Numerical Methods for PDEs
MATH 610 - 600
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Homework 3
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Exercise 1 **25%**

Consider the unit segment $K=[0,1]$ and let P be the set of functions that are piecewise quadratic over the intervals $[0,\frac{1}{2}]\cup[\frac{1}{2},1]$ and are of class $\mathcal{C}^1$ over K. Functions in P are continuous and their first derivatives are continuous at $\frac{1}{2}$. Let $\Sigma=\{\sigma_1,\ldots,\sigma_4\}$ be defined for $p\in P$ as $\sigma_1(p)=p(0)$, $\sigma_2(p)=p'(0)$, $\sigma_3(p)=p(1)$, $\sigma_4(p)=p'(1)$. Prove that the triple (K,P,Σ) is a finite element. Compute the shape functions associated with the degrees of freedom in Σ.

Exercise 2 **25%**

Let K be a nondegenerate triangle in $\mathbb{R}^2$. Let a_1, a_2, a_3 be the three vertices of K. Let $a_{ij}=a_{ji}$ denote the midpoint of the segment (a_i,a_j), $i,j\in\{1,2,3\}$. Let $\mathbb{P}^2$ be the set of the polynomial functions over K of total degree at most 2. Let $\Sigma=\{\sigma_1,\sigma_2,\sigma_3,\sigma_{12},\sigma_{23},\sigma_{31}\}$ be the functionals (or degrees of freedom) on $\mathbb{P}^2$ defined as

$$\sigma_i(p)=p(a_i),\ i\in\{1,2,3\}\qquad \sigma_{ij}(p)=p(a_i)+p(a_j)-2p(a_{ij}),\quad i,j=1,2,3,\quad i\neq j.$$

(a) Show that Σ is a unisolvent set for $\mathbb{P}^2$ (this means that any $p\in\mathbb{P}^2$ is uniquely determined by the values of the above degrees of freedom applied to p).

(b) Compute the "nodal" basis of $\mathbb{P}^2$ which corresponds to $\{\sigma_1,\ldots,\sigma_{31}\}$.

(c) Evaluate the entry m_{11} of the element mass matrix.

Hint. If you have problem computing the functions from (b) for a general triangle, derive them for a reference element.

Exercise 3 **50%**

Let $\Omega\subset\mathbb{R}^2$ be a bounded polygonal domain. Let $f,g\in L^2(\Omega)$ and consider the system for u and ϕ

$$\begin{aligned} -\Delta u-\phi&=f \qquad \text{in} \qquad \Omega,\\ u-\Delta\phi&=g \qquad \text{in} \qquad \Omega, \end{aligned}$$

supplemented by the boundary condition $u=\phi=0$ on $\partial\Omega$.

1. Consider the weak formulation of the above system: Seek $u\in\mathbb{V}$, $\phi\in\mathbb{W}$ such that

$$a((u,\phi),(v,\psi))=\int_\Omega f\ v+\int_\Omega g\ \psi,\qquad \forall(v,\psi)\in\mathbb{V}\times\mathbb{W}.$$

   Determine $\mathbb{V}$, $\mathbb{W}$ and the bilinear form $a(.,.)$ such that there exists a unique solution $(u,\phi)\in\mathbb{V}\times\mathbb{W}$.

2. Prove using the Lax-Milgram theorem that the weak formulation as indeed a unique solution and deduce a stability bound for the solution.

3. For a domain $\Omega = \Omega_d := (-d, d)^2$, show that there exists an absolute constant c such that

$$\|u\|_{L^2(\Omega)} \leqslant cd\|\nabla u\|,$$

for any function $u \in H_0^1(\Omega_d)$.

4. Consider the modified problem

$$\begin{aligned} -\Delta u + \phi = f \qquad & \text{in} \qquad \Omega_d, \\ u - \Delta \phi = g \qquad & \text{in} \qquad \Omega_d. \end{aligned}$$

Determine a new weak formulation and prove the existence and uniqueness of a solution provided that d is small enough. Derive a stability bound.