Numerical Methods
MATH 417 - 501
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First Name: ____________________ **Last Name:** ____________________

Homework 11

Exercise 1 **50% + 10% bonus (MATLAB)**

The deflection of a uniformly loaded, long rectangular plate under axial tension force and fixed ends, for small deflections, is governed by the second order differential equation. Let S represent the axial force and q the intensity of the uniform load. The deflection W along the elemental length is given by:

$$-W''(x) + \frac{S}{D}W(x) = \frac{qx}{2D}(l-x), \quad 0 < x < l, \quad W(0) = W(l) = 0, \tag{1}$$

where l is the length of the plate, and D is the flexural rigidity of the plate. Implement a finite difference approximation of (1) using $N+1$ nodes x_i

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[WFD,err] = rode(S,D,q,N),
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where WFD is a $N+1$ matlab array containing the approximations of $W(x_i)$, $i = 0, ..., N$ and err is the error

$$err = \max_{i=0,\ldots,N} |WFD[i] - W(x_i)|.$$

1. Take $q = 200\ lb/in^2$, $S = 100\ lb/in$, $D = 8.8\ 10^7\ lb\ in$, and $l = 50\ in$. The exact solution is given by: $a = \frac{Sl^2}{D}$, $b = \frac{ql^4}{2D}$, $t = x/l$ and

$$W(t) = \frac{b}{a}\left(-t^2 + t - \frac{2}{a} + \frac{2}{a\ \sinh(\sqrt{a})}[\sinh(\sqrt{a}t) + \sinh(\sqrt{a}(1-t))]\right).$$

   Run your implementation for $N =$ 25, 50, 100, 200. Provide a log-log plot of the error versus N and a graph of the approximate solution.

2. Consider now the problem (1) in the case when the right end of the plate is free, i.e. instead of the boundary condition $W(l) = 0$ now we have the condition $W'(l) = 0$. The exact solution in this case is:

$$W(t) = \frac{b}{a}\left(-t^2 + t - \frac{2}{a} + \frac{1}{a\ \cosh(\sqrt{a})}[\sqrt{a}\sinh(\sqrt{a}t) + 2\cosh(\sqrt{a}(1-t))]\right).$$

   Use a first order finite difference approximation $W'(l) = 0$ and run your new implementation for $N =$ 25, 50, 100, 200. Provide a log-log plot of the error versus N and a graph of the approximate solution. What do you conclude?

3. *Bonus question:* Proceed as in the previous step but with the following approximation

$$W'(l) = W'(x_N) \approx \frac{3W(x_N) - 4W(x_{N-1}) + W(x_{N-2})}{2h}$$

   where $h = l/N$. What do you conclude?

Exercise 2 **50% (MATLAB and BY HAND)**

Consider the following PDE modeling the temperature on a rode of length 1

$$\frac{\partial}{\partial t}y(x,t) - \frac{\partial^2}{\partial x^2}y(x,t) = 0 \qquad 0 < x < 1, \qquad t \geqslant 0.$$

The rode is initially at temperature 0 and heated on both side to the temperature 1:

$$y(x,0) = 0 \qquad 0 < x < 1, \qquad \text{and} \qquad y(0,t) = y(1,t) = 1 \qquad t > 0.$$

1. Given an integer N, let $h = 1/N$ and $x_i = ih$, $i = 0, ..., N$. Consider a centered second order difference for the approximation of $\frac{\partial^2}{\partial x^2}$ and show (by hand) that $Y_i(t)$, the approximation of $y(x_i, t)$ satisfies
$$\frac{d}{dt}\mathbf{Y}(t) + A\mathbf{Y} = \mathbf{F}, \qquad \mathbf{Y}(0) = \mathbf{0}, \tag{2}$$
where $\mathbf{Y}(t) = (Y_1(t), ..., Y_{N-1}(t))^t$. Explicitly determine the matrix A and the vector $\mathbf{F}$ (Note that the boundary points $Y_0(t)$ and $Y_N(t)$ are eliminated from the system).
2. Given a time step k, write the backward Euler scheme for the system of ODEs (2).
3. In matlab, implement the resulting fully discrete scheme for an adequate choice of k and N. Together with your implementation, return a graph containing the approximations $y(x,t)$ versus x at several time t. The time snapshots should be appropriately chosen to illustrate the temperature evolution in the rode.