Numerical Methods for PDEs — Homework 2
MATH 610 - 600 — Spring 2018
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Homework 2

Exercise 1 50%

1. Let f be a given real valued function in $L^2(0,1)$ and let $\alpha > 0$ be a given real number. We want to solve the problem: Find u such that

$$-u''(x) + \alpha\, u(x) = f(x), \qquad 0 < x < 1, \tag{1}$$

with the boundary conditions

$$u'(0) = 0, \qquad u(1) = 0. \tag{2}$$

   (a) Write (1) and (2) in a variational (weak) form.
   (be carful in specifying the correct functional spaces.)

   (b) Derive an energy (stability) estimate.

   (c) Assume that there exists at least one solution of the variational problem. Prove that the solution has to be unique. (take two solutions and find an equation for the difference.)

2. With the above notation, consider the problem: Find $u \in H^1(0,1)$, such that

$$\forall v \in H^1(0,1), \qquad \int_0^1 u'(x)v'(x)\,dx + \alpha \int_0^1 u(x)v(x)\,dx + u(0)v(0) = \int_0^1 f(x)\,v(x)\,dx.$$

   (a) Find the boundary value problem (strong formulation) that has this variational formulation.

   (b) Derive an energy (stability) estimate.

   (c) Assume that there exists at least one solution of the variational problem. Prove that the solution has to be unique. (take two solutions and find an equation for the difference.)

Exercise 2 50%

Consider the following two-point b.v.p.:

$$-u'' + u = 1, \quad \text{for } x \in (0,1), \; u'(0) = \beta_0, \; u'(1) + u(1) = \beta_1, \tag{3}$$

where β_0 and β_1 are given constants.
In the interval $(0,1)$ introduce the uniform grid $x_j = jh$, $h = 1/n$, $j = 0, \ldots, n$. Further, consider the space V_h of **piece-wise quadratic** function over this splitting (x_j, x_{j+1}), $j = 0, \ldots, n-1$.

1. (10 pts) Give the variational formulation of this problem.

2. (10 pts) Describe the space V for which the variational formulation of this problem is posed.

3. (20 pts) For degrees of freedom use the following functionals

$$v(x_j), \quad j = 0, \ldots, n, \quad \text{and} \quad \frac{1}{h}\int_{x_j}^{x_{j+1}} v\,dx, \quad j = 0, \ldots, n-1,$$

i.e. the function values at the grid points x_j and the mean values of v over (x_j, x_{j+1}). Give the formulas for the corresponding basis function over all intervals (x_j, x_{j+1}), i.e. find ϕ_i^j, $i = 1, 2, 3$, $j = 0, \ldots, n-1$ such that ϕ_i^j are quadratics over (x_j, x_{j+1}) and satisfy

- $\phi_1^j(x_j) = 1$, $\phi_1^j(x_{j+1}) = 0$, $\frac{1}{h}\int_{x_j}^{x_{j+1}} \phi_1^j dx = 0$ for $j = 0, \ldots, n-1$;
- $\phi_3^j(x_j) = 0$, $\phi_3^j(x_{j+1}) = 1$, $\frac{1}{h}\int_{x_j}^{x_{j+1}} \phi_3^j dx = 0$ for $j = 0, \ldots, n-1$;
- $\phi_2^j(x_j) = 0$, $\phi_2^j(x_{j+1}) = 0$, $\frac{1}{h}\int_{x_j}^{x_{j+1}} \phi_2^j dx = 1$ for $j = 0, \ldots, n-1$.

4. (20 pts) Compute the element "stiffness" and "mass" matrices corresponding to the basis defined in (3).

5. (20 pts) Assembly the global "stiffness" and "mass" matrices for the above problem.

6. (20 pts) Compute the right hand side of the Ritz system. Compute the global "stiffness" and "mass" matrices for a problem where the above boundary condition at $x = 0$ is replaced by the Dirichlet condition $u(0) = 0$ and write the Ritz system.