Exercise 1 **30%**

FEM for Two-Point Boundary Value Problems Using Finite Differences

Let $I = (0, 1)$ and consider the ODE

$$-(k(x)u'(x))' + u(x) = f(x), \qquad x \in I, \qquad u(0) = u(1) = 1,$$

where $k(x) = 1 + x$ and

$$f(x) = \begin{cases} x^2 \sin(\pi x) & 0 < x < 1/2 \\ \sqrt{x} & 1/2 < x < 1. \end{cases}$$

1. Write a second order finite difference scheme to approximate the solution of the above ODE. To check your implementation, pick randomly a non trivial u satisfying the boundary condition and compute the corresponding f. Then verify that you observe the asymptotic rate predicted by the theory.
   HINT: You do not need to define a lifting of the boundary conditions but observe that non vanishing boundary conditions only affect your right hand side.
2. Compute an approximation of the solution corresponding to $f(x) = \sin(4\pi x)$ (using sufficiently enough break points).
3. Same questions with the boundary conditions $u(0) = 0$ and $u'(1) = 0$.

Exercise 2 **70%**

FEM for Two-Point Boundary Value Problems Using Linear Elements

Write a program for solving two-point boundary value problems for second order ordinary differential equations by Ritz-Galerkin method using linear finite elements. Submit a report with graphs of the results, table with the error in discrete L^2- and maximum-norms, and comments.

Specifications

1. Use double precision. For solving the corresponding system of linear equations use the program from LAPACK that you discussed on your lab (or any other program you have).
2. Use 25, 50, 100, 200 linear finite elements. Plot the solution. In a table give the error in L^2, H^1, L^∞ and make also plots of the error norms versus the mesh size.

Computational examples - solve the following problems:

1. The deflection of a uniformly loaded, long rectangular plate under axial tension force and fixed ends, for small deflections, is governed by the second order differential equation. Let S represent the axial force and q the intensity of the uniform load. The deflection W along the elemental length is given by:

$$W''(x) - \frac{S}{D}W(x) = -\frac{qx}{2D}(l - x), \quad 0 < x < l, \quad W(0) = W(l) = 0, \tag{1}$$

where l is the length of the plate, and D is the flexural rigidity of the plate. Take $q = 200\ lb/in^2$, $S = 100\ lb/in.$, $D = 8.8\ 10^7\ lb\ in$, and $l = 50\ in$. Consider also the cases $S = 1000\ lb/in$ and $S = 10000\ lb/in$. The exact solution is given by: $a = \frac{Sl^2}{D}$, $b = \frac{ql^4}{2D}$, $t = x/l$ and

$$W(t) = \frac{b}{a}\left(-t^2 + t - \frac{2}{a} + \frac{2}{a\ \sinh(\sqrt{a})}[\sinh(\sqrt{a}t) + \sinh(\sqrt{a}(1-t))]\right).$$

2. Consider the problem (1) when the r.h.s. $\frac{qx}{2D}(l-x)$ is replaced by the constant $\frac{q}{2D}$. The exact solution for $Q = \frac{ql^2}{2D}$ is:

$$W(t) = \frac{Q}{a}\left(1 - \frac{1}{\sinh\sqrt{a}}\left(\sinh(\sqrt{a}t) + \sinh(\sqrt{a}(1-t))\right)\right).$$

3. Consider the problem (1) in the case when the right end of the plate is free, i.e. instead of the boundary condition $W(l) = 0$ now we have the condition $W'(l) = 0$. The exact solution in this case is:

$$W(t) = \frac{b}{a}\left(-t^2 + t - \frac{2}{a} + \frac{1}{a\ \cosh(\sqrt{a})}[\sqrt{a}\sinh(\sqrt{a}t) + 2\cosh(\sqrt{a}(1-t))]\right).$$

If the r.h.s. $\frac{qx}{2D}(l-x)$ is replaced by the constant $\frac{q}{2D}$ then the exact solution is (with $Q = \frac{ql^2}{2D}$):

$$W(t) = \frac{Q}{a}\left(1 + \frac{\sinh\sqrt{a}}{\cosh\sqrt{a}}\sinh(\sqrt{a}t) - \cosh(\sqrt{a}t)\right).$$

4. A thin rod made of three different materials with insulated lateral surface has ends kept at temperature 0 and $\frac{4}{\pi} + \frac{3}{2}$ degrees respectively. The steady state distribution of the temperature $u(x)$ is a solution to the problem: $-(ku')' = 0, \quad x \in (0,1), \quad u(0) = 0, \quad u(1) = \frac{4}{\pi} + \frac{3}{2}$, where $k(x) = 1$ for $x \in (0, \pi/6)$; $k(x) = 2$, for $x \in (\pi/6, \pi/4)$, and $k(x) = 3$ for $x \in (\pi/4, 1)$. The exact solution $u(x)$ is a piece-wise linear function defined as $\frac{12}{\pi}x$, $\frac{6}{\pi}x + 1$, and $\frac{4}{\pi}x + \frac{3}{2}$ in the corresponding intervals of definition of $k(x)$.