Partial Differential Equations
MATH 612
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Due February 19
Spring 2013

Assignement 1

Exercise 1 **15%**

1. Find the values $l \in \mathbb{R}$ for which
$$u(x) = |\log|x||^l$$
is weakly differentiable on $U = (-1, 1)$. Be rigourous in computing weak derivatives.
2. Determine the values $1 \leqslant p \leqslant \infty$ for which $u \in W^{1,p}(U)$.
3. Show that for all $l \in \mathbb{R}$
$$u(\mathbf{x}) = |\log|\mathbf{x}||^l$$
is weakly differentiable on $U = B^0(0,1) \subset \mathbb{R}^n$, $n \geqslant 2$. Be rigourous in computing the weak derivative.
4. Find the values $l \in \mathbb{R}$ for which $u(\mathbf{x})$ is in $W^{l,p}(U)$. Take care with the case $n = p$.

Exercise 2 **10%**

Let X denote a real Banach space and show that
$$u_j \to u \qquad \text{in } X \Longrightarrow \quad u_j \rightharpoonup u \qquad \text{in} X.$$

Exercise 3 **15%**

Show that if $u \in H^1(\mathbb{R})$, and u' denotes the weak derivative of u, then
$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h},$$
where the limit is in the $L^2(\mathbb{R})$ sense.

Exercise 4 **15%**

Suppose U is connected and $u \in W^{1,p}(U)$ satisfies
$$Du = 0, \qquad \text{a.e.} \quad \text{in} \quad U.$$
Prove that u is a constant a.e. in U.

Exercise 5 **15%**

Prove directly that $u \in W^{1,p}(0,1)$ for some $1 \leqslant p < \infty$, then
$$|u(x) - u(y)| \leqslant |x-y|^{1-1/p} \left(\int_0^1 |u'|^p dt \right)^{1/p}, \qquad a.e.\ x, y \in [0,1].$$

Exercise 6 **15%**

Integrate by parts to prove the interpolation inequality :

$$\int_U |Du|^2 \leqslant C \left(\int_U u^2 \right)^{1/2} \left(\int_U |D^2 u|^2 \right)^{1/2}$$

for all $u \in C_c^\infty(U)$. Assume ∂U is smooth, and prove tis inequality if $u \in H^2(U) \cap H_0^1(U)$.
(Hint : take $\{v_k\}_{k=1}^\infty \subset C_c^\infty(U)$ converging to u in $H_0^1(U)$ and $\{w_k\}_{k=1}^\infty \subset C^\infty(U)$ converging to u in $H^2(U)$.)

Exercise 7 **15%**

Let U be bounded with C^1 boundary. Show that in general a function $u \in L^p(U)$ $(1 \leqslant p < \infty)$ does not have a trace on ∂U. More precisely, prove that there does not exist a bounded linear operator

$$T : L^p(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\overline{U}) \cap L^p(U)$,