Ordinary Differential Equations
MATH 308H - 201
A. Bonito

February 21
Spring 2019

First Name: ____________________ **Last Name:** ____________________

Exam 1

- 75 minute individual exam;
- Answer the questions in the space provided. If you run out of space, continue onto the back of the page. Additional space is provided at the end;
- **Show and explain all work**;
- **Underline** the answer of each steps;
- The use of books, personal notes, **calculator**, cellphone, laptop, and communication with others is forbidden;
- By taking this exam, you agree to follow the university's code of academic integrity.

Ex 1	Ex 2	Ex 3	Ex 4	Total

Exercise 1 **25%**

Find the solution to

$$ty' + 2y = t^2 - t + 1, \qquad y(1) = \frac{1}{2}, \qquad t > 0.$$

Exercise 2 **25%**

Solve the differential equation
$$2x + y^2 + 2xyy' = 0.$$

Hint: You do not have to find explicit solutions but an algebraic relation only depending on x and $y(x)$.

<u>Exercise 3</u> **25%**

Solve the following initial value problem:

$$2y'' + 7y' - 4y = 0, \qquad y(0) = 9/4, \quad y'(0) = 0.$$

<u>Exercise 4</u> **25%**

We want to solve the following ODE

$$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t).$$

- Show that if $y_{p1}(t)$ satisfies

  $$y''_{p1} + p(t)y'_{p1} + q(t)y_{p1} = g_1(t)$$

  and $y_{p2}(t)$ satisfies

  $$y''_{p2} + p(t)y'_{p2} + q(t)y_{p2} = g_2(t),$$

  then $y_p(t) = y_{p1}(t) + y_{p2}(t)$ solve the original ODE.

- Use the above result to find the *general solution* of

  $$y'' + y = \cos(2t) + \sin(t).$$

Ordinary Differential Equations
MATH 308H - 201
A. Bonito

February 21
Spring 2019

Exam 1: solutions

Exercise 1 25%

First, we rewrite the ODE as

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

and look for an integrating factor. The latter solves

$$\mu' = \frac{2}{t}\mu,$$

i.e.

$$\ln(\mu) = \ln(t^2) + C$$

for any constant C. Chosing $C = 1$ and solving for μ yields

$$\mu = t^2.$$

Hence the original ODE becomes

$$\frac{d}{dt}(t^2 y) = t^3 - t^2 + t$$

or, after integrating from $t = 1$ and using the inital condition $y(1) = \frac{1}{2}$

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}.$$

Exercise 2 25%

Since

$$\frac{\partial}{\partial x}(2xy) = \frac{\partial}{\partial y}(2x + y^2)$$

the ODE is exact. Hence, we need to find $\psi(x, y)$ such that

$$\frac{\partial}{\partial x}\psi(x, y) = 2x + y^2; \qquad \frac{\partial}{\partial y}\psi(x, y) = 2xy.$$

Integrating both relations yield

$$\psi(x, y) = x^2 + xy^2 + C_1(y); \qquad \psi(x, y) = xy^2 + C_2(x)$$

so that choosing $C_1(y) = 0$ and $C_2(x) = x^2$ guarantees that the two expressions match and

$$\psi(x, y) = x^2 + xy^2.$$

Therefore the solutions are given by

$$x(x + y^2) = c$$

for any constant c.

Exercise 3 **25%**

Guessing $y(x) = e^{\lambda x}$ we obtain the characteristic equation

$$2\lambda^2 + 7\lambda - 4 = 0.$$

Two find the roots of the above polynomial, we compute the discriminant

$$\Delta = 7^2 - 4 \times 2 \times (-4) = 81 = 9^2$$

so that the two distinct roots are given by

$$\lambda_1 = 1/2 \qquad \text{and} \qquad \lambda_2 = -4.$$

Therefore, two linearly independent solutions read

$$y_1(x) = e^{x/2} \qquad \text{and} \qquad y_2(x) = e^{-4x}.$$

The general solution is given by

$$y(x) = C_1 e^{x/2} + C_2 e^{-4x}$$

where C_1, C_2 are constants to be determined using the initial conditions:

$$C_1 + C_2 = \frac{9}{4} \qquad \text{and} \qquad \frac{1}{2}C_1 - 4C_2 = 0.$$

Solving the above system leads to

$$C_1 = 2 \qquad \text{and} \qquad C_2 = \frac{1}{4}$$

so that the desired solution is given by

$$y(x) = 2e^{x/2} + \frac{1}{4}e^{-4x}.$$

Exercise 4 **25%**

- We use the notation
  $$L[y](t) := y'' + p(t)y' + q(t)y.$$
  We have that $L[y_{p1}] = g_1$ and $L[y_{p2}] = g_2$, so that the linearity of L implies
  $$L[y_p] = L[y_{p1} + y_{p2}] = L[y_{p1}] + L[y_{p2}] = g_1 + g_2.$$
  This proves that y_p satisfies the original ODE as claimed.
- - The homogeneous equation
    $$y'' + y = 0$$
    has $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ as two independent solutions.
  - We now try to find a particular solution to
    $$y'' + y = \cos(2t).$$
    In this aim, we consider the complex-valued ODE
    $$z'' + z = e^{2it}$$
    which becomes using $z(t) = u(t)e^{2it}$
    $$u'' + 4iu' - 3u = 1.$$

Then, we seek a particular solution to the above ODE of the form

$$u_{p1} = A$$

and find $u_p = -1/3$. In turn, we deduce

$$z_{p1}(t) = u_{p1}(t)e^{2it} = -1/3e^{2it}$$

and

$$y_{p1}(t) = \Re(z_{p1}(t)) = -1/3\cos(2t).$$

– We proceed similarly to construct a particular solution to

$$y'' + y = \sin(t).$$

We consider the complex-valued ODE

$$z'' + z = e^{it}$$

which becomes using $z(t) = u(t)e^{it}$

$$u'' + 2iu' = 1.$$

Then, we seek a particular solution to the above ODE of the form

$$u_{p2} = At$$

and find $u_{p2} = t/(2i) = -it/2$. In turn, we deduce

$$z_{p2}(t) = -it/3e^{it}$$

and

$$y_{p2}(t) = \Im(z_{p2}(t)) = -t/2\cos(t).$$

– Gathering the above result, we obtain that the general solution reads

$$y(t) = C_1\cos(t) + C_2\sin(t) - \frac{1}{3}\cos(2t) - \frac{t}{2}\cos(t),$$

for any constants C_1 and C_2.