Numerical Methods
MATH 417 - 501
A. Bonito

April 26
Spring 2018

First Name: ________________ **Last Name:** ________________

Exam 2

- 75 minute individual exam;
- Answer the questions in the space provided. If you run out of space, continue onto the back of the page. Additional space is provided at the end;
- **Show and explain all work**;
- **Underline** the answer of each steps;
- The use of books, personal notes, **calculator**, cellphone, laptop, and communication with others is forbidden;
- By taking this exam, you agree to follow the university's code of academic integrity.

Ex 1	Ex 2	Ex 3	Total

Exercise 1 **20%**

Replace the symbols W, X, Y, Z in the matlab code below to solve the linear system

$$Ax = b$$

when A is tri-diagonal, i.e. the coefficients satisfy $a_{ij} = 0$ whenever $|i - j| > 1$.

```
%%%% GaussTri %%%%
%% Input:  tridiagonal square matrix A (not checked)
%%         vector b of corresponding size (not checked)
%% Output: solution to Ax=b

function  x=GaussTri(A,b)

N=size(A,1);

for i=1:N-1
        %multiply the ith row by pivot
        p=1/A(i,i);
        A(i,W) = p*A(i,X);
        b(i) = p*b(i);

        % eliminate the ith column
        A(i+1,i+1)=A(i+1,i+1)-A(Y,i)*A(i,Y);
        b(i+1) = b(i+1)-A(Z,i)*b(i);
end

%last step
p = 1/A(N,N);
b(N)=p*b(N);

%once the matrix is upper triangular (with one on the diagonal)
%solve (the solution is stored in b)
for i=N-1:-1:1
         b(i)=b(i)-A(i,i+1)*b(i+1);
end

x=b;
%%%% END %%%%
```

Exercise 2 **40%**

Consider the following boundary value problem

$$\begin{aligned} -y''(x) + \sqrt{x}\, y(x) &= 1, \qquad 0 < x < 1; \\ y(0) = 0, \quad y(1) &= 1. \end{aligned}$$

1. Let N be a positive integer, $h = 1/N$ and $x_i = ih$ for $i = 0, ..., N$. Write a finite difference scheme to approximate the above boundary value problem based on a second order finite difference approximation of y''.
2. Determine the linear system satisfied by

   $$\mathbf{Y} = (Y_i)_{i=1}^{N-1},$$

   where Y_i are the approximations of $y(x_i)$ for $i = 1, ..., N-1$.

Exercise 3 **40%**

- Consider the following data $x_1 = 0$, $y_1 = 0$, $x_2 = 1$, $y_2 = -1$, $x_3 = 2$, $y_3 = 0$. Write the linear system to be solved to find the coefficients α, β for the polynomial $p(x) = \alpha x + \beta$ to be the best least-square fit of the data.
- Consider the following data $x_1 = 0$, $y_1 = 0$, $x_2 = \frac{1}{2}$, $y_2 = -1$, $x_3 = 1$, $y_3 = 0$. Write the linear system to be solved to find the coefficients α, β for the trigonometric polynomial $T(x) = \alpha \cos(\pi x) + \beta$ to be the best least-square fit of the data.

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Exam 2: solutions

Exercise 1 20%

```
%%%% GaussTri %%%%
%% Input:  tridiagonal square matrix A (not checked)
%%         vector b of corresponding size (not checked)
%% Output: solution to Ax=b

function  x=GaussTri(A,b)

N=size(A,1);

for i=1:N-1
        %multiply the ith row by pivot
        p=1/A(i,i);
        A(i,i+1) = p*A(i,i+1);
        b(i) = p*b(i);

        % eliminate the ith column
        A(i+1,i+1)=A(i+1,i+1)-A(i+1,i)*A(i,i+1);
        b(i+1) = b(i+1)-A(i+1,i)*b(i);
end

%last step
p = 1/A(N,N);
b(N)=p*b(N);

%once the matrix is upper triangular (with one on the diagonal)
%solve (the solution is stored in b)
for i=N-1:-1:1
         b(i)=b(i)-A(i,i+1)*b(i+1);
end

x=b;
%%%% END %%%%
```

Exercise 2 40%

1. The corresponding finite difference scheme reads

$$\frac{-Y_{i+1}+2Y_i-Y_{i-1}}{h^2}+\sqrt{x_i}\;Y_i=1, \qquad i=1,...,N-1;$$
$$Y_0=0; \qquad Y_N=1.$$

2. The linear system is $AY=F$ where

$$A=\begin{pmatrix} \frac{2}{h^2}+\sqrt{x_1} & -\frac{1}{h^2} & & & \\ -\frac{1}{h^2} & \frac{2}{h^2}+\sqrt{x_2} & -\frac{1}{h^2} & & \\ & & \ddots & & \\ & -\frac{1}{h^2} & \frac{2}{h^2}+\sqrt{x_{N-2}} & & -\frac{1}{h^2} \\ & & -\frac{1}{h^2} & & \frac{2}{h^2}+\sqrt{x_{N-1}} \end{pmatrix} \in \mathbb{R}^{N-1\times N-1}$$

$$
Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_{N-1} \end{pmatrix} \in \mathbb{R}^{N-1}, \qquad F = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 + \frac{1}{h^2}. \end{pmatrix} \in \mathbb{R}^{N-1},
$$

Exercise 3 **40%**

Consider the following data $x_1 = 0$, $y_1 = 0$, $x_2 = 1$, $y_2 = -1$, $x_3 = 2$, $y_3 = 0$.

- Ideally we would like that

  $$\beta = 0, \qquad \alpha + \beta = -1, \qquad 2\alpha + \beta = 0$$

  or

  $$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

  Whence, the least-square system to solve reads

  $$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}^T \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

  i.e.

  $$\begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

- We proceed similarly. Ideally we would like that

  $$\alpha + \beta = 0, \qquad \beta = -1, -\alpha + \beta = 0$$

  or

  $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

  Whence, the least-square system to solve reads

  $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}^T \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

  i.e.

  $$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$