A multigrid algorithm for an elliptic problem with a perturbed boundary condition

Andrea Bonito and Joseph E. Pasciak

Abstract We discuss the preconditioning of systems coupling elliptic operators in $\Omega \subset \mathbb{R}^d$, $d = 2,3$, with elliptic operators defined on hyper-surfaces. These systems arise naturally when physical phenomena are affected by geometric boundary forces, such as the evolution of liquid drops subject to surface tension. The resulting operators are sums of interior and boundary terms weighted by parameters. We investigate the behavior of multigrid algorithms suited to this context and demonstrate numerical results which suggest uniform preconditioning bounds that are level and parameter independent.

1 Introduction

There has been considerable interest in geometric differential equations in recent years as they play a crucial role in many applications. In this paper, we consider one aspect of developing efficient preconditioners for the systems of algebraic equations resulting from finite element approximation to these problems.

Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a bounded domain separated into two subdomains by an interface γ. We denote the subdomains by Ω_i, $i = 1,2$. This paper focuses on the study of an optimal multigrid algorithm for interactions between "bulk" perturbed elliptic operators and the surface Laplacian. Such variational problems involving interaction between diffusion operators on domains and surfaces appear in many different contexts, see, for example, [1, 2, 10, 15, 18, 20, 21].

Andrea Bonito
Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, e-mail: bonito@math.tamu.edu

Joseph E. Pasciak
Name, Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, e-mail: pasciak@math.tamu.edu

As an illustration, we now describe an application involving capillary flow [1]. We consider the evolution of two different fluids inside a domain Ω separated by a moving interface $\gamma(t)$, $t > 0$. The interface γ is described as the deformation of a smooth reference domain $\hat{\gamma}$. We denote by $\mathbf{x}(t) : \hat{\gamma} \to \gamma(t)$ the mapping relating the two interfaces. Typically, $\mathbf{x}(t)$ is bi-Lipschitz but we will require more smoothness on γ and therefore on $\mathbf{x}$. The fluids are assumed to be governed by the Stokes equations, i.e. the velocities $\mathbf{u}_i$ and the pressures p_i, $i = 1,2$, satisfy on each subdomain Ω_i:

$$\frac{\partial}{\partial t}\mathbf{u}_i - 2\mathrm{div}(D(\mathbf{u}_i)) + \nabla p_i = \mathbf{f}_i, \quad \mathrm{div}(\mathbf{u}_i) = 0, \qquad \text{on} \quad \Omega_i,$$

where $D(\mathbf{v}) := \frac{1}{2}((\nabla \mathbf{v}) + (\nabla \mathbf{v})^T)$ and $\{\mathbf{f}_i\}$ are given body forces. The surface tension effect appears together with the continuity of the velocity, i.e.,

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{u}_2, \quad \text{on} \quad \gamma, \\ (2D(\mathbf{u}_1) - p_1)\nu_1 + (2D(\mathbf{u}_2) - p_2)\nu_2 &= \alpha \Delta_\gamma \mathbf{x}, \quad \text{on} \quad \gamma, \end{aligned}$$

where ν_i are unit outward pointing normals, Δ_γ is the Laplace-Beltrami operator and $\alpha > 0$ is the surface tension coefficient. The term $\Delta_\gamma \mathbf{x}$ is the total vector curvature (sum of principal curvatures in the normal direction) [13]. In addition, the system of equations is supplemented by the interface motion relation

$$\dot{\mathbf{x}} = \mathbf{u} \quad \text{on} \quad \gamma, \tag{1}$$

where $\mathbf{u} = \mathbf{u}_1 = \mathbf{u}_2$ is the fluid velocity at the interface γ.

Following an original idea of Dziuk [11] in the context of purely geometric flows (see also [12]), Bänsch [1] proposes a first order scheme in time leading to a semi-implicit discretization of the curvature, thereby taking advantage of the stability property inherent to surface tension effects. It relies on an implicit Euler discretization of the interface motion (1)

$$\mathbf{x} \approx \mathbf{x}^{\mathrm{old}} + \tau\, \mathbf{u} \quad \text{on} \quad \gamma,$$

where τ is the time-stepping parameter. Injecting the above relation in the interface condition, we obtain an approximate interface relation

$$(D(\mathbf{u}_1) - p_1)\, \nu_1 + (D(\mathbf{u}_2) - p_2)\, \nu_2 \approx \tau \alpha \Delta_\gamma \mathbf{u} + \alpha \Delta_\gamma \mathbf{x}^{\mathrm{old}} \quad \text{on} \quad \gamma.$$

Hence, denoting by $\mathbf{u}$ the combined velocity, i.e., $\mathbf{u} = \mathbf{u}_i$ on Ω_i and by p the combined pressure, one arrives at a semi-discrete approximation in time: Given an interface position $\mathbf{x}^{\mathrm{old}} \in W^1_\infty(\hat{\gamma})$ and a previous velocity $\mathbf{u}^{\mathrm{old}} \in V^d$, seek $\mathbf{u} \in V^d$ and $p \in L_2(\Omega)$ such that for all $\mathbf{v} \in V^d$

$$\begin{aligned} \int_\Omega \mathbf{u}\cdot\mathbf{v} + 2\tau \int_\Omega D(\mathbf{u}) \cdot D(\mathbf{v}) - \tau \int_\Omega p \cdot \mathrm{div}(\mathbf{v}) + \alpha\tau^2 \int_\gamma \nabla_\gamma \mathbf{u} \cdot \nabla_\gamma \mathbf{v} \\ = \int_\Omega \mathbf{u}^{\mathrm{old}} \cdot \mathbf{v} + \tau \int_\Omega \mathbf{f} \cdot \mathbf{v} - \alpha\tau^2 \int_\gamma \nabla_\gamma \mathbf{x}^{\mathrm{old}} \cdot \nabla_\gamma \mathbf{v} \end{aligned} \tag{2}$$

and for all $q \in L_2(\Omega)$

$$\int_\Omega \mathrm{div}(\mathbf{u})\ q = 0. \tag{3}$$

Here V denotes the set of functions in $H^1(\Omega)$ whose trace are in $H^1(\gamma)$. The above weak formulation assumes, in addition, that γ is closed and avoids additional terms involving $\partial\gamma$. A more general variational formulation taking into account the possible intersection of γ with the $\partial\Omega$ is considered by Bänsch [1].

There are a variety of well known iterative methods for saddle-point problems whose efficiency depends on effective preconditioning of the velocity system and a Schur complement system [6, 8, 19]. This paper addresses the preconditioning of the velocity system. The efficient preconditioning of the Schur complement system is a topic of future research. Moreover, we report results for a simplified scalar system involving the form

$$A(u,v) := \alpha_0(u,v) + \alpha_1 D(u,v) + \alpha_2\ D_\gamma(u,v), u, v \in V, \tag{4}$$

where

$$(u,v) := \int_\Omega u\ v, \qquad D(u,v) := \int_\Omega \nabla u \cdot \nabla v$$

and

$$D_\gamma(u,v) := \int_\gamma \nabla_\gamma u \cdot \nabla_\gamma v.$$

Here α_i, $i = 0,1,2$ are nonnegative constants. From a preconditioning point of view, the problem of preconditioning the velocity system of (2) and that of (4) are more or less equivalent.

The goal of this paper is to investigate the behavior of multigrid algorithms applied to preconditioning the form $A(\cdot,\cdot)$. We shall demonstrate numerical results which suggest level and parameter independent convergence rates. In a subsequent manuscript [3], we shall provide theoretical results which guarantee such convergence in the case $\alpha_0 = 0$. Level and parameter independent convergence results for multigrid algorithms in the case when $\alpha_2 = 0$ have been considered before, see, e.g., [9]. The approach for the analysis in the case of $\alpha_0 = 0$ will also be described.

2 Preliminaries.

For theoretical purposes, we take $\alpha_0 = 0$ and restrict our attention to the case where Ω is a polygonal or polyhedral domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively, which has been triangulated with an initial coarse mesh. Moreover, we consider the case when γ coincides with Γ, the boundary of Ω. Clearly, $\Gamma = \cup\bar{\Gamma_j}$ where $\{\Gamma_j\}$ denotes the set of polygonal faces of Γ.

We assume that we have a nested sequence of globally refined partitioning of Ω into triangles or tetrahedra, i.e., $\mathscr{T}_j$, $j = 1,2,\ldots,J$. These are developed by uniform refinement of a coarse triangulation $\mathscr{T}_1$ of Ω and have a mesh size $h_j \approx \varepsilon^j$ for some

$\varepsilon \in (0,1)$. In particular, we assume there are positive constants C and c satisfying

$$c\varepsilon^j \le h_j \le C\varepsilon^j.$$

The corresponding multilevel spaces of piecewise linear continuous functions are denote by W_j. The functions in W_j with zero mean value on Γ are denoted by V_j and V_j restricted to Γ is denoted by M_j. Conceptually, we use $\theta_j^i := \varphi_j^i - |\Gamma|^{-1}\int_\Gamma \varphi_j^i$ as our computational basis for V_j, where $|\Gamma|$ denotes the measure of Γ and φ_j^i are the nodal basis associated to the subdivision j. Technically, this means that our basis functions no longer have compact support. However, because the form $A(\cdot,\cdot)$ kills constants (for $\alpha_0 = 0$), the stiffness matrix is still sparse. The action of the smoother is more or less local as discussed in Remark 4.7 in [4].

There is one fundamental difference between the cases of $\alpha_1 = 0$ and $\alpha_1 \neq 0$. In the first case, the form $A(\cdot,\cdot)$ is indefinite and hence, for uniqueness, one computes in the subspace of reduced dimension, V_J. It is natural to develop the multigrid analysis on the sequence $\{V_j\}$. Keeping track of the mean value is mostly an implementation issue, see, e.g., [4]. Moreover, standard smoothing procedures work provided that the smoother on V_j is based on the natural decompositions in the larger space W_j. An alternative point of view for the indefiniteness issue in the multigrid context is taken in [16, 17].

The analysis of the multigrid algorithm involves the interaction between the quadratic form $A(\cdot,\cdot)$ and a base inner product. The analysis of [3] involves the use of a boundary extension operator $E^j : M_j \to V_j$. Let $\{x_j^i\}$ denote the grid points of the mesh $\mathscr{T}_j$. Given a function in M_j, we first define $E_j : M_j \to V_j$ by setting

$$(E_j u)(x_j^i) = \begin{cases} u(x_j^i): & \text{if } x_j^i \in \Gamma, \\ 0: & \text{otherwise.} \end{cases}$$

We then set

$$E^j u = \sum_{\ell=1}^{j} E_\ell((q_\ell - q_{\ell-1})u). \tag{5}$$

Here q_ℓ, for $\ell > 0$ denotes the $L^2(\Gamma)$ projection onto M_ℓ and $q_0 \equiv 0$. Note that even though E^j is based on the telescoping decomposition

$$u|_\Gamma = \sum_{\ell=1}^{j} ((q_\ell - q_{\ell-1})u)_\Gamma,$$

the sum in (5) does not telescope.

The critical property of this extension is given in the following proposition proven in [3].

Proposition 1. *For $s = 0, 1$, the extension $E^j : M_j \to V_j$ satisfies*

$$\|E^j u\|_{H^s(\Omega)} \le C\|u\|_{s-1/2,\Gamma}, \qquad \textit{for all } u \in M_j.$$

This extension operator was proposed by [14] for developing computable boundary extension operators for domain decomposition preconditioners. The $s = 1$ case of the above theorem was also given there.

3 The Multigrid algorithm.

The analysis of the multigrid algorithm requires the use of a base inner product. We note that even though the operators appearing in the multigrid algorithm below are defined in terms of the base inner product, the base inner product disappears in the implementation as long as the smoothers are defined by Jacobi or Gauss Seidel iteration. We introduce the base norm (corresponding to $\alpha_0 = 0$):

$$|||u||| = \left[\alpha_1(\|u - E^J u\|^2_{L^2(\Omega)} + \|u\|^2_{-1/2,\Gamma}) + \alpha_2\|u\|^2_{L^2(\Gamma)}\right]^{1/2},$$

for $u \in V_J$. This is the diagonal of the inner product which we denote by $(((\cdot,\cdot)))$. This norm and inner product play a major role in the multigrid analysis in [3].

Following [5], we define the operators:

1. $A_j : V_j \to V_j$ is defined by

$$(((A_j v, \theta))) = A(v, \theta) \qquad \text{for all } v, \theta \in V_j.$$

2. $P_j : V \to V_j$ is defined by

$$A(P_j v, \theta) = A(v, \theta) \qquad \text{for all } v \in V, \theta \in V_j.$$

3. $\widehat{Q}_j : V_J \to V_j$ is defined by

$$(((\widehat{Q}_j v, \theta))) = (((v, \theta)))) \qquad \text{for all } v \in V_J, \theta \in V_j.$$

Along with these operators, we require a sequence of "smoothing" operators $R_j : V_j \to V_j$, $j = 2, 3, \ldots, J$. The smoothing iteration associated with R_j is the operator $S_j : V_j \times V_j \to V_J$ defined by $S_j(x, f) = x + R_j(f - A_j x)$. The adjoint of R_j with respect to the base inner product is denoted by R_j^t and we set $S_j^*(w, f) = w + R_j^t(f - A_j w)$. The solution $w = A_j^{-1} f$ is a fixed point of the smoother iteration and we find that for $x = A_j^{-1} f$, $(x - S_j(w, f)) = S_j(x - w, 0) = (I - R_j A_j)(x - w) \equiv K_j(x - w)$. Thus, K_j relates the error before smoothing to that after. Similarly, we define $K_j^* = I - R_j^t A_j$ and note that K_j^* is the $A(\cdot,\cdot)$ adjoint of K_j, i.e.,

$$A(K_j x, y) = A(x, K_j^* y) \qquad \text{for all } x, y \in V_j.$$

The multigrid algorithms can be defined abstractly in terms of the above operators. We include this definition for completeness as it is certainly classical. For simplicity, we shall consider the V-cycle algorithm. The definitions of other variants

such as the W-cycle or F-cycle algorithm are similar and their analysis follows along the same lines. We define the multigrid operator as a map $Mg_j : V_j \times V_j \to V_j$ given as follows:

Multigrid Algorithm ($Mg_j : V_j \times V_j \to V_j$)

(a) If $j = 1$, set $Mg_1(V,F) = A_1^{-1}F$.
(b) Otherwise, for $j = 2,3,\ldots,J$ define $Mg_j(W,F)$ from $Mg_{j-1}(\cdot,\cdot)$ by:

(i) $V = S_j(W,F)$ (Pre-smoothing).
(ii) $U = V + Mg_{j-1}(0, \widehat{Q}_{j-1}(F - A_j V))$ (Correction).
(iii) $Mg_j(W,F) = S_j^*(U,F)$ (Post-smoothing).

4 Multigrid analysis.

The goal of the computational results of this paper and the analysis of [3] is the demonstration that the natural multigrid algorithm applied to our parameter dependent problem converges uniformly independently of the parameters. We have developed a framework in [3] which allows the use of classical abstract multigrid theory to obtain parameter independent convergence. The key to this is the introduction of the base norm and the analysis of the related projector:

$$\pi_j u = E_j u + Q_j(u - E_j u).$$

Here Q_j denotes the projection onto the subspace of V_j consisting of functions vanishing on Γ. The base inner product and above projector work as long as α_1 and α_0 are of the same magnitude. This framework fails to provide uniform convergence estimates when $\alpha_1 \ll \alpha_0$.

There are two fundamental ingredients in the algorithm of the previous section. We have already discused the nested spaces $\{V_j\}$ and their natural imbeddings. The other ingredient is the smoothing iterations. These are naturally defined in terms of a subspace decomposition of V_j, i.e.,

$$V_j = \cup_i V_j^i, \qquad i = 1,\ldots N_j.$$

The above decomposition may or may not be a direct sum. This gives rise to two distinct smoothing algorithms, specifically, block Jacobi and block Gauss-Seidel smoothing, see [5, 7]. Either of these give rise to the operators R_j, S_j, K_j, R_j^t, S_j^* and K_j^*.

*Remark 1 (***Implementation).** Even though the algorithm of the previous section is defined in terms of operators involving the base inner product $(((\cdot,\cdot)))$, this inner product never appears in the implementation. In fact, the implementation of the resulting multigrid algorithm only requires the sparse stiffness matrices on each of the levels, a solver for the stiffness matrices corresponding to $j = 1$ and the smoother

subspaces, and a "prolongation" matrix which takes coefficients of the representation of a function $v_j \in V_j$ (in the basis for V_j) into the coefficients for v_j represented in the basis for V_{j+1}.

Our smoothers will be required to satisfy the following two conditions:

(C.1) For some $\omega \in (0,1]$ not depending on j,

$$A(K_j x, K_j x) \le A((I - \omega \lambda_j^{-1} A_j)x, x) \qquad \text{for all } x \in V_j, \tag{6}$$

where $\lambda_j := \sup_{u \in V_j} \frac{A(u,u)}{|||u|||^2}$.

(C.2) For some $\theta < 2$ not depending on j,

$$A(R_j v, R_j v) \le \theta(((R_j v, v))), \qquad \text{for all } v \in V_j. \tag{7}$$

These conditions are just (SM.1) and (SM.2) in [5].

To analyze the multigrid algorithm, we apply abstract results which can be found in [5]. Along with conditions (C.1) and (C.2) above, we introduce two additional conditions ((A.5) and (A.6) of [5]):

(C.3) There exists operators $\pi_j : V_J \to V_j$ (with $\pi_0 = \mathbf{0}$) satisfying

$$\sum_{j=1}^{J} \lambda_j |||(\pi_j - \pi_{j-1})v|||^2 \le C_a A(v,v), \qquad \text{for all } v \in V_J.$$

(C.4) There is an $\varepsilon_1 \in (0,1)$ and a positive constant C_{cs} such that for $v_j \in V_j$ and $w_\ell \in V_\ell$ with $\ell > j$,

$$A(v_j, w_\ell) \le C_{cs} \varepsilon_1^{\ell - j} A(v_j, v_j)^{1/2} (\lambda_\ell^{1/2} |||w_\ell|||).$$

The following theorem is Theorem 5.2 of [5].

Theorem 1. *Assume that conditions (C.1)-(C.4) hold. Then*

$$0 \le A(\mathscr{E}_J v, v) \le (1 - 1/C_M) A(v,v), \qquad \textit{for all } v \in V_J,$$

where

$$C_M = \left[\left(1 + \frac{C_a}{\omega}\right)^{1/2} + \left(\frac{C_{cs}\varepsilon_1}{1-\varepsilon_1}\right)\left(\frac{C_a \theta}{2-\theta}\right)^{1/2} \right].$$

We use standard Jacobi or Gauss-Siedel smoothing but with subspaces associated with the nodal decomposition on W_j. The analysis of these smoothers is classical once one verifies the following proposition.

Proposition 2. *Assume that $\alpha_0 = 0$, $0 < \alpha_1$ and $0 < \alpha_2 \le \alpha_1$. Then there is a constant C not depending on j, such that*

$$\sum_i A(v_j^i, v_j^i) \le C \lambda_j |||v_j|||^2 \qquad \textit{for all } v_J \in V_j.$$

Here $v_j = \sum_i v_j^i$ is the expansion of v_j into the finite element basis of V_j.

The above proposition implies [3] that (C.1) and (C.2) hold for the Gauss-Seidel smoother as well as a properly scaled Jacobi smoother. We first show [3] that

$$\left[\alpha_1(\|u-E^J u\|^2_{H^1(\Omega)} + \|u\|^2_{1/2,\Gamma}) + \alpha_2\|u\|^2_{H^1(\Gamma)}\right]^{1/2}$$

provides a norm that is equivalent to $A(u,u)$ for $u \in V_J$. This and Proposition 1 eventially lead to (C.3) and (C.4) (cf. [3]).

5 Numerical Results.

To illustrate the theory suggested in the previous sections, we report the results of numerical computations. We consider two simple domains in $\mathbb{R}^2$, the first being the unit square and the second being the disk of radius one (centered at the origin).

The case of the unit square fits the theory discussed earlier. The square boundary can be mapped via a piecewise smooth map to the circle. Moving the square so that it is centered about the origin, the map takes the point (x,y) on the boundary to the point of unit absolute value in the same direction.

The multigrid algorithm is variational and so $\lambda = 1$ is always the largest eigenvalue of the preconditioned system. The coarsest mesh in the multigrid algorithm was $h = 1/4$ and we used one forward and one reverse sweep of the Gauss-Siedel iteration as a pre and post smoother (4 sweeps per V-cycle iteration on each positive level). In Table 1, we report the condition number $K = 1/\lambda_0$[1] for the preconditioned multigrid algorithm when $\alpha_0 = 0$. We consider three cases corresponding to $(\alpha_1 = 1, \alpha_2 = 1)$, $(\alpha_1 = 1, \alpha_2 = .1)$, and $(\alpha_1 = 1, \alpha_2 = 0)$. In all cases, the numerical results illustrate the uniform convergence suggested by the theory given in [3].

Table 1 Condition numbers for the square, $\alpha_0 = 0$.

h	$\alpha_1 = 1, \alpha_2 = 1$	$\alpha_1 = 1, \alpha_2 = .1$	$\alpha_1 = 1, \alpha_2 = 0$
1/16	1.162	1.167	1.176
1/32	1.180	1.194	1.200
1/64	1.207	1.208	1.211
1/128	1.214	1.214	1.216
1/256	1.217	1.217	1.218
1/512	1.219	1.219	1.219

In the Table 2, we consider the case when $\alpha_0 = 1$. In this case, the multigrid algorithm is based on the original finite element spaces $\{W_j\}$. Again we report the condition numbers for three cases. The first is $\alpha_1 = 1, \alpha_2 = 0$ and corresponds to a

[1] λ_0 is the smallest eigenvalue of the preconditioned system.

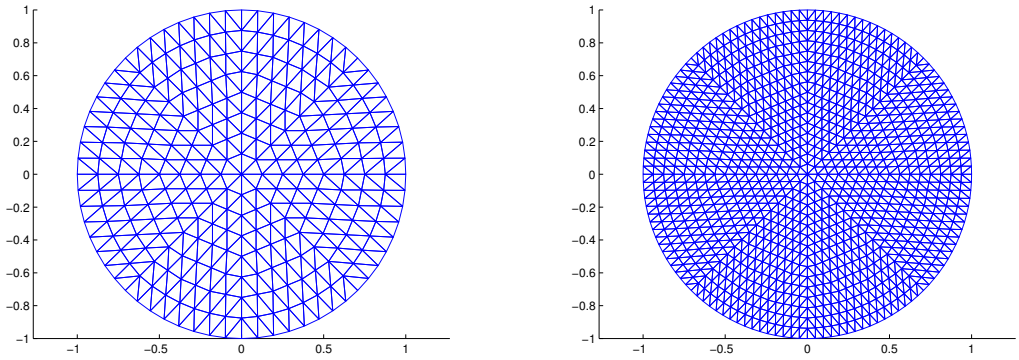

Fig. 1 The grids with 17^2 and 33^2 vertices.

uniformly elliptic second order problem. The second and third are singularly perturbed problems of the form $\alpha_1 = \tau$, $\alpha_2 = \tau^2$ with τ representing the time step size. Because of the lower order term, this case does not fit into the theory of [3].

Table 2 Condition numbers for the square, $\alpha_0 = 1$

h	$\alpha_1 = 1, \alpha_2 = 0$	$\alpha_1 = k$, $\alpha_2 = k^2$, $k = .1$	$\alpha_1 = k$, $\alpha_2 = k^2$, $k = .01$
$1/16$	1.173	1.144	1.078
$1/32$	1.198	1.180	1.133
$1/64$	1.209	1.200	1.172
$1/128$	1.215	1.210	1.195
$1/256$	1.218	1.215	1.208
$1/512$	1.219	1.219	1.214

As a final example, we consider the case when Ω is the unit disk. We set up a sequence of triangulations providing successively better approximations to Ω. The coarsest grid contained 5^2 vertices or $h \approx 1/2$. The meshes with 17^2 and 33^2 vertices are given in Figure 1. The resulting finite element spaces are no longer nested and the multigrid algorithm is no longer variational. Nonvariational multigrid algorithms for the surface Laplacian were investigated in [4]. We believe that the techniques in [3] and [4] can be combined to give rise to uniform convergence for the non-variational V-cycle multigrid algorithm. The computational results reported in Table 3 clearly illustrate uniform parameter independent convergence.

Table 3 Condition numbers for the disk, $\alpha_0 = 0$.

# vertices	$\alpha_1 = 1$, $\alpha_2 = 1$	$\alpha_1 = 1$, $\alpha_2 = .1$	$\alpha_1 = 1$, $\alpha_2 = 0$
17^2	1.206	1.250	1.304
33^2	1.286	1.316	1.360
65^2	1.348	1.364	1.397
129^2	1.395	1.399	1.421
257^2	1.432	1.426	1.437
513^2	1.463	1.448	1.456

6 Acknowledgments.

This work was supported in part by award number KUS-C1-016-04 made by King Abdulla University of Science and Technology (KAUST). It was also supported in part by the National Science Foundation through Grant DMS-0914977 and DMS-1216551.

References

1. E. Bänsch. Finite element discretization of the Navier-Stokes equations with a free capillary surface. *Numer. Math.*, 88(2):203–235, 2001.
2. A. Bonito, R. H. Nochetto, and M. S. Pauletti. Dynamics of biomembranes: effect of the bulk fluid. *Math. Model. Nat. Phenom.*, 6(5):25–43, 2011.
3. Andrea Bonito and Joseph E. Pasciak. Analysis of a multigrid algorithm for an elliptic problem with a perturbed boundary condition. Technical report. (in preparation).
4. Andrea Bonito and Joseph E. Pasciak. Convergence analysis of variational and non-variational multigrid algorithms for the Laplace-Beltrami operator. *Math. Comp.*, 81(279):1263–1288, 2012.
5. James Bramble and Xeujun Zhang. The analysis of multigrid methods. In P.C. Ciarlet and J.L. Lions, editors, *Handbook of numerical analysis, techniques of scientific computing (Part 3)*. Elsevier, Amsterdam, 2000.
6. James H. Bramble and Joseph E. Pasciak. A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems. *Math. Comp.*, 50(181):1–17, 1988.
7. James H. Bramble and Joseph E. Pasciak. The analysis of smoothers for multigrid algorithms. *Math. Comp.*, 58(198):467–488, 1992.
8. James H. Bramble, Joseph E. Pasciak, and Apostol T. Vassilev. Analysis of the inexact Uzawa algorithm for saddle point problems. *SIAM J. Numer. Anal.*, 34(3):1072–1092, 1997.
9. James H. Bramble, Joseph E. Pasciak, and Panayot S. Vassilevski. Computational scales of Sobolev norms with application to preconditioning. *Math. Comp.*, 69(230):463–480, 2000.
10. Q. Du, Ch. Liu, R. Ryham, and X. Wang. Energetic variational approaches in modeling vesicle and fluid interactions. *Physica D*, 238(9-10):923–930, 2009.
11. G. Dziuk. An algorithm for evolutionary surfaces. *Numer. Math.*, 58(6):603–611, 1991.
12. G. Dziuk and C. M. Elliott. Finite elements on evolving surfaces. *IMA J. Numer. Anal.*, 27(2):262–292, 2007.
13. David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

14. G. Haase, U. Langer, A. Meyer, and S. V. Nepomnyaschikh. Hierarchical extension operators and local multigrid methods in domain decomposition preconditioners. *East-West J. Numer. Math.*, 2(3):173–193, 1994.
15. S. Hysing. A new implicit surface tension implementation for interfacial flows. *Internat. J. Numer. Methods Fluids*, 51(6):659–672, 2006.
16. Young-Ju Lee, Jinbiao Wu, Jinchao Xu, and Ludmil Zikatanov. Robust subspace correction methods for nearly singular systems. *Math. Models Methods Appl. Sci.*, 17(11):1937–1963, 2007.
17. Young-Ju Lee, Jinbiao Wu, Jinchao Xu, and Ludmil Zikatanov. A sharp convergence estimate for the method of subspace corrections for singular systems of equations. *Math. Comp.*, 77(262):831–850, 2008.
18. M. S. Pauletti. *Parametric AFEM for geometric evolution equation and coupled fluid-membrane interaction*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–University of Maryland, College Park.
19. Torgeir Rusten and Ragnar Winther. A preconditioned iterative method for saddlepoint problems. *SIAM J. Matrix Anal. Appl.*, 13(3):887–904, 1992. Iterative methods in numerical linear algebra (Copper Mountain, CO, 1990).
20. Jin Sun Sohn, Yu-Hau Tseng, Shuwang Li, Axel Voigt, and John S. Lowengrub. Dynamics of multicomponent vesicles in a viscous fluid. *JOURNAL OF COMPUTATIONAL PHYSICS*, 229(1):119–144, JAN 1 2010.
21. Shawn W. Walker, Andrea Bonito, and Ricardo H. Nochetto. Mixed finite element method for electrowetting on dielectric with contact line pinning. *Interfaces Free Bound.*, 12(1):85–119, 2010.