Ordinary Differential Equations
MATH 308 - 523
A. Bonito

April 21
Spring 2015

First Name: ____________________ **Last Name:** ____________________

Exam 2

- 75 minute individual exam;
- Answer the questions in the space provided. If you run out of space, continue onto the back of the page. Additional space is provided at the end;
- **Show and explain all work**;
- **Underline** the answer of each steps;
- The use of books, personal notes, **calculator**, cellphone, laptop, and communication with others is forbidden;
- By taking this exam, you agree to follow the university's code of academic integrity.

Ex 1	Ex 2	Ex 3	Ex 4	Total

Some Laplace Transforms

f	$\mathcal{L}(f)$		f	$\mathcal{L}(f)$	
1	$\frac{1}{s}$	$s>0$	$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$	$s>0$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$	$s>-\alpha$	$e^{-\alpha t}\ t^n$	$\frac{n!}{(s+\alpha)^{n+1}}$	$s>-\alpha$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s>0$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s>0$
$e^{\alpha t}\sin(\omega t)$	$\frac{\omega}{(s-\alpha)^2+\omega^2}$	$s>\alpha$	$e^{\alpha t}\cos(\omega t)$	$\frac{s-\alpha}{(s-\alpha)^2+\omega^2}$	$s>\alpha$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$s>\lvert\omega\rvert$	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$s>\lvert\omega\rvert$
$H_\alpha(t)$	$\frac{e^{-\alpha s}}{s}$	$s>0$	$\delta_\alpha(t)$	$e^{-\alpha s}$	$s>-\infty$

Some Properties of the Laplace Transforms

Let $f, g : [0, +\infty) \to \mathbb{R}$ be piecewise continuous functions with piecewise continuous derivatives. Assume there exists $K \geqslant 0$ and $a_1,\ a_2 \in \mathbb{R}$ such that

$$|f(t)| \leqslant K e^{a_1 t}, \qquad |g(t)| \leqslant M e^{a_2 t}, \qquad \forall t \in [0, +\infty).$$

Then there holds

$(\imath.)$ $\mathcal{L}\left(\frac{d^n}{dt^n} f(t)\right)(s) = s^n \mathcal{L}\left(f(t)\right) - s^{n-1} f(0) - ... - s\frac{d^{n-2}}{dt^{n-2}} f(0) - \frac{d^{n-1}}{dt^{n-1}} f(0),$
$\forall s > a_1,\ (f \in C^{n-1}([0,\infty)),\ \frac{d^n}{dt^n} f$ piecewise continuous$)$

$(\imath\imath.)$ $\mathcal{L}\left(\int_0^t f(\tau)d\tau\right)(s) = \frac{1}{s}\mathcal{L}\left(f(t)\right)(s), \qquad \forall s > a_1,$

$(\imath\imath\imath.)$ $\mathcal{L}\left((-1)^n t^n f(t)\right)(s) = \frac{d^n}{ds^n}\mathcal{L}\left(f(t)\right)(s), \qquad \forall s > a_1,$

$(\imath v.)$ $\mathcal{L}\left(e^{-\alpha t} f(t)\right)(s) = \mathcal{L}\left(f(t)\right)(s+\alpha), \qquad \forall s > a_1 + \alpha,\ \alpha \geqslant 0,$

$(v.)$ $\mathcal{L}\left(H_\alpha(t) f(t-\alpha)\right)(s) = e^{-\alpha s}\mathcal{L}\left(f(t)\right)(s), \qquad \forall s > a_1,\ \alpha \geqslant 0,$

$(v\imath.)$ $\mathcal{L}\left((f * g)(t)\right)(s) = \mathcal{L}(f(t))(s) \cdot \mathcal{L}(g(t))(s), \qquad \forall s > \max(a_1, a_2).$

Exercise 1 **25%**

Solve the following initial value problem using the Laplace Transforms

$$y'' + y = \sin(t), \qquad y(0) = 1, \quad y'(0) = 1.$$

(Hint: use the convolution product and the identity $2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b)$.)

Exercise 2 **25%**

- Write the definition of the laplace transform of a function $f(t)$.
- *Without* using the table, compute the laplace transform of $g(t) = 5\delta_{3\pi}(t)$. Do not forget to mention for which values of s the laplace transform $\mathcal{L}(g)(s)$ is well defined.
- Find the solution of

$$y'' + 2y' + 3y = 5\delta_{3\pi}, \qquad y(0) = y'(0) = 0.$$

(you can use the table for this part.)

Exercise 3 **25%**

Solve the following system

$$\mathbf{y}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{y}$$

and draw the phase portrait.

Exercise 4 **25%**

Solve the following system

$$\mathbf{y}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{y}$$

and draw the phase portrait.

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Exam 2: solutions

Exercise 1 **25%**

The Laplace transform of both sides of the ODE yields

$$(s^2+1)Y - s - 1 = \mathcal{L}(\sin(t)(s),$$

where $Y(s) = \mathcal{L}(y)(s)$. Thus we obtain an expression for Y

$$Y = \frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{\mathcal{L}(\sin(t)(s)}{s^2+1}.$$

Noting that

$$\frac{1}{s^2+1} = \mathcal{L}(\sin(t))(s),$$

we obtain

$$Y = \frac{s}{s^2+1} + \frac{1}{s^2+1} + \mathcal{L}(\sin(t)(s)\mathcal{L}(\sin(t)(s).$$

It remains to compute the inverse Laplace transforms of the three terms in the right hand side of the above equation. Using the Laplace table for the two first terms we have

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t), \qquad \text{and} \qquad \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos(t).$$

For the last term, property (vi) yields

$$\mathcal{L}^{-1}\left(\mathcal{L}(\sin(t)(s)\mathcal{L}(\sin(t)(s)\right) = \sin(t) * \sin(t).$$

We compute

$$\sin(t) * \sin(t) = \int_0^t \sin(t-u)\sin(u)du = \frac{1}{2}\int_0^t (\cos(t-2u) - \cos(t)) \quad = \frac{1}{2}\sin(t) - \frac{t}{2}\cos(t).$$

Adding all the inverse Laplace transforms computed leads to

$$y(t) = \frac{3}{2}\sin(t) + \cos(t) - \frac{t}{2}\cos(t).$$

Exercise 2 **25%**

- $\mathcal{L}(f)(s) := \int_0^\infty f(t)e^{-st}dt$.
- We start from the definition of the laplace transform

  $$\mathcal{L}(g)(s) := 5\int_0^\infty \delta(t-3\pi)e^{-st}ds = 5e^{-3\pi\ s}$$

  which holds for any $s \in \mathbb{R}$.

- We take the laplace transform and use the previous item to arrive at

$$(s^2+2s+3)Y = 5e^{-3\pi s}$$

or

$$Y = \frac{5}{s^2+2s+3}e^{-3\pi s}.$$

We first find the Laplace transform inverse of

$$F(s) := \frac{5}{s^2+2s+3} = \frac{5}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2},$$

which is given by

$$\mathcal{L}^{-1}(F)(t) = \frac{5}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t).$$

Hence,using the formula

$$\mathcal{L}^{-1}(F(s)e^{-cs}) = \mathcal{L}^{-1}(F)(t-c)u_c(t)$$

we arrive at

$$y(t) = \mathcal{L}^{-1}(\frac{5}{s^2+2s+3}e^{-3\pi s})(t) = \frac{5}{\sqrt{2}}e^{-(t-3\pi)}\sin(\sqrt{2}(t-3\pi))u_{3\pi}(t).$$

Exercise 3 **25%**

We find the eigenvalues-eigenvectors of the matrix. The eigenvalues are found solving

$$\det\begin{pmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{pmatrix} = 0,$$

i.e.

$$\lambda^2 - 2\lambda - 3 = 0$$

or $\lambda_1 = 3$ and $\lambda_2 = -1$.
The eigenvector associated to λ_1 is given by

$$\begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 3\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

or

$$-5\xi_1 + \xi_2 = 0.$$

Hence all eigenvectors are given by

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \qquad \forall\alpha \neq 0.$$

The eigenvector associated to λ_2 is given by

$$\begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = -\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

or

$$-\xi_1 + \xi_2 = 0.$$

Hence all eigenvectors are given by

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \forall\alpha \neq 0.$$

Gathering the above results, we find that the general solution reads

$$\mathbf{y} = C_1\begin{pmatrix} 1 \\ 5 \end{pmatrix}e^{3t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t}.$$

The phase portrait is provided in Fig. 1.

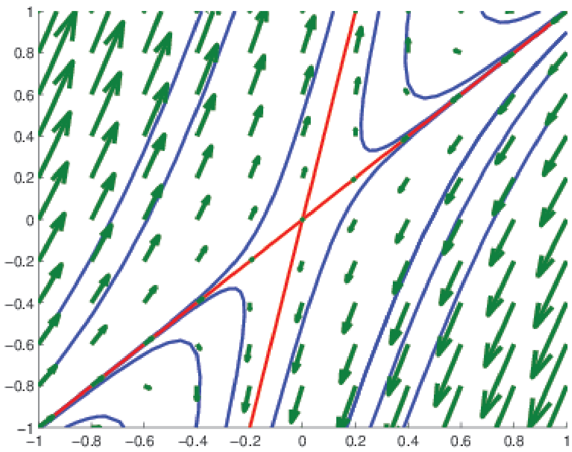

Figure 1: Phase portrait: The red lines correspond to the directions given by the eigenvectors

Exercise 4 **25%**

We find the eigenvalues-eigenvectors of the matrix. The eigenvalues are found solving

$$\det\begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} = 0,$$

i.e.

$$\lambda^2 - 2\lambda + 5 = 0$$

or $\lambda = 1 \pm 2i$.
We only consider $\lambda = 1 + 2i$ and find the associated eigenvector:

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = (1+2i)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

or

$$(2-2i)\xi_1 - 2\xi_2 = 0.$$

Hence all eigenvectors are given by

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1-i \end{pmatrix}, \qquad \forall \alpha \neq 0.$$

As a consequence two linearly independent solution are given by

$$\mathbf{y} = \Re\left(e^{(1+2i)t}\begin{pmatrix} 1 \\ 1-i \end{pmatrix}\right) = e^t\begin{pmatrix} \cos(2t) \\ \cos(2t)+\sin(2t) \end{pmatrix}$$

and

$$\mathbf{y} = \Im\left(e^{(1+2i)t}\begin{pmatrix} 1 \\ 1-i \end{pmatrix}\right) = e^t\begin{pmatrix} \sin(2t) \\ \sin(2t)-\cos(2t) \end{pmatrix}.$$

Gathering the above results, we find that the general solution reads

$$\mathbf{y} = e^t\left(C_1\begin{pmatrix} \cos(2t) \\ \cos(2t)+\sin(2t) \end{pmatrix} + C_2\begin{pmatrix} \sin(2t) \\ \sin(2t)-\cos(2t) \end{pmatrix}\right).$$

The phase portrait is provided in Fig. 2.

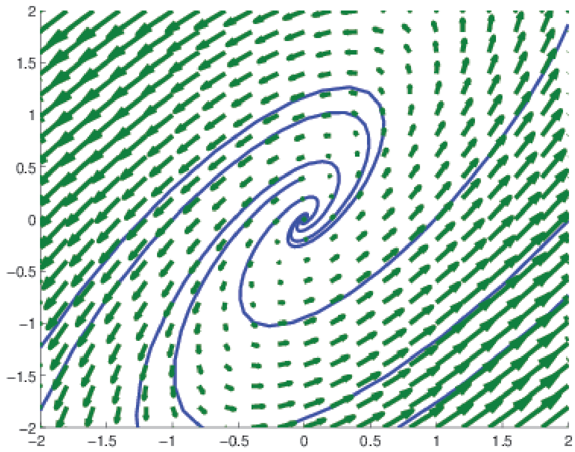

Figure 2: Phase portrait