A face penalty method for the three fields Stokes equation arising from Oldroyd-B viscoelastic flows

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Summary. We apply the continuous interior penalty method to the three fields Stokes problem. We prove an inf-sup condition for the proposed method leading to optimal a priori error estimates for smooth exact solutions. Moreover we propose an iterative algorithm for the separate solution of the velocities and the pressures on the one hand and the extra-stress on the other. The stability of the iterative algorithm is established.

1 Introduction

Numerical modeling of viscoelastic flows is of great importance for complex engineering applications involving foodstuff, blood, paints or adhesives. When considering viscoelastic flows, the velocity, pressure and stress must satisfy the mass and momentum equation, supplemented with a constitutive equation involving the velocity and stress. The simplest model is the so-called Oldroyd-B constitutive relation which can be derived from the kinetic theory of polymer dilute solutions, see for instance [1, 12]. The unknowns of the Oldroyd-B model are the velocity u, the pressure p, the extra-stress σ (the non Newtonian part of the stress due to polymer chains for instance) which must satisfy :

$$\rho\frac{\partial u}{\partial t} + \rho(u\cdot\nabla)u - 2\eta_s\ \nabla\cdot\epsilon(u) + \nabla p - \nabla\cdot\sigma = f, \qquad \nabla\cdot u = 0,$$
$$\sigma + \lambda\Big(\frac{\partial\sigma}{\partial t} + (u\cdot\nabla)\sigma - (\nabla u)\sigma - \sigma(\nabla u)^T\Big) - 2\eta_p\epsilon(u) = 0.$$

Here ρ is the density, f a force term, η_s and η_p are the solvent and polymer viscosities, λ the relaxation time, $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ the strain rate tensor, $(\nabla u)\sigma$ denotes the matrix-matrix product between ∇u and σ.

When solving viscoelastic flows with finite element methods, the following points should be addressed:

i) the presence of the quadratic term $(\nabla u)\sigma + \sigma(\nabla u)^T$ which prevents a priori estimates to be obtained and therefore existence to be proved for any data;
ii) the presence of a convective term $(u \cdot \nabla)\sigma$ which requires the use of numerical schemes suited to transport dominated problems;
iii) the finite element spaces used to approximate the velocity, the pressure and the extra-stress fields can not be chosen arbitrarily, an inf-sup condition has to be satisfied [10, 11, 14, 15];
iv) the case $\eta_s = 0$ which requires either a compatibility condition between the finite element spaces for u and σ or the use of adequate stabilization procedures.

In this paper, we will focus on points *iii*), *iv*) and propose an alternative to the EVSS method [2, 9, 13, 16]. We will consider the stationary linear problem, say $\rho = 0,\ \lambda = 0$ and $\eta_p = 0$. This is

$$\begin{aligned} -\nabla \cdot \sigma + \nabla p = f \quad & \text{in } \Omega, \qquad & \nabla \cdot u = 0 \quad & \text{in } \Omega, \\ \sigma - 2\eta_p \epsilon(u) = 0 \quad & \text{in } \Omega, \qquad & u = 0 \quad & \text{on } \partial\Omega. \end{aligned} \tag{1}$$

There are a vast number of finite element spaces satisfying the inf-sup condition for the pressure velocity coupling. For the extra-stress however the situation is much less clear even though the relation is trivial in the continuous case. A number of different stabilized methods have therefore been proposed in order to get a stable approximation using equal order approximation for the velocities and the extra-stress, see [7, 10, 11, 14, 15]. In this paper we propose to extend the recently introduced continuous interior penalty method (CIP), or Edge stabilization method, of [3, 5, 8] to the case of the three fields Stokes equation. The case of the Stokes-Darcy problem was treated in [6] and the generalized Oseen's problem was considered in [4]. Advantages of the present method is the unified way of stabilizing different phenomena: for each case the jump in the gradient, or the jump of the non-symmetric operator in question, over element faces is penalized in the L^2 sense. This yields a method with optimal convergence properties for all polynomial degrees that is completely flexible with respect to time-stepping schemes and which does not give rise to any artificial boundary conditions. The price to pay are some added couplings in the stiffness matrix since the penalty operator couples all the degrees of freedom in adjacent elements. However note that in the case of the three fields Stokes equation the stabilization only acts on the pressure and the velocities, hence keeping down the additional memory cost to a moderate factor of 1.5 in two and three space dimensions compared to (the unstable) standard Galerkin formulation. For more complex cases as those encountered in viscoelastic flows where also convection of the extra-stress has to be stabilized, on the other hand one must expect to pay a factor two in the case of two space dimensions and a factor three in the case of three space dimensions due to the fact that the stabilization has to act also on the extra-stress.

2 A finite element formulation

Let Ω be a bounded, polygonal (respectively polyhedral) and connected open set of $\mathbb{R}^d$, $d \geq 2$. We will use the notation $(.,.)$ for the $L^2(\Omega)$ scalar product for scalars, vectors, tensors and $\langle u, v\rangle_x = \int_x u \cdot v \, \mathrm{d}s$. Let $\mathcal{T}_h$ be a conforming triangulation of Ω, $\mathcal{E}$ be the set of interior faces in $\mathcal{T}_h$ and $[x]_e$ be the jump of the quantity x on the face e. We shall henceforth assume the local quasiuniformity of the mesh, we assume there exists a constant $C_q > 0$ such that for all $\mathcal{T}_h$ and all vertices $S_i \in \mathcal{T}_h$, we have

$$\max_{e \in \Omega_i} h_e \leq C_q \min_{e \in \Omega_i} h_e. \tag{2}$$

Here Ω_i denotes the macro-element formed by elements $K \in \mathcal{T}_h$ sharing vertex S_i. Let $\mathbf{W_h} = \{w_h : w_h|_K \in \mathbb{P}_k(K)\}$ and $\mathbf{V_h} = \mathbf{W_h} \cap H^1(\Omega)$ We introduce the interior penalty operators

$$j_p(p_h, q_h) = \gamma_p \sum_{e \in \mathcal{E}} \left\langle \frac{h^3}{\eta_p} [\nabla p_h], [\nabla q_h] \right\rangle_e \tag{3}$$

and

$$j_u(u_h, v_h) = \gamma_u \sum_{e \in \mathcal{E}} \langle 2\eta_p h_e [\nabla u_h], [\nabla v_h]\rangle_e + \gamma_b \left\langle \frac{\eta_p}{h} u_h, v_h \right\rangle_{\partial\Omega}, \tag{4}$$

where γ_p, γ_u and γ_b are positive constants to be determined. Moreover let us introduce the bilinear forms

$$a(\sigma_h, v_h) = (\sigma_h, \epsilon(v_h)) - \langle \sigma_h \cdot n, v_h\rangle_{\partial\Omega} \tag{5}$$

$$b(p_h, v_h) = -(p_h, \nabla \cdot v_h) + \langle p_h, v_h \cdot n\rangle_{\partial\Omega}. \tag{6}$$

The method we propose then takes the form, find $(u_h, \sigma_h, p_h) \in \mathbf{V_h^d} \times \mathbf{V_h^{d\times d}} \times \mathbf{V_h}$ such that

$$\begin{aligned} a(\sigma_h, v_h) + b(p_h, v_h) - b(q_h, u_h) - a(\tau_h, u_h) + \left(\frac{1}{2\eta_p}\sigma_h, \tau_h\right) + j_p(p_h, q_h) \\ + j_u(u_h, v_h) = (f, v_h), \quad \text{for all } (v_h, \tau_h, q_h) \in \mathbf{V_h^d} \times \mathbf{V_h^{d\times d}} \times \mathbf{V_h}. \end{aligned} \tag{7}$$

For ease of notation we will also consider the following compact form, introducing the variables $U_h = (u_h, \sigma_h, p_h)$ and $V_h = (v_h, q_h, \tau_h)$ and the finite element space $\mathbf{X_h} = \mathbf{V_h^d} \times \mathbf{V_h^{d\times d}} \times \mathbf{V_h}$

$$A(U_h, V_h) = a(\sigma_h, v_h) + b(p_h, v_h) - b(q_h, u_h) - a(\tau_h, u_h) + \left(\frac{1}{2\eta_p}\sigma_h, \tau_h\right)$$

and

$$J(U_h, V_h) = j_p(p_h, q_h) + j_u(u_h, v_h), \qquad F(V_h) = (f, v_h)$$

yielding the compact formulation find $U_h \in \mathbf{X_h}$ such that

$$A(U_h, V_h) + J(U_h, V_h) = F(V_h) \quad \text{for all } V_h \in \mathbf{X_h}.$$

Clearly, since $j_u(U, V_h) = 0$, this formulation is strongly consistent for $(u, \sigma, p) \in H^2(\Omega)^d \times H^1(\Omega)^{d\times d} \times H^2(\Omega)$.

3 The inf-sup condition

For the numerical scheme (7) to be well posed it is essential that there holds an inf-sup condition uniformly in the mesh size h.

Consider the triple norm given by

$$|||U|||^2 = |||(u,\sigma,p)|||^2 = \frac{1}{2\eta_p}\|\sigma\|^2_{0,\Omega} + 2\eta_p\|\epsilon(u)\|^2_{0,\Omega} + \frac{1}{2\eta_p}\|p\|^2_{0,\Omega}$$

and the following corresponding discrete triple norm

$$|||U_h|||^2_h = \frac{1}{2\eta_p}\|\sigma_h\|^2_{0,\Omega} + 2\eta_p\|\epsilon(u_h)\|^2_{0,\Omega} + \frac{1}{2\eta_p}\|p_h\|^2_{0,\Omega} + j_u(u_h,u_h) + j_p(p_h,p_h).$$

Then the following inf-sup condition is satisfied for the discrete form.

Theorem 1. *Assume that the mesh satisfies the local quasiuniformity condition (2). Then for the formulation (7) there holds for all* $U_h \in \mathbf{V_h^d} \times \mathbf{V_h^{d\times d}} \times \mathbf{V_h}$

$$|||U_h|||_h \leq \sup_{V_h \neq 0} \frac{A(U_h,V_h) + J(U_h,V_h)}{|||V_h|||_h}.$$

4 A priori error estimates

A priori error estimates follow from the previously proved inf-sup condition together with the proper continuities of the bilinear forms and the approximation properties of the finite element space.

Theorem 2. *Assume that the mesh satisfies the local quasiuniformity condition (2) and that all the components of* $U := (u,p,\sigma)$ *are in* $H^{k+1}(\Omega)$ *then there holds*

$$|||U - U_h|||_h \leq Ch^k \left({\eta_p}^{1/2}\|u\|_{k+1,\Omega} + \frac{1}{{\eta_p}^{1/2}} h\|\sigma\|_{k+1,\Omega} + \frac{1}{{\eta_p}^{1/2}} h\|p\|_{k+1,\Omega} \right)$$

where $k \geq 1$ *is the polynomial order of the finite element spaces.*

Theorem 3. *Assume that the mesh satisfies the local quasiuniformity condition (2) and that* $U := (u,p,\sigma) \in H^2(\Omega)^d \times H^1(\Omega) \times H^1(\Omega)^{d\times d}$ *then there holds*

$$|||U - U_h||| \leq Ch \left({\eta_p}^{1/2}\|u\|_{2,\Omega} + \frac{1}{{\eta_p}^{1/2}}\|\sigma\|_{1,\Omega} + \frac{1}{{\eta_p}^{1/2}}\|p\|_{1,\Omega} \right).$$

5 A stable iterative algorithm

A similar iterative method as in [9, 10] and [2] is presented. The aim of such an algorithm is to de-couple the velocity-pressure computation from the extra stress computation for solving (7).

Each subiteration of the iterative algorithm consists of two steps. Firstly, using the Navier-Stokes equation, the new approximation (u_h^n, p_h^n) is determined using the value of the extra stress at previous step σ_h^{n-1}. Then the new approximation σ_h^n is computed by using the constitutive relation using the value u_h^n. More precisely, assuming that $(u_h^{n-1}, \sigma_h^{n-1}, p_h^{n-1})$ is the known approximation of (u_h, σ_h, p_h) after $n-1$ steps. The first step consists on finding (u_h^n, p_h^n) such that

$$(A+J)((u_h^n, \sigma_h^{n-1}, p_h^n), (v_h, 0, q_h)) + K(u_h^n, u_h^{n-1}, v_h) = (f_h, v_h) \qquad \forall (v_h, p_h) \in \mathbf{V_h^d} \times \mathbf{V_h}, \tag{8}$$

and in the second step we find σ_h^n such that

$$A((u_h^n, \sigma_h^n, p_h^n), (0, \tau_h, 0)) = 0 \quad \forall \tau_h \in \mathbf{V_h^{d\times d}}. \tag{9}$$

Hereabove $K : H^1(\Omega)^d \times H^1(\Omega)^d \times H^1(\Omega)^d \to \mathrm{I\!R}$ is defined for all $u_1, u_2, v \in H^1(\Omega)^d$ by $K(u_1, u_2, v) := 2\eta_p\,(\epsilon(u_1 - u_2), \epsilon(v))$. The term $K(u_h^n, u_h^{n-1}, v)$, which vanishes at continuous level, has been added to (7) in (8) in order to obtain a stable iterative algorithm.

Lemma 1 (Stability). *Assume that the mesh satisfies the local quasiuniformity condition (2). Let $(u_h^n, \sigma_h^n, p_h^n)$ be the solution of (8),(9) with $f = 0$. There exists γ_u^* and $\gamma_b^* > 0$ independent of h such that for all $\gamma_p > 0$, $\gamma_u \geq \gamma_u^*$ and $\gamma_b \geq \gamma_b^*$, there exists a constant $C > 0$ such that*

$$\eta_p \left\|\epsilon(u_h^n)\right\|^2 + \frac{3}{16\eta_p}\left\|\sigma_h^n\right\|^2 + C\left\|p_h^n\right\|^2 \leq \eta_p \left\|\epsilon(u_h^{n-1})\right\|^2 + \frac{3}{16\eta_p}\left\|\sigma_h^{n-1}\right\|^2.$$

6 Preliminary numerical results

For all the numerical experiments we choose $\eta_p = 1$ $[Pa.s]$ and consider $\mathbb{P}_1$ approximations for the velocity, the pressure and the stress.

6.1 Poiseuille flow

Consider a rectangular pipe of dimensions $[0, L_1] \times [0, L_2]$ in the $x-y$ directions, where $L_1 = 0.15$ $[m]$ and $L_2 = 0.03$ $[m]$. The boundary conditions are the following. On the top and bottom sides ($y = 0$ and $y = L_2$), no-slip boundary conditions apply. On the inlet ($x = 0$) the velocity and the extra-stress are given by

$$\mathbf{u}(0,y) = \begin{pmatrix} u_x(y) \\ 0 \end{pmatrix}, \qquad \sigma(0,y) = \begin{pmatrix} 0 & \sigma_{xy}(y) \\ \sigma_{xy}(y) & 0 \end{pmatrix}, \tag{10}$$

with $u_x(y) = (L_2 + y)(L_2 - y)$ and $\sigma_{xy}(y) = -2\eta_p y$. On the outlet $(x = L_1)$ the velocity and the pressure are given by

$$\mathbf{u}(L_1,y) = \begin{pmatrix} u_x(y) \\ 0 \end{pmatrix}, \qquad p(L_1,y) \equiv 0.$$

The velocity and extra-stress must satisfy (10) in the whole pipe. Three unstructured meshes are used to check convergence (coarse: 50×10, intermediate: 100×20, fine: 200×40). In Fig. 1, the error in the L^2 norm of the velocity u, the pressure p and extra-stress components σ_{xx}, σ_{xy} is plotted versus the mesh size. Clearly order one convergence rate is observed for the pressure (in fact superconvergence is observed for the pressure) and the extra-stress whilst the convergence rate of the velocity is order two, this being consistent with theoretical predictions.

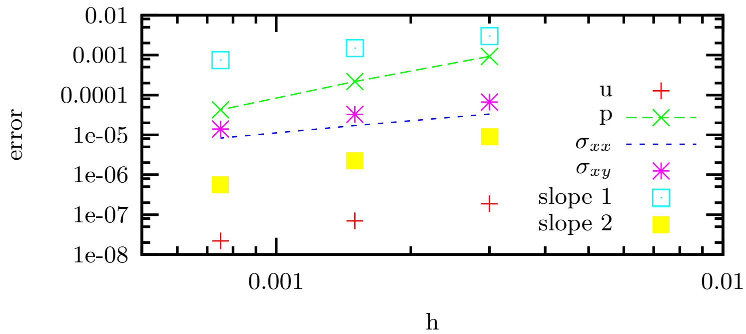

Fig. 1. Poiseuille flow: convergence orders.

6.2 The 4:1 planar contraction

Numerical results of computation in the 4:1 abrupt contraction flow case are presented and comparison with the EVSS (see for instance [2, 10]) method is performed. This test case underlines the importance of the stabilization of the constitutive equation. The symmetry of the geometry is used to reduce the computational domain by half, as shown in Fig. 2 (left). Zero Dirichlet boundary conditions are imposed on the walls, the Poiseuille velocity profile $u_x(y) = 64(L_0-y)(L_0+y)$ is imposed at the inlet with $L_0 = 0.025[m]$, natural boundary conditions on the symmetry axis and at the outlet of the domain.

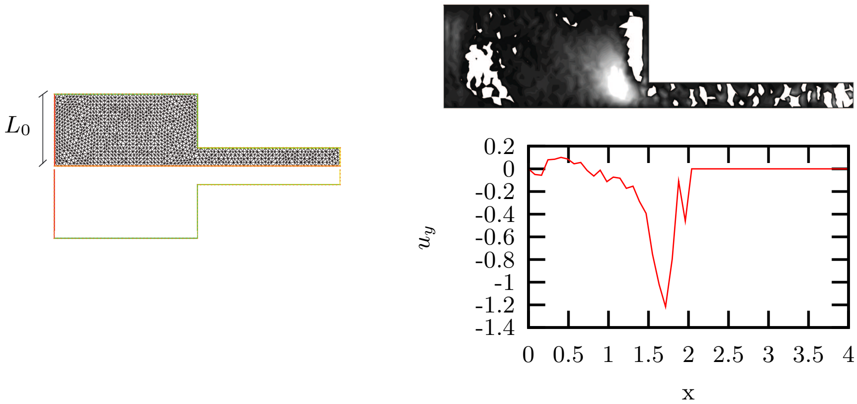

Fig. 2. (left) Computational domain for the 4:1 contraction, (right-top) 20 isovalues of the GLS method only for the pressure from -0.9 (black) to 0.06 (white) and (right-bottom) profile of $u_y(x, 0.025)$.

The results applying only GLS stabilization for the pressure are shown in Fig. 2 (right). Similar results obtained using the EVSS method (see [2] for a detailed description) and the CIP formulation are presented in Fig. 3.

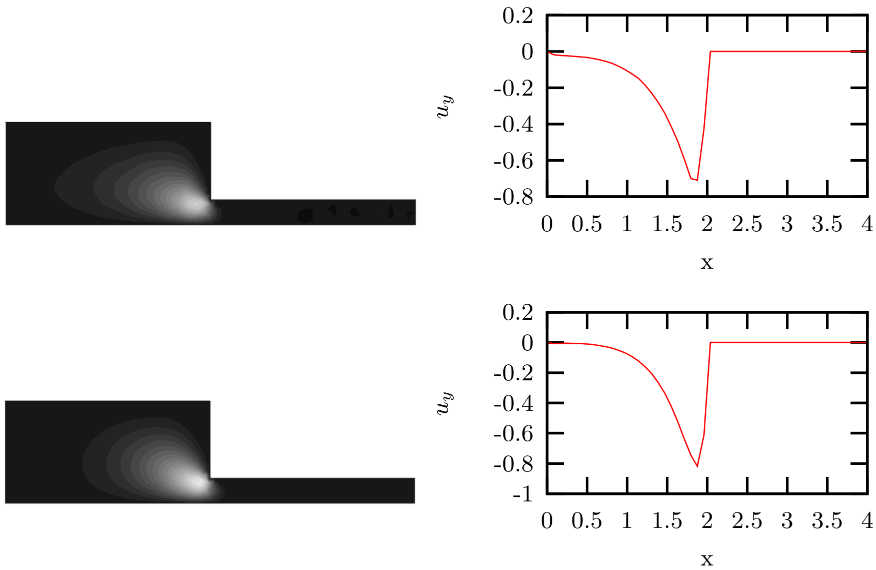

Fig. 3. Left column: 20 isovalues from -0.9 (black) to 0.06 (white), right column: profile of $u_y(x, 0.025)$ (top: EVSS, bottom: CIP).

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