A CONTINUOUS INTERIOR PENALTY METHOD FOR VISCOELASTIC FLOWS

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Abstract. In this paper we consider a finite element discretization of the Oldroyd-B model of viscoelastic flows. The method uses standard continuous polynomial finite element spaces for velocities, pressures and stresses. Inf-sup stability and stability for convection-dominated flows are obtained by adding a term penalizing the jump of the solution gradient over element faces. To increase robustness when the Deborah number is high we add a non-linear artificial viscosity of shock-capturing type. The method is analyzed on a linear model problem, optimal *a priori* error estimates are proven that are independent of the solvent viscosity η_s. Finally we demonstrate the performance of the method on some known benchmark cases.

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1. Introduction. The numerical computation of viscoelastic flows is a challenging problem that has received increasing attention during the last twenty years. The system takes the form of the incompressible Navier-Stokes equations coupled to a nonlinear hyperbolic equation for the extra stress. Several difficulties have to be handled simultaneously by the numerical method. First the inf-sup stability condition for the velocity/pressure coupling of the Navier-Stokes equation, second there is an inf-sup condition due to the coupling between the equation for the extra stress and the Stokes' type system of the Navier-Stokes' equations. These difficulties can be studied by considering the so called three field Stokes' equations [8, 14, 31, 41, 39, 40, 43, 3, 9].

Third the full viscoelastic system also features a transport term and a nonlinear coupling term in the equation for the stresses, the strength of this term is measured by a parameter that can be expressed in the non dimensional Deborah number. This results in the need to stabilize also the transport term and possibly account for instabilities induced by the nonlinear terms.

The existence of a slow steady viscoelastic flow has been proven by Renardy [38] in Hilbert spaces.

Picasso and Rappaz analyzed the stationary nonlinear case (without transport term in the extra stress equation), they proved *a priori* and a posteriori error estimates for the finite element approximation error provided the Deborah number is small. The extension to the time dependent problem has been treated in [6] and to a stochastic model in [5, 4]. Another theoretical approach to numerical methods for viscoelastic flows was proposed by Lozinski and Owens [30]. They proved an energy estimate and introduced a numerical model guaranteeing the positive definiteness of the stress tensor. Lee and Xu [29] also emphasized the importance of keeping the stress tensor positive definite during the computations and gave some guidelines on how to construct a finite element method that would satisfy this constraint.

There is a huge literature on finite element methods for viscoelastic flow, we

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refer for to the monograph by Owens and Philips for an overview [35]. For works on methods using mixed finite element methods we refer to the review article of Baaijens [1] and references therein and the more recent works by Ervin and coworkers [19, 20]. For methods using stabilized finite elements in a framework similar to ours we refer to the work of Behr and coworkers [32] and the work of Codina [15]. Finally, note that fully three dimensional free surface Oldroyd-B flows were successfully computed by Bonito et al. [7] using a Galerkin-Least-Square (GLS)/Elastic Viscous Split Stress procedure (EVSS).

In this paper we propose a method where all the instabilities are treated in a uniform fashion, by adding a term stabilizing the gradient jump over element faces. This type of method is a generalization of the interior penalty method for continuous approximation spaces proposed by Douglas and Dupont [17] for convection–diffusion problems. The analysis for high Peclet number problems was given by Burman and Hansbo in [12] and inf-sup stability for Stokes' systems using equal order interpolation was proven in [13]. Here we prove estimates for the interior penalty method applied to a linear model problem of viscoelastic flow showing that the discretization is stable and has quasioptimal convergence properties.

An outline of the paper is as follows: first we introduce the linear model problem and comment on the wellposedness. In section 2 we give the finite element formulation and in section 3 we prove an inf-sup condition. This leads to optimal a priori error estimates that are given in section 4. In section 5 we discuss an iterative solution algorithm decoupling the velocity/pressure computation from the stress computation and we prove that the iterations converge. In section 6 we introduce an additional nonlinear stabilization term drawing on earlier work on nonlinear hyperbolic conservation laws and nonlinear artificial viscosity. In section 7 finally we give some numerical examples demonstrating optimal convergence for smooth solutions in the nonlinear case, the effect of linear stabilization in the presence of singularities for the linear problem and finally the effect of nonlinear stabilization in a nonlinear case.

1.1. The Oldroyd-B model of viscoelastic flow and a linear model problem. Given η_s, $\eta_p > 0$ the solvent and polymer viscosities respectively and $\lambda > 0$ the relaxation time parameter of the fluid - the time for the stress to return to zero under constant-strain condition. Denoting by u the velocities, p the pressure and σ the extra stress-tensor, the Oldroyd-B model for viscoelastic flows takes the form

$$\begin{aligned} -\nabla\cdot(2\eta_s\epsilon(u)+\sigma)+\nabla p &= f \quad \text{in } \Omega, \\ \nabla\cdot u &= 0 \quad \text{in } \Omega, \\ \lambda((u\cdot\nabla\ )\ \sigma - g(\sigma,u)) + \sigma - 2\eta_p\epsilon(u) &= 0 \quad \text{in } \Omega, \\ u &= \beta \quad \text{on } \partial\Omega, \\ \sigma &= 0 \quad \text{on } \partial\Omega_{\text{in}}, \end{aligned} \tag{1.1}$$

with

$$g(\sigma,u) = \nabla u\ \sigma + \sigma(\nabla u)^T.$$

Here $\partial\Omega_{\text{in}} := \{x\in\partial\Omega : \beta\cdot\nu < 0\}$ and ν denotes the unit outer normal vector of Ω. Note that the term $(u\cdot\nabla\ )\ \sigma - g(\sigma,u)$ is a (stationary) Oldroyd type derivative, which is frame invariant. Refer to [2] for more precisions and other models.

The existence of a viscoelastic flow $(u, \sigma) \in H^3 \times H^2$ satisfying (1.1) with $\partial\Omega \in C^{1,1}$ has been proved by Renardy [37] in Hilbert spaces provided the data f is small enough in H^1. The extension of this result to Banach spaces has been treated by Fernández-Cara et al., see for instance [22].

The linear problem that we propose for the analysis is obtained from the stationary Oldroyd-B equation by dropping the non-linear terms, i.e. taking formally $g(\sigma, u) = 0$ and replacing $(u \cdot \nabla)\sigma$ by $(\beta \cdot \nabla)\sigma$ above, where the assumption on the vector field β will be specified later. This results in a system that resembles the known three field Stokes system, but with an advective term in the equation for the stresses. The equation for the stress tensor is therefore a pure transport equation. The analysis of the method will be performed on a linear model problem of the following form

$$
\begin{aligned}
-\nabla \cdot (2\eta_s \epsilon(u) + \sigma) + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
\lambda \left(\beta \cdot \nabla\ \right)\ \sigma + \sigma - 2\eta_p \epsilon(u) &= 0 \quad \text{in } \Omega, \\
u &= \beta \quad \text{on } \partial\Omega, \\
\sigma &= 0 \quad \text{on } \partial\Omega_{\text{in}}.
\end{aligned}
\tag{1.2}
$$

where $\beta \in H^1(\Omega)$, $\nabla \cdot \beta = 0$, and $\|\nabla\beta\|_{L^\infty(\Omega)} \leq M < \infty$. Existence and uniqueness for this problem was proved in [18] under sufficient conditions on M and λ. In this paper we will assume we are working in an Ω that is bounded convex polygonal and we have (at least) the additional regularity $\beta \in \mathcal{C}^1(\overline{\Omega})$,

$$
\|D^1\sigma\|_{L^2(\Omega)} + \|D^1 p\|_{L^2(\Omega)} + \|D^2 u\|_{L^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|\beta\|_{\mathcal{C}^1(\overline{\Omega})} \right), \tag{1.3}
$$

where C is a constant only depending on Ω, η_s, η_p and λ. A proof of this additional regularity in a smooth domain with small data can be found in [37, 22] for the more general problem (1.1).

2. A finite element formulation. We will use the notations $(u, v) = \int_\Omega u \cdot v$ and $\langle u, v \rangle_S = \int_S u \cdot v$, with S the boundary of the domain or a face of an element.

Let $\mathcal{T}_h$ denotes a conforming triangulation of Ω and let $\mathcal{E}_h$ denotes the set of interior faces in $\mathcal{T}_h$. We shall henceforth assume that the sequence of meshes $\{\mathcal{T}_h\}_{0<h<1}$ is quasiuniform.

Let $\mathbb{W}_h = \{w_h : w_h|_K \in P_k(K)\}$ and $\mathbb{V}_h = \mathbb{W}_h \cap H^1(\Omega)$. Let π_h be indifferently the L^2 projection onto $\mathbb{V}_h$, $\mathbb{V}_h^d$ or $\mathbb{V}_h^{d \times d}$. We introduce the interior penalty operators

$$
j_p(p_h, q_h) = \gamma_p \sum_{e \in \mathcal{E}_h} \left\langle \frac{h^3}{2\eta_p} [\nabla p_h], [\nabla q_h] \right\rangle_e, \tag{2.1}
$$

$$
j_u(u_h, v_h) = \gamma_u \sum_{e \in \mathcal{E}_h} \left\langle 2\eta_p h [\nabla u_h], [\nabla v_h] \right\rangle_e + \gamma_b \left\langle \frac{\eta_s + \eta_p}{h} u_h, v_h \right\rangle_{\partial\Omega}, \tag{2.2}
$$

and

$$
j_\sigma(\sigma_h, \tau_h) = \gamma_s \sum_{e \in \mathcal{E}_h} \left\langle h^2 \|\beta \cdot \nu\|_{L^\infty(e)} [\nabla \sigma_h], [\nabla \tau_h] \right\rangle_e, \tag{2.3}
$$

where γ_p, γ_u, γ_b,γ_s are positive constants and $[v]|_e$ denotes the jump of the quantity v over e. Moreover let us introduce the bilinear forms

$$a(u_h, v_h) = 2\eta_s(\epsilon(u_h), \epsilon(v_h)) - 2\eta_s \left\langle \epsilon(u_h) \cdot \nu, v_h \right\rangle_{\partial\Omega}, \tag{2.4}$$

$$b(p_h, v_h) = -(p_h, \nabla \cdot v_h) + \left\langle p_h, v_h \cdot \nu \right\rangle_{\partial\Omega}, \tag{2.5}$$

$$c(\sigma_h, v_h) = (\sigma_h, \epsilon(v_h)) - \left\langle \sigma_h \cdot \nu, v_h \right\rangle_{\partial\Omega}, \tag{2.6}$$

$$d(\sigma_h, \tau_h) = \frac{1}{2\eta_p}(\sigma_h + \lambda(\beta \cdot \nabla)\sigma_h, \tau_h) + \frac{\lambda}{2\eta_p} \left\langle \sigma_h, |\beta \cdot \nu \,|\, \tau_h \right\rangle_{\partial\Omega_{\text{in}}}. \tag{2.7}$$

The method we propose then takes the form, find $(u_h, \sigma_h, p_h) \in \mathbb{V}_h^d \times \mathbb{V}_h^{d\times d} \times \mathbb{V}_h$ such that

$$\begin{aligned} a(u_h, v_h) + b(p_h, v_h) - b(q_h, u_h) + c(\sigma_h, v_h) - c(\tau_h, u_h) + d(\sigma_h, \tau_h) \\ + j_p(p_h, q_h) + j_u(u_h, v_h) + j_\sigma(\sigma_h, \tau_h) = (f, v_h) + \left\langle \frac{\gamma_b(\eta_s + \eta_p)}{h}\beta, v_h \right\rangle_{\partial\Omega} \\ \forall (v_h, \tau_h, q_h) \in \mathbb{V}_h^d \times \mathbb{V}_h^{d\times d} \times \mathbb{V}_h. \end{aligned} \tag{2.8}$$

Note that in the above formulation the boundary conditions

$$u = \beta \quad \text{on } \partial\Omega, \qquad \text{resp.} \quad \sigma = 0 \quad \text{on} \quad \partial\Omega_{\text{in}},$$

are imposed weakly justifying the presence of the last term in (2.2), resp. (2.7).

For ease of notation we will also consider the following compact form, introducing the variables $U_h = (u_h, \sigma_h, p_h)$ and $V_h = (v_h, \tau_h, q_h)$ and the finite element space $\mathbb{X}_h = \mathbb{V}_h^d \times \mathbb{V}_h^{d\times d} \times \mathbb{V}_h$,

$$\begin{aligned} A(U_h, V_h) =& a(u_h, v_h) + b(p_h, v_h) - b(q_h, u_h) \\ &+ c(\sigma_h, v_h) - c(\tau_h, u_h) + d(\sigma_h, \tau_h) \end{aligned}$$

and

$$J(U_h, V_h) = j_p(p_h, q_h) + j_u(u_h, v_h) + j_\sigma(\sigma_h, \tau_h)$$

yielding the compact formulation find $U_h \in \mathbb{X}_h$ such that

$$A(U_h, V_h) + J(U_h, V_h) = (f, v_h) + \left\langle \frac{\gamma_b(\eta_s + \eta_p)}{h}\beta, v_h \right\rangle_{\partial\Omega} \qquad \forall\ V_h \in \mathbb{X}_h.$$

Clearly this formulation is strongly consistent for sufficiently smooth exact solutions as pointed out in the following lemma.

LEMMA 2.1 (Galerkin orthogonality). *Assume that* $U = (u, \sigma, p) \in H^2(\Omega)^d \times H^2(\Omega)^{d\times d} \times H^2(\Omega)$ *then the formulation (2.8) satisfies*

$$A(U - U_h, V_h) + J(U - U_h, V_h) = 0, \qquad \forall\ V_h \in \mathbb{X}_h.$$

Proof. Immediate since under the regularity assumption there holds $j_u(u, v_h) = \left\langle \frac{\gamma_b(\eta_s+\eta_p)}{h} u, v_h \right\rangle_{\partial\Omega}$, $j_p(p, q_h) = 0$ and $j_\sigma(\sigma, \tau_h) = 0$. □

REMARK 2.2. *The analysis proposed below may be extended in a straightforward manner to the case where the velocities are chosen in the affine H^1-conforming finite element space whereas pressures and stresses are approximated by piecewise constants. In this case only the jumps of discontinuous pressures have to be stabilized and the convection term of the stresses is discretized using standard upwind fluxes.*

3. The inf-sup condition. For the numerical scheme (2.8) to be wellposed it is essential that there holds an inf-sup condition uniformly in the meshsize h. In order to prove this we recall a lemma from [10] based on the Oswald interpolant [33, 26].

LEMMA 3.1 (Oswald interpolation). *Let $\pi_h^* : \mathbb{W}_h \to \mathbb{V}_h$ denote the Oswald interpolation operator on the finite element space. Then there exists a constant c_O independent of h such that*

$$\|h^s(\nabla y_h - \pi_h^* \nabla y_h)\|^2 \leq c_O \sum_{e \in \mathcal{E}_h} \left\langle h^{2s+1}[\nabla y_h], [\nabla y_h]\right\rangle_e, \quad 0 \leq s \leq 1.$$

Consider now the triple norm given by

$$|||U|||^2 = \frac{1}{2\eta_p}\|\sigma\|^2 + 2\eta_s\|\epsilon(u)\|^2 + \frac{1}{2\eta_p}\|p\|^2 + \frac{\lambda}{2\eta_p}\|\,|\beta\cdot\nu\,|^{\frac{1}{2}}\,\sigma\|^2_{L^2(\partial\Omega)},$$

where $U = (u, \sigma, p)$ and the following corresponding discrete triple norm

$$|||U_h|||_h^2 = |||U_h|||^2 + J(U_h, U_h).$$

When $\beta = 0$, the following discrete norm will be used

$$|||(u_h, \sigma_h, p_h)|||_*^2 = \frac{1}{2\eta_p}\|\sigma_h\|^2 + 2(\eta_s+\eta_p)\|\epsilon(u_h)\|^2 + \frac{1}{2\eta_p}\|p_h\|^2 + j_p(p_h, p_h) + j_u(u_h, u_h).$$

We will prove that the inf-sup condition is satisfied for the discrete form.

THEOREM 3.2 (Stability). *Assume that the mesh satisfies the quasiuniformity of the mesh. Then there exists two constants $\gamma_b^* > 0$ and c independent of h such that for all γ_p, γ_u, γ_s positive constants and for all $\gamma_b > \gamma_b^*$, there holds for all $U_h \in \mathbb{X}_h$*

$$|||U_h|||_h \leq c \sup_{\substack{V_h \in \mathbb{X}_h \\ V_h\ \neq 0}} \frac{A(U_h, V_h) + J(U_h, V_h)}{|||V_h|||_h}$$

REMARK 3.3. *When $\eta_s = 0$, the above inf-sup condition is not sufficient to control the velocity. However, if $\beta = 0$, using the same arguments a similar inf-sup condition holds for the discrete norm $|||.|||_*$ even when $\eta_s = 0$, see [3]. The additional control of $\|\epsilon(U_h)\|$ is obtained by testing with $\tau = \pi_h\epsilon(u_h)$ in* (2.8) *in a third step of the proof, see [3].*

Proof. [of Theorem 3.2] First we show that there exists $V_h = (v_h, \tau_h, q_h) \in \mathbb{X}_h$ and a constant c_1 independent of h and η_s such that

$$c_1|||U_h|||_h^2 \leq A(U_h, V_h) + J(U_h, V_h)$$

and then we show that there exists a constant c_2 independent of h and η_s such that $|||V_h|||_h \leq c_2|||U_h|||_h$, after which the claim follows. The first part is the most laborious. The proof is made in two steps. First we establish the coercivity of the bilinear form. Then we recover control of the L^2-norm of the pressure.

1) Choosing $(v_h, \tau_h, q_h) = (u_h, \sigma_h, p_h)$ immediately leads to the equality

$$\begin{aligned} A(U_h, U_h) = 2\eta_s\|\epsilon(u_h)\|^2 + \frac{1}{2\eta_p}\|\sigma_h\|^2 + \frac{\lambda}{2\eta_p}\left((\beta\cdot\nabla)\sigma_h, \sigma_h\right) \\ - \left\langle (2\eta_s\epsilon(u_h))\cdot\nu, u_h\right\rangle_{\partial\Omega} + \frac{\lambda}{2\eta_p}\left\langle \sigma_h, |\beta\cdot\nu\,|\,\sigma_h\right\rangle_{\partial\Omega_{\text{in}}}. \quad (3.1) \end{aligned}$$

Using an integration by part and since $\nabla \cdot \beta = 0$, it follows that

$$
\begin{aligned}
((\beta \cdot \nabla)\sigma_h, \sigma_h) &= \frac{1}{2} \left\langle (\beta \cdot \nu)\sigma_h, \sigma_h \right\rangle_{\partial\Omega} \\
&= \frac{1}{2} \left\langle |\beta \cdot \nu \,|\, \sigma_h, \sigma_h \right\rangle_{\partial\Omega \setminus \partial\Omega_{in}} - \frac{1}{2} \left\langle |\beta \cdot \nu \,|\, \sigma_h, \sigma_h \right\rangle_{\partial\Omega_{in}} .
\end{aligned}
$$

Moreover a Cauchy-Schwarz inequality and a trace inequality lead to

$$
\begin{aligned}
- c_t \left((2\eta_s)^{\frac{1}{2}} \|\epsilon(u_h)\|_{L^2(\Omega)} + \frac{1}{(2\eta_p)^{\frac{1}{2}}} \|\sigma_h\|_{L^2(\Omega)} \right) \| \frac{\eta_s + \eta_p}{h^{\frac{1}{2}}} u_h \|_{\partial\Omega} \\
\leq - \left\langle (2\eta_s \epsilon(u_h) + \sigma_h) \cdot \nu, u_h \right\rangle_{\partial\Omega}
\end{aligned}
$$

where c_t depends only on the trace inequality. Thus, for γ_b sufficiently large there exists a constant c_1 independent of h and η_s such that

$$
\begin{aligned}
c_1 \left(\frac{1}{2\eta_p} \|\sigma_h\|^2 + 2\eta_s \|\epsilon(u_h)\|^2 + \frac{\lambda}{2\eta_p} \| \, |\beta \cdot \nu \,|^{1/2} \, \sigma_h\|^2_{L^2(\partial\Omega)} + J(U_h, U_h) \right) \\
\leq A(U_h, U_h) + J(U_h, U_h).
\end{aligned}
$$

2) By the surjectivity of the divergence operator we know that there exists $v_p \in H_0^1(\Omega)$ such that $\nabla \cdot v_p = p_h$ and $\|v_p\|_{1,\Omega} \leq c\|p_h\|_{0,\Omega}$, where c is a constant independent of h and η_s. We now take $v_h = -\frac{1}{2\eta_p}\pi_h v_p$, $q_h = 0$ and $\tau_h = 0$ to obtain

$$
\begin{aligned}
-\tfrac{1}{2\eta_p} A((u_h, \sigma_h, p_h), (\pi_h v_p, 0, 0)) - \tfrac{1}{2\eta_p} j_u(u_h, \pi_h v_p) \geq -\epsilon_1 2\eta_s \|\epsilon(u_h)\|^2 \\
- \tfrac{2\eta_s}{16\epsilon_1 \eta_p^2} \|\epsilon(\pi_h v_p)\|^2 - \tfrac{\epsilon_2}{2\eta_p} \|\sigma_h\|^2 - \tfrac{1}{8\epsilon_2 \eta_p} \|\epsilon(\pi_h v_p)\|^2 + \tfrac{1}{2\eta_p} \|p_h\|^2 \\
+ \tfrac{1}{2\eta_p} (\nabla p_h - \pi^* \nabla p_h, v_p - \pi_h v_p) \\
+ \tfrac{1}{2\eta_p} \left\langle (2\eta_s \epsilon(u_h) + \sigma_h) \cdot \nu, \pi_h v_p - v_p \right\rangle_{\partial\Omega} - j_u(u_h, \tfrac{1}{2\eta_p} \pi_h v_p)
\end{aligned}
$$

for all $\epsilon_1 > 0$, $\epsilon_2 > 0$. We now note that the following inequalities holds

$$
\|\epsilon(\pi_h v_p)\|^2 \leq c_3 \|p_h\|^2, \tag{3.2}
$$

$$
\frac{1}{2\eta_p} (\nabla p_h - \pi^* \nabla p_h, v_p - \pi_h v_p) \leq \epsilon_3 j_p(p_h, p_h) + \frac{c_O}{16\epsilon_3 \eta_p \gamma_p} \|h^{-1}(v_p - \pi_h v_p)\|^2,
$$

for all $\epsilon_3 > 0$ and where c_O is given by Lemma 3.1. Using a standard interpolation result for the L^2-projection and the properties of the function v_p we have

$$
\|h^{-1}(v_p - \pi_h v_p)\|^2 \leq c_4 \|p_h\|^2.
$$

By a trace inequality followed by an inverse inequality and the above approximation result we have for the boundary term

$$
\begin{aligned}
\frac{1}{2\eta_p} \left\langle (2\eta_s \epsilon(u_h) + \sigma_h) \cdot \nu, \pi_h v_p - v_p \right\rangle_{\partial\Omega} \\
\leq 2\eta_s \epsilon_4 \|\epsilon(u_h)\|^2 + \frac{\epsilon_5}{2\eta_p} \|\sigma_h\|^2 + \left(\frac{2c_5 \eta_s}{16\epsilon_4 \eta_p^2} + \frac{c_5}{8\epsilon_5 \eta_p} \right) \|p_h\|^2,
\end{aligned}
$$

for all $\epsilon_4 > 0$, $\epsilon_5 > 0$ and where the constant c_5 depends on the trace inequality, the inverse inequality and the interpolation estimate. For the jump term on the velocities there holds

$$j_u(u_h, \frac{1}{2\eta_p}\pi_h v_p) \geq -\epsilon_6 j_u(u_h, u_h) - \frac{1}{4\epsilon_6} j_u(\frac{1}{2\eta_p}\pi_h v_p, \frac{1}{2\eta_p}\pi_h v_p),$$

for all $\epsilon_6 > 0$ and we note that by the trace inequality once again there holds

$$j_u(\frac{1}{2\eta_p}\pi_h v_p, \frac{1}{2\eta_p}\pi_h v_p) \leq c_6 \frac{1}{2\eta_p}\|p_h\|^2, \tag{3.3}$$

where c_6 is independent of h. Collecting terms we see that there holds

$$\begin{aligned} A((u_h, \sigma_h, p_h), (\tfrac{1}{2\eta_p}\pi_h v_p, 0, 0)) + J((u_h, \sigma_h, p_h), (\tfrac{1}{2\eta_p}\pi_h v_p, 0, 0)) \\ \geq (1 - \tfrac{2c_3\eta_s}{8\epsilon_1\eta_p} - \tfrac{c_3}{4\epsilon_2} - \tfrac{c_O c_4}{8\epsilon_3\gamma_p} - \tfrac{2c_5\eta_s}{8\epsilon_4\eta_p} - \tfrac{c_5}{4\epsilon_5} - \tfrac{c_6}{4\epsilon_6})\tfrac{1}{2\eta_p}\|p_h\|^2 \\ -(\epsilon_2 + \epsilon_5)\tfrac{1}{2\eta_p}\|\sigma_h\|^2 - (\epsilon_1 + \epsilon_4)\tfrac{2\eta_s}{2\eta_p}\|\epsilon(u_h)\|^2 - \epsilon_6 j_u(u_h, u_h) - \epsilon_3 j_p(p_h, p_h), \end{aligned}$$

for all ϵ_i, $i = 1, \ldots, 6$ positive constants.

Collecting the results of 1) and 2) we may conclude that there exists constants α and $c_\alpha > 0$ independent of h and η_s such that for all $\eta \geq 0$ taking $(v_h, \tau_h, q_h) = (u_h + \alpha\frac{1}{2\eta_p}\pi_h v_p, \sigma_h, p_h)$ yields

$$c_\alpha |||U_h|||_h^2 \leq A(U_h, V_h) + J(U_h, V_h).$$

To conclude we now need to show that

$$|||V_h|||_h \leq c|||U_h|||_h,$$

but this follows immediately by the triangle inequality and by recalling the inequalities (3.2) and (3.3) and using the stability of the L^2-projection. □

4. A priori error estimates. An a priori error estimates follow from the previously proved inf-sup condition together with the proper continuities of the bilinear forms and the approximation properties of the finite element space. We will start with the latter. Let $U = (u, p, \sigma)$.

LEMMA 4.1 (Interpolation). *Assume that all the components of U are in $H^{k+1}(\Omega)$. Then there exists a constant c independent of h and η_s such that there holds*

$$|||U - \pi_h U|||_h \leq ch^k \left((\eta_p + \eta_s)^{\frac{1}{2}}\|u\|_{k+1,\Omega} + \frac{1}{(2\eta_p)^{\frac{1}{2}}} h^{\frac{1}{2}}\|\sigma\|_{k+1,\Omega} + \frac{1}{(2\eta_p)^{\frac{1}{2}}} h\|p\|_{k+1,\Omega} \right)$$

for all $h < 1$, where $k \geq 1$ is the polynomial order of the finite element spaces.

Proof. The only part that has to be investigated is the convergence order of the jump terms. Let us denote by c a generic constant independent of h and η_s. Note that by the trace inequality we have

$$\begin{aligned} \langle 2\eta_p h[\nabla(u - \pi_h u)], [\nabla(u - \pi_h u)]\rangle_e + \left\langle \frac{(\eta_s + \eta_p)\gamma_b}{h} u - \pi_h u, u - \pi_h u \right\rangle_{\partial\Omega} \\ \leq c\left((\eta_s + \eta_p)\|\nabla(u - \pi_h u)\|_{0,K}^2 + (\eta_s + \eta_p)h^2\|\nabla(u - \pi_h u)\|_{1,K}^2\right). \end{aligned}$$

Summing over all $e \in \mathcal{E}_h$ yields by the quasi uniformity of the mesh,

$$j_u(u - \pi_h u, u - \pi_h u) \leq c\eta_p h^{2k} \|u\|^2_{k+1,\Omega}.$$

Similarly we obtain for the pressure

$$j_p(p - \pi_h p, p - \pi_h p) \leq c\frac{1}{\eta_p} h^{2k+2} \|p\|^2_{k+1,\Omega}$$

and the extra-stress

$$j_\sigma(\sigma - \pi_h \sigma, \sigma - \pi_h \sigma) \leq ch^{1+2k} \|\sigma\|^2_{k+1,\Omega}$$

The claim now follows as an immediate consequence of the approximation properties of the L^2 projection. □

THEOREM 4.2 (Convergence in the mesh dependent norm). *Assume that all the components of $U = (u, \sigma, p)$ are in $H^{k+1}(\Omega)$ and that the hypothesis of Theorem 3.2 are satisfied, then for all for all γ_p, γ_u, γ_s positive constants and for all $\gamma_b > \gamma_b^*$, there exists a constant c independent of h and η_s such that there holds*

$$|||U - U_h|||_h \leq ch^k \left((\eta_p + \eta_s)^{\frac{1}{2}} \|u\|_{k+1,\Omega} + \frac{1}{\eta_p^{\frac{1}{2}}} h^{\frac{1}{2}} \|\sigma\|_{k+1,\Omega} + \frac{1}{\eta_p^{\frac{1}{2}}} h \|p\|_{k+1,\Omega} \right)$$

for all $h < 1$, where $k \geq 1$ is the polynomial order of the finite element spaces.

REMARK 4.3. *Theorem 4.2 does not guarantee the control of $\|\epsilon(u - u_h)\|$ when $\eta_s = 0$. However, for $\eta_s = 0$ a similar results holds when $\beta = 0$ using the stronger norm $|||\cdot|||_*$:*

$$|||U - U_h|||_* \leq ch^k \left(\eta_p^{\frac{1}{2}} \|u\|_{k+1,\Omega} + \frac{1}{\eta_p^{\frac{1}{2}}} h \|\sigma\|_{k+1,\Omega} + \frac{1}{\eta_p^{\frac{1}{2}}} h \|p\|_{k+1,\Omega} \right),$$

see the Remark 3.3 and [3].

Proof. First note that by the triangle inequality we have

$$|||U - U_h|||_h \leq |||U - \pi_h U|||_h + |||U_h - \pi_h U|||_h.$$

The convergence of the first term is immediate by the lemma 4.1. For the second term we know that the inf-sup condition holds

$$|||U_h - \pi_h U|||_h \leq c \sup_{\substack{V_h \in \mathbb{V}_h \\ V_h \neq 0}} \frac{A(U_h - \pi_h U, V_h) + J(U_h - \pi_h U; V_h)}{|||V_h|||_h}.$$

By Galerkin orthogonality there holds

$$|||U_h - \pi_h U|||_h \leq c \sup_{\substack{V_h \in \mathbb{V}_h \\ V_h \neq 0}} \frac{A(U - \pi_h U, V_h) + J(U - \pi_h U; V_h)}{|||V_h|||_h}.$$

Now we may note that by a Cauchy-Schwarz inequality there exists a constant c independent of h and η_s such that

$$J(U - \pi_h U, V_h) \leq c|||U - \pi_h U|||_h \, |||V_h|||_h.$$

Considering now the standard Galerkin part we have

$$\begin{aligned} A(U-\pi_h U;V_h) \quad &= a(u-\pi_h u, v_h) + b(p-\pi_h p, v_h) - b(q_h, u-\pi_h u) \\ &+ c(\sigma-\pi_h\sigma, u_h) - c(\tau_h, u-\pi_h u) + d(\sigma-\pi_h\sigma, \tau_h). \end{aligned} \tag{4.1}$$

Let us now estimate all the terms in the right hand. For ease of notation let us denote by c a generic constant independent of h and η_s. All the terms in the right hand side of the above equation can be estimated as follows

$$\begin{aligned} a(u-\pi_h u, v_h) &\leq 2\eta_s \|\epsilon(u-\pi_h u)\| \|\epsilon(v_h)\| + 2\eta_s \|\epsilon(u-\pi_h u)\| c \|\frac{1}{h} v_h\|_{L^2(\partial\Omega)} \\ &\leq c|||U-\pi_h U|||\ |||V_h|||_h, \end{aligned}$$

$$\begin{aligned} b(p-\pi_h p, v_h) &\leq \frac{1}{(2\eta_p)^{\frac{1}{2}}} \|p-\pi_h p\| (2\eta_p)^{\frac{1}{2}} \|\nabla\cdot v_h - \pi^*\nabla\cdot v_h\| + |\langle p-\pi_h p, v_h\cdot\nu\rangle_{\partial\Omega}| \\ &\leq c|||U-\pi_h U|||_h |||V_h|||_h + \frac{Ch^{\frac{1}{2}}}{\eta_p^{\frac{1}{2}}} \|p-\pi_h p\|_{\partial\Omega} \frac{\eta_p^{\frac{1}{2}}}{h^{\frac{1}{2}}} \|v_h\|_{\partial\Omega} \\ &\leq c\left(|||U-\pi_h U|||_h + \frac{h^{k+1}}{\eta_p^{\frac{1}{2}}} \|p\|_{k+1} \right) |||V_h|||_h, \end{aligned}$$

$$\begin{aligned} b(q_h, u-\pi_h u) &\leq (q_h, \nabla\cdot(u-\pi_h u)) - (q_h, (u-\pi_h u)\cdot\nu) \\ &= (\nabla q_h - \pi^*\nabla q_h, u-\pi_h u) \leq c j_p(q_h,q_h) h^{-1} \eta_p^{\frac{1}{2}} \|u-\pi_h u\|_{0,\Omega} \\ &\leq c j_p(q_h,q_h) h^k \eta_p^{\frac{1}{2}} \|u\|_{k+1,\Omega}, \end{aligned}$$

$$\begin{aligned} c(\sigma-\pi_h\sigma, v_h) &\leq \|\sigma-\ \pi_h\sigma\|\, \|\epsilon(v_h)\| + |\langle(\sigma-\pi_h\sigma)\cdot\nu, v_h\rangle| \\ &\leq \frac{1}{(2\eta_p)^{\frac{1}{2}}} \|\sigma-\pi_h\sigma\|\, (2\eta_p)^{\frac{1}{2}} (\|\epsilon(v_h) - \pi^*\epsilon(v_h)\| + |h^{-\frac{1}{2}} v_h|_{\partial\Omega}) \\ &\leq c|||U-\pi_h U|||\ |||V_h|||_h, \end{aligned}$$

$$\begin{aligned} c(\tau_h, u-\pi_h u) &\leq \|\tau_h\|\, \|\epsilon(u-\pi_h u)\| + |\langle \tau_h\cdot\nu, u-\pi_h u\rangle| \\ &\leq \frac{1}{(2\eta_p)^{\frac{1}{2}}} \|\tau_h\|\, (2\eta_p)^{\frac{1}{2}} (\|\epsilon(u-\pi_h u)\| + |h^{-\frac{1}{2}}(u-\pi_h u)|_{\partial\Omega}) \\ &\leq c|||V_h|||_h\, h^k (2\eta_p)^{\frac{1}{2}} \|u\|_{k+1}. \end{aligned}$$

For the last term, we denote by β_h the L^2-projection of β onto

$$\left\{ v_h \in L^2(\Omega)^d; \forall K\in\mathcal{T}_h,\ v_h|_K \in \mathbb{P}^0(K) \right\}.$$

Since $\beta \in \mathcal{C}^{\frac{1}{2}}(\Omega)$ it holds

$$\|\beta-\beta_h\|_{L^\infty} \leq ch^{\frac{1}{2}} \|\beta\|_{\mathcal{C}^{\frac{1}{2}}(\Omega)}.$$

Thus we obtain

$$
\begin{aligned}
d(\sigma-\pi_h\sigma,\tau_h) &= \frac{\lambda}{2\eta_p}\left((\beta\cdot\nabla)(\sigma-\pi_h\sigma),\tau_h\ \right) - \frac{\lambda}{2\eta_p}\left\langle \sigma-\pi_h\sigma,(\beta\cdot\nu)\tau_h\right\rangle_{\partial\Omega_{\mathrm{in}}} \\
&\leq -\frac{\lambda}{2\eta_p}\left(\sigma-\pi_h\sigma,((\beta-\beta_h)\cdot\nabla)\tau_h\ \right) \\
&\quad -\frac{\lambda}{2\eta_p}\left(\sigma-\pi_h\sigma,(\beta_h\cdot\nabla)\tau_h\ -\pi^*(\ (\beta_h\cdot\nabla)\tau_h)\right) \\
&\quad +\frac{\lambda}{2\eta_p}\left\langle \sigma-\pi_h\sigma,|\beta\cdot\nu|^{\frac{1}{2}}\,\tau_h\right\rangle_{\partial\Omega} \\
&\leq \frac{c\lambda}{2\eta_p h^{\frac{1}{2}}}\|\sigma-\pi_h\sigma\|\left(\frac{1}{h^{\frac{1}{2}}}\|\beta-\beta_h\|_{L^\infty}\|\tau_h\| + \|\,|\beta\cdot n|^{\frac{1}{2}}\,\tau_h\|_{\partial\Omega}\right. \\
&\quad \left.\|h\,|\beta-\beta_h\,|\,[\nabla\tau_h]\|_{L^2(\Omega)} + \|h\,|\beta\,|\,[\nabla\tau_h]\|_{L^2(\Omega)}\right) \\
&\leq \frac{c}{h^{\frac{1}{2}}}||\sigma-\pi_h\sigma||\ |||V_h|||_h.
\end{aligned}
$$

The claim now follows by collecting the bounds obtained and by using Lemma 4.1. ∎

Theorem 4.4 (Convergence in the energy norm). *Assume that $u\in H^2(\Omega)^d$, $\sigma\in H^1(\Omega)^{d\times d}$ and $p\in H^1(\Omega)$ and that the hypothesis of Theorem 3.2 are satisfied, then for all for all γ_p, γ_u, γ_s positive constants and for all $\gamma_b>\gamma_b^*$, there exists a constant c independent of h and η_s such that*

$$
|||U-U_h||| \leq ch^{\frac{1}{2}}\left((\eta_p+\eta_s)^{\frac{1}{2}}h^{\frac{1}{2}}\|u\|_{2,\Omega} + \frac{1}{\eta_p^{\frac{1}{2}}}\|\sigma\|_{1,\Omega} + \frac{1}{\eta_p^{\frac{1}{2}}}h^{\frac{1}{2}}\|p\|_{1,\Omega}\right).
$$

Moreover, by regularity assumption (1.3) *we have*

$$
|||U-U_h||| \leq \tilde{c}h^{\frac{1}{2}},
$$

where the constant $\tilde{c}$ depends only on the problem data f, β, η_s, η_p, λ and Ω.

Remark 4.5. *As in Theorem 4.2, for $\eta_s=0$ a similar theorem holds when $\beta=0$ using the stronger norm $|||\cdot|||_*$:*

$$
|||U-U_h|||_* \leq ch\left(\eta_p^{\frac{1}{2}}\|u\|_{k+1,\Omega} + \frac{1}{\eta_p^{\frac{1}{2}}}\|\sigma\|_{k,\Omega} + \frac{1}{\eta_p^{\frac{1}{2}}}\|p\|_{k,\Omega}\right),
$$

see [3].

Proof. Very similar to the previous proof but using the decomposition

$$
|||U-U_h||| \leq |||U-\pi_h U||| + |||U_h-\pi_h U|||_h.
$$

Then the inf-sup condition is applied to the second term. However Galerkin orthogonality no longer holds exactly and we get

$$
|||U_h-\pi_h U|||_h \leq c \sup_{\substack{V_h\in\mathbb{V}_h\\ V_h\neq 0}} \frac{A(U-\pi_h U,V_h)+J(U-\pi_h U,V_h)-j_p(\pi_h p,q_h)-j_\sigma(\pi_h\sigma,\tau_h)}{|||V_h|||_h}.
$$

Clearly the only thing that differs compared to the previous case is the nonconsistence of the pressure and the extra-stress terms which may be treated as follows:

$$
j_p(\pi_h p,q_h) \leq j_p(\pi_h p,\pi_h p)|||V_h|||_h \leq ch\|p\|_{1,\Omega}|||V_h|||_h,
$$

where c is a generic constant independent of h and η_s. The last inequality is a consequence of the trace inequality and the stability of the L^2 projection applied in the following fashion:

$$\sum_{e\in\mathcal{E}_h} \left\langle h^3[\nabla\pi_h p], [\nabla\pi_h p]\right\rangle_e \leq c \sum_{K\in\mathcal{T}_h} h^2\|\nabla\pi_h p\|_K^2 \leq ch^2\|\nabla p\|^2.$$

The term for the extra-stress can be treated in the same way. □

5. A stable iterative algorithm. We propose an iterative algorithm allowing us to solve for the velocity pressure coupling independently of the extra stresses. A similar iterative method as in [21, 24] and [8] is presented. The interest of such an algorithm is to decouple the velocity-pressure computation from extra stress computation for solving (2.8).

Each subiteration of the iterative algorithm consists of two steps. Firstly, using the Navier-Stokes equation, the new approximation (u_h^n, p_h^n) is determined using the value of the extra stress at previous step σ_h^{n-1}. Then the new approximation σ_h^n is computed using the constitutive relation with the value u_h^n. More precisely, assuming $(u_h^{n-1}, \sigma_h^{n-1}, p_h^{n-1})$ be the known approximation of (u_h, σ_h, p_h) after $n-1$ steps. First step consists of finding (u_h^n, p_h^n) such as

$$\begin{aligned}(A+J)((u_h^n, \sigma_h^{n-1}, p_h^n), (v_h, 0, q_h)) &+ K(u_h^n, u_h^{n-1}, v_h)\\ = (f_h, v_h) &+ \left\langle \frac{\gamma_b(\eta_s+\eta_p)}{h}\beta, v_h\right\rangle_{\partial\Omega}, \qquad \forall (v_h, p_h) \in \mathbb{V}^d\times\mathbb{V}, \end{aligned} \tag{5.1}$$

and then finding σ_h^n such as

$$A((u_h^n, \sigma_h^n, p_h^n), (0, \tau_h, 0)) = 0, \qquad \forall \tau_h \in \mathbb{V}^{d\times d}. \tag{5.2}$$

Here $K : H^1(\Omega)^d \times H^1(\Omega)^d \times H^1(\Omega)^d \to \mathbb{R}$ is defined for all $u_1, u_2, v \in H^1(\Omega)^d$ by

$$K[u_1, u_2, v] := 2\eta_p\left(\epsilon(u_1 - u_2), \epsilon(v)\right).$$

The term $K(u_h^n, u_h^{n-1}, v)$, which vanish for $u_h^n = u_h^{n-1}$, has been added to (2.8) in (5.1) in order to obtain a stable iterative algorithm even when η_s vanishes.

THEOREM 5.1 (Stability of the iterative scheme). *Let $(u_h^n, \sigma_h^n, p_h^n)$ be the solution of (5.1),(5.2) with $f = 0$ and $\beta = 0$. There exists γ_u^* and $\gamma_b^* > 0$ independent of h such that for all $\gamma_p > 0$, $\gamma_u \geq \gamma_u^*$, $\gamma_b \geq \gamma_b^*$ and $\gamma_\sigma \geq 0$, there exists a constant $c > 0$ such that*

$$\eta_p\|\epsilon(u_h^n)\|^2 + \frac{3}{16\eta_p}\|\sigma_h^n\|^2 + c\|p_h^n\|^2 \leq \eta_p\|\epsilon(u_h^{n-1})\|^2 + \frac{3}{16\eta_p}\|\sigma_h^{n-1}\|^2.$$

Proof. The same arguments as in Theorem 3.2 are used in this proof. Taking $v_h = u_h^n$, $q = p_h^n$ in (5.1) and $\tau = \sigma_h^n$ in (5.2) it follows using the definitions of the bilinear forms (2.4), (2.5), (2.1), (2.2)

$$\begin{aligned}2(\eta_s+\eta_p)\|\epsilon(u_h^n)\|^2 &+ \frac{1}{2\eta_p}\|\sigma_h^n\|^2 + j_u(u_h^n, u_h^n) + j_p(p_h^n, p_h^n) + j_\sigma(\sigma_h^n, \sigma_h^n)\\ &= \left(2\eta_p\epsilon(u_h^{n-1}) - \sigma_h^{n-1}, \epsilon(u_h^n)\right) + (\sigma_h^n, \epsilon(u_h^n))\\ &\quad + \left\langle(\sigma_h^{n-1} - \sigma_h^n)\cdot\nu, u_h^n\right\rangle_{\partial\Omega} + \langle 2\eta_s\epsilon(u_h^n)\cdot\nu, u_h^n\rangle_{\partial\Omega}.\end{aligned} \tag{5.3}$$

A standard Young's inequality leads to

$$(\sigma_h^n, \epsilon(u_h^n)) \le \eta_p \|\epsilon(u_h^n)\|^2 + \frac{1}{4\eta_p}\|\sigma_h^n\|^2.$$

Since $\sigma_h^{n-1} = 2\eta_p \pi_h \epsilon(u_h^{n-1})$, using the orthogonality of the Galerkin approximation and a Young's inequality again we obtain

$$\begin{aligned}\big(2\eta_p \epsilon(u_h^{n-1}) - \sigma_h^{n-1}, \epsilon(u_h^n)\big) &= \big(2\eta_p \epsilon(u_h^{n-1}) - 2\eta_p \pi_h \epsilon(u_h^{n-1}), \epsilon(u_h^n)\big) - \pi_h^* \epsilon(u_h^n)) \\ &\le \eta_p \|\epsilon(u_h^{n-1})\|^2 + \frac{1}{8\eta_p}\|\sigma_h^{n-1}\|^2 + 3\eta_p \|\epsilon(u_h^n) - \pi_h^* \epsilon(u_h^n)\|^2.\end{aligned}$$

The last term in the inequality above can be estimated using Lemma 3.1

$$\begin{aligned}3\eta_p \|\epsilon(u_h^n) - \pi_h^* \epsilon(u_h^n)\|^2 &\le 3\eta_p c_O \sum_e \langle h_e[\epsilon(u_h^n)], [\epsilon(u_h^n)]\rangle_{\partial\Omega} \\ &\le \frac{3c_O}{2\gamma_u} 2\eta_p \gamma_u \sum_e \langle h_e[\nabla u_h^n], [\nabla u_h^n]\rangle_{\partial\Omega}\,.\end{aligned}$$

Using the trace inequality followed by an inverse estimate, we can prove there exists two constants $\tilde{c}_1,\ \tilde{c}_2 > 0$ such that

$$\begin{aligned}\left\langle(\sigma_h^{n-1} - \sigma_h^n)\cdot\nu, u_h^n\right\rangle_{\partial\Omega} &\le \frac{1}{16\eta_p}\left(\|\sigma_h^{n-1}\|^2 + \|\sigma_h^n\|^2\right) \\ &\quad + \frac{8\tilde{c}_1}{\gamma_b}\left\langle \frac{\eta_p\gamma_b}{h} u_h^n, u_h^n\right\rangle_{\partial\Omega}\end{aligned}$$

and

$$\langle 2\eta_s(\epsilon(u_h^n)\cdot\nu, u_h^n\rangle_{\partial\Omega} \le \eta_s \|\epsilon(u_h^n\|^2 + \frac{\tilde{c}_2}{\gamma_b}\left\langle \frac{\eta_p\gamma_b}{h} u_h^n, u_h^n\right\rangle_{\partial\Omega}$$

Collecting the inequalities above in (5.3) we obtain

$$\begin{aligned}&\eta_p \|\epsilon(u_h^n)\|^2 + \frac{3}{16\eta_p}\|\sigma_h^n\|^2 + j_u(u_h^n, u_h^n) + j_p(p_h^n, p_h^n) + j_\sigma(\sigma_h^n, \sigma_h^n) \\ &\le \eta_p \|\epsilon(u_h^{n-1})\|^2 + \frac{3}{16\eta_p}\|\sigma_h^{n-1}\|^2 \\ &+ \frac{3c_O}{2\gamma_u} 2\eta_p\gamma_u \sum_e \langle h_e[\nabla u_h^n], [\nabla u_h^n]\rangle_{\partial\Omega} + \frac{8\tilde{c}_1 + \tilde{c}_2}{\gamma_b}\left\langle \frac{(\eta_s + \eta_p)\gamma_b}{h} u_h^n, u_h^n\right\rangle_{\partial\Omega}.\end{aligned}$$

Now Choosing γ_u^* and γ_b^* such that

$$\gamma_u^* > \frac{3}{2} c_O, \quad \gamma_b^* > 8\tilde{c}_1 + \tilde{c}_2,$$

we obtain there exists a constant $c > 0$ independent of h and η_s such that

$$\begin{aligned}(\eta_s + \eta_p)\|\epsilon(u_h^n)\|^2 + \frac{3}{16\eta_p}\|\sigma_h^n\|^2 + c\left(j_u(u_h^n, u_h^n) + j_p(p_h^n, p_h^n) + j_\sigma(\sigma_h^n, \sigma_h^n)\right) \\ \le \eta_p \|\epsilon(u_h^{n-1})\|^2 + \frac{3}{16\eta_p}\|\sigma_h^{n-1}\|^2.\end{aligned} \tag{5.4}$$

It remains to prove there exists a constant $c > 0$ independent of h and η_s such that

$$\|p_h^n\|^2 \leq c\left((\eta_s + \eta_p)\|\epsilon(u_h^n)\|^2 + \frac{1}{2\eta_p}\|\sigma_h^n\|^2 + j_u(u_h^n, u_h^n) + j_p(p_h^n, p_h^n) + j_\sigma(\sigma_h, \sigma_h)\right). \tag{5.5}$$

In order to do that, using the surjectivity of the divergence operator once again, there exists $v_p^n \in H_0^1(\Omega)$ such that $\nabla \cdot v_p^n = p_h^n$ and $\|v_p^n\|_{1,\Omega} \leq \tilde{c}\|p_h^n\|_{0,\Omega}$, for a constant $\tilde{c}$ independent of h and η_s. Choosing now $v_h = -\pi_h \frac{1}{2\eta_p} v_p^n$ and $q_h = 0$ in (5.1) and using the same arguments as in Theorem 3.2 part 2) one can prove (5.5). ▯

6. Nonlinear stabilization terms, shock capturing. It is well known that in the case of nonlinear hyperbolic conservation laws, linear stabilization is insufficient to assure convergence to an entropy solution. This is due to the spurious oscillations that remain close to internal layers. Such oscillations destroy the monotonicity properties needed to pass to the limit in the entropy inequality. It was shown by Szepessy and Johnson [27] (see also [28]) that a nonlinear shock capturing term can be used to smooth the solution close to layers. This way, in the case of the Burgers' equation one may show that the sequence of finite element solutions converges to the unique entropy solution. More recently it was shown that a nonlinear viscosity based on the gradient jumps is alone sufficient to assure convergence of finite element approximations of the Burgers' equation using a discrete maximum principle [11]. Drawing from these experiences from the Burgers' equation we propose to add a stabilization term on the form

$$j_{sc}(u_h; \sigma_h, \tau_h) = \sum_{K \in \mathcal{T}_h} (\nu_K \nabla\sigma, \nabla\tau),$$

with $\nu_K = \gamma_{nl} h_K^2 \lambda \max_{e \in \partial K} \|[\nabla u]\|_e$. Note that this term is weakly consistent, clearly if $u \in [H^2(\Omega)]$ then $j_{sc}(u, \sigma_h, \tau_h) = 0$. The idea is to add an artificial viscosity that can control the non-linear term, but that vanishes at an optimal rate under mesh refinement. In particular the nonlinear stabilization term smears the extra stress field in zones where the velocity gradient exhibits large variations. As we shall see in the numerical section such an ad hoc nonlinear diffusion allows us to double the Deborah number that may be attained.

7. Numerical examples. Several tests are presented in this section. THe computations were performed using FreeFem++ [36]. We will restrict ourself to the case $k = 1$ corresponding to $\mathbb{P}^1$ approximations. Considering Poiseuille flow of a linearized Oldroyd-B fluid, convergence rates for problem (1.2) consistent with Theorems 4.2 and 4.4 will be presented. Then, it will be shown using the three-fields Stokes' system that similar results as when using a Galerkin-Least-Square (GLS)/Elastic Viscous Split Stress procedure (EVSS), see for instance [23, 8, 24]), are obtained on a contraction $4:1$ problem.

Finally, we will focus on problem (1.1) and it will be shown that the non-linear stabilization term allow to reach higher Deborah number on the flow past a cylinder in a channel problem. The stationary results provided in this section have been obtained by solving the evolutionary problem using the iterative procedure of Section 5 for each time step.

7.1. Poiseuille flow. Consider a rectangular pipe of dimensions $[0, L] \times [-\frac{H}{2}, \frac{H}{2}]$ in the $x-y$ directions, where $L = 0.15$ $[m]$ and $H = 0.03$ $[m]$. The boundary conditions are the following. On the top and bottom sides ($y = 0$ and $y = \frac{H}{2}$), no-slip boundary conditions apply. On the inlet ($x = 0$) the velocity and the extra-stress are given by

$$u(0,y) = \begin{pmatrix} u_x(y) \\ 0 \end{pmatrix}, \quad \sigma(0,y) = \begin{pmatrix} \sigma_{11}(y) & \sigma_{12}(y) \\ \sigma_{12}(y) & 0 \end{pmatrix},$$

with

$$u_x(y) = V(\frac{H}{2} - y)(\frac{H}{2} + y),\ \sigma_{11}(y) = 8\lambda\eta_p V^2 y^2 \text{ and } \sigma_{12}(y) = -2\eta_p V y. \tag{7.1}$$

On the outlet ($x = L$) the velocity and the pressure are given by

$$\mathbf{u}(L,y) = \begin{pmatrix} u_x(y) \\ 0 \end{pmatrix}, \qquad p(L_1, y) \equiv 0.$$

We set $\beta_x(y) = u_x(y)$ and $\beta_y(x,y) = 0$. The velocity and extra-stress must satisfy (7.1) in the whole pipe and we choose $V = 1\ m/s$, $\lambda = 1\ s$, $\eta_s = 0$ and $\eta_p = 1\ Pa\ s$. Three unstructured meshes are used to check convergence (coarse: 50×10, intermediate: 100×20, fine: 200×40). In Fig. 7.1, the error in the L^2 norm of the velocity u, the pressure p and extra-stress components σ_{11}, σ_{12} is plotted versus the mesh size. Clearly order one convergence rate is observed for the pressure (in fact superconvergence is observed for the pressure) and the extra-stress whilst the convergence rate of the velocity is order two, this being consistent with theoretical predictions.

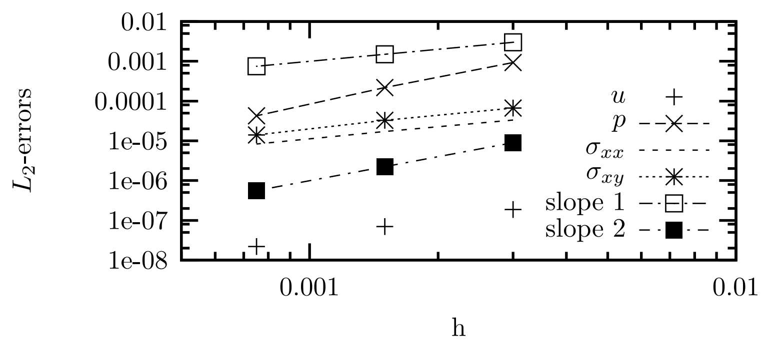

FIGURE 7.1. *Poiseuille flow: convergence orders.*

7.2. The 4:1 planar contraction. Numerical results of computation in the 4:1 abrupt contraction flow case are presented and comparison with the GLS/EVSS method is performed.

Let us recall briefly the EVSS procedure in the present context. Using the L^2 projection onto $\mathbb{V}_h^{d\times d}$ of $\epsilon(u_h)$, namely $\pi_h\epsilon(u_h)$, it is possible to take advantage of the term $2\eta_p\epsilon(u_h)$ present in the third equation of (1.2) to obtain control of the velocity gradient even when $\eta_s = 0$. Indeed, an iterative procedure is used to decoupled the computation of the velocity-pressure to the extra stress as in Section 5 and the term

$$2\eta_p\left(\epsilon(u_h), \epsilon(v_h)\right) - 2\eta_p\left(\pi_h\epsilon(u_h^{\text{previous}}), \epsilon(v_h)\right)$$

is added to the momentum equation (first equation of (1.2)). Here u_h^{previous} correspond to velocity on the previous step of the iterative method. Refer for instance to [23, 8, 24]

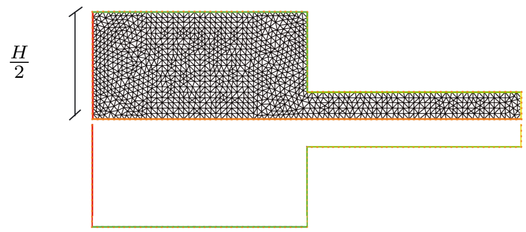

FIGURE 7.2. *Computational domain for the 4:1 contraction* (1233 *vertices and since* $k = 1$ *this corresponds to* $(2+1+3) \times 1233 = 7398$ *dof).*

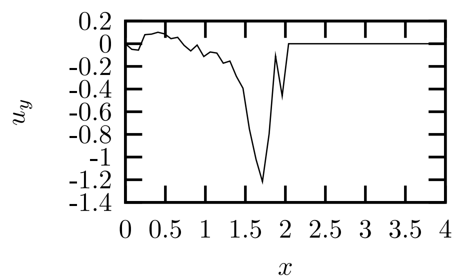

FIGURE 7.3. *(Left) 20 isovalues of the GLS method only for the pressure from -0.9 (black) to 0.06 (white) and (Right) profile of* $u_y(x, 0.025)$.

for a detailed description of the method. In addition, following the ideas of [25], the "reduced" GLS stabilization consists of stabilizing the pressure by adding the following term to the discrete problem

$$\sum_{e \in \epsilon_h} \frac{\alpha h^2}{\eta_p} \left(-\nabla \cdot (2\eta_s \epsilon(u_h) - p_h + \sigma_h) - f, \nabla q_h\right),$$

where $\alpha > 0$. Refer to [8] for a detailed description of those procedures as well as the link between them.

This test case underlines the importance of the stabilization of the constitutive equation. The symmetry of the geometry is used to reduce the computational domain by half, as shown in Fig. 7.2. Zero Dirichlet boundary conditions are imposed on the walls, the Poiseuille velocity profile (7.1) is imposed at the inlet with $V = 64\ m/s$, $H = 0.05[m]$ and $\beta \equiv 0$, natural boundary conditions on the symmetry axis and at the outlet of the domain. For all the computations presented in this subsection we choose, $\eta_s = 0\ Pa\ s$, $\eta_p = 1\ Pa\ s$, $\lambda = 0\ s$. The results applying only Galerkin Least Square (GLS) are shown in Fig. 7.3.

Similar results obtained using the EVSS method and the CIP ($\gamma_u = 0.1, \gamma_p = 0.1, \gamma_s = 0$) formulation are presented in Fig. 7.4. Note that the choice $\gamma_s = 0$ is possible since $\beta \equiv 0$ in that case.

The number of iterations N needed by the algorithm described in Section 5 to achieve (7.2) are provided in Figures 7.5 and 7.6. The robustness with respect to the stabilization parameter γ_p is clearly observed, see Figure 7.5. The algorithm is more dependent on the value of the parameter γ_u but provides reasonable number of iteration for $\gamma_u \in [5.10^{-2}, 10]$, see Figure 7.6.

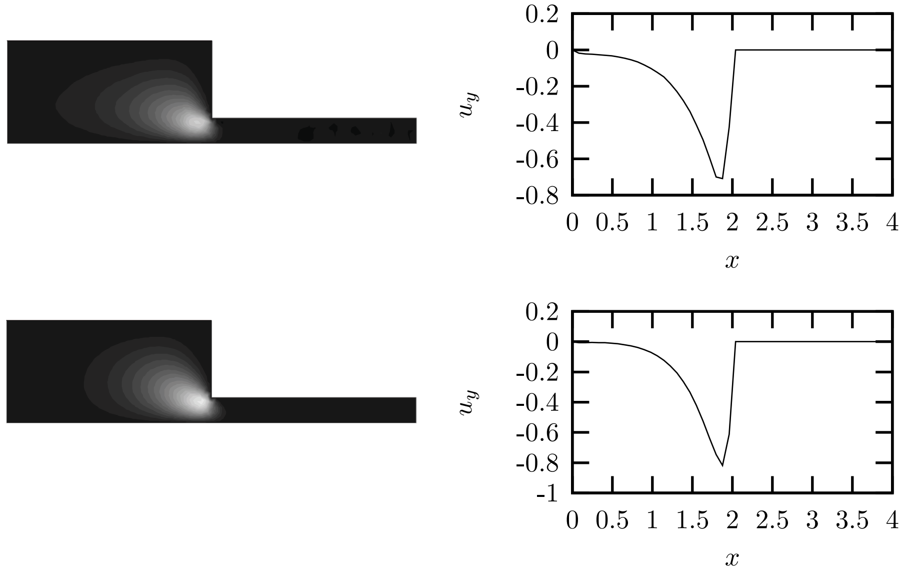

FIGURE 7.4. *Left column: 20 isovalues from -0.9 (black) to 0.06 (white), right column: profile of $u_y(x, 0.025)$ (top: EVSS, bottom: CIP).*

η_s	η_p	V	γ_p	γ_u	γ_σ
$0.59\ Pa\ s$	$0.41\ Pa\ s$	$3/8\ m/s$	0.1	0.1	0.1

TABLE 7.1
Flow past a cylinder; parameters used.

7.3. The flow past a cylinder in a channel. In this test case we consider the nonlinear problem (1.1). A cylinder of radius $R = 1[m]$ is placed at a distance $L_0 = 5[m]$ symmetrically in a $2D$ channel of width $H = 4[m]$ and length $L = 20[m]$. (see fig. 7.7). Boundary conditions are imposed as follows. Poiseuille flow at the inlet and outlet:

$$u(0,y) = u(L,y) = \begin{pmatrix} u_x(y) \\ 0 \end{pmatrix}, \qquad \sigma(0,y) = \sigma(L,y) = \begin{pmatrix} \sigma_{11}(y) & \sigma_{12}(y) \\ \sigma_{12}(y) & 0 \end{pmatrix},$$

where $u_x(y)$, $\sigma_{11}(y)$ and $\sigma_{12}(y)$ are given by (7.1). The pressure is imposed to be zero at the outflow, $p(y, L) = 0$. No-slip condition are imposed on the sphere and on the upper wall whilst the y component of the velocity is imposed to vanish on the lower wall. The parameters are chosen such as described in table 7.1. Due to the nonlineary of the considered problem, we propose to seek a solution of (1.1) as the stationary limit of the corresponding evolution problem. Therefore to (1.1) we add the time derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial \sigma}{\partial t}$. The time step used to reach the stationary state was chosen to be 0.01 for all the simulations and the algorithm stopped when

$$\frac{\|\nabla(u^n - u^{n-1})\|_{L^2(\Omega)}}{\|\nabla u^0\|_{L^2(\Omega)}} < 1.e^{-6}. \tag{7.2}$$

The characteristics of the three different meshes used are provided in table 7.2. We define the *Deborah* number, $De = \frac{\lambda V H^2}{6R}$, which will be the non-dimensional

γ_p	N
10^{-5}	23
10^{-3}	25
10^{-1}	26
1	26
10	26

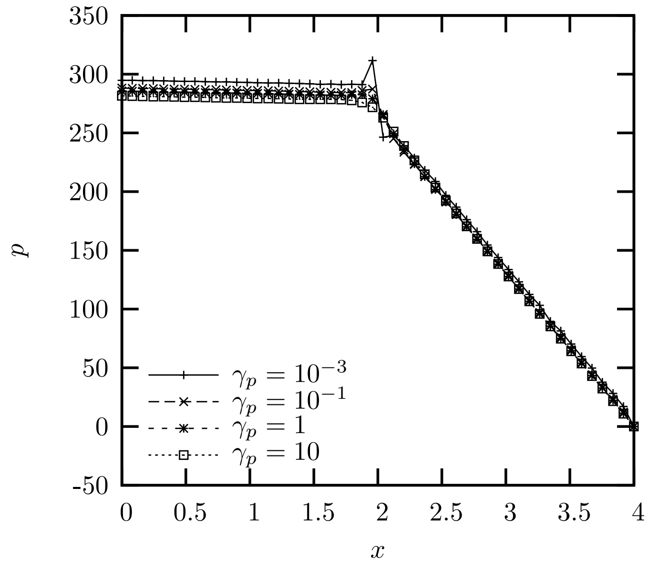

FIGURE 7.5. *Robustness of the number of iterations with respect to the parameter γ_p (when $\gamma_u = 0.1$)). (Left) Number of iterations for the algorithm to converge for several values of γ_p. (Right) Plot of some corresponding pressure profiles $p(x, 0.025)$.*

γ_u	N
5.10^{-2}	44
10^{-1}	26
1	7
5	4
10	3

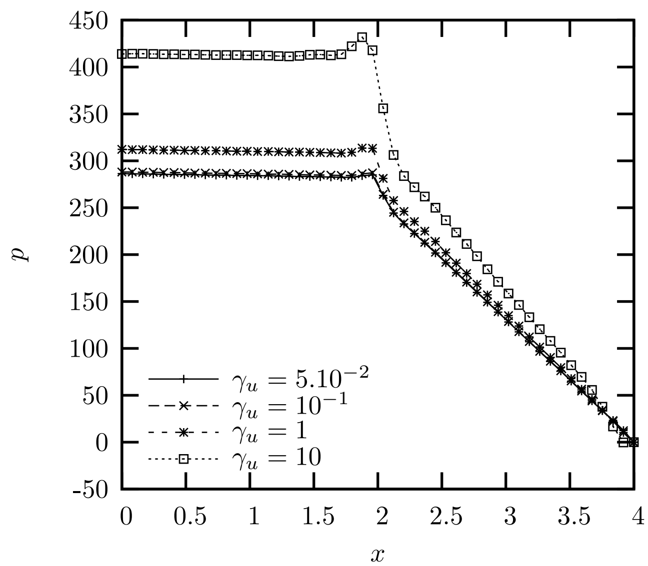

FIGURE 7.6. *Robustness of the number of iterations with respect to the parameter γ_u (when $\gamma_p = 0.1$)). (Left) Number of iterations for the algorithm to converge for several values of γ_u. (Right) Plot of some corresponding pressure profiles $p(x, 0.025)$.*

	coarse	intermediate (int)	fine
h $[m]$	0.5	0.25	0.125

TABLE 7.2
Flow past a cylinder; meshes used.

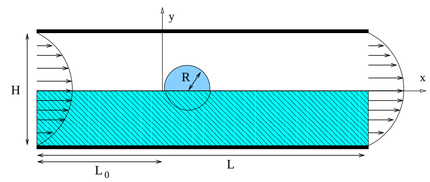

FIGURE 7.7. *Cylinder radius R placed symmetrically in a $2D$ planar channel of half width.*

De	coarse	int	fine
0.5	9.32	9.51	9.51
0.7	9.42	9.51	9.45
1	10.07	9.92	9.67
1.5	11.76	11.47	$\times\times\times$

TABLE 7.3

Flow past a cylinder; Drag factor F^* without non-linear stabilization.

parameter used to characterize the viscoelasticity of the fluid. We compare our results with results presented in the literature by means of the drag factor F^* defined by

$$F^* = \frac{F}{4\pi(\eta_s + \eta_p)V/(6H^2)},$$

where F is the drag on the cylinder. Drag factors when $\gamma_{\rm nl} = 0$ (no non-linear stabilization used) are shown in Table 7.3 ($\times\times\times$ means that the pseudo time-stepping scheme did not reach stationary state or the iterative scheme did not converge for the time step used). The values match those of [34, 16, 42] and those presented in [35, Tab. 9.1] (cf. Table 7.4).

We now turn on the non linear stabilization term and set $\gamma_{\rm nl} = 0.1$, drag factors are shown in table 7.4. For comparison we give the results obtained by Dou and Phan-Thien [16] and Sun et al. [42]. For further comparisons we refer to [35, Tab. 9.1]. When the nonlinear stabilization is added higher Deborah number can be reached. It is interesting to see that the behavior of the method changes. When no nonlinear stabilization is present the method can converge on coarse meshes, whereas no convergence is obtained on finer meshes, this typically indicates that the method is unstable: increasing the number of degrees of freedom makes the algorithm deteriorate. In the stabilized case the situation is the opposite, for high Deborah number the algorithm does not converge on coarse meshes due to divergence of the extra-stress. On fine enough meshes however the method does converge, indicating that in this case the increasing number of freedom may be used to resolve the nonlinear operator without losing stability. We also observed that the magnitude of the nonlinear stabilization decrease under refinement.

The profiles of the component σ_{11} of the extra-stress on the cylinder surface and on the wake of the cylinder at $De = 0.5$ (fig 7.8), $De = 1.5$ (fig 7.9), $De = 2.5$ (fig 7.10), $De = 3$ (fig 7.11) are reported. For low Deborah numbers the coarse mesh approximations underestimate the stresses. However, for high Deborah numbers, the stresses on the cylinder are strongly overestimated on coarse meshes, see Figures 7.9 and 7.10.

Acknowledgments. The authors are thankful to the anonymous referees for constructive comments.

De	coarse	int	fine	Dou and Phan-Thien [16]	Sun et al. [42]
0.5	8.99	9.35	9.47	9.59	9.48
0.7	8.99	9.29	9.39	9.64	9.38
1	9.59	9.66	9.60	10.1	9.49
1.5	12.6	11.8	10.76	11.7	10.1
2	29.7	14.6	12.22	-	-
2.5	93.9	19.6	13.65	-	-
3	$\times\times\times$	$\times\times\times$	15.12	-	-

TABLE 7.4

Flow past a cylinder; Drag factor with the non-linear stabilization. For comparison the results of references [16] and [42]

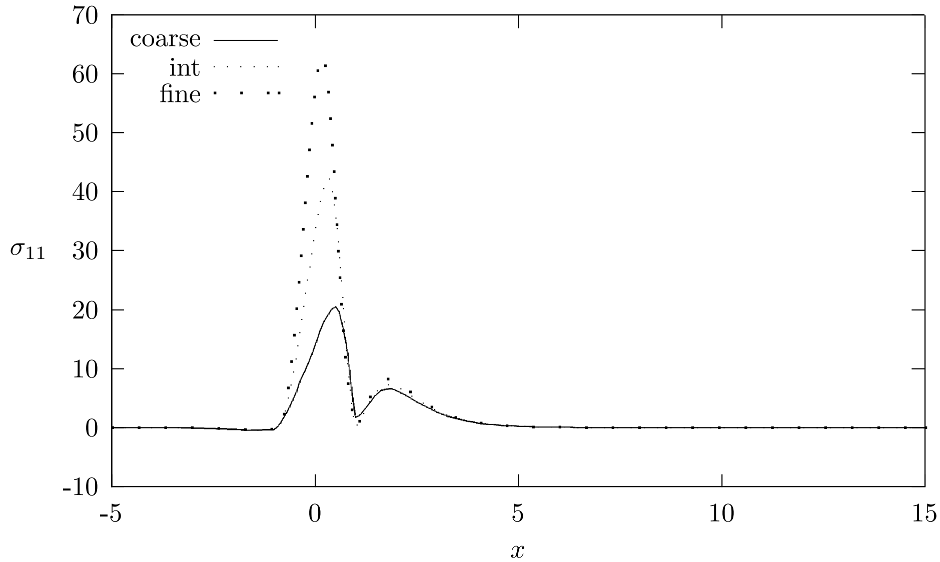

FIGURE 7.8. *Flow past a cylinder:* σ_{11}, $De = 0.5$.

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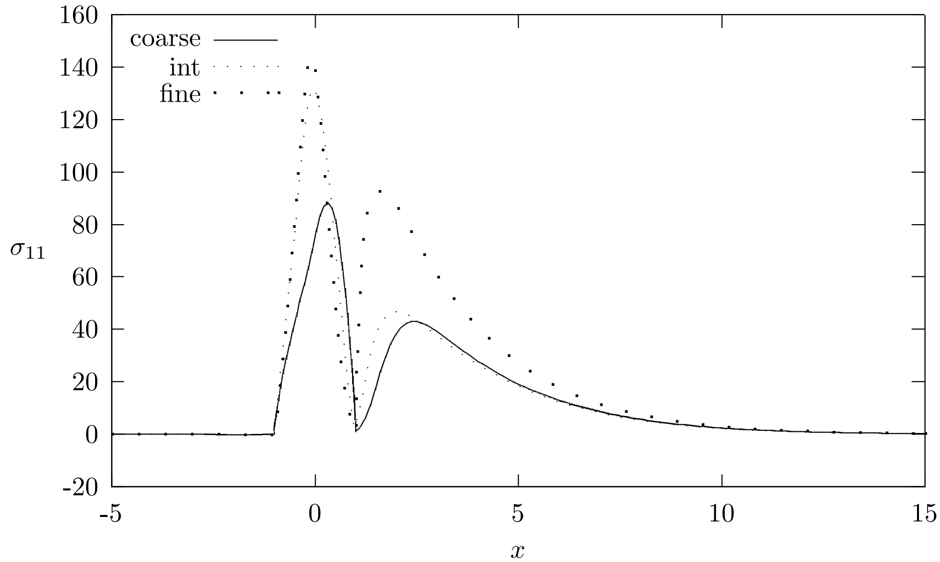

FIGURE 7.9. *Flow past a cylinder: σ_{11}, $De = 1.5$.*

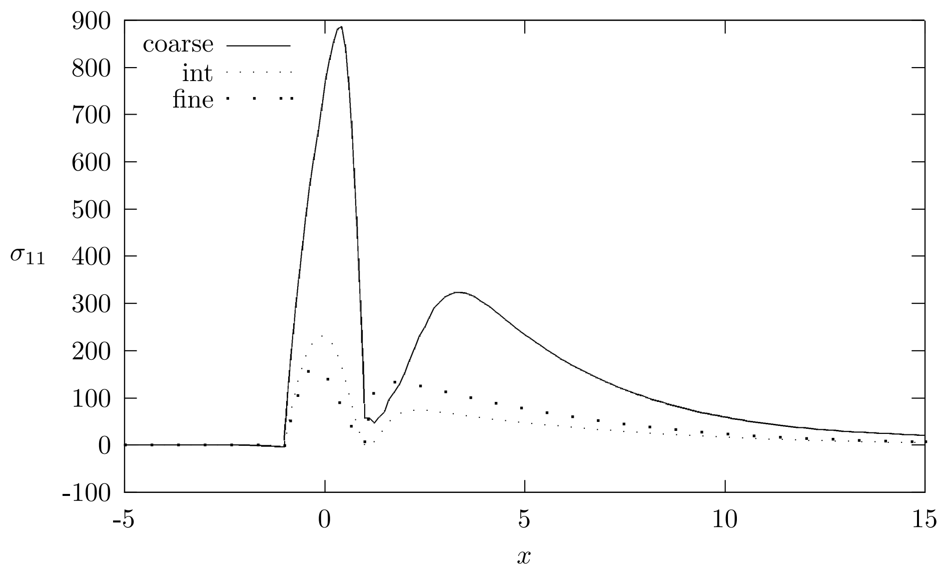

FIGURE 7.10. *Flow past a cylinder: σ_{11}, $De = 2.5$.*

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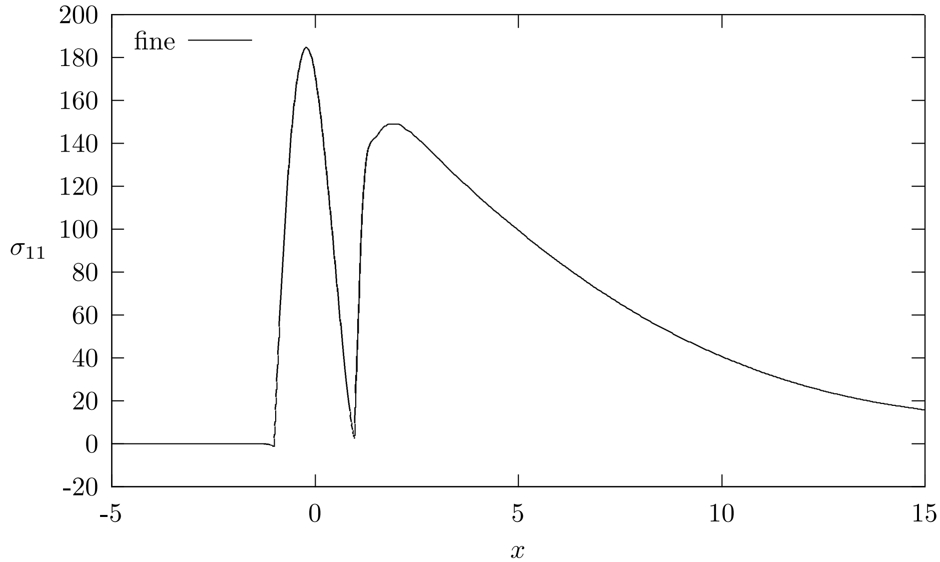

FIGURE 7.11. *Flow past a cylinder: σ_{11}, $De = 3$.*

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