Methods of Applied Mathematics II
MATH 603 - 600
A. Bonito

October 29th
Fall 2009

First Name: ____________________ **Last Name:** ____________________

Midterm

- 75 minute individual midterm;
- Answer the questions in the space provided. If you run out of space, continue onto the back of the page. Additional space is provided at the end;
- **Show and explain all work**;
- **Underline** the answer of each steps;
- The use of books, personal notes, **calculator**, cellphone, laptop, and communication with others is forbidden;
- By taking this midterm, you agree to follow the university's code of academic integrity.

Ex 1 (40%)	Ex 2 (30%)	Ex 3 (30%)	Ex 4 (B 10%)	Total

<u>Exercise 1</u> **40%**

Let $k > 0$ and $\phi : (0,1) \to \mathbb{R}$ be a smooth function. Consider the following heat equation

$$\frac{\partial}{\partial t} u(x,t) - k \frac{\partial^2}{\partial x^2} u(x,t) = 0, \qquad 0 < x < 1, \qquad t > 0$$

supplemented with the initial condition

$$u(x,0) = \phi(x), \qquad 0 < x < 1$$

and the boundary conditions

$$u(0,t) = \frac{\partial}{\partial x} u(1,t) = 0, \qquad t > 0.$$

1. Derive an energy estimate and deduce the uniqueness of a solution $u(x,t)$ to the above system ;
2. Using Fourier series, find the solution u in the particular case $\phi(x) = 1$;
3. Compute $\lim_{t\to\infty} u(t,x)$.

<u>Hint :</u>Explain precisely your work.

Exercise 2 **30%**

Find the singular value decomposition of

$$A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}.$$

Exercise 3 **30%**

Let $f(x) = e^x$ for $0 < x < 1$.

1. Find B_n and ω_n such that

$$e^x = \sum_{n=1}^{\infty} B_n \sin(\omega_n x), \qquad 0 < x < 1.$$

2. Find A_n such that

$$e^x = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(\omega_n x), \qquad 0 < x < 1$$

(same w_n's as in the first item).

3. Discuss the pointwise and uniform convergence of the above two series. Moreover discuss what happens at $x = 0$ in both cases.
4. Explain why

$$B_n = -\omega_n A_n, \qquad n \geqslant 1.$$

Exercise 4 **10% BONUS**

Find a function $u : \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}$ satisfying

$$\Delta u(x,y) = \frac{1}{(x^2+y^2)^{3/2}}.$$

Hint : Recall the formula for the Laplacian in polar coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

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Midterm: solutions

Exercise 1

1. Multiplying the PDE by u and integrating for $x = 0, ..., 1$ yields

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 + k\int_0^1 u_x^2 - uu_x|_{x=0}^{x=1} = 0,$$

where we also used an integration by parts. The boundary and initial conditions yield the energy estimate after integrating over time

$$\frac{1}{2}\int_0^1 u^2 + k\int_0^1 u_x^2 = \frac{1}{2}\int_0^1 \phi^2.$$

To deduce the uniqueness, set $w = u_1 - u_2$ where u_1, u_2 are two solutions. We readily obtain that w satisfies

$$\frac{\partial}{\partial t}w(x,t) - k\frac{\partial^2}{\partial x^2}w(x,t) = 0, \qquad 0 < x < 1, \qquad t > 0$$

and

$$w(x,0) = 0 \qquad 0 < x < 1, \qquad \text{and} \qquad w(0,t) = \frac{\partial}{\partial x}w(1,t) = 0, \qquad t > 0.$$

The energy estimate implies

$$\frac{1}{2}\int_0^1 w^2 + k\int_0^1 w_x^2 = 0$$

and as a consequence $w = 0$, i.e. $u_1 = u_2$.

2. Given the boundary conditions, we define for any $f : [0,1] \to \mathbb{R}$, an extension operator as follows

$$\widetilde{f} = \begin{cases} f(x) & 0 < x < 1, \\ f(1-x) & 1 < x < 2, \\ -f(x) & -2 < x < 0, \\ 4-\text{periodic}. \end{cases}$$

It is easy to see that if u satisfies the heat equation on for $0 < x < 1$, then $\widetilde{u}$ satisfies

$$\frac{\partial}{\partial t}\widetilde{u}(x,t) - k\frac{\partial^2}{\partial x^2}\widetilde{u}(x,t) = 0, \qquad \text{on} \quad \mathbb{R}, \qquad t > 0$$

together with the initial condition

$$\widetilde{u}(x,0) = \widetilde{\phi} \qquad \text{on} \quad \mathbb{R}.$$

Since the above extension is odd, we look for $U_n(t)$, $n \geqslant 1$, such that

$$\widetilde{u}(x,t) = \sum_{n=1}^{\infty} U_n(t)\sin\left(\frac{n\pi}{2}x\right).$$

We look for a solution u at least $C^2(0,1)$ and the choice of the extension ensures that $\widetilde{u}$ is $C^1(\mathbb{R})$ with second derivative pwc. As a consequence,

$$\widetilde{u}_{xx}(x,t) = \sum_{n=1}^{\infty} -\frac{n^2\pi^2}{4}U_n(t)\sin\left(\frac{n\pi}{2}x\right).$$

Plugging the expression for $\widetilde{u}$ in the PDE we obtain ODE's for $U_n(t)$

$$U_n' + \frac{kn^2\pi^2}{4} U_n = 0,$$

i.e.

$$U_n = C_n e^{-\frac{kn^2\pi^2}{4} t}$$

for some constants C_n. The initial condition $\widetilde{u} = \widetilde{\phi}$ yields

$$C_n = \frac{1}{2}\int_{-2}^{2} \widetilde{\phi} \sin\left(\frac{n\pi}{2} x\right) = \int_0^2 \sin\left(\frac{n\pi}{2} x\right) = \frac{2}{n\pi}\left(1 - (-1)^n\right).$$

Finally

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - (-1)^n\right) e^{-\frac{kn^2\pi^2}{4} t} \sin\left(\frac{n\pi}{2} x\right), \qquad 0 < x < 1.$$

3. $\lim_{t\to\infty} u(x,t) = 0$.

Exercise 2

In order to find V we compute

$$A_1 A_1^T = \begin{pmatrix} 10 & 20 \\ 20 & 40 \end{pmatrix}$$

and find the eigenpairs of the above matrix :

$$\lambda_1 = 50;\ v_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \lambda_2 = 0;\ v_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Note that only the fist eigenvector needs to be computed, the second eigenvector is in the orthogonal direction. Thus,

$$V = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Similarly,

$$W = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$$

is obtained after computing $A_1^T A_1$. Finally

$$D = V^T A_1 W = \begin{pmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{pmatrix}$$

so that the SVD of A_1 reads

$$A_1 = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}.$$

Exercise 3

1. In order to obtain a Fourier serie in sin one needs to consider the odd extension of e^x, $0 < x < 1$ about $x = 0$ together with the 2−periodic extension. This yields $\omega_n = n\pi$ and

$$B_n = 2\int_0^1 e^x \sin(n\pi x) = \frac{2n\pi\left(1 - (-1)^n e\right)}{1 + n^2\pi^2}.$$

2. Similarly but using an even extension leads to

$$A_n = 2\int_0^1 e^x \cos(n\pi x) = \frac{2\left((-1)^n e - 1\right)}{1 + n^2\pi^2}.$$

3. In the one hand, the first extension (item 1) is only pw continuous so that its Fourier serie converge pointwise to e^x, $0 < x < 1$. In the other hand, the second extension (item 2) is continuous with derivative pw continuous. We can therefore conclude that its Fourier serie is uniformly converging to e^x, $0 < x < 1$. Also

$$\sum_{n=1}^{\infty} \frac{2n\pi\,(1-(-1)^n e)}{1+n^2\pi^2} = \frac{e-e}{2} = 0$$

and

$$\frac{2\,(e-1)}{2} + \sum_{n=1}^{\infty} \frac{2\,((-1)^n e - 1)}{1+n^2\pi^2} = \frac{e+e}{2} = e.$$

4. We discussed in item 3 why the serie in cos is uniformly converging thereby justifying the derivation term by term

$$(e^x)' = \left(\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(\omega_n x)\right)' = \sum_{n=1}^{\infty} -\omega_n A_n \sin(\omega_n x).$$

Comparing with the serie in sin of $e^x = (e^x)'$ and invoking the uniqueness of the Fourier serie one gets

$$B_n = -\omega_n A_n, \qquad n \geqslant 1$$

(as predicted by items 1 and 2).

Exercise 4

Using the polar coordinates we rewrite the equation

$$\Delta u(r,\theta) = r^{-3}$$

and in fact look for a solution $u(r,\theta) = R(r)$. Computing the laplacian operator in polar coordinates yields

$$R'' + \frac{1}{r}R' = r^{-3}$$

or

$$(rR')' = r^{-2}.$$

Integrating twice the above equation leads to

$$R(r) = \frac{1}{r} + \tilde{C}\ln(r) + D$$

for some constants $\tilde{C}$, D. Finally we obtain

$$u(x,y) = \frac{1}{\sqrt{x^2+y^2}} + C\ln(x^2+y^2) + D,$$

where $C = \tilde{C}/2$.