Partial Differential Equations
MATH 612
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Due March 26
Spring 2013

Assignement 2

Exercise 1 **35%**

The aim of this exercise is to provide a generalization of the Lax-Milgram lemma.

Let W and V be Banach spaces with W reflexive. Let $B : V \times W \to \mathbb{R}$ be a bounded bilinear form and $F \in W^*$. Then there exists a unique solution of the following problem : Seek $u \in V$ such that

$$B(u, v) = F(v), \qquad \forall v \in W^*$$

if and only if the following two conditions hold

1. there exists $\alpha > 0$ such that
$$\inf_{w \in W} \sup_{v \in V} \frac{B(w, v)}{\|w\|_W \ \|v\|_V} \geqslant \alpha;$$
2. For all $v \in V$ we have the relation
$$(\forall w \in W, \qquad B(w, v) = 0) \Rightarrow (v = 0).$$

Prove the "if" statement and show that this generalizes the case where $V = W$ and B is coercive.
Hint : to prove the "if" statement prove first that a bounded linear operator $A : V \to W$ is bijective if the following two conditions holds (i) there exists a constant $\alpha > 0$ such that

$$\forall v \in V, \qquad \|Av\|_W \geqslant \alpha \|v\|_V$$

and (ii)

$$\forall w^* \in W^*, \qquad (A^* w^* = 0) \Rightarrow (w^* = 0).$$

Second, note that (ii) is equivalent to

$$\forall w^* \in W^*, \qquad (\langle w^*, Av \rangle_{W^*,W} = 0, \quad \forall v \in V) \Rightarrow (w^* = 0).$$

Exercise 2 **35%**

Let $\mathcal{U} \subset \mathbb{R}^n$ be open, bounded and with $\partial\mathcal{U} \in C^1$. Given $f \in L^2(\mathcal{U})$, consider the following second order elliptic PDE :

$$-\Delta u = f \quad \text{in} \quad \mathcal{U}, \qquad \frac{\partial}{\partial \nu} u = 0, \qquad \text{on} \quad \partial\mathcal{U}. \tag{1}$$

Show that for each $f \in L^2(\mathcal{U})$ such that $\int_{\mathcal{U}} f = 0$, there exists a solution $u \in H^1(\mathcal{U})$ of the boundary value problem (1). Morever, the weak solution is unique if we require in addition that it has vanishing mean value.

Exercise 3 **30%**

Let $u \in H^{(}\mathbb{R}^n)$ have compact support and be the weak solution of the semilinear PDE

$$-\Delta u + c(u) = f, \qquad \text{in} \quad \mathbb{R}^n,$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ is smooth, with $c(0) = 0$ and $c' \geqslant 0$. Prove that $u \in H^2(\mathbb{R}^n)$.
Hint : Mimic the proof of Thm 1 in section 6.3.1 in Evans but without the cutoff function.