NUMERICAL ANALYSIS OF THE LDG METHOD FOR LARGE DEFORMATIONS OF PRESTRAINED PLATES

ANDREA BONITO, DIANE GUIGNARD, RICARDO H. NOCHETTO, AND SHUO YANG

Abstract. A local discontinuous Galerkin (LDG) method for approximating large deformations of prestrained plates is introduced and tested on several insightful numerical examples in [8]. This paper presents a numerical analysis of this LDG method, focusing on the *free boundary* case. The problem consists of minimizing a fourth order bending energy subject to a nonlinear and nonconvex metric constraint. The energy is discretized using LDG and a discrete gradient flow is used for computing discrete minimizers. We first show Γ-convergence of the discrete energy to the continuous one. Then we prove that the discrete gradient flow decreases the energy at each step and computes discrete minimizers with control of the metric constraint defect. We also present a numerical scheme for initialization of the gradient flow, and discuss the conditional stability of it.

Keywords: Prestrained materials; metric constraint; local discontinuous Galerkin; reconstructed Hessian; discrete gradient flow; free boundary conditions

1. Introduction

Prestrained materials can develop internal stresses at rest, deform out of plane even without an external force, and exhibit nontrivial 3d shapes. This is a rich area of research with numerous applications for instance in nematic glasses [24, 25], natural growth of soft tissues [21, 30] and manufactured polymer gels [22, 23, 29].

Starting from 3d hyperelasticity, a geometric nonlinear and dimensionally reduced energy for isotropic prestrained plates where bending was the chief mechanism for deformation was proposed in [17] and derived rigorously via Γ-convergence in [6]. The bending energy reads

$$E(\mathbf{y}) = \frac{\mu}{12}\int_\Omega \left| g^{-\frac12}\mathrm{I\!I}[\mathbf{y}]g^{-\frac12}\right|^2 + \frac{\lambda}{2\mu+\lambda}\mathrm{tr}\left(g^{-\frac12}\mathrm{I\!I}[\mathbf{y}]g^{-\frac12}\right)^2 \tag{1.1}$$

and is subject to the nonlinear and nonconvex metric constraint

$$\mathrm{I}[\mathbf{y}](\mathbf{x}) = g(\mathbf{x}) \quad \text{a.e. in } \Omega, \tag{1.2}$$

where $\mathbf{y}:\Omega\to\mathbb{R}^3$ is the deformation of the midplane $\Omega\subset\mathbb{R}^2$, $g:\Omega\to\mathbb{R}^{2\times2}$ is a given symmetric positive definite matrix, and λ and μ are Lamé parameters of the material.

Date: December 8, 2021.

(Andrea Bonito) Department of Mathematics, Texas A&M University, College Station, TX 77843, USA. Email: bonito@tamu.edu.

(Diane Guignard) Department of Mathematics and Statistics, University of Ottawa, Ottawa, ON K1N 6N5, Canada. Email: dguignar@uottawa.ca.

(Ricardo H. Nochetto) Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742, USA. Email: rhn@umd.edu.

(Shuo Yang) Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Beijing 101408, China, and Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China. Email: shuoyang@bimsa.cn.

Hereafter, $\mathrm{I}[\mathbf{y}]$ and $\mathrm{II}[\mathbf{y}]$ denote respectively the first and second fundamental forms of the deformed plate $\mathbf{y}(\Omega)$, namely

$$\mathrm{I}[\mathbf{y}] := \nabla\mathbf{y}^T\nabla\mathbf{y} \quad \text{and} \quad \mathrm{II}[\mathbf{y}] := -\nabla\boldsymbol{\nu}^T\nabla\mathbf{y} = (\partial_{ij}\mathbf{y}\cdot\boldsymbol{\nu})_{i,j=1}^2, \tag{1.3}$$

where $\boldsymbol{\nu} := \frac{\partial_1\mathbf{y}\times\partial_2\mathbf{y}}{|\partial_1\mathbf{y}\times\partial_2\mathbf{y}|}$ is the unit normal vector to the surface $\mathbf{y}(\Omega)$. Moreover, $|\cdot|$ stands for the Frobenius norm. Given an immersible metric g, our goal is to construct a deformation $\mathbf{y}$ that minimizes (1.1) subject to (1.2).

In [8], we depart from the 3d elastic energy of prestrained plates based on the Saint-Venant Kirchhoff energy density for classical isotropic materials and derive formally the 2d energy (1.1) with a modified Kirchhoff-Love assumption. In the special case $g = I_2$ with I_2 the 2×2 identity matrix (i.e., when $\mathbf{y}$ is an isometry), thanks to the relations [2, 4, 11]

$$|\mathrm{II}[\mathbf{y}]| = |D^2\mathbf{y}| = |\Delta\mathbf{y}| = \mathrm{tr}(\mathrm{II}[\mathbf{y}]), \tag{1.4}$$

(1.1) and (1.2) reduce to the nonlinear Kirchhoff plate model: minimize the energy

$$E(\mathbf{y}) = \frac{\alpha}{2}\int_\Omega |D^2\mathbf{y}|^2, \qquad \alpha := \frac{\mu(\mu+\lambda)}{3(2\mu+\lambda)}, \tag{1.5}$$

subject to the isometry constraint $\nabla\mathbf{y}^T\nabla\mathbf{y} = I_2$ a.e. in Ω. A formal derivation of (1.5) can be traced back to Kirchhoff in 1850, and an ansatz-free rigorous derivation was carried out in the seminal work of Friesecke, James, and Müller [19] via Γ-convergence.

1.1. **Problem statement.** The presence of the highly nonlinear quantity $\mathrm{II}[\mathbf{y}]$ in the energy (1.1) is an obstacle to the design of efficient numerical algorithms. Since for a general g the relation (1.4) does not hold, the energy given in (1.1) cannot be reduced to (1.5). However, thanks to Proposition A.1 given in the appendix, we get an equivalent formulation by replacing the second fundamental form $\mathrm{II}[\mathbf{y}]$ in (1.1) by the Hessian $D^2\mathbf{y}$ of the deformation.

We intend to study the approximation of the following constrained minimization problem

$$\min_{\mathbf{y}\in\mathbb{A}} E(\mathbf{y}), \tag{1.6}$$

where

$$E(\mathbf{y}) := \frac{\mu}{12}\sum_{m=1}^3\int_\Omega \left|g^{-\frac12}D^2y_m g^{-\frac12}\right|^2 + \frac{\lambda}{2\mu+\lambda}\mathrm{tr}\left(g^{-\frac12}D^2y_m g^{-\frac12}\right)^2 \tag{1.7}$$

with $\mathbf{y} = (y_m)_{m=1}^3$, and where the set of *admissible* functions is

$$\mathbb{A} := \left\{\mathbf{y}\in[H^2(\Omega)]^3 : \nabla\mathbf{y}^T\nabla\mathbf{y} = g \quad \text{a.e. in } \Omega\right\}. \tag{1.8}$$

Throughout this work, we assume that g is *immersible* in $\mathbb{R}^3$, i.e., the admissible set $\mathbb{A}$ is not empty. In addition, we assume that $g\in[H^1(\Omega)\cap L^\infty(\Omega)]^{2\times 2}$.

1.2. **Numerical methods.** In the case $g = I_2$, (1.5) is discretized with Kirchhoff finite elements in [2] and symmetric interior penalty discontinuous Galerkin (SIPG) methods in [11]. For bilayer plates with isometry constraint, discretizations relying on Kirchhoff finite elements and on SIPG methods are proposed in [4, 3] and [10], respectively. In our previous computational work [8], we consider (1.5) with a general immersible $g \neq I_2$, introduce a *local discontinuous Galerkin* (LDG) approach in which the Hessian $D^2\mathbf{y}$ is replaced by a reconstructed Hessian $H_h(\mathbf{y}_h)$, and explore the performance of LDG computationally. The present manuscript provides a mathematical justification of several properties of the algorithms in [8], such as convergence, energy decrease and metric defect control.

For Kirchhoff finite elements, the discrete isometry constraint is imposed at the nodes of the mesh and Dirichlet boundary conditions are incorporated in the admissible set. They are based on polynomials with degree $k = 3$ and require the computation of a discrete gradient, which may complicate the implementation of the method.

In both the LDG and SIPG approaches, the pointwise metric constraint is relaxed by imposing it on average over the elements, and any prescribed boundary conditions are imposed weakly via the *Nitsche* approach, thereby allowing for more geometric flexibility. Furthermore, the method is well defined for polynomials with degree $k = 2$ which are implemented in most standard finite element libraries. For instance, we refer to the step-82 tutorial program [7] for an implementation of the reconstructed Hessian in the deal.ii [1] library.

Compared to SIPG, LDG is conceptually simpler in that it uses $H_h(\mathbf{y}_h)$ as a chief constituent of the method. Moreover, contrary to SIPG which requires sufficiently large stabilization parameters for stability, there is no such condition for LDG. Stability of LDG is ensured for any positive stabilization parameters, as proved in Theorem 3.1 below. We refer to [8, Section 3.1.1] for further comments on the comparison of SIPG and LDG.

The LDG method was originally proposed in [16]. Motivated by the lifting and discrete gradient operator introduced in [26, 27], the discrete Hessian

$$H_h(\mathbf{y}_h) := D_h^2\mathbf{y}_h - R_h([\nabla_h\mathbf{y}_h]) + B_h([\mathbf{y}_h]) \tag{1.9}$$

consists of three parts: the broken Hessian $D_h^2\mathbf{y}_h$, the lifting of the jumps of the broken gradient $[\nabla_h\mathbf{y}_h]$, and the lifting of the jumps $[\mathbf{y}_h]$ of $\mathbf{y}_h$ itself; a precise definition is given in (2.21) below. Lifting operators were initially introduced in [5] and further analyzed in [13, 14]. It is worth mentioning that similar discrete Hessians are used in [28] to study the convergence of dG for the bi-Laplacian and in [11] to prove the Γ-convergence for plates with isometry constraint. In the present work, $H_h(\mathbf{y}_h)$ is an integral part of the numerical method and not a mere theoretical device. A key property of $H_h(\mathbf{y}_h)$ is consistency with integration by parts which yields its weak convergence in $L^2(\Omega)$ (see Lemma 2.4 below).

1.3. **Discrete problem.** The LDG counterpart of the energy (1.7) reads

$$\begin{aligned} E_h(\mathbf{y}_h) :=& \frac{\mu}{12}\sum_{m=1}^{3}\int_\Omega \left|g^{-\frac12}\, H_h(y_{h,m})\, g^{-\frac12}\right|^2 \\ &+ \frac{\mu\lambda}{12(2\mu+\lambda)}\sum_{m=1}^{3}\int_\Omega \operatorname{tr}\left(g^{-\frac12}\, H_h(y_{h,m})\, g^{-\frac12}\right)^2 \\ &+ \frac{\gamma_1}{2}\|\mathrm{h}^{-\frac12}[\nabla_h\mathbf{y}_h]\|^2_{L^2(\Gamma_h^0)} + \frac{\gamma_0}{2}\|\mathrm{h}^{-\frac32}[\mathbf{y}_h]\|^2_{L^2(\Gamma_h^0)}, \end{aligned} \tag{1.10}$$

where $\mathbf{y}_h = (y_{h,m})_{m=1}^3 \in [\mathbb{V}_h^k]^3$ is the discrete deformation over a shape-regular mesh $\mathcal{T}_h$ of Ω (in the sense given in Subsection 2.1), $H_h(\mathbf{y}_h)$ is given in (1.9), $\gamma_0, \gamma_1 > 0$ are stabilization parameters, and Γ_h^0 is the skeleton of $\mathcal{T}_h$ defined in (2.1). Given a parameter $\varepsilon > 0$ so that $\varepsilon \to 0$ as $h \to 0$, the *discrete admissible set* is

$$\mathbb{A}_{h,\varepsilon}^k := \left\{\mathbf{y}_h \in [\mathbb{V}_h^k]^3 : \quad D_h(\mathbf{y}_h) \le \varepsilon\right\}, \tag{1.11}$$

where

$$D_h(\mathbf{y}_h) := \sum_{T\in\mathcal{T}_h}\left|\int_T \nabla\mathbf{y}_h^T\nabla\mathbf{y}_h - g\right| \tag{1.12}$$

is the *metric defect*, also called *prestrain defect* in what follows. Note that $\mathbb{A}_{h,\varepsilon}^k$ is nonconvex. Finally, the discrete counterpart of (1.6) reads

$$\min_{\mathbf{y}_h\in\mathbb{A}_{h,\varepsilon}^k} E_h(\mathbf{y}_h). \tag{1.13}$$

This is a nonconvex energy minimization due to (1.11) and is discussed in detail in Section 2.

1.4. **Contributions and outline.** In this article, we analyze the algorithms proposed in [8]. Following [2, 4, 11], we develop a Γ-convergence theory and show that (up to a subsequence) the discrete global minimizers of the discrete energy (1.13) converge to global minimizers of the continuous counterpart (1.6). We focus on the *free boundary* case (no Dirichlet boundary conditions imposed), which is not considered in previous numerical analysis works on large deformation of plates with metric constraint [2, 4, 11]. We then examine the discrete H^2-gradient flow with linearized metric constraint proposed in [8], and prove that the discrete energy decreases at each step while the metric defect is kept under control. Deformations in the *free boundary* case are defined up to rigid motions which requires the addition of an L^2 term in the gradient flow metric. Last but not least, we study the behavior of (a generalization of) the preprocessing algorithm proposed in [8] and designed to construct an initial deformation for the main gradient flow with a small prestrain defect. An error analysis is out of reach partly due to poor understanding of the nonconvex constraint (1.2) and lack of characterization of immersible metrics.

The rest of the article is organized as follows. In Section 2, we introduce the (*broken*) finite element spaces and prove preliminary key properties for discrete functions, such as Poincaré-Friedrichs type inequalities and a compactness result. The discrete Hessian operator H_h is discussed in Subsection 2.4 together with weak and strong convergence properties, bounds on the lifting operators, and an equivalence relation crucial to prove the coercivity of the discrete energy. In Section 3, we define the discrete problem briefly introduced in Subsection 1.3 above and investigate its properties. The proof of Γ-convergence of the discrete energy to the exact one is the content of Section 4. In Section 5, we recall the gradient flow scheme used in [8] to solve the discrete problem, prove its unconditional stability and show how the prestrain defect is controlled throughout the flow. The preprocessing algorithm is discussed in Section 6. The equivalence between the energy (1.1) and (1.7), where the second fundamental forms are replaced by Hessians is the subject of Appendix A. For completeness, we also discuss in Appendix C how the theory for *free boundary* conditions can be extended to settings where Dirichlet boundary conditions are imposed on a portion $\Gamma^D \neq \varnothing$ of the boundary $\partial\Omega$ and where the plate is subject to external forces.

The notations $A \lesssim B$ and $A \sim B$ used throughout stand for $A \leq CB$ and $cB \leq A \leq CB$, where c, C are constants independent of the discretization parameters h, ε and τ.

2. Discontinuous finite elements

2.1. **Subdivisions.** Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and consider a sequence $\{\mathcal{T}_h\}_{h>0}$ of shape-regular conforming partitions of Ω made of either triangles or quadrilaterals T of diameter $h_T \leq h$. Let $\mathcal{E}_h^0$ be the set of interior edges and Γ_h^0 be the interior skeleton of $\mathcal{T}_h$

$$\Gamma_h^0 := \{\mathbf{x} \in e \,:\, e \in \mathcal{E}_h^0\}. \tag{2.1}$$

For triangles, the family of meshes $\{\mathcal{T}_h\}_{h>0}$ is assumed to be regular in the sense of Ciarlet [15]: there exists a constant $\varrho > 0$ independent of h such that

$$\frac{h_T}{\rho_T} \leq \varrho \qquad \forall\, T \in \mathcal{T}_h, \tag{2.2}$$

where ρ_T denotes the diameter of the largest ball inscribed in T. For quadrilaterals, we assume that the elements are convex and that the subtriangles obtained by bisecting an element along each of its diagonals satisfy (2.2). Under these regularity conditions, there is an invertible affine (resp. bi-affine) mapping $F_T : \widehat{T} \to T$ that maps the reference unit triangle (resp. square) $\widehat{T}$ onto T and which satisfies

$$\|DF_T\|_{L^\infty(\hat{T})} \lesssim h_T, \qquad \|DF_T^{-1}\|_{L^\infty(T)} \lesssim h_T^{-1}, \tag{2.3}$$

where DF_T and DF_T^{-1} denote the Jacobian matrices of F_T and F_T^{-1}, respectively; see for instance [15, 20]. In order to simplify the notation, we use a mesh function h such that $h_T \lesssim \mathrm{h}|_T \lesssim h_T$ for all $T \in \mathcal{T}_h$ and $h_e \lesssim \mathrm{h}|_e \lesssim h_e$ for all $e \in \mathcal{E}_h$.

2.2. **Broken spaces.** Let $k \geq 0$ and $\mathbb{P}_k$ (resp. $\mathbb{Q}_k$) be the space of polynomials of total degree at most k (resp. of degree at most k in each variable). Each component of the deformation $\mathbf{y}$ is approximated by functions from the (*broken*) finite element space

$$\mathbb{V}_h^k := \left\{ v_h \in L^2(\Omega) : \; v_{h|T} \circ F_T \in \mathbb{P}_k \text{ (resp. } \mathbb{Q}_k) \quad \forall\, T \in \mathcal{T}_h \right\} \tag{2.4}$$

when $\mathcal{T}_h$ is made of triangles (resp. quadrilaterals). In view of the energy (1.10), we require from now on that $k \geq 2$. Throughout this work, functions with values in $\mathbb{R}^3$ are written with bold symbols and subindices indicate their components; for example, $y_{h,m} \in \mathbb{V}_h^k$, $m = 1, 2, 3$ are the components of $\mathbf{y}_h \in [\mathbb{V}_h^k]^3$. The broken gradient of a scalar function $v_h \in \mathbb{V}_h^k$ is given by $\nabla_h v_h$. We use a similar notation for other piecewise differential operators, for instance $D_h^2 v_h = \nabla_h \nabla_h v_h$ denotes the broken Hessian. For vector-valued functions these operators are computed component-wise.

We now introduce the jump and average operators. To this end, let $\mathbf{n}_e$ be a unit normal to $e \in \mathcal{E}_h^0$ (the orientation is chosen arbitrarily but is fixed once for all). For $v_h \in \mathbb{V}_h^k$ and $e \in \mathcal{E}_h^0$, let $v_h^{\pm}(\mathbf{x}) := \lim_{s \to 0^+} v_h(\mathbf{x} \pm s\mathbf{n}_e)$ for any $\mathbf{x} \in e$, and set

$$[v_h]|_e := v_h^- - v_h^+, \qquad \{v_h\}\,|_e := \frac{1}{2}(v_h^+ + v_h^-). \tag{2.5}$$

The jumps and averages of non-scalar functions are computed component-wise.

2.3. **Discrete Poincaré-Friedrichs type inequalities and compactness.** We introduce the mesh-dependent bilinear form $\langle \cdot, \cdot \rangle_{H_h^2(\Omega)}$ defined for any $v_h, w_h \in \mathbb{V}_h^k$ by

$$\begin{aligned} \langle v_h, w_h \rangle_{H_h^2(\Omega)} :=& (D_h^2 v_h, D_h^2 w_h)_{L^2(\Omega)} \\ &+ (\mathrm{h}^{-1}[\nabla_h v_h], [\nabla_h w_h])_{L^2(\Gamma_h^0)} + (\mathrm{h}^{-3}[v_h], [w_h])_{L^2(\Gamma_h^0)}, \end{aligned} \tag{2.6}$$

where Γ_h^0 is defined in (2.1). Hereafter, $(\cdot,\cdot)_{L^2(\varpi)}$ denotes the $L^2(\varpi) := L^2(\varpi; d\mathbf{x})$ inner product associate with the Lebesgue measure $d\mathbf{x}$ on $\mathbb{R}$ or $\mathbb{R}^2$ depending on whether ϖ is a measurable set of dimension 1 or 2. We also define

$$|v_h|_{H_h^2(\Omega)}^2 := \langle v_h, v_h \rangle_{H_h^2(\Omega)} \qquad \forall\, v_h \in \mathbb{V}_h^k. \tag{2.7}$$

Note the slight abuse of notation as $\langle \cdot, \cdot \rangle_{H_h^2(\Omega)}$ is not a scalar product; $|\cdot|_{H_h^2(\Omega)}$ is just a semi-norm. For vector-valued functions $\mathbf{v}_h, \mathbf{w}_h \in [\mathbb{V}_h^k]^3$, we define $\langle \mathbf{v}_h, \mathbf{w}_h \rangle_{H_h^2(\Omega)} := \sum_{m=1}^3 \langle v_{h,m}, w_{h,m} \rangle_{H_h^2(\Omega)}$ and similarly for $|\mathbf{v}_h|_{H_h^2(\Omega)}$.

In contrast to (2.6), we introduce the following scalar product and norm on $\mathbb{V}_h^k$

$$(v_h, w_h)_{H_h^2(\Omega)} := \langle v_h, w_h \rangle_{H_h^2(\Omega)} + (v_h, w_h)_{L^2(\Omega)}, \qquad \|v_h\|_{H_h^2(\Omega)}^2 := (v_h, v_h)_{H_h^2(\Omega)}. \tag{2.8}$$

They are critical to guarantee the unique solvability of the linear system arising in each step of the gradient flow algorithm of Section 5 and control of the metric defect (1.12).

To derive discrete Poincaré-Friedrichs inequalities, we rely on the smoothing interpolation operator $\Pi_h : \mathbb{E}(\mathcal{T}_h) := \prod_{T \in \mathcal{T}_h} H^1(T) \to \mathbb{V}_h^k \cap H^1(\Omega)$ defined by $\Pi_h := I_h \circ P_h$, where I_h is the Clément type interpolant proposed in [9] and P_h is an element-wise $L^2(T)$ projection onto the restriction $\mathbb{V}_h^k(T)$ of $\mathbb{V}_h^k$ to $T \in \mathcal{T}_h$. The domain $\mathbb{E}(\mathcal{T}_h)$ of Π_h is larger than $\mathbb{V}_h^k$ because the smoothing operator shall be employed on functions in $\nabla \mathbb{V}_h^k$. The latter are in general not in $\mathbb{V}_h^{k'}$ for any $k' \geq 0$ when $\mathbb{V}_h^k$ is based on quadrilateral elements; see (2.4).

Before embarking on the proof of the discrete Poincaré-Friedrichs inequalities, we record several properties of Π_h. For any $v \in \mathbb{E}(\mathcal{T}_h)$ we have

$$\|\Pi_h v\|_{L^2(\Omega)} \lesssim \|v\|_{L^2(\Omega)}, \tag{2.9}$$

$$\|\nabla \Pi_h v\|_{L^2(\Omega)} + \|\mathrm{h}^{-1}(v - \Pi_h v)\|_{L^2(\Omega)} \lesssim \|\nabla_h v\|_{L^2(\Omega)} + \|\mathrm{h}^{-\frac{1}{2}}[v]\|_{L^2(\Gamma_h^0)}, \tag{2.10}$$

and

$$\|\mathrm{h}^{-1}(\nabla_h v - \nabla \Pi_h v)\|_{L^2(\Omega)} \lesssim \|D_h^2 v\|_{L^2(\Omega)} + \|\mathrm{h}^{-\frac{1}{2}}[\nabla_h v]\|_{L^2(\Gamma_h^0)} + \|\mathrm{h}^{-\frac{3}{2}}[v]\|_{L^2(\Gamma_h^0)}. \tag{2.11}$$

Estimate (2.9) follows from the $L^2(\Omega)$ stability of I_h [9], estimate (2.10) is guaranteed by Lemma 2.1 in [11] and similar arguments can be used to derive (2.11).

We are now in position to derive the following discrete Poincaré-Friedrichs inequalities.

Lemma 2.1 (discrete Poincaré-Friedrichs inequalities). *For any $v \in \mathbb{E}(\mathcal{T}_h)$ there holds*

$$\Big\|v - \fint_\Omega v\Big\|_{L^2(\Omega)} \lesssim \|\nabla_h v\|_{L^2(\Omega)} + \|\mathrm{h}^{-\frac{1}{2}}[v]\|_{L^2(\Gamma_h^0)}, \tag{2.12}$$

where $\fint_\Omega$ stands for the average over Ω. Moreover, for any $v_h \in \mathbb{V}_h^k$ there holds

$$\|\nabla_h v_h\|_{L^2(\Omega)} + \|\nabla \Pi_h v_h\|_{L^2(\Omega)} \lesssim \|v_h\|_{L^2(\Omega)} + |v_h|_{H_h^2(\Omega)}. \tag{2.13}$$

Proof. We split the proof in several steps.

Step 1. Let $v \in \mathbb{E}(\mathcal{T}_h)$. The Cauchy-Schwarz inequality and a standard Poincaré-Friedrichs inequality for $\Pi_h v \in H^1(\Omega)$ yield

$$\begin{aligned} \Big\|v - \fint_\Omega v\Big\|_{L^2(\Omega)} &\leq \|v - \Pi_h v\|_{L^2(\Omega)} + \Big\|\Pi_h v - \fint_\Omega \Pi_h v\Big\|_{L^2(\Omega)} + \Big\| \fint_\Omega (v - \Pi_h v)\Big\|_{L^2(\Omega)} \\ &\lesssim \|v - \Pi_h v\|_{L^2(\Omega)} + \|\nabla \Pi_h v\|_{L^2(\Omega)}. \end{aligned}$$

Since $\mathrm{h}|_T \lesssim h_T \leq \operatorname{diam}(\Omega)$ for all $T \in \mathcal{T}_h$, the first estimate (2.12) then directly follows from (2.10) with a hidden constant that depends on Ω.

Step 2. We claim that for any $v_h \in \mathbb{V}_h^k$ there holds

$$\Big\|\nabla \Pi_h v_h - \fint_\Omega \nabla \Pi_h v_h\Big\|_{L^2(\Omega)} \lesssim \|D_h^2 v_h\|_{L^2(\Omega)} + \|\mathrm{h}^{-\frac{1}{2}}[\nabla_h v_h]\|_{L^2(\Gamma_h^0)} + \|\mathrm{h}^{-\frac{3}{2}}[v_h]\|_{L^2(\Gamma_h^0)}. \tag{2.14}$$

To see this, we first employ (2.12) on each component of $\nabla \Pi_h v_h \in [\mathbb{E}(\mathcal{T}_h)]^3$ to write

$$\Big\|\nabla \Pi_h v_h - \fint_\Omega \nabla \Pi_h v_h\Big\|_{L^2(\Omega)} \lesssim \|D_h^2 \Pi_h v_h\|_{L^2(\Omega)} + \|\mathrm{h}^{-\frac{1}{2}}[\nabla \Pi_h v_h]\|_{L^2(\Gamma_h^0)}.$$

Therefore, to obtain (2.14) it remains to show that

$$\|D_h^2 \Pi_h v_h\|_{L^2(\Omega)} + \|\mathrm{h}^{-\frac{1}{2}}[\nabla \Pi_h v_h]\|_{L^2(\Gamma_h^0)} \lesssim \|D_h^2 v_h\|_{L^2(\Omega)} + \|\mathrm{h}^{-\frac{1}{2}}[\nabla_h v_h]\|_{L^2(\Gamma_h^0)} + \|\mathrm{h}^{-\frac{3}{2}}[v_h]\|_{L^2(\Gamma_h^0)},$$

which can be deduced from standard scaling arguments and equivalence of norms on finite dimensional spaces. The details are omitted but we refer to the proof of Lemma 6.6 in [9] for additional information.

Step 3. We write $\nabla_h v_h = \nabla \Pi_h v_h + \nabla_h (v_h - \Pi_h v_h)$. On the one hand, we infer from (2.11) that

$$\|\nabla_h v_h - \nabla \Pi_h v_h\|_{L^2(\Omega)} \lesssim \|D_h^2 v_h\|_{L^2(\Omega)} + \|\mathrm{h}^{-\frac{1}{2}}[\nabla_h v_h]\|_{L^2(\Gamma_h^0)} + \|\mathrm{h}^{-\frac{3}{2}}[v_h]\|_{L^2(\Gamma_h^0)}.$$

On the other hand, (2.14) implies

$$\|\nabla\Pi_h v_h\|_{L^2(\Omega)} \lesssim \|D_h^2 v_h\|_{L^2(\Omega)} + \|\mathrm{h}^{-\frac{1}{2}}[\nabla_h v_h]\|_{L^2(\Gamma_h^0)} + \|\mathrm{h}^{-\frac{3}{2}}[v_h]\|_{L^2(\Gamma_h^0)} + \Big\| \fint_\Omega \nabla\Pi_h v_h \Big\|_{L^2(\Omega)},$$

whence, recalling the definition (2.7) of $|\cdot|_{H_h^2(\Omega)}$, we arrive at

$$\|\nabla_h v_h\|_{L^2(\Omega)} + \|\nabla\Pi_h v_h\|_{L^2(\Omega)} \lesssim |v_h|_{H_h^2(\Omega)} + \Big\| \fint_\Omega \nabla\Pi_h v_h \Big\|_{L^2(\Omega)}. \tag{2.15}$$

It remains to estimate $\| \fint_\Omega \nabla\Pi_h v_h\|_{L^2(\Omega)}$. If $\mathbf{n}_{\partial\Omega}$ denotes the outward unit normal vector to $\partial\Omega$, integrating by parts

$$\int_\Omega \nabla\Pi_h v_h = \int_{\partial\Omega} (\Pi_h v_h)\mathbf{n}_{\partial\Omega}$$

and combining Cauchy-Schwarz, trace and Young's inequalities, we obtain for any $\epsilon > 0$

$$\Big\| \fint_\Omega \nabla\Pi_h v_h \Big\|_{L^2(\Omega)} \lesssim \|\Pi_h v_h\|_{L^2(\partial\Omega)} \lesssim \epsilon\|\nabla\Pi_h v_h\|_{L^2(\Omega)} + \epsilon^{-1}\|\Pi_h v_h\|_{L^2(\Omega)}.$$

The desired estimate (2.13) follows from the $L^2(\Omega)$ stability (2.9) of Π_h and upon choosing ϵ sufficiently small so that the term $\epsilon\|\nabla\Pi_h v_h\|_{L^2(\Omega)}$ in the above estimate can be absorbed in the left-hand side of (2.15). This ends the proof. $\square$

We end this subsection with a compactness result for discrete balls

$$\big\{v_h \in \mathbb{V}_h^k \ : \ \|\nabla_h v_h\|_{L^2(\Omega)} + |v_h|_{H_h^2(\Omega)} \lesssim 1\big\}.$$

As we shall see, the discrete energy (1.10) provides control of the $|\cdot|_{H_h^2(\Omega)}$ semi-norm while a uniform bound for the broken $H^1(\Omega)$ semi-norm is guaranteed for functions in the discrete admissible set (1.11).

Lemma 2.2 (compactness)**.** *Assume that $\{v_h\}_{h>0} \subset \mathbb{V}_h^k$ is a sequence such that*

$$\|\nabla_h v_h\|_{L^2(\Omega)} + |v_h|_{H_h^2(\Omega)} \lesssim 1. \tag{2.16}$$

Then there exists $\overline{v} \in H^2(\Omega)$ with $\fint_\Omega \overline{v} = 0$ such that (up to a subsequence) $\overline{v}_h := v_h - \fint_\Omega v_h \to \overline{v}$ in $L^2(\Omega)$ and $\nabla_h\overline{v}_h \to \nabla\overline{v}$ in $[L^2(\Omega)]^3$ as $h \to 0$.

Proof. We let $c_h := \fint_\Omega v_h$ and invoke the Poincaré-Friedrichs inequality (2.12) to write

$$\|v_h - c_h\|_{L^2(\Omega)}^2 \lesssim \|\nabla_h v_h\|_{L^2(\Omega)}^2 + \|\mathrm{h}^{-\frac{1}{2}}[v_h]\|_{L^2(\Gamma_h^0)}^2 \lesssim 1.$$

This, together with the uniform boundedness assumption, implies

$$\|v_h - c_h\|_{L^2(\Omega)} + \|\nabla_h v_h\|_{L^2(\Omega)} + \|D_h^2 v_h\|_{L^2(\Omega)} \lesssim 1. \tag{2.17}$$

With this bound being established, the rest of the proof readily follows step 1 - step 3 of Proposition 5.1 in [11]; it is therefore only sketched here. The uniform bound (2.17) guarantees that $\overline{v}_h = v_h - c_h$ converges weakly (up to a subsequence) in $L^2(\Omega)$ to some $\overline{v}$. Setting $\overline{z}_h := \Pi_h v_h - \fint_\Omega \Pi_h v_h \in \mathbb{V}_h^k \cap H^1(\Omega)$, we invoke the Poincaré-Friedrichs inequality (2.12) coupled with the $H^1(\Omega)$ stability (2.10) of Π_h to deduce that $\overline{z}_h$ is uniformly bounded in $H^1(\Omega)$. As a consequence, $\overline{z}_h$ converges strongly (up to a subsequence) in $L^2(\Omega)$ to some $\overline{z} \in H^1(\Omega)$. To show that $\overline{v} = \overline{z}$, we note that $\|(v_h - c_h) - \overline{z}_h\|_{L^2(\Omega)} \to 0$ as $h \to 0$ because of the interpolation property (2.11), Poincaré-Friedrichs inequality (2.12) and the uniform boundedness (2.16); hence,

$$\|(v_h - c_h) - \overline{z}\|_{L^2(\Omega)} \le \|(v_h - c_h) - \overline{z}_h\|_{L^2(\Omega)} + \|\overline{z}_h - \overline{z}\|_{L^2(\Omega)} \to 0 \qquad \text{as } h \to 0.$$

The uniqueness of weak limits guarantees that $\overline{v} = \overline{z}$ and thus $v_h - c_h$ strongly converges (up to a subsequence) in $L^2(\Omega)$ to $\overline{v} \in H^1(\Omega)$. Repeating this argument for $\nabla_h\overline{v}_h$ yields that $\nabla_h\overline{v}_h$ converges strongly in $[L^2(\Omega)]^3$ (up to a subsequence) to $\nabla\overline{v}$ and $\overline{v} \in H^2(\Omega)$. $\square$

2.4. **Discrete Hessian.** The LDG approximation (1.10) of the elastic energy (1.7) relies on the discrete approximation $H_h(\mathbf{y}_h) \in \left[L^2(\Omega)\right]^{3\times2\times2}$ of the Hessian $D^2\mathbf{y}$ introduced in (1.9); we now give a precise definition. The convergence of E_h naturally depends on the convergence of the discrete Hessian towards $D^2\mathbf{y}$. The piecewise Hessian $D_h^2\mathbf{y}_h$ alone does not contain enough information and cannot be used as discrete approximation $H_h(\mathbf{y}_h)$. In fact, the jumps of $\mathbf{y}_h$ and $\nabla_h\mathbf{y}_h$ must be accounted for. This is the purpose of the lifting operators.

Let l_1, l_2 be two non-negative integers, and consider the *local lifting operators* $r_e : [L^2(e)]^2 \to [\mathbb{V}_h^{l_1}]^{2\times2}$ and $b_e : L^2(e) \to [\mathbb{V}_h^{l_2}]^{2\times2}$ defined for $e \in \mathcal{E}_h^0$ by

$$r_e(\boldsymbol{\phi}) \in [\mathbb{V}_h^{l_1}]^{2\times2} : \quad \int_\Omega r_e(\boldsymbol{\phi}) : \tau_h = \int_e \{\tau_h\}\,\mathbf{n}_e \cdot \boldsymbol{\phi} \qquad \forall\, \tau_h \in [\mathbb{V}_h^{l_1}]^{2\times2}, \tag{2.18}$$

$$b_e(\phi) \in [\mathbb{V}_h^{l_2}]^{2\times2} : \quad \int_\Omega b_e(\phi) : \tau_h = \int_e \{\operatorname{div}\tau_h\} \cdot \mathbf{n}_e \phi \qquad \forall\, \tau_h \in [\mathbb{V}_h^{l_2}]^{2\times2}; \tag{2.19}$$

note that $\operatorname{supp}(r_e(\boldsymbol{\phi})) = \operatorname{supp}(b_e(\phi)) = \omega_e$, the union of the two elements sharing e. These lifting operators are extended to $[L^2(e)]^{3\times2} = \left[[L^2(e)]^2\right]^3$ and $[L^2(e)]^3$, respectively, by component-wise applications. The *global lifting operators* are then given by

$$R_h := \sum_{e\in\mathcal{E}_h^0} r_e : [L^2(\Gamma_h^0)]^2 \to [\mathbb{V}_h^{l_1}]^{2\times2}, \qquad B_h := \sum_{e\in\mathcal{E}_h^0} b_e : L^2(\Gamma_h^0) \to [\mathbb{V}_h^{l_2}]^{2\times2}. \tag{2.20}$$

It is worth mentioning that this construction is simpler than the one in [11] for quadrilaterals, which had to be defined on $D_h^2\mathbb{V}_h$ for the method to match the interior penalty discretization. As a consequence, the weak convergence of the discrete Hessian considered in [11] towards its corresponding exact Hessian requires a restrictive assumption on the sequence of subdivisions (see Proposition 4.3 in [11]). This restriction is not needed in Lemma 2.4 below.

The following estimates for R_h and B_h can be found e.g. in [14, 11].

Lemma 2.3 (stability of lifting operators)**.** *For any $v_h \in \mathbb{V}_h^k$ and for any $l_1, l_2 \geq 0$ we have*

$$\|R_h([\nabla_h v_h])\|_{L^2(\Omega)} \lesssim \|\mathrm{h}^{-\frac{1}{2}}[\nabla_h v_h]\|_{L^2(\Gamma_h^0)}, \qquad \|B_h([v_h])\|_{L^2(\Omega)} \lesssim \|\mathrm{h}^{-\frac{3}{2}}[v_h]\|_{L^2(\Gamma_h^0)}.$$

As anticipated in (1.9), the discrete Hessian operator $H_h : \mathbb{V}_h^k \to \left[L^2(\Omega)\right]^{2\times2}$ is defined as

$$H_h(v_h) := D_h^2 v_h - R_h([\nabla_h v_h]) + B_h([v_h]). \tag{2.21}$$

The definition (2.20) together with integration by parts of $D_h^2 v_h$ yields weak convergence of $H_h(v_h)$. It also gives strong convergence of $H_h(v_h)$ provided v_h is the Lagrange interpolant of a given $v \in H^2(\Omega)$. These results are stated in Lemmas 2.4 and 2.5 below, whose proofs are postponed to Appendix B. Such results are rather standard for the discrete gradient operator in the LDG context [27] and extend to the discrete Hessian [28, 11].

Lemma 2.4 (weak convergence of H_h)**.** *Let $\{v_h\}_{h>0} \subset \mathbb{V}_h^k$ be such that $|v_h|_{H_h^2(\Omega)} \lesssim 1$ uniformly in h and $v_h \to v$ in $L^2(\Omega)$ as $h \to 0$ for some $v \in H^2(\Omega)$. Then for any polynomial degrees $l_1, l_2 \geq 0$ of R_h and B_h, we have*

$$H_h(v_h) \rightharpoonup D^2 v \quad \text{in } \left[L^2(\Omega)\right]^{2\times2} \qquad \text{as } h \to 0. \tag{2.22}$$

Lemma 2.5 (strong convergence of H_h)**.** *Let $v \in H^2(\Omega)$ and let $v_h := \mathcal{I}_h^k v \in \mathbb{V}_h^k \cap H^1(\Omega)$ be the Lagrange interpolant of v. Then for any polynomial degrees $l_1, l_2 \geq 0$ of R_h and B_h, we have the following strong convergences in $[L^2(\Omega)]^{2\times2}$*

$$D_h^2 v_h \to D^2 v, \qquad R_h([\nabla_h v_h]) \to 0, \qquad B_h([v_h]) \to 0 \qquad \text{as } h \to 0.$$

In particular,

$$H_h(v_h) \to D^2 v \quad \textit{in} \;\; [L^2(\Omega)]^{2\times 2} \qquad \textit{as } h \to 0. \tag{2.23}$$

We end this subsection by showing that the quantity $\|H_h(\cdot)\|_{L^2(\Omega)} + \|\mathrm{h}^{-\frac{1}{2}}[\nabla_h \cdot]\|_{L^2(\Gamma_h^0)} + \|\mathrm{h}^{-\frac{3}{2}}[\cdot]\|_{L^2(\Gamma_h^0)}$ is equivalent to the $|\cdot|_{H_h^2(\Omega)}$ semi-norm. The definition (2.21) of $H_h(v_h)$ and Lemma 2.3 (stability of lifting operators) readily imply

$$\int_\Omega |H_h(v_h)|^2 + \gamma_1 \sum_{e\in\mathcal{E}_h^0} \int_e \mathrm{h}^{-1}|[\nabla_h v_h]|^2 + \gamma_0 \sum_{e\in\mathcal{E}_h^0} \int_e \mathrm{h}^{-3}|[v_h]|^2 \lesssim |v_h|^2_{H_h^2(\Omega)}. \tag{2.24}$$

We now prove the converse and trickier inequality.

Lemma 2.6 (discrete H^2 semi-norm equivalence)**.** *For any stabilization parameters $\gamma_1, \gamma_0 > 0$ there exists a constant $C(\gamma_0,\gamma_1) > 0$ such that for any $v_h \in \mathbb{V}_h^k$ and any polynomial degrees $l_1, l_2 \geq 0$ there holds*

$$C(\gamma_0,\gamma_1)|v_h|^2_{H_h^2(\Omega)} \leq \int_\Omega |H_h(v_h)|^2 + \gamma_1 \sum_{e\in\mathcal{E}_h^0} \int_e \mathrm{h}^{-1}|[\nabla_h v_h]|^2 + \gamma_0 \sum_{e\in\mathcal{E}_h^0} \int_e \mathrm{h}^{-3}|[v_h]|^2. \tag{2.25}$$

Moreover, the constant $C(\gamma_0,\gamma_1)$ tends to 0 when γ_0 or γ_1 tends to 0.

Proof. We define

$$I_1 := \int_\Omega |H_h(v_h)|^2, \qquad I_2 := \gamma_1 \sum_{e\in\mathcal{E}_h^0} \int_e \mathrm{h}^{-1}|[\nabla_h v_h]|^2 + \gamma_0 \sum_{e\in\mathcal{E}_h^0} \int_e \mathrm{h}^{-3}|[v_h]|^2, \tag{2.26}$$

and prove a lower bound for I_1. The definition (2.21) of the discrete Hessian yields

$$\begin{aligned} I_1 &= \|D_h^2 v_h\|^2_{L^2(\Omega)} + \|B_h([v_h]) - R_h([\nabla_h v_h])\|^2_{L^2(\Omega)} + 2\int_\Omega D_h^2 v_h : \big(B_h([v_h]) - R_h([\nabla_h v_h])\big) \\ &\geq (1-\alpha^{-1})\|D_h^2 v_h\|^2_{L^2(\Omega)} + (1-\alpha)\|B_h([v_h]) - R_h([\nabla_h v_h])\|^2_{L^2(\Omega)}, \end{aligned}$$

where we used Young's inequality with $\alpha > 1$. Note that Lemma 2.3 (stability of the lifting operators) guarantees the existence of a constant C independent of h such that

$$\|B_h([v_h]) - R_h([\nabla_h v_h])\|^2_{L^2(\Omega)} \leq C\|\mathrm{h}^{-\frac{3}{2}}[v_h]\|^2_{L^2(\Gamma_h^0)} + C\|\mathrm{h}^{-\frac{1}{2}}[\nabla_h v_h]\|^2_{L^2(\Gamma_h^0)}.$$

Returning to the estimate for I_1, we thus arrive at

$$I_1 \geq (1-\alpha^{-1})\|D_h^2 v_h\|^2_{L^2(\Omega)} + (1-\alpha)C\|\mathrm{h}^{-\frac{3}{2}}[v_h]\|^2_{L^2(\Gamma_h^0)} + (1-\alpha)C\|\mathrm{h}^{-\frac{1}{2}}[\nabla_h v_h]\|^2_{L^2(\Gamma_h^0)}.$$

Combining this with I_2, we deduce that

$$I_1 + I_2 \geq \min\Big\{1-\alpha^{-1}, (1-\alpha)C + \gamma_0, (1-\alpha)C + \gamma_1\Big\}|v_h|_{H_h^2(\Omega)}.$$

Therefore, recalling that $\gamma_0, \gamma_1 > 0$, the assertion (2.25) holds with

$$C(\gamma_0,\gamma_1) := \min\big\{1-\alpha^{-1}, (1-\alpha)C + \gamma_0, (1-\alpha)C + \gamma_1\big\} \tag{2.27}$$

upon choosing $1 < \alpha < 1 + \min(\gamma_0,\gamma_1)/C$. ◻

3. Discrete energy and discrete admissible set

We now deal with the discrete energy $E_h(\mathbf{y}_h)$ defined in (1.10). Compared to the exact energy (1.7), the Hessians $D^2 y_k$ are replaced by the discrete Hessians $H_h(y_{h,k})$ and stabilization terms with parameters $\gamma_0, \gamma_1 > 0$ are included. The latter are motivated by the following coercivity result, which holds for any parameters $\gamma_0, \gamma_1 > 0$. Note that they are not required to be large enough unlike for the interior penalty method [11].

Theorem 3.1 (coercivity of E_h)**.** *Let $\mathbf{y}_h \in [\mathbb{V}_h^k]^3$ and let $\gamma_0, \gamma_1 > 0$. There holds*

$$|\mathbf{y}_h|^2_{H^2_h(\Omega)} \lesssim E_h(\mathbf{y}_h). \tag{3.1}$$

The hidden constant in the above estimate depends only on μ, g, and the constant $C(\gamma_0, \gamma_1)$ that appears in (2.25)*, and tends to infinity as γ_0 or $\gamma_1 \to 0$.*

Proof. Let $R(H_h)$ denote the range of $H_h : [\mathbb{V}_h^k]^3 \to [L^2(\Omega)]^{3\times2\times2}$. Because the metric $g(\mathbf{x})$ is SPD for a.e. $\mathbf{x} \in \Omega$, the quantity $\left(\int_\Omega |g^{-\frac{1}{2}} \cdot g^{-\frac{1}{2}}|^2\right)^{\frac{1}{2}} : R(H_h) \to \mathbb{R}$ is a norm in the finite dimensional space $R(H_h)$ and is thus equivalent to $\| |\cdot| \|_{L^2(\Omega)}$, where $|\cdot|$ is the Frobenius norm. Hence, there exists a constant $C > 0$ depending only on g such that

$$C\frac{\mu}{12}\|H_h(\mathbf{y}_h)\|^2_{L^2(\Omega)} + \frac{\gamma_1}{2}\|\mathrm{h}^{-\frac{1}{2}}[\nabla_h \mathbf{y}_h]\|^2_{L^2(\Gamma_h^0)} + \frac{\gamma_0}{2}\|\mathrm{h}^{-\frac{3}{2}}[\mathbf{y}_h]\|^2_{L^2(\Gamma_h^0)} \le E_h(\mathbf{y}_h).$$

Lemma 2.6 (discrete H^2 semi-norm equivalence) implies the desired estimate. $\square$

We now discuss the approximation $\mathbb{A}^k_{h,\varepsilon}$ of the admissible set $\mathbb{A}$ defined in (1.11). The pointwise metric constraint $\nabla\mathbf{y}^T\nabla\mathbf{y} = g$ is too strong to be imposed on a polynomial space. This leads to the definitions (1.12) and (1.11) of the metric defect $D_h(\mathbf{y}_h)$ and the discrete admissible set $\mathbb{A}^k_{h,\varepsilon}$, namely

$$D_h(\mathbf{y}_h) = \sum_{T\in\mathcal{T}_h}\left|\int_T \nabla\mathbf{y}_h^T\nabla\mathbf{y}_h - g\right|, \qquad \mathbb{A}^k_{h,\varepsilon} = \left\{\mathbf{y}_h \in [\mathbb{V}_h^k]^3 \ : \ D_h(\mathbf{y}_h) \le \varepsilon\right\},$$

for a positive number ε. The discrete counterpart of (1.6) finally reads $\min_{\mathbf{y}_h\in\mathbb{A}^k_{h,\varepsilon}} E_h(\mathbf{y}_h)$.

Recall that by assumption, g is immersible and so $\mathbb{A} \neq \emptyset$. The following lemma guarantees that $\mathbb{A}^k_{h,\varepsilon}$ is not empty provided that ε is sufficiently large.

Lemma 3.1 ($\mathbb{A}^k_{h,\varepsilon}$ is non-empty)**.** *Let $\mathbf{y} \in \mathbb{A}$ and let $\mathbf{y}_h := \mathcal{I}_h^k\mathbf{y} \in [\mathbb{V}_h^k]^3 \cap [H^1(\Omega)]^3$ be the Lagrange interpolant of $\mathbf{y}$. Then there exists a constant $C > 0$ depending only on the shape regularity of $\{\mathcal{T}_h\}_{h>0}$ and Ω such that*

$$D_h(\mathbf{y}_h) \le Ch\|\mathbf{y}\|^2_{H^2(\Omega)}.$$

In particular, $\mathbf{y}_h \in \mathbb{A}^k_{h,\varepsilon}$ provided $\varepsilon \ge Ch\|\mathbf{y}\|^2_{H^2(\Omega)}$.

Proof. We proceed as in Step 2 of Proposition 5.3 in [11] and compute

$$\left(\nabla_h\mathbf{y}_h^T\nabla_h\mathbf{y}_h - g\right) - \left(\nabla\mathbf{y}^T\nabla\mathbf{y} - g\right) = \nabla_h(\mathbf{y}_h - \mathbf{y})^T\nabla_h\mathbf{y}_h + \nabla\mathbf{y}^T\nabla_h(\mathbf{y}_h - \mathbf{y}). \tag{3.2}$$

Because $\mathbf{y} \in \mathbb{A}$, further algebraic manipulation yields

$$\nabla_h\mathbf{y}_h^T\nabla_h\mathbf{y}_h - g = \nabla_h(\mathbf{y}_h - \mathbf{y})^T\nabla\mathbf{y} + \nabla\mathbf{y}^T\nabla_h(\mathbf{y}_h - \mathbf{y}) + \nabla_h(\mathbf{y}_h - \mathbf{y})^T\nabla_h(\mathbf{y}_h - \mathbf{y})$$

whence, thanks to the interpolation estimate

$$\|\nabla_h(\mathbf{y} - \mathbf{y}_h)\|_{L^2(\Omega)} \lesssim h|\mathbf{y}|_{H^2(\Omega)},$$

we obtain

$$D_h(\mathbf{y}_h) \le \|\nabla_h \mathbf{y}_h^T \nabla_h \mathbf{y}_h - g\|_{L^1(\Omega)} \lesssim \left(\|\nabla \mathbf{y}\|_{L^2(\Omega)} + h|\mathbf{y}|_{H^2(\Omega)}\right) h|\mathbf{y}|_{H^2(\Omega)} \lesssim h\|\mathbf{y}\|_{H^2(\Omega)}^2,$$

which is the desired estimate. □

Lemma 2.2 (compactness) requires sequences uniformly bounded in $|\cdot|_{H_h^2(\Omega)}$ and in the $H^1(\Omega)$ broken semi-norm. The former stems from Theorem 3.1 (coercivity of E_h) for sequences with bounded energies. For the latter, we resort to the constraint encoded in the discrete admissible set $\mathbb{A}_{h,\varepsilon}^k$. This is the object of the next lemma.

Lemma 3.2 (gradient estimate)**.** *We have*

$$\|\nabla_h \mathbf{y}_h\|_{L^2(\Omega)}^2 \le \sqrt{2}(\varepsilon + \|g\|_{L^1(\Omega)}) \qquad \forall\, \mathbf{y}_h \in \mathbb{A}_{h,\varepsilon}^k. \tag{3.3}$$

Proof. It suffices to note that for any $T \in \mathcal{T}_h$

$$\frac{1}{2}\left(\int_T |\nabla \mathbf{y}_h|^2\right)^2 \le \sum_{i=1}^2 \left(\int_T |\partial_i \mathbf{y}_h|^2\right)^2 \le \sum_{i,j=1}^2 \left(\int_T \partial_i \mathbf{y}_h \cdot \partial_j \mathbf{y}_h\right)^2 = \left|\int_T \nabla \mathbf{y}_h^T \nabla \mathbf{y}_h\right|^2, \tag{3.4}$$

whence, taking advantage of the discrete constraint in $\mathbb{A}_{h,\varepsilon}^k$, we have

$$2^{-\frac{1}{2}}\|\nabla_h \mathbf{y}_h\|_{L^2(\Omega)}^2 \le \sum_{T\in\mathcal{T}_h}\left|\int_T \nabla \mathbf{y}_h^T \nabla \mathbf{y}_h\right| \le \sum_{T\in\mathcal{T}_h}\left|\int_T \nabla \mathbf{y}_h^T \nabla \mathbf{y}_h - g\right| + \sum_{T\in\mathcal{T}_h}\left|\int_T g\right| \le \varepsilon + \|g\|_{L^1(\Omega)}.$$

This ends the proof. □

In view of Lemma 3.1 ($\mathbb{A}_{h,\varepsilon}^k$ is non-empty), the existence of a solution to the minimization problem (1.13) follows from standard arguments. This is the object of the next proposition. Note that for this result, h and ε are fixed.

Proposition 3.1 (existence of discrete solutions)**.** *Let $h > 0$ and $\varepsilon > 0$ be such that $\mathbb{A}_{h,\varepsilon}^k \neq \varnothing$. Then there exists at least one solution to the minimization problem* (1.13)*.*

Proof. Let $0 \le m := \inf_{\mathbf{y}_h \in \mathbb{A}_{h,\varepsilon}^k} E_h(\mathbf{y}_h) < \infty$ and $\{\mathbf{y}_h^n\}_{n\ge1} \subset \mathbb{A}_{h,\varepsilon}^k$ be a minimizing sequence

$$\lim_{n\to\infty} E_h(\mathbf{y}_h^n) = m. \tag{3.5}$$

Because $E_h(\mathbf{y}_h^n + \mathbf{c}) = E_h(\mathbf{y}_h^n)$ and $D_h(\mathbf{y}_h^n + \mathbf{c}) = D_h(\mathbf{y}_h^n)$ for any constant vector $\mathbf{c} \in \mathbb{R}^3$, we can assume without loss of generality that $\int_\Omega \mathbf{y}_h^n = 0$. Combining estimate (2.12) of Lemma 2.1 (discrete Poincaré-Friedrichs inequalities), Lemma 3.2 (gradient estimate) and Theorem 3.1 (coercivity of E_h), we deduce that $\|\mathbf{y}_h^n\|_{L^2(\Omega)} \lesssim 1$. Because $[\mathbb{V}_h^k]^3$ is finite dimensional, we have that (up to a subsequence) $\{\mathbf{y}_h^n\}_{n\ge1}$ converges strongly in $[L^2(\Omega)]^3$ to some $\mathbf{y}_h^\infty \in [\mathbb{V}_h^k]^3$, and so in any norm. In turn, the continuity of the quadratic energy E_h and the prestrain defect D_h guarantee that

$$E_h(\mathbf{y}_h^\infty) = \lim_{n\to\infty} E_h(\mathbf{y}_h^n) = \inf_{\mathbf{y}_h \in \mathbb{A}_{h,\varepsilon}^k} E_h(\mathbf{y}_h)$$

and $\mathbf{y}_h^\infty \in \mathbb{A}_{h,\varepsilon}^k$. This proves that $\mathbf{y}_h^\infty$ is a solution to the minimization problem (1.13). □

4. Γ-convergence of E_h and convergence of global minimizers

The convergence of discrete global minimizers of problem (1.13) towards a global minimizer of (1.6) follows from the Γ-convergence of E_h towards E as $h \to 0$. The latter hinges on the so-called lim-inf and lim-sup properties and this section is devoted to the proof of these two properties. However, we start by stating the convergence of discrete global minimizers $\mathbf{y}_h$ satisfying $E_h(\mathbf{y}_h) \le \Lambda$ for a constant Λ independent of h; see Theorem 4.1 below. For the sake of brevity, the proof is omitted as it closely follows the one of [11, Theorem 5.1], where the deformations of single layer plates subject to an isometry constraint and Dirichlet boundary conditions are considered. We require that the prestrain parameter ε satisfies

$$\varepsilon \ge Ch\big(\|g\|_{L^1(\Omega)} + \Lambda\big), \tag{4.1}$$

where $C = C(g)$ is a constant only depending on the hidden constant in (2.12) and that in

$$|\mathbf{w}|^2_{H^2(\Omega)} \lesssim E(\mathbf{w}) \qquad \forall\, \mathbf{w} \in \mathbb{A}.$$

Note that the condition (4.1) on ε differs from the one that appears in [11, Theorem 5.1], which involves the boundary data but not the metric g.

Theorem 4.1 (convergence of global minimizers)**.** *Let $\{\mathbf{y}_h\}_{h>0} \subset \mathbb{A}^k_{h,\varepsilon}$ be a sequence of functions such that $E_h(\mathbf{y}_h) \le \Lambda$ for a constant Λ independent of h and let the prestrain defect parameter ε satisfy* (4.1)*. If $\mathbf{y}_h \in \mathbb{A}^k_{h,\varepsilon}$ is an almost global minimizer of E_h in the sense that*

$$E_h(\mathbf{y}_h) \le \inf_{\mathbf{w}_h \in \mathbb{A}^k_{h,\varepsilon}} E_h(\mathbf{w}_h) + \sigma,$$

where $\sigma, \varepsilon \to 0$ as $h \to 0$, then $\{\overline{\mathbf{y}}_h\}_{h>0}$ with $\overline{\mathbf{y}}_h := \mathbf{y}_h - \fint_\Omega \mathbf{y}_h$ is precompact in $[L^2(\Omega)]^3$ and every cluster point $\overline{\mathbf{y}}$ of $\overline{\mathbf{y}}_h$ belongs to $\mathbb{A}$ and is a global minimizer of E, namely $E(\bar{\mathbf{y}}) = \inf_{\mathbf{w}\in\mathbb{A}} E(\mathbf{w})$. Moreover, up to a subsequence (not relabeled), the energies converge

$$\lim_{h\to 0} E_h(\mathbf{y}_h) = E(\bar{\mathbf{y}}).$$

We now provide a proof of the lim-inf property.

Theorem 4.2 (lim-inf of E_h)**.** *Let the prestrain defect parameter $\varepsilon = \varepsilon(h) \to 0$ as $h \to 0$. Let $\{\mathbf{y}_h\}_{h>0} \subset \mathbb{A}^k_{h,\varepsilon}$ be a sequence of functions such that $E_h(\mathbf{y}_h) \lesssim 1$. Then there exists $\overline{\mathbf{y}} \in \mathbb{A}$ with $\fint_\Omega \overline{\mathbf{y}} = 0$ such that (up to a subsequence) the shifted sequence $\overline{\mathbf{y}}_h := \mathbf{y}_h - \fint_\Omega \mathbf{y}_h \in \mathbb{A}^k_{h,\varepsilon}$ satisfies $\overline{\mathbf{y}}_h \to \overline{\mathbf{y}}$, $\nabla_h \overline{\mathbf{y}}_h \to \nabla\overline{\mathbf{y}}$, $H_h(\overline{\mathbf{y}}_h) \rightharpoonup D^2\overline{\mathbf{y}}$ in $L^2(\Omega)$ as $h, \varepsilon \to 0$ and*

$$E(\overline{\mathbf{y}}) \le \liminf_{h\to 0} E_h(\overline{\mathbf{y}}_h).$$

Proof. We proceed in several steps.

Step 1: Accumulation point. Lemma 3.2 (gradient estimate), Theorem 3.1 (coercivity of E_h) and the uniform bound $E_h(\mathbf{y}_h) \lesssim 1$ guarantee that

$$\|\nabla_h \mathbf{y}_h\|^2_{L^2(\Omega)} + |\mathbf{y}_h|^2_{H^2_h(\Omega)} \lesssim \varepsilon + \|g\|_{L^1(\Omega)} + E_h(\mathbf{y}_h) \lesssim 1. \tag{4.2}$$

We can thus invoke Lemma 2.2 (compactness) with $v_h = y_{h,m}$ and deduce the existence of $\overline{\mathbf{y}} \in [H^2(\Omega)]^3$ with $\fint_\Omega \overline{\mathbf{y}} = 0$ such that $\overline{\mathbf{y}}_h \to \overline{\mathbf{y}}$ and $\nabla_h \overline{\mathbf{y}}_h \to \nabla\overline{\mathbf{y}}$ as $h \to 0$.

Step 2: Admissible deformation. We now show that $\overline{\mathbf{y}} \in \mathbb{A}$. Proceeding as in Step 4 of Proposition 5.1 in [11], which considers the case $g = I$, we have that

$$\|\nabla_h \overline{\mathbf{y}}_h^T \nabla_h \overline{\mathbf{y}}_h - g\|_{L^1(\Omega)} \lesssim h\big(\|D^2_h \overline{\mathbf{y}}_h\|_{L^2(\Omega)} \|\nabla_h \overline{\mathbf{y}}_h\|_{L^2(\Omega)} + \|\nabla g\|_{L^1(\Omega)}\big) + D_h(\overline{\mathbf{y}}_h) \lesssim h + \varepsilon, \tag{4.3}$$

where we used (4.2), the fact that $D_h^2\overline{\mathbf{y}}_h = D_h^2\mathbf{y}_h$, $\nabla_h\overline{\mathbf{y}}_h = \nabla_h\mathbf{y}_h$ and $D_h(\overline{\mathbf{y}}_h) = D_h(\mathbf{y}_h) \le \varepsilon$ for $\mathbf{y}_h \in \mathbb{A}_{h,\varepsilon}^k$. Hence, taking advantage of the relation (3.2) and the convergence $\nabla_h\overline{\mathbf{y}}_h \to \nabla\overline{\mathbf{y}}$ in $[L^2(\Omega)]^3$, we deduce that

$$\|\nabla\overline{\mathbf{y}}^T\nabla\overline{\mathbf{y}}-g\|_{L^1(\Omega)} \le \left(\|\nabla\overline{\mathbf{y}}\|_{L^2(\Omega)} + \|\nabla_h\overline{\mathbf{y}}_h\|_{L^2(\Omega)}\right)\|\nabla_h\overline{\mathbf{y}}_h-\nabla\overline{\mathbf{y}}\|_{L^2(\Omega)}+\|\nabla_h\overline{\mathbf{y}}_h^T\nabla_h\overline{\mathbf{y}}_h-g\|_{L^1(\Omega)} \to 0$$

as $h,\varepsilon \to 0$. This proves that $\nabla\overline{\mathbf{y}}^T\nabla\overline{\mathbf{y}} = g$ a.e. in Ω, and hence $\overline{\mathbf{y}} \in \mathbb{A}$.

Step 3: lim – inf *property.* Thanks to Lemma 2.4 (weak convergence of H_h) we have $H_h(\overline{y}_{h,m}) \rightharpoonup D^2\overline{y}_m$ as $h \to 0$ for $m = 1,2,3$, and so

$$g^{-\frac{1}{2}}H_h(\overline{y}_{h,m})g^{-\frac{1}{2}} \rightharpoonup g^{-\frac{1}{2}}D^2\overline{y}_m g^{-\frac{1}{2}} \qquad \text{as } h \to 0, \qquad m = 1,2,3. \tag{4.4}$$

Thus, the weak lower semi-continuity of the $L^2(\Omega)$ norm implies that

$$\int_\Omega |g^{-\frac{1}{2}}D^2\overline{y}_m g^{-\frac{1}{2}}|^2 \le \liminf_{h\to 0}\int_\Omega |g^{-\frac{1}{2}}H_h(\overline{y}_{h,m})g^{-\frac{1}{2}}|^2, \qquad m = 1,2,3.$$

A similar estimate for the trace terms in E_h and E can be derived. First note that $p : [L^2(\Omega)]^{2\times 2} \to \mathbb{R}$ defined by $p(F) := (\int_\Omega \mathrm{tr}(F)^2)^{\frac{1}{2}}$ is convex (it is a semi-norm) and satisfies

$$p(F_n) \to p(F) \qquad \text{when } F_n \to F \quad \text{strongly in } [L^2(\Omega)]^{2\times 2}. \tag{4.5}$$

In particular, p is lower semi-continuous with respect to the strong topology of $[L^2(\Omega)]^{2\times 2}$. Consequently it is also lower semi-continuous with respect to the weak topology and so

$$\int_\Omega \mathrm{tr}(g^{-\frac{1}{2}}D^2\overline{y}_m g^{-\frac{1}{2}})^2 \le \liminf_{h\to 0}\int_\Omega \mathrm{tr}(g^{-\frac{1}{2}}H_h(\overline{y}_{h,m})g^{-\frac{1}{2}})^2, \qquad m = 1,2,3,$$

follows from the weak convergence property (4.4).

It remains to use the fact that the remaining terms in E_h (namely the stabilization terms) are positive, to conclude that $E(\overline{\mathbf{y}}) \le \liminf_{h\to 0} E_h(\overline{\mathbf{y}}_h)$ as desired. □

We now discuss the lim-sup property. It turns out that in our context, we are able to construct a recovery sequence with strongly converging energies.

Theorem 4.3 (lim-sup of E_h)**.** *For any $\mathbf{y} \in \mathbb{A}$, there exists a recovery sequence $\{\mathbf{y}_h\}_{h>0} \subset \mathbb{A}_{h,\varepsilon}^k \cap [H^1(\Omega)]^3$ such that*

$$\mathbf{y}_h \to \mathbf{y} \quad \textit{in} \ \ [L^2(\Omega)]^3 \qquad \textit{as } h \to 0.$$

Moreover, for $\varepsilon \ge Ch\|y\|_{H^2(\Omega)}^2$, where C is the constant appearing in Lemma 3.1 ($\mathbb{A}_{h,\varepsilon}^k$ is non-empty), we have $E(\mathbf{y}) = \lim_{h\to 0} E_h(\mathbf{y}_h)$.

Proof. Consider the sequence $\mathbf{y}_h := \mathcal{I}_h^k\mathbf{y} \in [\mathbb{V}_h^k]^3 \cap [H^1(\Omega)]^3$ consisting of the Lagrange interpolants of $\mathbf{y}$ and note that, thanks to Lemma 3.1, $\mathbf{y}_h \in \mathbb{A}_{h,\varepsilon}^k$ since $\varepsilon \ge Ch\|\mathbf{y}\|_{H^2(\Omega)}^2$ by assumption. Moreover, the fact that $\mathbf{y}_h \to \mathbf{y}$ in $[L^2(\Omega)]^3$ as $h \to 0$ stems directly from interpolation estimates. Furthermore, Lemma 2.5 (strong convergence of H_h) applied to $v_h = y_{h,m}$, for $m = 1,2,3$, yields the convergence in norm

$$\lim_{h\to 0}\int_\Omega |g^{-\frac{1}{2}}H_h(y_{h,m})g^{-\frac{1}{2}}|^2 = \int_\Omega |g^{-\frac{1}{2}}D^2 y_m g^{-\frac{1}{2}}|^2, \qquad m = 1,2,3. \tag{4.6}$$

Similarly for the trace term, thanks to (4.5) and the strong convergence $g^{-\frac{1}{2}}H_h(y_{h,m})g^{-\frac{1}{2}} \to g^{-\frac{1}{2}}D^2 y_m g^{-\frac{1}{2}}$ in $[L^2(\Omega)]^{2\times 2}$ for $m = 1,2,3$ we have

$$\lim_{h\to 0}\int_\Omega \mathrm{tr}(g^{-\frac{1}{2}}H_h(y_{h,m})g^{-\frac{1}{2}})^2 = \int_\Omega \mathrm{tr}(g^{-\frac{1}{2}}D^2 y_m g^{-\frac{1}{2}})^2, \qquad m = 1,2,3.$$

Gathering these estimates together with the fact that the stabilization terms in E_h vanish in the limit $h \to 0$, according to Lemma 2.5, we have $E(\mathbf{y}) = \lim_{h\to 0} E_h(\mathbf{y}_h)$ as desired. □

5. Discrete gradient flow

We advocate a discrete H^2-gradient flow to determine local minimizers $\mathbf{y}_h \in \mathbb{A}^k_{h,\varepsilon}$ of $E_h(\mathbf{y}_h)$, which is driven by the Hilbert structure induced by the scalar product $(\mathbf{v}_h, \mathbf{w}_h)_{H^2_h(\Omega)}$ defined in (2.8). We now introduce this flow and discuss its crucial properties.

Given an initial guess $\mathbf{y}^0_h \in \mathbb{A}^k_{h,\varepsilon_0}$ and a pseudo time-step $\tau > 0$, we iteratively compute $\mathbf{y}^{n+1}_h := \mathbf{y}^n_h + \delta\mathbf{y}^{n+1}_h \in [\mathbb{V}^k_h]^3$ that minimizes the functional

$$\mathbf{y}_h \mapsto G_h(\mathbf{y}_h) := \frac{1}{2\tau}\|\mathbf{y}_h - \mathbf{y}^n_h\|^2_{H^2_h(\Omega)} + E_h(\mathbf{y}_h), \tag{5.1}$$

under the *linearized metric constraint* $\delta\mathbf{y}^{n+1}_h \in \mathcal{F}_h(\mathbf{y}^n_h)$ for the increment, where

$$\mathcal{F}_h(\mathbf{y}^n_h) := \left\{\mathbf{v}_h \in [\mathbb{V}^k_h]^3 : \ L_T(\mathbf{y}^n_h;\mathbf{v}_h) := \int_T \nabla\mathbf{v}^T_h \nabla\mathbf{y}^n_h + (\nabla\mathbf{y}^n_h)^T\nabla\mathbf{v}_h = 0 \quad \forall\, T \in \mathcal{T}_h\right\}. \tag{5.2}$$

We emphasize that (5.1) minimizes $E_h(\mathbf{y}_h)$ but penalizes the deviation of $\mathbf{y}^{n+1}_h$ from $\mathbf{y}^n_h$. In the formal limit $\tau \to 0$, the first variation of (5.1) becomes an ODE in $H^2_h(\Omega)$ against the variational derivative $\delta E_h(\mathbf{y}_h)$ of $E_h(\mathbf{y}_h)$. The first variation of (5.1) does indeed give the first optimality condition: $\delta\mathbf{y}^{n+1}_h \in \mathcal{F}_h(\mathbf{y}^n_h)$ satisfies the Euler-Lagrange system of equations

$$\tau^{-1}(\delta\mathbf{y}^{n+1}_h, \mathbf{v}_h)_{H^2_h(\Omega)} + a_h(\mathbf{y}^{n+1}_h, \mathbf{v}_h) = 0 \qquad \forall\, \mathbf{v}_h \in \mathcal{F}_h(\mathbf{y}^n_h), \tag{5.3}$$

where $a_h(\mathbf{y}_h, \mathbf{v}_h) = \delta E_h(\mathbf{y}_h)(\mathbf{v}_h)$ is given by

$$\begin{aligned} a_h(\mathbf{y}_h, \mathbf{v}_h) :=& \frac{\mu}{6}\sum_{m=1}^{3}\int_\Omega (g^{-\frac12}H_h(y_{h,m})g^{-\frac12}) : (g^{-\frac12}H_h(v_{h,m})g^{-\frac12}) \\ &+ \frac{\mu\lambda}{6(2\mu+\lambda)}\sum_{m=1}^{3}\int_\Omega \operatorname{tr}\left(g^{-\frac12}H_h(y_{h,m})g^{-\frac12}\right)\operatorname{tr}\left(g^{-\frac12}H_h(v_{h,m})g^{-\frac12}\right) \\ &+ \gamma_1\big(\mathrm{h}^{-1}[\nabla_h\mathbf{y}_h], [\nabla_h\mathbf{v}_h]\big)_{L^2(\Gamma^0_h)} + \gamma_0\big(\mathrm{h}^{-3}[\mathbf{y}_h], [\mathbf{v}_h]\big)_{L^2(\Gamma^0_h)}. \end{aligned} \tag{5.4}$$

We refer to [8] for an implementation of the method using Lagrange multipliers to enforce the constraint. For convenience in the analysis below, we rewrite (5.3) as

$$\tau^{-1}(\delta\mathbf{y}^{n+1}_h, \mathbf{v}_h)_{H^2_h(\Omega)} + a_h(\delta\mathbf{y}^{n+1}_h, \mathbf{v}_h) = -a_h(\mathbf{y}^n_h, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathcal{F}_h(\mathbf{y}^n_h). \tag{5.5}$$

Remark 5.1 (well-posedness). *Note that the bilinear form $a_h(\cdot,\cdot)$ has a nontrivial kernel in $\mathcal{F}_h(\mathbf{y}^n_h)$; it contains at least the constants. However, $\tau^{-1}(\cdot,\cdot)_{H^2_h(\Omega)} + a_h(\cdot,\cdot)$ is coercive and continuous on $\mathcal{F}_h(\mathbf{y}^n_h) \neq \emptyset$ thanks to the L^2 term in the H^2_h metric. Hence, the Lax-Milgram theory guarantees the existence and uniqueness of $\delta y^{n+1}_h \in \mathcal{F}_h(\mathbf{y}^n_h)$ satisfying* (5.5).

We now embark on the study of properties of the discrete gradient flow (5.1). We start with a simple observation: since $\mathbf{y}^{n+1}_h$ minimizes the functional G_h in (5.1), we deduce that $G_h(\mathbf{y}^{n+1}_h) \le G_h(\mathbf{y}^n_h)$ or equivalently $E_h(\mathbf{y}^{n+1}_h) + \frac{1}{2\tau}\|\delta\mathbf{y}^{n+1}_h\|^2_{H^2_h(\Omega)} \le E_h(\mathbf{y}^n_h)$. We first show an improved energy reduction at each step that hinges on the quadratic structure of G_h.

Proposition 5.1 (energy decay). *If $\delta\mathbf{y}^{n+1}_h \in \mathcal{F}_h(\mathbf{y}^n_h)$ solves* (5.5)*, then the next iterate $\mathbf{y}^{n+1}_h = \mathbf{y}^n_h + \delta\mathbf{y}^{n+1}_h$ satisfies*

$$E_h(\mathbf{y}^{n+1}_h) + \frac{1}{\tau}\|\delta\mathbf{y}^{n+1}_h\|^2_{H^2_h(\Omega)} \le E_h(\mathbf{y}^n_h). \tag{5.6}$$

Proof. It suffices to utilize the identity $a(a-b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2$ to write

$$
\begin{aligned}
a_h(\mathbf{y}_h^{n+1}, \delta\mathbf{y}_h^{n+1}) &= a_h(\mathbf{y}_h^{n+1}, \mathbf{y}_h^{n+1} - \mathbf{y}_h^n) \\
&= \frac{1}{2}a_h(\mathbf{y}_h^{n+1}, \mathbf{y}_h^{n+1}) - \frac{1}{2}a_h(\mathbf{y}_h^n, \mathbf{y}_h^n) + \frac{1}{2}a_h(\delta\mathbf{y}_h^{n+1}, \delta\mathbf{y}_h^{n+1}) \geq E_h(\mathbf{y}_h^{n+1}) - E_h(\mathbf{y}_h^n),
\end{aligned} \tag{5.7}
$$

and to replace the left-hand side by $-\frac{1}{\tau}\|\delta\mathbf{y}_h^{n+1}\|^2_{H_h^2(\Omega)}$ according to (5.5) for $\mathbf{v}_h = \delta\mathbf{y}_h^{n+1}$. □

Note that (5.6) gives a precise control of the energy decay provided $\delta\mathbf{y}_h^{n+1} \neq 0$. Moreover, upon summing (5.6) over $n = 0, 1, \ldots, N-1$ for $N \geq 1$, we get the estimate

$$
\frac{1}{\tau}\sum_{n=0}^{N-1} \|\delta\mathbf{y}_h^{n+1}\|^2_{H_h^2(\Omega)} + E_h(\mathbf{y}_h^N) \leq E_h(\mathbf{y}_h^0). \tag{5.8}
$$

The next result quantifies the prestrain defect of iterates obtained within the gradient flow starting from an initial deformation $\mathbf{y}_h^0 \in \mathbb{A}_{h,\varepsilon_0}^k$ for some ε_0.

Proposition 5.2 (control of metric defect)**.** *Let $\mathbf{y}_h^0 \in \mathbb{A}_{h,\varepsilon_0}^k$. Then, for any $N \geq 1$, the N^{th} iterate $\mathbf{y}_h^N$ of the gradient flow satisfies*

$$
D_h(\mathbf{y}_h^N) = \sum_{T\in\mathcal{T}_h} \left| \int_T (\nabla\mathbf{y}_h^N)^T \nabla\mathbf{y}_h^N - g \right| \leq \varepsilon_0 + c\tau E_h(\mathbf{y}_h^0), \tag{5.9}
$$

where $c > 0$ is the hidden constant in (2.13). *In particular, if $\mathbf{y}_h := \mathcal{I}_h^k \mathbf{y}^0$ is the Lagrange interpolant of some $\mathbf{y}^0 \in \mathbb{A}$, then*

$$
D_h(\mathbf{y}_h^N) \lesssim (h+\tau)\|\mathbf{y}^0\|^2_{H^2(\Omega)}. \tag{5.10}
$$

Proof. The argument follows verbatim that of [11, Lemma 3.4] and is therefore only sketched. We take advantage of the linearized metric constraint $L_T(\mathbf{y}_h^n; \delta\mathbf{y}_h^{n+1}) = 0$, encoded in (5.2) for $\mathbf{v}_h = \delta\mathbf{y}_h^{n+1}$, to obtain for all $n \geq 0$

$$
(\nabla_h\mathbf{y}_h^{n+1})^T \nabla_h\mathbf{y}_h^{n+1} - g = (\nabla_h\mathbf{y}_h^n)^T \nabla_h\mathbf{y}_h^n - g + (\nabla_h\delta\mathbf{y}_h^{n+1})^T \nabla_h\delta\mathbf{y}_h^{n+1}.
$$

Therefore, summing for $n = 0, ..., N-1$ and exploiting telescopic cancellation yield

$$
D_h(\mathbf{y}_h^N) = \sum_{T\in\mathcal{T}_h} \left| \int_T (\nabla\mathbf{y}_h^N)^T \nabla\mathbf{y}_h^N - g \right| \leq D_h(\mathbf{y}_h^0) + \sum_{n=0}^{N-1} \|\nabla_h\delta\mathbf{y}_h^{n+1}\|^2_{L^2(\Omega)}.
$$

Since $\mathbf{y}_h^0 \in \mathbb{A}_{h,\varepsilon_0}^k$, using (2.13) of Lemma 2.1 (discrete Poincaré-Friedrichs inequalities) implies

$$
D_h(\mathbf{y}_h^N) \leq \varepsilon_0 + c\sum_{n=0}^{N-1} \|\delta\mathbf{y}_h^{n+1}\|^2_{H_h^2(\Omega)}.
$$

To derive (5.9), we employ (5.8) along with $E_h(\cdot) \geq 0$ and realize that

$$
\sum_{n=0}^{N-1} \|\delta\mathbf{y}_h^{n+1}\|^2_{H_h^2(\Omega)} \leq \tau E_h(\mathbf{y}_h^0). \tag{5.11}
$$

On the other hand, if $\mathbf{y}^0 \in \mathbb{A}$, then (2.24) in conjunction with a trace inequality, and the local H^2-stability of the Lagrange interpolant imply

$$
E_h(\mathcal{I}_h^k\mathbf{y}^0) \lesssim \|\mathcal{I}_h^k\mathbf{y}^0\|^2_{H_h^2(\Omega)} \lesssim \|\mathbf{y}^0\|^2_{H^2(\Omega)}. \tag{5.12}
$$

This, together with Lemma 3.1 ($\mathbb{A}_{h,\varepsilon}^k$ is non-empty), gives the desired estimate (5.10). □

The control on the prestrain defect offered by Proposition 5.2 indicates that ε_0 should tend to zero as $h \to 0$. We discuss in Section 6 the delicate construction of initial deformations $\mathbf{y}_h^0 \in \mathbb{A}_{h,\varepsilon_0}^k$, so that $\varepsilon_0 \to 0$ as $h \to 0$.

We pointed out in Remark 5.1 that the bilinear form $a_h(\cdot,\cdot)$ has a nontrivial kernel in $\mathcal{F}_h(\mathbf{y}_h^n)$ containing the constant vectors and yet the variational problem (5.5) uniquely determines the increment $\delta\mathbf{y}_h^{n+1}$. This is reflected in Theorem 4.1 (convergence of global minimizers) where the sequence $\overline{\mathbf{y}}_h := \mathbf{y}_h - \fint_\Omega \mathbf{y}_h$, rather than $\mathbf{y}_h$, is precompact. We explore next that the gradient flow preserves deformation averages throughout the evolution.

Proposition 5.3 (evolution of averages). *Let $\mathbf{y}_h^0 \in [\mathbb{V}_h^k]^3$. Then all the iterates $\mathbf{y}_h^n$, $n \geq 1$, of the gradient flow* (5.3) *satisfy*

$$\int_\Omega \mathbf{y}_h^n = \int_\Omega \mathbf{y}_h^0. \tag{5.13}$$

Proof. It suffices to choose a constant test function $\mathbf{v}_h = \mathbf{c} \in \mathcal{F}_h(\mathbf{y}_h^n)$ in (5.3) to obtain

$$(\delta\mathbf{y}_h^{n+1}, \mathbf{c})_{L^2(\Omega)} = 0,$$

whence (5.13) follows immediately. □

In particular, Proposition 5.3 implies that $\fint_\Omega \mathbf{y}_h^n = \mathbf{0}$ for all the iterates if $\fint_\Omega \mathbf{y}_h^0 = \mathbf{0}$. The latter can easily be achieved by subtracting $\fint_\Omega \mathbf{y}_h^0$ from any initial guess $\mathbf{y}_h^0$ without affecting $E_h(\mathbf{y}_h^0)$ or $D_h(\mathbf{y}_h^0)$. In this case, the sequence $\{\mathbf{y}_h^N\}_{h>0}$ of outputs of the gradient flow (5.3) satisfies the assumption in Theorem 4.2 and is precompact without further shifting.

Energy decreasing gradient flow algorithms are generally not guaranteed to converge to global minimizers. We address this issue next upon showing that the gradient flow (5.3) reaches a local minimum $\mathbf{y}_h^\infty$ for E_h in the direction of the tangent plane $\mathcal{F}_h(\mathbf{y}_h^\infty)$. This requires, however, the following (possibly degenerate) *inf-sup condition*: for all $n \geq 0$, there exists a constant $\beta_h > 0$ independent of n but possibly depending on h such that

$$\inf_{\mu_h \in \Lambda_h} \sup_{\mathbf{v}_h \in [\mathbb{V}_h^k]^3} \frac{b_h(\mathbf{y}_h^n; \mathbf{v}_h, \mu_h)}{\|\mathbf{v}_h\|_{H_h^2(\Omega)} \|\mu_h\|_{L^2(\Omega)}} \geq \beta_h, \tag{5.14}$$

where $\Lambda_h := \{\mu_h \in [\mathbb{V}_h^0]^{2\times 2} : \mu_h^T = \mu_h\}$ is the set of Lagrange multipliers and the bilinear form $b_h(\mathbf{y}_h^n;\cdot,\cdot)$ is defined for any $(\mathbf{v}_h, \mu_h) \in [\mathbb{V}_h^k]^3 \times \Lambda_h$ by

$$b_h(\mathbf{y}_h^n; \mathbf{v}_h, \mu_h) := \sum_{T\in\mathcal{T}_h} \int_T \mu_h : \big(\nabla\mathbf{v}_h^T \nabla\mathbf{y}_h^n + (\nabla\mathbf{y}_h^n)^T \nabla\mathbf{v}_h\big).$$

The proof of (5.14) is an open problem, but experiments presented in [8] suggest its validity. We notice the mismatch between the function spaces $H_h^2(\Omega)$ and $L^2(\Omega)$, which are natural for E_h but not for b_h, and the fact that $\mathbf{y}_h^n$ is not known to belong to $[W_\infty^1(\Omega)]^3$ uniformly. We stress that an inf-sup condition similar to (5.14) is valid for an LDG scheme for bilayer plates provided the linearized metric constraint $L_T(\mathbf{y}_h, \mathbf{v}_h) = 0$ of (5.2) is enforced pointwise [12].

Proposition 5.4 (limit of gradient flow). *Fix $h > 0$ and assume that the inf-sup condition* (5.14) *holds . Let $\mathbf{y}_h^0 \in \mathbb{A}_{h,\varepsilon_0}$ be such that $E_h(\mathbf{y}_h^0) < \infty$ and let $\{\mathbf{y}_h^n\}_{n\geq 1} \subset \mathbb{A}_{h,\varepsilon}^k$ be the sequence produced by the discrete gradient flow* (5.3). *Then there exists $\mathbf{y}_h^\infty \in \mathbb{A}_{h,\varepsilon}^k$ such that (up to a subsequence) $\mathbf{y}_h^n \to \mathbf{y}_h^\infty$ as $n \to \infty$ and $\mathbf{y}_h^\infty$ is a local minimum for E_h in the direction $\mathcal{F}_h(\mathbf{y}_h^\infty)$, namely*

$$E_h(\mathbf{y}_h^\infty) \leq E_h(\mathbf{y}_h^\infty + \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathcal{F}_h(\mathbf{y}_h^\infty). \tag{5.15}$$

Proof. Thanks to the energy decay property (5.6) and the average conservation property (5.13), we have that $\sup_{n\geq 1} E_h(\mathbf{y}_h^n) \leq E_h(\mathbf{y}_h^0) < \infty$ and $\fint_\Omega \mathbf{y}_h^n = \fint_\Omega \mathbf{y}_h^0$ for all $n \geq 1$. Arguing as in the proof of Proposition 3.1 (existence of discrete solutions), we deduce that a subsequence (not relabeled) converges to some $\mathbf{y}_h^\infty \in \mathbb{A}_{h,\varepsilon}^k$ in any norm defined on $[\mathbb{V}_h^k]^3$.

It remains to prove (5.15). Since E_h is quadratic and convex, we infer that

$$E_h(\mathbf{y}_h^\infty + \mathbf{v}_h) \geq E_h(\mathbf{y}_h^\infty) + \delta E_h(\mathbf{y}_h^\infty)(\mathbf{v}_h).$$

Hence, to prove (5.15), it suffices to show that

$$a_h(\mathbf{y}_h^\infty, \mathbf{v}_h) = \delta E_h(\mathbf{y}_h^\infty)(\mathbf{v}_h) = 0 \qquad \forall\, \mathbf{v}_h \in \mathcal{F}_h(\mathbf{y}_h^\infty). \tag{5.16}$$

To this end, we take advantage of the inf-sup condition (5.14) to guarantee the existence of a unique $\lambda_h^{n+1} \in \Lambda_h$ such that $\delta\mathbf{y}_h^{n+1}$ in (5.3) satisfies

$$\tau^{-1}(\delta\mathbf{y}_h^{n+1}, \mathbf{v}_h)_{H_h^2(\Omega)} + a_h(\mathbf{y}_h^{n+1}, \mathbf{v}_h) + b_h(\mathbf{y}_h^n; \mathbf{v}_h, \lambda_h^{n+1}) = 0 \qquad \forall\, \mathbf{v}_h \in [\mathbb{V}_h^k]^3. \tag{5.17}$$

Now, from the estimate (5.11) on the increments $\delta\mathbf{y}_h^n$, we deduce that $\lim_{n\to\infty} \delta\mathbf{y}_h^n = 0$ and so taking the limit as $n \to \infty$ in (5.17) yields

$$\lim_{n\to\infty} b_h(\mathbf{y}_h^n; \mathbf{v}_h, \lambda_h^{n+1}) = -a_h(\mathbf{y}_h^\infty, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in [\mathbb{V}_h^k]^3. \tag{5.18}$$

This in conjunction with the inf-sup condition (5.14) yields

$$\sup_{n\geq 0} \|\lambda_h^{n+1}\|_{L^2(\Omega)} \leq \frac{1}{\beta_h} \sup_{n\geq 0} \sup_{\mathbf{v}_h \in [\mathbb{V}_h^k]^3} \frac{b_h(\mathbf{y}_h^n; \mathbf{v}_h, \lambda_h^{n+1})}{\|\mathbf{v}_h\|_{H_h^2(\Omega)}} < \infty.$$

This in turn implies the existence of $\lambda_h^\infty \in \Lambda_h$ such that (up to a subsequence) $\lambda_h^n \to \lambda_h^\infty$ in any norm defined on the finite dimensional space Λ_h. In particular, (5.18) becomes

$$\sum_{T\in\mathcal{T}_h} \lambda_h^\infty|_T : L_T(\mathbf{y}_h^\infty, \mathbf{v}_h) = \sum_{T\in\mathcal{T}_h} \int_T \lambda_h^\infty : (\nabla\mathbf{v}_h^T \nabla\mathbf{y}_h^\infty + (\nabla\mathbf{y}_h^\infty)^T \nabla\mathbf{v}_h) = -a_h(\mathbf{y}_h^\infty, \mathbf{v}_h)$$

for all $\mathbf{v}_h \in [\mathbb{V}_h^k]^3$. In particular, if $\mathbf{v}_h \in \mathcal{F}_h(\mathbf{y}_h^\infty)$, then $L_T(\mathbf{y}_h^\infty, \mathbf{v}_h) = 0$ for all $T \in \mathcal{T}_h$ according to (5.2). This implies (5.16) and ends the proof. $\square$

6. Preprocessing: initial data preparation

Propositions 5.1 (energy decay) and 5.2 (control of metric defect) guarantee that the gradient flow (5.5) constructs iterates $\mathbf{y}_h^n$ with decreasing energy $E_h(\mathbf{y}_h^n)$ (as long as the increments do not vanish) and with prestrain defect $D_h(\mathbf{y}_h^n)$ smaller than $\varepsilon = D_h(\mathbf{y}_h^0) + CE_h(\mathbf{y}_h^0)\tau$. Since $\tau = \mathcal{O}(h)$ in practice, we realize that the choice of the initial deformation $\mathbf{y}_h^0 \in [\mathbb{V}_h^k]^3$ dictates the size of ε, which must satisfy $\varepsilon \to 0$ as $h \to 0$ for Theorem 4.1 (convergence of global minimizers) to hold. The assumption $\mathbb{A} \neq \emptyset$ along with Lemma 3.1 ($\mathbb{A}_{h,\varepsilon}^k$ is non-empty) implies that such $\mathbf{y}_h^0$'s exist. Yet, their construction is a delicate issue, especially when $g \neq I_2$ as in the present study. This is the objective of this section.

Motivated by the numerical experiments presented in [8], we propose a *metric preprocessing* algorithm consisting of a discrete H^2-gradient flow for the preprocessing energy

$$E_h^p(\mathbf{y}_h) \coloneqq E_h^s(\mathbf{y}_h) + \sigma_h E_h^b(\mathbf{y}_h), \tag{6.1}$$

where

$$E_h^s(\mathbf{y}_h) \coloneqq \frac{1}{2}\int_\Omega \left|\nabla_h\mathbf{y}_h^T \nabla_h\mathbf{y}_h - g\right|^2 \tag{6.2}$$

is a (simplified) discrete stretching energy and

$$E_h^b(\mathbf{y}_h) \coloneqq \frac{1}{2}\left(\int_\Omega |g^{-\frac{1}{2}} H_h(\mathbf{y}_h) g^{-\frac{1}{2}}|^2 + \|\mathrm{h}^{-\frac{1}{2}}[\nabla_h\mathbf{y}_h]\|_{L^2(\Gamma_h^0)}^2 + \|\mathrm{h}^{-\frac{3}{2}}[\mathbf{y}_h]\|_{L^2(\Gamma_h^0)}^2\right) \tag{6.3}$$

denotes (part of) the discrete bending energy, and $\sigma_h \geq 0$ is a (small) parameter which may depend on h. Note that the stretching energy E_h^s controls the prestrain defect

$$D_h(\mathbf{y}_h) \leq \|(\nabla_h \mathbf{y}_h)^T \nabla_h \mathbf{y}_h - g\|_{L^1(\Omega)} \lesssim \|(\nabla_h \mathbf{y}_h)^T \nabla_h \mathbf{y}_h - g\|_{L^2(\Omega)} \approx E_h^s(\mathbf{y}_h)^{\frac{1}{2}} \tag{6.4}$$

for all $\mathbf{y}_h \in [\mathbb{V}_h^k]^3$, while the bending energy E_h^b controls the $H_h^2(\Omega)$ semi-norm in view of Lemma 2.6. As we shall see, for a pseudo time-step sufficiently small, the preprocessing gradient flow produces sequences of deformations $\{\mathbf{y}_h^n\}_{n\geq 0}$ so that $\{E_h^p(\mathbf{y}_h^n)\}_{n\geq 0}$ is decreasing as long as the increments do not vanish. Therefore, for any $\sigma_h \geq 0$ we have $D_h(\mathbf{y}_h^n) \lesssim E_h^p(\mathbf{y}_h^n)^{\frac{1}{2}}$ and if $\sigma_h > 0$ and the N-th iterate $\mathbf{y}_h^N$ satisfies $E_h^p(\mathbf{y}_h^N) \lesssim \sigma_h$, then both $D_h(\mathbf{y}_h^N) \lesssim \sigma_h^{\frac{1}{2}}$ and $E_h(\mathbf{y}_h^N) \lesssim E_h^b(\mathbf{y}_h^N) \lesssim 1$ for the (full) bending energy (1.10). The total energy E_h^p is inspired by the pre-asymptotic model reduction of [8] as well as the methodology of augmented Lagrangian [18]. We recall that σ_h scales like the square of the (three-dimensional) plate thickness and tends to 0 [8], which motivates the choice $\sigma_h \approx h^2$. In practice, however, the contribution of E_h^b to E_h^p makes negligible difference in computations, and the numerical experiments of [8] are done with $\sigma_h = 0$.

Furthermore, to cope with the non-quadratic nature of the stretching energy E_h^s, the gradient flow is linearized at the previous iterate and reads: Starting from any initial guess $\mathbf{y}_h^0 \in [\mathbb{V}_h^k]^3$, compute recursively $\mathbf{y}_h^{n+1} := \mathbf{y}_h^n + \delta\mathbf{y}_h^{n+1}$ where $\delta\mathbf{y}_h^{n+1} \in [\mathbb{V}_h^k]^3$ satisfies

$$\begin{aligned} \tau^{-1}(\delta\mathbf{y}_h^{n+1}, \mathbf{v}_h)_{H_h^2(\Omega)} + a_h^s(\mathbf{y}_h^n; \delta\mathbf{y}_h^{n+1}, \mathbf{v}_h) + \sigma_h a_h^b(\delta\mathbf{y}_h^{n+1}, \mathbf{v}_h) \\ = -a_h^s(\mathbf{y}_h^n; \mathbf{y}_h^n, \mathbf{v}_h) - \sigma_h a_h^b(\mathbf{y}_h^n, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in [\mathbb{V}_h^k]^3. \end{aligned} \tag{6.5}$$

Here, $(\cdot,\cdot)_{H_h^2(\Omega)}$ is defined in (2.8) and

$$a_h^s(\mathbf{y}_h^n; \mathbf{y}_h, \mathbf{v}_h) := \int_\Omega \left(\nabla_h \mathbf{v}_h^T \nabla_h \mathbf{y}_h + \nabla_h \mathbf{y}_h^T \nabla_h \mathbf{v}_h\right) : \left((\nabla_h \mathbf{y}_h^n)^T \nabla_h \mathbf{y}_h^n - g\right) \tag{6.6}$$

$$\begin{aligned} a_h^b(\mathbf{y}_h, \mathbf{v}_h) := &\int_\Omega \left(g^{-\frac{1}{2}} H_h(\mathbf{y}_h) g^{-\frac{1}{2}}\right) : \left(g^{-\frac{1}{2}} H_h(\mathbf{v}_h) g^{-\frac{1}{2}}\right) \\ &+ \left(\mathrm{h}^{-1}[\nabla_h \mathbf{y}_h], [\nabla_h \mathbf{v}_h]\right)_{L^2(\Gamma_h^0)} + \left(\mathrm{h}^{-3}[\mathbf{y}_h], [\mathbf{v}_h]\right)_{L^2(\Gamma_h^0)}, \end{aligned} \tag{6.7}$$

and we use τ to denote a pseudo time-step parameter. Note that to avoid an overload of symbols, we used the same notation as for the pseudo time-step in the main gradient flow. However, these two pseudo time-steps do not need to take the same value.

We start by showing that the variational problem (6.5) has a unique solution. In preparation, we note that the following discrete Sobolev inequality holds

$$\|v_h\|_{L^4(\Omega)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)} + \|\mathrm{h}^{-\frac{1}{2}}[v_h]\|_{L^2(\Gamma_h^0)} + \|v_h\|_{L^2(\Omega)} \qquad \forall\, v_h \in \mathbb{E}(\mathcal{T}_h). \tag{6.8}$$

To see this, we resort to the smoothing interpolation operator Π_h (see Section 2) and invoke an inverse inequality and a standard Sobolev inequality for $\Pi_h v_h \in H^1(\Omega)$ to deduce

$$\|v_h\|_{L^4(\Omega)} \lesssim \|v_h - \Pi_h v_h\|_{L^4(\Omega)} + \|\Pi_h v_h\|_{L^4(\Omega)} \lesssim \|\mathrm{h}^{-1}(v_h - \Pi_h v_h)\|_{L^2(\Omega)} + \|\Pi_h v_h\|_{H^1(\Omega)}.$$

The claim follows from the stability and approximability estimate (2.10) of Π_h. For later use, we record that the discrete Poincaré-Friedrichs inequality (2.13) and the discrete Sobolev inequality (6.8) applied to each component of $\nabla_h \mathbf{v}_h \in [\mathbb{E}(\mathcal{T}_h)]^{3\times 2}$ for $\mathbf{v}_h \in [\mathbb{V}_h^k]^3$ yield

$$\|\nabla_h \mathbf{v}_h\|_{L^4(\Omega)} \lesssim \|\mathbf{v}_h\|_{H_h^2(\Omega)}. \tag{6.9}$$

The next proposition concerns the form $a_h^s(\cdot;\cdot,\cdot)$ and is key to guarantee that the hypotheses of the Lax-Milgram Lemma are satisfied.

Lemma 6.1 (solvability of (6.5)). *There exists a constant C_p independent of h such that*

$$|a_h^s(\mathbf{z}_h;\mathbf{v}_h,\mathbf{w}_h)| \le C_p E_h^s(\mathbf{z}_h)^{\frac{1}{2}} \|\mathbf{v}_h\|_{H_h^2(\Omega)} \|\mathbf{w}_h\|_{H_h^2(\Omega)} \qquad \forall\, \mathbf{v}_h,\mathbf{w}_h,\mathbf{z}_h \in [\mathbb{V}_h^k]^3, \tag{6.10}$$

Moreover, for $\mathbf{z}_h \in [\mathbb{V}_h^k]^3$ and τ satisfying

$$\tau \le \left(1 + C_p E_h^s(\mathbf{z}_h)^{\frac{1}{2}}\right)^{-1}, \tag{6.11}$$

we have

$$\|\mathbf{v}_h\|^2_{H_h^2(\Omega)} \le \frac{1}{\tau}(\mathbf{v}_h,\mathbf{v}_h)_{H_h^2(\Omega)} + a_h^s(\mathbf{z}_h;\mathbf{v}_h,\mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in [\mathbb{V}_h^k]^3. \tag{6.12}$$

Consequently, there exists a unique solution to (6.5) *provided τ satisfies* (6.11) *with $\mathbf{z}_h = \mathbf{y}_h^n$.*

Proof. Let $\mathbf{v}_h,\mathbf{w}_h,\mathbf{z}_h \in [\mathbb{V}_h^k]^3$ and note that

$$|a_h^s(\mathbf{z}_h;\mathbf{v}_h,\mathbf{w}_h)| \le 2\sqrt{2}\|\nabla_h \mathbf{v}_h\|_{L^4(\Omega)} \|\nabla_h \mathbf{w}_h\|_{L^4(\Omega)} E_h^s(\mathbf{z}_h)^{\frac{1}{2}}. \tag{6.13}$$

Then (6.10) follows from the discrete Sobolev inequality (6.9). The estimate (6.12) results from taking $\mathbf{w}_h = \mathbf{v}_h$ in (6.10) and the pseudo time-step restriction (6.11). Thanks to (6.10), (6.12) and Lemma 2.6 (discrete H^2 semi-norm equivalence), the Lax-Milgram theory applies to guarantee the existence and uniqueness of a solution to (6.5). □

The main result of this section is next. It shows that the preprocessing energy $E_h^p(\mathbf{y}_h^{n+1})$ of the deformation $\mathbf{y}_h^{n+1}$ obtained after one step of the linearized gradient flow (6.5) is smaller than the energy $E_h^p(\mathbf{y}_h^n)$ of the previous iterate provided that the pseudo time-step satisfies $\tau \le \frac{1}{2} c_h(\mathbf{y}_h^n)$, where

$$c_h(\mathbf{y}_h^n) := \min\left\{\left(1 + C_p E_h^p(\mathbf{y}_h^n)^{\frac{1}{2}}\right)^{-1}, d_h(\mathbf{y}_h^n)^{-1}\right\} \tag{6.14}$$

with

$$d_h(\mathbf{y}_h^n) := \frac{C_p}{2} E_h^p(\mathbf{y}_h^n)^{\frac{1}{2}} + \frac{\widetilde{C}_p}{2}\Big(h_{\min}^{-1}\big(E_h^p(\mathbf{y}_h^n)+1\big)\big(E_h^p(\mathbf{y}_h^n)^{\frac{1}{2}} + \|g\|_{L^1(\Omega)}\big) + \sigma_h E_h^p(\mathbf{y}_h^n)\Big), \tag{6.15}$$

where $h_{\min} := \min_{T\in\mathcal{T}_h} h_T$ and $\widetilde{C}_p$ is a constant independent of n and h (to be determined in Proposition 6.1). While the above restriction on τ depends on $E_h^p(\mathbf{y}_h^n)$, we show in the subsequent Corollary 6.1 that $c_h(\mathbf{y}_h^n) \ge c$ for a constant c independent of h and n.

Proposition 6.1 (energy decay for prestrain preprocessing). *Let $\sigma_h \ge 0$. Let $\mathbf{y}_h^n \in [\mathbb{V}_h^k]^3$ and assume that $\tau \le \frac{1}{2}c_h(\mathbf{y}_h^n)$ where $c_h(\mathbf{y}_h^n)$ is defined in* (6.14). *If $\delta\mathbf{y}_h^{n+1} \in [\mathbb{V}_h^k]^3$ is the unique solution to* (6.5), *then the new iterate $\mathbf{y}_h^{n+1} := \mathbf{y}_h^n + \delta\mathbf{y}_h^{n+1}$ satisfies*

$$E_h^p(\mathbf{y}_h^{n+1}) + \frac{1}{2\tau}\|\delta\mathbf{y}_h^{n+1}\|^2_{H_h^2(\Omega)} \le E_h^p(\mathbf{y}_h^n). \tag{6.16}$$

Proof. Because $\tau \le \frac{1}{2}c_h(\mathbf{y}_h^n)$ and $E_h^s(\mathbf{y}_h^n) \le E_h^p(\mathbf{y}_h^n)$, τ satisfies the assumption (6.11) of Lemma 6.1 (solvability of (6.5)) and thus there exists a unique solution $\delta\mathbf{y}_h^{n+1} \in [\mathbb{V}_h^k]^3$ to (6.5). Next we take $\mathbf{v}_h = \delta\mathbf{y}_h^{n+1}$ in (6.5) to obtain

$$\tau^{-1}\|\delta\mathbf{y}_h^{n+1}\|^2_{H_h^2(\Omega)} + a_h^s(\mathbf{y}_h^n;\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1}) + \sigma_h a_h^b(\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1}) = 0 \tag{6.17}$$

and proceed in several steps. In contrast to Proposition 5.1 (energy decay), the main difficulty is that a_h^s is quadratic in its first argument.

Step 1: Energy relation. Since $a_h^s(\mathbf{y}_h^n;\cdot,\cdot)$ is bilinear and symmetric, arguing as in (5.7) yields

$$a_h^s(\mathbf{y}_h^n;\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1}) = \frac{1}{2}a_h^s(\mathbf{y}_h^n;\mathbf{y}_h^{n+1},\mathbf{y}_h^{n+1}) - \frac{1}{2}a_h^s(\mathbf{y}_h^n;\mathbf{y}_h^n,\mathbf{y}_h^n) + \frac{1}{2}a_h^s(\mathbf{y}_h^n;\delta\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1}). \tag{6.18}$$

Furthermore, using the identity $(a-b)b = \frac{1}{2}a^2 - \frac{1}{2}b^2 - \frac{1}{2}(a-b)^2$, we have

$$\begin{aligned}\frac{1}{2}a_h^s(\mathbf{y}_h^n;\mathbf{y}_h^{n+1},\mathbf{y}_h^{n+1}) - \frac{1}{2}a_h^s(\mathbf{y}_h^n;\mathbf{y}_h^n,\mathbf{y}_h^n) &= \int_\Omega W_h^n : \big((\nabla_h\mathbf{y}_h^n)^T\nabla_h\mathbf{y}_h^n - g\big)\\ &= E_h^s(\mathbf{y}_h^{n+1}) - E_h^s(\mathbf{y}_h^n) - \frac{1}{2}\|W_h^n\|_{L^2(\Omega)}^2,\end{aligned}$$

where

$$W_h^n := (\nabla_h\mathbf{y}_h^{n+1})^T\nabla_h\mathbf{y}_h^{n+1} - (\nabla_h\mathbf{y}_h^n)^T\nabla_h\mathbf{y}_h^n. \tag{6.19}$$

Therefore, we are able to express $a_h^s(\mathbf{y}_h^n;\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1})$ in terms of energies as

$$a_h^s(\mathbf{y}_h^n;\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1}) = E_h^s(\mathbf{y}_h^{n+1}) - E_h^s(\mathbf{y}_h^n) + \frac{1}{2}a_h^s(\mathbf{y}_h^n;\delta\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1}) - \frac{1}{2}\|W_h^n\|_{L^2(\Omega)}^2.$$

Similarly, noting that $E_h^b(\mathbf{v}_h) = \frac{1}{2}a_h^b(\mathbf{v}_h,\mathbf{v}_h)$ for any $\mathbf{v}_h \in [\mathbb{V}_h^k]^3$ is quadratic, we obtain

$$a_h^b(\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1}) = E_h^b(\mathbf{y}_h^{n+1}) - E_h^b(\mathbf{y}_h^n) + \frac{1}{2}a_h^b(\delta\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1}) \ge E_h^b(\mathbf{y}_h^{n+1}) - E_h^b(\mathbf{y}_h^n).$$

Using these two relations in (6.17), we arrive at

$$E_h^p(\mathbf{y}_h^{n+1}) - E_h^p(\mathbf{y}_h^n) + \tau^{-1}\|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)}^2 \le R_h^n, \tag{6.20}$$

where

$$R_h^n := \frac{1}{2}\|W_h^n\|_{L^2(\Omega)}^2 - \frac{1}{2}a_h^s(\mathbf{y}_h^n;\delta\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1}). \tag{6.21}$$

Step 2: Bounds for R_h^n. We now prove the estimate

$$|R_h^n| \le d_h(\mathbf{y}_h^n)\|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)}^2 \tag{6.22}$$

with $d_h(\mathbf{y}_h^n)$ defined in (6.15). We first apply the continuity property (6.10) of a_h^s to get

$$|a_h^s(\mathbf{y}_h^n;\delta\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1})| \le C_p E_h^p(\mathbf{y}_h^n)^{\frac{1}{2}}\|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)}^2. \tag{6.23}$$

Then we note that W_h^n can be equivalently written as

$$W_h^n = (\nabla_h\delta\mathbf{y}_h^{n+1})^T\nabla_h\mathbf{y}_h^n + (\nabla_h\mathbf{y}_h^n)^T\nabla_h\delta\mathbf{y}_h^{n+1} + (\nabla_h\delta\mathbf{y}_h^{n+1})^T\nabla_h\delta\mathbf{y}_h^{n+1}$$

whence, resorting to the discrete Sobolev inequality (6.9), we obtain

$$\|W_h^n\|_{L^2(\Omega)}^2 \lesssim \big(\|\nabla_h\mathbf{y}_h^n\|_{L^4(\Omega)}^2 + \|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)}^2\big)\|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)}^2. \tag{6.24}$$

To derive (6.22) we estimate $\|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)}^2$ in terms of $\|\nabla_h\mathbf{y}_h^n\|_{L^4(\Omega)}^2$. To this end, we note that $\tau \le \frac{1}{2}c_h(\mathbf{y}_h^n) < (1 + C_pE_h^s(\mathbf{y}_h^n)^{\frac{1}{2}})^{-1}$ and apply the coercivity estimate (6.12) together with the positivity of $a_h^b(\cdot,\cdot)$ and the gradient flow equation (6.5) satisfied by $\delta\mathbf{y}_h^{n+1}$ to derive

$$\begin{aligned}\|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)}^2 &\le \tau^{-1}\|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)}^2 + a_h^s(\mathbf{y}_h^n;\delta\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1}) + \sigma_h a_h^b(\delta\mathbf{y}_h^{n+1},\delta\mathbf{y}_h^{n+1})\\ &= -a_h^s(\mathbf{y}_h^n;\mathbf{y}_h^n,\delta\mathbf{y}_h^{n+1}) - \sigma_h a_h^b(\mathbf{y}_h^n,\delta\mathbf{y}_h^{n+1}).\end{aligned}$$

The H^2 semi-norm equivalence estimates (2.24) and (2.25) show that $E_h^b(\cdot) \sim a_h^b(\cdot,\cdot) \sim |\cdot|_{H_h^2(\Omega)}^2$ on $[\mathbb{V}_h^k]^3$. Hence, from the continuity property (6.13) of a_h^s and (6.9), we infer that

$$\|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)} \lesssim E_h^p(\mathbf{y}_h^n)^{\frac{1}{2}}\big(\|\nabla_h\mathbf{y}_h^n\|_{L^4(\Omega)} + \sigma_h^{\frac{1}{2}}\big).$$

Inserting this into (6.24) gives

$$\|W_h^n\|_{L^2(\Omega)}^2 \lesssim \Big(\big(1+E_h^p(\mathbf{y}_h^n)\big)\|\nabla_h \mathbf{y}_h^n\|_{L^4(\Omega)}^2 + \sigma_h E_h^p(\mathbf{y}_h^n)\Big)\|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)}^2.$$

We tackle $\|\nabla_h \mathbf{y}_h^n\|_{L^4(\Omega)}$ via the inverse inequality $\|\nabla_h \mathbf{y}_h^n\|_{L^4(\Omega)}^2 \lesssim h_{\min}^{-1}\|\nabla_h \mathbf{y}_h^n\|_{L^2(\Omega)}^2$ and

$$\|\nabla_h \mathbf{y}_h^n\|_{L^2(\Omega)}^2 \lesssim E_h^s(\mathbf{y}_h^n)^{\frac{1}{2}} + \|g\|_{L^1(\Omega)} \lesssim E_h^p(\mathbf{y}_h^n)^{\frac{1}{2}} + \|g\|_{L^1(\Omega)},$$

a relation directly following from (3.4). Altogether, we get

$$\|W_h^n\|_{L^2(\Omega)}^2 \le \widetilde{C}_p\Big(h_{\min}^{-1}\big(E_h^p(\mathbf{y}_h^n)+1\big)\big(E_h^p(\mathbf{y}_h^n)^{\frac{1}{2}} + \|g\|_{L^1(\Omega)}\big) + \sigma_h E_h^p(\mathbf{y}_h^n)\Big)\|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)}^2$$

for some constant $\widetilde{C}_p$ independent of n and h. This together with (6.23) yields (6.22).

Step 3: Conditional energy decay. Substituting (6.22) into (6.20) we observe that

$$E_h^p(\mathbf{y}_h^{n+1}) + \big(\tau^{-1} - d_h(\mathbf{y}_h^n)\big)\|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)}^2 \le E_h^p(\mathbf{y}_h^n). \tag{6.25}$$

The desired energy decay is obtained upon realizing that $d_h(\mathbf{y}_h^n) \le c_h(\mathbf{y}_h^n)^{-1} \le (2\tau)^{-1}$, which follows from the definition (6.14) of $c_h(\mathbf{y}_h^n)$ and the assumption on τ. □

Corollary 6.1 (uniform energy decay)**.** *Let $\mathbf{y}_h^0 \in [\mathbb{V}_h^k]^3$ and assume that $\tau \le \frac{1}{2}c_h(\mathbf{y}_h^0)$. The sequence of successive iterates $\mathbf{y}_h^{n+1} := \mathbf{y}_h^n + \delta\mathbf{y}_h^{n+1}$, $n \ge 0$, where $\delta\mathbf{y}_h^{n+1} \in [\mathbb{V}_h^k]^3$ satisfies* (6.5)*, is well defined. Furthermore, the sequence $\{c_h(\mathbf{y}_h^n)\}_{n\ge 0}$ is nondecreasing and there holds*

$$E_h^p(\mathbf{y}_h^{N+1}) + \frac{1}{2\tau}\sum_{n=0}^{N}\|\delta\mathbf{y}_h^{n+1}\|_{H_h^2(\Omega)}^2 \le E_h^p(\mathbf{y}_h^0) \qquad \forall\, N \ge 0. \tag{6.26}$$

Proof. We proceed by induction to prove that for every $n \ge 0$, $\delta\mathbf{y}_h^{n+1}$ is well defined and $c_h(\mathbf{y}_h^{n+1}) \ge c_h(\mathbf{y}_h^n)$. We start with $n = 0$. In that case, by assumption $\tau \le \frac{1}{2}c_h(\mathbf{y}_h^0)$ and Lemma 6.1 and Proposition 6.1 guarantee that $\delta\mathbf{y}_h^1 \in [\mathbb{V}_h^k]^3$ is well defined and

$$E_h^p(\mathbf{y}_h^1) \le E_h^p(\mathbf{y}_h^0).$$

From the expression (6.14) of $c_h(\mathbf{y}_h^n)$, which increases as $E_h^p(\mathbf{y}_h^n)$ decreases, we also deduce that $c_h(\mathbf{y}_h^1) \ge c_h(\mathbf{y}_h^0)$. For the induction step, we assume that $\{\delta\mathbf{y}_h^j\}_{j=1}^n$ is well defined and $c_h(\mathbf{y}_h^j) \ge c_h(\mathbf{y}_h^{j-1})$, $j = 1,..,n$, which implies $\tau \le \frac{1}{2}c_h(\mathbf{y}_h^0) \le \frac{1}{2}c_h(\mathbf{y}_h^n)$. Therefore, Lemma 6.1 (solvability of (6.5)) and Proposition 6.1 (energy decay for prestrain processing) again guarantee that $\delta\mathbf{y}_h^{n+1} \in [\mathbb{V}_h^k]^3$ is well defined and

$$E_h^p(\mathbf{y}_h^{n+1}) \le E_h^p(\mathbf{y}_h^n) \qquad \Longrightarrow \qquad c_h(\mathbf{y}_h^{n+1}) \ge c_h(\mathbf{y}_h^n).$$

This is the desired property for $n+1$ and concludes the induction argument.

Finally, since the condition $\tau \le \frac{1}{2}c_h(\mathbf{y}_h^0) \le \frac{1}{2}c_h(\mathbf{y}_h^n)$ holds for all $n = 1,\dots,N$, (6.16) is valid, whence summing (6.16) over n yields (6.26). □

We finish this section by relating the initial deformation $\mathbf{y}_h^0$ for the main gradient flow (5.3) with the output of the preprocessing gradient flow (6.5).

Remark 6.1 (choice of σ_h)**.** *Under the assumptions of Corollary 6.1 (uniform energy decay), the preprocessing gradient flow produces a sequence of deformations $\{\widetilde{\mathbf{y}}_h^n\}_{n\ge 0}$ with decreasing preprocessing energy $E_h^p(\widetilde{\mathbf{y}}_h^n)$. We assume that the n_h-th iterate of the preprocessing gradient flow, denoted $\widetilde{\mathbf{y}}_h^{n_h}$, is such that*

$$E_h^p(\widetilde{\mathbf{y}}_h^{n_h}) \lesssim \sigma_h.$$

Since σ_h scales like the square of the (three-dimensional) plate thickness, according to (6.1) *and the pre-asymptotic analysis of* [8]*, a natural choice is $\sigma_h \approx h^2$. Regardless of this scaling, in view of* (6.4)*, the prestrain defect of $\widetilde{\mathbf{y}}_h^{n_h}$ satisfies*

$$D_h(\widetilde{\mathbf{y}}_h^{n_h}) \lesssim E_h^s(\widetilde{\mathbf{y}}_h^{n_h})^{\frac{1}{2}} \lesssim \sigma_h^{\frac{1}{2}}, \qquad i.e., \qquad \widetilde{\mathbf{y}}_h^{n_h} \in \mathbb{A}^k_{h,c\sigma_h^{\frac{1}{2}}},$$

for a suitable constant $c > 0$. Moreover, we have

$$E_h^b(\widetilde{\mathbf{y}}_h^{n_h}) \lesssim \sigma_h^{-1} E_h^p(\widetilde{\mathbf{y}}_h^{n_h}) \lesssim 1.$$

This implies that $E_h(\widetilde{\mathbf{y}}_h^{n_h})$ is also uniformly bounded, due to the continuity of E_h and the coercivity of E_h^b. As a consequence, the main gradient flow (5.3) *with initial deformation $\mathbf{y}_h^0 = \widetilde{\mathbf{y}}_h^{n_h}$ produces iterates $\mathbf{y}_h^n$ satisfying $D_h(\mathbf{y}_h^n) \lesssim \sigma_h^{\frac{1}{2}} + \tau E_h(\mathbf{y}_h^0)$ thanks to Proposition 5.2 (control of metric defect) and $E_h(\mathbf{y}_h^n) \lesssim 1$ thanks to* (5.8)*. In particular, if* (5.3) *leads to an almost global minimizer $\mathbf{y}_h^{N_h}$ of the energy E_h, then the sequence $\{\mathbf{y}_h^{N_h}\}_{h>0}$ satisfies the uniform boundedness assumption of Theorem 4.1 (convergence of global minimizers).*

Appendix A. Equivalence between energies (1.1) and (1.7)

The following proposition, first shown in [8, Proposition 1], justifies the replacement of the highly nonlinear reduced bending energy (1.1) involving the second fundamental form $\mathrm{I\!I}[\mathbf{y}]$ of the deformation $\mathbf{y}$ by the quadratic energy (1.7) involving the Hessian $D^2\mathbf{y}$. This is critical for the design of the numerical scheme. We sketch the proof for completeness.

Proposition A.1 (equivalence of (1.1) and (1.7))**.** *If $g \in [H^1(\Omega) \cap L^\infty(\Omega)]^{2\times 2}$ is SPD a.e. in Ω and $\mathbf{y} = (y_m)_{m=1}^3 \in [H^2(\Omega)]^3$ satisfies $\nabla\mathbf{y}^T\nabla\mathbf{y} = g$ a.e. in Ω, then there exist two non-negative functions $f_1, f_2 \in L^2(\Omega)$ depending only on g and its partial derivatives such that*

$$\left|g^{-\frac{1}{2}}\,\mathrm{I\!I}[\mathbf{y}]\,g^{-\frac{1}{2}}\right|^2 = \sum_{m=1}^3 \left|g^{-\frac{1}{2}}\,D^2 y_m\,g^{-\frac{1}{2}}\right|^2 + f_1 \qquad \text{a.e. in } \Omega, \tag{A.1}$$

and

$$\mathrm{tr}\left(g^{-\frac{1}{2}}\,\mathrm{I\!I}[\mathbf{y}]\,g^{-\frac{1}{2}}\right)^2 = \sum_{m=1}^3 \left|\mathrm{tr}\left(g^{-\frac{1}{2}}\,D^2 y_m\,g^{-\frac{1}{2}}\right)\right|^2 + f_2 \qquad \text{a.e. in } \Omega. \tag{A.2}$$

Proof. Since $g \in [H^1(\Omega) \cap L^\infty(\Omega)]^{2\times 2}$ and $\mathbf{y} \in [H^2(\Omega)]^3$ with $\nabla\mathbf{y}^T\nabla\mathbf{y} = g$, we have

$$|\nabla\mathbf{y}|^2 = \mathrm{tr}(g) \in L^\infty(\Omega) \qquad \text{and} \qquad \boldsymbol{\nu} = \frac{\partial_1\mathbf{y}\times\partial_2\mathbf{y}}{|\partial_1\mathbf{y}\times\partial_2\mathbf{y}|} = \frac{\partial_1\mathbf{y}\times\partial_2\mathbf{y}}{\sqrt{\det(g)}}.$$

As a consequence, we deduce the regularity properties $\nabla\mathbf{y} \in [H^1(\Omega) \cap L^\infty(\Omega)]^{3\times 2}$, $\boldsymbol{\nu} \in [H^1(\Omega) \cap L^\infty(\Omega)]^3$, and $\mathrm{I\!I}[\mathbf{y}] \in L^2(\Omega)^{2\times 2}$. In addition, for $i,j \in \{1,2\}$, we represent $\partial_{ij}\mathbf{y}$ in terms of the basis $\{\partial_1\mathbf{y}, \partial_2\mathbf{y}, \boldsymbol{\nu}\}$ of $\mathbb{R}^3$ and the Christoffel symbols Γ_{ij}^l of the surface $\mathbf{y}(\Omega)$ as follows:

$$\partial_{ij}\mathbf{y} = \sum_{l=1}^2 \Gamma_{ij}^l\,\partial_l\mathbf{y} + \mathrm{I\!I}_{ij}[\mathbf{y}]\,\boldsymbol{\nu} \qquad \text{a.e. in } \Omega;$$

we recall that the symbols Γ_{ij}^l are intrinsic quantities that depend only on g and its derivatives but not on $\mathbf{y}$. To prove (A.1), write $a = g^{\frac{1}{2}}$ and note the validity of the expression

$$\left(\left(aD^2y_ka\right)_{ij}\right)_{k=1}^3 = \left(a\mathrm{I\!I}[\mathbf{y}]a\right)_{ij}\boldsymbol{\nu} + \sum_{m,n=1}^2 a_{im}\left(\sum_{l=1}^2 \Gamma_{mn}^l\partial_l\mathbf{y}\right)a_{nj}. \tag{A.3}$$

Since $\boldsymbol{\nu}$ is orthogonal to $\{\partial_1\mathbf{y}, \partial_2\mathbf{y}\}$ and $\boldsymbol{\nu}^T\boldsymbol{\nu} = 1$, taking the square of the l^2-norm on both sides of (A.3) yields

$$\sum_{k=1}^{3}\left(aD^2y_ka\right)_{ij}^2 = \left(a\mathrm{I\!I}[\mathbf{y}]a\right)_{ij}^2 + f_{ij},$$

where f_{ij} does not depend explicitly on $\mathbf{y}$ but only on g and its derivatives. This concludes the proof of (A.1). The proof of (A.2) is similar. $\square$

Appendix B. Proofs of Lemma 2.4 and Lemma 2.5

We follow [11]. We stress that the mesh assumptions of Lemma 2.4 (weak convergence of H_h) are less restrictive and its proof is simpler than [11, Proposition 4.3] due to the simpler structure of the lifting operators R_h and B_h of (2.20).

Proof of Lemma 2.4. Let $\phi \in [C_0^\infty(\Omega)]^{2\times 2}$. We integrate by parts twice to write

$$\int_\Omega H_h(v_h) : \phi = \int_\Omega D_h^2 v_h : \phi - R_h([\nabla_h v_h]) : \phi + B_h([v_h]) : \phi = T_1 + T_2 + T_3 + T_4 + T_5,$$

with

$$T_1 := \int_\Omega v_h \operatorname{div}(\operatorname{div}\phi),$$

$$T_2 := -\int_\Omega R_h([\nabla_h v_h]) : (\phi - \mathcal{I}_h\phi), \quad T_3 := \int_\Omega B_h([v_h]) : (\phi - \mathcal{I}_h\phi),$$

$$T_4 := \sum_{e\in\mathcal{E}_h^0}\int_e [\nabla_h v_h]\cdot\{\phi - \mathcal{I}_h\phi\}\,\mathbf{n}_e, \quad T_5 := -\sum_{e\in\mathcal{E}_h^0}\int_e [v_h]\left\{\operatorname{div}(\phi - \mathcal{I}_h\phi)\right\}\cdot\mathbf{n}_e.$$

Here, $\mathcal{I}_h\phi \in [\mathbb{V}_h^l \cap H_0^1(\Omega)]^{2\times 2}$ denotes the Lagrange interpolant of ϕ of degree $\min\{l_1, l_2\}$, where l_1 and l_2 are the polynomial degrees of R_h and B_h. We treat each term T_i separately. By assumption $v_h \to v \in H^2(\Omega)$ in $[L^2(\Omega)]^3$ as $h\to 0$, whence we have

$$T_1 \to \int_\Omega v \operatorname{div}(\operatorname{div}\phi) = -\int_\Omega \nabla v\cdot\operatorname{div}\phi = \int_\Omega D^2 v : \phi \quad \text{as } h\to 0.$$

For T_2, we invoke the assumed uniform boundedness $|v_h|_{H_h^2(\Omega)} \le C$ and Lemma 2.3 (stability of lifting operators) to get

$$|T_2| \lesssim \|\mathrm{h}^{-\frac{1}{2}}[\nabla_h v_h]\|_{L^2(\Gamma_h^0)}\|\phi - \mathcal{I}_h\phi\|_{L^2(\Omega)} \le C\|\mathcal{I}_h\phi - \phi\|_{L^2(\Omega)} \to 0 \qquad \text{as } h\to 0.$$

Similarly, we have $T_3 \to 0$ as $h\to 0$. To estimate T_4, we start with a scaled trace inequality

$$\|\mathcal{I}_h\phi - \phi\|_{L^2(e)} \lesssim \|\mathrm{h}^{-\frac{1}{2}}(\mathcal{I}_h\phi - \phi)\|_{L^2(\omega(e))} + \|\mathrm{h}^{\frac{1}{2}}\nabla(\mathcal{I}_h\phi - \phi)\|_{L^2(\omega_e)}, \tag{B.1}$$

and recall that ω_e is the union of the two elements adjacent to $e\in\mathcal{E}_h^0$ and that the shape regularity property guarantees that $h_e \approx h_T$ for $T\subset\omega_e$. This, together with the assumption $|v_h|_{H_h^2(\Omega)} \le C$ and the shape regularity of $\{\mathcal{T}_h\}_{h>0}$, yields

$$|T_4| \lesssim \left(\sum_{e\in\mathcal{E}_h^0}\|\mathrm{h}^{-\frac{1}{2}}[\nabla_h v_h]\|_{L^2(e)}^2\right)^{\frac{1}{2}}\left(\|\mathcal{I}_h\phi - \phi\|_{L^2(\Omega)} + \|\mathrm{h}\nabla(\mathcal{I}_h\phi - \phi)\|_{L^2(\Omega)}\right) \to 0 \qquad \text{as } h\to 0.$$

Similarly, $T_5 \to 0$ as $h\to 0$. Gathering the above relations for $T_1, ..., T_5$, we obtain $\int_\Omega H_h(v_h) : \phi \to \int_\Omega D^2 v : \phi$ as $h\to 0$, which is the desired weak convergence property. $\square$

Proof of Lemma 2.5. We split the proof into three steps.

Step 1: strong convergence of the broken Hessian. We recall that for $k' \geq 1$, the Lagrange interpolation operator $\mathcal{I}_h^{k'} v$ is locally H^2 stable

$$\|D^2 \mathcal{I}_h^{k'} w\|_{L^2(T)} \lesssim |w|_{H^2(T)} \qquad \forall\, w \in H^2(T), \quad \forall\, T \in \mathcal{T}_h, \tag{B.2}$$

and satisfies the following approximation estimates for $0 \leq m \leq k'+1$

$$\|w - \mathcal{I}_h^{k'} w\|_{H^m(T)} \lesssim h_T^{k'+1-m} |w|_{H^{k'+1}(T)} \qquad \forall\, w \in H^{k'+1}(T). \tag{B.3}$$

These estimates are less standard and somewhat more intricate for subdivisions made of quadrilaterals; we refer to Section 9 of [11] for their proofs.

We now argue by density. Let $v^\epsilon \in C^\infty(\Omega)$ be a smooth mollifier of v such that $v^\epsilon \to v$ in $H^2(\Omega)$ as $\epsilon \to 0$. We also set $v_h^\epsilon := \mathcal{I}_h^k v^\epsilon$ and write $v_h - v = v_h - v_h^\epsilon + v_h^\epsilon - v^\epsilon + v^\epsilon - v$ so that employing (B.2), (B.3) and summing over $T \in \mathcal{T}_h$ yield

$$\|D_h^2 v_h - D^2 v\|_{L^2(\Omega)} \leq C \left(|v - v^\epsilon|_{H^2(\Omega)} + h |v^\epsilon|_{H^3(\Omega)} \right)$$

for a constant C independent of h and ϵ because $k \geq 2$. Therefore, for every $\eta > 0$, we choose ϵ sufficiently small so that $C|v - v^\epsilon|_{H^2(\Omega)} \leq \eta/2$ and then h sufficiently small so that $Ch|v^\epsilon|_{H^3(\Omega)} \leq \eta/2$ to arrive at

$$\|D_h^2 v_h - D^2 v\,\|_{L^2(\Omega)} \leq \eta.$$

This shows the strong convergence of $D_h^2 v_h$ towards $D^2 v$ in $[L^2(\Omega)]^{2\times 2}$ as $h \to 0$.

Step 2: strong convergence of lifting operators. We now prove that $R_h([\nabla_h v_h]) \to 0$ but omit dealing with $B_h([v_h]) \to 0$, whose proof follows the same idea. Lemma 2.3 (stability of lifting operators) implies

$$\|R_h([\nabla_h v_h])\|_{L^2(\Omega)} \lesssim \|\mathrm{h}^{-\frac{1}{2}} [\nabla_h v_h]\|_{L^2(\Gamma_h^0)} = \|\mathrm{h}^{-\frac{1}{2}} [\nabla_h (v_h - v)]\|_{L^2(\Gamma_h^0)}$$

because $[\nabla v]|_e = 0$ for $e \in \mathcal{E}_h^0$. Thanks to the scaled trace inequality (B.1), the property $\mathcal{I}_h^k(v_h - v) = 0$, and the interpolation estimates (B.3), we obtain for any $e \in \mathcal{E}_h^0$

$$\begin{aligned}
\|\mathrm{h}^{-\frac{1}{2}} [\nabla_h (v_h - v)]\|_{L^2(e)}^2 &\lesssim h_e^{-2} \|\nabla_h (v_h - v)\|_{L^2(\omega_e)}^2 + \|D_h^2 (v_h - v)\|_{L^2(\omega_e)}^2 \\
&\lesssim \|\mathrm{h}^{-2} \nabla_h (v_h - v - \mathcal{I}_h^k (v_h - v))\|_{L^2(\omega_e)}^2 + \|D_h^2 (v_h - v)\|_{L^2(\omega_e)}^2 \\
&\lesssim \|D_h^2 v_h - D^2 v\|_{L^2(\omega_e)}^2.
\end{aligned}$$

Summing over all interior edges and using the shape-regularity of $\{\mathcal{T}_h\}_{h>0}$, we find that

$$\|R_h([\nabla_h v_h])\|_{L^2(\Omega)} \lesssim \|D_h^2 v_h - D^2 v\,\|_{L^2(\Omega)} \to 0 \qquad \text{as } h \to 0,$$

which is the desired property.

Step 3: strong convergence of discrete Hessian. The strong convergence (2.23) of the reconstructed Hessian $H_h(v_h)$ to $D^2 v$ follows from the definition (2.21) of $H_h(v_h)$ and the strong convergence of $D_h^2 v_h$, $R_h([\nabla_h v_h])$, and $B_h([v_h])$ established in Steps 1 and 2. ☐

Appendix C. Dirichlet boundary conditions and forcing term

We have considered so far *free boundary* conditions. In this case, a physically necessary assumption is that any external forcing $\mathbf{f} \in [L^2(\Omega)]^3$ has zero average, i.e., $\fint_\Omega \mathbf{f} = 0$, for otherwise there is no equilibrium configuration. To see this, suppose that a non-zero external force $\mathbf{f}$ is added to the discrete energy (1.10) as well as to the right-hand side of the discrete gradient flow (5.5). Repeating the proof of Proposition 5.3 (evolution of averages), one can easily show that each step of the gradient flow yields

$$\int_\Omega \delta \mathbf{y}_h^{n+1} = \tau \int_\Omega \mathbf{f}.$$

Consequently, for $\|\delta \mathbf{y}_h^{n+1}\|_{L^2(\Omega)} \to 0$ as $n \to \infty$ it is necessary that $\int_\Omega \mathbf{f} = 0$. In previous sections we assume, for simplicity of presentation, that $\mathbf{f} = \mathbf{0}$ but the theory extends to $\int_\Omega \mathbf{f} = 0$.

In this section, we prescribe Dirichlet boundary conditions on a portion $\Gamma^D \neq \varnothing$ of the boundary $\partial\Omega$, namely

$$\mathbf{y} = \boldsymbol{\varphi} \quad \text{and} \quad \nabla \mathbf{y} = \Phi \quad \text{on } \Gamma^D, \tag{C.1}$$

where $\boldsymbol{\varphi} \in [H^1(\Omega)]^3$ and $\Phi \in [H^1(\Omega)]^{3\times 2}$ is such that $\Phi^T \Phi = g$ a.e. in Ω. In this case, we redefine the admissible set $\mathbb{A}$ as

$$\mathbb{A} := \mathbb{A}(\boldsymbol{\varphi}, \Phi) := \left\{ \mathbf{y} \in \mathbb{V}(\boldsymbol{\varphi}, \Phi) : \nabla \mathbf{y}^T \nabla \mathbf{y} = g \quad \text{a.e. in } \Omega \right\}, \tag{C.2}$$

where $\mathbb{V}(\boldsymbol{\varphi}, \Phi)$ is the affine manifold

$$\mathbb{V}(\boldsymbol{\varphi}, \Phi) := \left\{ \mathbf{y} \in [H^2(\Omega)]^3 : \mathbf{y}_{|\Gamma^D} = \boldsymbol{\varphi}, \nabla \mathbf{y}_{|\Gamma^D} = \Phi \right\}. \tag{C.3}$$

Furthermore, we also subtract the term $\int_\Omega \mathbf{f} \cdot \mathbf{y}$ from the energy $E(\mathbf{y})$ defined in (1.7), where $\mathbf{f} \in [L^2(\Omega)]^3$ is a given forcing function. We resort to a Nitsche approach to impose the essential boundary conditions (C.1). As a consequence, they do not need to be included as a strong constraint in the discrete counterpart of the admissible set $\mathbb{A}$. This turns out to be an advantage for the analysis of the method [11].

Let $\mathcal{E}_h := \mathcal{E}_h^0 \cup \mathcal{E}_h^b$ be the set of edges of the subdivision $\mathcal{T}_h$ decomposed into interior edges $\mathcal{E}_h^0$ and boundary edges $\mathcal{E}_h^b$. We assume that the Dirichlet boundary Γ^D is compatible with $\mathcal{T}_h$, $h > 0$, in the sense that $\Gamma^D = \{\mathbf{x} \in e \ : e \in \mathcal{E}_h^D\}$ for some $\mathcal{E}_h^D \subset \mathcal{E}_h^b$. The set of *active edges*, across which jumps and averages will be computed, and associated skeleton are denoted by $\mathcal{E}_h^a := \mathcal{E}_h^0 \cup \mathcal{E}_h^D$ and $\Gamma_h^a := \Gamma_h^0 \cup \Gamma^D$. For interior edges $e \in \mathcal{E}_h^0$, jumps and averages are defined (component-wise) by (2.5). For Dirichlet boundary edges $e \in \mathcal{E}_h^D$, we define averages by $\{\mathbf{v}_h\}|_e := \mathbf{v}_h$ and $\{\nabla_h \mathbf{v}_h\}|_e := \nabla_h \mathbf{v}_h$, while jumps are given by

$$[\mathbf{v}_h]|_e := \mathbf{v}_h - \boldsymbol{\varphi}, \quad [\nabla_h \mathbf{v}_h]|_e := \nabla_h \mathbf{v}_h - \Phi. \tag{C.4}$$

To simplify the notation, we define

$$\mathbb{V}_h^k(\boldsymbol{\varphi}, \Phi) := \left\{ \mathbf{v}_h \in [\mathbb{V}_h^k]^3 : \ [\mathbf{v}_h], [\nabla_h \mathbf{v}_h] \text{ given by (C.4) for all } e \in \mathcal{E}_h^D \right\}. \tag{C.5}$$

We insist that $\mathbb{V}_h^k(\cdot,\cdot)$ coincides with $[\mathbb{V}_h^k]^3$ but the former carries the notion of boundary jumps. In addition, for $\mathbf{v}_h \in \mathbb{V}_h^k(\boldsymbol{\varphi}, \Phi)$, $\|[\mathbf{v}_h]\|_{L^2(\Gamma^D)} \to 0$ and $\|[\nabla_h \mathbf{v}_h]\|_{L^2(\Gamma^D)} \to 0$ imply $\mathbf{v}_h \to \boldsymbol{\varphi}$ and $\nabla_h \mathbf{v}_h \to \Phi$ in $L^2(\Gamma^D)$ as $h \to 0$, thereby relating $\mathbb{V}_h^k(\boldsymbol{\varphi}, \Phi)$ and $\mathbb{V}(\boldsymbol{\varphi}, \Phi)$.

The definitions (2.18) and (2.19) of lifting operators for interior edges $e \in \mathcal{E}_h^0$ extend trivially to boundary edges $e \in \mathcal{E}_h^D$, in which case ω_e reduces to a single element. The discrete energy $E_h(\mathbf{y}_h)$ then reads as (1.10) upon (i) replacing $\mathcal{E}_h^0$ by $\mathcal{E}_h^a$ in the definition (2.20) of lifting operators, which affects the discrete Hessian operator (2.21); (ii) replacing

Γ_h^0 by Γ_h^a in the stabilization terms of E_h and subtracting the forcing term $\int_\Omega \mathbf{f}\cdot\mathbf{y}_h$; and (iii) replacing the discrete admissible set by

$$\mathbb{A}_{h,\varepsilon}^k := \{\mathbf{y}_h \in \mathbb{V}_h^k(\boldsymbol{\varphi},\Phi) \;:\; D_h(\mathbf{y}_h) \le \varepsilon\}.$$

Finally, we note that for non-homogeneous Dirichlet data, $|\cdot|_{H_h^2(\Omega)}$ defined in (2.7) is no longer a semi-norm on $\mathbb{V}_h^k(\boldsymbol{\varphi},\Phi)$ since the boundary data are encoded in the jumps across $e\in\mathcal{E}_h^D$, but it is a norm on $\mathbb{V}_h^k(\mathbf{0},\mathbf{0})$ by definition.

All statements and proofs presented earlier extend to Dirichlet boundary conditions with minor modifications. To be concise, we summarize below the key differences between Dirichlet and *free boundary* conditions.

- **Discrete Poincaré-Friedrichs inequality**: For any $\mathbf{v}_h\in\mathbb{V}_h^k(\boldsymbol{\varphi},\Phi)$ we have
$$\|\mathbf{v}_h\|_{L^2(\Omega)} + \|\nabla_h\mathbf{v}_h\|_{L^2(\Omega)} \lesssim |\mathbf{v}_h|_{H_h^2(\Omega)} + \|\boldsymbol{\varphi}\|_{H^1(\Omega)} + \|\Phi\|_{H^1(\Omega)}; \tag{C.6}$$
see [11]. In contrast to (2.13), the term $\|\mathbf{v}_h\|_{L^2(\Omega)}$ is not needed to bound $\|\nabla_h\mathbf{v}_h\|_{L^2(\Omega)}$.
- **Weak convergence of discrete Hessian**: Let $\{\mathbf{v}_h\}_{h>0}\subset\mathbb{V}_h^k(\boldsymbol{\varphi},\Phi)$ satisfy $\|\mathbf{v}_h\|_{H_h^2(\Omega)}\lesssim 1$ uniformly in h and $\mathbf{v}_h\to\mathbf{v}$ in $L^2(\Omega)$ as $h\to 0$ for some $\mathbf{v}\in[H^2(\Omega)]^3$. We proceed as in Lemma 2.4 (weak convergence of H_h) given in Appendix B to prove that $H_h(\mathbf{v}_h)\rightharpoonup D^2\mathbf{v}$ in $L^2(\Omega)$, except that integrating $\int_\Omega H_h(\mathbf{v}_h):\boldsymbol{\phi}$ by parts twice gives the extra term
$$T_6 := \sum_{e\in\mathcal{E}_h^D}\int_e(\mathbf{v}_h-\boldsymbol{\varphi})\cdot\{\operatorname{div}\mathcal{I}_h\boldsymbol{\phi}\}\,\mathbf{n}_e, \qquad \boldsymbol{\phi}\in[C_0^\infty(\Omega)]^{3\times2\times2}.$$
Its convergence is standard by uniform boundedness of $|\mathbf{v}_h|_{H_h^2(\Omega)}$ and the trace inequality.
- **Strong convergence of discrete Hessian**: Let $\mathbf{v}\in[H^2(\Omega)]^3$ satisfy $\mathbf{v}=\boldsymbol{\varphi}$ and $\nabla\mathbf{v}=\Phi$ on Γ^D. The proof of Lemma 2.5 (strong convergence of H_h) follows as in Appendix B.
- **Coercivity**: The analogue of Theorem 3.1 (coercivity of H_h) involves the boundary data and external forcing term, namely for any $\mathbf{y}_h\in\mathbb{V}_h^k(\boldsymbol{\varphi},\Phi)$ and any $\gamma_0,\gamma_1>0$ we have
$$|\mathbf{y}_h|_{H_h^2(\Omega)}^2 \lesssim E_h(\mathbf{y}_h) + \|\boldsymbol{\varphi}\|_{H^1(\Omega)}^2 + \|\Phi\|_{H^1(\Omega)}^2 + \|\mathbf{f}\|_{L^2(\Omega)}^2; \tag{C.7}$$
the proof is similar to [11, Lemma 2.3]. Note that now $E_h(\mathbf{y}_h)$ is bounded from below but not necessarily by zero [11].
- **Compactness**: In contrast to Lemma 2.2 (compactness), we do not need to consider a *shifted* sequence with vanishing mean value. If a sequence $\{\mathbf{v}_h\}_{h>0}\subset\mathbb{V}_h^k(\boldsymbol{\varphi},\Phi)$ satisfies $|\mathbf{v}_h|_{H_h^2(\Omega)}\lesssim 1$, then there exists $\mathbf{v}\in\mathbb{V}(\boldsymbol{\varphi},\Phi)$ such that, up to a subsequence, $\mathbf{v}_h\to\mathbf{v}$ in $[L^2(\Omega)]^3$ and $\nabla_h\mathbf{v}_h\to\nabla\mathbf{v}$ in $[L^2(\Omega)]^{3\times2}$ as $h\to 0$. A proof of this statement follows along the lines of [11, Proposition 5.1], thanks to (C.6). Therefore, Proposition 3.1 and Theorems 4.1 and 4.2 are valid without removing the mean of $\mathbf{v}_h$.
- **Convergence of forcing term**: The addition of the forcing term in the energy does not affect Theorem 4.2 (lim-inf of E_h) and Theorem 4.3 (lim-sup of E_h) because $\int_\Omega\mathbf{f}\cdot\mathbf{y}_h\to\int_\Omega\mathbf{f}\cdot\mathbf{y}$ when $\mathbf{y}_h\to\mathbf{y}$ in $[L^2(\Omega)]^3$ as $h\to 0$.
- **Gradient flow**: For *free boundary* conditions, the gradient flow metric $\|\cdot\|_{H_h^2(\Omega)}$ contains an L^2 term to guarantee solvability of (5.5) (see Remark 5.1) and control of the average of iterates (see Proposition 5.3). In contrast, since $|\cdot|_{H_h^2(\Omega)}$ defined in (2.7) is a norm on $\mathbb{V}_h^k(\mathbf{0},\mathbf{0})$ when Dirichlet boundary conditions are imposed, the extra L^2 term is no longer

needed. The counterpart of the gradient flow of Section 5 reads: given $\mathbf{y}_h^0 \in \mathbb{A}_{h,\varepsilon_0}^k$ and $\tau > 0$, iteratively compute $\mathbf{y}_h^{n+1} := \mathbf{y}_h^n + \delta\mathbf{y}_h^{n+1} \in \mathbb{V}_h^k(\boldsymbol{\varphi}, \Phi)$ with $\delta\mathbf{y}_h^{n+1} \in \mathcal{F}_h(\mathbf{y}_h^n)$ satisfying

$$\tau^{-1}\langle \delta\mathbf{y}_h^{n+1}, \mathbf{v}_h\rangle_{H_h^2(\Omega)} + a_h(\delta\mathbf{y}_h^{n+1}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)} - a_h(\mathbf{y}_h^n, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathcal{F}_h(\mathbf{y}_h^n), \quad (C.8)$$

where the tangent space is given by

$$\mathcal{F}_h(\mathbf{y}_h^n) := \left\{ \mathbf{v}_h \in \mathbb{V}_h^k(\mathbf{0},\mathbf{0}) : \int_T \nabla\mathbf{v}_h^T \nabla\mathbf{y}_h^n + (\nabla\mathbf{y}_h^n)^T \nabla\mathbf{v}_h = 0 \quad \forall\, T \in \mathcal{T}_h \right\}$$

and $\langle\cdot,\cdot\rangle_{H_h^2(\Omega)}$ is defined in (2.6). Note that the Dirichlet data are implicitly contained in $a_h(\mathbf{y}_h^n, \mathbf{v}_h)$ through the liftings of the boundary data that appear in $H_h(\mathbf{y}_h)$.

For Dirichlet boundary conditions, the counterpart of the control of defect (5.9) reads

$$D_h(\mathbf{y}_h^n) \le \varepsilon_0 + c\tau\big(E_h(\mathbf{y}_h^0) + \widetilde{c}\big). \quad (C.9)$$

Here $c > 0$ is the hidden constant of (C.6), which depends only on Ω and Γ^D, while $\widetilde{c} \ge 0$ depends only on μ, g, $\|\boldsymbol{\varphi}\|_{H^1(\Omega)}$, $\|\Phi\|_{H^1(\Omega)}$, $\|\mathbf{f}\|_{L^2(\Omega)}$ and the constant $C(\gamma_0,\gamma_1)$ that appears in (2.25) (with $\mathcal{E}_h^0$ replaced by $\mathcal{E}_h^a$). The proof relies on (C.6) and (C.7) to deal with the forcing term; the proof is similar to [11, Lemma 3.4].

- **Preprocessing**: The prestrain defect of the iterates $\mathbf{y}_h^n$ produced by (C.8) is controlled by (C.9). In this case, the energy $E_h(\mathbf{y}_h^0)$ is also affected by how well $\mathbf{y}_h^0$ satisfies the prescribed boundary conditions as E_h contains the terms $(\nabla_h\mathbf{y}_h^0 - \Phi)$ and $(\mathbf{y}_h^0 - \boldsymbol{\varphi})$ in the discrete Hessian and the stabilization terms. Therefore, since flat surfaces are stationary for the *metric preprocessing* step [8, Section 3.3], we first solve the bi-Laplacian problem

$$\Delta^2\widehat{\mathbf{y}} = \widehat{\mathbf{f}} \quad \text{in } \Omega, \qquad \nabla\widehat{\mathbf{y}} = \Phi \quad \text{on } \Gamma^D, \qquad \widehat{\mathbf{y}} = \boldsymbol{\varphi} \quad \text{on } \Gamma^D, \quad (C.10)$$

where typically $\widehat{\mathbf{f}} = \mathbf{0}$. Note that this vector-valued problem is well-posed and gives, in general, a non-flat surface $\widehat{\mathbf{y}}(\Omega)$. Using the LDG method with boundary conditions imposed *à la Nitsche* to approximate the solution $\widehat{\mathbf{y}} \in \mathbb{V}(\boldsymbol{\varphi},\Phi)$ of (C.10), we thus solve:

$$\widehat{\mathbf{y}}_h \in \mathbb{V}_h^k(\boldsymbol{\varphi},\Phi): \quad c_h(\widehat{\mathbf{y}}_h, \mathbf{v}_h) = (\widehat{\mathbf{f}}, \mathbf{v}_h)_{L^2(\Omega)} \qquad \forall\, \mathbf{v}_h \in \mathbb{V}_h^k(\mathbf{0},\mathbf{0}). \quad (C.11)$$

Here

$$\begin{aligned} c_h(\mathbf{w}_h, \mathbf{v}_h) &:= \big(H_h(\mathbf{w}_h), H_h(\mathbf{v}_h)\big)_{L^2(\Omega)} \\ &\quad + \widehat{\gamma}_1\big(\mathrm{h}^{-1}[\nabla_h\mathbf{w}_h], [\nabla_h\mathbf{v}_h]\big)_{L^2(\Gamma_h^a)} + \widehat{\gamma}_0\big(\mathrm{h}^{-3}[\mathbf{w}_h], [\mathbf{v}_h]\big)_{L^2(\Gamma_h^a)}, \end{aligned} \quad (C.12)$$

where $\widehat{\gamma}_0$ and $\widehat{\gamma}_1$ are positive stabilization parameters that may not necessarily be the same as their counterparts γ_0 and γ_1 used in the definition (1.10) of E_h. Then $\widehat{\mathbf{y}}_h$ satisfies (approximately) the given boundary conditions on Γ^D and $\widehat{\mathbf{y}}_h(\Omega)$ is, in general, non-flat. To generate a deformation with small prestrain defect, we may then apply a *metric preprocessing* step similar to the one proposed in Section 6 starting from $\widetilde{\mathbf{y}}_h^0 = \widehat{\mathbf{y}}_h \in \mathbb{V}_h^k(\boldsymbol{\varphi},\Phi)$. Then all the results of Section 6 extend to Dirichlet boundary conditions upon replacing $\|\cdot\|_{H_h^2(\Omega)}$ by $|\cdot|_{H_h^2(\Omega)}$ and, wherever appropriate, $\mathcal{E}_h^0$ and Γ_h^0 by $\mathcal{E}_h^a$ and Γ_h^a, respectively. In particular, the boundary conditions satisfied by the initial deformation $\widetilde{\mathbf{y}}_h^0 \in \mathbb{V}_h^k(\boldsymbol{\varphi},\Phi)$ are approximately preserved during the *metric preprocessing* step (6.5). The latter consists of seeking increments $\delta\mathbf{y}_h^n \in \mathbb{V}_h^k(\mathbf{0},\mathbf{0})$ minimizing the linearized stretching energy using the metric

$$(D_h^2 v_h, D_h^2 w_h)_{L^2(\Omega)} + (\mathrm{h}^{-1}[\nabla_h v_h], [\nabla_h w_h])_{L^2(\Gamma_h^a)} + (\mathrm{h}^{-3}[v_h], [w_h])_{L^2(\Gamma_h^a)}.$$

Moreover, Remark 6.1 (choice of σ_h) applies and leads to an output $\widetilde{\mathbf{y}}_h^{n_h} \in \mathbb{V}_h^k(\boldsymbol{\varphi},\Phi)$ such that $E_h^b(\widetilde{\mathbf{y}}_h^{n_h}) \lesssim 1$ and $D_h(\widetilde{\mathbf{y}}_h^{n_h}) \lesssim \sigma_h^{\frac{1}{2}}$ provided that $E_h^p(\widetilde{\mathbf{y}}_h^{n_h}) \lesssim \sigma_h$.

Acknowledgment

Andrea Bonito and Diane Guignard were partially supported by the NSF Grant DMS-1817691 and DMS-2110811. Ricardo H. Nochetto and Shuo Yang were partially supported by the NSF Grants DMS-1411808 and DMS-1908267.

References

1. W. Bangerth, R. Hartmann, and G. Kanschat, *deal.II – a general purpose object oriented finite element library*, ACM Trans. Math. Softw. **33** (2007), no. 4, 24/1–24/27.
2. S. Bartels, *Finite element approximation of large bending isometries*, Numer. Math. **124** (2013), no. 3, 415–440.
3. S. Bartels, A. Bonito, A.H. Muliana, and R.H. Nochetto, *Modeling and simulation of thermally actuated bilayer plates*, J. Comput. Phys. **354** (2018), 512–528.
4. S. Bartels, A. Bonito, and R.H. Nochetto, *Bilayer plates: Model reduction, Γ-convergent finite element approximation, and discrete gradient flow*, Comm. Pure Appl. Math. **70** (2017), no. 3, 547–589.
5. F. Bassi, S. Rebay, G. Mariotti, S. Pedinotti, and M. Savini, *A high-order accurate discontinuous finite element method for inviscid and viscous turbomachinery flows*, Proceedings of the 2nd European Conference on Turbomachinery Fluid Dynamics and Thermodynamics, 1997, pp. 99–109.
6. K. Bhattacharya, M. Lewicka, and M. Schäffner, *Plates with incompatible prestrain*, Arch. Rational Mech. Anal. **221** (2016), no. 1, 143–181.
7. A. Bonito and D. Guignard, *The step-82 tutorial program: solving the fourth-order biharmonic equation using a lifting operator approach*, deal.II library (2021), URL `https://www.dealii.org/developer/doxygen/deal.II/step_82.html`.
8. A. Bonito, D. Guignard, R.H. Nochetto, and S. Yang, *LDG approximation of large deformations of prestrained plates*, J. Comput. Phys. **448** (2022), 110719.
9. A. Bonito and R.H. Nochetto, *Quasi-optimal convergence rate of an adaptive discontinuous Galerkin method*, SIAM J. Numer. Anal. **48** (2010), no. 2, 734–771.
10. A. Bonito, R.H. Nochetto, and D. Ntogkas, *Discontinuous Galerkin approach to large bending deformation of a bilayer plate with isometry constraint*, J. Comput. Phys. **423** (2020), 109785.
11. ———, *DG approach to large bending deformations with isometry constraint*, Math. Models Methods Appl. Sci. **31** (2021), no. 1, 133–175.
12. A. Bonito, R.H. Nochetto, and S. Yang, *LDG approximation of large deformations of bilayer plates*, (In preparation).
13. F. Brezzi, G. Manzini, D. Marini, P. Pietra, and A. Russo, *Discontinuous finite elements for diffusion problems*, Atti Convegno in onore di F. Brioschi (Milano 1997), Istituto Lombardo, Accademia di Scienze e Lettere (1999), 197–217.
14. ———, *Discontinuous Galerkin approximations for elliptic problems*, Numer. Methods Partial Differential Equations **16** (2000), no. 4, 365–378.
15. P.G. Ciarlet, *Basic error estimates for elliptic problems*, Handbook of Numerical Analysis, Vol. II, Handb. Numer. Anal., II, North-Holland, Amsterdam, 1991, pp. 17–351.
16. B. Cockburn and C.-W. Shu, *The local discontinuous Galerkin method for time-dependent convection-diffusion systems*, SIAM J. Numer. Anal. **35** (1988), no. 6, 2440–2463.
17. E. Efrati, E. Sharon, and R. Kupferman, *Elastic theory of unconstrained non-euclidean plates*, J. Mech. Phys. Solids **57** (2009), no. 4, 762–775.
18. M. Fortin and R. Glowinski, *Augmented Lagrangian methods: Applications to the numerical solution of boundary-value problems*, Elsevier Science, 2000.
19. G. Friesecke, R.D. James, and S. Müller, *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity*, C.R. Math. **55** (2002), no. 11, 1461–1506.
20. V. Girault and P.-A. Raviart, *Finite element methods for Navier–Stokes equations: Theory and algorithms*, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986.
21. A. Goriely and M. Ben Amar, *Differential growth and instability in elastic shells*, Phys. Rev. Lett. **94** (2005), no. 19, 198103.
22. J. Kim, J.A. Hanna, R.C. Hayward, and C.D. Santangelo, *Thermally responsive rolling of thin gel strips with discrete variations in swelling*, Soft Matter **8** (2012), no. 8, 2375–2381.
23. Y. Klein, E. Efrati, and E. Sharon, *Shaping of elastic sheets by prescription of non-euclidean metrics*, Science **315** (2007), no. 5815, 1116–1120.

24. C.D. Modes, K. Bhattacharya, and M. Warner, *Disclination-mediated thermo-optical response in nematic glass sheets*, Phys. Rev. E **81** (2010), no. 6, 060701.
25. ______, *Gaussian curvature from flat elastica sheets*, Proc. Royal Soc. **467** (2010), no. 2128, 1121–1140.
26. D.A. Di Pietro and A. Ern, *Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier–Stokes equations*, Math. Comp. **79** (2010), no. 271, 1303–1330.
27. ______, *Mathematical aspects of discontinuous Galerkin methods*, Mathématiques et Applications, Springer Berlin Heidelberg, 2011.
28. T. Pryer, *Discontinuous Galerkin methods for the p-biharmonic equation from a discrete variational perspective*, Electron. Trans. Numer. Anal. **41** (2014), 328 – 349.
29. Z.L. Wu, M. Moshe, J. Greener, H. Therien-Aubin, Z. Nie, E. Sharon, and E. Kumacheva, *Three-dimensional shape transformations of hydrogel sheets induced by small-scale modulation of internal stresses*, Nat. Commun. **4** (2013), 1586.
30. A. Yavari, *A geometric theory of growth mechanics*, J. Nonlinear Sci. **20** (2010), no. 6, 781–830.