A REMARK ON A THEOREM OF DORE CONCERNING L^p MAXIMAL REGULARITY

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Abstract. The aim of this note is to show that for the necessary conditions in Theorems 2.1 and 2.2 in [3], the operator A does not have to be densely defined. Evenmore, when the space X is reflexive this assumption becomes a conclusion. An application to the Stokes problem is given.

1. Introduction

Let X be a real or complex Banach space, $A : \mathcal{D}(A) \subset X \to X$ be a linear closed (not necessarily densely defined) operator, $p \in [1, \infty[$ and $0 < T < \infty$. We say that A possesses the L^p maximal regularity property (MRp) on the interval I (with $I = [0, T]$ or $I = [0, \infty[)$ if for every $f \in L^p(I; X)$, there exists one and only one $u \in W^{1,p}(I; X) \cap L^p(I; \mathcal{D}(A))$ ($\mathcal{D}(A)$ endowed with the graph norm) satisfying the problem

$$u' = Au + f, \quad \text{in } L^p(I; X), \qquad u(0) = 0. \tag{1.1}$$

In [3, Theorem 2.1 and 2.2], Dore proved that if the operator A is densely defined (and $p > 1$) a necessary condition for A to possess MRp on I is the existence of $\delta \geq 0$ and $C > 0$ such that

$$\begin{aligned} &\{\lambda \in \mathbb{C} : \mathcal{R}e\,(\lambda) \geq \delta\} \subset \rho(A), \\ \text{and} \qquad &\mathcal{R}e\,(\lambda) \geq \delta \implies \left\|(\lambda - A)^{-1}\right\| \leq \frac{C}{1 + |\lambda|}, \end{aligned} \tag{1.2}$$

where $\delta = 0$ in case $I = [0, \infty[$ (Theorem 2.1), and $\delta > 0$ in the case $I = [0, T]$ (Theorem 2.2). The aim of this note is to observe that the density of the domain $\mathcal{D}(A)$ is not needed and evenmore that when the space X is reflexive this assumption becomes a conclusion in Theorems 2.1 and 2.2 in [3]. Since the proofs in [3] are only sketched we give a detailed proof for the sake of completeness.

We recall that the observation that MRp implies that A generates an analytic semigroup (in case A generates a $\mathcal{C}_0$ semigroup) goes back to Sobolevskii [11], see also [14, Theorem III.1.3].

2. Main results

Theorem 2.1. *Let X be a complex Banach space and A be a linear closed operator possessing MRp for some $p \in [1, \infty[$ on the interval $I = [0, \infty[$ (resp. the interval $I = [0, T]$ for some $T \in]0, \infty[)$. Then there exist $\delta \geq 0$ ($\delta = 0$ in case $I = [0, \infty[)$ and $C > 0$ such that (1.2) holds.*

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Proposition 2.2. *Let X be a real or complex Banach space and A be a linear closed operator possessing MRp for some $p \in [1,\infty[$ on the interval $I = [0,\infty[$ (resp. the interval $I = [0,T]$ for some $T \in]0,\infty[$). Let Y be a reflexive Banach space contained in X with continuous imbeddings. Suppose that for every $f \in L^p(I;Y)$ the solution u of (1.1) satisfies $Au \in L^p(I;Y)$, then the set $\{x \in \mathcal{D}(A); Ax \in Y\}$ has to be dense in Y. In particular, if X is a reflexive Banach space, then A has to be densely defined.*

Remark 2.3. It follows from Theorem 2.1 that the operator A is sectorial in the sense of Lunardi [9, Definition 2.0.1]. Therefore it generates an analytic semigroup $\{e^{tA} : t \geq 0\}$ (not necessary strongly continuous) [9, Proposition 2.1.1] and [10]. As a consequence, Theorems 2.3-2.5 in [3] even hold when A is not densely defined. This is clear from the proofs of Theorems 2.3-2.5 given in [3].

Remark 2.4. Under the assumptions of Theorem 2.1, it follows from Theorem 2.4 in [3] that the necessary conditions of Weis [15] (see also [2, 1]), namely

$$\left\{\lambda\,(\lambda - A)^{-1}; \mathcal{R}e\,(\lambda) = \delta,\ \lambda \neq 0\right\} \text{ is } \mathcal{R}\text{-bounded,}$$

where δ is as in (1.2), holds. Indeed, thanks to Theorem 2.4 in [3], there exists $\delta \geq 0$ such that $A - \delta$ possesses MRp on $[0,\infty[$. Thus the familly of operators $\{is(is - A),\ s \in \mathbb{R}, s \neq 0\}$ is a L^p multiplier, which implies its $\mathcal{R}$-boundedness.

3. Proof of Theorem 2.1 and Proposition 2.2

3.1. **Proof of Theorem 2.1.** First observe (as in [3]) that it is sufficient to prove the theorem where (1.2) is replaced by

$$\begin{aligned} &\{\lambda \in \mathbb{C} : \mathcal{R}e\,(\lambda) > \delta\} \subset \rho(A), \\ \text{and} \qquad &\mathcal{R}e\,(\lambda) > \delta \implies \left\|(\lambda - A)^{-1}\right\| \leq \frac{C}{1 + |\lambda|}. \end{aligned} \tag{3.1}$$

This is obvious when $\delta > 0$. In the case $\delta = 0$ ($I = [0,\infty[$), it follows from

$$\left\|(\lambda - A)^{-1}\right\| \geq \textit{ spectral radius}\left((\lambda - A)^{-1}\right) = \frac{1}{dist(\lambda, \rho(A)^C)}, \tag{3.2}$$

(see e.g. [8, Chapter III, Problem 6.16] or [9, Proposition A.0.3 and Corollary A.0.4]), that the first part of (1.2) holds. The estimate is a consequence of the continuity of the resolvent.

Let $p \in [1,\infty[$. As in [3], we denote by $\mathcal{M}$ the operator in $L^p(I;X)$ such that $\mathcal{M}(f) = u$, where u is the solution of (1.1) and observe that it follows from the closed graph theorem that there exists $C_1 > 0$ such that

$$\|\mathcal{M}(f)\|_{L^p(I;X)} + \left\|(\mathcal{M}(f))'\right\|_{L^p(I;X)} \leq C_1 \|f\|_{L^p(I;X)}, \tag{3.3}$$

for every $f \in L^p(I;X)$.

Case $I = [0,T]$

Let us prove the surjectivity of $(\lambda - A)$ for $\mathcal{R}e\,(\lambda) > \delta_1 > 0$, where δ_1 is such that for every $\lambda \in \mathbb{C}$ with $\mathcal{R}e\,(\lambda) > \delta_1$, it holds

$$\left\{ \begin{array}{c} C_1(\mathcal{R}e\,(\lambda))^{1-1/p} e^{-\mathcal{R}e(\lambda)T} \left(\frac{e^p - 1}{p}\right)^{1/p} \leq 1/2, \\ \text{and} \qquad \mathcal{R}e\,(\lambda)\, e^{-\mathcal{R}e(\lambda)T} \leq 1, \end{array} \right. \tag{3.4}$$

and where C_1 is the constant in (3.3). Let $\lambda \in \mathbb{C}$ with $\mathcal{R}e\,(\lambda) >?\delta_1$. Define $\phi_\lambda \in L^p(I, \mathbb{C})$ such that

$$\phi_\lambda(t) = \begin{cases} e^{\lambda t} & \text{for } 0 \le t \le \frac{1}{\mathcal{R}e(\lambda)}, \\ 0 & \text{for } \frac{1}{\mathcal{R}e(\lambda)} < t \le T, \end{cases} \tag{3.5}$$

and for $x \in X$ set

$$R_\lambda x = \mathcal{R}e\,(\lambda) \int_0^T e^{-\lambda t} \mathcal{M}(\phi_\lambda x)(t) dt. \tag{3.6}$$

Using integration by parts, it follows for $x \in X$ that

$$\int_0^T e^{-\lambda t} \mathcal{M}(\phi_\lambda x)'(t) dt = e^{-\lambda T} \mathcal{M}(\phi_\lambda x)(T) + \lambda \int_0^T e^{-\lambda t} \mathcal{M}(\phi_\lambda x)(t) dt. \tag{3.7}$$

Thus, since A is closed,

$$\begin{aligned} e^{-\lambda T} \mathcal{M}(\phi_\lambda x)(T) + \lambda \int_0^T e^{-\lambda t} \mathcal{M}(\phi_\lambda x)(t) dt \\ = A \int_0^T e^{-\lambda t} \mathcal{M}(\phi_\lambda x)(t) dt + \int_0^T e^{-\lambda t} \phi_\lambda x \; dt. \end{aligned}$$

Hence, multiplying by $\mathcal{R}e\,(\lambda)$ and using definition (3.5) of ϕ_λ, it holds

$$(\lambda - A) R_\lambda(x) - x = -\mathcal{R}e\,(\lambda)\, e^{-\lambda T} \mathcal{M}(\phi_\lambda x)(T). \tag{3.8}$$

Moreover, we have

$$\|\mathcal{M}(\phi_\lambda x)(T)\|_X \le \int_0^T \|\mathcal{M}(\phi_\lambda x)'(t) dt\| \le T^{1-1/p} \|\mathcal{M}(\phi_\lambda x)'\|_{L^p(I;X)}. \tag{3.9}$$

Going back to (3.8) and using (3.3), we obtain

$$\|(\lambda - A) R_\lambda(x) - x\|_X \le C_1 \mathcal{R}e\,(\lambda)\, T^{1-1/p} e^{-\mathcal{R}e(\lambda) T} \|\phi_\lambda x\|_{L^p(I;X)}.$$

A simple calculation shows that

$$\|\phi_\lambda x\|_{L^p(I;X)} = \left(\frac{e^p - 1}{p \mathcal{R}e\,(\lambda)} \right)^{1/p} \|x\|_X. \tag{3.10}$$

Thus, using (3.4), we obtain

$$\|(\lambda - A) R_\lambda(x) - x\|_X \le 1/2 \|x\|_X. \tag{3.11}$$

We have proved that for $\lambda \in \mathbb{C}$ with $\mathcal{R}e\,(\lambda) > \delta_1$, the operator $B_\lambda = (\lambda - A) R_\lambda - I : X \to X$ is bounded with $\|B_\lambda\| \le 1/2$. Hence, $I + B_\lambda$ is invertible and

$$S_\lambda = R_\lambda (I + B_\lambda)^{-1} : X \to X \tag{3.12}$$

is bounded. Moreover we have

$$(\lambda - A) S_\lambda = I,$$

which implies that $\lambda - A$ is surjective.

We proceed differently from [3] for the injectivity of $(\lambda - A)$. Let $\delta_2 > 0$ be such that for every $\lambda \in \mathbb{C}$ with $\mathcal{R}e\,(\lambda) > \delta_2$ we have

$$\int_0^T |1 + t\lambda|^p e^{p \mathcal{R}e(\lambda) t} dt > C_1^p \int_0^T e^{p \mathcal{R}e(\lambda) t} dt, \tag{3.13}$$

where C_1 is the constant in (3.3). Such δ_2 exists since

$$\lim_{\mathcal{R}e(\lambda)\to\infty} \frac{\int_0^T t e^{p\mathcal{R}e(\lambda)t}dt}{\int_0^T e^{p\mathcal{R}e(\lambda)t}dt} = T,$$

hence

$$\lim_{\mathcal{R}e(\lambda)\to\infty} \frac{\int_0^T |1+t\lambda|^p e^{p\mathcal{R}e(\lambda)t}dt}{\int_0^T e^{p\mathcal{R}e(\lambda)t}dt} \geq \lim_{\mathcal{R}e(\lambda)\to\infty} |\lambda| \frac{\int_0^T t e^{p\mathcal{R}e(\lambda)t}dt}{\int_0^T e^{p\mathcal{R}e(\lambda)t}dt} = \infty.$$

Suppose $x \in \mathcal{N}(\lambda - A) \setminus \{0\}$. Set $u(t) = te^{\lambda t}x$, $0 \leq t \leq T$, then u satisfies (1.1) (note that $x \in \mathcal{D}(A)$) with $f(t) = e^{\lambda t}x$, $0 \leq t \leq T$. From (3.3) we obtain

$$\int_0^T \left|(1+t\lambda)e^{\lambda t}\right|^p \, dt \leq C_1^p \int_0^T \left|e^{\lambda t}\right|^p \, dt$$

Hence

$$\frac{\int_0^T |1+t\lambda|^p e^{p\mathcal{R}e(\lambda)t}dt}{\int_0^T e^{p\mathcal{R}e(\lambda)t}dt} \leq C_1^p.$$

Thus using (3.13) we have proved $\mathcal{R}e\,(\lambda) \leq \delta_2$. Hence $(\lambda - A)$ is injective for $\mathcal{R}e\,(\lambda) > \delta_2$.

The first part of condition (1.2) is then proved with $\delta = \max(\delta_1, \delta_2)$ and $(\lambda - A)^{-1} = S_\lambda$ defined by (3.12) for $\mathcal{R}e\,(\lambda) > \delta$. Let us prove the second part of condition (1.2). Using the definition (3.12) of S_λ, it follows

$$\|S_\lambda\| \leq \|R_\lambda\| \left\|(I+B_\lambda)^{-1}\right\|.$$

Thus, since $\|B_\lambda\| \leq 1/2$, it suffices to prove there exists C_2 independent of λ such that for $\mathcal{R}e\,(\lambda) > \delta$ we have

$$\|R_\lambda\| \leq \frac{C_2}{1+|\lambda|}. \tag{3.14}$$

Let $\lambda \in \mathbb{C}$ with $\mathcal{R}e\,(\lambda) > \delta$. Using the integration by part (3.7), definition (3.6) of R_λ and the estimate (3.9), it follows

$$\|R_\lambda x\|_X \leq \frac{\mathcal{R}e\,(\lambda)}{|\lambda|} \left(\int_0^T e^{-\mathcal{R}e(\lambda)t} \|\mathcal{M}(\phi_\lambda x)'(t)\|_X \, dt + e^{-\mathcal{R}e(\lambda)T} T^{1-1/p} \|\mathcal{M}(\phi_\lambda x)'\|_{L^p(I;X)} \right).$$

Using Hölder inequality and estimate (3.3), it holds

$$\|R_\lambda x\|_X \leq \frac{C_1 \mathcal{R}e\,(\lambda)}{|\lambda|} \left(\left(\frac{1}{\mathcal{R}e\,(\lambda)\, p'} \right)^{1/p'} + e^{-\mathcal{R}e(\lambda)T} T^{1/p'} \right) \|\phi_\lambda x\|_{L^p(I;X)},$$

where p' satisfies $1/p + 1/p' = 1$ (resp. $\|R_\lambda x\|_X \leq \frac{C_1 \mathcal{R}e(\lambda)}{|\lambda|} \|\phi_\lambda x\|_{L^1(I;X)}$ when $p = 1$). Using estimation (3.10) and relation (3.4) we obtain

$$\|R_\lambda\| \leq \frac{C_1}{|\lambda|} \left(\frac{e^p - 1}{p} \right)^{1/p} \left(\left(\frac{1}{p'} \right)^{1/p'} + T^{1/p'} \right), \tag{3.15}$$

(resp. $\|R_\lambda\| \leq \frac{C_1}{|\lambda|}(e-1)$ when $p=1$) . Moreover, same arguments lead to

$$\begin{aligned} \|R_\lambda\| &= \left\| \mathcal{R}e\,(\lambda) \int_0^T e^{-\lambda t} \mathcal{M}(\phi_\lambda x)(t)dt \right\|_X \\ &\leq C_1 \left(\frac{1}{p'}\right)^{1/p'} \left(\frac{1}{p}\right)^{1/p} (e^p-1)^{1/p}, \end{aligned} \tag{3.16}$$

(resp. $\|R_\lambda\| \leq C_1(e-1)$ when $p=1$). Finally (3.15) and (3.16) prove (3.14) for $1<p<\infty$ (resp. for $p=1$).

Case $I=[0,\infty[$

Let $\lambda \in \mathbb{C}$ such that $\mathcal{R}e\,(\lambda)>0$. Define $\phi_\lambda \in L^p(I,\mathbb{C})$ such that

$$\phi_\lambda(t) = \begin{cases} e^{\lambda t} & \text{for } 0 \leq t \leq \frac{1}{\mathcal{R}e(\lambda)}, \\ 0 & \text{for } t > \frac{1}{\mathcal{R}e(\lambda)}, \end{cases} \tag{3.17}$$

and for $x \in X$, set

$$R_\lambda x = \mathcal{R}e\,(\lambda) \int_0^\infty e^{-\lambda t} \mathcal{M}(\phi_\lambda x)(t)dt, \tag{3.18}$$

which is well defined since $\mathcal{M}(\phi_\lambda x) \in L^p(I;X)$. Moreover, the integration by part (3.7) holds for $0<T<\infty$. Thus, since $\mathcal{M}(\phi_\lambda x)' \in L^p(I;X)$ and using (3.9), one obtains

$$R_\lambda x = \frac{\mathcal{R}e\,(\lambda)}{\lambda} \int_0^\infty e^{-\lambda t} \mathcal{M}(\phi_\lambda x)'(t)dt. \tag{3.19}$$

In the present case $I=[0,\infty[$, we claim that R_λ is the inverse of $(\lambda-A)$ for $\mathcal{R}e\,(\lambda)>0$. Obviously R_λ is a right inverse of $(\lambda-A)$, *i.e.* for all $x \in X$ it follows

$$(\lambda-A)R_\lambda x = x.$$

The fact that R_λ is also a left inverse of $(\lambda-A)$ can be deduced as in [3] from the fact that for $x \in \mathcal{D}(A)$ it holds

$$R_\lambda x \in \mathcal{D}(A) \qquad \text{and} \qquad R_\lambda A x = A R_\lambda x. \tag{3.20}$$

We shall use the following Lemma.

Lemma 3.1. *Let E be a Banach space. Let $\mathcal{A}:\mathcal{D}(\mathcal{A}) \subset E \to E$ be a linear closed operator and $\mathcal{B} \in \mathcal{L}(E)$ with $0 \in \rho(\mathcal{B})$. Assume that $\forall u \in \mathcal{D}(\mathcal{A})$ it holds*

$$\mathcal{B}u \in \mathcal{D}(\mathcal{A}) \quad \textit{and} \quad \mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u. \tag{3.21}$$

Assume moroever that for all $f \in E$, there exists one and only one $u \in \mathcal{D}(\mathcal{A})$ such that

$$u = \mathcal{B}\mathcal{A}u + \mathcal{B}f. \tag{3.22}$$

Then the following holds. For all $f \in \mathcal{D}(\mathcal{A})$, if $v \in \mathcal{D}(\mathcal{A})$ satisfies

$$v = \mathcal{B}\mathcal{A}v + \mathcal{B}\mathcal{A}f, \tag{3.23}$$

then $v = \mathcal{A}u$.

Proof. Let $f \in \mathcal{D}(\mathcal{A})$, $u \in \mathcal{D}(\mathcal{A})$ satisfying (3.22) and $v \in \mathcal{D}(\mathcal{A})$ satisfying (3.23). From (3.21) and (3.22), it follows that $\mathcal{B}\mathcal{A}u \in \mathcal{D}(\mathcal{A})$ and we have

$$\mathcal{A}u = \mathcal{A}\mathcal{B}\mathcal{A}u + \mathcal{A}\mathcal{B}f.$$

Moreover, since $\mathcal{B} \in \mathcal{L}(E)$, it holds

$$\mathcal{B}\mathcal{A}u = \mathcal{B}\mathcal{A}\mathcal{B}\mathcal{A}u + \mathcal{B}\mathcal{A}\mathcal{B}f.$$

Using (3.23) and (3.21) we have

$$\mathcal{B}v = \mathcal{B}\mathcal{B}\mathcal{A}v + \mathcal{B}\mathcal{B}\mathcal{A}f = \mathcal{B}\mathcal{A}\mathcal{B}v + \mathcal{B}\mathcal{A}\mathcal{B}f.$$

By uniqueness of the solution of (3.22) we find

$$\mathcal{B}\mathcal{A}u = \mathcal{B}v,$$

and since $0 \in \rho(\mathcal{B})$, $\mathcal{A}u = v$. □

In order to prove (3.20) it suffices to show that for $x \in \mathcal{D}(A)$ it holds

$$\mathcal{M}(\phi_\lambda Ax) = A\mathcal{M}(\phi_\lambda x), \tag{3.24}$$

where ϕ_λ is defined by (3.17). Let $E = L^p(I,X)$. Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset E \to E$, where

$$\mathcal{D}(\mathcal{A}) = L^p(I, \mathcal{D}(A)) \cap W^{1,p}(I,X),$$

and for $u \in \mathcal{D}(\mathcal{A})$

$$\mathcal{A}u = (I + A)u.$$

Let $\mathcal{B} : E \to E$ defined for $u \in E$ by

$$\mathcal{B}u(t) = \int_0^t e^{-(t-s)}u(s)ds, \qquad t \in I. \tag{3.25}$$

Clearly, $0 \in \rho(\mathcal{B})$, $\mathcal{B} \in \mathcal{L}(E)$ and satisfies (3.21). Moreover, since $A : \mathcal{D}(A) \to X$ is closed then same holds for $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to E$. Let $x \in \mathcal{D}(A)$ and define $u = \mathcal{M}(\phi_\lambda x) \in \mathcal{D}(\mathcal{A})$. It follows that u satisfies

$$u + u' = (I + A)u + \phi_\lambda x, \text{ in } L^p(I;X), \qquad u(0) = 0,$$

and using the definiton of the operators $\mathcal{A}$ and $\mathcal{B}$ it follows that

$$u = \mathcal{B}\mathcal{A} + \mathcal{B}\phi_\lambda x.$$

The same reasoning also leads for $v = \mathcal{M}(\mathcal{A}\phi_\lambda x)$ to

$$v = \mathcal{B}\mathcal{A} + \mathcal{B}\mathcal{A}\phi_\lambda x.$$

Lemma 3.1 ensures that $v = \mathcal{A}u$. Thus $\mathcal{M}(I+A\phi_\lambda) = (I+A)\mathcal{M}(\phi_\lambda)$, which implies (3.24).

At this point, we have proved that for $\lambda \in \mathbb{C}$ with $\mathcal{R}e\,(\lambda) > 0$, $R_\lambda : X \to \mathcal{D}(A)$ is the inverse of $(\lambda - A)$. Let us now prove the second part of condition (1.2). Let $x \in X$ and $\lambda \in \mathbb{C}$ with $\mathcal{R}e\,(\lambda) > 0$. We have

$$\begin{aligned} \|R_\lambda x\|_X &= \left\| \frac{\mathcal{R}e\,(\lambda)}{\lambda} \int_0^\infty e^{-\lambda t}\mathcal{M}(\phi_\lambda x)'(t)dt \right\|_X \\ &\leq \frac{\mathcal{R}e\,(\lambda)}{|\lambda|} \int_0^\infty e^{-\mathcal{R}e(\lambda)t} \left\|\mathcal{M}(\phi_\lambda x)'(t)\right\|_X \, dt. \end{aligned}$$

Hölder inequality and estimate (3.3) lead to

$$\|R_\lambda x\|_X \leq C_1 \frac{\mathcal{R}e\,(\lambda)}{|\lambda|} \left(\frac{1}{\mathcal{R}e\,(\lambda)\,p'}\right)^{1/p'} \|\phi_\lambda y\|_{L^p(I,X)},$$

where $p' \in \mathbb{R}$ satisfies $1/p + 1/p' = 1$ (resp. $\|R_\lambda x\|_X \le C_1 \frac{\mathcal{R}e(\lambda)}{|\lambda|} \|\phi_\lambda y\|_{L^1(I,X)}$ when $p = 1$). A simple calculation shows that

$$\|\phi_\lambda x\|_{L^p(I,X)} = (\mathcal{R}e\,(\lambda))^{-1/p} \left(\frac{e^p - 1}{p}\right)^{1/p} \|x\|_X .$$

Hence, we obtain

$$\|R_\lambda x\|_X \le \frac{C_1}{|\lambda|} \left(\frac{1}{p'}\right)^{1/p'} \left(\frac{1}{p}\right)^{1/p} (e^p - 1)^{1/p} \|x\|_X , \tag{3.26}$$

(resp. $\|R_\lambda x\|_X \le \frac{C_1}{|\lambda|}(e-1)\|x\|_X$ when $p = 1$). Moreover, by using the same arguments we have

$$\begin{aligned} \|R_\lambda x\|_X &= \left\| \mathcal{R}e\,(\lambda) \int_0^\infty e^{-\lambda t} \mathcal{M}(\phi_\lambda x)(t) dt \right\|_X \\ &\le C_1 \left(\frac{1}{p'}\right)^{1/p'} \left(\frac{1}{p}\right)^{1/p} (e^p - 1)^{1/p} \|x\|_X . \end{aligned} \tag{3.27}$$

Combining relations (3.26) and (3.27) we find

$$\|R_\lambda\| \le \frac{C_3}{1 + |\lambda|},$$

where C_3 is a constant not depending on λ.

3.2. **Proof of Proposition 2.2.** First notice that the case where X is a real Banach space follows from the case where X is a complex Banach space. Indeed, let $A_\mathbb{C} = A + iA : \mathcal{D}(A_\mathbb{C}) \subset X_\mathbb{C} \to X_\mathbb{C}$, where

$$\mathcal{D}(A_\mathbb{C}) = \{z \in \mathbb{C}\ ; z = x + iy,\ x, y \in \mathcal{D}(A)\}$$

and

$$X_\mathbb{C} = \{z \in \mathbb{C}\ ; z = x + iy,\ x, y \in X\}.$$

Clearly $A_\mathbb{C}$ is a linear closed operator possessing MRp for some $p \in [1, \infty[$ on the interval $I = [0, \infty[$ (resp. the interval $I = [0, T]$ for some $T \in]0, \infty[$) and if $A_\mathbb{C} : \mathcal{D}(A_\mathbb{C}) \subset X_\mathbb{C} \to X_\mathbb{C}$ is densely defined the same holds for $A : \mathcal{D}(A) \subset X \to X$.

Thus, let us consider X be a complex Banach space and the operator B be the part of A in Y in the sense of [9, p.40], namely

$$\mathcal{D}(B) = \{x \in \mathcal{D}(A) : Ax \in Y\} \quad \text{and} \quad B : \mathcal{D}(B) \subset Y \to Y,\ Bx = Ax. \tag{3.28}$$

It is easy to verify that the operator B is closed in Y. Consider the equation for $f \in L^p(I; Y)$

$$u' = Bu + f, \quad \text{in } L^p(I; Y), \qquad u(0) = 0.$$

The uniqueness in $L^p(I; X)$ implies the uniqueness in $L^p(I; Y)$ and our assumption implies MRp holds for B in $L^p(I; Y)$. The estimate (1.2) ensured by Theorem 2.1 implies that the assumptions of Corollary 2 in [8] are satisfied and thus B is densely defined.

4. Some examples

4.1. **The one dimensional heat equation.** The following example illustrates how extra regularity in space can not always be obtained directly even in a simple case. Let $X = L^2(0,1)$, $\mathcal{D}(A) = H^2(0,1) \cap H^1_0(0,1)$ and $A : \mathcal{D}(A) \subset L^2(0,1) \to L^2(0,1)$ defined by $A = \frac{d^2}{dx^2}$. It is well known that A possesses the MR2 on $[0,\infty[$. Considering the reflexive Banach space $Y = H^1(0,1)$ and $B : \mathcal{D}(B) \subset Y \to Y$ defined by (3.28). It is easy to see that $\mathcal{D}(B) = H^3(0,1) \cap H^1_0(0,1)$ which is not dense in $H^1(0,1)$. As a consequence of Proposition 2.2, there is a $f \in L^2(0,\infty; H^1(0,1))$ such that $\frac{\partial^2 \mathcal{M}(f)}{\partial x^2}$ does not belongs to $L^2(0,\infty; H^1(0,1))$.

4.2. **The Stokes problem.** Let $1 < p,\ r < \infty$, $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded, connected open set with boundary $\partial\Omega$ of class $\mathcal{C}^\infty$, and let $T > 0$. Let $A_r = P_r \Delta : \mathcal{D}(A_r) \subset \mathcal{H}_r \to \mathcal{H}_r$ be the Stokes operator [4, 5, 6], where

$$\mathcal{D}(A_r) = \left\{ v \in W^{2,r}(\Omega;\mathbb{R}^d) \cap W^{1,r}_0(\Omega;\mathbb{R}^d) \mid \nabla \cdot v = 0 \right\},$$

$$\mathcal{H}_r = \left\{ v \in L^r(\Omega;\mathbb{R}^d)\ ;\ \nabla \cdot v = 0,\ v \cdot n = 0 \text{ on } \partial\Omega, \text{ hold weakly} \right\},$$

is provided with the norm of $L^r(\Omega;\mathbb{R}^d)$ and

$$P_r : L^r(\Omega;\mathbb{R}^d) \to \mathcal{H}_r \qquad 1 < r < \infty,$$

is the Helmoltz-Weyl projector. Solonnikov [13, Theorem 15, Section 17] proved that $A_r : \mathcal{D}(A_r) \subset \mathcal{H}_r \to \mathcal{H}_r$ possesses the MRp on the interval $[0,T]$. Let $Y = W^{1,r}(\Omega;\mathbb{R}^d) \cap \mathcal{H}_r$. We claim that there exists $f \in L^p(0,T;Y)$ such that $A_r\mathcal{M}(f)$ does not belongs to $L^p(I;Y)$. Indeed, suppose for contradiction that this is not the case and consider $B_r : \mathcal{D}(B_r) \subset Y \to Y$ be the part of A_r in Y defined by

$$\mathcal{D}(B_r) = \{x \in \mathcal{D}(A_r) : A_r x \in Y\} \quad \text{and} \quad B_r : \mathcal{D}(B_r) \subset Y \to Y,\ B_r x = A_r x.$$

The operator A_r is closed (see [7]), Y is reflexive (as a closed subspace of a reflexive Banach space) and is continuously imbedded in $\mathcal{H}_r$, thus Proposition 2.2 would imply that

$$\overline{\mathcal{D}(B_r)}^Y = Y. \tag{4.1}$$

It is known that

$$D(B_r) = W^{3,r}(\Omega;\mathbb{R}^d) \cap W^{1,r}_0(\Omega;\mathbb{R}^d) \cap \mathcal{H}_r \subset W^{1,r}_0(\Omega;\mathbb{R}^d) \cap \mathcal{H}_r,$$

see [12, Theorem 1.5.1 in Chapter III] or [5, Theorem 6.1]. Hence, since $W^{1,r}_0(\Omega;\mathbb{R}^d) \cap \mathcal{H}_r$ is closed in Y but $W^{1,r}_0(\Omega;\mathbb{R}^d) \cap \mathcal{H}_r \neq Y$ (consider for instance $u = constant \in Y$, with $u \not\equiv 0$), there is a contradiction with (4.1).

However, for $Y = W^{1,r}_0(\Omega) \cap \mathcal{H}_r$, since $W^{1,r}_0(\Omega) \cap \mathcal{H}_r = \mathcal{D}((-A_r)^{1/2})$, see [14, Theorem III.2.6], the part of A_r in Y possesses the MRp.

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