MATHEMATICAL ANALYSIS OF A SIMPLIFIED HOOKEAN DUMBBELLS MODEL ARISING FROM VISCOELASTIC FLOWS

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On the occasion of the seventieth birthday of Prof. G. Da Prato

Abstract. A stochastic model corresponding to a simplified Hookean dumbbells viscoelastic fluid is considered, the convective terms being disregarded. Existence on a fixed time interval is proved provided the data are small enough, using the implicit function theorem and a maximum regularity property for a three fields Stokes problem.

1. Introduction

Modeling of viscoelastic flows is of great importance for complex engineering applications involving blood, paints or adhesives. In the traditional macroscopic approach the unknowns are the velocity, the pressure and the extra-stress satisfying the mass and momentum equations supplemented with a so-called constitutive equation. This constitutive equation between the velocity and the stress can be either differential or integral and can be justified by a kinetic theory [5, 33].

The simplest macroscopic example is the Oldroyd-B model which can be derived from to the mesoscopic Hookean dumbbells model. The stochastic dumbbells model corresponds to a dilute solution of liquid polymer, that is a newtonian solvent with non interacting polymer chains. The polymer chains are modeled by dumbbells, two beads connected with elastic springs, see Fig. 1.

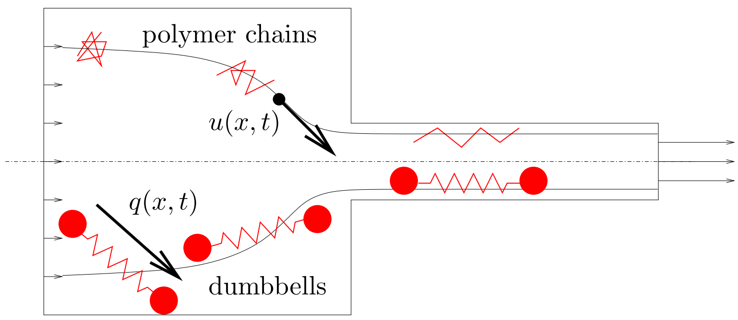

Figure 1. The mesoscopic dumbbells model for a dilute solution of liquid polymer.

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The mass and momentum conservation laws lead to the following partial differential equations for the velocity u, the pressure p and the extra-stress σ

$$\rho\left(\frac{\partial u}{\partial t}+(u\cdot\nabla)u\right)-\nabla\cdot(2\eta_s\epsilon(u)+\sigma)+\nabla p=f, \tag{1.1}$$

$$\nabla\cdot u=0. \tag{1.2}$$

Here ρ is the density, f a force term, η_s is the solvent viscosity and $\epsilon(u) := \frac{1}{2}\left(\nabla u+(\nabla u)^T\right)$ is the symmetric part of the velocity gradient. On the other hand, the Newton law for the beads leads to the following stochastic differential equation for the dimensionless spring elongation q

$$dq=\left(-(u\cdot\nabla)q)+(\nabla u)q-\frac{1}{2\lambda}\mathsf{F}(q)\right)dt+\frac{1}{\sqrt{\lambda}}\,dB, \tag{1.3}$$

where λ is the relaxation time, F is the force due to the elastic spring and B is a vector of independent Wiener processes modeling the thermal agitation and collisions with the solvent molecules. The transport term $(u\cdot\nabla)q$ in (1.3) corresponds to the fact that the trajectories of the dumbbells center of mass are those of the liquid particles. The term $(\nabla u)q$ takes into account the drag force due to the beads. The extra-stress σ is then obtained by the mean of the closure equation

$$\sigma=\frac{\eta_p}{\lambda}(\mathbb{E}(q\otimes\mathsf{F}(q))-I), \tag{1.4}$$

with η_p the polymer viscosity. The case $\mathsf{F}(q)=q$, namely Hookean dumbbells, leads using formal stochastic calculus to the Oldroyd-B model where the extra-stress σ satisfies

$$\sigma+\lambda\left(\frac{\partial\sigma}{\partial t}+(u\cdot\nabla)\sigma-(\nabla u)\sigma-\sigma(\nabla u)^T\right)=2\eta_p\epsilon(u). \tag{1.5}$$

The FENE dumbbells model (see [5, 33] for a detailed description) is a more realistic model corresponding to $F(q)=\frac{q}{1-q^2/b}$, where $b>0$ depends on the number of monomer units of a polymer chain. In that case, there is no equivalent constitutive relation for the extra-stress, but closure approximations (such as FENE-P, see for instance [5, 33]) have been derived. These approximations can have significant impact on rheological prediction (see for instance [2, 10, 26]). Recently, due to increasing computational resources, the equations (1.1)-(1.4) have been solved numerically to obtain more realistic results [9, 8, 11, 26, 29, 28, 7]. For a review of numerical methods used in viscoelastic flows we refer for instance to [34, 4, 27].

We will focus in this paper on the stochastic description of the simplest dumbbells model, namely the Hookean dumbbells model $F(q)=q$. Although the Hookean dumbbells model is too simple to reproduce experiment such as shear thinning for instance, it already contains some of the mathematical and numerical difficulties included in the kinetic theory. Indeed, this model is formally equivalent to the Oldroyd-B model and consequently, to the major difficulties already present in the macroscopic model, we must add those coming from stochastic modeling. These difficulties are:

i. The presence of the quadratic term $(\nabla u)q$ which prevents a priori estimates to be obtained and therefore existence to be proved for any data;
ii. The presence of the convective term $(u\cdot\nabla)q$ which requires an adequate mathematical analysis [30] and the use of numerical schemes suited to transport dominated problems;

iii. The Wiener process in (1.3) requires a specific mathematical treatment.

Concerning the analysis of macroscopic viscoelastic models, a large amount of publications can be found. The existence of slow steady viscoelastic flow has been proved in [35, 3]. For the time-dependent case, existence of solutions locally in time and, for small data, globally in time has been proved in [22] in Hilbert spaces. Extensions to Banach spaces and a review can be found in [16]. Finally, existence for any data has been proved in [31] for a corotational Oldroyd model only.

On the other hand, only a few papers pertaining to the kinetic theory have been published. The complete analysis and numerical analysis of a one dimensional FENE shear flows can be found in [23, 24]. Well posedeness of the dumbbell model in three space dimensions has been proved for nonlinear elastic dumbbells in [15].

In this paper, the mathematical analysis is proposed for a simplified time-dependent Hookean dumbbells problem in two or three space dimensions. More precisely, we focus on item i. and iii. above, thus remove the convective terms. The reason for considering the time-dependent Hookean dumbbells problem without convection is motivated by the fact that this simplified problem corresponds to the correction step in the splitting algorithm described in [7, 21] for solving viscoelastic flows with complex free surfaces using macroscopic and/or mesoscopic models. The consequence when removing convective terms is that the implicit function theorem can be used to prove existence and convergence results, whenever the data are small enough, thus the techniques presented in [6] are extended to a stochastic framework. Our existence result is obtained as in [6] by invoking a maximum regularity property for the three fields Stokes.

The outline of the paper is as follows. The simplified Hookean dumbbells problem is introduced in the next section. Then, in section 3, mathematical existence of a solution is proved in Banach spaces.

2. A simplified Hookean dumbbells model

Let D be a bounded, connected open set of $\mathbb{R}^d$, $d = 2$ or 3 with boundary ∂D of class $\mathcal{C}^2$, and let $T > 0$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete filtered probability space. The filtration $\mathcal{F}_t$ upon which the Brownian process B is defined is completed with respect to $\mathcal{P}$ and is assumed to be right continuous on $[0, T]$. We assume also that the space Ω is rich enough to accommodate a random vector $q_0 : \Omega \to \mathbb{R}^d$ such that

$$\begin{cases} q_0 \text{ is independent of } B \text{ and } (q_0)_i \text{ is independent of } (q_0)_j, 1 \le i \ne j \le d, \\ \text{and } \mathbb{E}(q_0) = 0, \ \mathbb{E}(q_0 \otimes q_0) = I. \end{cases} \tag{2.1}$$

In fact, q_0 is an initial condition for the dumbbells elongation q which corresponds to the equilibrium state since the conditions $\mathbb{E}(q_0) = 0$ and $\mathbb{E}(q_0 \otimes q_0) = I$ lead to a vanishing initial extra-stress. These conditions could be relaxed to yield constant initial stresses with respect to the space variable $x \in D$.

We consider the following problem. Given initial conditions $u_0 : D \to \mathbb{R}^d$, $q_0 : \Omega \to \mathbb{R}^d$ satisfying (2.1), a force term $f : D \times [0, T] \to \mathbb{R}^d$, constant solvent and polymer viscosities $\eta_s > 0$, $\eta_p > 0$, a constant relaxation time $\lambda > 0$, find the velocity $u : D \times [0, T] \to \mathbb{R}^d$, the pressure $p : D \times [0, T] \to \mathbb{R}$ and the dumbbells elongation vector $q : D \times [0,T] \times \Omega \to \mathbb{R}^d$ such that

$$dq - \left((\nabla u)q - \frac{1}{2\lambda} q \right) dt - \frac{1}{\sqrt{\lambda}}\, dB = 0 \qquad \text{in } D \times (0,T) \times \Omega, \tag{2.2}$$

$$\rho \frac{\partial u}{\partial t} - \nabla \cdot \left(2\eta_s \epsilon(u) + \frac{\eta_p}{\lambda} \left(\mathbb{E}(q \otimes q) - I \right) \right) + \nabla p = f \qquad \text{in } D \times (0,T), \tag{2.3}$$

$$\nabla \cdot u = 0 \qquad \text{in } D \times (0,T), \tag{2.4}$$

$$u\,(.,0) = u_0 \qquad \text{in } D, \tag{2.5}$$

$$q(.,0,.) = q_0 \qquad \text{in } D \times \Omega, \tag{2.6}$$

$$u = 0 \qquad \text{on } \partial D \times (0,T). \tag{2.7}$$

Remark 2.1. Equations (2.2) and (2.6) are notations for

$$q(x,t,\omega) - q_0(t,\omega) - \int_0^t \left((\nabla u(x,s)) q(x,s,\omega) - \frac{1}{2\lambda} q(x,s,\omega) \right) ds - \frac{1}{\sqrt{\lambda}} B(t,\omega) = 0,$$

where $(x,t,\omega) \in D \times [0,T] \times \Omega$.

System (2.2)-(2.7) formally contains the simplified Oldroyd-B problem studied in [6]. Indeed, using Îto's formula, one obtains that the variance

$$V := \mathbb{E}(q \otimes q)$$

satisfies the deterministic equation

$$\frac{\partial V}{\partial t} = \left(\nabla u - \frac{I}{2\lambda} \right) V + V \left(\nabla u - \frac{I}{2\lambda} \right)^T + \frac{1}{\lambda} I \qquad \text{in } D \times [0,T], \tag{2.8}$$

see for instance problem 6.1 (p.355) in [25]. Thus setting

$$\sigma := \frac{\eta_p}{\lambda} \left(V - I \right), \tag{2.9}$$

equation (2.8) corresponds to the constitutive equation of the simplified Oldroyd-B model without convective terms

$$\sigma + \lambda \left(\frac{\partial \sigma}{\partial t} - (\nabla u)\sigma - \sigma (\nabla u)^T \right) = 2\eta_p \epsilon(u), \quad \sigma(0) = 0. \tag{2.10}$$

In Remark 3.12 the link between the solution of the Oldroyd-B problem studied in [6] and that of system (2.2)-(2.7) will be made precise from the mathematical viewpoint.

One of the difficulties of problem (2.2)-(2.7) is that we have to deal with stochastic processes with value in Banach spaces. Indeed in classical textbooks, Îto formula (see for instance Theorem 3.3, chapter IV, in [36]), relation (2.8) as well as classical existence and uniqueness results for linear stochastic differential equation (Theorem 2.1, chapter IX, in [36]) are not presented in this context. Hence, we will split the dumbbells elongation into two components

$$q = q^S + q^D, \tag{2.11}$$

where $q^S : \Omega \times [0,T] \to \mathbb{R}^d$ is the solution at equilibrium (that is to say when $u = 0$) and obviously $q^D : \Omega \times [0,T] \times D \to \mathbb{R}^d$ is the discrepancy with respect to the equilibrium. The stochastic differential equation satisfied by q^S is

$$dq^S = -\frac{1}{2\lambda} q^S dt + \frac{1}{\sqrt{\lambda}} dB, \qquad q^S(0) = q_0, \tag{2.12}$$

while the equation satisfied by q^D

$$\frac{\partial q^D}{\partial t} = \nabla u \; (q^D + q^S) - \frac{1}{2\lambda} q^D, \qquad q^D(0) = 0, \tag{2.13}$$

is a differential equation with a stochastic forcing term.

In [6] we have proved that the extra-stress σ solution of the simplified Oldroyd-B problem (2.10) was in spaces $W^{1,q}(0,T;W^{1,r}(D;\mathbb{R}^{d\times d}_{sym}))$ or in spaces $h^{1+\mu}([0,T];W^{1,r}(D;\mathbb{R}^{d\times d}_{sym}))$, where the little Hölder spaces $h^\mu([0,T];E)$ are closed subset of the classical Hölder spaces $\mathcal{C}^\mu([0,T];E)$ and are defined for all Banach space E and for all $0<\mu<1$ by

$$h^\mu([0,T];E) := \{f \in \mathcal{C}^\mu([0,T];E); \lim_{\delta\to 0} \sup_{t,s\in I,|t-s\,|<\delta} \frac{\|f(t)-f(s)\|_E}{|t-s\,|^\mu} = 0\},$$

see for instance [32]. Since, according to [36] (Theorem 2.2 p. 26, Corollary 2.6 p 28 and Theorem 2.7 p. 29), a Brownian motion can not be expected in a more regular space than $L^\gamma(\Omega;\mathcal{C}^{\mu'}(0,T))$, $\mu' < \frac{1}{2}$ and $2 \le \gamma < \infty$, the use of Sobolev spaces in time is not appropriate. Moreover, the reason for using little Hölder spaces is that in a stochastic context, it is more convenient to deal with separable spaces; the spaces $h^k([0,T];E)$ provided with the norm of $C^\mu(0,T;E)$ are separable Banach spaces if E is separable and for all $0<\mu<\mu'<1$ we have $C^{\mu'} \subset h^\mu$ (see again [32]). We will use the notation $h^\mu_0([0,T];E)$ for the restriction of functions of $h^\mu([0,T];E)$ vanishing at the origin. For simplicity, the notation will be abridged as follows whenever there is no possible confusion. For $1<r,\gamma<+\infty$, $0<\mu<1$ the space L^r denotes $L^r(D;\mathbb{R})$ or $L^r(D;\mathbb{R}^d)$. Also, $h^\mu(L^r)$ stands for $h^\mu([0,T];L^r(D;\mathbb{R}))$ or $h^\mu([0,T];L^r(D;\mathbb{R}^d))$ and $L^\gamma(h^\mu(L^r))$ for $L^\gamma(\Omega;h^\mu([0,T];L^r(D;\mathbb{R})))$ or $L^\gamma(\Omega;h^\mu([0,T];L^r(D;\mathbb{R}^d)))$. The same notation applies for higher order spaces such as $W^{1,r}$, $h^{1+\mu}(W^{1,r})$ and $L^\gamma(h^{1+\mu}(W^{1,r}))$.

The implicit function theorem will be used to prove that (2.2)-(2.7) admits a unique solution

$$u \in h^{1+\mu}(L^r) \cap h^\mu(W^{2,r}\cap H^1_0), \quad q \in L^\gamma(h^\mu(W^{1,r})), \quad p \in h^\mu(W^{1,r}\cap L^2_0), \tag{2.14}$$

with $2\le\gamma<\infty$, $0<\mu<1/2$ and $d<r<\infty$ whenever the data f, u_0 are small enough in appropriate spaces. It will be shown that $h^\mu([0,T];W^{1,r}(D)) \subset \mathcal{C}([0,T]\times\overline{D})$, which implies in particular, that a process $q \in L^\gamma(h^\mu(W^{1,r}))$ has a continuous sample path for almost each realization.

3. Existence of the simplified Hookean dumbbells model

We introduce, as in [16], the Helmholtz-Weyl projector [17, 18, 19] defined by

$$P_r : L^r(D;\mathbb{R}^d) \to \mathcal{H}_r, \quad 1<r<\infty, \tag{3.1}$$

where $\mathcal{H}_r$ is the completion of the divergence free $C^\infty_0(D)$ vector fields with respect to the norm of L^r. The space $\mathcal{H}_r$ can be characterized as follows (again see for instance [18])

$$\mathcal{H}_r = \left\{v \in L^r(D;\mathbb{R}^d)\ ;\ \nabla\cdot v = 0,\ v\cdot n = 0 \text{ on } \partial D \text{ in a weak sense}\right\}.$$

Since D is of class $\mathcal{C}^2$, there exists a constant C such that for all $f \in L^r$

$$\|P_r f\|_{L^r(D)} \le C\|f\|_{L^r(D)}. \tag{3.2}$$

We define $A_r := -P_r\Delta : \mathcal{D}_{A_r} \to \mathcal{H}_r$ the Stokes operator, where

$$\mathcal{D}_{A_r} := \left\{ v \in W^{2,r}(D;\mathbb{R}^d) \cap W_0^{1,r}(D;\mathbb{R}^d) \mid \nabla \cdot v = 0 \right\}. \tag{3.3}$$

It is well known (see [20]) for D of class $\mathcal{C}^2$, that the operator A_r equipped with the usual norm of $L^r(D;\mathbb{R}^d)$ is closed and densely defined in $\mathcal{H}_r$. Moreover, the graph norm of A_r is equivalent to the $W^{2,r}$ norm.

With the above definitions, (u, q) is said to be a solution of (2.2)-(2.7) if

$$u \in h^{1+\mu}(\mathcal{H}_r) \cap h^{\mu}(\mathcal{D}_{A_r}), \quad q \in L^{\gamma}(h^{\mu}(W^{1,r})),$$

q adapted to $(\mathcal{F}_t)$, with $0 < \mu < 1/2$, $d < r < \infty$, $2 \le \gamma < \infty$ and satisfies

$$dq - \left(\nabla u q - \frac{1}{2\lambda} q\right) dt - \frac{1}{\sqrt{\lambda}}\, dB = 0 \qquad \text{in } D \times (0,T) \times \Omega, \tag{3.4}$$

$$\rho \frac{\partial u}{\partial t} + \eta_s A_r u - \frac{\eta_p}{\lambda} P_r \nabla \cdot (\mathbb{E}(q \otimes q) - I) = P_r f \qquad \text{in } D \times (0,T), \tag{3.5}$$

$$u(.,0) = u_0 \qquad \text{in } D, \tag{3.6}$$

$$q(.,0) = q_0 \qquad \text{in } \Omega. \tag{3.7}$$

We will assume that the source term is $f \in h^{\mu}(L^r)$, the initial data are $u_0 \in \mathcal{D}_{A_r}$ and $q_0 \in L^{\gamma}(\Omega)$ satisfying (2.1) together with the following compatibility condition

$$-A_r u_0 + P_r f(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}},$$

where

$$E_{\mu,\infty} := (\mathcal{H}_r, \mathcal{D}_{A_r})_{\mu,\infty} = \left\{ x \in \mathcal{H}_r; \sup_{t>0} \|t^{1-\mu} A_r e^{-tA_r} x\|_{L^r(D)} < +\infty \right\}$$

is a Banach space endowed with the norm

$$\|x\|_{E_{\mu,\infty}} := \|x\|_{L^r(D)} + \sup_{t>0} \|t^{1-\mu} A_r e^{-tA_r} x\|_{L^r(D)}.$$

We refer to [37, 14] for more details.

We can now state the main result of this paper.

Theorem 3.1. *Let $d \ge 2$, let $D \subset \mathbb{R}^d$ be a bounded, connected open set with boundary of class $\mathcal{C}^2$ and let $T > 0$. Assume $0 < \mu < \frac{1}{2}$, $2 \le \gamma < \infty$ and $d < r < \infty$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ a complete filtered probability space with $\mathcal{F}_t$ right continuous for all $t \in [0,T]$ and upon which the Brownian process $B \in L^{\gamma}(h^{\mu})$ and the initial condition $q_0 \in L^{\gamma}$ are defined. Moreover we assume q_0 satisfies (2.1). Then there exists $\delta_0 > 0$ such that for every $f \in h^{\mu}(L^r)$, $u_0 \in \mathcal{D}_{A_r}$ satisfying*

$$-\eta_s A_r u_0 + P_r f(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}},$$

and

$$\|P_r f - P_r f(0)\|_{h^{\mu}(L^r)} + \|u_0\|_{W^{2,r}} + \| - \eta_s A_r + P_r f(0)\|_{\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}} \le \delta_0,$$

there exists exactly one solution of (3.4)-(3.7). Moreover, the mapping

$$(P_r f, u_0) \mapsto (u(f, u_0), q(f, u_0))$$

is analytic.

Using the well known properties of the Helmholtz-Weyl projector [17, 18, 19], we obtain the following result.

Corollary 3.2. *Under the assumptions of the above theorem, there exists a unique solution (u, q, p) of (2.2)-(2.7) with the regularity (2.14).*

The vector field $q^S : \Omega \times [0, T] \times D \to \mathbb{R}^d$ is now more precisely defined. Given $q_0 \in L^\gamma(\Omega)$ satisfying (2.1), $q^S \in L^\gamma(\Omega; h^\mu([0, T]; \mathbb{R}^d))$ is the unique solution (up to indistinguishability and ensured by Theorem 2.1, chapter IX, in [36]) of (2.12). Moreover, (using equation (6.8) section 5.6 of [25]) we obtain a relation for the covariance of q^S:

$$\mathbb{E}(q^S(s) \otimes q^S(t)) = e^{-\frac{|t-s|}{2\lambda}} I, \quad \forall s, t \in [0, T]. \tag{3.8}$$

Eventhough q^S does not depend on $x \in D$, we will consider when needed q^S as an element of $L^\gamma(h^\mu(W^{1,r}))$.

Using (2.11) and (2.13) problem (3.4)-(3.7) can be rewritten as, find

$$u \in h^{1+\mu}(\mathcal{H}_r) \cap h^\mu(\mathcal{D}_{A_r}), \quad q^D \in L^\gamma(h^{1+\mu}(W^{1,r}) \cap h_0^\mu(W^{1,r})),$$

with $2 \le \gamma < \infty,\ 0 < \mu < 1/2,\ d < r < \infty$, such that

$$\frac{\partial q^D}{\partial t} + \frac{1}{2\lambda} q^D - S_d(u, q^D) = (\nabla u) q^S \qquad \text{in } D \times (0, T) \times \Omega, \tag{3.9}$$

$$\begin{aligned} &\rho \frac{\partial u}{\partial t} + \eta_s A_r u \\ &- \frac{\eta_p}{\lambda} P_r \nabla \cdot \big(\mathbb{E}(q^D \otimes q^S + q^S \otimes q^D) + S_c(u, q^D)\big) = P_r f \end{aligned} \qquad \text{in } D \times (0, T), \tag{3.10}$$

$$u\,(.,0) = u_0 \qquad \text{in } D, \tag{3.11}$$

with

$$S_d(u, q^D) := (\nabla u) q^D, \tag{3.12}$$

$$S_c(u, q^D) := \mathbb{E}(q^D \otimes q^D). \tag{3.13}$$

It will be shown S_d and S_c are well defined in appropriate spaces.

In order to prove Theorem 3.1, we shall introduce the mapping $F : Y \times X \to Z$, where

$$\begin{aligned} Y =& \big\{(P_r f, u_0), \text{ such that } (f, u_0) \in h^\mu(L^r) \times \mathcal{D}_{A_r} \text{ and} \\ & - \eta_s A_r u_0 + P_r f(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}} \big\}, \\ W =& \big\{ w \in L^\gamma(h^\mu(W^{1,r})); w \text{ adapted to } (\mathcal{F}_t)_{t \in [0,T]} \big\}, \quad Z = W \times Y, \\ X =& \big\{(u, q) \in h^{1+\mu}(\mathcal{H}_r) \cap h^\mu(\mathcal{D}_{A_r}) \times L^\gamma(h^{1+\mu}(W^{1,r}) \cap h_0^\mu(W^{1,r})); \\ & q \text{ adapted to } (\mathcal{F}_t)_{t \in [0,T]} \big\}, \end{aligned}$$

and for $y = (P_r f, u_0) \in Y$ and $x = (u, q^D) \in X$

$$F(y, x) = \begin{pmatrix} \dfrac{\partial q^D}{\partial t} + \dfrac{1}{2\lambda} q^D - S_d(u, q^D) - (\nabla u) q^S \\ \rho \dfrac{\partial u}{\partial t} + \eta_s A_r u - \frac{\eta_p}{\lambda} P_r \nabla \cdot \big(\mathbb{E}(q^D \otimes q^S + q^S \otimes q^D) + S_c(u, q^D)\big) - P_r f \\ u\,(.,0) - u_0 \end{pmatrix}, \tag{3.14}$$

with $q^S \in L^\gamma(h^\mu(W^{1,r}))$ defined by (2.12). Then problem (3.9)-(3.11) can be reformulated as: given $q_0 \in L^\gamma(\Omega)$ satisfying (2.1), $q^S \in L^\gamma(h^\mu(W^{1,r}))$ defined by (2.12) and $y = (P_r f, u_0) \in Y$, find $x = (u, q^D) \in X$ such that

$$F(y, x) = 0 \text{ in } Z. \tag{3.15}$$

The aim is to use the implicit function theorem by proving

- the spaces X, Y, W and Z equipped with appropriate norms are Banach spaces,
- F is a well defined, real analytic mapping,
- $F(0,0)=0$ and the Fréchet derivative $D_xF(0,0)$ is an isomorphism from X to Z.

This establishes the existence part in the conclusion of Theorem 3.1. The uniqueness part is treated separately.

The space X is equipped with the norm $\|\cdot\|_X$ defined for $x=(u,q^D)\in X$ by

$$\|x\|_X = \|u,q^D\|_X = \|u\|_{h^{1+\mu}(L^r)} + \|u\|_{h^\mu(W^{2,r})} + \|q^D\|_{L^\gamma(h^{1+\mu}(W^{1,r}))}.$$

Since X is a closed subspace of $h^{1+\mu}(\mathcal{H}_r)\cap h^\mu(\mathcal{D}_{A_r})\times L^\gamma(h^\mu(W^{1,r}))$, it becomes a Banach space. The space Y is equipped with the norm $\|\cdot\|_Y$ defined for $y=(P_rf,u_0)\in Y$ by

$$\begin{aligned}\|y\|_Y &= \|P_rf,u_0\|_Y\\ &= \|P_rf-P_rf(0)\|_{h^\mu(L^r)} + \|u_0\|_{W^{2,r}} + \|-\eta_sA_ru_0+P_rf(0)\|_{\overline{\mathcal{D}_{A_r}}^{E_\mu,\infty}}.\end{aligned}$$

As a consequence of the continuity of the linear mapping

$$(P_rf,u_0)\longmapsto -\eta_sA_ru_0+P_rf(0)$$

from $h^\mu(\mathcal{H}_r)\times\mathcal{D}_{A_r}$ (equipped with the product norm) to L^r and of to the completeness of $\overline{\mathcal{D}_{A_r}}^{E_\mu,\infty}$, the space $(Y,\|.\|_Y)$ is a closed subspace of $h^\mu(\mathcal{H}_r)\times\mathcal{D}_{A_r}\times W^{1,r}$ and thus a Banach space. Similarly W is a Banach space equipped with the induced norm of $L^\gamma(h^\mu(W^{1,r}))$. The space Z is equipped with the product norm and becomes a Banach space .

The following Lemma ensures the space $h^\mu(W^{1,r})$ is a Banach algebra and as a Corollary, it will be deduced the function $F: Y\times X\to Z$ is well defined and analytic.

Lemma 3.3. *For all $0<\mu<1$ and $r>d$, the space $h^\mu(W^{1,r})\subset\mathcal{C}^0([0,T]\times\overline{D})$ is a Banach algebra. Moreover, there exists a constant C such that for all $u,v\in h^\mu(W^{1,r})$ the following inequality holds*

$$\|uv\|_{h^\mu(W^{1,r})}\le C\|u\|_{h^\mu(W^{1,r})}\|v\|_{h^\mu(W^{1,r})},$$

where $(uv)(x,t):=u(x,t)v(x,t)$ for all $(x,t)\in D\times[0,T]$.

Proof. Let $u,v\in h^\mu(W^{1,r})$. Let $0\le s<t\le T$, using the triangle inequality it follows,

$$\|u(t)v(t)-u(s)v(s)\|_{W^{1,r}}\le\|u(t)(v(t)-v(s))\|_{W^{1,r}}+\|(u(t)-u(s))v(s)\|_{W^{1,r}}.$$

Since $h^\mu\subset_> L^\infty$ we also have $h^\mu(E)\subset_> L^\infty(E)$ for all Banach space E. Moreover, since $W^{1,r}$ is a Banach algebra for $r>d$ (see [1]), we obtain

$$\begin{aligned}\|u(t)v(t)-u(s)v(s)\|_{W^{1,r}}\le &C(\|u\|_{h^\mu(W^{1,r})}\|v(t)-v(s)\|_{W^{1,r}}\\ &+\|u(t)-u(s)\|_{W^{1,r}}\|v\|_{h^\mu(W^{1,r})}),\end{aligned}\tag{3.16}$$

where C_1 is a constant independent of u and v. Thus, we find

$$\begin{aligned}\|u(t)v(t)-u(s)v(s)\|_{W^{1,r}}\le &C_2(\|u\|_{h^\mu(W^{1,r})}\|v\|_{h^\mu(W^{1,r})}\,|t-s\,|^\mu\\ &+\|u\|_{h^\mu(W^{1,r})}\,|t-s\,|^\mu\,\|v\|_{h^\mu(W^{1,r})}),\end{aligned}$$

where C_2 is a constant independent of u and v. Hence, $u \cdot v \in \mathcal{C}^0(W^{1,r})$ and

$$\|uv\|_{h^\mu(W^{1,r})} := \sup_{t\in[0,T]} \|u(t)v(t)\|_{W^{1,r}} + \sup_{\substack{t,s\in[0,T]\\ t\neq s}} \frac{\|u(t)v(t)-u(s)v(s)\|_{W^{1,r}}}{|t-s\,|^\mu}$$
$$\leq C\|u\|_{h^\mu(W^{1,r})}\|v\|_{h^\mu(W^{1,r})}.$$

Moreover, from (3.16) we also deduce

$$\lim_{\delta\to 0} \sup_{|t-s\,|<\delta} \frac{\|u(t)v(t)-u(s)v(s)\|_{W^{1,r}}}{|t-s\,|^\mu} = 0,$$

which ensures $u \cdot v \in h^\mu(W^{1,r})$ and ends the proof of the lemma. □

The same arguments can be used to prove

Corollary 3.4. *Let $u,\ v \in h^{1+\mu}(W^{1,r})$, $0<\mu<1$, $d<r<\infty$, then the product $u \cdot v$ belongs to $h^{1+\mu}(W^{1,r})$ and there exists a constant C such that*

$$\|uv\|_{h^{1+\mu}(W^{1,r})} \leq C\|u\|_{h^{1+\mu}(W^{1,r})}\|v\|_{h^{1+\mu}(W^{1,r})}.$$

Corollary 3.5. *Let $x_1 = (u^1,q^1)$, $x_2 = (u^2,q^2) \in X$, then*

$$b_d(x_1,x_2) := (\nabla u^1)q^2 \in W,$$
$$b_c(x_1,x_2) := P_r\nabla\cdot\mathbb{E}(q^1\otimes q^2) \in h^\mu(\mathcal{H}_r).$$

Moreover, the corresponding bilinear mappings $b_d : X\times X \to W$ and $b_c : X\times X \to h^\mu(\mathcal{H}_r)$ are continuous, that is, there exist two constants C_1, C_2 such that for all $x_1, x_2 \in X$ we have

$$\|b_d(x_1,x_2)\|_W \leq C_1\|x_1\|_X\|x_2\|_X,$$
$$\|b_c(x_1,x_2)\|_{h^\mu(\mathcal{H}_r)} \leq C_2\|x_1\|_X\|x_2\|_X.$$

Proof. Let (u^1,q^1), $(u^2,q^2) \in X$. Using Lemma 3.3 it follows for almost $\omega \in \Omega$

$$(\nabla u^1)q^2(\omega) \in h^\mu(W^{1,r})$$

and

$$\|(\nabla u^1)q^2(\omega)\|_{h^\mu(W^{1,r})} \leq C\|u^1\|_{h^\mu(W^{1,r})}\|q^2(\omega)\|_{h^\mu(W^{1,r})},$$

where C is a constant independent of (u^1,q^1) and (u^2,q^2). Hence,

$$\|b_d((u^1,q^1),(u^2,q^2))\|_W \leq C\|(u^1,q^1)\|_X\|(u^2,q^2)\|_X.$$

This ensures $b_d((u^1,q^1),(u^2,q^2)) \in L^\gamma(h^\mu(W^{1,r}))$. Similarly, for almost $\omega \in \Omega$ using Lemma 3.3 it follows that for almost all $w \in \Omega$

$$q^1(\omega)\otimes q^2(\omega) \in h^\mu(W^{1,r})$$

and there exists a constant C independent of (u^1,q^1) and (u^2,q^2) such that for almost all $\omega \in \Omega$

$$\|q^1(\omega)\otimes q^2(\omega)\|_{h^\mu(W^{1,r})} \leq C\|q^1(\omega)\|_{h^\mu(W^{1,r})}\|q^2(\omega)\|_{h^\mu(W^{1,r})}.$$

Using Bochner's Theorem (see chapter 5 of [38]) we have $\Omega \ni \omega \mapsto q^1(\omega)\otimes q^2(\omega) \in h^\mu(W^{1,r})$ is Bochner integrable and $b_c((u^1,q^1),(u^2,q^2)) \in h^\mu(\mathcal{H}_r)$. Moreover, the Cauchy-Schwarz inequality implies

$$\|b_c((u^1,q^1),(u^2,q^2))\|_{h^\mu(L^r)} \leq C\|u^1,q^1\|_X\|u^2,q^2\|_X.$$

□

Remark 3.6. The mappings $S_d : X \to W$ and $S_c : X \to h^\mu(W^{1,r})$ can be characterized for all $x \in X$ by $S_d(x) = b_d(x, x)$ and $S_c(x) = b_c(x, x)$. Thus, in virtue of Proposition 5.4.1 in [13], the mappings S_d and S_c are well defined and even analytic in their respective spaces.

Remark 3.7. Using similar arguments, we also have for $(u, q^D) \in X$

$$\int_0^\cdot (\nabla u(s)) q^D(s) ds \in L^\gamma(h^{1+\mu}(W^{1,r}) \cap h_0^\mu(W^{1,r})).$$

Lemma 3.8. *The mapping $F : Y \times X \to Z$ is well defined and analytic. Moreover, for $x = (u, q^D) \in X$ and $q^S \in L^\gamma(h^\mu(W^{1,r}))$ defined by (2.12) its Fréchet derivative in $(0, 0)$, $D_xF(0, 0)$ is given by*

$$D_xF(0,0)x = \begin{pmatrix} \dfrac{\partial q^D}{\partial t} + \dfrac{1}{2\lambda} q^D - (\nabla u) q^S \\ \rho \dfrac{\partial u}{\partial t} + \eta_s A_r u - \frac{\eta_p}{\lambda} P_r \nabla \cdot \mathbb{E}(q^D \otimes q^S + q^S \otimes q^D) \\ u(0) \end{pmatrix}$$

Proof. In order to study the property of the mapping $F : Y \times X \to Z$ we rewrite it as follows

$$F(y, x) = L_1 y + L_2 x - \begin{pmatrix} S_d(x) \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{\eta_p}{\lambda} P_r \nabla \cdot S_c(x) \\ 0 \end{pmatrix}, \tag{3.17}$$

where $L_1 : Y \to Z$, $L_2 : X \to Z$ are bounded linear operator defined for $y = (P_r f, u_0) \in Y$ and $x = (u, q^D) \in X$ by

$$L_1 y = \begin{pmatrix} 0 \\ -P_r f \\ u_0 \end{pmatrix}, \quad L_2 x = \begin{pmatrix} \dfrac{\partial q^D}{\partial t} + \dfrac{1}{2\lambda} q^D - (\nabla u) q^S \\ \rho \dfrac{\partial u}{\partial t} + \eta_s A_r u - \frac{\eta_p}{\lambda} P_r \nabla \cdot \mathbb{E}\left(q^D \otimes q^S + q^S \otimes q^D\right) \\ u(0) \end{pmatrix}$$

and $S_d : X \to W$, $S_c : X \to h^\mu(\mathcal{H}_r)$ are well defined and analytic (see Remark 3.6). Clearly, the first two terms in (3.17) are also analytic. Thus, $F : Y \times X \to Z$ is analytic.

Moreover for $x \in X$

$$D_xF(0,0)x = L_2 x,$$

which completes the proof. $\square$

In order to use the implicit function theorem, it remains to check that $D_xF(0, 0)$ is an isomorphism from X to Z. Therefore, we have to check that, for $w \in W$ and $(f, u_0) \in Y$ there exists a unique $(u, q^D) \in X$ such that

$$\begin{cases} \dfrac{\partial q^D}{\partial t} + \dfrac{1}{2\lambda} q^D - (\nabla u) q^S = w, \\ \rho \dfrac{\partial u}{\partial t} + \eta_s A_r u - \dfrac{\eta_p}{\lambda} P_r \nabla \cdot \mathbb{E}(q^D \otimes q^S + q^S \otimes q^D) = f, \\ u(0) = u_0. \end{cases} \tag{3.18}$$

Lemma 3.9. *Given $w \in W$, $(f, u_0) \in Y$ and q^S defined by (2.12), there exists a unique $(u, q^D) \in X$ solution of (3.18).*

Proof. Solving the first equation of the above system, it follows for $t \in [0,T]$ and for almost all $\omega \in \Omega$

$$q^D(t) = \int_0^t e^{-\frac{t-s}{2\lambda}} (\nabla u(s) q^S(s) + w(s))\, ds, \tag{3.19}$$

The aim is to use equation (3.19) in the third equation of (3.18) in order to obtain a relation for u. For $t \in [0,T]$ we have

$$\mathbb{E}(q^D(t) \otimes q^S(t)) = \int_0^t e^{-\frac{t-s}{2\lambda}} \left(\mathbb{E}(\nabla u(s) q^S(s) \otimes q^S(t)) + \mathbb{E}(w(s) \otimes q^S(s))\right) ds.$$

Using (3.8) we obtain for the first term in the right hand side of the above equation

$$\int_0^t e^{-\frac{t-s}{2\lambda}} \mathbb{E}(\nabla u(s) q^S(s) \otimes q^S(t)) ds = \int_0^t e^{-\frac{t-s}{\lambda}} \nabla u(s) ds.$$

Using same arguments for the term $\mathbb{E}(q^S(t) \otimes q^D(t))$, we obtain

$$\begin{aligned} \mathbb{E}\left(q^D(t) \otimes q^S(t) + q^S(t) \otimes q^D(t)\right) &= \int_0^t e^{-\frac{t-s}{\lambda}} (\nabla u(s) + (\nabla u(s))^T) ds \\ &+ \int_0^t e^{-\frac{t-s}{2\lambda}} \mathbb{E}(w(s) \otimes q^S(t) + q^S(t) \otimes w(s)) ds. \end{aligned} \tag{3.20}$$

Going back to (3.18) it follows that u satisfies

$$\rho \frac{\partial u}{\partial t} + \eta_s A_r u + k * A_r u = P_r f + P_r \nabla \cdot g, \qquad u(0) = u_0, \tag{3.21}$$

where $k \in \mathcal{C}^\infty([0,T])$ is defined for $t \in [0,T]$ by $k(t) := \frac{\eta_p}{\lambda} e^{-\frac{t}{\lambda}}$, $g \in h^\mu(W^{1,r})$ is defined for $t \in [0,T]$ by

$$g(t) := \frac{\eta_p}{\lambda} \int_0^t e^{-\frac{t-s}{2\lambda}} \mathbb{E}(w(s) \otimes q^S(t) + q^S(t) \otimes w(s)) ds,$$

and $k * A_r u$ denotes the convolution in time of the kernel k with $A_r u$. The right hand side of (3.21) is well defined using Corollary 3.5. Moreover we have $g(0) = 0$ and then since $(f, u_0) \in Y$ we obtain that the compatibility condition

$$-\eta_s A_r u_0 + P_r f(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}$$

is satisfied. It suffices now to apply Lemma 13 in Appendix A of [6], which ensures the existence and uniqueness of $u \in h^{1+\mu}(\mathcal{H}_r) \cap h^\mu(\mathcal{D}_{A_r})$. Going back to (3.19) with a given $u \in L^\gamma(h^\mu(W^{1,r}))$, using the regularity of w and Remark 3.7, we find there exists an unique $q^D \in L^\gamma(h^{1+\mu}(W^{1,r}) \cap h_0^\mu(W^{1,r}))$ adapted to the filtration (up to indistinguishability). □

We are now in position to prove next Lemma.

Lemma 3.10. *Let $d \geq 2$, let $D \subset \mathbb{R}^d$ be a bounded, connected open set with boundary of class $\mathcal{C}^2$ and let $T > 0$. Assume $0 < \mu < \frac{1}{2}$, $2 \leq \gamma < \infty$ and $d < r < \infty$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ a complete filtered probability space with $\mathcal{F}_t$ right continuous for all $t \in [0,T]$ and upon which the Brownian process $B \in L^\gamma(h^\mu)$ and the initial condition $q_0 \in L^\gamma$ are defined. Moreover we assume q_0 satisfies (2.1) and let $q^S \in L^\gamma(h^\mu(W^{1,r}))$ satisfying (2.12). Then there exists $\delta_0 > 0$ such that for every $f \in h^\mu(L^r)$, $u_0 \in \mathcal{D}_{A_r}$ satisfying*

$$-\eta_s A_r u_0 + P_r f(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}},$$

and

$$\|P_r f - P_r f(0)\|_{h^\mu(L^r)} + \|u_0\|_{W^{2,r}} + \| - \eta_s A_r + P_r f(0)\|_{\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}} \leq \delta_0,$$

there exists exactly one solution of (3.9)-(3.11). Moreover, the mapping

$$(P_r f, u_0) \mapsto (u(f,u_0), q^D(f,u_0))$$

is analytic.

Proof. We apply the implicit function theorem to (3.15). From Lemma 3.8, F is well defined and analytic, $F(0,0)=0$ and from Lemma 3.9 $D_x F(0,0)$ is an isomorphism from X to Z. Therefore, we can apply the implicit function theorem in the analytic case (see for instance Thm 4.5.4 chapter 4 p. 56 of [12]). Thus there exists $\delta_0 > 0$ and $\varphi : Y \to X$ analytic such that for all $y := (P_r f, u_0) \in Y$ with $\|y\|_Y < \delta_0$ we have $F(y, \varphi(y)) = 0$.

Let us now check the uniqueness. Assume $(u, q^D) \in X$ satisfying (3.9)-(3.11). Using a standard result on system of ordinary differential equation we obtain q^D satisfies for $t \in [0,T]$

$$q^D(t) = \Phi(t) \int_0^t \Phi(s)^{-1} (\nabla u(s)) q^S(s) ds, \tag{3.22}$$

where $\Phi : D \times [0,T] \to \mathbb{R}^{d\times d}$ is the fundamental matrix satisfying

$$\Phi(t,x) = I + \int_0^t \left(\nabla u(s) - \frac{I}{2\lambda} \right) \Phi(s,x) ds.$$

Using a fixed point theorem in $h^\mu(W^{1,r})$ (see [13]), we obtain $\Phi \in h^{1+\mu}([0,T]; W^{1,r}(D;\mathbb{R}^{d\times d}))$. Then by reversing time, it is also possible to show Φ^{-1} also belongs to $h^{1+\mu}([0,T]; W^{1,r}(D;\mathbb{R}^{d\times d}))$. Moreover, using representation (3.22), relation (3.8), Corollary 3.5, Remarks 3.7 and Corollary 3.4, it follows

$$\sigma_1 := \mathbb{E}(q^D \otimes q^S) = \Phi \int_0^{\cdot} e^{-\frac{t-s}{2\lambda}} \Phi^{-1}(s,.) \nabla u(s,.) ds \in h^{1+\mu}(W^{1,r}),$$

$$\sigma_2 := \mathbb{E}(q^S \otimes q^D) = \int_0^{\cdot} e^{-\frac{t-s}{2\lambda}} (\nabla u(s,.))^T (\Phi^{-1}(s,.))^T ds\ \Phi^T \in h^{1+\mu}(W^{1,r})$$

and

$$\begin{aligned} &\sigma_3 := \mathbb{E}(q^D \otimes q^D) \\ &= \Phi \int_0^{\cdot} \int_0^{\cdot} \Phi(s,.)^{-1} (\nabla u(s,.)) e^{-\frac{|s-k|}{2\lambda}} (\nabla u(k,.))^T (\Phi^{-1}(k,.))^T ds\, dk\ \Phi^T \in h^{1+\mu}(W^{1,r}), \end{aligned}$$

where $.^T$ denotes the transposition. Thus, it follows

$$\frac{\partial \sigma_1}{\partial t} = -\frac{1}{\lambda}\sigma_1 + (\nabla u)\sigma_1 + \nabla u,$$

$$\frac{\partial \sigma_2}{\partial t} = -\frac{1}{\lambda}\sigma_2 + \sigma_2(\nabla u)^T + (\nabla u)^T$$

and

$$\frac{\partial \sigma_3}{\partial t} = -\frac{1}{\lambda}\sigma_3 + (\nabla u)\sigma_3 + \sigma_3(\nabla u)^T + (\nabla u)\sigma_2 + \sigma_1(\nabla u)^T.$$

Finally, setting $\sigma = \frac{\eta_p}{\lambda}(\sigma_1 + \sigma_2 + \sigma_3) \in h^{1+\mu(W^{1,r})}$ we obtain

$$\frac{\lambda}{\eta_p}\left(\frac{\partial \sigma}{\partial t} - (\nabla u)\sigma - \sigma(\nabla u)^T \right) + \frac{1}{\eta_p}\sigma = \left(\nabla u + (\nabla u)^T \right),$$

which correspond to (2.8) with definition (2.9). Then, using Theorem 1 of [6], one obtains the existence of exactly one solution $u \in h^{1+\mu}(\mathcal{H}_r) \cap h^{\mu}(\mathcal{D}_{A_r})$. Going back to (3.22), we obtain the uniqueness (up to indistinguishability) of $q^D \in L^{\gamma}(h^{1+\mu}(W^{1,r}) \cap h_0^{\mu}(W^{1,r}))$ adapted to the filtration. □

Corollary 3.11. *Under the assumptions of the above Lemma, there exists a unique* $(u, q^D, p) \in X \times h^{\mu}(W^{1,r} \cap L_0^2)$*, satisfying*

$$\frac{\partial q^D}{\partial t} + \frac{1}{2\lambda} q^D - (\nabla u) q^D = (\nabla u) q^S \qquad \textit{in } D \times (0,T) \times \Omega, \tag{3.23}$$

$$\begin{aligned} &\rho \frac{\partial u}{\partial t} - 2\eta_s \nabla \cdot \epsilon(u) \\ &- \frac{\eta_p}{\lambda} \nabla \cdot \mathbb{E}((q^S + q^D) \otimes (q^S + q^D)) + \nabla p = f \end{aligned} \qquad \textit{in } D \times (0,T), \tag{3.24}$$

$$u(.,0) = u_0 \qquad \textit{in } D,, \tag{3.25}$$

with q^D *adapted to* $(\mathcal{F}_t)_{t \geq 0}$.

Let us go back to the proof of Theorem 3.1.

Proof. (of Theorem 3.1) Let $y := (P_r f, u_0) \in Y$. We proved in Lemma 3.10 that $x := (u(y), q^D(y))$ is unique in X but $q(y) := q^S + q^D(y)$ is only in $L^{\gamma}(h^{\mu}(W^{1,r}))$ because of the regularity imposed by q^S. Obviously, q is unique (up to indistinguishability) and the mapping $Y \ni y \mapsto x(y) \in X$ is analytic. □

Remark 3.12. In the proof of uniqueness (in Lemma 3.10), we proved that $V := \mathbb{E}(q \otimes q)$ satisfies (2.8) with $V \in h^{1+\mu}(W^{1,r})$. Setting $\sigma := \frac{\eta_p}{\lambda}(V - I)$, we have $\sigma \in h^{1+\mu}(W^{1,r})$ and satisfies (1.5). Theorem 1 in [6] ensures $(u, \sigma) \in h^{1+\mu}(\mathcal{H}_r) \cap h^{\mu}(\mathcal{D}_{A_r}) \times h^{1+\mu}(W^{1,r})$ coincides with the unique solution of the Oldroyd-B problem. This previous argument is based on the equivalence between the Oldroyd-B model and the Hookean dumbbells one. This fact being established, the existence and uniqueness could also be directly ensured by Theorem 3.2 and 3.3 of [6], but our approach is more general. Indeed, we only need that the linearized problem have a unique solution thus the original problem does not need to have a deterministic equivalent. In that case, existence still holds but uniqueness is only ensured by the implicit function theorem. This is, there exists a neighborhood $\mathcal{V} \times \mathcal{U} \subset Y \times X$ of $(u, q^D) = (0, 0)$ and an analytic mapping $\varphi : Y \to \mathcal{U}$ such that $(y, x) \in \mathcal{V} \times \mathcal{U}$ and $F(y, x) = 0$ is equivalent to $y \in \mathcal{V}$ and $x = \varphi(y)$.

Remark 3.13. Since we proved $\mathbb{E}(q \otimes q) \in h^{1+\mu}(W^{1,r})$, assuming $f \in h^{1+\mu}(L^r)$, some compatibility conditions and using the same arguments as in [6], it is possible to prove the existence of a solution of (2.2)-(2.7) satisfying $u \in h^{2+\mu}(\mathcal{H}_r) \cap h^{1+\mu}(\mathcal{D}_{A_r})$ and $q \in L^{\gamma}(h^{\mu}(W^{1,r}))$ (q still remains in $L^{\gamma}(h^{\mu}(W^{1,r}))$ because of the Brownian motion).

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