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FINITE ELEMENT ANALYSIS OF A SIMPLIFIED STOCHASTIC HOOKEAN DUMBBELLS MODEL ARISING FROM VISCOELASTIC FLOWS*

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Abstract. A simplified stochastic Hookean dumbbells model arising from viscoelastic flows is considered, the convective terms being disregarded. A finite element discretization in space is proposed. Existence of the numerical solution is proved for small data, so as a priori error estimates, using an implicit function theorem and a regularity results obtained in [8] for the solution of the continuous problem.

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Introduction

Numerical modeling of viscoelastic flows is of great importance for complex engineering applications involving blood, paints or adhesives. In the traditional macroscopic approach the unknowns are the velocity, the pressure and the extra-stress satisfying the mass and momentum equations supplemented with a so-called constitutive equation. This constitutive equation between the velocity and the stress can be either differential or integral and can be justified by a kinetic theory [7, 45].

The simplest macroscopic example is the Oldroyd-B model which can be derived from to the mesoscopic Hookean dumbbells model. The stochastic dumbbells model corresponds to a dilute solution of liquid polymer, that is a newtonian solvent with non interacting polymer chains. The polymer chains are modeled by dumbbells, two beads connected with elastic springs, see Fig. 1.

The mass and momentum conservation laws lead to the following partial differential equations for the velocity u, the pressure p and the extra-stress σ

$$\rho\left(\frac{\partial u}{\partial t}+(u\cdot\nabla)u\right)-\nabla\cdot(2\eta_s\epsilon(u)+\sigma)+\nabla p=f, \tag{0.1}$$

$$\nabla\cdot u=0. \tag{0.2}$$

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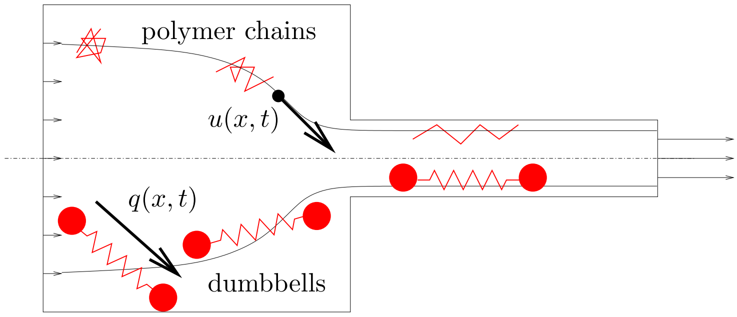

FIGURE 1. The mesoscopic dumbbells model for a dilute solution of liquid polymer.

Here ρ is the density, f a force term, η_s is the solvent viscosity and $\epsilon(u) := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$ is the symmetric part of the velocity gradient. On the other hand, the Newton law for the beads leads to the following stochastic differential equation for the dimensionless spring elongation q

$$dq = \left(-(u\cdot\nabla)q) + (\nabla u)q - \frac{1}{2\lambda}\mathsf{F}(q)\right)dt + \frac{1}{\sqrt{\lambda}}\,dB, \tag{0.3}$$

where λ is the relaxation time, F is the force due to the elastic spring and B is a vector of independent Wiener processes modeling the thermal agitation and collisions with the solvent molecules. The transport term $(u\cdot\nabla)q$ in (0.3) corresponds to the fact that the trajectories of the dumbbells center of mass are those of the liquid particles. The term $(\nabla u)q$ takes into account the drag force due to the beads. The extra-stress σ is then obtained by the mean of the closure equation

$$\sigma = \frac{\eta_p}{\lambda}(\mathbb{E}(q\otimes\mathsf{F}(q)) - I), \tag{0.4}$$

with η_p the polymer viscosity. The case $\mathsf{F}(q) = q$, namely Hookean dumbbells, leads to the Oldroyd-B model where the extra-stress σ satisfies

$$\sigma + \lambda\left(\frac{\partial\sigma}{\partial t} + (u\cdot\nabla)\sigma - (\nabla u)\sigma - \sigma(\nabla u)^T\right) = 2\eta_p\epsilon(u). \tag{0.5}$$

The FENE dumbbells model (see [7,45] for a detailed description) is a more realistic model corresponding to $F(q) = \frac{q}{1-q^2/b}$, where $b > 0$ depends on the number of monomer units of a polymer chain. The goal of the FENE model is to take into account the finite extensibility of the polymer chains. In that case, there is no equivalent constitutive relation for the extra-stress, but closure approximations (such as FENE-P, see for instance [7,45]) have been derived. These approximations can have significant impact on rheological prediction, see for instance [1,15,35]. Recently, due to increasing computational resources, equations (0.1)-(0.4) have been solved numerically to obtain more realistic results [12,14,16,35,38,39]. For a review of numerical methods used in viscoelastic flows we refer for instance to [3,36,46].

The kinetic theory can also be formulated by introducing the probability density $f(x,q,t)$ of the spring elongation which must satisfy a Fokker-Planck equation. We refer to [18,23,24] for numerical experiments and [6,50] for a mathematical analysis. This deterministic approach seems to be inappropriate when considering more complex kinetic models involving chains [36], although recent advances are encouraging [54].

We will focus in this paper on the stochastic description of the simplest dumbbells model, namely the Hookean dumbbells model $F(q) = q$. Although the Hookean dumbbells model is too simple to reproduce experiments such as shear thinning for instance, it already contains some numerical difficulties included in the kinetic theory. At the continuous level, the model is formally equivalent to the Oldroyd-B model but the equivalence does not hold anymore when considering finite element discretizations. Thus, to the major difficulties already present in the macroscopic model, we must add those coming from stochastic modeling. These difficulties are:

i. The presence of the quadratic term $(\nabla u)q$ which prevents a priori estimates to be obtained and therefore existence and convergence to be proved for any data;
ii. The presence of the convective term $(u \cdot \nabla)q$ which requires an adequate mathematical analysis [40] and the use of numerical schemes suited to transport dominated problems;
iii. The case $\eta_s = 0$ which require either a compatibility condition between the finite element spaces for u, q and p or the use of adequate stabilization procedures, such as EVSS for instance.
iv. The Wiener process in (0.3) requires efficient procedures such as variance reduction to be considered, see for instance [11, 16, 32, 36].

Concerning the analysis and numerical analysis of macroscopic viscoelastic models, a large amount of publications can be found. The existence of slow steady viscoelastic flow has been proved in [2, 49]. For the time-dependent case, existence of solutions locally in time and, for small data, globally in time has been proved in [31] in Hilbert spaces. Extensions to Banach spaces and a review can be found in [26]. Finally, existence for any data has been proved in [41] for a corotational Oldroyd model only. Convergence of finite element methods for the linear three fields Stokes problem have been studied for instance in [13, 27, 28, 51]. Convergence of continuous and discontinuous finite element methods for steady state viscoelastic fluids have been presented in [4, 25, 44, 52], provided the solution of the continuous problem is smooth and small enough. Extension to time-dependent problems have been proposed in [5, 21, 22, 43].

On the other hand, only a few papers pertaining to the kinetic theory have been published. The complete analysis and numerical analysis of a one dimensional FENE shear flows can be found in [33, 34]. The well posedeness of the dumbbells model in three space dimensions has been proved for nonlinear elastic dumbbells in [20]. Finally, an analysis of the Fokker-Plank equation has been performed in [6, 50].

In this paper, a finite element discretization in space is considered and the numerical analysis is proposed for a simplified time-dependent Hookean dumbbells problem in two space dimensions. More precisely, we focus on item i. and iv. above, assume $\eta_s > 0$ and remove the convective terms. The reason for removing the convective terms is motivated by the fact that this simplified problem corresponds to the correction step in the splitting algorithm described in [10, 30] for solving viscoelastic flows with complex free surfaces. The consequence when removing convective terms is that the implicit function theorem can be used to prove convergence results, whenever the data are small enough. Existence and regularity has already been proved in [8], this regularity being sufficient to prove convergence of a finite element discretization in space.

The outline of the paper is as follows. The continuous problem and its finite elements scheme are described in the next section. Then, some notations and the results of [8] are presented. Finally, existence of the finite element solution and a priori error estimates are established.

1. The simplified Hookean dumbbells problem and its finite element approximation in space

Let D be a bounded, connected open set of $\mathbb{R}^d$, $d = 2$ or 3 with boundary ∂D of class $\mathcal{C}^2$, and let $T > 0$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete filtered probability space. The filtration $\mathcal{F}_t$ upon which the Brownian process B is defined is completed with respect to $\mathcal{P}$ and is assumed to be right continuous on $[0, T]$.

Given the initial conditions $q_0 : \Omega \to \mathbb{R}^d$, $u_0 : D \to \mathbb{R}^d$, a force term f, constant solvent and polymer viscosities $\eta_s > 0$, $\eta_p > 0$, a constant relaxation time $\lambda > 0$, we are searching for the velocity $u : D \times (0, T) \to \mathbb{R}^d$, the pressure $p : D \times (0,T) \to \mathbb{R}$ and the elongation vector $q : D \times (0,T) \times \Omega \to \mathbb{R}^d$, which must satisfy

$$dq - \left((\nabla u) q - \frac{1}{2\lambda} q \right) dt - \frac{1}{\sqrt{\lambda}}\, dB = 0 \qquad \text{in } D \times (0,T) \times \Omega, \tag{1.1}$$

$$\rho \frac{\partial u}{\partial t} - \nabla \cdot \left(2\eta_s \epsilon(u) + \frac{\eta_p}{\lambda} \left(\mathbb{E}(q \otimes q) - I \right) \right) + \nabla p = f \qquad \text{in } D \times (0,T), \tag{1.2}$$

$$\nabla \cdot u = 0 \qquad \text{in } D \times (0,T), \tag{1.3}$$

$$u\,(.,0) = u_0 \qquad \text{in } D, \tag{1.4}$$

$$q(.,0,.) = q_0 \qquad \text{in } D \times \Omega, \tag{1.5}$$

$$u = 0 \qquad \text{on } \partial D \times (0,T). \tag{1.6}$$

Remark 1.1. Equations (1.1) and (1.5) are notations for

$$q(x,t,\omega) - q_0(t,\omega) + \int_0^t \left((\nabla u(x,s)) q(x,s,\omega) - \frac{1}{2\lambda} q(x,s,\omega) \right) ds - \frac{1}{\sqrt{\lambda}} B(t,\omega) = 0,$$

where $(x,t,\omega) \in D \times [0,T] \times \Omega$.

Due to the regularity of the Brownian process, little Hölder spaces will be used. They are closed subset of the classical Hölder spaces $\mathcal{C}^\mu([0,T];E)$ and are defined for all Banach space E and for all $0 < \mu < 1$ by

$$h^\mu([0,T];E) := \{ f \in \mathcal{C}^\mu([0,T];E); \lim_{\delta \to 0} \sup_{t,s \in I, |t-s| < \delta} \frac{\| f(t) - f(s) \|_E}{|t-s|^\mu} = 0 \}.$$

Provided with the norm of $\mathcal{C}^\mu([0,T];E)$, little Hölder spaces are Banach spaces and are separable Banach spaces assuming E is a separable Banach space, see for instance [42] for more details. We will use the notation $h_0^\mu([0,T];E)$ for the restriction of functions of $h^\mu([0,T];E)$ vanishing at the origin. For simplicity, the notation will be abridged as follows whenever there is no possible confusion. For $d < r < \infty$, the space L^r denotes $L^r(D;\mathbb{R})$ or $L^r(D;\mathbb{R}^d)$. Also, for $0 < \mu < 1/2$ and $2 \leq \gamma < \infty$, $h^\mu(L^r)$ stands for $h^\mu([0,T];L^r(D;\mathbb{R}))$ or $h^\mu([0,T];L^r(D;\mathbb{R}^d))$ and $L^\gamma(h^\mu(L^r))$ for $L^\gamma(\Omega; h^\mu([0,T];L^r(D;\mathbb{R})))$ or $L^\gamma(\Omega; h^\mu([0,T];L^r(D;\mathbb{R}^d)))$. The same notation applies for higher order spaces such as $W^{1,r}$, $h^{1+\mu}(W^{1,r})$ and $L^\gamma(h^{1+\mu}(W^{1,r}))$.

The implicit function theorem has been used in [8] to prove that (1.1)-(1.6) admits a unique solution

$$u \in h^{1+\mu}(L^r) \cap h^\mu(W^{2,r}), \qquad p \in h^\mu(W^{1,r} \cap L_0^r), \qquad q \in L^\gamma(h^\mu(W^{1,r})), \tag{1.7}$$

with $2 \leq \gamma < \infty$, $d < r < \infty$ and $0 < \mu < 1/2$, for any data (f, u_0) small enough in appropriate spaces and assuming the space Ω is rich enough to accommodate a given random vector $q_0 \in L^\gamma(\Omega)$ such that

$$\begin{cases} q_0 \text{ is independent of } B \text{ and } (q_0)_i \text{ is independent of } (q_0)_j, 1 \leq i \neq j \leq d, \\ \text{and } \mathbb{E}(q_0) = 0,\ \mathbb{E}(q_0 \otimes q_0) = I. \end{cases} \tag{1.8}$$

Since $h^\mu([0,T];W^{1,r}(D)) \subset \mathcal{C}([0,T] \times \overline{D})$, let us notice that in particular, a process $q \in L^\gamma(h^\mu(W^{1,r}))$ has a continuous sample path for almost each realization $\omega \in \Omega$.

In this paper we assume that the above existence result still holds when D is a convex polygon in $\mathbb{R}^2$. The key point to prove this result when D is a convex polygon is to prove that the negative Stokes operator $-A_r$ is still the generator of an analytic semi-group, see for instance [29]. We did not find such a result in the literature, therefore we will make this assumption and prove convergence of the finite element scheme. It should be noted that the corresponding property is true in stationary case for some $r > 2$ depending on the angles of the polygon, see [47].

Let us introduce the finite element approximation in space for D, a convex polygon in $\mathbb{R}^2$. For any $h > 0$, let $\mathcal{T}_h$ be a decomposition of D into triangles K with diameter h_K less than h, regular in the sense of [19]. We consider V_h, R_h and Q_h the finite element spaces for the velocity, dumbbells elongation and pressure, respectively defined by :

$$\begin{aligned}
V_h &:= \{v_h \in \mathcal{C}^0(\overline{\Omega};\mathbb{R}^d); v_h\mid_K \in (\mathbb{P}_1)^d \quad \forall K \in \mathcal{T}_h\} \cap H^1_0(\Omega;\mathbb{R}^d),\\
R_h &:= \{r_h \in \mathcal{C}^0(\overline{\Omega};\mathbb{R}^d); r_h\mid_K \in (\mathbb{P}_1)^d \quad \forall K \in \mathcal{T}_h\},\\
Q_h &:= \{s_h \in \mathcal{C}^0(\overline{\Omega};\mathbb{R}); s_h\mid_K \in \mathbb{P}_1 \quad \forall K \in \mathcal{T}_h\} \cap L^2_0(\Omega;\mathbb{R}).
\end{aligned}$$

We denote i_h the $L^2(D)$ projection onto V_h, R_h or Q_h and introduce the following stabilized finite element discretization in space of (1.1)-(1.6). Given f, u_0, q_0 find

$$\begin{aligned}
(u_h, q_h, p_h) : \Omega \times (0,T) &\longrightarrow V_h \times R_h \times Q_h,\\
(\omega, t) &\longmapsto (u_h(t), q_h(\omega,t), p_h(t)),
\end{aligned}$$

such that $u_h(0) = i_h u_0$, $q_h(0) = q_0$ and such that the following weak formulation holds in $(0,T)\times\Omega$:

$$\begin{aligned}
\rho\Big(\frac{\partial u_h}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(u_h), \epsilon(v_h)\Big) - \Big(p_h, \nabla\cdot v_h\Big) + \frac{\eta_p}{\lambda}\Big(\mathbb{E}(q_h\otimes q_h) - I, \epsilon(v_h)\Big) - \Big(f, v_h\Big)&\\
+ \Big(\nabla\cdot u_h, s_h\Big) + \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}(\nabla p_h, \nabla s_h)_K + (q_h(t), r_h) - (1, r_h)q_0&\\
+ \Big(\int_0^t \Big(\frac{1}{2\lambda}q_h(k) - (\nabla u_h(k))q_h(k)\Big)\,dk, r_h\Big) - \frac{1}{\sqrt{\lambda}}(1, r_h)B = 0,& \quad (1.9)
\end{aligned}$$

for all $(v_h, r_h, s_h) \in V_h \times R_h \times Q_h$. Here $\alpha > 0$ is a dimensionless stabilization parameter and $(\cdot,\cdot)$ (respectively $(\cdot,\cdot)_K$) denotes the $L^2(D)$ (resp. $L^2(K)$) scalar product for scalars, vectors and tensors.

Using an implicit function theorem taken from [17], existence and convergence will be proved for small data f and u_0, the difficulty being again due to the fact that no a priori estimates are available due to the nonlinear term $(\nabla u_h)q_h$. Therefore, we will proceed as in the continuous problem, see [8]. More precisely, it will be proven that the linearized problem in the neighborhood of the equilibrium state $u_h = 0$, $q_h = q^S$ is well posed (q^S will be defined in the next section).

The above nonlinear finite element scheme is closely linked to the Oldroyd-B scheme studied in a previous paper [9]. However, the numerical schemes are not equivalent, therefore the analysis has to be done again. Moreover, it should be noted that in this paper the case $\eta_s = 0$ is not considered, therefore some of the stabilization terms present in [12] are not included in the finite element formulation (1.9).

2. Preliminaries on the continuous problem

In this section, notations and results from [8] are recalled to the reader.

The proof of the existence of a solution (u, q, p) satisfying (1.1)-(1.6) with the regularity (1.7) is based on the splitting

$$q = q^S + q^D.$$

The equilibrium state q^S is a stochastic process independent of the space variable $x \in D$ and satisfies

$$dq^S = -\frac{1}{2\lambda}q^S dt + \frac{1}{\sqrt{\lambda}}dB, \qquad q^S(0) = q_0, \tag{2.1}$$

whilst q^D is the discrepancy with respect to the equilibrium q^S. The unknown function q^D satisfies a differential equation with a stochastic forcing term

$$\frac{\partial q^D}{\partial t} - \nabla u\ (q^D + q^S) + \frac{1}{2\lambda} q^D = 0, \qquad q(0) = 0. \tag{2.2}$$

Let $(X, \|.\|_X)$ be the Banach space defined by

$$X = \left\{(u,q) \in h^{1+\mu}(L^r) \cap h^\mu(W^{2,r}) \times L^\gamma(h^{1+\mu}(W^{1,r}) \cap h_0^\mu(W^{1,r})); q \text{ adapted to } (\mathcal{F}_t)_{t\in[0,T]}\right\},$$

and let $\|.\|_X$ be the product norm. Since existence (and uniqueness) of $q^S \in L^\gamma(\Omega; h^\mu([0,T]))$ is ensured by classical results on stochastic differential equations, existence an uniqueness of problem (1.1)-(1.6) for small data rise from existence and uniqueness of $(u, q^D, p) \in X \times h^\mu(W^{1,r} \cap L_0^r)$ solution of

$$\begin{aligned}
&\frac{\partial q^D}{\partial t} - \nabla u\ (q^D + q^S) + \frac{1}{2\lambda} q^D = 0 && \text{in } D \times [0,T] \times \Omega, && (2.3)\\
&\rho \frac{\partial u}{\partial t} - \nabla \cdot \epsilon(u) - \frac{\eta_p}{\lambda} \nabla \cdot \big(\mathbb{E}((q^D + q^S) \otimes (q^D + q^S) - I\big) + \nabla p = f && \text{in } D \times (0,T), && (2.4)\\
&\nabla \cdot u = 0 && \text{in } D \times (0,T), && (2.5)\\
&u\,(.,0) = u_0 && \text{in } D, && (2.6)\\
&q(.,0,.) = 0 && \text{in } D \times \Omega, && (2.7)\\
&u = 0 && \text{on } \partial D \times (0,T). && (2.8)
\end{aligned}$$

More precisely, given $q^S \in L^\gamma(\Omega; h^\mu([0,T]))$ and $(f, u_0) \in Y$ small enough, there exists a unique $(u, q^D, p) \in X \times h^\mu(W^{1,r} \cap L_0^r)$ solution (2.3)-(2.8). The mapping

$$\begin{aligned}
Y &\longrightarrow X,\\
(f, u_0) &\longmapsto (u(f,u_0), q^D(f,u_0))
\end{aligned}$$

being analytic (therefore continuous). The space for the data Y is a subset of $h^\mu(L^r) \times W^{2,r}$, which will be defined precisely later in this section. The above result is based on properties of the linearized problem: given $q_0 \in L^\gamma(\Omega)$ satisfying (1.8), $(f_1, u_0) \in Y$, $f_2 \in U$ and $w \in W$, find $(\tilde{u}, \tilde{q}^D, \tilde{p}) \in X \times h^\mu(W^{1,r} \cap L_0^r)$ such that

$$\begin{aligned}
&\rho \frac{\partial \tilde{u}}{\partial t} - 2\eta_s\ \nabla \cdot \epsilon(\tilde{u}) && &&\\
&\quad - \frac{\eta_p}{\lambda} \nabla \cdot \big(\mathbb{E}(\tilde{q}^D \otimes q^S + q^S \otimes \tilde{q}^D) + f_2\big) + \nabla \tilde{p} = f_1 && \text{in } D \times (0,T), && (2.9)\\
&\nabla \cdot \tilde{u} = 0 && \text{in } D \times (0,T), && (2.10)\\
&\frac{\partial \tilde{q}^D}{\partial t} + \frac{1}{2} \tilde{q}^D - (\nabla \tilde{u}) q^S = w && \text{in } D \times (0,T) \times \Omega, && (2.11)\\
&\tilde{u}(\cdot,0) = u_0 && \text{in } D, && (2.12)\\
&\tilde{q}^D(\cdot,0) = 0 && \text{in } \Omega, && (2.13)\\
&\tilde{u} = 0 && \text{on } \partial\Omega \times (0,T), && (2.14)
\end{aligned}$$

where $q^S \in L^\gamma(\Omega; h^\mu([0,T]))$ is defined by (2.1),

$$U = \left\{f_2 \in h^\mu(W^{1,r}); \nabla \cdot f_2(0) = 0\right\}$$

and

$$W = \{w \in L^\gamma(h^\mu(W^{1,r})); w \text{ adapted to } (\mathcal{F}_t)_{t\in[0,T]}\}$$

are Banach spaces endowed with the norm of $h^\mu(W^{1,r})$ and $L^\gamma(h^\mu(W^{1,r}))$ respectively.

The space Y will be now defined. Let us introduce the Helmholtz-Weyl projector $P_r : L^r \to \mathcal{H}_r$, where $\mathcal{H}_r$ is the completion of the divergence free $C_0^\infty(D)$ vector fields with respect to the norm of L^r. The Stokes operator $A_r = -P_r\Delta$ with domain $\mathcal{D}_{A_r} = W^{2,r} \cap W_0^{1,r} \cap \mathcal{H}_r$ and range $\mathcal{H}_r$ will be necessary to characterize the space for the data Y. For this purpose, let

$$E_{\mu,\infty} = (\mathcal{H}_r, \mathcal{D}_{A_r})_{\mu,\infty} = \left\{ x \in \mathcal{H}_r; \sup_{t>0} \|t^{1-\mu} A_r e^{-tA_r} x\|_{L^r(D)} < +\infty \right\}$$

be a Banach space endowed with the norm

$$\|x\|_{E_{\mu,\infty}} := \|x\|_{L^r(D)} + \sup_{t>0} \|t^{1-\mu} A_r e^{-tA_r} x\|_{L^r(D)}.$$

We will consider the data (f, u_0) belonging to Y defined by

$$Y = \{(f, u_0) \in h^\mu(L^r) \times \mathcal{D}_{A_r} \text{ such that } -\eta_s A_r u_0 + P_r f(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}\},$$

provided with the norm $\|.\|_Y$ defined by

$$\|f, u_0\|_Y = \|f\|_{h^\mu(L^r)} + \|u_0\|_{W^{2,r}} + \| - \eta_s A_r u_0 + P_r f(0)\|_{\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}}.$$

Finally, it should be noted that the solution $(\tilde{u}, \tilde{p}) \in h^{1+\mu}(L^r) \cap h^\mu(W^{2,r} \cap W_0^{1,r}) \times h^\mu(W^{1,r} \cap L_0^r)$ of (2.9)-(2.14) satisfies

$$\rho\frac{\partial \tilde{u}}{\partial t} - \nabla\cdot(2\eta_s\epsilon(\tilde{u}) - 2k * \epsilon(\tilde{u})) + \nabla\tilde{p} = f_1 + \frac{\eta_p}{\lambda}\nabla\cdot f_2 + \nabla\cdot g, \quad \nabla\tilde{u} = 0, \quad \tilde{u}(.,0) = u_0, \tag{2.15}$$

where $k \in \mathcal{C}^\infty([0,T])$ is defined for $t \in [0,T]$ by $k(t) := \frac{\eta_p}{\lambda}e^{-\frac{t}{\lambda}}$, $g \in h_0^\mu(\mathcal{H}_r)$ is defined for $t \in [0,T]$ by

$$g(t) := \frac{\eta_p}{\lambda}\int_0^t e^{-\frac{t-s}{2\lambda}}\mathbb{E}\left(w(s)\otimes q^S(t) + q^S(t)\otimes w(t)\right)ds, \tag{2.16}$$

and $k * \epsilon(\tilde{u})$ is the convolution in time of the kernel k with $\epsilon(\tilde{u})$. Moreover, there exists a constant $C > 0$ independent of f_1, f_2, u_0 and w such that

$$\|\frac{\partial\tilde{u}}{\partial t}\|_{h^\mu(L^r)} + \|A_r\tilde{u}\|_{h^\mu(L^r)} + \|\tilde{p}\|_{h^\mu(W^{1,r})} \le C\left(\|f_1, u_0\|_Y + \|f_2\|_{h^\mu(W^{1,r})} + \|w\|_{L^\gamma(h^\mu(W^{1,r}))}\right). \tag{2.17}$$

In this paper, we will assume the results presented in this section still hold when D is a convex polygon. Once again, the key point to prove these results when D is a convex polygon is to prove that the negative Stokes operator $-A_r$ is still the generator of an analytic semi-group, see for instance [29].

3. The finite element approximation and a priori error estimates

In order to prove that the solution of the nonlinear finite element discretization (1.9) exists and converges to that of (1.1)-(1.6), we introduce $X_h \subset X$ defined by

$$X_h = L^2(V_h) \times L^2(L^\infty(R_h)),$$

provided with the norm $\|.\|_{X_h}$ defined for all $x_h = (u_h, q_h) \in X_h$ by

$$\|x_h\|^2_{X_h} = 2\eta_s \int_0^T \|\epsilon(u_h(t))\|^2_{L^2(D)} dt + \int_\Omega \sup_{t\in[0,T]} \|q_h(\omega,t)\|^2_{L^2(D)}\, d\mathcal{P}(\omega).$$

The splitting $q_h = q^S + q^D_h$ will also be used for space discretization (remember q^S does not depend on the space variable and satisfies (2.1)) where $q^D_h \in L^2(L^\infty(R_h))$ satisfies

$$(q^D_h(t), r_h) + \left(\int_0^t \left(\frac{1}{2\lambda} q^D_h(k) - (\nabla u_h(k))(q^S(k) + q^D_h(k))\right) dk, r_h\right) = 0, \tag{3.1}$$

for all $r_h \in R_h$, a.e in $(0,T)$ and a.e in Ω.

It will be shown that there exists a unique $(u_h, q^D_h) \in X_h$ converging to $(u, q^D) \in X$ and thus a unique (u_h, q_h) converging to (u, q). For this purpose, the discrete problem corresponding to the unknowns (u_h, q^D_h, p_h) will be written in the abstract framework of [17]. Using the splitting $q_h = q^S + q^D_h$, we rewrite the solution of (1.9) as the following fixed point problem. Given $y = (f, u_0) \in Y$, find $x_h = (u_h, q^D_h) \in X_h$ such that

$$x_h = \mathsf{T}_h\Big(y, S_c(x_h), S_d(x_h)\Big), \tag{3.2}$$

where

$$\begin{aligned} S_c : L^2(H^1) \times L^2(L^\infty(L^2)) &\to L^2(L^2) \\ x_h = (u_h, q^D_h) &\longmapsto S_c(x_h) = \mathbb{E}(q^D_h \otimes q^D_h), \end{aligned}$$

$$\begin{aligned} S_d : L^2(H^1) \times L^2(L^\infty(L^2)) &\to L^2(L^2(L^2)) \\ x_h = (u_h, q^D_h) &\longmapsto S_d(x_h) = (\nabla u_h) q^D_h. \end{aligned}$$

The linear operator $\mathsf{T}_h : Y \times L^2(L^2) \times L^2(L^2(L^2)) \longrightarrow X_h$ is defined as follow

$$(f_1, u_0, f_2, w) \longmapsto \mathsf{T}_h(f_1, u_0, f_2, w) := (\tilde{u}_h, \tilde{q}^D_h) \in X_h, \tag{3.3}$$

where for almost all $t \in (0,T)$ and almost all $\omega \in \Omega$

$$(\tilde{u}_h, \tilde{q}^D_h, \tilde{p}_h) : (\omega, t) \longmapsto (\tilde{u}_h(t), \tilde{q}^D_h(\omega, t), \tilde{p}_h(t)) \in V_h \times R_h \times Q_h$$

satisfies $\tilde{u}_h(0) := i_h u_0$ and

$$\begin{aligned} \rho\Big(\frac{\partial \tilde{u}_h}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(\tilde{u}_h), \epsilon(v_h)\Big) - \Big(\tilde{p}_h, \nabla\cdot v_h\Big) + \frac{\eta_p}{\lambda}\Big(\mathbb{E}(\tilde{q}^D_h \otimes q^S + q^S \otimes \tilde{q}^D_h) + f_2, \epsilon(v_h)\Big) \\ - \Big(f_1, v_h\Big) + \Big(\nabla\cdot\tilde{u}_h, s_h\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}_h, \nabla s_h\Big)_K \\ (\tilde{q}^D_h(t), r_h) + \left(\int_0^t \left(\frac{1}{2\lambda}\tilde{q}^D_h(k) - (\nabla\tilde{u}_h(k)) q^S(k)\right) dk, r_h\right) - (w, r_h) = 0, \end{aligned} \tag{3.4}$$

for all $(v_h, r_h, s_h) \in V_h \times R_h \times Q_h$, a.e. in $(0,T)$ and a.e. in Ω.

It should be noticed that, given $y = (f, u_0) \in Y$ sufficiently small, the solution $x(y) := (u(y), q^D(y)) \in X$ of the continuous Hookean dumbbells problem (2.3)-(2.8) also satisfies a fixed point problem, namely

$$x(y) = \mathsf{T}\Big(y, S_c(x(y)), S_d(x(y))\Big). \tag{3.5}$$

Here the operator T is defined by

$$\begin{aligned} &\mathsf{T} : Y \times U \times W \to X \\ &(f_1, u_0, f_2, w) \to \mathsf{T}(f_1, u_0, f_2, w) \underset{\text{def.}}{=} (\tilde{u}, \tilde{q}^D), \end{aligned}$$

where $(\tilde{u}, \tilde{q}^D, \tilde{p}) \in X \times h^\mu(W^{1,r} \cap L_0^2)$ satisfy (2.9)-(2.14). Problem (3.5) is well defined since it has been proved that for $x = (u, q^D) \in X$ we have $S_c(x) \in h^\mu(W^{1,r})$ and $S_d(x) \in W$ (see Remark 3.6 in [8]). Moreover, since $q^D(0) = 0$, it follows that $S_c(x) = \mathbb{E}(q^D \otimes q^D)$ vanishes at time $t = 0$ and thus $S_c(x) \in U$ for $x \in X$.

The elongation vector $\tilde{q}^D$ can be eliminated from (3.4) and the next Lemma provides the equation satisfied by $\tilde{u}$. This equation being a discrete approximation of (2.15).

Lemma 3.1. *Let $\gamma \geq 2$, $0 < \mu < 1/2$ and $r > 2$. Let $(f_1, u_0) \in Y$, $f_2 \in L^2(L^2)$, $w \in L^2(L^2(L^2))$ and let $q^S \in L^\gamma(h^\mu(W^{1,r}))$ be defined by (2.1). Then problem (3.4) admits a unique solution $(\tilde{u}_h, \tilde{q}_h^D) \in X_h$. Moreover, $(\tilde{u}_h, \tilde{p}_h)$ satisfies*

$$\begin{aligned} \rho\Big(\frac{\partial \tilde{u}_h}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(\tilde{u}_h), \epsilon(v_h)\Big) - \Big(\tilde{p}_h, \nabla\cdot v_h\Big) + 2\Big(k * i_h\epsilon(u_h), i_h\epsilon(v_h)\Big) \\ + \Big(\nabla\cdot\tilde{u}_h, s_h\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}_h, \nabla s_h\Big)_K = \Big(f_1, v_h\Big) + \Big(-\frac{\eta_p}{\lambda} f_2 + i_h g, \epsilon(v_h)\Big), \end{aligned} \tag{3.6}$$

where $k \in \mathcal{C}^\infty([0,T])$ is defined by $k(t) := \frac{\eta_p}{\lambda} e^{-t/\lambda}$ and where $g \in L^\gamma(h^\mu(W^{1,r}))$ is defined by (2.16) and where i_h is the $L^2(D)$ projection onto $R_h \otimes R_h$.

Proof. In order to prove the existence (and uniqueness) of a solution $(\tilde{u}_h, \tilde{q}_h^D)$, we will write (3.4) using a basis of R_h. Hence, let us introduce φ_i^n, $n = 1,\ 2$, $i = 1, \ldots, P$ an orthonormal basis of R_h where P is the number of nodes of the mesh. Let $\tilde{q}_{h,i}^{D,n}$ be the components of q_h^D and $\tilde{u}_{h,i}^n$ be those of u_h with respect to the given basis φ_i^n. We have

$$\begin{aligned} \tilde{q}_h^D(\omega, t, x) &= \sum_{n=1}^{2}\sum_{i=1}^{P} q_{h,i}^{D,n}(\omega, t)\varphi_i^n(x), \\ \tilde{u}_h(t, x) &= \sum_{n=1}^{2}\sum_{i=1}^{P} u_{h,i}^n(t)\varphi_i^n(x). \end{aligned}$$

Choosing $v_h = 0,\ s_h = 0,\ r_h = \varphi_i^n$ in (3.4) we have

$$\tilde{q}_{h,i}^{D,n}(t) = \int_0^t e^{-\frac{t-s}{2\lambda}} \left(\sum_{k=1}^{2} \frac{\partial \tilde{u}_h^n}{\partial x_k}(s) q^{S,k}(s) + w^n, \varphi_i^n\right) ds, \quad n = 1,\ 2,\ i = 1, \ldots, P,$$

a.e in $(0, T)$, a.e. in Ω and with $w = (w^1, w^2)^T$. The definition of the $L^2(D)$ projection i_h implies for $t \in [0, T]$

$$\tilde{q}_h^D(t) = \int_0^t e^{-\frac{t-s}{2\lambda}} \left(i_h(\nabla\tilde{u}_h)q^S + i_h w\right) ds, \tag{3.7}$$

a.e. in Ω. Going back to (3.4) we find that $(\tilde{u}_h, \tilde{p}_h)$ satisfies

$$\rho\Big(\frac{\partial \tilde{u}_h}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(\tilde{u}_h), \epsilon(v_h)\Big) - \Big(\tilde{p}_h, \nabla\cdot v_h\Big) + 2\Big(k * i_h\epsilon(u_h), \epsilon(v_h)\Big)$$
$$+ \Big(\nabla\cdot\tilde{u}_h, s_h\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}_h, \nabla s_h\Big)_K = \Big(f_1, v_h\Big) + \Big(-\frac{\eta_p}{\lambda} f_2 + i_h g, \epsilon(v_h)\Big).$$

Using the property of the L^2-projection

$$(i_h\epsilon(\tilde{u}_h), \epsilon(v_h) - i_h\epsilon(v_h)) = 0 \qquad \forall v_h \in V_h,$$

we obtain (3.6). Thus, problem (3.4) is equivalent to (3.6) and (3.7). Existence (and uniqueness) of $\tilde{u}_h \in \mathcal{C}^1([0,T]; V_h)$ satisfying (3.6) is ensured by a standard argument on Stokes system (see for instance [48]) and a contraction mapping theorem (see for instance [37] or Appendix A in [9]). Finally, since $q^S \in L^\gamma(h^\mu(W^{1,r}))$, equation (3.7) ensures the existence (and uniqueness) of $\tilde{q}_h^D \in L^\gamma(\mathcal{C}^1(R_h))$ thus in $L^2(L^\infty(R_h))$. □

Remark 3.2. We proved in the previous Lemma that $\tilde{q}_h^D$ belongs to $L^\gamma(\mathcal{C}^1(R_h))$, thus (3.1) can be rewritten

$$(\frac{\partial q_h^D}{\partial t}, r_h) + (\frac{1}{2\lambda} q_h^D, r_h) - ((\nabla u_h)(q^S + q_h^D), r_h) = 0 \quad \forall r_h \in R_h, \qquad \text{a.e. in } \Omega. \tag{3.8}$$

We have the following stability and convergence result.

Lemma 3.3. *The operator* T_h *is well defined and uniformly bounded with respect to* h*: there exists* $C_1 > 0$ *such that for all* $h > 0$ *and for* $(f_1, u_0) \in Y$*,* $f_2 \in L^2(L^2)$*,* $w \in L^2(L^2(L^2))$ *we have*

$$||\mathsf{T}_h(f_1, u_0, f_2, w)||_{X_h} \le C_1\Big(||f_1, u_0||_Y + ||f_2||_{L^2(L^2)} + ||w||_{L^2(L^2(L^2))}\Big). \tag{3.9}$$

Moreover, there exists $C_2 > 0$ *such that for all* $h > 0$ *and for all* $(f_1, u_0, f_2, w) \in Y \times U \times W$*, we have*

$$||(\mathsf{T} - \mathsf{T}_h)(f_1, u_0, f_2, w)||_{X_h} \le C_2 h\Big(||f_1, u_0||_Y + ||f_2||_U + ||w||_W\Big). \tag{3.10}$$

Proof. Let us use the notation $(\tilde{u}_h, \tilde{q}_h^D) := \mathsf{T}_h(f_1, u_0, f_2, w)$, where $(\tilde{u}_h, \tilde{q}_h^D) \in X_h$ satisfies (3.6). From Lemma 1 in [53], we have

$$\int_0^T \int_0^T e^{-\frac{t-s}{\lambda}} (i_h\epsilon(\tilde{u}_h(s)), i_h\epsilon(\tilde{u}_h(t)))\, ds\; dt \ge 0. \tag{3.11}$$

Therefore, choosing $v_h = u_h(t)$ in (3.6), there exists a constant C independent of $f_1,\ f_2,\ g$ such that

$$||\tilde{u}_h||_{L^2(H^1)} \le C\Big(||f_1, u_0||_Y + ||f_2||_{L^2(L^2)} + ||i_h g||_{L^2(L^2)}\Big), \tag{3.12}$$

where $g \in h_0^\mu(W^{1,r})$ is defined by (2.16). Moreover since i_h is bounded in $L^2(D)$, using the continuous embedding $h^\mu([0,T]) \subset_{>} L^\infty([0,T])$ and a Cauchy-Schwarz inequality, we have

$$\|i_h g\|_{L^2(L^2)} \le C\|q^S\|_{L^2(h^\mu)}\|w\|_{L^2(L^2(L^2))} \tag{3.13}$$

where C is a constant independent of $h,\ w,\ q^S$ and g.

On the other hand from (3.7) we have

$$\|\tilde{q}_h^D\|_{L^2(L^\infty(L^2))} \le C\left(\|\tilde{u}_h\|_{L^2(H^1)} + \|w\|_{L^2(L^2(L^2))}\right), \tag{3.14}$$

where C is a constant independent of h, f_1, u_0, f_2 and w. Thus (3.13) in (3.12) and (3.14) leads to (3.9). The proof of (3.10) is provided in Appendix A. □

Our goal is now to prove that (1.9) has a unique solution (u_h, q_h) converging to that of (2.3)-(2.8). Since q^S does not depend on $x \in D$, it suffices to show that (3.2) has a unique solution (u_h, q_h^D) converging to (u, q^D) solution of (1.1)-(1.6). For this purpose, we use, as in [9, 47], an abstract framework and write (1.9) as the following problem : given $y = (f, u_0) \in Y$, find $x_h = (u_h, q_h^D) \in X_h$ such that

$$F_h(y, x_h) = 0, \tag{3.15}$$

where $F_h : Y \times X_h \to X_h$ is defined by

$$F_h(y, x_h) = x_h - \mathsf{T}_h\Big(y, S_c(x_h), S_d(i_h x)\Big).$$

In order to prove existence and convergence, we use Theorem 2.1 of [17]. The mapping $F_h : Y \times X_h \to X_h$ is $\mathcal{C}^1$. We first prove that the scheme is consistent and that $D_x F$ is locally Lipschitz.

Lemma 3.4. *Let $y := (f, u_0) \in Y$ be sufficiently small, let $x(y) := (u(y), q^D(y)) \in X$ be the solution of (2.3)-(2.8). Then, there exists a constant C_1 such that for all $0 < h < 1$, for all $y \in Y$ we have*

$$\|F_h(y, i_h x(y))\|_{X_h} \le C_1 h\Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2\Big). \tag{3.16}$$

Moreover, there exists a constant C_2 such that for all $h > 0$, for all $y \in Y$, for all $z \in X_h$ we have

$$\|D_x F_h(y, i_h x(y)) - D_x F_h(y, z)\|_{\mathcal{L}(X_h)} \le \frac{C_2}{h}\|i_h x(y) - z\|_{X_h}. \tag{3.17}$$

Proof. From the definition of F_h we have

$$\begin{aligned} F_h(y, i_h x) &= i_h x - x - \mathsf{T}_h\Big(y, S_c(i_h x), S_d(i_h x)\Big) \\ &= (i_h x - x) + \mathsf{T}_h(0, 0, S_c(i_h x) - S_c(x), S_d(x) - S_d(i_h x)) + (\mathsf{T} - \mathsf{T}_h)(y, S_c(x), S_d(x)) \end{aligned}$$

so that,

$$\begin{aligned} \frac{1}{3}\|F_h(y, i_h x)\|_{X_h}^2 \le& \|i_h x - x\|_{X_h}^2 + \|\mathsf{T}_h(0, 0, S_c(i_h x) - S_c(x), S_d(x) - S_d(i_h x))\|_{X_h}^2 \\ &+ \|(\mathsf{T} - \mathsf{T}_h)(y, S_c(x), S_d(x))\|_{X_h}^2. \end{aligned}$$

Using standard interpolation results for the first term of the right hand side, Lemma 3.3 for the second and third terms, it follows that

$$\begin{aligned} \|F_h(y, i_h x)\|_{X_h}^2 \le C\Big(&h^2\|x\|_X^2 + \|S_c(x) - S_c(i_h x)\|_{L^2(L^2)}^2 + \|S_d(x) - S_d(i_h x)\|_{L^2(L^2(L^2))}^2 \\ &+ h^2\|y\|_Y^2 + h^2\|S_c(x)\|_{h^\mu(W^{1,r})} + h^2\|S_d(x)\|_{L^2(h^\mu(W^{1,r}))}^2\Big), \end{aligned} \tag{3.18}$$

C being independent of h and y. Proceeding as in Corollary 3.5 in [8], we have

$$\|S_c(x)\|_{h^\mu(W^{1,r})} + \|S_d(x)\|_{L^2(h^\mu(W^{1,r}))} = \|\frac{\lambda}{\eta_p}\mathbb{E}(q^D \otimes q^D)\|_{h^\mu(W^{1,r})} + \|(\nabla u)q^D\|_{L^2(h^\mu(W^{1,r}))} \le C\|x\|_X^2, \tag{3.19}$$

C being independent of h and y. On the other hand, we also have

$$\begin{aligned} S_d(x) - S_d(i_h x) &= (\nabla u) q^D - (\nabla i_h u) i_h q^D \\ &= (\nabla (u - i_h u)) q^D + (\nabla i_h u)(q^D - i_h q^D) \end{aligned}$$

so that, using a Cauchy-Schwarz inequality

$$\|S_d(x) - S_d(i_h x)\|^2_{L^2(L^2(L^2))} \leq C\|x - i_h x\|_{X_h} \left(\|q^D\|_{L^2(L^\infty(L^\infty))} + \|\nabla i_h u\|_{L^2(L^\infty)}\right)$$

C being independent of h and y. Standard interpolation results lead to

$$\|x - i_h x\|_{X_h} \leq C_1 h \|x\|_X$$

and

$$\|\nabla i_h u\|_{L^2(L^\infty)} \leq \|\nabla u\|_{L^2(L^\infty)} + \|\nabla (u - i_h u\|_{L^2(L^\infty)} \leq C_2 \|u\|_{h^\mu(W^{2,r})},$$

for $0 < h < 1$ and where C_1, C_2 are constants independent of h and x. Thus we obtain

$$\|S_d(x) - S_d(i_h x)\|_{L^2(L^2(L^2))} \leq Ch\|x\|_X, \tag{3.20}$$

C being independent of h and y. Similarly, we obtain

$$\|S_c(x) - S_c(i_h x)\|_{L^2(L^2)} \leq Ch\|x\|_X. \tag{3.21}$$

Finally, (3.19), (3.20) and (3.21) in (3.18) yields (3.16).

Let us now prove (3.17). Let $z = (v, r) \in X_h$, let $\tilde{z} := (\tilde{v}, \tilde{r}) \in X_h$, we have

$$\Big(D_x F_h(y, i_h x) - D_x F_h(y, z)\Big)\tilde{z} = -\mathsf{T}_h\Big(0, 0, (DS_c(i_h x) - DS_c(z))\tilde{z}, (DS_d(i_h x) - DS_d(z))\tilde{z}\Big).$$

Using Lemma 3.3 we obtain

$$\begin{aligned} &\| \left(D_x F_h(y, i_h x) - D_x F_h(y, z)\right) \tilde{z}\|_{X_h} \\ &\qquad \leq C \left(\|(DS_c(i_h x) - DS_c(z))\tilde{z}\|_{L^2(L^2)} + \|(DS_d(i_h x) - DS_d(z))\tilde{z}\|_{L^2(L^2(L^2))}\right), \end{aligned} \tag{3.22}$$

C being independent of h and y. Using Cauchy-Schwarz inequality, there exists a constant C independent of h and y such that

$$\|(DS_d(i_h x) - DS_d(z))\tilde{z}\|_{L^2(L^2(L^2))} \leq C\Big(\|\nabla(i_h u - v)\|_{L^2(L^\infty)}\|\tilde{r}\|_{L^2(L^\infty(L^2))} + \|\nabla \tilde{v}\|_{L^2(L^\infty)}\|i_h q^D - r\|_{L^2(L^\infty(L^2))}\Big).$$

A classical inverse inequality yields

$$\|(DS_d(i_h x) - DS_d(z))\tilde{z}\|_{L^2(L^2(L^2))} \leq \frac{\tilde{C}}{h}\Big(\|\nabla(i_h u - v)\|_{L^2(L^2)}\|\tilde{r}\|_{L^2(L^\infty(L^2))} + \|\nabla \tilde{v}\|_{L^2(L^2)}\|i_h q^D - r\|_{L^2(L^\infty(L^2))}\Big),$$

$\tilde{C}$ being independent of h and y, so that we finally have

$$\|(DS_d(i_h x) - DS_d(z))\tilde{z}\|_{L^2(L^2(L^2))} \leq \frac{\tilde{C}}{h}\|i_h x - z\|_{X_h}\|\tilde{z}\|_{X_h}. \tag{3.23}$$

Similarly, we obtain

$$\begin{aligned}\|(DS_c(i_hx)-DS_c(z))\tilde{z}\|_{L^2(0,T;L^2(D))} &\le C\|i_hq-r\|_{L^2(L^\infty(L^\infty))}\|\tilde{r}\|_{L^2(L^\infty(L^2))}\\ &\le \frac{\tilde{C}}{h}\|i_hx-z\|_{X_h}\|\tilde{z}\|_{X_h}. \end{aligned} \tag{3.24}$$

Inequalities (3.23) and (3.24) in (3.22) yields (3.17). □

Before proving existence of a solution to (3.15) we still need to check that $D_xF_h(y,i_hx)$ is invertible.

Lemma 3.5. *Let $y := (f,u_0) \in Y$ be sufficiently small, let $x(y) := (u(y), q^D(y)) \in X$ be the solution of (2.3)-(2.8). Then, for y sufficiently small, for all $h \le 1$ we have*

$$\|D_xF_h(y,i_hx(y))^{-1}\|_{\mathcal{L}(X_h)} \le 2.$$

Proof. By definition of F_h, we have

$$D_xF_h(y,i_hx) = I - \mathsf{T}_h(0,0,DS_c(i_hx),DS_d(i_hx)),$$

so that we can write

$$D_xF_h(y,i_hx) = I - G_h \quad \text{with} \quad G_h := \mathsf{T}_h(0,0,DS_c(i_hx),DS(i_hx)).$$

If we prove that $\|G_h\|_{\mathcal{L}(X_h)} \le 1/2$ for y sufficiently small, then $D_xF_h(y,i_hx)$ is invertible and

$$\|D_xF_h(y,i_hx)^{-1}\|_{\mathcal{L}(X_h)} \le 2.$$

Let $z := (v,\tau) \in X_h$. Using Lemma 3.3 we have

$$\|G_h(z)\|_{X_h} \le C_1\left(\|DS_c(i_hx)z\|_{L^2(L^2)} + \|DS_d(i_hx)z\|_{L^2(L^2(L^2))}\right),$$

C_1 being independent of y and h. Proceeding as in the proof of Lemma 3.4, we have

$$\|DS_d(i_hx)z\|_{L^2(L^2(L^2))} \le C_2\Big(\|\nabla u\|_{L^2(W^{1,r})}\|\tau\|_{L^2(L^\infty(L^2))} + \|\nabla v\|_{L^2(L^2)}\|q^D\|_{L^2(L^\infty(W^{1,r}))}\Big),$$

C_2 being independent of y, h and z. Hence,

$$\|G_h(z)\|_{X_h} \le C_3\|x\|_X\|z\|_{X_h},$$

where C_3 is independent of y, h and z. From Lemma 3.10 in [8], the mapping $y \to x(y)$ is continuous, thus if $||y||_Y$ is sufficiently small we have $\|x\|_X \le 1/(2C_3)$ so that

$$\|G_h(z)\|_{X_h} \le \frac{1}{2}\|z\|_{X_h}.$$

□

We can now prove existence of a solution to the finite element scheme (1.9) and convergence to the solution of (1.1)-(1.6).

Theorem 3.6. *Let $y := (f, u_0) \in Y$ be sufficiently small, let $x(y) := (u(y), q^D(y)) \in X$ be the solution of (2.3)-(2.8). Then, for y and h sufficiently small, there exists a unique $x_h(y) = (u_h(y), q_h^D(y))$ in the ball of X_h centered at $i_h x(y)$ with radius ζh, satisfying*

$$F_h(y, x_h(y)) = 0.$$

Moreover, the mapping $y \to x_h(y)$ is continuous and there exists $C > 0$ independent of h and y such that the following a priori error estimate holds

$$\|x(y) - x_h(y)\|_{X_h} \le Ch. \tag{3.25}$$

In order to prove the above theorem, we will use the following abstract result.

Lemma 3.7 (Th. 2.1 of [17])**.** *Let Y and Z be two real Banach spaces with norms $\|.\|_Y$ and $\|.\|_Z$ respectively. Let $G : Y \to Z$ be a $\mathcal{C}^1$ mapping and $v \in Y$ be such that $DG(v) \in \mathcal{L}(Y; Z)$ is an isomorphism. We introduce the notations*

$$\begin{aligned}
\epsilon &= \|G(v)\|_Z,\\
\gamma &= \|DG(v)^{-1}\|_{\mathcal{L}(Y;Z)},\\
L(\alpha) &= \sup_{x \in \overline{B}(v,\alpha)} \|DG(v) - DG(x)\|_{\mathcal{L}(Y;Z)},
\end{aligned}$$

with $\overline{B}(v, \alpha) = \{y \in Y; \|v - y\|_Y \le \alpha\}$, and we are interested in finding $u \in Y$ such that

$$G(u) = 0. \tag{3.26}$$

We assume that $2\gamma L(2\gamma\epsilon) \le 1$. Then Problem (3.26) has a unique solution u in the ball $\overline{B}(v, 2\gamma\epsilon)$ and, for all $x \in \overline{B}(v, 2\gamma\epsilon)$, we have

$$\|x - u\|_Y \le 2\gamma \|G(x)\|_Z.$$

Proof of Theorem 3.6. We apply Lemma 3.7 with $Y = X_h,\ Z = X_h,\ G = F_h$ and $v = i_h x(y)$. According to Lemma 3.4 there exists a constant C_1 independent of y and h such that

$$\epsilon = \|F_h(y, i_h x(y))\|_{X_h} \le C_1 h\Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2\Big).$$

According to Lemma 3.5, for $\|y\|_Y$ sufficiently small

$$\gamma = D_x F_h(y, i_h x(y))_{\mathcal{L}(X_h)} \le 2.$$

According to Lemma 3.4, there is a constant C_2 independent of y and h such that

$$L(\alpha) = \sup_{x \in \overline{B}(i_h x(y),\alpha)} \|DF_h(i_h x(y)) - DF_h(x)\|_{\mathcal{L}(X_h)} \le \frac{C_2}{h}\alpha.$$

Hence, we have

$$\begin{aligned}
2\gamma L(2\gamma\epsilon) &\le 2.2\frac{C_2}{h}\left(2.2C_1 h\Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2\Big)\right)\\
&= 16C_1C_2\Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2\Big).
\end{aligned}$$

Using the continuity of the mapping $y \to x(y)$ it follows that, for sufficiently small $\|y\|_Y$ then $2\gamma L(2\gamma\epsilon) < 1$ and Lemma 3.7 applies. There exists a unique $x_h(y)$ in the ball $\overline{B}(i_h x(y), 2\gamma\epsilon)$ such that

$$F_h(y, x_h(y)) = 0$$

and we have

$$\|i_h x(y) - x_h(y)\|_{X_h} \leq 4C_1 h.$$

It suffices to use the triangle inequality

$$\|x(y) - x_h(y)\|_{X_h} \leq \|x(y) - i_h x(y)\|_{X_h} + \|i_h x(y) - x_h(y)\|_{X_h},$$

and standard interpolation results to obtain (3.25). The fact that the mapping $y \to (x_h(y))$ is continuous is a direct consequence of the implicit function theorem. □

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Appendix A. Proof of (3.10) in Lemma 3.3

Let

$$\begin{aligned} e_u &= \tilde{u} - \tilde{u}_h = \Pi_u + C_u, \qquad \Pi_u = \tilde{u} - i_h \tilde{u}, \qquad C_u = i_h \tilde{u} - \tilde{u}_h, \\ e_p &= \tilde{p} - \tilde{p}_h = \Pi_p + C_p, \qquad \Pi_p = \tilde{p} - i_h \tilde{p}, \qquad C_p = i_h \tilde{p} - \tilde{p}_h, \end{aligned}$$

where $(\tilde{u}_h, \tilde{p}_h)$ solve (3.6) and $(\tilde{u}, \tilde{p})$ solve (2.15) Using the triangle inequality we have

$$\|e_u\|_{L^2(H^1)} \leq \|\Pi_u\|_{L^2(H^1)} + \|C_u\|_{L^2(H^1)}.$$

Using classical interpolation results, we obtain

$$\|\Pi_u\|_{L^2(H^1)} \leq Ch\|\tilde{u}\|_{L^2(H^2)}.$$

We now estimate $\|C_u\|_{L^2(H^1)}$. The solution of (2.15) satisfies

$$\rho\Big(\frac{\partial \tilde{u}}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(\tilde{u}), \epsilon(v_h)\Big) - \Big(\tilde{p}, \nabla\cdot v_h\Big) + 2\Big(k * \epsilon(\tilde{u}), \epsilon(v_h)\Big) - \Big(f_1, v_h\Big) + \Big(\nabla\cdot\tilde{u}, s_h\Big) + \Big(\frac{\eta_p}{\lambda} f_2 - g, \epsilon(v_h)\Big) = 0$$

for all $(v_h, s_h) \in V_h \times Q_h$. Subtracting (3.6) to the above equation, it follows that

$$\begin{aligned} \rho\Big(\frac{\partial e_u}{\partial t}, v_h\Big) &+ 2\eta_s\Big(\epsilon(e_u), \epsilon(v_h)\Big) - \Big(e_p, \nabla\cdot v_h\Big) + 2\Big(k * (\epsilon(\tilde{u}) - i_h\epsilon(\tilde{u}_h)), \epsilon(v_h)\Big) \\ &+ \Big(\nabla\cdot e_u, r_h\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla e_p - \nabla\tilde{p}, \nabla r_h\Big)_K - \Big(g - i_h g, \epsilon(v_h)\Big) = 0, \quad \text{(A.1)} \end{aligned}$$

for all $(v_h, r_h) \in V_h \times Q_h$. On the other hand, from the definition of i_h (the L^2 projection onto the finite element spaces), C_u and Π_u, we have

$$\begin{aligned} \Big(k * i_h\epsilon(C_u), i_h\epsilon(C_u)\Big) &= \Big(k * i_h\epsilon(C_u), \epsilon(C_u)\Big) \\ &= \Big(k * (\epsilon(\tilde{u}) - i_h\epsilon(\tilde{u}_h)) + k * (i_h\epsilon(\tilde{u}) - \epsilon(\tilde{u})) - k * i_h\epsilon(\Pi_u), i_h\epsilon(C_u)\Big). \end{aligned}$$

Hence, we obtain

$$
\begin{aligned}
\rho\Big(\frac{\partial C_u}{\partial t}, C_u\Big) + 2\eta_s\Big(\epsilon(C_u), \epsilon(C_u)\Big) - \Big(C_p, \nabla\cdot C_u\Big) + 2\Big(k * i_h\epsilon(C_u), i_h\epsilon(C_u)\Big) \\
+ \Big(\nabla\cdot C_u, C_p\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla C_p, \nabla C_p\Big)_K \\
= \rho\Big(\frac{\partial(e_u - \Pi_u)}{\partial t}, C_u\Big) + 2\eta_s\Big(\epsilon(e_u - \Pi_u), \epsilon(C_u)\Big) - \Big(e_p - \Pi_p, \nabla\cdot C_u\Big) \\
+ 2\Big(k * (\epsilon(\tilde{u}) - i_h\epsilon(\tilde{u}_h)), \epsilon(C_u)\Big) - 2\Big(k * (\epsilon(\tilde{u}) - i_h\epsilon(\tilde{u})), \epsilon(C_u)\Big) - 2\Big(k * i_h\epsilon(\Pi_u), \epsilon(C_u)\Big) \\
+ \Big(\nabla\cdot(e_u - \Pi_u), C_p\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla(e_p - \Pi_p), \nabla C_p\Big)_K
\end{aligned}
\tag{A.2}
$$

From the definition of i_h again, we obviously have

$$
\Big(\frac{\partial \Pi_u}{\partial t}, C_u\Big) = 0,
$$

so that, using (A.1), (A.2) yields

$$
\begin{aligned}
&\rho\Big(\frac{\partial C_u}{\partial t}, C_u\Big) + 2\eta_s\Big(\epsilon(C_u), \epsilon(C_u)\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla C_p, \nabla C_p\Big)_K + 2\Big(k * i_h\epsilon(C_u), i_h\epsilon(C_u)\Big) \\
&= -2\eta_s\Big(\epsilon(\Pi_u), \epsilon(C_u)\Big) + \Big(\Pi_p, \nabla\cdot C_u\Big) - \Big(\nabla\cdot(\Pi_u), C_p\Big) - \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\Pi_p, \nabla C_p\Big)_K \\
&\quad + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}, \nabla C_p\Big)_K + 2\Big(k * (\epsilon(\tilde{u}) - i_h\epsilon(\tilde{u})), \epsilon(C_u)\Big) \\
&\quad + 2\Big(k * i_h\epsilon(\Pi_u), \epsilon(C_u)\Big) + (g - i_h g, \epsilon(C_u)) \\
&= I_1 + \cdots + I_8.
\end{aligned}
\tag{A.3}
$$

It now remains to bound the terms $I_1, ..., I_8$ in the above equality. Using Cauchy-Schwarz and Young's inequalities, we have

$$
\begin{aligned}
I_1 &= -2\eta_s\Big(\epsilon(\Pi_u), \epsilon(C_u)\Big) \\
&\leq 2\eta_s||\epsilon(\Pi_u)||_{L^2(\Omega)}||\epsilon(C_u)||_{L^2(\Omega)} \\
&\leq 5\eta_s||\epsilon(\Pi_u)||^2_{L^2(\Omega)} + \frac{\eta_s}{5}||\epsilon(C_u)||^2_{L^2(\Omega)}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
I_2 = \Big(\Pi_p, \nabla\cdot C_u\Big) &\leq \frac{5}{4\eta_s}||\Pi_p||^2_{L^2(\Omega)} + \frac{\eta_s}{5}||\nabla\cdot C_u||^2_{L^2(\Omega)} \\
&\leq \frac{5}{4\eta_s}||\Pi_p||^2_{L^2(\Omega)} + \frac{\eta_s}{5}||\epsilon(C_u)||^2_{L^2(\Omega)}.
\end{aligned}
$$

An integration by parts yields, since $\Pi_u = 0$ on $\partial\Omega$

$$I_3 = \Big(\nabla\cdot(\Pi_u), C_p\Big) = -\Big(\Pi_u, \nabla C_p\Big) = -\sum_{K\in\mathcal{T}_h}\Big(\Pi_u, \nabla C_p\Big)_K$$
$$\le \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{12\eta_p}||\nabla C_p||^2_{L^2(K)} + \sum_{K\in\mathcal{T}_h}\frac{3\eta_p}{\alpha h_K^2}||\Pi_u||^2_{L^2(K)}.$$

Again, Cauchy-Schwarz and Young's inequalities yield

$$I_4 = -\sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\Pi_p, \nabla C_p\Big)_K \le \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{12\eta_p}||\nabla C_p||^2_{L^2(K)} + \frac{3\alpha h^2}{4\eta_p}||\nabla\Pi_p||^2_{L^2(\Omega)}$$

and

$$I_5 = \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}, \nabla C_p\Big)_K \le \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{12\eta_p}||\nabla C_p||^2_{L^2(K)} + \frac{3\alpha h^2}{4\eta_p}\left\|\nabla\tilde{p}\right\|^2_{L^2(\Omega)}.$$

Since $k\in\mathcal{C}^\infty([0,T])\subset_> L^\infty([0,T])$, using Cauchy-Schwarz and Young's inequalities yield

$$I_6 = -2\Big(k*(i_h\epsilon(\tilde{u})-\epsilon(\tilde{u})), \epsilon(C_u)\Big) \le \frac{5}{\eta_s}\|k\|^2_{L^\infty(0,T)}\|i_h\epsilon(\tilde{u})-\epsilon(\tilde{u})\|^2_{L^2(D)} + \frac{\eta_s}{5}\|\epsilon(C_u)\|^2_{L^2(D)}$$

and

$$I_7 = -2\Big(k*i_h\epsilon(\Pi_u), \epsilon(C_u)\Big) \le \frac{5}{\eta_s}\|k\|^2_{L^\infty}\|\epsilon(\Pi_u)\|^2_{L^2(D)} + \frac{\eta_s}{5}\|\epsilon(C_u)\|^2_{L^2(D)},$$

where in the last inequality we used the stability of $i_h : L^2(D)\to L^2(D)$. Finally, a Cauchy-Schwarz and Young's inequalities again yields

$$I_8 = (g - i_h g, \epsilon(C_u)) \le \frac{5}{4\eta_s}\|g - i_h g\|^2_{L^2(D)} + \frac{\eta_s}{5}\|\epsilon(C_u)\|^2_{L^2(D)}.$$

The above estimates of $I_1, ..., I_8$ in (A.3) yield

$$\rho\Big(\frac{\partial C_u}{\partial t}, C_u\Big) + \frac{1}{2}2\eta_s\Big(\epsilon(C_u), \epsilon(C_u)\Big) + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla C_p, \nabla C_p\Big)_K + 2\Big(k*i_h\epsilon(C_u), i_h\epsilon(C_u)\Big)$$
$$\le C\Big(||\epsilon(\Pi_u)||^2_{L^2(\Omega)} + ||\Pi_p||^2_{L^2(\Omega)} + \sum_{K\in\mathcal{T}_h}\frac{1}{h_K^2}\|\Pi_u\|^2_{L^2(K)} + h^2||\nabla\Pi_p||^2_{L^2(\Omega)} + h^2||\nabla\tilde{p}||^2_{L^2(\Omega)}\Big)$$

where C depends only on ρ, η_s, η_p, k and α. Time integration for $0\le s\le T$ yields

$$\frac{\rho}{2}\|C_u(s)\|^2_{L^2(\Omega)} + \eta_s\int_0^s\|\epsilon(C_u)\|^2_{L^2(\Omega)} + 2\int_0^s\Big((k*i_h\epsilon(C_u))(s), i_h\epsilon(C_u(s))\Big)ds$$
$$\le \frac{\rho}{2}\|C_u(0)\|^2_{L^2(\Omega)} + C\int_0^s\Big(||\epsilon(\Pi_u)||^2_{L^2(\Omega)} + ||\Pi_p||^2_{L^2(\Omega)} + ||\Pi_\sigma||^2_{L^2(\Omega)}$$
$$+ \sum_{K\in\mathcal{T}_h}\frac{1}{h_K^2}\|\Pi_u\|^2_{L^2(K)} + h^2||\nabla\Pi_p||^2_{L^2(\Omega)} + h^2||\nabla\tilde{p}||^2_{L^2(\Omega)}.$$

Using standard interpolation results and (3.11), we obtain

$$\|\epsilon(C_u)\|^2_{L^2(L^2)} \le Ch^2\Big(\|\tilde{u}\|^2_{h^\mu(W^{2,r})} + \|\tilde{p}\|^2_{h^\mu(W^{1,r})} + \|\nabla u_0\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(W^{1,r}(D))}\Big),$$

where C does not depend on h, f_1, f_2, u_0 and g. Then, since

$$\|g\|_{L^2(W^{1,r})} \leq C\|q^S\|_{L^2(\Omega;L^\infty([0,T])}\|w\|_{L^2(L^2(W^{1,r}))},$$

using continuous embeddings between interpolation spaces (see [42]) and (2.17) we obtain

$$||\epsilon(C_u)||_{L^2(L^2)} \leq Ch\Big(||f_1, u_0||_Y + ||f_2||_U + ||w||_W\Big),$$

where C does not depend on h, f_1, f_2, u_0 and w. Thus

$$||\epsilon(e_u)||_{L^2(L^2)} \leq Ch\Big(||f_1, u_0||_Y + ||f_2||_U + ||w||_W\Big). \tag{A.4}$$

It remains to prove that

$$\|\tilde{q}^D - \tilde{q}_h^D\|_{L^2(L^\infty(L^2))} \leq Ch\Big(||y||_Y + ||f_2||_U + ||w||_W\Big), \tag{A.5}$$

where C does not depend on h, f_1, f_2, u_0 and w. The solution $(\tilde{u}, \tilde{q}^D) := \mathsf{T}(y, f_2, w)$ satisfies for $t \in [0, T]$

$$\tilde{q}^D(t) = \int_0^t e^{-\frac{t-s}{2\lambda}} \left((\nabla\tilde{u})q^S + w\right) ds.$$

Hence, (A.5) follows by substracting (3.7) to the above equation.

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