Numerical Methods for PDEs
MATH 610 - 600
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Homework 5
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Homework 5: Goal oriented and Nitsche

Exercise 1 **50%**

Let $\Omega = (0,1) \times (0,1)$, $f \in C^0(\overline{\Omega})$ and $q \in \mathbb{R}$ with $q \geqslant 0$. Consider the boundary value problem

$$-\Delta u + qu = f \qquad \text{in } \Omega; \qquad u = 0 \qquad \text{on } \partial\Omega.$$

We are interested in approximating the quantity $\alpha := \int_{\partial\Omega} \mathbf{n} \cdot \nabla u$ where $\mathbf{n}$ is the outward unit normal of Ω.

1. The boundary problem has a weak formulation: Find $u \in \mathbb{V}$ such that

   $$\forall v \in \mathbb{V}: \qquad a(u,v) = L(v).$$

   Identify $\mathbb{V}$, $a(u,v)$ and $L(v)$. Show that there exists a unique solution $u \in \mathbb{V}$ satisfying the above weak formulation.
2. Let $\{\mathcal{T}_h\}_{0<h<1}$ be a sequence of conforming shape-regular subdivisions of Ω such that $\operatorname{diam}(T) \leqslant h$, for all $T \in \mathcal{T}_h$ and define

   $$\mathbb{V}_h := \left\{ v \in C^0(\overline{\Omega}) \cap \mathbb{V} \quad | \quad \forall T \in \mathcal{T}_h, \quad v|_T \quad \text{is linear} \right\}.$$

   Write the weak formulation satisfied by the finite element approximation $u_h \in \mathbb{V}_h$ of u. Prove that the function u_h exists and is unique.
3. Assume from now on that $u \in H^2(\Omega)$. Derive the error estimate

   $$\|u - u_h\|_{H^1(\Omega)} \leqslant c_1 h \|u\|_{H^2(\Omega)},$$

   where c_1 is a constant independent of h and u.
   Hint: you can use without proof the fact that there exists a constant C independent of h such that for any $v \in H^2(\Omega)$

   $$\inf_{v_h \in \mathbb{V}_h} \|v - v_h\|_{\mathbb{V}} \leqslant Ch\|v\|_{H^2(\Omega)}.$$

4. Show that for the constant function $w(\mathbf{x}) = 1$ we have

   $$\alpha = a(u,w) - L(w).$$

   Now let $\alpha_h := a(u_h, w) - L(w)$. Using the previous parts, show that when $q > 0$ there holds

   $$|\alpha - \alpha_h| \leqslant c_2 h^2 \|u\|_{H^2(\Omega)},$$

   where c_2 is a constant independent of h and u. What can you say about $|\alpha - \alpha_h|$ when $q = 0$?

Exercise 2 **50%**

Given a quasi-uniform and shape regular sequence of triangulations $\{\mathcal{T}_h\}_{h>0}$ of a polygonal domain $\Omega \subset \mathbb{R}^2$, consider the finite element space

$$\mathbb{V}_h := \left\{ v \in C^0(\overline{\Omega}) \quad | \quad \forall T \in \mathcal{T}_h, \quad v|_T \quad \text{is linear} \right\}.$$

- Show that there exists a constant C independent of h such that for every edge $e \subset \partial\Omega$ of the triangle $K \in \mathcal{T}_h$
$$\int_e \partial_\nu w_h v_h \leqslant C \left(\int_K |\nabla w_h|^2 \right)^{1/2} \left(\frac{1}{h} \int_e |v_h|^2 \right)^{1/2}.$$
Here ν denotes the outward pointing normal on $\partial\Omega$.

- Deduce that given $f \in L^2(\Omega)$, there exists a unique $u_h \in \mathbb{V}_h$ defined by
$$\int_\Omega \nabla u_h \cdot \nabla v_h - \int_{\partial\Omega} \partial_\nu u_h v_h + \frac{\alpha}{h} \int_{\partial\Omega} u_h v_h = \int_\Omega f v_h, \qquad \forall v_h \in \mathbb{V}_h,$$
proovided α is large enough.