Ordinary Differential Equations
MATH 308 - 523
A. Bonito

March 12
Spring 2015

First Name: ____________________ **Last Name:** ____________________

Exam 1

- 75 minute individual exam;
- Answer the questions in the space provided. If you run out of space, continue onto the back of the page. Additional space is provided at the end;
- **Show and explain all work**;
- **Underline** the answer of each steps;
- The use of books, personal notes, **calculator**, cellphone, laptop, and communication with others is forbidden;
- By taking this exam, you agree to follow the university's code of academic integrity.

Ex 1	Ex 2	Ex 3	Ex 4	Ex 5	Total

Exercise 1 **20%**

Consider the following ODE

$$\frac{dy}{dx} + y^2 \ \sin(x) = 0.$$

suplemented by the boundary condition $y(0) = -1$.

1. Find the solution $y(x)$.

2. Determine for which x this solution is well defined.

3. Same questions but when $y(0) = 0$.

Exercise 2 **20%**

Consider the following ODE

$$\cos(t)\ y' + \sin(t)y = 1, \qquad y(5\pi/4) = 0.$$

Solve for y. *Hint:* $\int \tan(t) = -\ln|\cos(t)| + C$ and $\int 1/\cos^2(t) = \tan(t) + C$.

<u>Exercise 3</u> **20%**

Solve the differential equation

$$2x + y^2 + 2xyy' = 0.$$

<u>Hint:</u> You do not have to find explicit solutions but an algebraic relation only depending on x and $y(x)$.

Exercise 4 **15%**

Discuss the existence and uniqueness of a solution to the following ordinary differential equation (JUSTIFY your argumentation)

$$e^x y''(x) - \frac{y'(x)}{x-3} + y(x) = \ln x, \qquad y(1) = 1,\ y'(1) = 1.$$

Exercise 5 **25%**

Find the solution of

$$6y'' - y' - y = 2e^{\frac{1}{2}t}, \qquad y(0) = 0, \qquad y'(0) = 1.$$

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Exam 1: solutions

Exercise 1 **20%**

1. Since $y(0) \neq 0$, we divide by y^2 to obtain the following separable equation
$$\frac{1}{y^2}\frac{d}{dx}y = -\sin(x).$$
It suffices to take the anti-derivative with respect to x and use the substitution rule
$$\int^x \frac{1}{y^2}\frac{d}{dx}y = \int^y \frac{1}{y^2}$$
to arrive at
$$-\frac{1}{y} = \cos(x) + C,$$
where C is a constant. To solve for the constant, we use the initial coniditon $y(0) = -1$ and find
$$C = 0.$$
Solving for y, we obtain
$$y(x) = -\frac{1}{\cos(x)}.$$
2. $y(x)$ is well defined for $-\pi/2 < x < \pi/2$.
3. When $y(0) = 0$ then the ODE implies $\frac{d}{dx}y(0) = 0$ and thus
$$y(x) = 0$$
which is defined for all $x \in \mathbb{R}$.

Exercise 2 **20%**

This is a linear and first order ODE. The function
$$\mu(t) := 1/\cos(t)$$
is an integrating factor so that we rewrite the ODE
$$\frac{d}{dt}(y(t)/\cos(t)) = 1/\cos^2(t).$$
Integrating yields
$$y(t)/\cos(t) = \tan(t) + C.$$
Using the initial condition we find $C = -\tan(5\pi/4) = -1$ so that
$$y(t) = \cos(t)(\tan(t) - 1).$$

Exercise 3 **20%**

Since

$$\frac{\partial}{\partial x}(2xy) = \frac{\partial}{\partial y}(2x + y^2)$$

the ODE is exact. Hence, we need to find $\psi(x, y)$ such that

$$\frac{\partial}{\partial x}\psi(x, y) = 2x + y^2; \qquad \frac{\partial}{\partial y}\psi(x, y) = 2xy.$$

Integrating both relations yield

$$\psi(x, y) = x^2 + xy^2 + C_1(y); \qquad \psi(x, y) = xy^2 + C_2(x)$$

so that choosing $C_1(y) = 0$ and $C_2(x) = x^2$ guarantees that the two expressions match and

$$\psi(x, y) = x^2 + xy^2.$$

Therefore the solutions are given by

$$x(x + y^2) = c$$

for any constant c.

Exercise 4 **15%**

First we rewrite the ODE to fit the existence/uniqueness theorem framework dividing by e^x

$$y''(x) - \underbrace{\frac{e^{-x}}{x-3}}_{p(x)} y'(x) + \underbrace{e^{-x}}_{q(x)} y(x) = \underbrace{e^{-x}\ln(x)}_{g(x)}.$$

Note that $p(x)$ is continuous for all $x \neq 3$, $q(x)$ is continuous for all $x \in \mathbb{R}$ and $g(x)$ is continuous for all $x > 0$. Therefore, recalling that we are interested in the solution starting at $x = 1$ (c.f. initial condition), there exists a unique solution $y(x)$ for all $0 < x < 3$.

Exercise 5 **25%**

Guessing $y(t) = e^{rt}$ we derive the characterisitc equation

$$6r^2 - r - 1 = 0.$$

Its discrimant is $\Delta = 25$ so that the two real roots are given by

$$r_1 = \frac{1}{2}, \qquad r_2 = -\frac{1}{3}$$

so that

$$y_1(t) = e^{\frac{1}{2}t}, \qquad y_2(t) = e^{-\frac{1}{3}t}.$$

As a consequence, the general solution is given by

$$y(t) = C_1 e^{\frac{1}{2}t} + C_2 e^{-\frac{1}{3}t}$$

for any constants C_1 and C_2.
We now find a particular solution $y_p(t) = v_p(t)e^t$, where $v_p(t)$ satisfies

$$6v''(t) + 5v'(t) = 2.$$

Hence guessing $v_p(t) = At$ we find

$$v_p(t) = \frac{2}{5}t.$$

Therefore, we obtain

$$y_p(t) = \frac{2}{5} t e^{\frac{1}{2}t}$$

so that the general solution reads

$$y(t) = C_1 e^{\frac{1}{2}t} + C_2 e^{-\frac{1}{3}t} + \frac{2}{5} t e^{\frac{1}{2}t}.$$

We now solve for C_1 and C_2 to accomodate for the initial conditions:

$$0 = y(0) = C_1 + C_2, \qquad 1 = y'(0) = \frac{1}{2} C_1 - \frac{1}{3} C_2 + \frac{1}{5}$$

which yields

$$C_1 = -\frac{24}{5}, \qquad C_2 = \frac{24}{5}.$$

Finally the solution is given by

$$y(t) = -\frac{24}{5} e^{\frac{1}{2}t} + \frac{24}{5} e^{-\frac{1}{3}t} + \frac{2}{5} t e^{\frac{1}{2}t}.$$