ADAPTIVE FINITE ELEMENTS: TAKE HOME EXAM

Matlab Project

Download the compressed tar file matlab_project.tbz and extract it using the command

tar -xvjf matlab_project.tbz .

The main file AFEM.m is a matlab implementation of an adaptive finite element method to approximate the solution of

$$\begin{aligned} -\Delta u &= 0 \qquad \text{in} \qquad \Omega \\ u &= g \qquad \text{on} \qquad \partial\Omega. \end{aligned}$$

Here

$$\Omega = [-1,1] \times [-1,1] \setminus ([0,1] \times [0,-1])$$

and the Dirichlet data g is given (in polar coordinates at $(0,0)$) by

$$g(r,\phi) = r^{2/3} \sin(\frac{2}{3}\phi).$$

The adaptive method is based on the discussion we had in class (Chapter 1) except that the routine MARK selects elements for refinement using two different strategies:

(1) Mark all the elements $K \in \mathcal{T}$ such that

$$\eta_{\mathcal{T}}(U,K) \geq 0.5 \max_{K' \in \mathcal{T}} \eta_{\mathcal{T}}(U,K').$$

(2) Uniform refinement, i.e. mark all the triangles.

The variable “refinement” select which strategy is used. Both strategies update the vector VE containing the index of triangle marked for refinement. Try it out by running the command AFEM (inside matlab). You will get at each iteration a plot of the solution followed by the convergence rate.

You are asked to create a new marking strategy “adaptive_dorfler” based on the Dörfler marking strategy: Select $\mathcal{M}$ a subset of triangles with *minimal cardinality* satisfying

$$\eta_{\mathcal{T}}(U,\mathcal{M}) := \left(\sum_{K \in \mathcal{M}} \eta^2_{\mathcal{T}}(U,K) \right)^{1/2} \geq \theta\ \eta_{\mathcal{T}}(U)$$

for some $\theta \in (0,1)$. To do this, modify the file “Dorfler.m” defining the routine

$$\text{VE} = \text{Dorfler}(\eta,\theta).$$

The parameter η is a vector collecting the estimator for each triangle (computed in the routine AFEM, so nothing to do) and θ is the Dorfler parameter.

The following matlab routines might be useful:

- sum(V): returns the sum of all the element of V.
- [out,indices] = sort(in,’descend’) stores in “out” the elements of “in” sorted by decreasing order. The vector “indices” stores the index mapping from *out* to *in*, i.e. out(j) = in(indices(j)).

After coding the Dörfler marking strategy, provide a discussion on

(1) The influence on the parameter θ.
(2) The comparison between the "uniform", "adaptive_residual", "adaptive_dorfler" cases.

For each item, provide the solutions and meshes at relevant iterations (not all) and the associated rate of convergence.

Reaction Diffusion Problem

The notations introduced in Chapter 1 in class are used in the following. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and consider the following problem

$$-\Delta u + a\ u = f \qquad \text{in} \qquad \Omega$$
$$\frac{\partial u}{\partial n} = g \qquad \text{on} \qquad \partial\Omega$$

where $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ and $a \in L^\infty(\Omega)$ with $\inf_{x\in\Omega} a(x) \geq A$ for some $A > 0$.

(1) Extend the estimator discussed in class to this case and prove the upper bound
$$|||u - U_k||| \preceq \eta_{\mathcal{T}_k}(U_k),$$
where
$$|||v||| := \left(||a^{1/2}v||^2_{L^2(\Omega)} + ||\nabla v||^2_{L^2(\Omega)}\right)^{1/2}.$$
(2) Consider the adaptive loop
$$SOLVE \to ESTIMATE \to MARK \to REFINE$$
and prove the error reduction property
$$|||u - U_k|||^2 = |||u - U_{k-1}|||^2 - |||U_k - U_{k-1}|||^2.$$
(3) Assume that the estimator satisfies the reduction property
$$\eta^2_{\mathcal{T}_k}(V_k) \leq (1+\delta)\{\eta^2_{\mathcal{T}_{k-1}}(V_{k-1}) - \lambda\eta^2_{\mathcal{T}_{k-1}}(V_{k-1}, \mathcal{M})\} + (1+\delta^{-1})C|||V_k - V_{k-1}|||^2,$$
$$\forall V_k \in \mathbb{V}(\mathcal{T}_k), \qquad \forall V_{k-1} \in \mathbb{V}(\mathcal{T}_{k-1})$$
where $\delta > 0$, $0 < \lambda < 1$ and C is a constant only depending on the data, the domain Ω, and the shape regularity constant of the initial mesh. Prove that following contraction property holds
$$|||u - U_k|||^2 + \gamma\eta^2_{\mathcal{T}_k}(U_k) \leq \alpha\left(|||u - U_{k-1}|||^2 + \gamma\eta^2_{\mathcal{T}_{k-1}}(U_{k-1})\right),$$
for some $\gamma > 0$ and $0 < \alpha < 1$.