Ordinary Differential Equations
MATH 308 - 512
A. Bonito

February 21
Spring 2019

First Name: ____________ **Last Name:** ____________

Exam 1

- 75 minute individual exam;
- Answer the questions in the space provided. If you run out of space, continue onto the back of the page. Additional space is provided at the end;
- **Show and explain all work**;
- **Underline** the answer of each steps;
- The use of books, personal notes, **calculator**, cellphone, laptop, and communication with others is forbidden;
- By taking this exam, you agree to follow the university's code of academic integrity.

Ex 1	Ex 2	Ex 3	Ex 4	Total

Exercise 1 **25%**

Find the solution to

$$ty' + 2y = t^2 - t + 1, \qquad y(1) = \frac{1}{2}, \qquad t > 0.$$

Exercise 2 **25%**

Solve the differential equation

$$2x + y^2 + 2xyy' = 0.$$

Hint: You do not have to find explicit solutions but an algebraic relation only depending on x and $y(x)$.

Exercise 3 **25%**

Solve the following initial value problem:

$$2y'' + 7y' - 4y = 0, \qquad y(0) = 9/4, \quad y'(0) = 0.$$

Exercise 4 **25%**

Find the general solution to

$$y'' + y = \sin(x).$$

Ordinary Differential Equations
MATH 308 - 512
A. Bonito

March 12
Spring 2019

Exam 1: solutions

Exercise 1 **25%**

First, we rewrite the ODE as

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

and look for an integrating factor. The latter solves

$$\mu' = \frac{2}{t}\mu,$$

i.e.

$$\ln(\mu) = \ln(t^2) + C$$

for any constant C. Chosing $C = 1$ and solving for μ yields

$$\mu = t^2.$$

Hence the original ODE becomes

$$\frac{d}{dt}(t^2 y) = t^3 - t^2 + t$$

or, after integrating from $t = 1$ and using the inital condition $y(1) = \frac{1}{2}$

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}.$$

Exercise 2 **25%**

Since

$$\frac{\partial}{\partial x}(2xy) = \frac{\partial}{\partial y}(2x + y^2)$$

the ODE is exact. Hence, we need to find $\psi(x, y)$ such that

$$\frac{\partial}{\partial x}\psi(x, y) = 2x + y^2; \qquad \frac{\partial}{\partial y}\psi(x, y) = 2xy.$$

Integrating both relations yield

$$\psi(x, y) = x^2 + xy^2 + C_1(y); \qquad \psi(x, y) = xy^2 + C_2(x)$$

so that choosing $C_1(y) = 0$ and $C_2(x) = x^2$ guarantees that the two expressions match and

$$\psi(x, y) = x^2 + xy^2.$$

Therefore the solutions are given by

$$x(x + y^2) = c$$

for any constant c.

Exercise 3 **25%**

Guessing $y(x) = e^{\lambda x}$ we obtain the characteristic equation

$$2\lambda^2 + 7\lambda - 4 = 0.$$

Two find the roots of the above polynomial, we compute the discriminant

$$\Delta = 7^2 - 4 \times 2 \times (-4) = 81 = 9^2$$

so that the two distinct roots are given by

$$\lambda_1 = 1/2 \qquad \text{and} \qquad \lambda_2 = -4.$$

Therefore, two linearly independent solutions read

$$y_1(x) = e^{x/2} \qquad \text{and} \qquad y_2(x) = e^{-4x}.$$

The general solution is given by

$$y(x) = C_1 e^{x/2} + C_2 e^{-4x}$$

where C_1, C_2 are constants to be determined using the initial conditions:

$$C_1 + C_2 = \frac{9}{4} \qquad \text{and} \qquad \frac{1}{2}C_1 - 4C_2 = 0.$$

Solving the above system leads to

$$C_1 = 2 \qquad \text{and} \qquad C_2 = \frac{1}{4}$$

so that the desired solution is given by

$$y(x) = 2e^{x/2} + \frac{1}{4}e^{-4x}.$$

Exercise 4 **25%**

The characteristic equation reads

$$r^2 + 1 = 0,$$

and has roots $r = \pm i$. Therefore a fundamental set of solutions is given by

$$y_1(x) = \Re(e^{ix}) = \cos(x), \qquad y_2(x) = \Im(e^{ix}) = \sin(x).$$

To find a particular solution, we writing the ODE in the complex plan

$$z'' + z = e^{ix}.$$

A particular solution of the original ODE corresponds to the imaginary part of a particular solution of the above ODE. Using the change of variable $z(x) = u(x)e^{ix}$ yields

$$u'' + 2iu' = 1. \tag{1}$$

Therefore a particular solution $u_p(x)$ is sought in the form

$$u_p(x) = Ax^3 + Bx^2 + Cx$$

where A, B, and C are complex constants. Plugging u_p in (1) leads to

$$6Aix^2 + (6A + 4Bi)x + 2Ci + 2B = 1$$

so that

$$A = 0, \qquad B = 0, \qquad \text{and} \qquad C = -i/2.$$

Hence,

$$u_p(x) = -\frac{ix}{2}, \qquad \text{and} \qquad z_p(x) = -\frac{ix}{2}e^{ix}.$$

A particular solution to the original ODE is given by

$$y_p(x) = \mathrm{Im}(-\frac{ix}{2}e^{ix}) = -\frac{x}{2}\cos(x).$$