11. Lecture 11

11.1. **Differentiation via Polynomial Interpolation.** Suppose we want a differentiation formula of the form

$$f'(x) = \sum_{i=0}^{n} \alpha_i f(x_i),$$

with $\{x_0, ..., x_n\}$ distinct in $[a, b]$.

Let $p \in \mathbb{P}^n$ be the polynomial interpolating f at $x_0, ..., x_n$, i.e.

$$p(x) = \sum_{i=0}^{n} l_i(x) f(x_i),$$

where $l_i(x)$ are the Lagrange polynomials (see Section 3.2). Then $p'(x) = \sum_{i=0}^{n} l_i'(x) f(x_i)$ should approximate $f'(x)$. This is indeed the case at $x = x_i$, $i = 0, ..., n$ but it is less clear what happen when x is not an interpolation point.

Theorem 11.1. *Assume that $f \in C^{n+1}[a, b]$, $\{x_0, ..., x_n\}$ distinct in $[a, b]$ and $p \in \mathbb{P}^n$ interpolates f at $x_0, ..., x_n$. Then, there exists $\xi_j \in [a, b]$ such that*

$$f'(x_j) - p'(x_j) = \frac{f^{(n+1)}(\xi_j)}{(n+1)!} \prod_{l \neq j} (x_j - x_l).$$

Proof. For $x \notin \{x_0, ..., x_n\}$, define

$$\Theta(x) := \frac{f(x) - p(x)}{\prod_{i=0}^{n}(x - x_i)} (n+1)! \qquad .$$

Note that Theorem 4.1 guarantees that

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

so

$$\Theta(x) = f^{(n+1)}(\xi_x)$$

Since we do not know what happens to ξ_x when $x \to x_j$, we cannot use this representation further but it shows that

$$\Theta(x) \in \text{Range}(f^{(n+1)}), \qquad \text{for } x \in [a, b].$$

Instead, we evaluate

$$\lim_{x \to x_j} \Theta(x) = \lim_{x \to x_j} \frac{(f(x) - p(x))(n+1)!}{\prod_{i=0}^{n}(x - x_i)}.$$

This is a 0/0 type of indetermination so we can use L'Hospital's rule

$$\lim_{x \to x_j} \Theta(x) = \lim_{x \to x_j} \frac{(f'(x) - p'(x))(n+1)!}{\left(\prod_{i=0}^{n}(x - x_i)\right)'}.$$

We now compute the denominator

$$\left(\prod_{i=0}^{n} (x - x_i) \right)' = \sum_{i=0}^{n} \prod_{l \neq i} (x - x_l).$$

Each term of the above sum has a factor $(x-x_j)$ except when $i=j$ and so the only one that is not vanishing when $x \to x_j$ is the *ith* factor. This implies that

$$\lim_{x\to x_j}\left(\prod_{i=0}^{n}(x-x_i)\right)' = \prod_{l\neq j}(x_j - x_l)$$

and thus

$$\lim_{x\to x_j}\Theta(x) = \frac{(f'(x_j)-p'(x_j))(n+1)!}{\prod_{l\neq j}(x_j-x_l)}.$$

Now, $f^{(n+1)}$ is continuous on $[a,b]$ by assumption and we have already seen that

$$\Theta(x) \in \mathrm{Range}(f^{(n+1)}), \qquad \text{for } x \in [a,b].$$

In particular,

$$\lim_{x\to x_j}\Theta(x) \in \mathrm{Range}(f^{(n+1)})$$

or, there exists $\xi_j \in [a,b]$ with

$$\lim_{x\to x_j}\Theta(x) = \xi_j.$$

Thus

$$f^{(n+1)}(\xi_j) = \frac{(f'(x_j)-p'(x_j))(n+1)!}{\prod_{l\neq j}(x_j-x_l)},$$

which is the desired estimate after simple algebraic manipulations. □

Remark 11.1 (Continuity of ξ_x). Within the proof of the above theorem, we actually showed that the function

$$x \mapsto f^{(n+1)}(\xi_x)$$

is continuous on $[a,b]$ provided $f \in C^{(n+1)}[a,b]$. This fact will be used later.

Example 11.1 (3 points differentiation scheme)**.** *Use polynomial interpolation to derive an approximation to the derivative of the form*

$$f'(x) \approx af(x) + bf(x-h) + cf(x-2h).$$

We first compute the lagrange basis for the interpolation points $\{x, x-h, x-2h\}$ *(we use* t *for the variable for a fixed* x*)*

$$l_0(t) = \frac{(t-(x-h))(t-(x-2h))}{(x-(x-h))(x-(x-2h))} = \frac{t^2-(2x-3h)t+(x-h)(x-2h)}{2h^2};$$
$$l_1(t) = \frac{(t-x)(t-(x-2h))}{((x-h)-x)((x-h)-(x-2h))} = \frac{t^2-(2x-2h)t+x(x-2h)}{-h^2};$$
$$l_2(t) = \frac{(t-x)(t-(x-h))}{(x-2h-x)(x-2h-(x-h))} = \frac{t^2-(2x-h)t+x(x-h)}{2h^2}.$$

The associated interpolant is

$$p(t) = l_0(t)f(x) + l_1(t)f(x-h) + l_2(t)f(x-2h)$$

so that

$$f'(x) \approx l_0'(x)f(x) + l_1'(x)f(x-h) + l_2'(x)f(x-2h).$$

This means

$$\begin{aligned} f'(x) &\approx \frac{2x-(2x-3h)}{2h^2}f(x) + \frac{2x-(2x-2h)}{-h^2}f(x-h) + \frac{2x-(2x-h)}{2h^2}f(x-2h) \\ &= \frac{3}{2h}f(x) - \frac{2}{h}f(x-h) + \frac{1}{2h}f(x-2h). \end{aligned}$$

The error term is

$$\begin{aligned} f'(x) - \left(\frac{3}{2h}f(x) - \frac{2}{h}f(x-h) + \frac{1}{2h}f(x-2h)\right) &= \frac{f'''(\xi_0)}{6}(x-(x-h))(x-(x-2h)) \\ &= \frac{f'''(\xi_0)}{3}h^2. \end{aligned}$$