A DG APPROACH TO HIGHER ORDER ALE FORMULATIONS IN TIME

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Abstract. We review recent results [10, 9, 8] on time-discrete discontinuous Galerkin (dG) methods for advection-diffusion model problems defined on deformable domains and written on the Arbitrary Lagrangian Eulerian (ALE) framework. ALE formulations deal with PDEs on deformable domains upon extending the domain velocity from the boundary into the bulk with the purpose of keeping mesh regularity. We describe the construction of higher order in time numerical schemes enjoying stability properties independent of the arbitrary extension chosen. Our approach is based on the validity of Reynolds' identity for dG methods which generalize to higher order schemes the Geometric Conservation Law (GCL) condition. Stability, a priori and a posteriori error analyses are briefly discussed and illustrated by insightful numerical experiments.

Key words. ALE formulations, moving domains, domain velocity, material derivative, discrete Reynolds' identities, dG-methods in time, stability, geometric conservation law.

AMS(MOS) subject classifications. 65M12, 65M15, 65M50, 65M60.

1. Introduction. Problems governed by partial differential equations (PDEs) on deformable domains $\Omega_t \subset \mathbb{R}^d$, which change in time $0 \leq t \leq T < \infty$, are of fundamental importance in science and engineering, especially for space dimensions $d \geq 2$. A typical example is fluid structure interaction problems. They are of particular relevance in the design of many engineering systems, (e.g., aircrafts and bridges) as well as to the analysis of several biological phenomena (e.g., blood flow in arteries).

However, the mathematical understanding of such methods is still precarious even when the deformation of the boundary $\partial\Omega_t$ of Ω_t is prescribed a priori and thus known, instead of more realistic free boundary problems. One obstacle encountered when dealing numerically with such problems is the possibility of excessive mesh distortion. The *Arbitrary Lagrangian Eulerian (ALE)* approach, introduced in [16, 26, 27], is a way to overcome

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this difficulty. Its main idea is that the mesh boundary is deformed according to the prescribed boundary velocity $\mathbf{w}$, but an arbitrary, yet adequate extension is used to perform the bulk deformation. The extension of $\mathbf{w}$ from $\partial\Omega_t$ to Ω_t can be performed using various techniques such as solving for a suitable boundary value problem with Dirichlet boundary condition $\mathbf{w}$; see [19, 35, 22, 32] and the references therein. This extension induces a map $\mathcal{A}_t : \Omega_0 \to \Omega_t$, the so-called *ALE map*, with the key property that

$$\mathbf{w}(\mathbf{x}, t) = \frac{d}{dt}\mathcal{A}_t(\mathbf{y}), \qquad \mathbf{x} = \mathcal{A}_t(\mathbf{y}).$$

The ALE velocity $\mathbf{w}$ is unrelated to the advection coefficient $\mathbf{b}$ inherent to the underlying system and is mostly dictated by the geometric principle of preserving mesh regularity. In contrast, the pure Lagrangian approach consists of mesh deformation velocities given by $\mathbf{w} = \mathbf{b}$, whereas $\mathbf{w} = \mathbf{0}$ corresponds to the pure Eulerian approach. In the latter case, $\Omega_t = \Omega_0$ for all $t \in [0, T]$, and thus the domain does not change in time. Hence, the ALE is a generalization of both the Lagrangian and the Eulerian approaches.

This paper is a review of our recent results [10, 9, 8] on the design, stability, and error control of higher order in time ALE formulations for a linear advection-diffusion model problem defined on time dependent domains based on the discontinuous Galerkin (dG) approach. In particular, we discuss higher order in time, unconditionally stable numerical methods within the ALE framework, which seems to be lacking in the current literature. In the current paper, we intend to:

- Introduce the major difficulties caused by ALE formulations on deformable domains;
- Emphasize the key ideas leading to unconditionally stable higher order in time ALE formulations and point out the importance of such schemes for efficient error control;
- Compare our results with the existing literature and highlight the novel aspects of our analysis.

As already pointed out by other authors (see for example [5, 21, 20, 22, 31]), time-discretization is the main obstruction for the design of unconditionally stable higher order ALE-schemes. This is why we examine in [10, 9, 8], and review here, the critical issue of time-discrete schemes without space discretization. This prevents additional technicalities in dealing with new tools developed to handle the domain motion. In addition, it guarantees that no unnecessary CFL type restrictions are required by our techniques.

However, we note that understanding the effect of the finite element discretization in space is an important problem. Extending our analysis to fully discrete schemes is not straightforward, yet plausible as the required regularity on the ALE map in space in our time-discrete approach is compatible with a C^0 finite element framework.

1.1. The ALE Formulation. As in [5, 7, 21, 20, 22, 31], we consider the following time dependent diffusion-advection model problem defined on moving domains:

$$\begin{cases} \partial_t u + \nabla_{\mathbf{x}} \cdot (\mathbf{b}u) - \mu \Delta_{\mathbf{x}} u = f & \mathbf{x} \in \Omega_t,\ t \in [0,T] \\ u(\mathbf{x},t) = 0 & \mathbf{x} \in \partial\Omega_t,\ t \in [0,T] \\ u(\mathbf{x},0) = u_0(\mathbf{x}) & \mathbf{x} \in \Omega_0, \end{cases} \tag{1.1}$$

where $\mu > 0$ is a constant diffusion parameter, $\mathbf{b}$ is a (divergence-free) convective velocity, f is a forcing term, and u_0 is the initial condition.

In order to rewrite (1.1) in the ALE framework, we consider for $t = 0$, Ω_0 as the reference domain assumed to have Lipschitz boundary $\partial\Omega_0$, and let $\Omega_t \subset \mathbb{R}^d$ be the corresponding moving domain at time $t \in (0,T]$. Let $\{\mathcal{A}_t\}_{t\in[0,T]}$ be a family of maps with $\mathcal{A}_0 = \mathrm{I}_d$ the identity map, such that $\Omega_t = \mathcal{A}_t(\Omega_0)$, $t \in [0,T]$. In other words, for $t \in [0,T]$, each $\mathbf{y}$ from the reference domain Ω_0 is mapped through $\mathcal{A}_t$ to the corresponding $\mathbf{x} \in \Omega_t$, i.e., the map $\mathcal{A}_t$ is given by:

$$\mathcal{A}_t : \Omega_0 \subseteq \mathbb{R}^d \to \Omega_t \subseteq \mathbb{R}^d, \quad \mathbf{x}(\mathbf{y},t) = \mathcal{A}_t(\mathbf{y}).$$

We frequently regard $\mathcal{A}_t$ as a space-time function $\mathcal{A}(\mathbf{y},t) := \mathcal{A}_t(\mathbf{y})$, and we refer to $\mathbf{y} \in \Omega_0$ as the ALE coordinate and $\mathbf{x} = \mathbf{x}(\mathbf{y},t)$ as the spatial or Eulerian coordinate. Hereafter, we say that $\{\mathcal{A}_t\}_{t\in[0,T]}$ is a *family of ALE maps* if the following two conditions are satisfied [10]:

- Regularity: $\mathcal{A}(\cdot,\cdot) \in \mathbf{W}^1_\infty\big((0,T);\mathbf{W}^1_\infty(\Omega_0)\big)$;
- Injectivity: there exists a constant $\lambda > 0$ such that for all $t \in [0,T]$,

$$\|\mathcal{A}_t(\mathbf{y}_1) - \mathcal{A}_t(\mathbf{y}_2)\| \geq \lambda \|\mathbf{y}_1 - \mathbf{y}_2\|, \qquad \forall\, \mathbf{y}_1,\ \mathbf{y}_2 \in \Omega_0, \tag{1.2}$$

for some norm $\|\cdot\|$ in $\mathbb{R}^d$. The regularity assumption implies that $\mathcal{A}_t$ is Lipschitz continuous, whereas the combination with injectivity assumption gives that $\mathcal{A}_t : \Omega_0 \to \Omega_t$ is invertible with Lipschitz inverse, i.e., $\mathcal{A}_t$ is bi-Lipschitz and thus a homeomorphism. This implies that $v := \hat{v} \circ \mathcal{A}_t^{-1} \in H^1_0(\Omega_t)$ if and only if $\hat{v} \in H^1_0(\Omega_0)$, [21, Proposition 1].

Using these notations, problem (1.1) is defined in the space-time domain:

$$\mathcal{Q}_T := \big\{(\mathbf{x},t) \in \mathbb{R}^d \times \mathbb{R} : t \in [0,T],\ \mathbf{x} = \mathcal{A}_t(\mathbf{y}),\ \mathbf{y} \in \Omega_0\big\}.$$

The *ALE velocity* $\widehat{\mathbf{w}} : \Omega_0 \times [0,T] \to \mathbb{R}^d$ in the ALE frame is given by

$$\widehat{\mathbf{w}}(\mathbf{y},t) := \partial_t \mathbf{x}(\mathbf{y},t),$$

and we indicate by $\mathbf{w} : \mathcal{Q}_T \to \mathbb{R}^d$ the corresponding function on the Eulerian frame. We use ∂_t to denote the usual weak partial derivative in time holding the space variable $\mathbf{x}$ constant. Given a function $g : \mathcal{Q}_T \to \mathbb{R}$, we denote by $D_t g$ the *ALE time-derivative*, namely the time-derivative keeping the ALE coordinate $\mathbf{y}$ fixed:

$$(D_t g)(\mathbf{x}, t) := (\partial_t g)(\mathcal{A}_t(\mathbf{y}), t).$$

The derivation of the ALE formulation of (1.1) is based on the next lemma, proved in [10].

LEMMA 1.1 (Leibnitz formula in $W_1^1(\mathcal{Q}_T)$). *Let $g \in W_1^1(\mathcal{Q}_T)$ and $\{\mathcal{A}_t\}_{t\in[0,T]}$ be a family of ALE maps. Then, $D_t g \in L^1(\mathcal{Q}_T)$ and*

$$D_t g = \partial_t g + \mathbf{w} \cdot \nabla_{\mathbf{x}} g. \tag{1.3}$$

Leibnitz formula (1.3) relates the usual time-derivative with the corresponding ALE time-derivative through the ALE velocity $\mathbf{w}$ and it is a justification of the chain rule for weak ALE time-derivatives. Using (1.3), (1.1) is equivalently written in the ALE framework as follows:

$$\begin{cases} D_t u + (\mathbf{b} - \mathbf{w}) \cdot \nabla_{\mathbf{x}} u - \mu \Delta_{\mathbf{x}} u = f & \text{in } \mathcal{Q}_T, \\ u = 0 & \text{on } \partial\mathcal{Q}_T, \\ u(\cdot, 0) = u_0 & \text{in } \Omega_0. \end{cases} \tag{1.4}$$

Before setting problem (1.4) in its variational form, we introduce some further notation. For any domain D of $\mathbb{R}^m$, $m = d$ or $d + 1$, we denote by $W_r^\ell(D)$ the standard Sobolev spaces with integrability $1 \le r \le \infty$ and differentiability $0 \le \ell < \infty$. We use the notation $L^r(D)$ when $\ell = 0$ and $H^\ell(D)$ when $r = 2$ and $\ell \ge 1$. With $H_0^1(D)$ we denote the subspace of $H^1(D)$ consisting of functions with vanishing trace and equipped with the norm $\|\nabla_{\mathbf{x}} v\|_{L^2(D)}$; we denote its dual by $H^{-1}(D)$. We indicate with $\langle \cdot, \cdot \rangle_D$ both the $H_0^1 - H^{-1}$ duality pairing and the L^2-inner product in D, depending on the context. Spaces of vector-valued functions are written in bold-face. For $Y = W_r^\ell$, $\ell \ge 0, 1 \le r \le \infty$, H_0^1, or H^{-1}, we define the spaces

$$L^2(Y; \mathcal{Q}_T) := \Big\{ v : \mathcal{Q}_T \to \mathbb{R} : \int_0^T \|v(t)\|^2_{Y(\Omega_t)} \, dt < \infty \Big\}.$$

We define accordingly the spaces $C(Y; \mathcal{Q}_T)$ of continuous functions with values in Y, and set

$$\begin{aligned} L^\infty(\mathrm{div}; \mathcal{Q}_T) := \{ &\mathbf{c} : \mathcal{Q}_T \to \mathbb{R}^d : \\ &\operatorname{ess\,sup}_{t\in(0,T)} \big(\|\mathbf{c}(t)\|_{\mathbf{L}^\infty(\Omega_t)} + \|\nabla_{\mathbf{x}} \cdot \mathbf{c}(t)\|_{L^\infty(\Omega_t)} \big) < \infty \}. \end{aligned}$$

To simplify the notation we omit writing the dependency in $\mathcal{Q}_T$ when there is no confusion.

A *non-conservative* weak ALE formulation for problem (1.4) reads as follows: seek $u \in L^2(H^1_0;\ \mathcal{Q}_T) \cap H^1(L^2; \mathcal{Q}_T)$ satisfying $u(\cdot, 0) = u_0$ and such that for all $\upsilon \in L^2(H^1_0)$ and $\tau, t \in [0, T]$ with $\tau < t$,

$$\begin{aligned}(1.5)\quad &\int_\tau^t \langle D_t u, \upsilon\rangle_{\Omega_s}\, ds + \int_\tau^t \langle (\mathbf{b} - \mathbf{w}) \cdot \nabla_{\mathbf{x}} u, \upsilon\rangle_{\Omega_s}\, ds \\ &+ \int_\tau^t \langle (\nabla_{\mathbf{x}} \cdot \mathbf{b}) u, \upsilon\rangle_{\Omega_s}\, ds + \mu \int_\tau^t \langle \nabla_{\mathbf{x}} u, \nabla_{\mathbf{x}} \upsilon\rangle_{\Omega_s}\, ds = \int_\tau^t \langle f, \upsilon\rangle_{\Omega_s}\, ds.\end{aligned}$$

It is known that (1.5) admits a unique solution, provided that $u_0 \in H^1_0(\Omega_0)$, $f \in L^2(\mathcal{Q}_T)$ and $\mathbf{b} \in L^\infty(\mathrm{div}; \mathcal{Q}_T)$, [10]. This regularity on the data of the problem guarantees that $u \in H^1(\mathcal{Q}_T) \subset C(L^2; \mathcal{Q}_T)$, which implies the further regularity $\Delta_{\mathbf{x}} u,\ D_t u \in L^2(\mathcal{Q}_T)$; cf., [10].

A conservative weak formulation for (1.4) can be obtained using Reynolds' identities, which can be regarded as weak versions of Reynolds' Transport Theorem [10].

LEMMA 1.2 (Reynolds' identities). *Let $\{\mathcal{A}_t\}_{t\in[0,T]}$ be a family of ALE maps. For any $\upsilon \in W^1_1(\mathcal{Q}_T)$ there holds*

$$(1.6)\qquad \frac{d}{dt}\int_{\Omega_t} \upsilon\, d\mathbf{x} = \int_{\Omega_t} (D_t \upsilon + \upsilon \nabla_{\mathbf{x}} \cdot \mathbf{w})\, d\mathbf{x}.$$

In particular, for $w,\ \upsilon \in H^1(\mathcal{Q}_T)$ we have

$$(1.7)\qquad \frac{d}{dt}\int_{\Omega_t} \upsilon w\, d\mathbf{x} = \int_{\Omega_t} w(D_t \upsilon + \upsilon \nabla_{\mathbf{x}} \cdot \mathbf{w})\, d\mathbf{x} + \int_{\Omega_t} \upsilon D_t w\, d\mathbf{x}.$$

As we shall see in the sequel, and is thoroughly discussed in [10, 9], Reynolds' identities (1.6), (1.7) play a significant role in the stability and error analysis of dG schemes for (1.4).

Inserting (1.7) into (1.5) give the *conservative* weak formulation for problem (1.1):

(1.8)

$$\begin{aligned}&\langle u(t), \upsilon(t)\rangle_{\Omega_t} + \int_\tau^t \langle \nabla_{\mathbf{x}} \cdot \big[(\mathbf{b} - \mathbf{w})u\big], \upsilon\rangle_{\Omega_s}\, ds + \mu \int_\tau^t \langle \nabla_{\mathbf{x}} u, \nabla_{\mathbf{x}} \upsilon\rangle_{\Omega_s}\, ds \\ &- \int_\tau^t \langle u, D_t \upsilon\rangle_{\Omega_s}\, ds = \langle u(\tau), \upsilon(\tau)\rangle_{\Omega_\tau} + \int_\tau^t \langle f, \upsilon\rangle_{\Omega_s}\, ds, \quad \forall \upsilon \in H^1_0(\mathcal{Q}_T).\end{aligned}$$

It is clear that non-conservative and conservative weak formulations (1.5) and (1.8) are equivalent at the continuous level. Moreover, setting $\upsilon = u$ in either (1.5) or (1.8), it is possible to prove the following stability result for the PDE (1.1) [21, 10]:

$$\begin{aligned}(1.9)\qquad &\|u(t)\|^2_{L^2(\Omega_t)} + \mu \int_\tau^t \|\nabla_{\mathbf{x}} u(s)\|^2_{L^2(\Omega_s)}\, ds \\ &\qquad \le \|u(\tau)\|^2_{L^2(\Omega_\tau)} + \frac{1}{\mu}\int_\tau^t \|f(s)\|^2_{H^{-1}(\Omega_s)}\, ds,\end{aligned}$$

for $0 \le \tau \le t \le T$. Note that in particular, estimate (1.9) does not involve any constants depending on the particular choice of the ALE map $\mathcal{A}_t$ and exhibits monotone behavior of the norm $\|u(t)\|_{L^2(\Omega_t)}$ provided $f \equiv 0$. This is expected, as the original problem (1.1) is independent of the family of the ALE maps. However, this property is not guaranteed anymore after time discretization. In fact, at the discrete level, the arbitrary extension of the ALE map, not only may influence and pollute the stability of the corresponding discrete scheme, but it may also lead to schemes where conservative and non-conservative formulations are no-longer equivalent. We discuss this further in the sequel.

We say that a numerical method for problem (1.4) is *ALE-free stable* with respect to the energy norm if it reproduces (1.9); otherwise, if (1.9) is valid with a constant multiplying the right-hand side that depends on $\mathcal{A}_t$, we say that the method is *ALE stable*. In both cases, we say that the method is *stable*. ALE-free stable schemes are most desirable for problem (1.4) because they enjoy the same stability properties as the continuous problem.

1.2. Existing Literature. Second order ALE methods for advection-dominated diffusion problems on moving domains are discussed in the literature for finite difference and finite volume schemes [19, 17, 18, 14]. A numerical scheme is said to satisfy the GCL if it is able to reproduce exactly a constant solution. GCL was introduced in [36, 24, 17] for finite volume schemes as a minimum criterion for unconditional stability. However, the *GCL is not a necessary condition* for numerical schemes to be ALE-free stable. For example, Geuzaine, Grandmont & Farhat [23] propose second order finite volume ALE schemes which enjoy the same stability properties as the continuous problem without satisfying the GCL.

On the other hand, the only provable ALE-free stable scheme for (1.4) based on finite element (FE) discretization in space, and without time-step constraints (unconditional stability), is the backward Euler method [7, 21, 20, 22, 31]. In particular, Formaggia & Nobile [21] consider a conservative backward Euler FE scheme for (1.4) which satisfies the GCL and prove that this scheme is ALE-free stable. Moreover, they study a non-conservative backward Euler FE scheme for (1.4) that fails to satisfy the GCL [21]. This method is not equivalent to the corresponding conservative scheme and it turns out that its stability estimate requires a time-step restriction depending on the ALE map. In addition, the derivation of the stability estimate relies on a Gronwall-type argument which entails constants depending exponentially on the $L^\infty-$norm of the domain velocity $\mathbf{w}$. This does not reflect the stability properties of the continuous problem (1.4). A priori error analysis for the backward Euler FE methods is provided by Gastaldi, [22], Boffi & Gastaldi, [7], Nobile, [31], and Badia & Codina, [5].

Besides first order schemes, there are also second order FE schemes based on the Crank-Nicolson method and the backward differentiation formula (BDF) method of second order; see Formaggia & Nobile [20], Boffi & Gastaldi [7], and Badia & Codina [5]. One consequence of these studies is that the *GCL condition is not sufficient* to guarantee an ALE-free stable scheme, and similar stability issues as for the non-conservative backward Euler FE scheme are observed. In fact, numerical simulations show that the monotonicity of $\|u(t)\|_{L^2(\Omega_t)}$ does not hold at the discrete level [20]. At a theoretical level, this effect appears when the ALE velocity $\mathbf{w}$ is treated as an extra advection for the method. Hence, the stability bound requires a Gronwall-type argument which yields time-step constraints and stability constants depending on the ALE map. This is despite the fact that (1.9) is insensitive to $\mathbf{w}$.

The generalization of the GCL condition proposed in our recent work [10, 9] is a *sufficient* condition to guarantee ALE-free stable schemes (independent of the accuracy of the method). In fact, we propose a class of ALE-free schemes of any desired accuracy based on discontinuous Galerkin (dG) methods in time. These seem to be the first schemes having this property.

At this point, we also mention the work by Mackenzie & Mekwi [29], who propose an adaptive $\theta-$method time integrator and prove it is ALE-free stable. However, the statement about asymptotic second order accuracy hinges on heuristics. This method chooses the parameter θ in each time-step, depending on the domain velocity $\mathbf{w}$, so as to satisfy the GCL.

Moreover, it is worth mentioning that this discussion is about a priori error analysis. We are unaware of any a posteriori error estimates for ALE methods.

1.3. A dG Approach. As already alluded to in the previous subsection, the existing FE methods and their analysis share the following common features (1.4):

- All weak formulations have test functions with vanishing material derivative, which seems not adequate when the space and time are tangled together.
- All time discretizations are defined pointwise, which makes it difficult, if not impossible, to reproduce at the discrete level the subtle cancellations leading to the stability estimate (1.9).
- For second order schemes, the discrete stability is obtained via a discrete Gronwall-type argument, which does not reflect the stability properties of the continuous problem.
- Despite the fact that the role of the GCL for stability and accuracy is not clear, for FE schemes the GCL is closely related to quadrature in time; in fact, for test functions with vanishing material derivative, the GCL is equivalent to a discrete version of Reynolds' identity (1.7).

In view of these observations, we realize that the main obstruction in the design of ALE-free stable schemes is the discretization in time. Hence, we consider in [10, 9, 8] time-discrete schemes based on dG methods of any order. The reasons for this specific choice are:

- The backward Euler method, the only known ALE-free stable method, is a dG method.
- dG methods couple time and space in a natural variational way and allow for test functions with non-vanishing material derivative. As we shall see in the sequel, the variational structure of the methods suggests a class of ALE-free stable schemes of any order in time [10].
- dG methods are suitable for time and space adaptivity. In this context, adaptivity is an essential tool for capturing disparate space and time scales efficiently and to cope with the nonlinear interaction between the approximation of the moving domain and the approximation of the solution to (1.1) (defined on the approximate domain).

In [10], we propose dG methods of any order for problems defined on time-dependent domains and the ALE framework, and we study their stability properties. We also derive practical algorithms by enforcing appropriate quadrature in time. The first family of these algorithms are the so-called Reynolds' methods and correspond to quadratures that keep valid the Reynolds' identity at the discrete level. It can be shown that, for piecewise polynomial in time ALE maps, Reynolds' methods are ALE-free stable and lead to optimal order error bounds (both a priori and a posteriori), without any time-step restrictions. The second family of methods is the well known Runge-Kutta-Radau (RKR) methods, that result from dG methods of order $q+1$ by approximating integrals in time by the Radau quadrature with $q+1$ nodes. These methods are also ALE-free stable, but subject to an ALE-time step constraint (conditional stability). We perform an a priori error analysis for these methods in [9] and an a posteriori error analysis in [8]. Our work extends the analysis of dG methods of any order for non-moving domains [37, Chapter 12] to time-dependent domains within the ALE framework. We also refer to [25] for the implementation of first-order dG methods in the context of fluid-structure interactions. dG methods have been considered earlier by Chrysafinos & Walkington in [15] within a pure Lagrangian approach for advection-dominated diffusion problems but with a purpose distinct from ours. More precisely, in our approach, the ALE velocity $\mathbf{w}$ does not play the role of an advective velocity, while in the approach of Chrysafinos & Walkington, the ALE velocity $\mathbf{w}$ is designed to compensate for large advections $\mathbf{b}$, and thus is chosen to satisfy $\mathbf{w} \approx \mathbf{b}$. For more details on a comparison between [15] and our work, we refer to [10].

In this paper we review the results of [10, 9, 8] about stability and error analysis of dG methods for ALE formulations. The main contributions of [10] regarding stability are as follows:

- Propositions of dG methods of any order in time with exact integration for problems defined on time-dependent domains and the ALE framework. These methods lead to ALE-free stability at the nodes $t = t_n$

(nodal stability) and ALE stability for all $t \in [0, T]$ (global stability), both without time-step constraints (unconditional stability).

- Generalization of GCL to higher order in time ALE schemes. The variational structure of the dG methods allow us to impose appropriate quadrature in time that lead to a discrete Reynolds' identity for piecewise polynomial ALE maps in time . The chosen quadrature leads to the practical Reynolds' methods for which we show ALE-free nodal stability and ALE global stability, both without time-step constraints.
- Stability study of dG methods with Radau quadrature, which is the natural quadrature for problems defined on time-independent domains. For these methods, we prove ALE-free nodal stability and ALE global stability, both with an ALE time-constraint, but for any ALE map with piecewise W^2_∞ regularity in time and global W^1_∞ in space ALE maps.

The main contributions of [9, 8] related to error control are the following:

- Introduction of a novel ALE projection and study of its properties. The ALE projection extends the usual dG projection to time-dependent domains and the ALE framework. It is one of the main ingredients in the a priori error analysis.
- Introduction of a novel dG reconstruction in the ALE framework and study of its properties. This reconstruction is a generalization of the dG reconstruction in [30], proposed by Makridakis & Nochetto for time-independent domains. It is one of the main ingredients leading to optimal order a posteriori error bounds.
- Proof of optimal order a priori error estimates for dG of any degree with exact integration and Reynolds' quadrature, the latter provided that the ALE map is a continuous piecewise polynomial in time. These estimates are valid without time-step restrictions. In addition, we prove optimal order a priori error estimates for dG with Radau quadrature, but under a mild time-step restriction depending on μ and $\mathcal{A}_t$. The first a priori error bounds for dG-type methods applied to (1.1) were derived by Jamet [28], but without imposing any quadrature in the involved integrals, and thus leading to non-practical algorithms.
- Proof of optimal order a posteriori error estimates for dG of any order with exact integration and for Reynolds' methods, the latter provided that $\mathcal{A}_t$ is a continuous piecewise polynomial in time. These estimates are valid without any time-step restrictions, and are obtained using the same stability PDE techniques as for the continuous problem through the dG reconstruction on the ALE framework. As already mentioned, there seems to be no a-posteriori error estimates available in the current literature for problems defined on moving domains.
- Piecewise polynomial approximation of the ALE map. Given the domain velocity $\mathbf{w}$ at the boundary $\partial\Omega_t$, we approximate $\mathbf{w}$ on the boundary by a piecewise polynomial of degree q (to match with the dG accuracy) in the case of Reynolds' methods. We reconstruct a perturbed ALE map $\tilde{\mathcal{A}}_t$

by time integration and suitable extension inside the domain. Then, we obtain optimal order error bounds by invoking a PDE perturbation argument in which we evaluate the additional geometrical error committed by the approximation of the ALE map.

As discussed in [10, Section 6], our results can be extended to problems (1.1) with advections with non-vanishing divergence. This is a prototype problem for the practically more interesting and theoretically more challenging Navier-Stokes equations on time-dependent domains. Some preliminary estimates for these equations and the Crank-Nicolson FE method that satisfies the GCL have been derived by Nobile in [31]. In [34], Quarteroni & Formaggia use the same method in simulations of their model of the cardiovascular system, based on Navier-Stokes equations on moving domains.

1.4. Organization of the paper. The paper is organized as follows. In Section 2 we review results from [10, 9, 8] regarding the stability and the error analysis for discontinuous Galerkin (dG) methods in time of any order $q \geq 0$ for problem (1.4) and assuming exact integration in time. In Section 3 we consider practical algorithms that are obtained from the dG methods by applying Reynolds' quadrature, and we discuss the relation between Reynolds' quadrature and the GCL. Finally, Section 4 is devoted to Runge-Kutta-Radau (RKR) methods. We present, without proofs, the main results of [10, 9] on stability and a priori error analysis, and we explain the importance of RKR methods.

2. Discontinuous Galerkin method in Time: Exact Integration. In this section, we review some recent results from [10, 9] related to the stability and a priori error analysis for discontinuous Galerkin (dG) methods for problem (1.5) (or (1.8)) defined on time dependent domains. We also report without proofs the main results of the a posteriori error analysis developed in [8]. We discuss exact integration in time and we emphasize the main ideas and the key ingredients leading to efficient (in terms of stability) practical algorithms.

2.1. The dG Method & Stability. The time discretization of (1.5) or (1.8) starts with a partition $0 =: t_0 < t_1 < \cdots < t_N =: T$ of $[0, T]$. For $0 \leq n \leq N-1$, we let $I_n := (t_n, t_{n+1}]$ and $k_n := t_{n+1} - t_n$ be the subintervals and variable time-steps, respectively. We also let $k := \max_{0 \leq n \leq N-1} k_n$ and

$$\mathcal{Q}_n := \{(\mathbf{x}, t) \in \mathcal{Q}_T : \, t \in I_n\}.$$

For $q \geq 0$, the discrete space $\mathcal{V}_q$ associated with the dG method in time of order $q+1$ is [10]

$$\begin{aligned} \mathcal{V}_q := \{V : \mathcal{Q}_T \to \mathbb{R} : \, V|_{I_n} = \sum_{j=0}^{q} \varphi_j t^j \text{ where } \varphi_j \in L^2(H_0^1) \\ \text{with } D_t \varphi_j = 0, \, 0 \leq j \leq q\}. \end{aligned} \tag{2.1}$$

Note that the definition of the discrete space (2.1) is a natural generalization of the corresponding dG space for problems defined on moving domains and written on the ALE framework. Indeed, when the domain does not undergo deformations, i.e., $\mathcal{A}_t = \mathrm{I}_d$ is the identity map for all $t \in [0,T]$, $\mathcal{V}_q$ is reduced to the standard dG space [37, 1]. Moreover, the definition of $\mathcal{V}_q$ ensures that whenever $V \in \mathcal{V}_q$ is considered on the reference domain, $\widehat{V}(\mathbf{y},t) := V\big(\mathcal{A}_t(\mathbf{y}),t\big)$, $\widehat{V}$ is piecewise polynomial of degree at most q with coefficients in H_0^1. In the dG analysis it is customary to consider the space

$$\mathcal{V}_q(I_n) := \{V : \mathcal{Q}_n \to \mathbb{R} : \; V = W|_{\mathcal{Q}_n},\, W \in \mathcal{V}_q\}, \quad 0 \le n \le N-1,$$

consisting of restrictions to $\mathcal{Q}_n$ of functions in $\mathcal{V}_q$.

The dG approximation U to u via the non-conservative ALE formulation is defined as follows [10]: seek $U \in \mathcal{V}_q$ such that

$$U(\cdot,0) = u_0 \;\text{ in }\; \Omega_0, \tag{2.2}$$

and for $0 \le n \le N-1$,

$$\begin{aligned} &\int_{I_n} \langle D_t U, V\rangle_{\Omega_t}\, dt + \langle U(t_n^+) - U(t_n), V(t_n^+)\rangle_{\Omega_{t_n}} \\ &+ \int_{I_n} \langle (\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}} U, V\rangle_{\Omega_t}\, dt + \mu \int_{I_n} \langle \nabla_{\mathbf{x}} U, \nabla_{\mathbf{x}} V\rangle_{\Omega_t}\, dt \\ &= \int_{I_n} \langle f, V\rangle_{\Omega_t}\, dt, \quad \forall V \in \mathcal{V}_q(I_n). \end{aligned} \tag{2.3}$$

Similarly, the conservative dG formulation based on (1.8) reads as [10]

$$\begin{aligned} &\langle U(t_{n+1}), V(t_{n+1})\rangle_{\Omega_{t_{n+1}}} - \langle U(t_n), V(t_n^+)\rangle_{\Omega_{t_n}} \\ &+ \int_{I_n} \langle \nabla_{\mathbf{x}}\cdot\big((\mathbf{b}-\mathbf{w})U\big), V\rangle_{\Omega_t}\, dt + \mu \int_{I_n} \langle \nabla_{\mathbf{x}} U, \nabla_{\mathbf{x}} V\rangle_{\Omega_t}\, dt \\ &- \int_{I_n} \langle U, D_t V\rangle_{\Omega_t}\, dt = \int_{I_n} \langle f, V\rangle_{\Omega_t}\, dt, \quad \forall V \in \mathcal{V}_q(I_n). \end{aligned} \tag{2.4}$$

We emphasize that the above choice of discrete space $\mathcal{V}_q$, containing (time) discrete functions with non-vanishing material derivative, implies that the non-conservative formulation (2.3) and the conservative formulation (2.4) remain equivalent at the discrete level as well, and any $q \ge 0$ [10]. In contrast to most pointwise methods, we also note that the dG method produces approximations defined in the deformable domain Ω_t for all times $t \in [0,T]$. The latter consistency on the domain in which the dG approximation is defined, is critical for the stability analysis, as first observed by Pironneau, Liou, and Tezduyar, [33]. In particular, defining the dG method as in (2.3) or (2.4) provides the same coupling between time and space present at the continuous level, thereby giving rise to desirable stability properties for the time-discrete methods, [10]. Finally, as in the definition of the discrete space, we have that both (2.3) and (2.4) reduce to the standard dG method in time when the domain does not change.

The well posedness of the approximation $U \in \mathcal{V}_q$ is not straightforward as in the cases of time-independent domains. The coupling of time and space in the definition of the dG method on deformable domains forces us to consider (2.3) (or (2.4)) as a time-space problem. On the contrary, for the corresponding case of non-moving domains, proving the existence of the time discrete dG approximation is equivalent to the existence of the solution of an elliptic problem with homogeneous Dirichlet boundary conditions. We refer to [10, Proposition 3.1] for details on the existence of $U \in \mathcal{V}_q$ satisfying (2.2)-(2.3) (or (2.2)-(2.4)). We would like to mention though, that the key relation for the proof is the discrete Reynolds' identity

$$\begin{aligned}\int_{I_n} \Big(\langle D_t V, V\rangle_{\Omega_t} - \langle \mathbf{w}\cdot\nabla_{\mathbf{x}} V, V\rangle_{\Omega_t}\Big) dt \\ = \frac{1}{2}\|V(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} - \frac{1}{2}\|V(t_n^+)\|^2_{L^2(\Omega_{t_n})},\end{aligned} \tag{2.5}$$

valid for every $V \in \mathcal{V}_q(I_n)$ [10, Lemma 3.1]. This estimate is instrumental to prove the following:

Theorem 2.1 (nodal stability with exact integration [10]). *The dG solution $U \in \mathcal{V}_q$ satisfies for $0 \le m < n \le N$:*

$$\begin{aligned}\|U(t_n)\|^2_{L^2(\Omega_{t_n})} + \sum_{j=m}^{n-1} \|U(t_j^+) - U(t_j)\|^2_{L^2(\Omega_{t_j})} + \mu\int_{t_m}^{t_n} \|\nabla_{\mathbf{x}} U(t)\|^2_{L^2(\Omega_t)}\, dt \\ \le \|U(t_m)\|^2_{L^2(\Omega_{t_m})} + \frac{1}{\mu}\int_{t_m}^{t_n} \|f(t)\|^2_{H^{-1}(\Omega_t)}\, dt.\end{aligned} \tag{2.6}$$

The stability estimate (2.6) is the discrete version of (1.9). In particular, it is free from any constants depending on the ALE map, and for $f \equiv 0$ and $m = n-1$, it implies the discrete monotonicity property of the L^2-norm:

$$\|U(t_n)\|_{L^2(\Omega_{t_n})} \le \|U(t_{n-1})\|_{L^2(\Omega_{t_{n-1}})}, \qquad 1 \le n \le N.$$

This property, also valid for the continuous problem, was not observed in [5, 7, 20].

In the remaining stability analysis of [10] and the error analyses of [9, 8] with exact integration, there appear constants depending explicitly on $\nabla_{\mathbf{y}}\mathcal{A}_t$, the space differential of the ALE map. These constants are of the form

$$A_n \sim \|\nabla_{\mathbf{y}}\mathcal{A}_{t_n\to t}\|_{\mathbf{L}^\infty(I_n;\mathbf{L}^\infty(\Omega_{t_n}))}\|(\nabla_{\mathbf{y}}\mathcal{A}_{t_n\to t})^{-1}\|_{\mathbf{L}^\infty(I_n;\mathbf{L}^\infty(\Omega_{t_n}))}$$

and

$$B_n \sim \|\nabla_{\mathbf{y}}\mathcal{A}_{t_n\to t}\|_{\mathbf{W}^1_\infty(I_n;\mathbf{L}^\infty(\Omega_{t_n}))},$$

$0 \le n \le N$. Here, we do not give precise definitions of A_n, B_n, but rather point out the required regularity on the ALE map, when A_n or B_n appear in the estimates. The regularity assumptions on the family of the ALE map ensures that the determinant $\det \mathbf{J}_{\mathcal{A}_t}$ of the Jacobian matrix $\mathbf{J}_{\mathcal{A}_t} := \nabla_{\mathbf{y}} \mathcal{A}_t$ is positive and bounded away from 0 and ∞, uniformly for $t \in [0, T]$, [10]. Thus, A_n, B_n are well defined and bounded. It is to be emphasized that $A_n, B_n = \mathcal{O}(1)$ are local and do not involve exponentials of either geometric quantities or T; refer to [10] for details. In the rest of the paper we use the notation $\lesssim$ to indicate absolute constants depending only on the polynomial degree q, the space dimension d as well as constants arising due to the application of the Poincaré inequality.

Next, we state the stability result for $\|U(t)\|_{L^2(\Omega_t)}$ for every $t \in [0, T]$.

THEOREM 2.2 (global stability with exact integration [10]). *Let $f \in L^2(\mathcal{Q}_T)$ and $\{\mathcal{A}_t\}_{t\in[0,T]}$ be a family of ALE maps. Then, the dG approximation $U \in \mathcal{V}_q$ satisfies for $0 \le n \le N$:*

$$
\begin{aligned}
\sup_{t\in[0,t_n]} \|U(t)\|^2_{L^2(\Omega_t)} \lesssim \max_{0\le j\le n-1} \big\{A_j(1 + F_j k_j)\big\}\times \\
\Big(\|U(0)\|^2_{L^2(\Omega_0)} + \frac{1}{\mu}\int_0^{t_n} \|f(t)\|^2_{H^{-1}(\Omega_t)}\,dt\Big) \\
+ \max_{0\le j\le n-1} A_j k_j \int_{I_j} \|f(t)\|^2_{L^2(\Omega_t)}\,dt,
\end{aligned}
\tag{2.7}
$$

with

$$
F_j := B_j + \frac{\|\mathbf{b} - \mathbf{w}\|^2_{L^\infty(\mathcal{Q}_j)}}{\mu} \quad 0 \le j \le n. \tag{2.8}
$$

As in the case of non-moving domains [37], to prove an estimate of the form (2.7), we first need to derive a relation between the discrete approximation U and its material derivative $D_t U$. This is possible for the dG approximation in deformable domains as well, because $U \in \mathcal{V}_q$ is a piecewise polynomial of degree at most q when viewed on the reference domain. Therefore, finite-dimensional arguments, such as inverse inequalities, [13, Chapter 4, Lemma 4.5.3], can be applied to relate U with $D_t U$. For details on the proof of Theorem 2.2, we refer to [10]. The stability estimate (2.7) is valid for any choice of the time-steps k_n. However, in contrast to the nodal stability estimate (2.6), estimate (2.7) involves ALE constants. In fact, the global estimate (2.7) suggests that the monotonicity property of $\|U(t)\|_{L^2(\Omega_t)}$ does not hold for all $t \in [0, T]$, but only at the breakpoints t_n. This fact is also observed numerically and is reported in Figure 1.

2.2. The ALE Projection. Since u does not belong in general to $\mathcal{V}_q$, the derivation of optimal a priori error estimates is achieved by introducing an adequate projection $Pu \in \mathcal{V}_q$ of u. Then, the error $e := u - U$ is

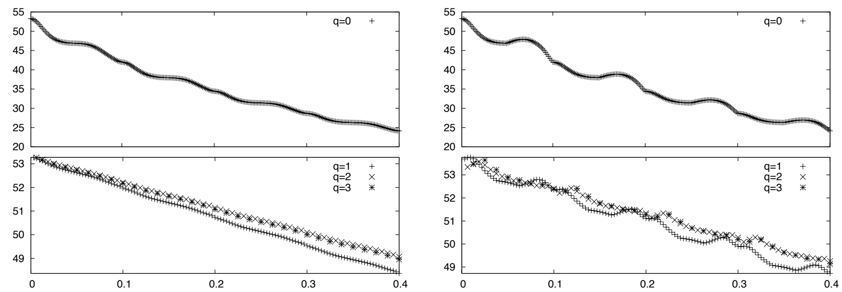

FIG. 1. *Evolution of* $\|U(t_n)\|_{L^2(\Omega_{t_n})}$ *(left) and* $\max_{t\in I_n} \|U(t)\|_{L^2(\Omega_t)}$ *(right) for* $q = 0$ *with* 2^8 *uniform time steps (top) and* $q = 1, 2, 3$ *with respectively* 2^7*,* 2^6*,* 2^5 *uniform time steps (bottom). The space discretization is fine enough not to influence the time discretization. The reference domain is* $\Omega_0 := (0, 1) \times (0, 1)$*, the time interval is* $[0, 0.4]$*, the diffusivity is* $\mu = 0.01$*, the domain velocity* $\mathbf{w}$ *is the* L^2-*projection over piecewise polynomials of degree* q *of the time derivative of the map* $(\mathbf{y}, t) \mapsto \mathbf{y}(2-\cos(20\pi\, t))$*, with* $\mathbf{y} \in \Omega_0$*,* $t \in (0, 0.4)$*, and the forcing is* $f = 0$ *[21]. The ALE map* $\mathcal{A}_t$ *is obtained by integrating* $\mathbf{w}$ *in each time interval* I_n*, thereby enforcing continuity at the nodes. All schemes display monotone* $\|U(t)\|_{L^2(\Omega_t)}$ *when restricted to the breakpoints* $t = t_n$*, as predicted by Theorems 2.1 and 3.1 below, the backward Euler scheme (*$q = 0$*) being much more dissipative than the others (*$q > 0$*). Oscillations of the ALE map destroy this monotonicity property over the whole time interval, thereby corroborating Theorems 2.2 and 3.2.*

decomposed as $e := \rho + \Theta$ with $\rho := u - Pu$ and $\Theta := Pu - U \in \mathcal{V}_q$. Selecting Pu so that ρ has optimal decay in targeted norms (see Proposition 2.2), the a priori error analysis boils down to proving optimal order a priori error estimates for $\Theta \in \mathcal{V}_q$. This is achieved using the stability results of the previous section. We now define Pu.

DEFINITION 2.1 (ALE projection [9]). *For* $q > 0$*, the ALE projection* $Pu \in \mathcal{V}_q$ *of* $u \in C(H_0^1; \mathcal{Q}_T)$ *is defined as follows:*

$$Pu(\cdot, 0) = u(0), \quad in \quad \Omega_0, \tag{2.9}$$

and for $0 \le n \le N - 1$*,*

$$Pu(\cdot, t_{n+1}) = u(\cdot, t_{n+1}), \quad in \quad \Omega_{t_{n+1}} \tag{2.10}$$

and

$$\int_{I_n} \langle Pu - u, V\rangle_{\Omega_t}\, dt = 0, \quad \forall\, V \in \mathcal{V}_{q-1}(I_n), \tag{2.11}$$

while for $q = 0$*, the latter condition is void.* We refer to Pu as the "ALE projection" of u since, according to (2.11), it is an L^2-type projection over the space-time strip Q_n and therefore defined through the ALE map $\mathcal{A}_t$. More precisely, considering $\Omega_{t_{n+1}}$, $0 \le n \le N - 1$ as the reference domain and associating to every function $g : \mathcal{Q}_T \to \mathbb{R}$ the function $\widehat{g} : \Omega_{t_{n+1}} \times [0,T] \to \mathbb{R}$ defined by $\widehat{g}(\mathbf{y},t) := g\big(\mathcal{A}_{t_{n+1}\to t}(\mathbf{y}),t\big)$, (2.11) is equivalent to

$$\int_{I_n} \langle (\widehat{Pu} - \hat{u}) \det \mathbf{J}_{\mathcal{A}_{t_{n+1}\to t}}, \widehat{V}\rangle_{\Omega_{t_{n+1}}} = 0, \quad \forall \widehat{V} \in \widehat{\mathcal{V}}_{q-1}(I_n), \tag{2.12}$$

with

$$\widehat{\mathcal{V}}_q(I_n) := \big\{\widehat{V} :\ V \in \mathcal{V}_q(I_n)\big\}. \tag{2.13}$$

Equality (2.12) reveals that the operator P is not a pure time operator, as for fixed domains, but it rather depends on the particular family of the ALE maps. Alternatively,we could replace Ω_t by Ω_{t_n} (or $\Omega_{t_{n+1}}$) in (2.11):

$$\int_{I_n} \langle (\widehat{Pu} - \hat{u}), \widehat{V}\rangle_{\Omega_{t_n}} = 0, \quad \forall \widehat{V} \in \widehat{\mathcal{V}}_{q-1}(I_n).$$

In this case, P would be a projection purely in time and the interaction between space and time be avoided. However, the choice of (2.9)-(2.11) is the only one that leads to the following property.

Lemma 2.1 (key property of P [9]). *For $0 \le n \le N$, the error $\rho := u - Pu$ satisfies*

$$\begin{aligned}\int_{I_n} \langle D_t\rho, V\rangle_{\Omega_t}\, dt + \langle \rho(t_n^+) - \rho(t_n), V(t_n^+)\rangle_{\Omega_{t_n}} \\ + \int_{I_n} \langle \rho\nabla_{\mathbf{x}}\cdot \mathbf{w}, V\rangle_{\Omega_t}\, dt = 0,\ \forall V \in \mathcal{V}_q(I_n).\end{aligned} \tag{2.14}$$

The proof of (2.14) is based on Reynolds' identity (1.7), as well as the definition (2.9)-(2.11) of P. At this point, we emphasize that the orthogonality property (2.14) is essential for the a priori error analysis, as it allows the cancelation of the term $D_t\rho$, which is not expected to be of optimal order of accuracy, as well as the cancelation of the discontinuities $\rho(t_n^+) - \rho(t_n)$ for which an optimal order a priori error bounds are not evident. An orthogonality property similar to (2.14) is used in cases of time independent domains and the corresponding dG projection, [37, 1]. Actually, having at hand (2.14), the a priori error bounds for the numerical scheme (2.3) can be obtained following the same steps as for non-moving domains. In that respect, the definition of the ALE projection P is a natural generalization of the corresponding dG projection in time independent domains; cf., e.g., [37, 1].

We next prove the *well posedness* of the ALE projection. The proof of uniqueness is rather easy. Indeed, if $V_1, V_2 \in \mathcal{V}_q$ satisfy (2.10), then $V_1 - V_2 = (t_{n+1} - t)V$ with $V \in \mathcal{V}_{q-1}(I_n)$; this in conjunction with (2.11) leads to $V \equiv 0$, or, $V_1 \equiv V_2$. However, the proof of the existence is technical and requires additional regularity on the ALE map. In contrast to time independent domains, the condition $u \in C(H_0^1; \mathcal{Q}_n)$ does not imply in general that $(Pu)(t) \in H_0^1(\Omega_t)$ for all $t \in I_n$. As we have shown in [9, Remark 3.1], if the spatial regularity of $\mathcal{A}_t$ is only $\mathbf{W}^1_\infty$, then the H_0^1-norm of $(Pu)(t)$ may blow-up for some $t \in I_n$. The space-time tangling is responsible once again for this difficulty and is a natural consequence of the movement of the domain in time. The ALE projection is not a pure time projection and inherits such a entanglement.

A sufficient condition on the ALE map that guarantees the existence of a $Pu \in \mathcal{V}_q$, whenever $u \in C(H_0^1; \mathcal{Q}_T)$, is the following:

$$\mathcal{A}_{t_{n+1}\to t} \in \mathbf{L}^\infty\big(I_n; \mathbf{W}^2_\infty(\Omega_{t_{n+1}})\big), \qquad 0 \le n \le N-1. \tag{2.15}$$

Let $\mathcal{W}_q$ and $\mathcal{W}_q(I_n)$ be defined as $\mathcal{V}_q$ and $\mathcal{V}_q(I_n)$, respectively, with the difference that H_0^1 is replaced by L^2. Then, the existence of the ALE projection can be established using the next auxiliary lemma:

LEMMA 2.2. *Let $p_n : \mathbb{R} \to \mathbb{R}$ be a non-zero and non-negative polynomial over I_n of degree s, $s \le q$. Then, for all $w_n \in C(L^2; \mathcal{Q}_n)$ there exists a unique $W_n \in \mathcal{W}_{q-s}(I_n)$ such that*

$$\int_{I_n} p_n(t)\langle W_n(t), V(t)\rangle_{\Omega_t}\, dt = \int_{I_n} \langle w_n(t), V(t)\rangle_{\Omega_t}\, dt, \tag{2.16}$$

for all $V \in \mathcal{W}_{q-s}(I_n)$. If, in addition, the family of ALE maps satisfies (2.15) *and $w_n \in C(H_0^1; \mathcal{Q}_n)$, then $W_n \in \mathcal{V}_{q-s}(I_n)$.*

Proof. The proof follows the lines of Proposition 3.1 in [9]. More precisely, for the proof of the first assertion, we take $\Omega_{t_{n+1}}$, $0 \le n \le N-1$, as the reference domain, and we let $\widehat{\mathcal{W}}_q(I_n) := \big\{\widehat{W} : W \in \mathcal{W}_q(I_n)\big\}$. Then, since p_n is a non-negative polynomial of degree $s \le q$, $\widehat{\mathcal{W}}_{q-s}(I_n)$ is a Hilbert space, with respect to the inner product $(\cdot,\cdot)_n : [\widehat{\mathcal{W}}_{q-s}(I_n)]^2 \to \mathbb{R}$, defined as:

$$(\widehat{W}, \widehat{V})_n := \int_{I_n} p_n(t)\langle \widehat{W} \det \mathbf{J}_{\mathcal{A}_{t_{n+1}\to t}}, \widehat{V}\rangle_{\Omega_{t_{n+1}}}, \quad \forall \widehat{W}, \widehat{V} \in \widehat{\mathcal{W}}_{q-s}(I_n).$$

The assumptionn $w_n \in C(L^2; \mathcal{Q}_n)$ implies that $\widehat{w}_n \in C\big(I_n; L^2(\Omega_{t_{n+1}})\big)$. Hence, by Riesz representation Theorem, there exists a unique $\widehat{W}_n \in \widehat{\mathcal{W}}_{q-s}(I_n)$ such that

$$(\widehat{W}_n, V)_n = \int_{I_n} \langle \widehat{w}_n \det \mathbf{J}_{\mathcal{A}_{t_{n+1}\to t}}, \widehat{V}\rangle_{\Omega_{t_{n+1}}}, \quad \forall \widehat{V} \in \widehat{\mathcal{W}}_{q-s}(I_n). \tag{2.17}$$

Since (2.17) is equivalent to (2.16), we deduce that there exists a unique $W_n \in \mathcal{W}_{q-s}(I_n)$ satisfying (2.16).

For the proof of the second claim, we write $\widehat{W}_n = \sum_{j=0}^{q-s} \widehat{W}_{n,j}(t_{n+1}-t)^j$ with $\widehat{W}_{n,j} \in L^2(\Omega_{t_{n+1}})$, $0 \le j \le q-s$, because $\widehat{W}_n \in \widehat{\mathcal{W}}_{q-s}$. Next, we rewrite (2.17) as

$$\int_{\Omega_{t_{n+1}}} \upsilon \int_{I_n} \big(p_n(t)\widehat{W}_n - \widehat{w}_n\big)(t_{n+1}-t)^i \det \mathbf{J}_{\mathcal{A}_{t_{n+1}\to t}} dt\, d\mathbf{y} = 0,$$

for all $v \in L^2(\Omega_{t_{n+1}})$ and $0 \le i \le q-s$. For a.e. $\mathbf{y} \in \Omega_{t_{n+1}}$, the above equation is equivalent to the algebraic system for $\widehat{\mathbf{W}}_n := (\widehat{W}_{n,j})_{j=0}^{q-s}$:

$$\mathbf{A}_n(\mathbf{y})\widehat{\mathbf{W}}_n(\mathbf{y}) = \widehat{\mathbf{w}}_n(\mathbf{y}),$$

with matrix

$$\mathbf{A}_n(\mathbf{y})_{i,j} := \int_{I_n} p_n(t)(t_{n+1}-t)^{(i-1)+(j-1)} \det \mathbf{J}_{\mathcal{A}_{t_{n+1}\to t}}(\mathbf{y},t)\,dt$$

and right-hand side

$$\widehat{\mathbf{w}}_n(\mathbf{y})_i := \int_{I_n} \widehat{w}_n(\mathbf{y},t)(t_{n+1}-t)^{i-1} \det \mathbf{J}_{\mathcal{A}_{t_{n+1}\to t}}(\mathbf{y},t)\,dt,$$

for $1 \le i,j \le q-s+1$. The additional regularity (2.15) of $\mathcal{A}_t$ yields that $\mathbf{A}_n$ is Lipschitz continuous and invertible, and that $\mathbf{A}_n^{-1}$ is also Lipschitz (see Step 4 in the proof of [9, Proposition 3.1]), Therefore, using that $\widehat{\mathbf{w}}_n \in \mathbf{H}_0^1(\Omega_{t_{n+1}})$, we conclude that $\widehat{\mathbf{W}}_n = \mathbf{A}_n^{-1}\widehat{\mathbf{w}}_n \in \mathbf{H}_0^1(\Omega_{t_{n+1}})$. This, completes the proof of the second claim. ☐

Using Lemma 2.2 we deduce:

Proposition 2.1 (existence of the ALE projection [9]). *Let $\mathcal{A}_t$ satisfy* (2.15)*. Then, for every $u \in C(H_0^1;\mathcal{Q}_T)$ there exists a unique $Pu \in \mathcal{V}_q$ satisfying* (2.9)-(2.11).

Proof. Finding $Pu \in \mathcal{V}_q$ satisfying (2.9)-(2.11) is equivalent to finding $W_n \in \mathcal{V}_{q-1}(I_n)$, $0 \le n \le N-1$, such that

$$\begin{aligned}\int_{I_n} &(t_{n+1}-t)\langle W_n(t), V(t)\rangle_{\Omega_t}\,dt \\ &= \int_{I_n} \langle u(t) - u(\mathcal{A}_{t\to t_{n+1}}(\cdot), t_{n+1}), V(t)\rangle_{\Omega_t}\,dt, \quad \forall V \in V_{q-1}(I_n).\end{aligned}$$

The asserted claim of the proposition follows immediately from Lemma 2.2 with $p_n(t) := t_{n+1}-t$, $s := 1$ and $w_n := u - u(\mathcal{A}_{t\to t_{n+1}}(\cdot), t_{n+1}) \in C(H_0^1;\mathcal{Q}_n)$. ☐

We finish the subsection by stating, without proof, the approximation and stability properties of Pu. The proof of the next proposition is based on comparing the ALE projection with the standard dG projection [37, 1]; details can be found in [9].

Proposition 2.2 (approximation properties & stability of the ALE projection [9]). *Let Pu be the ALE projection defined in* (2.9)-(2.11)*. If the family of ALE maps satisfies* (2.15)*, then, for $t \in I_n$, we have*

$$\|(u-Pu)(t)\|_{L^2(\Omega_t)}^2 \le C_n k_n^{2j+1} \int_{I_n} \|D_t^{j+1}u(t)\|_{L^2(\Omega_t)}^2\,dt, \tag{2.18}$$

$$\begin{aligned}\|\nabla_{\mathbf{x}}(u - Pu)(t)\|^2_{\mathbf{L}^2(\Omega_t)} &\le D_n k_n^{2j+1} \times \\ &\int_{I_n} \left(\|D_t^{j+1} u(t)\|^2_{L^2(\Omega_t)} + \|\nabla_{\mathbf{x}} D_t^{j+1} u(t)\|^2_{\mathbf{L}^2(\Omega_t)} \right) dt,\end{aligned} \tag{2.19}$$

for $0 \le n \le N-1$ and $0 \le j \le q$, where C_n depends on A_n and D_n depends on A_n and $M_n := \|\mathcal{A}_{t_n \to t}\|_{\mathbf{L}^\infty\left(I_n; \mathbf{W}^2_\infty(\Omega_{t_n})\right)}$. In addition, the following stability bounds are valid for P:

$$\int_{I_n} \|D_t^j Pu(t)\|^2_{L^2(\Omega_t)} \, dt \lesssim C_n \int_{I_n} \|D_t^j u(t)\|^2_{L^2(\Omega_t)} \, dt. \tag{2.20}$$

2.3. A Priori Error Analysis. We briefly present now, without proofs, the main results of [9] for the numerical method (2.3) assuming exact integration in time.

As already mentioned in the previous subsection, to obtain an optimal order a priori error bound, we split the error $e = u - U = \rho + \Theta$. The interpolation error $\rho = u - Pu$ is estimated a priori through the approximation properties (2.18), (2.19). On the other hand, because of the key property (2.14), $\Theta = Pu - U \in \mathcal{V}_q$ satisfies:

$$\begin{aligned}\int_{I_n} \langle D_t \Theta, V \rangle_{\Omega_t} \, dt &+ \langle \Theta(t_n^+) - \Theta(t_n), V(t_n^+) \rangle_{\Omega_{t_n}} \\ &+ \int_{I_n} \langle (\mathbf{b} - \mathbf{w}) \cdot \nabla_{\mathbf{x}} \Theta, V \rangle_{\Omega_t} \, dt + \mu \int_{I_n} \langle \nabla_{\mathbf{x}} \Theta, \nabla_{\mathbf{x}} V \rangle_{\Omega_t} \, dt \\ &= \int_{I_n} \langle \rho(\mathbf{b} - \mathbf{w}) - \mu \nabla_{\mathbf{x}} \rho, \nabla_{\mathbf{x}} V \rangle_{\Omega_t} \, dt, \quad \forall V \in \mathcal{V}_q(I_n),\end{aligned} \tag{2.21}$$

and an optimal order a priori error bound can be established for Θ by applying the stability result (2.6) to (2.21).

Theorem 2.3 (a priori error estimate for dG in time dependent domains [9]). *If the family of the ALE maps satisfies* (2.15)*, then the following estimate holds:*

$$\begin{aligned}&\max_{0 \le n \le N} \|(u - U)(t_n)\|^2_{L^2(\Omega_{t_n})} + \mu \int_0^T \|\nabla_{\mathbf{x}}(u - U)(t)\|^2_{\mathbf{L}^2(\Omega_t)} \, dt \\ &\le \frac{1}{\mu} \sum_{n=0}^{N-1} C_n k_n^{2q+2} \sup_{t \in I_n} \|(\mathbf{b} - \mathbf{w})(t)\|^2_{\mathbf{L}^\infty(\Omega_t)} \int_{I_n} \|D_t^{q+1} u(t)\|^2_{L^2(\Omega_t)} \, dt \\ &+ \mu \sum_{n=0}^{N-1} D_n k_n^{2q+2} \int_{I_n} \left(\|D_t^{q+1} u(t)\|^2_{L^2(\Omega_t)} + \|\nabla_{\mathbf{x}} D_t^{q+1} u(t)\|^2_{\mathbf{L}^2(\Omega_t)} \right) dt,\end{aligned}$$

with u the solution of (1.4) *and U the dG solution of* (2.3)*, and where C_n, D_n, $0 \le n \le N-1$, are constants proportional to those in* (2.18),(2.19).

2.4. The ALE Reconstruction. For the a posteriori error analysis, we follow the reconstruction technique proposed by Akrivis, Makridakis & Nochetto in [2, 30, 3, 4] for time discrete schemes on time independent domains. The main idea is to introduce a continuous approximation $U_{\mathrm{R}} \in \mathcal{V}_{q+1}$ of the dG approximation U of (2.3), called the reconstruction of U, with the following properties:

- $U_{\mathrm{R}}(t_n) = U(t_n), \quad 0 \leq n \leq N$.
- U_{R} satisfies a perturbation of the original problem (1.4).
- $U_{\mathrm{R}} - U$ is of optimal order of accuracy.

Then, the error $e = u - U$ splits into $u - U_{\mathrm{R}}$ and $U_{\mathrm{R}} - U$ and the final a posteriori error bound is obtained using the stability properties of the continuous equation (1.4).

Following [30], one of the key points to derive a posteriori error estimations is the definition of an appropriate reconstruction of the discrete solution U. Extending the work presented in [30] to moving domains relies mainly on the principle that integration by parts is replaced by Reynolds' identity (1.7). In particular, the reconstruction $U_{\mathrm{R}} \in \mathcal{V}_{q+1}$ of U is defined as follows: For $0 \leq n \leq N-1$,

$$U_{\mathrm{R}}(t_n^+) = U(t_n) \quad \text{in } \Omega_{t_n} \tag{2.22}$$

and

$$\begin{aligned}\int_{I_n} \langle D_t U_{\mathrm{R}}, V\rangle_{\Omega_t}\, dt + \int_{I_n} \langle U_{\mathrm{R}} \nabla_{\mathbf{x}} \cdot \mathbf{w}, V\rangle_{\Omega_t}\, dt &= \int_{I_n} \langle D_t U, V\rangle_{\Omega_t}\, dt \\ &+ \int_{I_n} \langle U \nabla_{\mathbf{x}} \cdot \mathbf{w}, V\rangle_{\Omega_t}\, dt \\ &+ \langle U(t_n^+) - U(t_n), V(t_n^+)\rangle_{\Omega_{t_n}}, \quad \forall V \in \mathcal{V}_q(I_n).\end{aligned} \tag{2.23}$$

Using Lemma 2.2, it is possible to prove that the reconstruction U_{R}, defined through (2.22)-(2.23) is well defined, provided that the family of the ALE maps $\{\mathcal{A}_t\}_{t\in[0,T]}$ satisfies (2.15). The detailed proof of the well posedness of U_{R}, as well as its properties, are discussed in [8].

2.5. A Posteriori Error Analysis. The function U_{R} is instrumental to derive the following bound.

THEOREM 2.4 (a posteriori error estimate for dG in time dependent domains [8]). *Let U_{R} denote the reconstruction of U defined in* (2.22)-(2.23). *If the family of ALE maps satisfies* (2.15), *then, the following a* *posteriori error estimate holds true:*

$$
\begin{aligned}
&\max_{0\le t\le T}\Big\{\|(u-U_R)(t)\|^2_{L^2(\Omega_t)}+\mu\int_0^t\Big[\|\nabla_{\mathbf{x}}(u-U)(s)\|^2_{L^2(\Omega_s)}\\
&+\frac{1}{2}\|\nabla_{\mathbf{x}}(u-U_{\mathrm{R}})(s)\|^2_{L^2(\Omega_s)}\Big]\,ds\Big\}\\
&\le \mu\int_0^T\|\nabla_{\mathbf{x}}(U-U_{\mathrm{R}})(s)\|^2_{L^2(\Omega_s)}\,ds\\
&+\frac{4}{\mu}\int_0^t\|\mathcal{E}(s)\|^2_{H^{-1}(\Omega_s)}\,ds+\frac{4}{\mu}\int_0^T\|(f-\Pi_q f)(s)\|^2_{H^{-1}(\Omega_s)}\,ds
\end{aligned}
\tag{2.24}
$$

with

$$
\begin{aligned}
\mathcal{E} :=\ &\nabla_{\mathbf{x}}\cdot\big[(\mathbf{b}-\mathbf{w})(U-U_{\mathrm{R}})\big]+\big[U_{\mathrm{R}}\nabla_{\mathbf{x}}\cdot\mathbf{w}-\Pi_q(U_{\mathrm{R}}\nabla_{\mathbf{x}}\cdot\mathbf{w})\big]\\
&+\Big[\nabla_{\mathbf{x}}\cdot\big[(\mathbf{b}-\mathbf{w})U\big]-\Pi_q\big(\nabla_{\mathbf{x}}\cdot\big[(\mathbf{b}-\mathbf{w})U\big]\big)\Big]+\mu\big[\Pi_q(\Delta_{\mathbf{x}}U)-\Delta_{\mathbf{x}}U\big],
\end{aligned}
$$

and $\Pi_q : L^2(L^2;\mathcal{Q}_T)\to\mathcal{W}_q$ *an* L^2-*type projection.* □

We omit giving the precise definition of Π_q, as well as the proof of Theorem 2.4, and we refer to [8] for details. We emphasize though, that estimate (2.24) is of the same form as the corresponding one in time independent domains. The additional terms, appearing in the error indicator $\mathcal{E}$, reflect the geometry of the problem. Note also that because time and space are tangled together, the term $\Pi_q(\Delta_{\mathbf{x}}U)-\Delta_{\mathbf{x}}U$ does not vanish for moving domains, which creates additional difficulties from computational point of view; this issue is discussed in [8]. In addition, in contrast to time-independent domains, for $0\le n\le N-1$, the difference $U_{\mathrm{R}}-U$ in $\mathcal{Q}_n$, cannot be expressed in terms of the jump estimators

$$
J_n(\mathbf{x},t):=U\big(\mathcal{A}_{t_n}\circ\mathcal{A}_t^{-1}(\mathbf{x}),t_n^+\big)-U\big(\mathcal{A}_{t_n}\circ\mathcal{A}_t^{-1}(\mathbf{x}),t_n\big),\quad (\mathbf{x},t)\in\mathcal{Q}_n,
$$

for $q>0$. More precisely, for time-independent domains, it can be proven that [30]

$$
(U_{\mathrm{R}}-U)(\mathbf{x},t):=\frac{t-t_n}{k_n}J_n(\mathbf{x},t),\quad (\mathbf{x},t)\in\mathcal{Q}_n. \tag{2.25}
$$

This is not true for deformable domains due to the tangling of space and time, except for $q=0$. It can be proven though, [8], that the difference $U_{\mathrm{R}}-U$ in the $L^2(L^2;\mathcal{Q}_n)$ and $L^2(H^1_0;\mathcal{Q}_n)$ can be bounded above by the $L^2(L^2;\mathcal{Q}_n)$ and $L^2(H^1_0;\mathcal{Q}_n)$ norm of the jump estimator J_n multiplied by local constants depending on the ALE map. This is something also observed numerically, as depicted in Figure 2. Finally, we mention that the a priori error estimate of Theorem 2.3 implies the optimal decay of the left hand side of (2.24).

2.6. Numerical Experiment. We now check the optimal decay of the proposed a posteriori error estimator. The initial domain is the unit square, $\Omega_0 := (-1,1)\times(-1,1)$, and it is deformed according to the ALE map $\mathcal{A}_t(\mathbf{y}) := \mathbf{y}(1+\frac{1}{2}\,T_{11}(t))$ for $t \in (0,99)$, where $T_n(t) := \cos(n\arccos(t))$ is the nth Chebychev polynomial of first kind. We set $\mu = 1.0$ and the exact solution is manufactured to be $u(\mathbf{x},t) := \exp(x_1\ t)\sin(x_2\ t)$ with $\mathbf{x} := (x_1, x_2) \in \Omega_t$. We consider a naive time adaptivity consisting in the following steps (Dörfler strategy):

- We start with a subdivision of 8 uniform intervals of the time interval $(0, 0.99)$ and compute the space time dG solution.
- We compute the error estimator provided in Theorem 2.4 with the exception of the term involving $\Pi_q(\Delta_{\mathbf{x}}U)-\Delta_{\mathbf{x}}U$, as it is not directly implementable with continuous finite element in space (see above discussion). For later reference we call the resulting quantity the total estimator.
- Using this error estimate, we select a smaller subset of time intervals responsible for 10% of the total error estimation.
- We bisect these intervals into two equal parts and repeat the process with this new subdivision of $(0, 0.99)$.

Figure 2 displays the errors in the $\ell^\infty(L^2)$ and the $L^2(H^1)$-norms together with the computed total estimator and the total jump estimator $\Big(\sum_{i=0}^{N-1}\int_{I_n}(\|J_n(t)\|^2_{L^2(\Omega_t)} + \|\nabla_{\mathbf{x}} J_n(t)\|^2_{L^2(\Omega_t)})\,dt\Big)^{1/2}$, all against the number of time steps used for the computation for different schemes ($q = 0,1,2,3$). The space discretization is chosen not to influence the error. The computational rate of convergence is roughly $q+1$ in each case, hence optimal, as depicted in Figure 2.

We also point out that the adaptive algorithm refines systematically at the end of the interval where the motion is more oscillatory. Figure 3 depicts the Chebyshev polynomial used for the domain evolution together with the subdivision chosen by the algorithm at different refinement cycles for $q = 2$.

2.7. Comparison of A Priori and A Posteriori Analyses. We now explain that the a priori and a posteriori analyses are dual versions of one another:

- **A priori error analysis**
  - Find $Pu \in \mathcal{V}_q$, the ALE projection of u, such that $u - Pu$ is of optimal order of accuracy in $L^\infty(L^2)$ and $L^\infty(H^1_0)$.
  - Use the stability of the numerical method (2.3) to bound $Pu - U$ a priori, because $Pu - U \in \mathcal{V}_q$ satisfies an error equation at the discrete level.

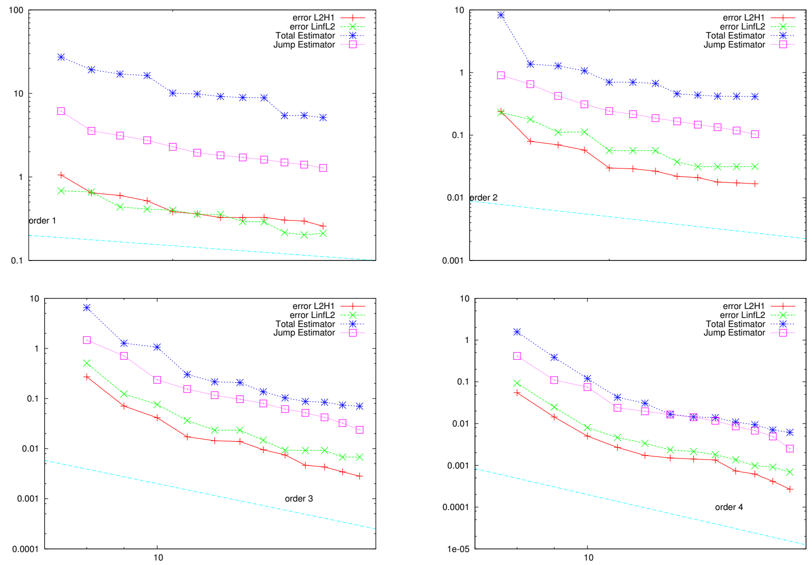

FIG. 2. *Errors in the dG approximation together with the total and jump a posteriori estimators are depicted for* $q = 0$ *(top left),* $q = 1$ *(top right),* $q = 2$ *(bottom left) and* $q = 3$ *(bottom right). The sequence of time steps is determined using the adaptive strategy described above. The reference domain is* $\Omega_0 := (-1,1) \times (-1,1)$ *and it is deformed according to the ALE map* $\mathcal{A}_t(\mathbf{y}) := \mathbf{y}(1 + \frac{1}{2}\, T_{11}(t))$ *for* $t \in (0, 0.99)$*, where* $T_n(t) := \cos(n \arccos(t))$ *is the nth Chebychev polynomial of first kind. The discretization in space is chosen sufficiently fine, not to influence the time discretization error. All quantities exhibit an approximate decay of* $O(N^{-(q+1)})$.

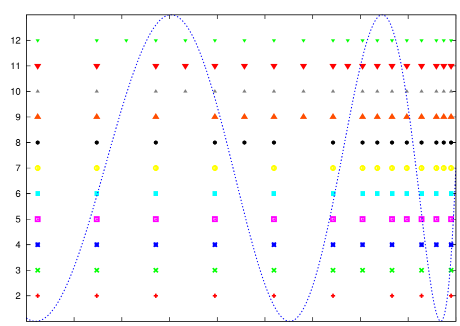

FIG. 3. *Subdivision of the time interval during the adaptive procedure (from bottom to top) for* $q = 2$*. Initially 8 intervals are considered and the algorithm select for refinement the smallest amount of intervals contributing for* 10% *of the total estimator. The Chebychev polynomial used for the domain deformation is plotted in bold line for comparison.*

- Estimate the error $u - U$ via $u - Pu$ and $Pu - U$.

- **A posteriori error analysis**
  - Find $U_{\mathrm{R}} \in \mathcal{V}_{q+1}$, the reconstruction of U, such that $U_{\mathrm{R}} - U$ is of optimal order of accuracy in $L^\infty(L^2)$ and $L^\infty(H_0^1)$.
  - Use the stability of the continuous PDE (1.4) to bound $u - U_{\mathrm{R}}$ a posteriori, because $u - U_{\mathrm{R}} \in C(H_0^1)$ satisfies an error equation at the continuous level.
  - Estimate the error $u - U$ via $U_{\mathrm{R}} - U$ and $u - U_{\mathrm{R}}$.

A priori analysis	**A posteriori analysis**
$\mathcal{V}_q$	$C(H_0^1)$
Pu	U_{R}
$Pu - U$	$u - U_{\mathrm{R}}$

3. Practical Algorithms: Reynolds' Methods. In the previous section, we reviewed the results of [10, 9, 8] related to dG methods of any order in time within the ALE framework. These methods enjoy the same stability properties as the continuous problem (1.4) and lead to optimal order a priori and a posteriori error bounds. However, the proposed methods are not practical since to implement them we need to employ appropriate quadrature in time. This raises the following questions:

> *Does there exist a quadrature in time that when applied to the numerical scheme* (2.3) *(or equivalently to* (2.4)*) leads to similar stability properties and error bounds as those of the previous section? If so, how to construct such a quadrature?*

The key observation for quadrature is to preserve the Reynolds' identity (2.5) [10]. This is possible, provided the ALE map is a continuous piecewise polynomial in time. The associated quadrature is then called *Reynolds' quadrature.*

In the sequel, we briefly present stability results and a priori error estimates of [10, 9] for Reynolds' methods and polynomial in time ALE maps. At the end of the section, we describe how we handle cases of non-polynomial ALE maps.

3.1. Reynolds' Quadrature & Stability. We assume that the ALE map is a continuous piecewise polynomial of degree $\le q'$ in time. Since the key ingredient for the validity of the stability setimate (2.6) is Reynolds' identity (2.5), we use quadratures in time of sufficiently high order to keep (2.5) valid. We refer to such quadratures as Reynolds' quadratures chosen so that for $t \in I_n$, $0 \le n \le N-1$, the integrals

$$\int_{\Omega_t} D_t V W \, d\mathbf{x}, \qquad \int_{\Omega_t} (\nabla_{\mathbf{x}} \cdot \mathbf{w}) V W \, d\mathbf{x}, \tag{3.1}$$

appearing in (2.5) are computed exactly for $V, W \in \mathcal{V}_q(I_n)$. As shown in [10], the integrals in (3.1) are polynomials of degree

$$p := 2q + dq' - 1 \tag{3.2}$$

in time if $q' \geq 1$. If $q' = 0$, the second term in (3.1) vanishes and the first term is a polynomial degree p in time, provided $q \geq 1$, and vanishes otherwise. Taking into account these observations, we propose the following definition of Reynolds' quadratures:

Definition 3.1 (Reynolds' Quadrature [10]). *We say that a quadrature Q on $(0,1]$ with positive weights ω_j and nodes τ_j, $j = 0, 1, \ldots, r$, is a* Reynolds' quadrature *if it is exact for polynomials of degree p defined in (3.2). The corresponding quadrature in $I_n = (t_n, t_{n+1}]$, $0 \leq n \leq N-1$, is denoted by Q_n and the corresponding weights $\{\omega_{n,j}\}_{j=0}^r$ and quadrature points $\{t_{n,j}\}_{j=0}^r$ in $I_n = (t_n, t_{n+1}]$ are given by*

$$\omega_{n,j} = k_n \omega_j, \quad t_{n,j} = t_n + k_n \tau_j, \quad 0 \leq j \leq r.$$

Applying a Reynolds' quadrature to (2.5) we obtain the discrete Reynolds' identity [10, Lemma 4.2]

$$\begin{aligned} \frac{1}{2}\|V(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} - \frac{1}{2}\|V(t_n^+)\|^2_{L^2(\Omega_{t_n})} \\ = Q_n\big(\langle D_t V - \mathbf{w} \cdot \nabla_{\mathbf{x}} V, V\rangle_{\Omega_t}\big). \end{aligned} \tag{3.3}$$

For $q = 0$, (3.3) is the geometric conservation law (GCL) appearing in [21, 20, 31, 22, 7]. In that respect, (3.3) may be regarded as a generalization of the GCL to higher order dG methods $q > 0$ and test functions with non-vanishing material derivative.

In addition, applying a Reynolds' quadrature to the non-conservative dG formulation (2.3), we get

$$\begin{aligned} & Q_n\big(\langle D_t U, V\rangle_{\Omega_t}\big) + \langle U(t_n^+) - U(t_n), V(t_n^+)\rangle_{\Omega_{t_n}} \\ & + Q_n\big(\langle(\mathbf{b} - \mathbf{w}) \cdot \nabla_{\mathbf{x}} U, V\rangle_{\Omega_t}\big) + \mu Q_n\big(\langle \nabla_{\mathbf{x}} U, \nabla_{\mathbf{x}} V\rangle_{\Omega_t}\big) \\ & = Q_n\big(\langle f, V\rangle_{\Omega_t}\big), \quad \forall V \in \mathcal{V}_q(I_n), \end{aligned} \tag{3.4}$$

whereas the conservative dG formulation (2.4) can be written as

$$\begin{aligned} \langle U(t_{n+1}), V(t_{n+1})\rangle_{\Omega_{t_{n+1}}} - \langle U(t_n), V(t_n^+)\rangle_{\Omega_{t_n}} \\ + Q_n\big(\langle \nabla_{\mathbf{x}} \cdot \big((\mathbf{b} - \mathbf{w})U\big), V\rangle_{\Omega_t}\big) + \mu Q_n(\langle \nabla_{\mathbf{x}} U, \nabla_{\mathbf{x}} V\rangle_{\Omega_t}) \\ - Q_n(\langle U, D_t V\rangle_{\Omega_t}) = Q_n(\langle f, V\rangle_{\Omega_t}), \quad \forall V \in \mathcal{V}_q(I_n). \end{aligned} \tag{3.5}$$

If Q_n is the Reynolds' quadrature, then we can prove that the non-conservative and conservative formulations (3.4) and (3.5) are equivalent. Moreover, for $q = 0$ and the mid-point integration rule, (3.5) reduces to the unconditionally stable backward Euler method proposed by Formaggia & Nobile in [20]. We refer to [10] for a detail discussion of these topics, as well as the proof of the next stability result (compare estimate (3.6) below with estimate (2.6)).

THEOREM 3.1 (nodal stability with Reynolds' quadrature [10]). *Let $f \in C(H^{-1}; \mathcal{Q}_T) \cap L^2(\mathcal{Q}_T)$ and the ALE map $\mathcal{A}_t$ be a continuous piecewise polynomial in time of degree q'. Let $U \in \mathcal{V}_q$ be the solution of problem* (3.4) *or* (3.5)*, together with* (2.2)*, using a Reynolds' quadrature Q_n over I_n. If $0 \le m < n \le N$, then*

$$\begin{aligned}
&\|U(t_n)\|^2_{L^2(\Omega_{t_n})} + \sum_{j=m}^{n-1} \|U(t_j^+) - U(t_j)\|^2_{L^2(\Omega_{t_j})} \\
&\qquad + \mu \sum_{j=m}^{n-1} Q_j(\|\nabla_{\mathbf{x}} U(t)\|^2_{L^2(\Omega_t)}) \\
&\qquad \le \|U(t_m)\|^2_{L^2(\Omega_{t_m})} + \frac{1}{\mu} \sum_{j=m}^{n-1} Q_j(\|f(t)\|^2_{H^{-1}(\Omega_t)}).
\end{aligned} \tag{3.6}$$

Theorem 3.1 provides an unconditional stability estimate of the discrete $L^2(H^1)-$norm. A similar estimate for the continuous $L^2(H^1)-$norm can be obtained using the equivalence of the discrete and continuous norms in conjunction with (3.6) [10, Lemma 4.3, Theorem 4.2]. The stability estimate in the continuous energy norm is needed for the derivation of optimal order a priori error bounds for the numerical scheme (3.4) or (3.5).

Finally, following similar arguments as for the proof of Theorem 2.2 and accounting for the quadrature error for the terms that are not integrated exactly in (3.4) or (3.5), it is possible to derive a stability estimate in the whole time interval I_n [10, Theorem 4.3].

THEOREM 3.2 (global stability with Reynolds' quadrature [10]). *Let $f \in C(L^2)$ and the ALE map $\mathcal{A}_t$ be a continuous piecewise polynomial of degree q'. Then the solution $U \in \mathcal{V}_q$ of either* (3.4) *or* (3.5)*, together with* (2.2)*, satisfies for $1 \le n \le N$, the following stability result*

$$\begin{aligned}
\sup_{t\in[0,t_n]} \|U(t)\|^2_{L^2(\Omega_t)} &\lesssim \max_{0\le j\le n-1} \{A_j(1 + k_j F_j)\} \times \\
&\quad \Big(\|U(0)\|^2_{L^2(\Omega_0)} + \frac{1}{\mu} \sum_{j=0}^{n-1} Q_j(\|f(t)\|^2_{H^{-1}(\Omega_t)})\Big) \\
&\quad + \max_{0\le j\le n-1} k_j A_j Q_j \Big(\|f(t)\|^2_{L^2(\Omega_t)}\Big),
\end{aligned} \tag{3.7}$$

where F_j is defined in (2.8).

Note that estimate (3.7) is the discrete analogue (in terms of quadrature) of estimate (2.7).

3.2. Error Analysis for Polynomial ALE maps. Since the stability estimate (3.6) is valid for a Reynolds' quadrature, we expect to be able to prove optimal order a priori error estimates with the aid of the ALE projection Pu (see Definition 2.1), provided that the ALE map is a continuous piecewise polynomial of degree q' in time.

Indeed, in this case, the error $\Theta = Pu - U$ satisfies the equation

$$\begin{aligned} &Q_n(\langle D_t\Theta, V\rangle_{\Omega_t}) + \langle \Theta(t_n^+) - \Theta(t_n), V(t_n^+)\rangle_{\Omega_{t_n}} \\ &+ Q_n(\langle (\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}}\Theta, V\rangle_{\Omega_t}) + \mu Q_n(\langle \nabla_{\mathbf{x}}\Theta, \nabla_{\mathbf{x}}V\rangle_{\Omega_t}) \\ &= Q_n(\langle (\mathbf{b}-\mathbf{w})\rho, \nabla_{\mathbf{x}}V\rangle_{\Omega_t}) - \mu Q_n(\langle \nabla_{\mathbf{x}}\rho, \nabla_{\mathbf{x}}V\rangle_{\Omega_t}) \\ &+ E_n(\langle (\mathbf{b}-\mathbf{w})u, \nabla_{\mathbf{x}}V\rangle_{\Omega_t}) - \mu E_n(\langle \nabla_{\mathbf{x}}u, \nabla_{\mathbf{x}}V\rangle_{\Omega_t}) + E_n(\langle f, V\rangle_{\Omega_t}), \end{aligned} \tag{3.8}$$

for all $V \in \mathcal{V}_q(I_n)$ and where $E_n(\cdot) := \int_{I_n} \cdot - Q_n(\cdot)$ denotes the quadrature error over I_n. Equation (3.8) is similar to the error equation (2.21), except that $\int_{I_n}$ is replaced by Q_n and the last three terms on the right-hand side reflect the effect of quadrature.

The quadrature error that appears on the right-hand side of (3.8) has to be of order $q+1$ (the order of the dG method). To achieve this, we need Reynolds' quadrature satisfying

$$q \le r \le p - q. \tag{3.9}$$

We point out that Reynolds' quadrature satisfying (3.9) do exist. For example, we could use $r+1$ Radau or Gauss quadrature points with $r = q + [\frac{dq'}{2}]$, where $[\cdot]$ denotes the integer part [9, Section 2].

We next state a result analogous to Theorem 2.3, corresponding to Reynolds' quadratures and piecewise polynomial ALE maps.

THEOREM 3.3 (a priori error estimate with Reynolds' quadrature [9]). *Let the ALE map $\mathcal{A}_t$ be a piecewise polynomial in time of degree q' and satisfy* (2.15). *Let U be the solution of* (2.2)-(3.4) *with Q_n a Reynolds' quadrature satisfying* (3.9). *Then the following a priori error estimate holds*

$$\begin{aligned} &\max_{0\le n\le N} \|(u-U)(t_n)\|^2_{L^2(\Omega_{t_n})} \\ &\qquad + \frac{\mu}{2}\sum_{n=0}^{N-1} Q_n(\|\nabla_{\mathbf{x}}(u-U)(t)\|^2_{\mathbf{L}^2(\Omega_t)}) \le \sum_{i=1}^{5}\mathcal{E}_i^2, \end{aligned} \tag{3.10}$$

with

$$\mathcal{E}_1^2 := \frac{1}{\mu}\sum_{n=0}^{N-1} C_n k_n^{2q+2} \sup_{t\in I_n} \|(\mathbf{b}-\mathbf{w})(t)\|^2_{\mathbf{L}^\infty(\Omega_t)} \int_{I_n} \|D_t^{q+1}u(t)\|^2_{L^2(\Omega_t)}\,dt$$

$$\mathcal{E}_2^2 := \frac{\mu}{2}\sum_{n=0}^{N-1} D_n k_n^{2q+2} \int_{I_n} \Big(\|D_t^{q+1}u(t)\|^2_{L^2(\Omega_t)} + \|\nabla_{\mathbf{x}} D_t^{q+1}u(t)\|^2_{\mathbf{L}^2(\Omega_t)} \Big)\, dt$$

$$\mathcal{E}_3^2 := \frac{1}{\mu}\sum_{n=0}^{N-1} G_{n,q+1} k_n^{2q+2} \sum_{i=0}^{q+1} \int_{I_n} \|D_t^q \Big((\mathbf{b}-\mathbf{w})u \Big)(t)\|^2_{\mathbf{L}^2(\Omega_t)}\, dt$$

$$\mathcal{E}_4^2 := \mu \sum_{n=0}^{N-1} G_{n,j+1} k_n^{2j+2} \sum_{i=0}^{j+1} \int_{I_n} \|\nabla_{\mathbf{x}} D_t^i u(t)\|^2_{\mathbf{L}^2(\Omega_t)}\, dt$$

$$\mathcal{E}_5^2 := \frac{1}{\mu}\sum_{n=0}^{N-1} G_{n,q+1} k_n^{2q+2} \sum_{i=0}^{q+1} \int_{I_n} \|D_t^i f(t)\|^2_{H^{-1}(\Omega_t)}\, dt,$$

and constants C_n, D_n, $0 \le n \le N-1$, *proportional to those in* (2.18) *and* (2.19)*, respectively, and*

$$G_{n,q+1} := A_n B_{n,q+1}, \qquad B_{n,q+1} := \|\nabla_{\mathbf{y}} \mathcal{A}_{t_n \to t}\|_{W^{q+1}_\infty(I_n; L^\infty(\Omega_{t_n}))}. \tag{3.11}$$

Note that estimate (3.10) holds for any choice of the time-steps k_n (unconditional a priori error bound). Also, as discussed in the previous subsection, it can be proven that the continuous and discrete energy norms are equivalent. Thus, an a priori error estimate similar to (3.10) can be derived for the continuous energy norm as well.

The derivation of optimal order *a posteriori error estimates* is a very interesting and non-trivial question. Despite the fact that the definition (2.22) and (2.23) of the reconstruction remains the same for Reynolds' methods, the analysis is not a direct generalization of the dG methods with exact integration in time. Since the main error analysis is performed at the continuous level instead of the discrete, several projections must be used to handle the terms that are not integrated exactly in (3.4). It turns out that they are not pure time-projections, as it happens for the ALE projection and the reconstruction, which adds additional technical difficulties and makes the analysis tedious. It is however possible to prove rigorously optimal order a posteriori error bounds for dG methods with Reynolds' quadrature.

3.3. Non-Polynomial ALE maps. The previous subsection was devoted to stability and error analysis for practical Reynolds' algorithms for problem (1.4) written on the ALE framework, but under the assumption that the ALE map is continuous piecewise polynomial in time. Since this is not always realistic, the following question arises:

> *How to apply the previous analysis to non-polynomial in time ALE maps?*

The answer to this question is briefly discussed below and details are given in [9, Section 5].

We approximate the domain velocity $\widehat{\mathbf{w}}$ in the ALE frame by $\widehat{\mathbf{W}}$ using a piecewise polynomial in time of order q, i.e., $\widehat{\mathbf{W}} \in \widehat{\mathcal{V}}_q(I_n)$. This creates a new family $\{\tilde{\mathcal{A}}_t\}_{t\in[0,T]}$ of ALE maps that corresponds to $\widehat{\mathbf{W}}$ and defined by $\tilde{\mathcal{A}}_0 = I_d$ and for $0 \le n \le N-1$,

$$\tilde{\mathcal{A}}_t(\mathbf{y}) = \tilde{\mathcal{A}}_{t_n}(\mathbf{y}) + \int_{t_n}^{t} \widehat{\mathbf{W}}(\mathbf{y}, s)\, ds, \qquad \tilde{\mathbf{x}}(\mathbf{y}, t) := \tilde{\mathcal{A}}_t(\mathbf{y}). \tag{3.12}$$

For every $t \in [0, T]$, $\widehat{\mathbf{W}}$ also creates a perturbed domain $\tilde{\Omega}_t = \tilde{\mathcal{A}}_t(\Omega_0)$. Since $\widehat{\mathbf{W}}$ is a piecewise polynomial in time of degree q, the definition (3.12) of $\tilde{\mathcal{A}}_t$ implies that $\tilde{\mathcal{A}}_t$ is a continuous piecewise polynomial in time of degree $q+1$.

The idea is to define the discrete dG space and the dG method with Reynolds' quadrature with solution $\widetilde{U}$ with respect to the perturbed domain $\tilde{\Omega}_t$ (and the perturbed ALE map $\tilde{\mathcal{A}}_t$). Since the solution u of (1.4) is defined in Ω_t and $\widetilde{U}$ in $\tilde{\Omega}_t$, which are different but close domains, the key point of the analysis is to write u with respect to $\tilde{\mathcal{A}}_t$ and denote it $\tilde{u}$. Then, it is possible to prove, using a perturbation argument, that $\tilde{u}$ satisfies an equation of the form (1.4) with a defect of optimal order of accuracy. The defect includes geometric quantities, due to the approximation of Ω_t by $\tilde{\Omega}_t$. Enforcing the geometrical defect to be of optimal order of accuracy in time, entails approximating $\widehat{\mathbf{w}}$ by a piecewise polynomial of degree q. Finally, proceeding as in the previous subsection, an a priori error estimate similar to (3.10) is derived in [9]. As expected, the upper bound contains additional error terms due to the domain approximation. The computational rates of convergence depicted in Figure 4 corroborate theory.

4. Practical Algortithms: Runge-Kutta-Radau Methods.

In the previous section, we used Reynolds' quadrature to approximate the integrals in time, appearing in dG method (2.3) (or (2.4)). Using such quadratures leads to unconditional stability and a priori error bounds. However, Reynolds' quadratures are dimensional dependent and become computationally more intensive for higher dimensions. Indeed, a Reynolds' quadrature integrates exactly polynomials of degree $p = 2q + dq' - 1$, where q' is the degree of the polynomial ALE map (see (3.2)).

4.1. Runge-Kutta-Radau (RKR) Methods.

We now review the results of [10, 9] related to RKR methods, which in the ALE framework can be obtained from (2.3) by applying the Radau quadrature with $q+1$ nodes. Such a quadrature is enough for dG methods to be unconditionally stable when the domain is not moving and give optimal order a priori error estimates [37, Chapter 12]. Notice that $q+1$ Radau nodes integrate exactly polynomials of degree $\le 2q$, which compares favorably with p in (3.2) for $q' \ge 1$. Moreover, if the domain does not move ($q' = 0$), then Reynolds' quadrature is exact for polynomials of degree $p = 2q - 1$ which can be realized with $q+1$ Radau nodes.

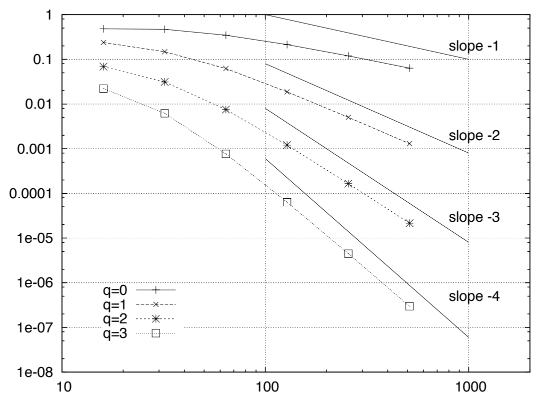

FIG. 4. *Error in the $\ell^\infty(L^2)-$norm against the number N of uniform timesteps is depicted for $q = 0, 1, 2, 3$. The reference domain is $\Omega_0 := (-1, 1) \times (-1, 1)$ and it is deformed according to the ALE map $\mathcal{A}_t(\mathbf{y}) := \mathbf{y}(1 + \frac{1}{2}\ T_{11}(t))$ for $t \in (0, 0.99)$, where $T_n(t) := \cos(n \arccos(t))$ is the nth Chebychev polynomial of first kind. We set $\mu = 1.0$ and the exact solution is manufactured to be $u(\mathbf{x}, t) := \exp(x_1\ t) \sin(x_2\ t)$ with $\mathbf{x} := (x_1, x_2) \in \Omega_t$. We take $\widehat{\mathbf{W}}$ to be the L^2-projection of the ALE velocity $\widehat{\mathbf{w}}$ onto $\widehat{\mathcal{V}}_q$ and compute $\tilde{A}_t$ according to (3.12). Over each interval I_n, a Reynolds' quadrature based on $q+1+[d(q+1))/2]$ Radau points is used, cf. (3.9). The discretization in space is chosen sufficiently fine not to influence the time discretization error. All schemes exhibit the optimal $\mathcal{O}(N^{-(q+1)})$ order of convergence.*

As discussed in [10], the use of the Radau quadrature with $q+1$ nodes leads to stable practical numerical methods, subject to a *mild constraint on the time-steps depending on the ALE map (conditional stability).* In [9], we were also able to prove that RKR methods on the ALE framework lead to optimal order a priori error bounds, but under the same time-step restriction as for the stability. The reason for this time-step constraint is the violation of Reynolds' identity when using $q + 1$ Radau quadrature points for the approximation of the integrals in (2.5). However, it is to be emphasized that the time-step restriction is *not a CFL condition*, as the considered numerical schemes are only discrete in time, i.e., the space is continuous. Despite the fact that RKR methods in the ALE framework lead to conditional stability, these methods have some advantages in comparison with Reynolds' methods. Inevitably, a natural question arises:

> *Which family of methods is more appropriate for problems defined on time dependent domains and the ALE framework: Reynolds' or RKR methods?'*

Reynolds' methods are more appropriate when dealing with highly oscillatory ALE maps, because they lead to ALE-free stable schemes. On the contrary, the minimal complexity of RKR methods makes them more appropriate in cases of non-oscillatory maps, when the time-step requirement is less restrictive and in practice unnoticeable.

We continue now with a brief description, without proofs, of the main results of [10, 9] regarding the analysis of RKR methods in the ALE framework. Our analysis is, in some sense, related to the one by Badia & Codina, [5], who proposed first and second order accurate BDF schemes in the ALE framework for time dependent domains. Their schemes do not satisfy the GCL and are stable and optimally accurate under a time-step constraint similar to ours.

Let ω_j and τ_j, $0 \le j \le q$, be the weights and nodes, respectively, for the Radau quadrature rule Q^q in $(0,1]$ and let $\{\omega_{n,j}\}_{j=0}^q$ and $\{\tau_{n,j}\}_{j=0}^q$ be those for I_n, $0 \le n \le N-1$. Using such a quadrature in (2.3), say Q_n^q, the RKR method in the non-conservative ALE framework reads as:

$$\begin{aligned} &Q_n^q(\langle D_t U, V\rangle_{\Omega_t}) + \langle U(t_n^+) - U(t_n), V(t_n^+)\rangle_{\Omega_{t_n}} \\ &\qquad + Q_n^q(\langle(\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}} U, V\rangle_{\Omega_t}) + \mu Q_n^q(\langle \nabla_{\mathbf{x}} U, \nabla_{\mathbf{x}} V\rangle_{\Omega_t}) \\ &\qquad = Q_n^q(\langle f, V\rangle_{\Omega_t}), \quad \forall\, V \in \mathcal{V}_q(I_n), \end{aligned} \tag{4.1}$$

with $U(\cdot,0) = u_0$ in Ω_0. We point out that for RKR methods, conservative and non-conservative formulations are no longer equivalent. This is because, in contrast to Reynolds' quadrature Q_n, Radau quadrature Q_n^q does not integrate exactly the terms appearing in Reynolds' identity (2.5). Nevertheless, similar stability results and a priori error bounds are valid for both RKR methods.

To compensate for the extra variational crime, introduced due to the violation of Reynolds' identity, we need to impose an extra local-time regularity condition on the family of the ALE maps $\{\mathcal{A}_t\}_{t\in[0,T]}$:

$$B_{n,2} := \|D\mathcal{A}_{t_n\to t}\|_{\mathbf{W}^2_\infty(I_n;\mathbf{L}^\infty(\Omega_{t_n}))} < \infty. \tag{4.2}$$

Using similar arguments as in the proof of Theorem 3.1 and the Bramble-Hilbert Theorem to bound the quadrature error of the terms appearing in Reynolds' identity, it is possible to prove the following theorem:

Theorem 4.1 (conditional nodal stability for RKR [10]). *Let $f \in C(H^{-1};\mathcal{Q}_T) \cap L^2(\mathcal{Q}_T)$ and (4.2) be valid. If*

$$A_n(1+B_{n,2})k_n \lesssim \mu, \quad \forall\, 0 \le n < N, \tag{4.3}$$

then the solution $U \in \mathcal{V}_q$ of problem (2.2) and (4.1) satisfies, for $0 \le m < n \le N$,

$$\begin{aligned} &\|U(t_n)\|^2_{L^2(\Omega_{t_n})} + \sum_{j=m}^{n-1} \|U(t_j^+) - U(t_j)\|^2_{L^2(\Omega_{t_j})} \\ &\qquad + \mu \sum_{j=m}^{n-1} Q_j(\|\nabla_{\mathbf{x}} U(t)\|^2_{\Omega_{t_j}}) \\ &\qquad \le \|U(t_m)\|^2_{L^2(\Omega_{t_m})} + \frac{2}{\mu}\sum_{j=m}^{n-1} Q_j(\|f(t)\|^2_{H^{-1}(\Omega_t)}). \end{aligned} \tag{4.4}$$

Figure 5 documents the behavior of $\|U(t_n)\|_{L^2(\Omega_{t_n})}$ for RKR methods of order $0 \leq q \leq 3$ and the same oscillatory case of [21], already displayed in Figure 1.

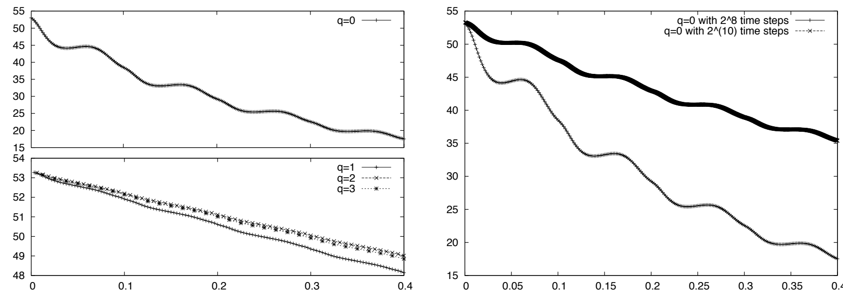

FIG. 5. *Evolution of* $\|U(t_n)\|_{L^2(\Omega_{t_n})}$ *for* $q = 0$ *with* 2^8 *uniform time steps (top-left), for* $q = 1, 2, 3$ *with* 2^7, 2^6, 2^5 *uniform time steps respectively (bottom-left), and for* $q = 0$ *with* 2^8 *(upper curve) and* 2^{10} *(lower curve) uniform time steps (right). The space discretization is fine enough not to influence the time discretization. The reference domain is* $\Omega_0 := (0,1) \times (0,1)$*, the time interval is* $[0, 0.4]$*, the diffusivity is* $\mu = 0.01$*, the domain velocity* $\mathbf{w}$ *is the* L^2-*projection over piecewise polynomials of degree* q *of the time derivative of the map* $(\mathbf{y}, t) \mapsto \mathbf{y}(2 - \cos(20\pi t))$*, with* $\mathbf{y} \in \Omega_0$*,* $t \in (0, 0.4)$*, and the forcing is* $f = 0$*. The ALE map* $\mathcal{A}_t$ *is obtained by integrating* $\mathbf{w}$ *in each time interval* I_n*, enforcing continuity at the nodes. Monotonicity of* $\|U(t_n)\|_{L^2(\Omega_{t_n})}$ *for* $q = 0$ *is sensitive to the time step size (conditional stability), a property of RKR methods proved in Theorem 4.1 for all* $q \geq 0$ *(see right). Stability of higher order RKR methods (*$q > 0$*) is less sensitive to the time steps (bottom-left).*

The numerical experiments depicted in Figure 5 illustrate that for $f \equiv 0$, the monotonicity of $\|U(t_n)\|_{L^2(\Omega_{t_n})}$ is retained for $q = 0$ provided that the time-step constraint (4.3) is enforced. However, in this particular example, it seems that the case $q > 0$ does not require any time step constraints to exhibit the monotone behavior.

Global stability in the whole time-interval I_n is also valid for RKR methods, as it happens for dG methods with exact integration and Reynolds' quadrature, provided that the time-steps are chosen so that (4.3) is satisfied.

Regarding the a priori error analysis, the error $\Theta = Pu - U$ satisfies an equation similar to (3.8) with the additional quadrature error terms $E_n(\langle D_t Pu, V\rangle_{\Omega_t})$ and $E_n(\nabla_{\mathbf{x}} \cdot \mathbf{w} Pu, V\rangle_{\Omega_t})$. These terms arise because RKR methods violate Reynolds' identity. Estimating these quadrature errors leads to derivatives $D_t^i Pu$ for $0 \leq i \leq q + 1$, which are handled via the stability bounds in (2.20) for the ALE projection Pu. Thus, proceeding as in the proof of Theorem 3.3, and using the stability estimate (4.4), we managed in [9] to prove the following:

THEOREM 4.2 (a priori error estimate for RKR methods [9]). *Let $U \in \mathcal{V}_q$ be the solution of* (4.1) *with $q+1$ Radau quadrature points and let $\mathcal{A}_t$ satisfy* (4.2). *If the time-steps satisfy* (4.3), *then the following error estimate for $u-U$ is valid:*

$$\max_{0\le n\le N} \|(u-U)(t_n)\|^2_{L^2(\Omega_{t_n})} + \frac{\mu}{2}\sum_{n=0}^{N-1} Q_n\big(\|\nabla_{\mathbf{x}}(u-U)(t)\|^2_{\mathbf{L}^2(\Omega_t)}\big)$$

$$\le \mathcal{E}(u,f,\mathcal{A}_t,\mathbf{b}) + \frac{1}{\mu}\sum_{n=0}^{N-1} C_n G_{n,q+1} k_n^{2q+2} \times \tag{4.5}$$

$$\sum_{i=0}^{q+1}\int_{I_n}\Big(1+\|D_t^{q+1-i}\nabla_{\mathbf{x}}\mathbf{w}(t)\|^2_{\mathbf{L}^\infty(\Omega_t)}\Big)\|D_t^i u(t)\|^2_{L^2(\Omega_t)}\,dt$$

where $\mathcal{E}(u,f,\mathcal{A}_t,\mathbf{b})$ denotes the right-hand side of (3.10) *(with proportionality constants), Pu is the ALE projection defined in* (2.10)-(2.11) *and $C_n, G_{n,q+1}$ are as in* (2.18) *and* (3.11), *respectively.*

Figure 6 displays an optimal rate of convergence $q+1$ for RKR methods with $0 \le q \le 3$ and the same experiment as in Figure 4, for Reynolds' quadrature.

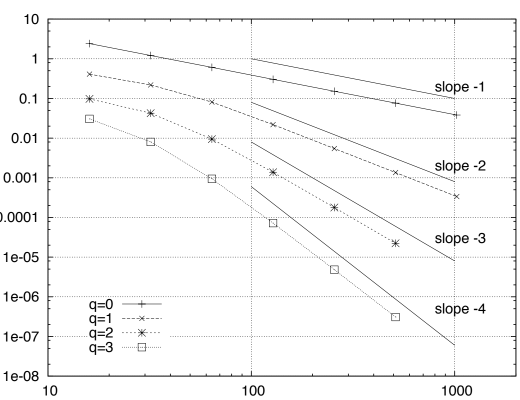

FIG. 6. *The error $\max_{0\le n\le N}\|(u-U)(t_n)\|^2_{L^2(\Omega_{t_n})}$ versus the number of uniform timesteps N is depicted for $q=0,1,2,3$. The reference domain is $\Omega_0 := (-1,1)\times(-1,1)$ and it is deformed according to the ALE map $\mathcal{A}_t(\mathbf{y}) := \mathbf{y}(1+\frac{1}{2}\,T_{11}(t))$ for $t\in(0,0.99)$, where $T_n(t) := \cos(n\arccos(t))$ is the nth Chebychev polynomial of first kind. We set $\mu = 1.0$ and the exact solution is manufactured to be $u(\mathbf{x},t) := \exp(x_1\,t)\sin(x_2\,t)$ with $\mathbf{x} := (x_1,x_2)\in\Omega_t$. We take $\widehat{\mathbf{W}}$ to be the L^2-projection of the ALE velocity $\widehat{\mathbf{w}}$ onto $\widehat{\mathcal{V}}_q$ and compute $\tilde{\mathcal{A}}_t$ according to* (3.12). *Over each interval I_n, a quadrature based on $q+1$ Radau points is used. The discretization in space is chosen not to influence the time discretization error. As predicted by estimate* (4.5), *all schemes exhibit the optimal $\mathcal{O}(N^{-(q+1)})$ order of convergence.*

4.2. Implicit-Explicit Runge-Kutta (IERK) Method. Finally, we mention that similar stability and a priori error estimates can be established for the IERK method of first order:

$$\begin{aligned} &\langle (U_{n+1} - U_n) + k_n\big((\mathbf{b} - \mathbf{w})(t_n) \cdot \nabla_{\mathbf{x}} U_{n+1}\big), V\rangle_{\Omega_{t_n}} \\ &+ \mu k_n \langle \nabla_{\mathbf{x}} U_{n+1}, \nabla_{\mathbf{x}} V\rangle_{\Omega_{t_n}} = k_n \langle f(t_n), V\rangle_{\Omega_{t_n}}, \quad \forall V \in \mathcal{V}_0(I_n), \end{aligned} \tag{4.6}$$

where $U_{n+1} := U(\mathcal{A}_{t_n \to t_{n+1}}(\cdot), t_{n+1})$ for $U \in \mathcal{V}_0(I_n)$. Method (4.6) can be obtained by approximation of the integrals in (2.3) with the left-side rectangle quadrature. Despite the fact that (4.6) is not a RKR method, the error analysis for RKR methods in the ALE framework is applicable to (4.6) as well [10]. This method is natural for free-boundary problems, [6, 11, 12], because it is implicit with respect to the approximation U, but explicit with respect to the moving domain. The latter is beneficial whenever we do not know in advance $\Omega_{t_{n+1}}$ at step n, while the implicit nature of the method in U helps avoiding any CFL condition. Rigorous error analysis for the IERK method (4.6) appears for the first time in [10].

We conclude with an application of IERK to *fluid-membrane interaction* due to A. Bonito, R.H. Nochetto, and M.S. Pauletti [12]. The deformable domain Ω_t contains an incompressible fluid governed by the Navier-Stokes equation

$$\begin{aligned} \rho D_t \mathbf{v} - \operatorname{div}\Sigma(\mathbf{v}, p) &= 0 \qquad \text{in } \Omega_t, \\ \operatorname{div} \mathbf{v} &= 0 \qquad \text{in } \Omega_t, \end{aligned} \tag{4.7}$$

where $\mathbf{v}$ is the fluid velocity, p is its pressure, $\Sigma(\mathbf{v}, p) = -pI + \mu D(\mathbf{v})$ is the Cauchy stress tensor, $D(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$ is the symmetric part of the gradient, and μ is the viscosity. The membrane $\partial\Omega_t$ is governed by the *Canham-Helfrich energy*

$$J(\partial\Omega_t) = \frac{1}{2} \int_{\partial\Omega_t} H^2 + \lambda \left(\int_{\partial\Omega_t} 1 - \int_{\partial\Omega_0} 1 \right). \tag{4.8}$$

where H stands for the *mean curvature* of $\partial\Omega_t$ and λ is the Lagrange multiplier enforcing area conservation. The fluid and membrane interact through the boundary condition, which represents a balance of forces at the interface $\partial\Omega_t$:

$$\Sigma \boldsymbol{\nu} = \kappa\, \delta J(\partial\Omega_t), \tag{4.9}$$

where κ is the membrane bending rigidity coefficient and $\delta J(\partial\Omega_t)$ is the variational (or shape) derivative. The latter obeys the expression

$$\delta J(\partial\Omega_t) = \Big(\Delta_{\partial\Omega_t} H + \frac{1}{2} H^3 - 2KH + \lambda H \boldsymbol{\nu} \Big) \boldsymbol{\nu}, \tag{4.10}$$

where $\Delta_{\partial\Omega_t}$ is the Laplace-Beltrami operator on $\partial\Omega_t$ and K is the Gaussian curvature of $\partial\Omega_t$. It is important to notice that $\delta J(\partial\Omega_t)$ is a vector field perpendicular to $\partial\Omega_t$ because $\boldsymbol{\nu}$ is the unit normal to $\partial\Omega_t$. The discretization of (4.7)–(4.10) consists of Taylor-Hood finite elements coupled with IERK. Figure 7 displays the complex behavior of the fluid membrane and quite noticeable inertial effects.

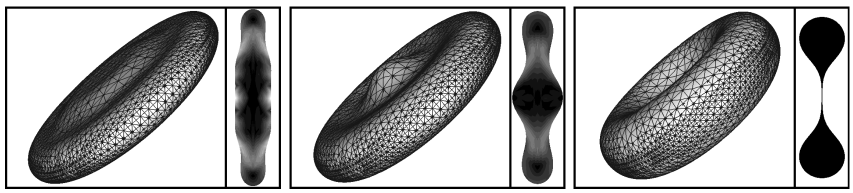

FIG. 7. *Evolution of a fluid membrane with initial axisymmetric ellipsoidal shape of aspect ratio* $5\times5\times1$ *and final shape similar to a red blood cell. Each frame shows the membrane mesh and a symmetry cut along a big axis. The fluid flow is quite complex, creating first a bump in the middle and next moving towards the circumference and producing a depresion in the center with flat pinching profile. The inertial effects are due to unrealistic physical parameters.*

REFERENCES

[1] G. Akrivis and Ch. Makridakis. Galerkin time-stepping methods for nonlinear parabolic equations. *M2AN Math. Model. Numer. Anal.*, 38(2):261–289, 2004.

[2] G. Akrivis, Ch. Makridakis, and R.H. Nochetto. A posteriori error estimates for the Crank-Nicolson method for parabolic equations. *Math. Comp.*, 75(254):511–531, 2006.

[3] G. Akrivis, Ch. Makridakis, and R.H. Nochetto. Optimal order a posteriori error estimates for a class of Runge-Kutta and Galerkin methods. *Numer. Math.*, 114(1):133–160, 2009.

[4] G. Akrivis, Ch. Makridakis, and R.H. Nochetto. Galerkin and Runge-Kutta methods: unified formulation, a posteriori error estimates and nodal superconvergence. *Numer. Math.*, 118(3):429–456, 2011.

[5] S. Badia and R. Codina. Analysis of a stabilized finite element approximation of the transient convection-diffusion equation using an ALE framework. *SIAM J. Numer. Anal.*, 44(5):2159–2197 (electronic), 2006.

[6] E. Bänsch. Finite element discretization of the Navier-Stokes equations with a free capillary surface. *Numer. Math.*, 88(2):203–235, 2001.

[7] D. Boffi and L. Gastaldi. Stability and geometric conservation laws for ALE formulations. *Comput. Methods Appl. Mech. Engrg.*, 193(42-44):4717–4739, 2004.

[8] A. Bonito, I. Kyza, and R.H. Nochetto. Time-discrete higher order ALE formulations: A posteriori error analysis. In preparation.

[9] A. Bonito, I. Kyza, and R.H. Nochetto. Time-discrete higher order ALE formulations: A priori error analysis. To appear in Numer. Math.

[10] A. Bonito, I. Kyza, and R.H. Nochetto. Time-discrete higher order ALE formulations: Stability. To appear in SIAM J. Numer. Anal.

[11] A. Bonito, R.H. Nochetto, and M.S. Pauletti. Parametric FEM for geometric biomembranes. *J. Comput. Phys.*, 229(9):3171–3188, 2010.

[12] A. Bonito, R.H. Nochetto, and M.S. Pauletti. Dynamics of biomembranes: Effect of the bulk fluid. *Math. Model. Nat. Phenom.*, 6(5):25–43, 2011.

[13] S.C. Brenner and L.R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[14] S Brogniez, A Rajasekharan, and Ch. Farhat. Provably stable and time-accurate extensions of runge-kutta schemes for cfd computations on moving grids. *Internat. J. Numer. Methods Fluids*, 69(7):1249–1270, 2012.

[15] K. Chrysafinos and N.J. Walkington. Lagrangian and moving mesh methods for the convection diffusion equation. *M2AN Math. Model. Numer. Anal.*, 42(1):25–55, 2008.

[16] J. Donéa, S. Giuliani, and J.P. Halleux. An arbitrary lagrangian-eulerian finite element method for transient dynamic fluid-structure interactions. *Comput. Methods Appl. Mech. Engrg.*, 33(1-3):689 – 723, 1982.

[17] Ch. Farhat, P. Geuzaine, and C. Grandmont. The discrete geometric conservation law and the nonlinear stability of ALE schemes for the solution of flow problems on moving grids. *J. Comput. Phys.*, 174(2):669–694, 2001.

[18] Ch. Farhat and Ph. Geuzaine. Design and analysis of robust ALE time-integrators for the solution of unsteady flow problems on moving grids. *Internat. J. Numer. Methods Fluids*, 193(39–41):4073–4095, 2004.

[19] Ch. Farhat, M. Lesoinne, and N. Maman. Mixed explicit/implicit time integration of coupled aeroelastic problems: three-field formulation, geometric conservation and distributed solution. *Internat. J. Numer. Methods Fluids*, 21(10):807–835, 1995. Finite element methods in large-scale computational fluid dynamics (Tokyo, 1994).

[20] L. Formaggia and F. Nobile. A stability analysis for the arbitrary Lagrangian Eulerian formulation with finite elements. *East-West J. Numer. Math.*, 7(2):105–131, 1999.

[21] L. Formaggia and F. Nobile. Stability analysis of second-order time accurate schemes for ALE-FEM. *Comput. Methods Appl. Mech. Engrg.*, 193(39-41):4097–4116, 2004.

[22] L. Gastaldi. A priori error estimates for the arbitrary Lagrangian Eulerian formulation with finite elements. *East-West J. Numer. Math.*, 9(2):123–156, 2001.

[23] P. Geuzaine, C. Grandmont, and Ch. Farhat. Design and analysis of ale schemes with provable second-order time-accuracy for inviscid and viscous flow simulations. *Journal of Computational Physics*, 191(1):206 – 227, 2003.

[24] H. Guillard and Ch. Farhat. On the significance of the geometric conservation law for flow computations on moving meshes. *Comput. Methods Appl. Mech. Engrg.*, 190(11-12):1467–1482, 2000.

[25] P. Hansbo, J. Hermansson, and T. Svedberg. Nitsche's method combined with space-time finite elements for ALE fluid-structure interaction problems. *Comput. Methods Appl. Mech. Engrg.*, 193(39-41):4195–4206, 2004.

[26] C.W. Hirt, A.A. Amsden, and J.L. Cook. An arbitrary Lagrangian-Eulerian computing method for all flow speeds [J. Comput. Phys. **14** (1974), no. 3, 227–253]. *J. Comput. Phys.*, 135(2):198–216, 1997. With an introduction by L. G. Margolin, Commemoration of the 30th anniversary {of J. Comput. Phys.}.

[27] T.J.R. Hughes, W.K. Liu, and T.K. Zimmermann. Lagrangian-Eulerian finite element formulation for incompressible viscous flows. *Comput. Methods Appl. Mech. Engrg.*, 29(3):329–349, 1981.

[28] P. Jamet. Galerkin-type approximations which are discontinuous in time for parabolic equations in a variable domain. *SIAM J. Numer. Anal.*, 15(5):912–928, 1978.

[29] J. A. Mackenzie and W. R. Mekwi. An unconditionally stable second-order accurate ALE-FEM scheme for two-dimensional convection-diffusion problems. *IMA J. Numer. Anal.*, 32(3):888–905, 2012.

[30] Ch. Makridakis and R.H. Nochetto. A posteriori error analysis for higher order dissipative methods for evolution problems. *Numer. Math.*, 104(4):489–514, 2006.

[31] F. Nobile. *Numerical approximation on fluid-structure interaction problems with application to haemodynamics.* PhD thesis, École Polytechnique Fédérale de Lausanne, 2001.

[32] L.A. Ortega and G. Scovazzi. A geometrically-conservative, synchronized, flux-corrected remap for arbitrary lagrangian-eulerian computations with nodal finite elements. *J. Comput. Phys.*, 230(17):6709–6741, 2011.
[33] O. Pironneau, J. Liou, and T. Tezduyar. Characteristic-Galerkin and Galerkin/least-squares space-time formulations for the advection-diffusion equations with time-dependent domains. *Comput. Methods Appl. Mech. Engrg.*, 100(1):117–141, 1992.
[34] A. Quarteroni and L. Formaggia. *Mathematical modelling and numerical simulation of the cardiovascular system*, volume XII of *Handb. Numer. Anal.* North-Holland, Amsterdam, 2004.
[35] A. Quarteroni, M. Tuveri, and A. Veneziani. Computational vascular fluid dynamics: problems, models and methods. *Comput. Visual. Sci.*, 2(4):163–197, 2000.
[36] P.D. Thomas and C.K. Lombard. The geometric conservation law–a link between finite-difference and finite–volume methods for flow computation on moving grids. *AIAA Paper 78–1208*, 1978.
[37] V. Thomée. *Galerkin finite element methods for parabolic problems*, volume 25 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, second edition, 2006.