}$$
$$= \int_\Omega f^{n-1/2}(\mathbf{x})v(\mathbf{x})\ d\mathbf{x} + \int_\Omega E^{n-1/2}(\mathbf{x})\ v(\mathbf{x})\ d\mathbf{x},$$

where

$$E^{n-1/2}(\mathbf{x}) := \frac{1}{\tau}(u(\mathbf{x},t_n) - u(\mathbf{x},t_{n-1})) - \frac{1}{2}\left(\frac{\partial u}{\partial t}(\mathbf{x},t_n)) + \frac{\partial u}{\partial t}(\mathbf{x},t_{n-1}))\right).$$

5. Use the Taylor expansion formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}\int_a^x (x-t)^2 f'''(t)dt$$

to prove

$$E^{n-1/2}(\mathbf{x}) = \frac{\tau}{8}\left(\frac{\partial^2}{\partial t^2}u(\mathbf{x},t_n) - \frac{\partial^2}{\partial t^2}u(\mathbf{x},t_{n-1})\right) + \frac{1}{6\tau}\int_{t_{n-1}}^{t_n}(t_{n-1/2}-t)^2\frac{\partial^3}{\partial t^3}u(\mathbf{x},t)\ dt,$$

where $t_{n-1/2} := \frac{1}{2}(t_n + t_{n-1})$. *Hint:* Apply the Taylor formula twice: once with $a = t_{n-1}$ and $x = t_{n-1/2}$ and once with $a = t_n$ and $x = t_{n-1/2}$.

6. Deduce the following bound for $E^{n-1/2}$

$$\|E^{n-1/2}\|^2_{L^2(\Omega)} \leqslant C\tau^3\|\frac{\partial^3}{\partial t^3}u\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))},$$

where C is a constant independent of N.

7. Denote by $e^n(\mathbf{x}) := u(.,t_n) - u^n(.)$, $n = 1,...,N$, the errors and prove *using the results obtained in the previous steps* that there exists a constant C independent of N such that

$$\left(\sup_{1\leqslant n\leqslant N}\|e^n\|^2_{L^2(\Omega)} + 2\sum_{n=1}^N \tau\|\nabla(\frac{e^n+e^{n-1}}{2})\|^2_{L^2(\Omega)}\right)^{1/2} \leqslant C\tau^2\|\frac{\partial^3}{\partial t^3}u\|_{L^2(0,T;L^2(\Omega))}.$$

Exercise 2 **50%**

Let $\Omega \subset \mathbb{R}^2$ be a bounded with smooth boundary. Given f smooth, consider the problem of finding $u \in H^1_0(\Omega)$ such that

$$a(u,v) := \int_\Omega \nabla u\cdot\nabla v = \int_\Omega fv =: F(v) \qquad \forall v \in H^1_0(\Omega).$$

Let $\mathcal{T}_h$ be a triangulation of Ω, where each triangle at the boundary has at most one curved side matching $\partial\Omega$. We assume that there exists $\rho > 0$ such that for each triangle $T \in \mathcal{T}_h$ we can find two concentric circular discs D_1 and D_2 such that

$$D_1 \subset T \subset D_2 \qquad \text{and} \qquad \frac{\text{diam}(D_2)}{\text{diam}(D_1)} \leqslant \rho. \tag{4}$$

We define the finite element space $\mathbb{V}_h$ by

$$\mathbb{V}_h := \left\{v \in C^0(\overline{\Omega})\ :\ v|_T \in \mathbb{P}_1 \qquad \text{and vanishes at boundary nodes}\right\}.$$

Notice that $\mathbb{V}_h \not\subset \mathbb{H}^1_0(\Omega)$ because these functions do not vanish on $\partial\Omega$. Still, the finite element approximation $u_h \in \mathbb{V}_h$ is defined as satisfying

$$a(u_h,v_h) = F(v_h), \qquad \forall v_h \in \mathbb{V}_h.$$

We start by showing the coercivity of $a(.,.)$ on $\mathbb{V}_h \times \mathbb{V}_h$.

- Let $T \in \mathcal{T}_h$ be a triangle with a curved edge e on the boundary of Ω. Assume that $v \in W^{3,\infty}(T)$ (all third order derivatives are in $L^\infty(T)$) and $w_h \in \mathbb{V}_h$. Show that there exists a constant C independent of h such that

$$\left| \int_e \frac{\partial v}{\partial \nu} w_h \right| \leqslant C h_e^3 \mathrm{diam}(D_2)^{-1} \|v\|_{W^{3,\infty}(T)} \|\nabla w_h\|_{L^2(T)},$$

where $h_e = |e|$, D_2 is the disc associated with T in (4) and ν is the outward pointing normal to Ω. .
Hint: Note that the integrant vanishes at the end points of the curved edge so that you can use (without proof) that

$$\int_e \frac{\partial v}{\partial \nu} w_h \leqslant C h_e^3 \left| \frac{\partial v}{\partial \nu} w_h \right|_{W^{2,\infty}(e)}.$$

However, you will need to prove that

$$\|w_h\|_{W^{1,\infty}(D_2)} \leqslant C \mathrm{diam}(D_2)^{-1} \|w_h\|_{H^1(T)}.$$

- Assume that h is small enough so that $h_e \leqslant 2\mathrm{diam}(T) < 2\mathrm{diam}(D_2)$ to deduce that

$$\left| \int_{\partial\Omega} \frac{\partial v}{\partial \nu} w_h \right| \leqslant C h^{3/2} \|v\|_{W^{3,\infty}(\Omega)} \|w_h\|_{H^1(\Omega)}. \tag{5}$$

- Show that for h sufficiently small, there exists a constant C independent of h such that

$$a(v_h, v_h) \geqslant C \|v_h\|_{H^1(\Omega)}^2, \qquad \forall v_h \in \mathbb{V}_h.$$

Hint: use (5) with v smooth such that $\frac{\partial}{\partial \nu} v = 1$ together with the estimate (seen in class)

$$C\|v_h\|_{H^1(\Omega)} \leqslant \|\nabla v_h\| + \left| \int_{\partial\Omega} v_h \right|$$

for a constant C only depending on Ω.

We can no proceed similarly to the non-conforming case to deduce error estimates.

- Prove that for every $w_h \in \mathbb{V}_h$

$$a(u - u_h, w_h) = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} w_h$$

and in particular for a constant C independent of h

$$\sup_{w_h \in \mathbb{V}_h} \frac{a(u - u_h, w_h)}{\|\nabla w_h\|} \leqslant C h^{3/2} \|u\|_{W^{3,\infty}(\Omega)}.$$

- Show that there exist constants C_1 and C_2 only depending on ρ such that

$$\|\nabla(u - u_h)\| \leqslant C_1 \left(\inf_{v_h \in \mathbb{V}_h} \|\nabla(u - v_h)\| + \sup_{w_h \in \mathbb{V}_h} \frac{a(u - u_h, w_h)}{\|\nabla w_h\|} \right) \leqslant C_2 h \|u\|_{W^{3,\infty}(\Omega)}.$$

Hint: you can use standard interpolation estimates without proving them.