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Take Home Exam

Exercise 1 20%

Let K be a polyedron in \mathbb{R}^d , $d \geq 1$. Let $h = \text{diam}(K)$ and define

$$\hat{K} = \{\hat{\mathbf{x}} = \mathbf{x}/\text{diam}(K), \quad \mathbf{x} \in K\}.$$

Show that there exists a constant c solely depending on \hat{K} such that for any $v \in H^1(K)$,

$$\|v\|_{L^2(\partial K)} \leq c \left(h^{-1/2} \|v\|_{L^2(K)} + h^{1/2} \|\nabla v\|_{L^2(K)} \right).$$

Exercise 2 80%

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, $\beta, \mu \in C^0(\bar{\Omega})$ and $f \in L^2(\Omega)$. Assume that for some positive constants μ_0, μ_1, β_1 , there holds $0 < \mu_0 \leq \mu(x) \leq \mu_1$ and $|\beta(x)| \leq \beta_1$ for all $x \in \Omega$. In addition, we suppose that $\text{div}(\beta) = 0$. The outward pointing unit normal to Ω is denoted by $\boldsymbol{\nu}$ (defined a.e.) and we consider a decomposition of the domain boundary $\Gamma := \partial\Omega$ in two disjoint pieces $\Gamma = \Gamma_1 \cup \Gamma_2$. We propose to analyze a *stabilized* numerical method for the following convection-diffusion problem : Seek $u : \Omega \rightarrow \mathbb{R}$ such that

$$-\text{div}(\mu \nabla u) + \beta \cdot \nabla u = f, \quad \text{in } \Omega$$

together with the boundary conditions

$$u = 0 \quad \text{on } \Gamma_1 \quad \text{and} \quad \mu \frac{\partial}{\partial \boldsymbol{\nu}} u = 0 \quad \text{on } \Gamma_2.$$

1. Derive an adequate weak formulation of the above problem : Seek $u \in X$ such that

$$a(u, v) = F(v), \quad \forall v \in X.$$

2. Show that for all $v, w \in H_0^1(\Omega)$, there exist a constant $M > 0$ such that

$$a(v, v) \geq \mu_0 \|\nabla v\|_{L^2(\Omega)}^2 \quad |a(w, v)| \leq M \|\nabla w\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

and deduce the existence and uniqueness of a weak solution.

3. Given a quasi-uniform triangulation \mathcal{T}_h of Ω (h denotes the largest outer circle diameter), we set for $k \geq 1$

$$\mathbb{V}_h := \{v_h \in C^0(\bar{\Omega}) \mid v_h|_T \in \mathbb{P}^k(T), \quad \forall T \in \mathcal{T}_h\}$$

and define the approximate bilinear form on $\mathbb{V}_h \times \mathbb{V}_h$

$$a_h(v_h, w_h) := a(v_h, w_h) + \alpha h \int_{\Omega} \nabla w_h \cdot \nabla v_h, \quad \forall w_h, v_h \in \mathbb{V}_h.$$

Consider the finite element approximation $u_h \in \mathbb{V}_h$ of $u \in X$ defined as satisfying

$$a_h(u_h, v_h) = F(v_h) \quad \forall v_h \in \mathbb{V}_h.$$

Show that $a_h(v_h, v_h) \geq \mu_h \|\nabla v_h\|_{L^2(\Omega)}^2$ with $\mu_h := \mu_0 + \alpha h$. Deduce the existence and uniqueness of $u_h \in \mathbb{V}_h$.

4. Show that for all $v_h \in \mathbb{V}_h$

$$\mu_h \|\nabla(v_h - u_h)\|_{L^2(\Omega)}^2 \leq a_h(v_h, v_h - u_h) - F(v_h - u_h),$$

deduce that

$$\|\nabla(v_h - u_h)\|_{L^2(\Omega)} \leq \frac{1}{\mu_h} \sup_{w_h \in \mathbb{V}_h} \frac{|a_h(v_h, w_h) - a(v_h, w_h)|}{\|\nabla w_h\|_{L^2(\Omega)}} + \frac{M}{\mu_h} \|\nabla(v_h - u_h)\|_{L^2(\Omega)}$$

and therefore

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq \inf_{v_h \in \mathbb{V}_h} \left\{ \frac{1}{\mu_h} \sup_{w_h \in \mathbb{V}_h} \frac{|a_h(v_h, w_h) - a(v_h, w_h)|}{\|\nabla w_h\|_{L^2(\Omega)}} + \left(1 + \frac{M}{\mu_h}\right) \|\nabla(v_h - u_h)\|_{L^2(\Omega)} \right\}.$$

(Strang's lemma).

5. Prove that for all $v_h \in \mathbb{V}_h$, there holds

$$\sup_{w_h \in \mathbb{V}_h} \frac{|a_h(v_h, w_h) - a(v_h, w_h)|}{\|\nabla w_h\|_{L^2(\Omega)}} \leq \alpha h \|\nabla v_h\|_{L^2(\Omega)}.$$

6. Recall that there exists a constant C such that (interpolation estimates)

$$\|\nabla(I_{\mathcal{T}_h} u - u)\|_{L^2(\Omega)} \leq C_1 |u|_{H^{k+1}(\Omega)}$$

provided $u \in H^{k+1}(\Omega)$ (which is assumed from now on). Deduce that for h sufficiently small, there exist a constant C_2 such that

$$\|\nabla(I_{\mathcal{T}_h} u)\|_{L^2(\Omega)} \leq C_2 \|\nabla u\|_{L^2(\Omega)}.$$

7. Show the existence of a constant C_3 such that

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq C_3 \left\{ \left(1 + \frac{M}{\mu_h}\right) |u|_{H^{k+1}(\Omega)} h^k + \frac{\alpha}{\mu_h} \|\nabla u\|_{L^2(\Omega)} h \right\}.$$