6. Lecture 6

The following example, shows the limitation of higher order interpolation methods.

Example 6.1 (No Convergence)**.** *Let $p \in \mathbb{P}^n$ interpolate $f(x) = x^{-\alpha}$ (α defined below) on the interval $[1/2, 2]$ using $n+1$ distinct nodes $\{x_i\}$ in $[1/2, 2]$. By the error Theorem 4.1*

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

so

$$|f(x) - p(x)| = \frac{\max_{\xi \in [1/2,2]} |f^{(n+1)}(\xi)|}{(n+1)!} \prod_{i=0}^{n} |x - x_i|.$$

Without further information on the $\{x_i\}$, we can only conclude

$$|x - x_i| \leq 3/2.$$

Also

$$\begin{aligned} f'(x) &= -\alpha x^{-(\alpha+1)}, \\ f''(x) &= \alpha(\alpha+1) x^{-(\alpha+2)}, \\ &\vdots \\ f^{(n+1)}(x) &= (-1)^{n+1} \alpha(\alpha+1) \cdot \ldots \cdot (\alpha+n) x^{-(\alpha+n+1)}. \end{aligned}$$

Suppose $\alpha \leq 1$, then

$$\frac{|f^{(n+1)}(\xi)|}{(n+1)!} \leq \frac{\alpha(\alpha+1) \cdot \ldots. \cdot (\alpha+n)}{1 \cdot 2 \cdot \ldots \cdot (n+1)} |\xi|^{-(\alpha+n+1)} \leq |\xi|^{-(\alpha+n+1)}.$$

In addition, $|\xi|^{-(\alpha+n+1)}$ is a decreasing function in ξ so that

$$\max_{\xi \in [1/2,2]} |\xi|^{-(\alpha+n+1)} = 2^{\alpha+n+1}.$$

Gathering the above estimates, we obtain the bound

$$|f(x) - p(x)| \leq 2^{\alpha+n+1} (3/2)^{n+1} = 2^\alpha \cdot 3^{n+1}.$$

Therefore, no convergence can be guaranteed (unless the interpolation points are chosen strategically).

Piecewise polynomial interpolation is a way to circumvent this issue.

6.1. **Piecewise Polynomial Interpolation.**

Let $I = [a, b]$ be the interval where we want to construct an interpolant and $N > 0$ an integer. Set $z_i := a + (b-a)\frac{i}{N}$ with constitutes a uniform partition of $[a, b]$ with spacing $h = \frac{b-a}{N}$, i.e.

$$a = z_0 < z_1 < \ldots < z_N = b$$

and $z_i - z_{i-1} = h$. On each subinterval $I_i = (z_{i-1}, z_i)$, we use low order approximation.

Case 1: $\mathbb{P}^0$ interpolation. Choose $\tilde{z}_i \in I_i$ and set $f_h(x)$ be defined on each subinterval by

$$f_h(x)|_{I_i} = f(\tilde{z}_i).$$

This is a polynomial interpolation in $\mathbb{P}^0$ on each subinterval, i.e. f_0 is piecewise constant. Using the error Theorem 4.1 on each subinterval I_i:

$$|f(x) - f_h(x)| = |f'(\xi_x)||(x - \tilde{z}_i)| \leq \sup_{\xi \in I_i} |f'(\xi)|h$$

for every $x \in I_i$. As a consequence, the piecewise interpolation converges as the $h \to 0$, i.e. the number of intervals increase to infinity ($N \to \infty$).

Example 6.2 ($x^{-\alpha}$)**.** *We return to the example above. In this case, $a = 1/2$, $b = 2$ and $z_i = \frac{1}{2} + \frac{3}{2}\frac{i}{N}$, $i = 0, ..., N$ with $h = \frac{3}{2N}$. For $x \in I_i$, we have*

$$|f(x) - f_h(x)| \leq \alpha h \sup_{\xi \in I_i} \xi^{-\alpha-1} \leq \alpha h 2^{\alpha+1}.$$

This implies that

$$\max_{x \in [1/2,2]} |f(x) - f_0(x)| \leq \alpha h 2^{\alpha+1}.$$

Notice that the interpolant $f_h(x)$ of $f(x)$ is *not continuous*. We sometimes write $f_h(x) \in C^{-1}(1/2, 2)$.

Case 2: Continuous $\mathbb{P}^1$ interpolation. We construct f_h, continuous on the entire interval and in $\mathbb{P}^1$ on each interval. To do this, we use the endpoints of I_i as the interpolation nodes, i.e. $x_i = z_i$ and

$$f_h(x) = f(x_{i-1})\frac{(x_i - x)}{h} + f(x_i)\frac{(x - x_{i-1})}{h}$$

for $x \in I_i$. Notice that in particular, $f_h(x_{i-1}) = f(x_{i-1})$ and $f_h(x_i) = f(x_i)$, which implies the resulting piecewise polynomial f_h is *continuous* because $f_h(x_i) = f(x_i)$ from I_i and I_{i+1} for $i = 1, ..., N-1$ (so we do not need to worry that I_i and I_{i+1} defines $f_h(x_i)$).

Remark 6.1 (Discontinuous piecewise linears). Continuity was forced by choosing the endpoints as interpolation nodes. If instead you used interior nodes, you will end up with a discontinuous approximation (in general).

On each I_i we use again error representation provided by Theorem 4.1:

$$|f(x) - f_h(x)| \leq \frac{|f^{(2)}(\xi_x)|}{2}|(x - x_{i-1})(x - x_i)|, \qquad x \in I_i.$$

Now $|(x - x_{i-1})(x - x_i)| = (x_i - x)(x - x_{i-1})$ which achieves its maximum at $x = \frac{1}{2}(x_i + x_{i-1})$ with a value of $\frac{h^2}{4}$. Therefore, for $x \in I_i$

$$|f(x) - f_h(x)| \leq \sup_{\xi \in I_i} \frac{|f^{(2)}(\xi|}{8} h^2$$

and for $x \in I$

$$|f(x) - f_h(x)| \leq \sup_{\xi \in I} \frac{|f^{(2)}(\xi|}{8} h^2.$$

Notice that again the error is converging but this time the rate is quadratic (instead of linear).

Example 6.3. $x^{-\alpha}$ *For the above example, this is*

$$|f(x) - f_h(x)| \leq \alpha(\alpha+1)\frac{h^2}{8}\max_{\xi\in[1/2,2]}\xi^{-\alpha-2} \leq \alpha(\alpha+1)2^{\alpha-1}h^2.$$

The following theorem gather both cases discussed

Theorem 6.1. *Let f be defined on $[a,b]$ with $f \in C^1[a,b]$. Set $z_i := a + ih$, $i = 0,...,N$ with $h = (b-a)/N$. If f_h is the piecewise constant approximation of f then*

$$|f_h(x) - f(x)| \leq \|f'\|_{L^\infty(a,b)}h.$$

Moreover, if $f \in C^2[a,b]$ and f_h is the continuous, piecewise linear approximation of f then

$$|f_h(x) - f(x)| \leq \|f''\|_{L^\infty(a,b)}\frac{h^2}{8}.$$

Here $\|v\|_{L^\infty(a,b)} := \max_{\xi\in[a,b]}|v(\xi)|$.