REGULARITY OF THE MAXWELL EQUATIONS IN HETEROGENEOUS MEDIA AND LIPSCHITZ DOMAINS

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Abstract. This note establishes regularity estimates for the solution of the Maxwell equations in Lipschitz domains with non-smooth coefficients and minimal regularity assumptions. The argumentation relies on elliptic regularity estimates for the Poisson problem with non-smooth coefficients.

1. Introduction

The purpose of this note is to prove regularity estimates for the solution of the Maxwell equations in Lipschitz domains with non-smooth coefficients and minimal regularity assumptions. More precisely, given a Lipschitz domain Ω, we are interested in the time harmonic Maxwell system,

$$\nabla\times\mathbf{E} - \mathrm{i}\omega\mu\mathbf{H} = 0 \quad \text{and} \quad \nabla\times\mathbf{H} + \mathrm{i}\omega\varepsilon\mathbf{E} = \mathbf{J}, \tag{1.1}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, $\mathbf{J}$ is a given (divergence-free) current density, ε is the tensor-valued electrical permittivity of the material, and μ is the tensor-valued magnetic permeability. The tensor fields $\mathbf{x} \mapsto \varepsilon(\mathbf{x})$ and $\mathbf{x} \mapsto \mu(\mathbf{x})$ are assumed to be piecewise smooth and uniformly positive definite. The Maxwell system (1.1) must be supplemented with boundary conditions. In this work, we assume that $\mathbf{E}$ satisfies the perfect conductor boundary condition , i.e.,

$$\mathbf{E}\times\mathbf{n}|_\Gamma = 0, \tag{1.2}$$

where $\mathbf{n}$ is the outer unit normal of Ω and Γ is the boundary of Ω. Eliminating the magnetic field from (1.1), the electric field satisfies the following system:

$$\nabla\times\left(\mu^{-1}\nabla\times\mathbf{E}\right) - \omega^2\varepsilon\mathbf{E} = i\omega\mathbf{J}, \quad \nabla\cdot(\varepsilon\mathbf{E}) = 0, \quad \mathbf{E}\times\mathbf{n}|_\Gamma = 0. \tag{1.3}$$

If the electric field is eliminated instead, we obtain

$$\nabla\times\left(\varepsilon^{-1}\nabla\times\mathbf{H}\right) - \omega^2\mu\mathbf{H} = \nabla\times\left(\varepsilon^{-1}\mathbf{J}\right), \quad \nabla\cdot(\mu\mathbf{H}) = 0, \quad (\mu\mathbf{H})\cdot\mathbf{n}|_\Gamma = 0, \tag{1.4}$$

where the boundary condition $(\mu\mathbf{H})\cdot\mathbf{n}|_\Gamma = 0$ is a consequence of (1.2).

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Establishing regularity estimates for (1.3) and (1.4) requires studying the following model problem

$$\nabla\times\left(\mu^{-1}\nabla\times\mathbf{F}\right)=\mathbf{g},\qquad \nabla\cdot(\varepsilon\mathbf{F})=0,\qquad \mathbf{F}\times\mathbf{n}|_\Gamma=0. \tag{1.5}$$

In the above formulation, $\mathbf{g}$ is assumed to belong to $\mathbf{L}^2(\Omega)$ with $\nabla\cdot\mathbf{g}=0$. However, for numerical considerations, see Bonito and Guermond [1] and Bonito et al. [2], it is advantageous to consider the following mixed formulation of the above problem:

$$\nabla\times\left(\mu^{-1}\nabla\times\mathbf{F}\right)+\varepsilon\nabla p=\mathbf{g},\qquad \nabla\cdot(\varepsilon\mathbf{F})=0,\qquad \mathbf{F}\times\mathbf{n}|_\Gamma=0,\quad p|_\Gamma=0, \tag{1.6}$$

where this time it is not necessary to require that $\mathbf{g}$ be solenoidal. The additional scalar field p accommodates the possible non-zero divergence of $\mathbf{g}$. Of course $p=0$ if $\mathbf{g}$ is divergence free.

The main result (Theorem 5.1) established in this paper is that, under very mild assumptions on the fields μ and ε, there is $\tau(\varepsilon,\mu)$ (possibly les than $\frac{1}{2}$ for Lipschitz boundaries) so that the mapping $\mathbf{g}\longmapsto(\mathbf{F},\nabla\times\mathbf{F})$ is continuous from $\mathbf{L}^2(\Omega)$ to $\mathbf{H}^s(\Omega)\times\mathbf{H}^s(\Omega)$ for all $0\le s<\tau(\varepsilon,\mu)$. Theorem 5.1 relies on the following two embedding estimates established in Proposition 4.1 and Proposition 4.2, respectively: There are constants $c(s,\varepsilon)$, $c(s,\mu)$ so that

$$\|\mathbf{F}\|_{\mathbf{H}^s(\Omega)}\le c(s,\varepsilon)\left(\|\nabla\times\mathbf{F}\|_{\mathbf{L}^2(\Omega)}+\|\nabla\cdot(\varepsilon\mathbf{F})\|_{H^{s-1}(\Omega)}\right),\qquad \forall s\in[0,\tau(\varepsilon)) \tag{1.7}$$

holds for all smooth vector fields $\mathbf{F}$ with zero tangent trace, and

$$\|\mathbf{G}\|_{\mathbf{H}^s(\Omega)}\le c(s,\mu)\|\nabla\times\mathbf{G}\|_{\mathbf{L}^2(\Omega)},\qquad \forall s\in[0,\tau(\mu)) \tag{1.8}$$

holds for all smooth vector fields $\mathbf{G}$ such that $\nabla\cdot(\mu\mathbf{G})=0$ and $(\mu\mathbf{G}\cdot)\mathbf{n}|_\Gamma=0$. The estimate (1.7) is of particular interest when approximating the Maxwell equations with Lagrange finite elements and when using a stabilization technique that requires controlling the divergence of the electric field in $\mathbf{H}^{s-1}(\Omega)$ with $s\in(0,\frac{1}{2})$, see e.g. Bonito and Guermond [1]. The estimates (1.7)-(1.8) are also useful to establish compactness on the electric field and its curl. More precisely, assuming that $\mathbf{F}$ solves (1.5) and upon setting $\mathbf{G}=\mu^{-1}\nabla\times\mathbf{F}$, we observe that $(\mu\mathbf{G}\cdot)\mathbf{n}|_\Gamma=0$, $\nabla\cdot(\mu\mathbf{G})=0$, and (1.8) implies that $\mathbf{G}$ is a member of $\mathbf{H}^s(\Omega)$, which in turn, under mild assumptions on the multiplier μ, implies that $\nabla\times\mathbf{F}$ is in $\mathbf{H}^s(\Omega)$. Estimates similar to (1.7)-(1.8) have been obtained by Jochmann [12] using different norms to control $\mathbf{F}$ in (1.7). More precisely, the right hand side of (1.7) is replaced therein by a norm in an interpolation space between $\mathbf{H}_{0,\mathrm{curl}}(\Omega)$ and $\mathbf{L}^2(\Omega)$. Although this alternative estimate entails less regularity on $\nabla\times\mathbf{F}$, it seems to us that the interpolation norm may not have a characterization as clear as that in (1.7).

To the best of our knowledge, the results stated in Theorem 4.1, Proposition 4.1, Proposition 4.2, and Theorem 5.1 are new in the range $s\in(0,\frac{1}{2})$. In particular Theorem 4.1, Proposition 4.1, and Proposition 4.2 generalize the now well-known fact, established in particular in Costabel [3], that $\mathbf{H}_{0,\mathrm{curl}}(\Omega)\cap\mathbf{H}_{\mathrm{div}}(\Omega)$ and $\mathbf{H}_{\mathrm{curl}}(\Omega)\cap\mathbf{H}_{0,\mathrm{div}}(\Omega)$ are continuously embedded in $\mathbf{H}^{\frac{1}{2}}(\Omega)$. The proofs of Proposition 4.1 and Proposition 4.2 use regularity estimates on the Laplace equation with non-smooth coefficients supplemented with either Dirichlet or Neumann data. These regularity estimates are stated in Theorem 3.1. The estimates in Theorem 3.1 are not new, and may be found scattered in the literature in various guises. We nevertheless have included the proof of this theorem in the paper to make it self-contained. For instance, Savaré [16] has proved similar results for Dirichlet data by assuming some global integrability of the right-hand side of the Laplace equation and assuming that the multiplier is piecewise constant over two sub-domains. Later, Jochmann [11] removed the extra integrability assumption, considered finitely many sub-domains and mixed Dirichlet-Neumann boundary conditions. His proof technique is based on local maps and requires some mild regularity on the boundary of the domain (each map is Lipschitz and its Jacobian is piecewise $\mathcal{C}^{0,\frac{1}{2}}$) when dealing with mixed boundary condition. Following the arguments proposed by Meyers [15] and Jochmann [11], we provide in Theorem 3.1 a regularity result for both types of boundary conditions assuming only Lipschitz regularity on the boundary of the domain and piecewise smoothness on the multiplier. The proof uses the regularity results of Jerison and Kenig [10] on Lipschitz domains for the Laplace equation.

The paper is organized as follows. We introduce some notation and prove preliminary results on multipliers in §2. Regularity properties of the Laplace equation with non-smooth coefficients are discussed in §3 and collected in Theorem 3.1. We establish embedding results in §4; these results are stated in Proposition 4.1 and Proposition 4.2 and are used to prove regularity estimates on the Maxwell system. Finally §5 focuses on the Maxwell system with non-smooth coefficients, e.g. electrical conductivity, magnetic permeability, or electrical permittivity. The main result of this section is Theorem 5.1. The main thrust for the present work is our ongoing research program to establish convergence estimates for the approximation of the Maxwell system using H^1-conforming Lagrange finite elements in the spirit of Bonito and Guermond [1], Bonito et al. [2].

2. Preliminaries

The objective of this section is to introduce notation and recall key results regarding multipliers. Although some of these results are somewhat standard, we provide proofs for the sake of completeness.

2.1. **Notation.** Henceforth Ω is a bounded, simply-connected Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$, that we assume to be partitioned into M connected Lipschitz subdomains $\Omega_1, \cdots, \Omega_M$. The boundary of the domain is assumed to be connected and is denoted Γ, i.e., $\Gamma := \partial\Omega$, the interface between the subdomains $\Omega_1, \cdots, \Omega_M$ is denoted Σ, i.e.,

$$\Sigma := \bigcup_{i \neq j} \Gamma_i \cap \Gamma_j. \tag{2.1}$$

Let $E \subset \Omega$ be a non-empty connected open Lipschitz subset of Ω. We denote $(\cdot, \cdot)_E$ the inner product in $L^2(E)$ for scalar-valued fields, in $\mathbf{L}^2(E)$ for vector-valued field, and in $\mathbb{L}^2(E)$ for $d \times d$ tensor-valued fields. The subscript is omitted if the domain of integration is Ω. The subspaces of $L^2(E)$ and $H^1(E)$ composed of the functions with zero average over E are denoted $\dot{L}^2(E)$ and $\dot{H}^1(E)$ respectively. Owing to the Poincaré and Poincaré-Friedrichs inequalities, we equip $H^1_0(E)$ and $\dot{H}^1(E)$ with the following norms:

$$\|u\|_{H^1_0(E)} := \|\nabla u\|_{\mathbf{L}^2(E)}, \qquad \|u\|_{\dot{H}^1(E)} := \|\nabla u\|_{\mathbf{L}^2(E)}. \tag{2.2}$$

The norm of $\big(H^1_0(E)\big)'$ is then defined by

$$\|F\|_{(H^1_0(E))'} := \sup_{0 \neq u \in H^1_0(E)} \frac{\langle F, u\rangle_{(H^1_0(E)', H^1_0(E)}}{\|\nabla u\|_{\mathbf{L}^2(E)}}, \tag{2.3}$$

and the norm of $\big(H^1(E)\big)'$ is defined similarly.

We define the Sobolev spaces $H^s(E)$, $\dot{H}^s(E)$ for $0 < s < 1$, by using the real interpolation method (K-method) between $L^2(E)$ and $H^1(E)$ and between $\dot{L}^2(E)$ and $\dot{H}^1(E)$, respectively; see for instance Lions and Peetre [14] or Tartar [17, Chapter 22]. We also define $H^s_0(E)$ by interpolation between $L^2(E)$ and $H^1_0(E)$, so that for any $0 \le s < \frac{1}{2}$, the spaces $H^s_0(E)$ and $H^s(E)$ coincide (cf. Lions and Magenes [13, Thm 11.1] or Grisvard [8, Cor. 1.4.4.5]). For the sake of conciseness, we denote

$$\mathcal{H}^s(E) := \begin{cases} H^s_0(E) & \text{for Dirichlet boundary conditions,} \\ \dot{H}^s(E) & \text{for Neumann boundary conditions,} \end{cases} \tag{2.4}$$

for $s \in [0,1]$, and $\mathcal{H}^s(E) := (\mathcal{H}^{-s}(E))'$ for $s \in [-1,0]$. Note that when we use $\mathcal{H}^1(E) = \dot{H}^1(E)$, the elements of the dual space $\mathcal{H}^s(E)$, $s \in [-1,0)$, cannot be identified with distributions in $(\mathcal{D}(E))'$ in general. For instance, for any $g \in L^2(\partial E)$ and $s \in [-1,-\frac{1}{2})$, the linear form $\mathcal{H}^{-s}(E) \ni p \mapsto \int_{\partial E} gp$ is in $\mathcal{H}^s(E)$ but it cannot be represented by a distribution in $(\mathcal{D}(E))'$. The above definitions naturally extend to vector fields and tensor fields, and in these cases we use bold and blackboard symbols to avoid confusion. For any $s \in [0,1]$, we abusively denote $\mathcal{H}^{1+s}(E)$ the following spaces:

$$\mathcal{H}^{1+s}(E) = \begin{cases} \{p \in H^1_0(E),\ \nabla p \in \mathbf{H}^s(E)\} & \text{if } \mathcal{H}^1(E) = H^1_0(E) \\ \{p \in \dot{H}^1(E),\ \nabla p \in \mathbf{H}^s(E)\} & \text{if } \mathcal{H}^1(E) = \dot{H}^1(E), \end{cases} \tag{2.5}$$

and we equip $\mathcal{H}^{1+s}(E)$ with the following norm:

$$\|p\|_{\mathcal{H}^{1+s}(E)} := \|\nabla p\|_{\mathbf{H}^s(E)}. \tag{2.6}$$

The Poincaré constant over each sudomain Ω_i is denoted C_{Ω_i}, i.e.,

$$\forall \mathbf{u} \in \mathbf{H}^1_0(\Omega_i), \qquad \|\mathbf{u}\|_{L^2(\Omega_i)} \le C_{\Omega_i} \|\nabla \mathbf{u}\|_{\mathbb{L}^2(\Omega_i)}, \tag{2.7}$$

and we set

$$C_\Omega := \max_{1 \le i \le M} C_{\Omega_i}. \tag{2.8}$$

The norm of the natural injection from $\mathbf{H}^s(\Omega_i)$ to $\mathbf{H}^s_0(\Omega_i)$ is denoted D_{s,Ω_i} for all $s \in [0,\frac{1}{2})$ and all $i = 1,\dots,M$, i.e.,

$$\|\mathbf{v}\|_{\mathbf{H}^s_0(\Omega_i)} \le D_{s,\Omega_i} \|\mathbf{v}\|_{\mathbf{H}^s(\Omega_i)}, \qquad \forall \mathbf{v} \in \mathbf{H}^s(\Omega_i). \tag{2.9}$$

In addition, we set

$$D_{s,\Omega} := \max(\max_{1 \le i \le M} D_{s,\Omega_i}, 1). \tag{2.10}$$

Assuming that X and Y are two Banach spaces, $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators $X \to Y$ equipped with their natural norm, $\|\cdot\|_{X\to Y}$. In the rest of the paper we use the generic notation c for constants. The value of c may change at each occurrence.

2.2. **Multiplier s.** We now introduce notation to stipulate the regularity that we require on the tensor fields ε and μ. In the rest of the paper we assume that all the tensors are symmetric. We then define

$$\mathbb{W}^{1,\infty}_{\Sigma}(\Omega) := \left\{ \nu \in \mathbb{L}^{\infty}(\Omega) \mid \nabla(\nu_{|\Omega_i}) \in \mathbb{L}^{\infty}(\Omega_i)^d,\ i = 1, \cdots, M \right\}, \tag{2.11}$$

where the norm of tensors is defined to be the norm induced by the Euclidean norm. For all ν in $\mathbb{W}^{1,\infty}_{\Sigma}(\Omega)$ we define $\nu_{\max} \in \mathbb{R}$ such that

$$\nu_{\max} = \|\lambda_{\max}(\nu)\|_{\mathbb{L}^{\infty}(\Omega)}, \tag{2.12}$$

where $\lambda_{\max}(\nu)$ is the largest eigenvalue of ν. We also define $\Lambda_\nu \in \mathbb{R}$ by

$$\Lambda_\nu := \frac{\max_{i=1,\cdots,M} \|\nabla(\nu_{|\Omega_i})\|_{\mathbb{L}^{\infty}(\Omega_i)^d}}{\nu_{\max}}, \quad \text{if} \quad \nu_{\max} \neq 0, \quad \Lambda_\nu := 0 \quad \text{otherwise}.$$

Given a tensor field ν in $\mathbb{W}^{1,\infty}_{\Sigma}(\Omega)$, we call multiplier $\mathcal{E}_\nu$ associated with ν the linear operator $\mathcal{E}_\nu : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ so that

$$(\mathcal{E}_\nu(\mathbf{u}))(\mathbf{x}) := \nu(\mathbf{x})\mathbf{u}(\mathbf{x}) \text{ for a.e. } \mathbf{x} \text{ in } \Omega, \quad \forall \mathbf{u} \in \mathbf{L}^2(\Omega). \tag{2.13}$$

The key result of this section is the following

Proposition 2.1. *Let $\nu \in \mathbb{W}^{1,\infty}_{\Sigma}(\Omega)$. Then $\mathcal{E}_\nu \in \mathcal{L}(\mathbf{H}^s(\Omega), \mathbf{H}^s(\Omega))$ for every $s \in \left[0, \frac{1}{2}\right)$ and*

$$\|\mathcal{E}_\nu\|_{\mathbf{H}^s(\Omega)\to\mathbf{H}^s_0(\Omega)} \leq \nu_{\max} N_{s,\nu}, \quad \textit{where} \quad N_{s,\nu} := D_{s,\Omega}(2(1 + C^2_\Omega \Lambda^2_\nu))^{\frac{s}{2}}. \tag{2.14}$$

Moreover, the following holds for all $r \in [0, \frac{1}{2})$,

$$\|\mathcal{E}_\nu\|_{\mathbf{H}^s(\Omega)\to\mathbf{H}^s_0(\Omega)} \leq \nu_{\max} N^{\frac{s}{r}}_{r,\nu}, \quad \forall s \in [0, r]. \tag{2.15}$$

Proof. Let $0 \leq s < \frac{1}{2}$ and consider $\mathbf{u} \in \mathbf{H}^s(\Omega)$. We set $\mathbf{u}_i := \mathbf{u}_{|\Omega_i}$ for $i = 1, \cdots M$. Then owing to Lemma 2.1 below, $\mathbf{u}_i \in \mathbf{H}^s(\Omega_i)$ for all $i = 1, \ldots, M$. This in turn implies that $\mathbf{u}_i \in \mathbf{H}^s_0(\Omega_i)$ since $0 \leq s < \frac{1}{2}$. We now proceed by using the K-interpolation theory. For any $\mathbf{u}_i \in [\mathbf{L}^2(\Omega_i), \mathbf{H}^1_0(\Omega_i)]_s = \mathbf{H}^s_0(\Omega)$, we set

$$\mathcal{K}(t, \mathbf{u}_i, \mathbf{L}^2(\Omega_i), \mathbf{H}^1_0(\Omega_i)) := \inf_{\mathbf{v}\in\mathbf{H}^1_0(\Omega_i)} \left\{ \|\mathbf{u}_i - \mathbf{v}\|^2_{\mathbf{L}^2(\Omega_i)} + t^2 \|\mathbf{v}\|^2_{\mathbf{H}^1_0(\Omega_i)} \right\}.$$

Then asserting that $\mathbf{u}_i$ is in $\mathbf{H}^s_0(\Omega_i)$ is equivalent to saying that the mapping $\mathbb{R}^+ \ni t \mapsto \mathcal{K}(t, \mathbf{u}_i, \mathbf{L}^2(\Omega_i), \mathbf{H}^1_0(\Omega_i))$ is in $L^1\left(\mathbb{R}^+, \frac{dt}{t^{1+2s}}\right)$. For any $t > 0$, we define $\mathbf{u}_{i,t} \in \mathbf{H}^1_0(\Omega_i)$ so that $(\mathbf{u}_{i,t}, \mathbf{v}_i)_{\Omega_i} + t^2 (\nabla\mathbf{u}_{i,t}, \nabla\mathbf{v}_i)_{\Omega_i} = (\mathbf{u}_i, \mathbf{v}_i)_{\Omega_i}$, for all $\mathbf{v}_i \in \mathbf{H}^1_0(\Omega_i)$. This definition implies that

$$\|\mathbf{u}_i - \mathbf{u}_{i,t}\|^2_{\mathbf{L}^2(\Omega_i)} + t^2\|\mathbf{u}_{i,t}\|^2_{\mathbf{H}^1_0(\Omega_i)} = \mathcal{K}(t, \mathbf{u}_i, \mathbf{L}^2(\Omega_i), \mathbf{H}^1_0(\Omega_i)).$$

Now we estimate $\mathcal{K}(t, \mathcal{E}_\nu\mathbf{u}, \mathbf{L}^2(\Omega), \mathbf{H}^1_0(\Omega))$. For this purpose we define $\mathbf{u}_t$ by $\mathbf{u}_{t|\Omega_i} = \mathbf{u}_{i,t}$. Since every $\mathbf{u}_{i,t}$ vanishes on Σ, we have $\mathbf{u}_t \in \mathbf{H}^1_0(\Omega)$, $\mathcal{E}_\nu\mathbf{u}_t \in \mathbf{H}^1_0(\Omega)$

with the following two estimates:

$$
\begin{aligned}
\|\mathcal{E}_\nu(\mathbf{u}-\mathbf{u}_t)\|^2_{\mathbf{L}^2(\Omega)} &= \sum_{i=1}^M \|\nu(\mathbf{u}_i-\mathbf{u}_{i,t})\|^2_{\mathbf{L}^2(\Omega_i)} \le \nu^2_{\max}\sum_{i=1}^M \|\mathbf{u}_i-\mathbf{u}_{i,t}\|^2_{\mathbf{L}^2(\Omega_i)},\\
\|\mathcal{E}_\nu \mathbf{u}_t\|^2_{\mathbf{H}^1_0(\Omega)} := \|\nabla\left(\mathcal{E}_\nu \mathbf{u}_t\right)\|^2_{\mathbb{L}^2(\Omega)} &\le 2\sum_{i=1}^M \|\nu\nabla \mathbf{u}_{i,t}\|^2_{\mathbb{L}^2(\Omega_i)} + \|(\nabla\nu)\mathbf{u}_{i,t}\|^2_{\mathbb{L}^2(\Omega_i)}\\
&\le 2\nu^2_{\max}\left(1+C^2_\Omega\Lambda^2_\nu\right)\sum_{i=1}^M \|\nabla \mathbf{u}_{i,t}\|^2_{\mathbb{L}^2(\Omega_i)}
\end{aligned}
$$

where we used the Poincaré inequality on every Ω_i in the second estimate. Combining the above two inequalities and setting $\alpha^2 := 2\left(1+C^2_\Omega\Lambda^2_\nu\right)$ gives

$$
\begin{aligned}
\mathcal{K}(t,\mathcal{E}_\nu\mathbf{u},\mathbf{L}^2(\Omega),\mathbf{H}^1_0(\Omega)) &\le \|\mathcal{E}_\nu(\mathbf{u}-\mathbf{u}_{\alpha t})\|^2_{\mathbf{L}^2(\Omega)} + t^2\|\mathcal{E}_\nu\mathbf{u}_{\alpha t}\|^2_{\mathbf{H}^1_0(\Omega)}\\
&\le \nu^2_{\max}\sum_{i=1}^M\left(\|\mathbf{u}_i-\mathbf{u}_{i,\alpha t}\|^2_{\mathbf{L}^2(\Omega_i)} + \alpha^2t^2\|\nabla\mathbf{u}_{i,\alpha t}\|^2_{\mathbb{L}^2(\Omega_i)}\right)\\
&\le \nu^2_{\max}\sum_{i=1}^M \mathcal{K}(\alpha t,\mathbf{u}_i,\mathbf{L}^2(\Omega_i),\mathbf{H}^1_0(\Omega_i)).
\end{aligned}
$$

As a result $\mathcal{E}_\nu\mathbf{u}\in\mathbf{H}^s_0(\Omega)$, and using $\|\mathbf{u}_i\|_{\mathbf{H}^s_0(\Omega_i)} \le D_{s,\Omega_i}\|\mathbf{u}_i\|_{\mathbf{H}^s(\Omega_i)}$ we deduce that

$$
\begin{aligned}
\nu^{-2}_{\max}\|\mathcal{E}_\nu\mathbf{u}\|^2_{\mathbf{H}^s_0(\Omega)} &\le \sum_{i=1}^M\int_0^\infty \mathcal{K}(\alpha t,\mathbf{u}_i,\mathbf{L}^2(\Omega_i),\mathbf{H}^1_0(\Omega_i))t^{-1-2s}\mathrm{d}t \le \alpha^{2s}\sum_{i=1}^M\|\mathbf{u}_i\|^2_{\mathbf{H}^s_0(\Omega_i)}\\
&\le \alpha^{2s}\sum_{i=1}^M D^2_{s,\Omega_i}\|\mathbf{u}_i\|^2_{\mathbf{H}^s(\Omega_i)} \le N^2_{s,\nu}\sum_{i=1}^M\|\mathbf{u}_i\|^2_{\mathbf{H}^s(\Omega_i)}.
\end{aligned}
$$

Then we finally obtain (2.14) by using Lemma 2.1. The inequality (2.15) directly follows from the re-interpolation formula $\mathbf{H}^s(\Omega) = \left[\mathbf{L}^2(\Omega),\mathbf{H}^r(\Omega)\right]_{\frac{s}{r}}$, upon noticing that $N_{0,\nu}=1$, This completes the proof. $\square$

Lemma 2.1. *The following holds for all $s\in[0,1]$ and for all $\mathbf{v}\in\mathbf{H}^s(\Omega)$,*

$$
\sum_{i=1}^M \|\mathbf{v}_{|\Omega_i}\|^2_{\mathbf{H}^s(\Omega_i)} \le \|\mathbf{v}\|^2_{\mathbf{H}^s(\Omega)}. \tag{2.16}
$$

Proof. The result is evident for $s=0$ and $s=1$. Let us now consider $s\in(0,1)$, and let $\mathbf{v}$ be a member of $\mathbf{H}^s(\Omega)$. Recall that

$$
\begin{aligned}
\|\mathbf{v}\|_{\mathbf{H}^s(\Omega)} &:= \left(\int_0^\infty \mathcal{K}(t,\mathbf{v},\mathbf{L}^2(\Omega),\mathbf{H}^1(\Omega))^2 t^{-1-2s}\mathrm{d}t\right)^{\frac{1}{2}},\\
\mathcal{K}(t,\mathbf{v},\mathbf{L}^2(\Omega),\mathbf{H}^1(\Omega))^2 &:= \inf_{\mathbf{w}\in\mathbf{H}^1(\Omega)}\left(\|\mathbf{v}-\mathbf{w}\|^2_{\mathbf{L}^2(\Omega)} + t^2\|\mathbf{w}\|^2_{\mathbf{H}^1(\Omega)}\right).
\end{aligned}
$$

For all $t\in\mathbb{R}_+$, let us denote $\mathbf{v}_t$ the function in $\mathbf{H}^1(\Omega)$ that achieves the infimum in the definition of $\mathcal{K}(t,\mathbf{v},\mathbf{L}^2(\Omega),\mathbf{H}^1(\Omega))$, i.e., $-t^2\Delta\mathbf{v}_t + t^2\mathbf{v}_t + (\mathbf{v}_t-\mathbf{v}) = 0$ over Ω

with homogeneous Neumann boundary condition. Then

$$\begin{aligned}\sum_{i=1}^{M} \|\mathbf{v}_{|\Omega_i}\|^2_{\mathbf{H}^s(\Omega_i)} &= \sum_{i=1}^{M} \int_0^\infty \mathcal{K}(t, \mathbf{v}_{|\Omega_i}, \mathbf{L}^2(\Omega_i), \mathbf{H}^1(\Omega_i))^2 t^{-1-2s} \mathrm{d}t \\ &\leq \sum_{i=1}^{M} \int_0^\infty \left(\|\mathbf{v}_{|\Omega_i} - \mathbf{v}_{t|\Omega_i}\|^2_{\mathbf{L}^2(\Omega_i)} + t^2 \|\mathbf{v}_{t|\Omega_i}\|^2_{\mathbf{H}^1(\Omega_i)} \right) t^{-1-2s} \mathrm{d}t \\ &= \int_0^\infty \left(\sum_{i=1}^{M} \|\mathbf{v}_{|\Omega_i} - \mathbf{v}_{t|\Omega_i}\|^2_{\mathbf{L}^2(\Omega_i)} + t^2 \|\mathbf{v}_{t|\Omega_i}\|^2_{\mathbf{H}^1(\Omega_i)} \right) t^{-1-2s} \mathrm{d}t \\ &= \int_0^\infty \mathcal{K}(t, \mathbf{v}, \mathbf{L}^2(\Omega), \mathbf{H}^1(\Omega))^2 t^{-1-2s} \mathrm{d}t := \|\mathbf{v}\|^2_{\mathbf{H}^s(\Omega)}.\end{aligned}$$

This completes the proof. □

3. Non-constant coefficient Laplace equation

We establish regularity estimates for the Laplace equation with non-constant coefficients in this section.

3.1. The main result.

Let $\mathbb{v}$ be a tensor field in $\mathbb{W}^{1,\infty}_{\Sigma}(\Omega)$ and assume that

$$\exists \mathbb{v}_{\min} > 0 \text{ such that } \boldsymbol{\xi}^T \mathbb{v} \boldsymbol{\xi} \geq \mathbb{v}_{\min} \boldsymbol{\xi}^T \boldsymbol{\xi} \text{ a.e. in } \Omega, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d. \tag{3.1}$$

Consider the following problem: given $f \in \mathcal{H}^{-1}(\Omega)$, find $p \in \mathcal{H}^1(\Omega)$ such that,

$$\forall q \in \mathcal{H}^1(\Omega), \qquad (\mathbb{v}\nabla p, \nabla q) = \langle f, q \rangle_{\mathcal{H}^{-1}(\Omega), \mathcal{H}^1(\Omega)}. \tag{3.2}$$

The existence and uniqueness of a solution to the above problem is ensured by the Lax-Milgram lemma since

$$\forall p \in \mathcal{H}^1(\Omega), \qquad \mathbb{v}_{\min} \|p\|^2_{\mathcal{H}^1(\Omega)} = \mathbb{v}_{\min} (\nabla p, \nabla p) \leq (\mathbb{v}\nabla p, \nabla p).$$

We re-write the above problem (3.2) in the symbolic form $-\Delta^{\mathcal{H}^1}_{\mathbb{v}} p = f$.

The objective of this section is to prove the following theorem which is a variant of a result from Jochmann [11].

Theorem 3.1. *Let $\mathbb{v} \in \mathbb{W}^{1,\infty}_{\Sigma}(\Omega)$ be satisfying (3.1). There exists $\tau \in (0, \frac{1}{2})$, only depending on $\mathbb{v}$, Ω, and the partition $\{\Omega_i\}_{i=1}^M$ such that, for every $s \in [0, \tau)$ and every $f \in \mathcal{H}^{s-1}(\Omega)$, the solution $p \in \mathcal{H}^1(\Omega)$ of the problem (3.2) is in $\mathcal{H}^{1+s}(\Omega)$ and satisfies the estimate*

$$\|p\|_{\mathcal{H}^{s+1}(\Omega)} \leq c \|f\|_{\mathcal{H}^{s-1}(\Omega)}, \tag{3.3}$$

where c depends only on Ω, $\mathbb{v}$, the partition $\{\Omega_i\}_{i=1}^M$, and s.

Remark 3.1 (Extensions to general multipliers). Theorem 3.1 holds in the general case when $\mathbb{v}$ is a multiplier of $\mathbf{H}^s(\Omega)$ satisfying (3.1), i.e., $\mathcal{E}_{\mathbb{v}} \in \mathcal{L}(\mathbf{H}^s(\Omega), \mathbf{H}^s(\Omega))$. The proof proposed in §3.3 extends readily for any multiplier such that $\mathcal{E}_{\mathbb{v}} \in \mathcal{L}(\mathbf{H}^s(\Omega), \mathbf{H}^s(\Omega))$.

We postpone the proof of Theorem 3.1 to §3.3. We will use a technique similar to that in Jochmann [11], where the author proves the result for a more general class of spaces, but requires some additional regularity conditions on the boundary Γ and assumes (2.14). Since it was not clear to us whether or not the additional regularity condition stated in Jochmann [11] was needed for the pure Dirichlet and pure Neumann problems, we re-prove the result here and show that the Lipschitz condition is sufficient. We also derive an almost explicit admissible range for τ; however, our predicted admissible range may not be robust with respect to the contrast of ν.

3.2. **Key lemmas.** Using the same notation as in Jochmann [11], we introduce the operators $\mathcal{J} \in \mathcal{L}(\mathcal{H}^{-1}(\Omega), \mathcal{H}^1(\Omega))$ and $\mathcal{S} \in \mathcal{L}(\mathbf{L}^2, \mathcal{H}^{-1}(\Omega))$ defined as follows:

$$\forall f \in \mathcal{H}^{-1}(\Omega),\ \forall q \in \mathcal{H}^1(\Omega), \qquad (\nabla(\mathcal{J}f), \nabla q) = \langle f, q\rangle_{\mathcal{H}^{-1}(\Omega),\mathcal{H}^1(\Omega)}\,. \tag{3.4}$$

and

$$\forall \mathbf{F} \in \mathbf{L}^2,\ \forall q \in \mathcal{H}^1(\Omega), \qquad \langle \mathcal{S}\mathbf{F}, q\rangle_{\mathcal{H}^{-1}(\Omega),\mathcal{H}^1(\Omega)} := (\mathbf{F}, \nabla q)\,. \tag{3.5}$$

Note that $\mathcal{J}$ is well defined owing to the definition of $\mathcal{H}^1(\Omega)$ and the Lax-Milgram lemma.

Lemma 3.1. *For all $s \in [0,1]$ and for all $\mathbf{F} \in \mathbf{H}^s_0(\Omega)$, we have*

$$\mathcal{S}\mathbf{F} \in \mathcal{H}^{s-1}(\Omega) \text{ and } \|\mathcal{S}\mathbf{F}\|_{\mathcal{H}^{s-1}(\Omega)} \le \|\mathbf{F}\|_{\mathbf{H}^s_0(\Omega)}. \tag{3.6}$$

Proof. This is again a standard interpolation argument. We start with $s = 0$ and $\mathbf{F} \in \mathbf{L}^2(\Omega)$. Then the following series of bounds holds for all $p \in \mathcal{H}^1(\Omega)$:

$$\langle \mathcal{S}\mathbf{F}, p\rangle_{\mathcal{H}^{-1}(\Omega),\mathcal{H}^1(\Omega)} = (\mathbf{F}, \nabla p) \le \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}\|\nabla p\|_{\mathbf{L}^2(\Omega)} = \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}\|p\|_{\mathcal{H}^1(\Omega)},$$

which leads to

$$\|\mathcal{S}\mathbf{F}\|_{\mathcal{H}^{-1}(\Omega)} \le \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}. \tag{3.7}$$

When $s = 1$ and $\mathbf{F} \in \mathbf{H}^1_0(\Omega)$, the following estimates hold for all $p \in \mathcal{H}^1(\Omega)$:

$$\langle \mathcal{S}\mathbf{F}, p\rangle_{\mathcal{H}^{-1}(\Omega),\mathcal{H}^1(\Omega)} = (\mathbf{F}, \nabla p) = -\,(\nabla\cdot\mathbf{F}, p) \le \|\nabla\cdot\mathbf{F}\|_{L^2(\Omega)}\|p\|_{\mathcal{H}^0(\Omega)}.$$

Using the fact that

$$\|\nabla\cdot\mathbf{F}\|^2_{L^2(\Omega)} \le \|\nabla\cdot\mathbf{F}\|^2_{L^2(\Omega)} + \|\nabla\times\mathbf{F}\|^2_{\mathbf{L}^2(\Omega)} = \|\nabla\mathbf{F}\|^2_{L^2(\Omega)} = \|\mathbf{F}\|^2_{\mathbf{H}^1_0(\Omega)},$$

and recalling that $\mathcal{S}\mathbf{F} = -\nabla\cdot\mathbf{F}$ and $\int_\Omega \mathcal{S}\mathbf{F} = 0$, since $\mathbf{F} \in \mathbf{H}^1_0(\Omega)$, we infer that

$$\sup_{0\neq p\in\mathcal{H}^0(\Omega)} \frac{\langle \mathcal{S}\mathbf{F}, p\rangle_{\mathcal{H}^{-1}(\Omega),\mathcal{H}^1(\Omega)}}{\|p\|_{\mathcal{H}^0(\Omega)}} = \|\mathcal{S}\mathbf{F}\|_{\mathcal{H}^0} = \|\mathcal{S}\mathbf{F}\|_{\dot{L}^2(\Omega)} = \|\mathcal{S}\mathbf{F}\|_{L^2(\Omega)} \le \|\mathbf{F}\|_{\mathbf{H}^1_0(\Omega)}. \tag{3.8}$$

We conclude by the using the Riesz-Thorin Theorem. □

Lemma 3.2. *For all $r \in [0, \frac{1}{2})$, there is $K := K(\Omega, r)$ such that the following holds for all $f \in \mathcal{H}^{r-1}(\Omega)$,*

$$\mathcal{J}f \in \mathcal{H}^{1+r}(\Omega) \text{ and } \|\mathcal{J}f\|_{\mathcal{H}^{1+r}(\Omega)} \le K\|f\|_{\mathcal{H}^{r-1}(\Omega)}. \tag{3.9}$$

and for all $s \in [0, r]$ and all $f \in \mathcal{H}^{s-1}(\Omega)$,

$$\mathcal{J}f \in \mathcal{H}^{1+s}(\Omega) \text{ and } \|\mathcal{J}f\|_{\mathcal{H}^{1+s}(\Omega)} \le K^{\frac{s}{r}}\|f\|_{\mathcal{H}^{s-1}(\Omega)}. \tag{3.10}$$

Proof. The result is proved by using a standard interpolation technique. We first establish the estimate for $s = 0$. Taking $f \in \mathcal{H}^{-1}(\Omega)$ and using the definition of $\mathcal{J}$ together with the norm in $\mathcal{H}^1(\Omega)$ gives

$$\|\mathcal{J}f\|^2_{\mathcal{H}^1(\Omega)} = \langle f, \mathcal{J}f\rangle_{\mathcal{H}^{-1}(\Omega),\mathcal{H}^1(\Omega)} \le \|f\|_{\mathcal{H}^{-1}(\Omega)}\|\mathcal{J}f\|_{\mathcal{H}^1(\Omega)},$$

thereby leading to

$$\forall f \in \mathcal{H}^{-1}(\Omega),\ \|\mathcal{J}f\|_{\mathcal{H}^1(\Omega)} \le \|f\|_{\mathcal{H}^{-1}(\Omega)}. \tag{3.11}$$

We must distinguish two cases for $r < \frac{1}{2}$ depending whether $\mathcal{H}^1(\Omega) = H^1_0(\Omega)$ or $\mathcal{H}^1(\Omega) = \dot{H}^1(\Omega)$. If $\mathcal{H}^1(\Omega) = H^1_0(\Omega)$, then a standard result from Jerison and Kenig [10] (cf. Theorem 0.5) implies that there exists K only depending on Ω and r such that

$$\|\mathcal{J}f\|_{H^{1+r}(\Omega)} \le K\|f\|_{H^{r-1}(\Omega)}, \qquad \forall f \in \mathcal{H}^{r-1}(\Omega) = (H^{1-r}_0(\Omega))'.$$

The Neumann boundary case, $\mathcal{H}^1(\Omega) = \dot{H}^1(\Omega)$, does not seem to be as clear as the Dirichlet case. It appears however to be a by-product of Theorem 3 in Savaré [16]; it is proved therein that $\mathcal{J}f \in H^{1+r}(\Omega)$ for any $r \in [0, \frac{1}{2})$, i.e., (abusing the notation) there exists K only depending on Ω and r such that

$$\|\nabla\mathcal{J}f\|_{H^r(\Omega)} \le K\|f\|_{(H^{1-r}(\Omega))'}, \qquad \forall f \in \mathcal{H}^{r-1}(\Omega) = (H^{1-r}(\Omega))'.$$

In conclusion, for any $0 < r < \frac{1}{2}$ there exists $K = K(r, \Omega)$ such that (3.9) holds. The estimate (3.10) is obtained by interpolation using (3.11) and (3.9). $\square$

Remark 3.2. Owing to the property $\mathcal{J}\mathcal{S}\nabla u = u$ for all $u \in \mathcal{H}^1(\Omega)$, we infer that $\|\mathcal{J}\|_{\mathcal{H}^{r-1}\to\mathcal{H}^{r+1}}\|\mathcal{S}\|_{H^r_0\to\mathcal{H}^{r-1}} \ge 1$, which in turn implies the following lower bound $K(\Omega, r) \ge \|\mathcal{J}\|_{\mathcal{H}^{r-1}\to\mathcal{H}^{r+1}} \ge 1$.

3.3. **Proof of Theorem 3.1.** We want to use a perturbation argument à la Meyers [15]. Let $k > 0$ be a positive number yet to be chosen. Let $f \in \mathcal{H}^{-1}(\Omega)$ and let $p \in \mathcal{H}^1(\Omega)$ be the solution to (3.2). Let us start by observing that the following holds in the distribution sense if $f \in (H^1_0(\Omega))'$:

$$f = -\nabla\cdot(\mathbb{v}\nabla p) = -k\Delta p + \nabla\cdot((k\mathbb{I} - \mathbb{v})\nabla p) = -\Delta(kp) + \nabla\cdot((\mathbb{I} - \frac{1}{k}\mathbb{v})\nabla(kp)),$$

where $\mathbb{I} \in \mathbb{R}^{d\times d}$ is the identity matrix. This representation must be modified as follows to account for boundary conditions (in particular to account for Neumann boundary conditions, i.e., when $f \in (H^1(\Omega))'$):

$$f = \mathcal{S}(\mathbb{v}\nabla p) = k\mathcal{S}\nabla p - \mathcal{S}((k\mathbb{I} - \mathbb{v})\nabla p) = \mathcal{S}\nabla(kp) - \mathcal{S}((\mathbb{I} - \frac{1}{k}\mathbb{v})\nabla(kp)).$$

Upon setting $\bar{\mathbb{v}} := \mathbb{I} - \frac{1}{k}\mathbb{v}$ and $q = kp$, and using that $\mathcal{J}\mathcal{S}\nabla$ is the identity operator in $\mathcal{H}^1(\Omega)$, we arrive at

$$q - \mathcal{J}\left(\mathcal{S}(\bar{\mathbb{v}}\nabla q)\right) = \mathcal{J}f.$$

Let us denote $\mathcal{Q} := \mathcal{J}\mathcal{S}\mathcal{E}_{\bar{\mathbb{v}}}\nabla$ and let us assume for a moment that we can establish that $\mathcal{Q}$ is a bounded operator from $\mathcal{H}^{s+1}(\Omega)$ to $\mathcal{H}^{s+1}(\Omega)$ and that the norm of $\mathcal{Q}$ in $\mathcal{L}(\mathcal{H}^{s+1}(\Omega), \mathcal{H}^{s+1}(\Omega))$ is less than 1, say $\|\mathcal{Q}\|_{\mathcal{H}^{s+1}\to\mathcal{H}^{s+1}} < 1$. Then

$$k\|p\|_{\mathcal{H}^{s+1}(\Omega)} = \|q\|_{\mathcal{H}^{s+1}(\Omega)} \le \frac{\|\mathcal{J}\|}{1 - \|\mathcal{Q}\|}\|f\|_{\mathcal{H}^{s-1}(\Omega)}$$

and the conclusion follows readily. In summary, the crux of the matter consists of proving that $\|\mathcal{Q}\|_{\mathcal{H}^{s+1}\to\mathcal{H}^{s+1}} < 1$.

Since q is in $\mathcal{H}^{1+s}(\Omega)$, we infer that $\nabla q \in \mathbf{H}^s(\Omega)$. The hypothesis $s < \frac{1}{2}$ together with Proposition 2.1 implies that $\mathcal{E}_{\bar{\mathbb{v}}}\nabla q \in \mathbf{H}^s_0(\Omega)$. Using Lemma 3.1, we infer that $\mathcal{S}\mathcal{E}_{\bar{\nu}}\nabla q \in \mathcal{H}^{s-1}(\Omega)$ so that Lemma 3.2 yields $\mathcal{Q}q = \mathcal{J}\mathcal{S}\mathcal{E}_{\bar{\nu}}\nabla q \in \mathcal{H}^{s+1}(\Omega)$. In addition, we have

$$\begin{aligned}\|\mathcal{Q}\|_{\mathcal{H}^{s+1}\to\mathcal{H}^{s+1}} &\le \|\mathcal{J}\|_{\mathcal{H}^{s-1}\to\mathcal{H}^{s+1}}\|\mathcal{S}\|_{H_0^s\to\mathcal{H}^{s-1}}\|\mathcal{E}_{\bar{\nu}}\|_{\mathbf{H}^s\to\mathbf{H}_0^s}\\ &\le K^{\frac{s}{r}}\bar{\nu}_{\max}N_{r,\bar{\nu}}^{\frac{s}{r}}\end{aligned}$$

where $\bar{\nu}_{\max}$ is the L^∞-norm over Ω of the largest eigenvalue of $\bar{\nu}$, as defined in (2.12). Then by choosing $k = \frac{1}{2}(\nu_{\min}+\nu_{\max})$, we have $\bar{\nu}_{\max} \le (1 - 2\nu_{\min}/(\nu_{\max}+\nu_{\min}))$, we consequently have the following bound

$$\|\mathcal{Q}\|_{\mathcal{H}^{s+1}\to\mathcal{H}^{s+1}} \le \frac{\nu_{\max}-\nu_{\min}}{\nu_{\max}+\nu_{\min}}K^{\frac{s}{r}}N_{r,\bar{\nu}}^{\frac{s}{r}},$$

which implies that $\|\mathcal{Q}\|_{\mathcal{H}^{s+1}\to\mathcal{H}^{s+1}} < 1$ for all $s \in [0,\tau)$ where

$$\tau := r\min\left(1, \frac{\log\left(\frac{\nu_{\max}+\nu_{\min}}{\nu_{\max}-\nu_{\min}}\right)}{\log(KN_{r,\bar{\nu}})}\right).$$

Note that $KN_{r,\bar{\nu}} \ge 1$ owing to Remark 3.2 and definitions (2.10), (2.14). Observe also that $\tau \in (0, \frac{1}{2})$. Finally we arrive at

$$\|p\|_{\mathcal{H}^{s+1}(\Omega)} \le \frac{K^{\frac{s}{r}}}{1 - \frac{\nu_{\max}-\nu_{\min}}{\nu_{\max}+\nu_{\min}}K^{\frac{s}{r}}N_{r,\bar{\nu}}{}^{\frac{s}{r}}}\|f\|_{\mathcal{H}^{s-1}(\Omega)}.$$

This concludes the proof of Theorem 3.1.

4. $\mathbf{H}^s$ EMBEDDINGS

We prove in this section two embedding results that are used in §5 to establish regularity estimates on the Maxwell problem (1.6). The main results of this section are Propositions 4.1 and 4.2. Both these results are consequences of Theorem 4.1, which by itself is an improvement of Theorem 2 from Costabel [3].

4.1. **Notations.** In the rest of the paper we use the following spaces to characterize the regularity of vector fields:

$$\mathbf{H}_{\mathrm{curl}}(\Omega) := \left\{\mathbf{F} \in \mathbf{L}^2(\Omega) \mid \nabla\times\mathbf{F} \in \mathbf{L}^2(\Omega)\right\}, \tag{4.1}$$

$$\mathbf{H}_{\mathrm{div}}(\Omega) := \left\{\mathbf{F} \in \mathbf{L}^2(\Omega) \mid \nabla\cdot\mathbf{F} \in \mathbf{L}^2(\Omega)\right\}. \tag{4.2}$$

Let s be a real number in the range $[0, \frac{1}{2})$. We consider the following space equipped with its canonical norm:

$$\mathbf{Z}^{1-s}(\Omega) := \left\{\mathbf{F} \in \mathbf{L}^2(\Omega) \mid \nabla\times\mathbf{F} \in \mathbf{H}^{-s}(\Omega),\ \nabla\cdot\mathbf{F} \in H^{-s}(\Omega)\right\}. \tag{4.3}$$

Let $E : \mathbf{H}^{s+\frac{1}{2}}(\Gamma) \longrightarrow \mathbf{H}^{1+s}(\Omega)$ be one right-inverse of the trace operator γ : $\mathbf{H}^{1+s}(\Omega) \ni \mathbf{v} \longrightarrow \gamma(\mathbf{v}) := \mathbf{v}_{|\Gamma} \in \mathbf{H}^{s+\frac{1}{2}}(\Gamma)$, $s \in [0, \frac{1}{2})$. The existence of right-inverses is guaranteed by the continuity and surjectivity of the trace operator γ, see e.g. Jerison and Kenig [10, Thm 3.1]. The tangential trace of a function $\mathbf{v} \in \mathbf{H}^{-s}(\Omega)$ with $\nabla\times\mathbf{v} \in \mathbf{H}^{-s}(\Omega)$ is defined as an element of $\mathbf{H}^{-s-\frac{1}{2}}(\Gamma)$, $s \in [0, \frac{1}{2})$, by

$$\begin{aligned}\langle \mathbf{v}\times\mathbf{n}, \boldsymbol{\psi}\rangle_{\mathbf{H}^{-s-\frac{1}{2}}(\Gamma),\mathbf{H}^{s+\frac{1}{2}}(\Gamma)} &:= \langle\nabla\times\mathbf{v}, E(\boldsymbol{\psi})\rangle_{\mathbf{H}^{-s}(\Omega),\mathbf{H}^{s}(\Omega)}\\ &\quad - \langle\mathbf{v}, \nabla\times E(\boldsymbol{\psi})\rangle_{\mathbf{H}^{-s}(\Omega),\mathbf{H}^{s}(\Omega)}, \qquad \forall\boldsymbol{\psi} \in \mathbf{H}^{s+\frac{1}{2}}(\Gamma)\end{aligned} \tag{4.4}$$

for all $\boldsymbol{\psi} \in \mathbf{H}^{s+\frac{1}{2}}(\Gamma)$. Note that the above definition is consistent with the usual tangential traces when $\mathbf{v} \in \mathbf{H}_{\mathrm{curl}}(\Omega)$, and it is independent of the extension that is chosen owing to the density of $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ in $\mathbf{H}_0^{1+s}(\Omega)$. In addition, the following estimate holds

$$\|\mathbf{v}\times\mathbf{n}\|_{\mathbf{H}^{-s-\frac{1}{2}}(\Gamma)} \le c\left(\|\mathbf{v}\|_{\mathbf{H}^{-s}(\Omega)} + \|\nabla\times\mathbf{v}\|_{\mathbf{H}^{-s}(\Omega)}\right). \tag{4.5}$$

Similarly, for $\mathbf{v} \in H^{-s}(\Omega)$ with $\nabla\cdot\mathbf{v} \in H^{-s}(\Omega)$, we define $\mathbf{v}\cdot\mathbf{n} \in H^{-s-\frac{1}{2}}(\Gamma)$ by

$$\begin{aligned}\langle \mathbf{v}\cdot\mathbf{n}, \psi\rangle_{H^{-s-\frac{1}{2}}(\Gamma),H^{s+\frac{1}{2}}(\Gamma)} &:= \langle \nabla\cdot\mathbf{v}, E(\psi)\rangle_{H^{-s}(\Omega),H^{s}(\Omega)} \\ &\quad + \langle \mathbf{v}, \nabla E(\boldsymbol{\psi})\rangle_{\mathbf{H}^{-s}(\Omega),\mathbf{H}^{s}(\Omega)}, \qquad \forall\psi \in H^{s+\frac{1}{2}}(\Gamma).\end{aligned} \tag{4.6}$$

Moreover, the normal trace $\mathbf{v}\cdot\mathbf{n}$ satisfies

$$\|\mathbf{v}\cdot\mathbf{n}\|_{H^{-s-\frac{1}{2}}(\Gamma)} \le c\left(\|\mathbf{v}\|_{\mathbf{H}^{-s}(\Omega)} + \|\nabla\cdot\mathbf{v}\|_{H^{-s}(\Omega)}\right). \tag{4.7}$$

The above arguments show that it is legitimate to consider the following subspaces of $Z^{1-s}(\Omega)$:

$$\mathbf{Z}_T^{1-s}(\Omega) := \left\{\mathbf{v} \in \mathbf{Z}^{1-s}(\Omega) \mid \mathbf{v}\cdot\mathbf{n}_{|\Gamma} = 0\right\}, \tag{4.8}$$

$$\mathbf{Z}_N^{1-s}(\Omega) := \left\{\mathbf{v} \in \mathbf{Z}^{1-s}(\Omega) \mid \mathbf{v}\times\mathbf{n}_{|\Gamma} = 0\right\}. \tag{4.9}$$

4.2. **Case of constant coefficients.** The exponent $1 - s$ in the definitions of $\mathbf{Z}^{1-s}(\Omega)$, $\mathbf{Z}_N^{1-s}(\Omega)$, $\mathbf{Z}_T^{1-s}(\Omega)$ is meant to reflect the fact $\mathbf{Z}_N^{1-s}(\Omega)$, $\mathbf{Z}_T^{1-s}(\Omega)$ embed in $\mathbf{H}^{1-s}(\Omega)$ when the boundary of Ω is smooth with $s = 0$ and $s = 1$. Theorem 2 in Costabel [3] asserts that $\mathbf{Z}_T^{1}(\Omega)$ and $\mathbf{Z}_N^{1}(\Omega)$ are continuously embedded in $\mathbf{H}^{\frac{1}{2}}(\Omega)$ when the boundary is Lipschitz. The objective of this section is to extend this result by showing that the embedding in $\mathbf{H}^{\frac{1}{2}}(\Omega)$ holds for $\mathbf{Z}_T^{\frac{1}{2}+}(\Omega)$ and $\mathbf{Z}_N^{\frac{1}{2}+}(\Omega)$, where $\frac{1}{2}+$ is any number in $[0, \frac{1}{2})$.

Theorem 4.1. *For all $s \in [0, \frac{1}{2})$, there is $c > 0$ so that the following embedding estimates hold:*

$$\|\mathbf{u}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} \le c\left(\|\nabla\times\mathbf{u}\|_{\mathbf{H}^{-s}(\Omega)} + \|\nabla\cdot\mathbf{u}\|_{H^{-s}(\Omega)} + \|\mathbf{u}\times\mathbf{n}\|_{\mathbf{L}^2(\Gamma)}\right), \tag{4.10}$$

$$\|\mathbf{u}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} \le c\left(\|\nabla\times\mathbf{u}\|_{\mathbf{H}^{-s}(\Omega)} + \|\nabla\cdot\mathbf{u}\|_{H^{-s}(\Omega)} + \|\mathbf{u}\cdot\mathbf{n}\|_{L^2(\Gamma)}\right). \tag{4.11}$$

Moreover, for all $\mathbf{u} \in \mathbf{Z}^{1-s}(\Omega)$, the following conditions are equivalent:

(i) $\mathbf{u}\times\mathbf{n} \in \mathbf{L}^2(\Gamma)$,
(ii) $\mathbf{u}\cdot\mathbf{n} \in L^2(\Gamma)$.

In order to prove Theorem 4.1, we introduce $\mathcal{B}$, an open ball containing $\bar{\Omega}$, and we set $\Gamma_0 = \partial\mathcal{B}$ and $\mathcal{O} := \mathcal{B}\backslash\bar{\Omega}$. Then we establish the following lemma:

Lemma 4.1. *For any $s \in [0, \frac{1}{2})$ and for any $g \in H^{-s-\frac{1}{2}}(\Gamma)$, there exists $\chi \in \dot{H}^{1-s}(\mathcal{O})$ such that*

$$\langle \nabla\chi, \nabla\psi\rangle_{\mathbf{H}^{-s}(\mathcal{O}),\mathbf{H}^{s}(\mathcal{O})} = \langle g, \psi\rangle_{H^{-s-\frac{1}{2}}(\Gamma),H^{s+\frac{1}{2}}(\Gamma)}, \qquad \forall\psi \in \dot{H}^{1+s}(\mathcal{O}) \tag{4.12}$$

$$\|\chi\|_{\dot{H}^{1-s}(\mathcal{O})} \le c\|g\|_{H^{-s-\frac{1}{2}}(\Gamma)}, \tag{4.13}$$

where c is a constant that only depends on s and Γ. If in addition, $\langle g, 1\rangle_{H^{-s-\frac{1}{2}}(\Gamma),H^{s+\frac{1}{2}}(\Gamma)} = 0$, then (4.12) holds for any $\psi \in H^{1+s}(\mathcal{O})$.

Proof. Owing to the closed range Theorem, proving (4.12) is equivalent to proving the following inf-sup condition:

$$\inf_{0\neq\psi\in\dot{H}^{1+s}(\mathcal{O})}\ \sup_{0\neq\phi\in\dot{H}^{1-s}(\mathcal{O})} \frac{\langle\nabla\phi,\nabla\psi\rangle_{\mathbf{H}^{-s}(\mathcal{O}),\mathbf{H}^{s}(\mathcal{O})}}{\|\phi\|_{\dot{H}^{1-s}(\mathcal{O})}\|\psi\|_{\dot{H}^{1+s}(\mathcal{O})}} \geq \alpha,$$

for some $\alpha > 0$. Using the notation of §3 with $\mathcal{H}^s(\mathcal{O}) = \dot{H}^s(\mathcal{O})$, and defining the operator $\mathcal{S} : \mathbf{L}^2(\mathcal{O}) \to (\dot{H}^1(\mathcal{O}))'$ by

$$\forall\mathbf{f}\in\mathbf{L}^2(\mathcal{O}),\ \forall q\in\dot{H}^1(\mathcal{O}),\qquad \langle\mathcal{S}\mathbf{f},q\rangle_{(\dot{H}^1(\mathcal{O}))',\dot{H}^1(\mathcal{O})} = (\mathbf{f},\nabla q)_{\mathcal{O}}\,,$$

we infer that

$$\langle\nabla\phi,\nabla\psi\rangle_{\mathbf{H}^{-s}(\mathcal{O}),\mathbf{H}^{s}(\mathcal{O})} = \langle\mathcal{S}\nabla\psi,\phi\rangle_{\mathcal{H}^{s-1}(\mathcal{O}),\mathcal{H}^{1-s}(\mathcal{O})}\,,$$

for all $\phi\in\dot{H}^{1-s}(\mathcal{O})$ and all $\psi\in\dot{H}^{1+s}(\mathcal{O})$. As a result, the following holds for all $\psi\in\dot{H}^{s+1}(\mathcal{O})$:

$$\sup_{0\neq\phi\in\dot{H}^{1-s}} \frac{\langle\nabla\phi,\nabla\psi\rangle_{\mathbf{H}^{-s}(\mathcal{O}),\mathbf{H}^{s}(\mathcal{O})}}{\|\phi\|_{\dot{H}^{1-s}(\mathcal{O})}} = \|\mathcal{S}\nabla\psi\|_{\mathcal{H}^{s-1}(\mathcal{O})}, \tag{4.14}$$

Since $\mathcal{O}$ is a Lipschitz domain, we define the operator $\mathcal{J} : (\dot{H}^1(\mathcal{O}))' \to \dot{H}^1(\mathcal{O})$ by

$$\forall f\in(\dot{H}^1(\mathcal{O}))',\ \forall q\in\dot{H}^1(\mathcal{O}),\qquad (\nabla\mathcal{J}f,\nabla q)_{\mathcal{O}} = \langle f,q\rangle_{(\dot{H}^1(\mathcal{O}))',\dot{H}^1(\mathcal{O})}\,,$$

and we have $\mathcal{J}\mathcal{S}\nabla\psi = \psi$ for all $\psi\in\dot{H}^1(\mathcal{O})$. Then Lemma 3.2 implies that

$$\|\nabla\psi\|_{\mathbf{H}^s(\mathcal{O})} = \|\psi\|_{\mathcal{H}^{1+s}(\mathcal{O})} = \|\mathcal{J}\mathcal{S}\nabla\psi\|_{\mathcal{H}^{1+s}(\mathcal{O})} \leq K(\mathcal{O},s)\|\mathcal{S}\nabla\psi\|_{\mathcal{H}^{s-1}(\mathcal{O})}. \tag{4.15}$$

Combining (4.14) and (4.15) and using the Poincaré-Friedrichs inequality in $\mathcal{O}$ leads to the inf-sup condition with $\alpha^{-1} = K(\mathcal{O},r)$. This in turn implies the existence of a function χ satisfying (4.12) and the estimate (4.13) with $c = \alpha^{-1}$ (see for instance Ern and Guermond [6, Lemma A.42]). If in addition g satisfies the condition

$$\langle g,1\rangle_{H^{-s-\frac12}(\Gamma),H^{s+\frac12}(\Gamma)} = 0,$$

the definition (4.12) holds for all $\psi\in H^{1+s}(\mathcal{O})$, i.e.,

$$\forall\psi\in H^{1+s}(\mathcal{O}),\qquad \langle\nabla\chi,\nabla\psi\rangle_{\mathbf{H}^{-s}(\mathcal{O}),\mathbf{H}^{s}(\mathcal{O})} = \langle g,\psi\rangle_{H^{-s-\frac12}(\Gamma),H^{s+\frac12}(\Gamma)}\,, \tag{4.16}$$

since the left-hand side of (4.12) only involves gradients. □

Let D be an open Lipschitz domain in $\mathbb{R}^d$. We denote by $E_D : L^1(D) \longrightarrow L^1(\mathbb{R}^d)$ the zero extension operator, i.e.,

$$\forall F\in L^1(D),\qquad E_DF(\mathbf{x}) = \begin{cases} F(\mathbf{x}) & \text{if } \mathbf{x}\in D,\\ 0 & \text{elsewhere.}\end{cases} \tag{4.17}$$

We use the same definition for vector-valued functions in $\mathbf{L}^1(D)$.

Proof of Theorem 4.1. The proof is similar to that of Theorem 2 in Costabel [3]. Given $\mathbf{u}\in\mathbf{Z}^{1-s}(\Omega)$, we build an extension of $\nabla\times\mathbf{u}$ in $\mathbf{H}^{-s}(\mathbb{R}^d)$ in order to be able to construct $\mathbf{w}\in\mathbf{H}^{1-s}(\Omega)$ such that $\mathbf{u}-\mathbf{w}$ is curl-free. Then we use results from Jerison and Kenig (cf. Jerison and Kenig [9]) to obtain some regularity on $\mathbf{u}-\mathbf{w}$.

Consider $\mathbf{u} \in \mathbf{Z}^{1-s}(\Omega)$. By definition $\nabla{\times}\mathbf{u} \in \mathbf{H}^{-s}(\Omega)$ and $\nabla{\cdot}\nabla{\times}\mathbf{u} = 0$, so that $(\nabla{\times}\mathbf{u}){\cdot}\mathbf{n}$ is well-defined as an element of $H^{-s-\frac{1}{2}}(\Gamma)$, owing to (4.6). Note also that this normal trace satisfies

$$\langle(\nabla{\times}\mathbf{u}){\cdot}\mathbf{n}, 1\rangle_{H^{-s-\frac{1}{2}}(\Gamma), H^{s+\frac{1}{2}}(\Gamma)} = 0.$$

Thus we can apply Lemma 4.1, and there exists $\chi \in H^{1-s}(\mathcal{O})$ such that

$$\langle\nabla\chi, \nabla\psi\rangle_{\mathbf{H}^{-s}(\mathcal{O}),\mathbf{H}^{s}(\mathcal{O})} = \langle -(\nabla{\times}\mathbf{u}){\cdot}\mathbf{n}, \psi\rangle_{H^{-s-\frac{1}{2}}(\Gamma), H^{s+\frac{1}{2}}(\Gamma)}, \qquad \forall\psi \in H^{1+s}(\mathcal{O}).$$

We now set $\mathbf{f} = E_\Omega\nabla{\times}\mathbf{u} + E_{\mathcal{O}}\nabla\chi$. Since $s < \frac{1}{2}$, $\mathbf{H}^s(\Omega) = \mathbf{H}^s_0(\Omega)$ and $\mathbf{H}^s(\mathcal{B}) = \mathbf{H}^s_0(\mathcal{B})$, $\mathbf{f}$ can also be seen as an element of $\mathbf{H}^{-s}(\mathbb{R}^d)$, i.e., the following holds:

$$\langle\mathbf{f}, \Psi\rangle_{\mathbf{H}^{-s}(\mathbb{R}^d),\mathbf{H}^{s}(\mathbb{R}^d)} := \left\langle\nabla{\times}\mathbf{u}, \Psi_{|\Omega}\right\rangle_{\mathbf{H}^{-s}(\Omega),\mathbf{H}^{s}(\Omega)} + \left\langle\nabla\chi, \Psi_{|\mathcal{O}}\right\rangle_{\mathbf{H}^{-s}(\mathcal{O}),\mathbf{H}^{s}(\mathcal{O})},$$

for all $\Psi \in \mathbf{H}^s(\mathbb{R}^d)$. Moreover, since the restrictions $\mathbf{H}^s(\mathbb{R}^d) \to \mathbf{H}^s(\Omega)$ and $\mathbf{H}^s(\mathbb{R}^d) \to \mathbf{H}^s(\mathcal{O})$ are continuous with norm 1, combining (4.13) and (4.7) leads to

$$\begin{aligned}\|\mathbf{f}\|_{\mathbf{H}^{-s}(\mathbb{R}^d)} &\le \|\nabla{\times}\mathbf{u}\|_{\mathbf{H}^{-s}(\Omega)} + \|\nabla\chi\|_{\mathbf{H}^{-s}(\mathcal{O})}\\ &\le \|\nabla{\times}\mathbf{u}\|_{\mathbf{H}^{-s}(\Omega)} + c\|(\nabla{\times}\mathbf{u}){\cdot}\mathbf{n}\|_{H^{-s-\frac{1}{2}}(\Gamma)}\\ &\le c\|\nabla{\times}\mathbf{u}\|_{\mathbf{H}^{-s}(\Omega)},\end{aligned}$$

Owing to the definition of the trace $(\nabla{\times}\mathbf{u}){\cdot}\mathbf{n}$ and the definition of χ we infer that, the following hold for all $\phi \in \mathcal{C}^\infty_0(\mathbb{R}^d)$:

$$\begin{aligned}\langle\mathbf{f}, \nabla\phi\rangle_{\mathbf{H}^{-s}(\mathbb{R}^d),\mathbf{H}^{s}(\mathbb{R}^d)} &= \left\langle\nabla{\times}\mathbf{u}, \nabla\phi_{|\Omega}\right\rangle_{\mathbf{H}^{-s}(\Omega),\mathbf{H}^{s}(\Omega)} + \left\langle\nabla\chi, \nabla\phi_{|\mathcal{O}}\right\rangle_{\mathbf{H}^{-s}(\mathcal{O}),\mathbf{H}^{s}(\mathcal{O})},\\ &= \left\langle\nabla{\times}\mathbf{u}, \nabla\phi_{|\Omega}\right\rangle_{\mathbf{H}^{-s}(\Omega),\mathbf{H}^{s}(\Omega)} - \langle(\nabla{\times}\mathbf{u}){\cdot}\mathbf{n}, \phi\rangle_{H^{-s-\frac{1}{2}}(\Gamma), H^{s+\frac{1}{2}}(\Gamma)},\\ &= \left\langle\nabla{\times}\mathbf{u}, \nabla\phi_{|\Omega}\right\rangle_{\mathbf{H}^{-s}(\Omega),\mathbf{H}^{s}(\Omega)} - \left\langle\nabla{\times}\mathbf{u}, \nabla\phi_{|\Omega}\right\rangle_{\mathbf{H}^{-s}(\Omega),\mathbf{H}^{s}(\Omega)}\\ &\quad - \langle\nabla{\cdot}\nabla{\times}\mathbf{u}, \phi\rangle_{H^{-s}(\Omega),H^{s}(\Omega)} = 0,\end{aligned}$$

implying that $\nabla{\cdot}\mathbf{f} = 0$. As a result there exists $\boldsymbol{\Phi} \in \mathbf{H}^{2-s}(\mathbb{R}^d)$ such that

$$-\Delta\boldsymbol{\Phi} = \mathbf{f} \text{ with } \nabla{\cdot}\boldsymbol{\Phi} = 0 \text{ and } \|\boldsymbol{\Phi}\|_{\mathbf{H}^{2-s}(\mathbb{R}^d)} \le c\|\mathbf{f}\|_{\mathbf{H}^{-s}(\mathbb{R}^d)}.$$

Setting $\mathbf{w} := \nabla{\times}\boldsymbol{\Phi}$, we infer that

$$\|\mathbf{w}\|_{\mathbf{H}^{1-s}(\mathbb{R}^d)} \le c\|\mathbf{f}\|_{\mathbf{H}^{-s}(\mathbb{R}^d)}, \qquad \nabla{\times}\mathbf{w} = \mathbf{f}, \qquad \nabla{\cdot}\mathbf{w} = 0.$$

This in turn implies that

$$\nabla{\times}\mathbf{w}_{|\Omega} = \nabla{\times}\mathbf{u} \in \mathbf{H}^{-s}(\Omega), \qquad \nabla{\cdot}\mathbf{w}_{|\Omega} = 0, \qquad \|\mathbf{w}_{|\Omega}\|_{\mathbf{H}^{1-s}(\Omega)} \le c\|\mathbf{f}\|_{\mathbf{H}^{-s}(\mathbb{R}^d)}.$$

We set now $\mathbf{z} := \mathbf{u} - \mathbf{w}_{|\Omega}$. Using the fact that Ω is simply-connected together with $\nabla{\times}\mathbf{z} = 0$, we infer from Girault and Raviart [7, Theorem 2.9] that there exists $v \in H^1(\Omega)$ such that $\mathbf{z} = \nabla v$. We now split v to be able to apply a regularity result on a homogeneous Laplace equation with non homogeneous boundary conditions. Let $E_\Omega\nabla{\cdot}\mathbf{u}$ be the zero extension of $\nabla{\cdot}\mathbf{u}$. Clearly $E_\Omega\nabla{\cdot}\mathbf{u} \in H^{-s}(\mathcal{B})$ since

$$\langle E_\Omega\nabla{\cdot}\mathbf{u}, \psi\rangle_{H^{-s}(\mathcal{B}),H^{s}(\mathcal{B})} := \left\langle\nabla{\cdot}\mathbf{u}, \psi_{|\Omega}\right\rangle_{H^{-s}(\Omega),H^{s}(\Omega)}, \qquad \forall\psi \in H^s_0(\mathcal{B}),$$

and

$$\|E_\Omega\nabla{\cdot}\mathbf{u}\|_{H^{-s}(\mathcal{B})} \le \|\nabla{\cdot}\mathbf{u}\|_{H^{-s}(\Omega)},$$

since $s < \frac{1}{2}$. Let $p \in H^1_0(\mathcal{B})$ be the solution of $-\Delta p = -E_\Omega \nabla{\cdot}\mathbf{u}$. Then elliptic regularity implies that $p \in H^{2-s}(\mathcal{B})$ (see for instance Jerison and Kenig [10, Theorem 0.5]) and the following estimate holds:

$$\|\nabla p\|_{\mathbf{H}^{1-s}(\mathcal{B})} \le c\|E_\Omega \nabla{\cdot}\mathbf{u}\|_{H^{-s}(\mathcal{B})} \le c'\|\nabla{\cdot}\mathbf{u}\|_{H^{-s}(\Omega)}.$$

Finally, let us define $r := v - p_{|\Omega}$. By definition of $\mathbf{w}$ and p, $\Delta r = 0$ in $H^{-s}(\Omega)$, and we arrive at the following decomposition:

$$\mathbf{u} = (\mathbf{w} + \nabla p)_{|\Omega} + \nabla r.$$

Let us assume that (i) holds, i.e., $\mathbf{u}{\times}\mathbf{n} \in \mathbf{L}^2(\Gamma)$. Since $\mathbf{w}_{|\Omega} \in \mathbf{H}^{1-s}(\Omega)$, we have $\mathbf{w}{\times}\mathbf{n} \in \mathbf{H}^{\frac{1}{2}-s}(\Gamma) \subset \mathbf{L}^2(\Gamma)$. Similarly, $p_{|\Omega} \in H^{2-s}(\Omega)$ so that $\nabla p{\times}\mathbf{n} \in \mathbf{H}^{\frac{1}{2}-s}(\Gamma) \subset \mathbf{L}^2(\Gamma)$. As a result, we have $\nabla r{\times}\mathbf{n}_{|\Gamma} = (\mathbf{u}{\times}\mathbf{n} - \mathbf{w}{\times}\mathbf{n} - \nabla p{\times}\mathbf{n})_{|\Gamma} \in \mathbf{L}^2(\Gamma)$, which together with $r_{|\Gamma} \in L^2(\Gamma)$ implies $r_{|\Gamma} \in H^1(\Gamma)$. Thus we have

$$\Delta r = 0 \text{ in } \Omega, \qquad r_{|\Gamma} \in H^1(\Gamma).$$

Consequently $r \in H^{\frac{3}{2}}(\Omega)$ and the following estimate holds since Γ is connected

$$\|\nabla r\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} \le c\|r\|_{H^1(\Gamma)} \le c'\|\nabla r{\times}\mathbf{n}\|_{\mathbf{L}^2(\Gamma)}.$$

Hence, because $\mathbf{u} = (\mathbf{w} + \nabla p)_{|\Omega} + \nabla r$ with $\mathbf{w}_{|\Omega} \in \mathbf{H}^{1-s}(\Omega)$, $\nabla p_{|\Omega} \in H^{1-s}(\Omega)$, and $\nabla r \in \mathbf{H}^{\frac{1}{2}}(\Omega)$, we deduce that $\mathbf{u} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$. Note also that $(\mathbf{w}{\cdot}\mathbf{n} + \nabla p{\cdot}\mathbf{n})_{|\Gamma} \in L^2(\Gamma)$ and $r_{|\Gamma} \in H^1(\Gamma)$, which implies that $\mathbf{u}{\cdot}\mathbf{n}_{|\Gamma} = (\mathbf{w}{\cdot}\mathbf{n} + \nabla p{\cdot}\mathbf{n} + \nabla r{\cdot}\mathbf{n})_{|\Gamma} \in L^2(\Gamma)$ thereby proving (ii). The proof of the converse implication is similar, we leave the details to the reader; in particular, one must use the fact that the scalar field $r := v - p_{|\Omega}$ is such that

$$\Delta r = 0 \text{ in } \Omega, \qquad \mathbf{n}{\cdot}\nabla r \in L^2(\Gamma),$$

which again implies $r \in H^{\frac{3}{2}}(\Omega)$, (see for instance Jerison and Kenig [9] or Costabel [3, Lemma 1]). In summary, we have proved that (i) and (ii) are equivalent, and that both these assumptions imply $\mathbf{u} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$.

Using $s < \frac{1}{2}$ and gathering all the previous estimates, we end up with:

$$\begin{aligned}
\|\mathbf{u}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} &\le c\left(\|\mathbf{w}\|_{\mathbf{H}^{1-s}(\Omega)} + \|\nabla p\|_{\mathbf{H}^{1-s}(\Omega)} + \|\nabla r\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)}\right) \\
&\le c\left(\|\mathbf{w}\|_{\mathbf{H}^{1-s}(\Omega)} + \|\nabla p\|_{\mathbf{H}^{1-s}(\Omega)} + \|\mathbf{u}{\times}\mathbf{n}\|_{\mathbf{L}^2(\Gamma)} + \|\mathbf{w}{\times}\mathbf{n}\|_{\mathbf{L}^2(\Gamma)} + \|\nabla p{\times}\mathbf{n}\|_{\mathbf{L}^2(\Gamma)}\right) \\
&\le c\left(\|\mathbf{w}\|_{\mathbf{H}^{1-s}(\Omega)} + \|\nabla p\|_{\mathbf{H}^{1-s}(\Omega)} + \|\mathbf{u}{\times}\mathbf{n}\|_{\mathbf{L}^2(\Gamma)}\right) \\
&\le c\left(\|\mathbf{f}\|_{\mathbf{H}^{-s}(\mathbb{R}^d)} + \|E_\Omega \nabla{\cdot}\mathbf{u}\|_{H^{-s}(\mathcal{B})} + \|\mathbf{u}{\times}\mathbf{n}\|_{\mathbf{L}^2(\Gamma)}\right) \\
&\le c\left(\|\nabla{\times}\mathbf{u}\|_{\mathbf{H}^{-s}(\Omega)} + \|\nabla{\cdot}\mathbf{u}\|_{H^{-s}(\Omega)} + \|\mathbf{u}{\times}\mathbf{n}\|_{\mathbf{L}^2(\Gamma)}\right),
\end{aligned}$$

which is the desired result. The inequality involving $\mathbf{u}{\cdot}\mathbf{n}$ is obtained similarly. □

Remark 4.1. *During the review of this paper it has been brought to our attention that an alternative proof of Theorem 4.1 can be done by invoking arguments from Costabel and McIntosh [4]. Using an argument from Costabel and McIntosh [4] one can show that any* $\mathbf{u} \in \mathbf{Z}^{1-s}(\Omega)$ *has a decomposition* $\mathbf{u} = \mathbf{w} + \nabla p$, *where* $\mathbf{w} \in \mathbf{H}^{1-s}(\Omega)$, *$p$ is harmonic and* $p \in H^{1-s}(\Omega)$. *The regularity result then reduces to the case* $s = 0$ *since* $\nabla p \in \mathbf{Z}^0(\Omega)$.

4.3. **Case of non-constant coefficients.** Throughout §4 and §5 we assume that the tensor fields $\mathbb{e}$ and $\mathbb{\mu}$ satisfy the following property:

Assumption 4.1. $\mathbb{e}, \mathbb{\mu} \in \mathbb{W}^{1,\infty}_{\Sigma}(\Omega)$ *and there exist* $\mathbb{e}_{\min}$, $\mathbb{\mu}_{\min} > 0$ *such that*

$$\boldsymbol{\xi}^T \mathbb{e} \boldsymbol{\xi} \geq \mathbb{e}_{\min} \boldsymbol{\xi}^T \boldsymbol{\xi} \quad a.e.\ in\ \Omega, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d,$$
$$\boldsymbol{\xi}^T \mathbb{\mu} \boldsymbol{\xi} \geq \mathbb{\mu}_{\min} \boldsymbol{\xi}^T \boldsymbol{\xi} \quad a.e.\ in\ \Omega, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d.$$

The analysis of the regularity of Maxwell problem (1.6) requires introducing the following two spaces for $0 < s < \frac{1}{2}$:

$$\mathbf{Y}^s(\Omega) := \left\{ \mathbf{F} \in \mathbf{L}^2(\Omega) \mid \nabla\times\mathbf{F} \in \mathbf{H}^{-s}(\Omega),\ \nabla\cdot(\mathbb{\mu}\mathbf{F}) = 0,\ \mathbb{\mu}\mathbf{F}\cdot\mathbf{n}_{|\Gamma} = 0 \right\}, \tag{4.18}$$

$$\mathbf{X}^s(\Omega) := \left\{ \mathbf{F} \in \mathbf{L}^2(\Omega) \mid \nabla\times\mathbf{F} \in \mathbf{H}^{-s}(\Omega),\ \nabla\cdot(\mathbb{e}\mathbf{F}) \in H^{s-1}(\Omega),\ \mathbf{F}\times\mathbf{n}_{|\Gamma} = 0 \right\}. \tag{4.19}$$

We define the following semi-norms in $\mathbf{X}^s(\Omega)$ and $\mathbf{Y}^s(\Omega)$:

$$|\mathbf{F}|^2_{\mathbf{X}^s(\Omega)} := \|\nabla\times\mathbf{F}\|^2_{\mathbf{H}^{-s}(\Omega)} + \|\nabla\cdot(\mathbb{e}\mathbf{F})\|^2_{H^{s-1}(\Omega)}, \quad |\mathbf{F}|_{\mathbf{Y}^s(\Omega)} := \|\nabla\times\mathbf{F}\|_{\mathbf{H}^{-s}(\Omega)}. \tag{4.20}$$

Two embedding results for $\mathbf{X}^s$ and $\mathbf{Y}^s(\Omega)$ are established in this section.

Proposition 4.1. *Let Assumption 4.1 hold. There exists $\tau_{\mathbb{e}} > 0$, only depending on Ω and $\mathbb{e}$, such that, for any $s \in [0, \tau_{\mathbb{e}})$, $\mathbf{X}^s(\Omega)$ is continuously embedded in $\mathbf{H}^s(\Omega)$, and there is $c > 0$ so that*

$$\|\mathbf{F}\|_{\mathbf{H}^s(\Omega)} \leq c\,|\mathbf{F}|_{\mathbf{X}^s(\Omega)}, \qquad \forall \mathbf{F} \in \mathbf{X}^s(\Omega). \tag{4.21}$$

Proof. Owing to Assumption 4.1, we can apply Theorem 3.1 with $\mathbb{\nu} = \mathbb{e}$ and $\mathcal{H}^1(\Omega) = H^1_0(\Omega)$. Let $\tau_{\mathbb{e}} < \frac{1}{2}$ be the parameter defined in Theorem 3.1. Let us consider $\mathbf{F} \in \mathbf{X}^s(\Omega)$ with $s \in [0, \tau_{\mathbb{e}})$. We define $p \in H^1_0(\Omega)$ such that

$$(\nabla p, \nabla q) = (\mathbf{F}, \nabla q), \qquad \forall q \in H^1_0(\Omega),$$

and we set $\mathbf{w} := \mathbf{F} - \nabla p$. This definition implies that $\mathbf{w} \in \mathbf{Z}^{1-s}(\Omega)$ since $\nabla\times\mathbf{w} = \nabla\times\mathbf{F}$ and $\nabla\cdot\mathbf{w} = 0$. Observing also that $\mathbf{w}\times\mathbf{n}_{|\Gamma} = 0$ and applying Theorem 4.1, we deduce that $\mathbf{w} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$ and $\|\mathbf{w}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} \leq c\|\nabla\times\mathbf{w}\|_{\mathbf{H}^{-s}(\Omega)}$. In addition, $s < \frac{1}{2}$ and $\nabla\times\mathbf{w} = \nabla\times\mathbf{F}$, imply that

$$\|\mathbf{w}\|_{\mathbf{H}^s(\Omega)} \leq c\|\nabla\times\mathbf{F}\|_{\mathbf{H}^{-s}(\Omega)}. \tag{4.22}$$

Moreover, since $\mathbf{w} \in \mathbf{H}^s(\Omega)$, Proposition 2.1 ensures that $\mathbb{e}\mathbf{w} \in \mathbf{H}^s(\Omega)$; as a result, $\nabla\cdot(\mathbb{e}\mathbf{w}) \in H^{s-1}(\Omega)$ and

$$\|\nabla\cdot(\mathbb{e}\mathbf{w})\|_{H^{s-1}(\Omega)} \leq c\|\mathbb{e}\mathbf{w}\|_{H^s(\Omega)} \leq c'\|\mathbf{w}\|_{H^s(\Omega)} \leq c''\|\nabla\times\mathbf{F}\|_{\mathbf{H}^{-s}(\Omega)}. \tag{4.23}$$

Let us now turn our attention to p. In view of the following equality

$$(\mathbb{e}\nabla p, \nabla q) = (\mathbb{e}\mathbf{F}, \nabla q) - (\mathbb{e}\mathbf{w}, \nabla q), \qquad q \in H^1_0(\Omega),$$

and upon introducing the linear form $f : \mathcal{H}^1(\Omega) \ni q \mapsto (\mathbb{e}\mathbf{F} - \mathbb{e}\mathbf{w}, \nabla q)$, i.e., $f = -\nabla\cdot(\mathbb{e}(\mathbf{F} - \mathbf{w}))$, we infer that p solves:

$$(\mathbb{e}\nabla p, \nabla q) = \langle f, q\rangle_{\mathcal{H}^{-1}(\Omega),\mathcal{H}(\Omega)}, \qquad \forall q \in \mathcal{H}^1(\Omega).$$

The definition of $\mathbf{X}^s(\Omega)$ implies that $\nabla\cdot(\mathbb{e}\mathbf{F}) \in H^{s-1}(\Omega)$. This, together with (4.23), implies that $\|f\|_{\mathcal{H}^{s-1}(\Omega)} \leq c|\mathbf{F}|_{\mathbf{X}^s(\Omega)}$. Applying Theorem 3.1, we infer that $p \in H^{1+s}(\Omega)$ and

$$\|\nabla p\|_{\mathbf{H}^s(\Omega)} \leq c|\mathbf{F}|_{\mathbf{X}^s(\Omega)}. \tag{4.24}$$

Using (4.22), (4.24), and recalling the definition $\mathbf{F} = \mathbf{w} + \nabla p$, we conclude that $\mathbf{F} \in \mathbf{H}^s(\Omega)$ and there exists a constant c that only depends on Ω, ε, and s such that

$$\|\mathbf{F}\|_{\mathbf{H}^s(\Omega)} \leq c\,|\mathbf{F}|_{\mathbf{X}^s(\Omega)}\,. \tag{4.25}$$

This concludes the proof. □

Proposition 4.2. *Let Assumption 4.1 hold. There exists* τ_μ *only depending on* Ω *and* μ *such that, for any* $s \in [0, \tau_\mu)$ *and any* $t \in \left[0, \frac{1}{2}\right)$*, the space* $\mathbf{Y}^t(\Omega)$ *is continuously embedded in* $\mathbf{H}^s(\Omega)$*.*

Proof. We proceed as in the proof of Proposition 4.1. We consider $\mathbf{F} \in \mathbf{Y}^t(\Omega)$ and we want to decompose $\mathbf{F}$ as follows:

$$\mathbf{F} = \mathbf{w} + \nabla p,$$

where $\mathbf{w}$ is a regular part and p is the solution of an elliptic system with discontinuous coefficients. We first focus on the construction of $\mathbf{w}$. We introduce

$$\begin{aligned}\mathbf{H}_{0,\mathrm{curl}}(\Omega) &:= \{\mathbf{G} \in \mathbf{H}_{\mathrm{curl}}(\Omega) \mid \mathbf{G}\times\mathbf{n} = 0\}\,,\\ \mathbf{H}_{\mathrm{div}=0}(\Omega) &:= \{\mathbf{G} \in \mathbf{H}_{\mathrm{div}}(\Omega) \mid \nabla\cdot\mathbf{G} = 0\}\,.\end{aligned}$$

Owing to (4.10) with $s = 0$ (see also Costabel [3, Theorem 2]) and the Lax-Milgram lemma, there exists a unique $\mathbf{G} \in \mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H}_{\mathrm{div}=0}(\Omega)$ such that the following holds:

$$(\nabla\times\mathbf{G}, \nabla\times\mathbf{f}) = (\mathbf{F}, \nabla\times\mathbf{f})\,, \qquad \forall \mathbf{f} \in \mathbf{H}_{0,\mathrm{curl}}(\Omega) \cap \mathbf{H}_{\mathrm{div}=0}(\Omega)\,. \tag{4.26}$$

Since the above definition only involves $\nabla\times\mathbf{f}$, we infer that the following holds also

$$(\nabla\times\mathbf{G}, \nabla\times\mathbf{f}) = (\mathbf{F}, \nabla\times\mathbf{f})\,, \qquad \forall \mathbf{f} \in \mathbf{H}_{0,\mathrm{curl}}(\Omega)\,.$$

Setting $\mathbf{w} := \nabla\times\mathbf{G}$, the above equality implies that $\nabla\times\mathbf{w} = \nabla\times\mathbf{F}$. The equality $\nabla\times\mathbf{w} = \nabla\times\mathbf{F}$ first holds in the distribution space $\boldsymbol{\mathcal{D}}'(\Omega)$, and then in $\mathbf{H}^{-t}(\Omega)$ taking advantage of the regularity $\nabla\times\mathbf{F} \in \mathbf{H}^{-t}(\Omega)$. We have $\mathbf{w} \in \mathbf{L}^2(\Omega)$ with $\nabla\cdot\mathbf{w} = 0$ and $\nabla\times\mathbf{w} \in \mathbf{H}^{-t}(\Omega)$, i.e. $\mathbf{w} \in \mathbf{Z}^{1-t}(\Omega)$. Moreover, the condition $\mathbf{G}\times\mathbf{n}_{|\Gamma} = 0$ implies $\mathbf{w}\cdot\mathbf{n}_{|\Gamma} = 0$. Then Theorem 4.1 implies that $\mathbf{w} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$ and

$$\|\mathbf{w}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} \leq c\|\nabla\times\mathbf{w}\|_{\mathbf{H}^{-t}(\Omega)} = c\|\nabla\times\mathbf{F}\|_{\mathbf{H}^{-t}(\Omega)}. \tag{4.27}$$

The equality $\nabla\times\mathbf{w} = \nabla\times\mathbf{F}$ yields the existence of $p \in H^1(\Omega)$ such that $\mathbf{F} = \mathbf{w} + \nabla p$, see e.g. Girault and Raviart [7, Theorem 2.9]. Up to an additive constant, we assume that $p \in \dot{H}^1(\Omega)$ and we now derive $H^s(\Omega)$ estimates. The definition of p together with the assumption $\nabla\cdot(\mu\mathbf{F}) = 0$ implies that

$$(\mu\nabla p, \nabla q) = (\mu\mathbf{F}, \nabla q) - (\mu\mathbf{w}, \nabla q) = -(\mu\mathbf{w}, \nabla q)\,, \qquad \forall q \in H^1(\Omega).$$

As a result, we have

$$(\mu\nabla p, \nabla q) = -(\mathcal{S}(\mu\mathbf{w}), q), \qquad \forall q \in \dot{H}^1(\Omega).$$

Proposition 2.1 ensures that $\mu\mathbf{w} \in \mathbf{H}^s(\Omega)$ for all $s < \frac{1}{2}$, and Lemma 3.1 implies that $\mathcal{S}(\mu\mathbf{w}) \in \mathcal{H}^{s-1}(\Omega)$, so that

$$\|\mathcal{S}(\mu\mathbf{w})\|_{\mathcal{H}^{s-1}(\Omega)} \leq \|\mu\mathbf{w}\|_{\mathbf{H}^s_0(\Omega)} \leq \mu_{\max} N_{s,\mu}\|\mathbf{w}\|_{\mathbf{H}^s(\Omega)} \leq c\|\mathbf{w}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)}.$$

We now can apply Theorem 3.1 with $\mathcal{H}^1(\Omega) = \dot{H}^1(\Omega)$ and $\nu = \mu$. Let τ_{μ} be the parameter defined in Theorem 3.1. Then $p \in H^{1+s}(\Omega)$ for all $s \in [0, \tau_{\mu})$ and there is a constant c so that

$$\|\nabla p\|_{\mathbf{H}^s(\Omega)} \leq c\|\mathcal{S}(\mu\mathbf{w})\|_{\mathcal{H}^{s-1}(\Omega)} \leq c'\|\mathbf{w}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)}. \tag{4.28}$$

Recalling that and $\mathbf{F} = \mathbf{w} + \nabla p$, we finally conclude that $\mathbf{F} \in \mathbf{H}^s$, and owing to (4.27) and (4.28) we obtain the following estimate:

$$\|\mathbf{F}\|_{\mathbf{H}^s(\Omega)} \leq c\|\mathbf{w}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} \leq c'\|\nabla{\times}\mathbf{F}\|_{\mathbf{H}^{-t}(\Omega)} = c'\,|\mathbf{F}|_{\mathbf{Y}^t(\Omega)}\,. \tag{4.29}$$

This concludes the proof. □

Remark 4.1. The counterpart of Proposition 4.1 and Proposition 4.2 when ε are μ are constant or smooth tensor fields is that $\mathbf{X}^s(\Omega)$ and $\mathbf{Y}^s(\Omega)$ are continuously embedded in $\mathbf{H}^{\frac{1}{2}}(\Omega)$. There is loss of regularity when the fields ε are μ are discontinuous. See also Costabel et al. [5].

5. Application to Maxwell problem

We turn our attention in this section to the Maxwell problems mentioned in the introduction. Using Theorem 3.1, we establish a priori estimates for the following problem: Given $\mathbf{g}$ and b, find $(\mathbf{E}, p)$ so that

$$\nabla{\times}(\mu^{-1}\nabla{\times}\mathbf{E}) + \epsilon\nabla p = \mathbf{g}, \quad \nabla{\cdot}(\varepsilon\mathbf{E}) = b, \quad \mathbf{E}{\times}\mathbf{n}|_\Gamma = 0, \qquad p|_\Gamma = 0. \tag{5.1}$$

5.1. **Notation and preliminaries.** If b is nonzero, we define $\tilde{p} \in H^1_0(\Omega)$ so that

$$(\varepsilon\nabla\tilde{p}, \nabla r) = (b, r), \qquad \forall r \in H^1_0(\Omega), \tag{5.2}$$

and we set $\mathbf{E} := \widetilde{\mathbf{E}} + \nabla\tilde{p}$. Then the vector field $\widetilde{\mathbf{E}}$ solves the Maxwell system (5.1) with $b = 0$. Note in particular that $\nabla\tilde{p} \in \mathbf{H}^s(\Omega)$ if $b \in H^{s-1}(\Omega)$, $s \in [0, \tau_{\varepsilon})$ (see Theorem 3.1). We now consider that $b = 0$ in the rest of the paper.

Let us consider the following space:

$$\mathbf{X}^s_{\mathrm{div}=0}(\Omega) := \{\mathbf{F} \in \mathbf{X}^s(\Omega) \mid \nabla{\cdot}(\varepsilon\mathbf{F}) = 0\}\,, \tag{5.3}$$

equipped with the canonical norm $\|\mathbf{F}\|^2_{\mathbf{X}^s_{\mathrm{div}=0}(\Omega)} := \|\mathbf{F}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla{\times}\mathbf{F}\|^2_{\mathbf{H}^{-s}(\Omega)}$. We anticipate (see proof of Theorem 5.1) that Problem (5.1) can be reformulated as follows: Find $\mathbf{E} \in \mathbf{X}^0_{\mathrm{div}=0}(\Omega)$ and $p \in H^1_0(\Omega)$ such that the following hold:

$$\left(\mu^{-1}\nabla{\times}\mathbf{E}, \nabla{\times}\mathbf{F}\right) = (\mathbf{g}, \mathbf{F})\,, \qquad \forall\mathbf{F} \in \mathbf{X}^0_{\mathrm{div}=0}(\Omega). \tag{5.4}$$

$$(\epsilon\nabla p, \nabla r) = (\nabla{\cdot}\mathbf{g}, r), \qquad \forall r \in H^1_0(\Omega). \tag{5.5}$$

Let us denote $A : \mathbf{L}^2(\Omega) \ni g \mapsto E \in \mathbf{X}^0_{\mathrm{div}=0}(\Omega)$ the solution map for the model problem Problem (5.4). We deduce an existence result as an immediate consequence of Proposition 4.1, i.e., the linear operator A is well defined.

Proposition 5.1. *Let the Assumption* (4.1) *hold. Problem* (5.4) *has a unique solution* $\mathbf{E} := A\mathbf{g}$ *in* $\mathbf{X}^0_{div=0}(\Omega)$ *for any* $\mathbf{g} \in \mathbf{L}^2(\Omega)$ *and there is a constant* c*, independent of* $\mathbf{g}$*, so that*

$$\|A\mathbf{g}\|_{\mathbf{X}^0(\Omega)} \leq c\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{5.6}$$

Proof. This is a direct application of the Lax-Milgram lemma. Indeed, for any $\mathbf{F} \in \mathbf{X}^0_{\mathrm{div}=0}(\Omega)$, Proposition 4.1 implies that

$$\|\mathbf{F}\|^2_{\mathbf{X}^0(\Omega)} = \|\mathbf{F}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\times\mathbf{F}\|^2_{\mathbf{L}^2(\Omega)} \leq c\,|\mathbf{F}|^2_{\mathbf{X}^0(\Omega)} + \|\nabla\times\mathbf{F}\|^2_{\mathbf{L}^2}.$$

After observing that $|\mathbf{F}|_{\mathbf{X}^0(\Omega)} = \|\nabla\times\mathbf{F}\|_{\mathbf{L}^2}$, since $\mathbf{F} \in \mathbf{X}^0_{\mathrm{div}=0}(\Omega)$, we conclude that

$$\|\mathbf{F}\|^2_{\mathbf{X}^0(\Omega)} \leq c\|\nabla\times\mathbf{F}\|^2_{\mathbf{L}^2} \leq c\mu_{\max}\left(\mu^{-1}\nabla\times\mathbf{F}, \nabla\times\mathbf{F}\right),$$

and the bilinear form $\left(\mu^{-1}\nabla\times\mathbf{F}, \nabla\times\mathbf{G}\right)$ is coercive in $\mathbf{X}^0_{\mathrm{div}=0}(\Omega)$. The rest of the proof is standard. □

5.2. **Regularity of the Maxwell problem.** We now establish regularity estimates for the solution of the Maxwell problem (5.4).

Theorem 5.1. *Let the regularity Assumption 4.1 hold. There exist τ_ε, τ_μ, depending only on Ω, ε, and μ so that,*

$$\mathbf{g} \in \mathbf{L}^2(\Omega) \mapsto A\mathbf{g} \in \mathbf{H}^s(\Omega) \text{ is continuous for all } s \in [0, \tau_\varepsilon), \tag{5.7}$$

$$\mathbf{g} \in \mathbf{L}^2(\Omega) \mapsto \nabla\times A\mathbf{g} \in \mathbf{H}^s(\Omega) \text{ is continuous for all } s \in [0, \tau_\mu). \tag{5.8}$$

Proof. By applying Proposition 4.1 and Proposition 5.1, we infer that

$$\|A\mathbf{g}\|_{\mathbf{H}^s(\Omega)} \leq c\,|A\mathbf{g}|_{\mathbf{X}^s(\Omega)} \leq c\|\nabla\times A\mathbf{g}\|_{\mathbf{L}^2(\Omega)} \leq c'\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)},$$

which proves (5.7).

We now prove (5.8). We first establish that there exists $p \in H^1_0(\Omega)$ so that $\nabla\times\left(\mu^{-1}\nabla\times A\mathbf{g}\right) = \mathbf{g} - \varepsilon\nabla p$. Let $\mathbf{F} \in \boldsymbol{\mathcal{C}}^\infty_0(\Omega)$ and let $q \in H^1_0(\Omega)$ be so that

$$(\varepsilon\nabla q, \nabla r) = (\varepsilon\mathbf{F}, \nabla r), \qquad \forall r \in H^1_0(\Omega),$$

and set $\mathbf{w} := \mathbf{F} - \nabla q$. The definition of q implies that $\mathbf{w}\times\mathbf{n}|_\Gamma = 0$, $\nabla\cdot(\varepsilon\mathbf{w}) = 0$, and $\nabla\times\mathbf{w} = \nabla\times\mathbf{F} \in \mathbf{L}^2(\Omega)$. As a result, $\mathbf{w}$ is a member of $\mathbf{X}^0(\Omega)$. This in turn implies that

$$\left(\mu^{-1}\nabla\times A\mathbf{g}, \nabla\times\mathbf{F}\right) = \left(\mu^{-1}\nabla\times A\mathbf{g}, \nabla\times\mathbf{w}\right) = (\mathbf{g}, \mathbf{w}) = (\mathbf{g}, \mathbf{F} - \nabla q).$$

Now let us define $p \in H^1_0(\Omega)$ so that

$$(\varepsilon\nabla p, \nabla r) = (\mathbf{g}, \nabla r), \qquad \forall r \in H^1_0(\Omega).$$

Then,

$$\left(\mu^{-1}\nabla\times A\mathbf{g}, \nabla\times\mathbf{F}\right) = (\mathbf{g}, \mathbf{F}) - (\varepsilon\nabla p, \nabla q) = (\mathbf{g}, \mathbf{F}) - (\varepsilon\mathbf{F}, \nabla p).$$

Since $\mathbf{F}$ is an arbitrary member of $\boldsymbol{\mathcal{C}}^\infty_0(\Omega)$, the above equality implies that

$$\nabla\times\left(\mu^{-1}\nabla\times A\mathbf{g}\right) + \varepsilon\nabla p = \mathbf{g}, \qquad \text{in } (\boldsymbol{\mathcal{D}}(\Omega))'.$$

The equality actually holds in $\mathbf{L}^2(\Omega)$ since $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and $p \in H^1_0(\Omega)$, and

$$\|\nabla\times\left(\mu^{-1}\nabla\times A\mathbf{g}\right)\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + \varepsilon_{\max}\|\nabla p\|_{\mathbf{L}^2(\Omega)} \leq (1 + \varepsilon_{\max}\varepsilon^{-1}_{\min})\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

In conclusion $\mu^{-1}\nabla\times A\mathbf{g}$ is a member of $\mathbf{H}_{\mathrm{curl}}(\Omega)$.

Now let us observe that since $\nabla\times A\mathbf{g}$ is a member of $\mathbf{H}_{\mathrm{div}}(\Omega)$, the condition $\nabla\cdot(\nabla\times A\mathbf{g}) = 0$ together with the boundary condition $A\mathbf{g}\times\mathbf{n}|_\Gamma = 0$ implies that $(\nabla\times A\mathbf{g})\cdot\mathbf{n}|_\Gamma = 0$. Moreover it is clear that $\nabla\cdot(\mu(\mu^{-1}\nabla\times A\mathbf{g})) = 0$. In conclusion,

$\mu^{-1}\nabla\times A\mathbf{g}$ is a member of $\mathbf{Y}^0(\Omega)$. Using Proposition 4.2 with $t=0$, we infer that there is $\tau_\mu > 0$ so that the following holds for all $s \in [0, \tau_\mu)$:

$$\begin{aligned}
\|\mu^{-1}\nabla\times A\mathbf{g}\|_{\mathbf{H}^s(\Omega)} &\leq c\|\mu^{-1}\nabla\times A\mathbf{g}\|_{\mathbf{Y}^0(\Omega)} \\
&= c\left(\|\mu^{-1}\nabla\times A\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + \|\nabla\times(\mu^{-1}\nabla\times A\mathbf{g})\|_{\mathbf{L}^2(\Omega)}\right) \\
&\leq c'\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

We conclude by using the fact that $\mathcal{E}_\mu$ is a continuous mapping from $\mathbf{H}^s(\Omega)$ to $\mathbf{H}^s(\Omega)$ for all $s \leq \tau_\mu < \frac{1}{2}$. $\square$

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