# SUGGESTED METHOD OF UNDETERMINED COEFFICIENTS

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Given $a, b, c \in \mathbb{R}$ with $a \neq 0$, this is brief note discusses how to determine a particular solution of

$$ay'' + by' + cy = g$$

when $g(t) = P_n(t)$ or $g(t) = P_n(t)e^{\beta t}$ or $g(t) = P_n(t)\cos(\omega t)$ or $g(t) = P_n(t)\sin(\omega t)$ and $P_n(t) = \alpha_n t^n + \alpha_{n-1}t^{n-1} + ... + a_1 t + a_0$ is a polynomial of degree $n \geq 0$.

## 1. Case: $g(t) = P_n(t)$

In order to match a polynomial of degree $n$ on the right hand side, a particular solution of

$$ay'' + by' + cy = P_n$$

should read

$$y_p(t) = \begin{cases} Q_n(t) & \text{if } c \neq 0, \\ Q_{n+1}(t) & \text{if } c = 0 \quad \text{and } b \neq 0, \\ Q_{n+2}(t) & \text{if } c = 0 \quad \text{and } b = 0. \end{cases}$$

Here $Q_n$, $Q_{n+1}$ and $Q_{n+2}$ are polynomials of degree $n$, $n+1$ and $n+2$ respectively. As noted in class (check it!), we can actually restrict the last two to be $Q_{n+1}(t) = tQ_n(t)$ and $Q_{n+2}(t) = t^2 Q_n(t)$.

## 2. Case $g(t) = P_n(t)e^{\beta t}$

To deal with this case, we advocate the change of variable

$$y(t) = v(t)e^{\beta t}$$

so that the equation satisfied by $v(t)$ is (check it!)

$$av'' + p'(\beta)v' + p(\beta)v = P_n,$$

where $p(t) = at^2 + bt + c$ is the characteristic polynomial.

Applying the previous method we choose

$$v_p(t) = \begin{cases} Q_n(t) & \text{if } p(\beta) \neq 0, \\ tQ_n(t) & \text{if } p(\beta) = 0 \quad \text{and } p'(\beta) \neq 0, \\ t^2 Q_n(t) & \text{if } p(\beta) = 0 \quad \text{and } p'(\beta) = 0. \end{cases}$$

Once $v_p(t)$ is found, we deduce $y_p$ according to the formula

$$y_p(t) = v_p(t)e^{\beta t}.$$

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*Date*: March 23, 2015.

## 3. Case $g(t) = P_n(t)\cos(\omega t)$

In order to find a particular solution of

$$ay''(t) + by'(t) + cy(t) = P_n(t)\cos(\omega t)$$

for $\omega \neq 0$, we first note that

$$P_n(t)\cos(\omega t) = \mathrm{Re}(P_n(t)e^{i\omega t})$$

and if $z(t) = u(t) + iw(t)$ is a complex valued function satisfying

$$az''(t) + bz'(t) + cz(t) = P_n(t)e^{i\omega t},$$

then $y(t) = \mathrm{Re}(z(t))$ (check it!).

Therefore, we proceed as in the previous section and use the (complex valued) function $v(t)$ defined by

$$z(t) = v(t)e^{i\omega t},$$

which satisfies

$$av'' + p'(i\omega)v' + p(i\omega)v = P_n.$$

Hence we can choose

$$v_p(t) = \begin{cases} Q_n(t) & \text{if } p(i\omega) \neq 0, \\ tQ_n(t) & \text{otherwise,} \end{cases}$$

where $Q_n(t)$ is a polynomial of degree $n$ with complex coefficients. As a consequence

$$z_p(t) = v_p(t)e^{i\omega t}$$

and

$$y_p(t) = \mathrm{Re}\left(v_p(t)e^{i\omega t}\right).$$

## 4. Case $g(t) = P_n(t)\sin(\omega t)$

Proceed as in the previous section upon noting that

$$P_n(t)\sin(\omega t) = \mathrm{Im}(P_n(t)e^{i\omega t})$$

so that

$$y_p(t) = \mathrm{Im}\left(v_p(t)e^{i\omega t}\right).$$