PREFACE VOLUME 22

ANDREA BONITO AND RICARDO H. NOCHETTO

This and the preceding volumes are devoted to the numerical approximation of *geometric partial differential equations* (GPDEs). Before describing the objectives and contents of this project, it is appropriate to explain the meaning of GPDEs and the reasons why they deserve two volumes of the Handbook of Numerical Analysis. GPDEs are governing equations of natural, social, and economic phenomena where geometry play a prominent to dominant role. Examples abound from interfaces and free boundaries in fluids and solids — such as modeling of surface tension and bending effects involving second fundamental forms — to the development of defects in director and line fields in liquid crystal modeling, motion, merging and splitting of droplets within incompressible fluids, total variation minimization in imaging science, and shape morphing and extrapolation, just to name a few. These problems possess an intrinsic mathematical beauty and pose a formidable challenge both in analysis and computation. Besides their overwhelming mathematical richness, GPDEs are ubiquitous in many scientific, engineering and industrial applications, such as fluid and solid mechanics, materials science, biology, chemistry, astrophysics, plasma physics, imaging, and computer animation.

The last three decades have witnessed the development of powerful algorithms and corresponding numerical analysis for the description and computation of interfaces. The level set and phase field methods have joined the more traditional front tracking techniques and, together with the advent of ever more powerful and versatile computers, have allowed for the simulation and understanding of rather complex phenomena involving interfaces. It is fair to assert that, besides theory and experimentation, mathematical modeling and computation have established themselves as the third pillar of scientific inquiry. A well designed computational model can replace a very expensive or even unrealizable experimental setting, and give new insight into the theoretical developments of a specific discipline. Because of their technical complexity and practical relevance, GPDEs are a chief example.

The purpose of this two-volume contribution is to provide a missing reference book that portrays the state-of-the-art on basic algorithms for GPDEs and their analysis, along with their impacts in a wide variety of areas of science and engineering. Since this field has grown tremendously over the last few years, the selection of topics and authors is a formidable task. Our intention is to present different and complementary approaches ranging from fundamental numerical analysis of basic algorithms to scientific computation and exciting applications. The choice of authors reflects their expertise on various aspects of this ambitious and multifaceted project.

This second volume consists of eight chapters which encompass numerical analysis of GPDEs, algorithm design, analysis and simulation of interfaces, and shape morphing. A brief description follows.

- *Numerical optimization and control.* Chapter 1, by Allaire, Dapogny and Jouve, is an introduction to shape and topology optimization. The Hadamard shape derivative and resulting shape optimization methods are discussed along with different discretizations; they are the "body-fitted" geometric optimization and the level-set methods using either Eulerian or Lagrangian representations. Chapter 2, by Mérigot and Thibert, presents numerical algorithms for optimal transport problems along with their analysis. Their similarities, differences and connections with the theory of Kantorovich duality are highlighted. Chapter 3, by Hintermüller and Keil, discusses optimal control of geometric partial differential arising from interface problems. To overcome the inherent lack of smoothness, typical of interface problems, Yosida-type approximations are put forward together with sharp and diffuse interface representations.
- *Gradient Flows.* Chapter 4, by Carrillo, Matthes and Wolfram, reviews Lagrangian numerical methods for specific Wasserstein gradients flows. The presentation focuses on nonlinear Fokker-Plank equations, the Keller-Segel model and a fourth order thin film equations.
- *Nematic Liquid Crystals.* Chapter 5, by Borthagaray and Walker, provides an overview of the three basic continuum models for nematic liquid crystals, namely the Oseen-Frank, Ericksen and Landau-deGennes models. Several numerical methods are presented, along with their Γ-convergence, with emphasis on the Ericksen model and extensions to the Landau-deGennes model for uniaxial liquid crystals. A computational study illustrating physical properties and comparisons with the standard Landau-deGennes model is included.
- *Total Variation.* Chapter 6, by Chambolle and Pock, reviews several discretization techniques available for approximating total variation problems with emphasis on grayscale images in 2d. Finite difference and finite element methods are compared in terms of both theoretical error estimates and practical issues such as reconstruction quality.
- *Drops and Bubbles.* Chapter 7, by Turek and Mierka, describes interface tracking and interface capturing numerical algorithms for the simulation of multi-phase flow. Benchmark numerical simulations of three-dimensional rising bubbles and three dimensional Taylor bubbles are provided.
- *Isogeometric Analysis.* Chapter 8, by Hughes, Sangalli, Takacs, and Toshniwal, reviews three different methods to construct C^1 isogeometric spaces. They are instrumental for the approximation of higher order geometric partial differential equations.

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