Numerical Methods | April 11
MATH 417 - 501 | Spring 2018
A. Bonito

First Name: ____________________ **Last Name:** ____________________

Homework 9

Exercise 1 **35% (MATLAB)**

Implement the following conjugate-gradient algorithm in matlab:

- *Inputs:* $A \in \mathbb{R}^{n\times n}$, $\mathbf{b}, \mathbf{x}^0 \in \mathbb{R}^n$.
- *Initialization*
  Compute:
  $$\mathbf{r}^0 = \mathbf{b} - A\mathbf{x}^0, \qquad \mathbf{w}^1 = -\mathbf{r}^0, \qquad \mathbf{z}^1 = A\mathbf{w}^1$$
  and
  $$\alpha^1 = \frac{(\mathbf{r}^0)^T\mathbf{w}^1}{(\mathbf{w}^1)^T\mathbf{z}^1}, \qquad \mathbf{x}^1 = \mathbf{x}^0 + \alpha^1\mathbf{w}^1.$$
- *Main loop*
  For $i = 1, 2, 3, \ldots$, compute:
  $$\mathbf{r}^i = \mathbf{r}^{i-1} - \alpha^i\mathbf{z}^i,$$
  (if $\|r^i\|_2 < tol$ stop),
  $$\beta^i = \frac{(\mathbf{r}^i)^T\mathbf{z}^i}{(\mathbf{w}^i)^T\mathbf{z}^i}, \qquad \mathbf{w}^{i+1} = -\mathbf{r}^i + \beta^i\mathbf{w}^i, \qquad \mathbf{z}^{i+1} = A\mathbf{w}^{i+1}$$
  and
  $$\alpha^{i+1} = \frac{(\mathbf{r}^i)^T\mathbf{w}^{i+1}}{(\mathbf{w}^{i+1})^T\mathbf{z}^{i+1}}, \qquad \mathbf{x}^{i+1} = \mathbf{x}^i + \alpha^{i+1}\mathbf{w}^{i+1}$$
- *Outputs:* $\mathbf{x}^i$ and $\|\mathbf{r}^i\|$

Consider now the following linear systems

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = (a_{ij})_{i,j=1}^n, \qquad a_{ij} = \begin{cases} 2 & i = j \\ -1 & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbf{b} = (b_i)_{i=1}^n, \qquad b_i = 1, \qquad \mathbf{x}^0 = (x_i^0)_{i=1}^n, \qquad x_i^0 = 0.$$

Together with your matlab code, report the number of iterations for the conjugate gradient algorithm to reach a tolerance of 10^{-8} as well as the tolerance reached for $n = 5, 10, 20, 40, 80, 160$.

Exercise 2 35% (MATLAB)

You will learn in this exercise how to employ a least-square method to denoise data. Consider the function

$$g(x) = 1.7\sin(2\pi x) + 0.47\sin(4\pi x) + 0.73\sin(6\pi x) + 0.8\sin(8\pi x)$$

measured for $x \in (0, 3)$ at 51 equi-distributed points, this is $x_i = \frac{3i}{50}$ for $i = 0, ..., 50$. The measurements are polluted with random noise. This means that you are actually given (in matlab) the vector $\mathbf{F}$:

```
g = inline('1.7*sin(2*pi*x)+ 0.47*sin (4*pi*x) + 0.73*sin(6*pi*x) + 0.8*sin(8*pi*x)');
F = g(linspace(0,3,51)).' + .4*randn(51,1);
```

Write a least-square method to to find the coefficients α_j, $j = 1, ..., 8$ in

$$g_{LS}(x) = \sum_{j=1}^{8} \alpha_j \sin(j\pi x),$$

so that

$$\sum_{i=1}^{50} (F_i - g_{LS}(x_i))^2$$

is minimized. You can use matlab backslash routine to solve the linear solver.
Hand out your matlab code as well as two plots: one of (x_i, F_i) together with $(x, g_{LS}(x))$ and the other one of $(x, g(x))$ together with $(x, g_{LS}(x))$.

Exercise 3 30% (BY HAND)

Recall the conjugate gradient algorithm:

- *Inputs:* $A \in \mathbb{R}^{n\times n}$, $\mathbf{b}, \mathbf{x}^0 \in \mathbb{R}^n$.
- *Initialization*
  Compute:
  $$\mathbf{r}^0 = \mathbf{b} - A\mathbf{x}^0, \qquad \mathbf{w}^1 = -\mathbf{r}^0, \qquad \mathbf{z}^1 = A\mathbf{w}^1$$
  and
  $$\alpha^1 = \frac{(\mathbf{r}^0)^T\mathbf{w}^1}{(\mathbf{w}^1)^T\mathbf{z}^1}, \qquad \mathbf{x}^1 = \mathbf{x}^0 + \alpha^1\mathbf{w}^1.$$
- *Main loop*
  For $i = 1, 2, 3, ...,$ compute:
  $$\mathbf{r}^i = \mathbf{r}^{i-1} - \alpha^i\mathbf{z}^i,$$
  (if $\|r^i\|_2 < tol$ stop),
  $$\beta^i = \frac{(\mathbf{r}^i)^T\mathbf{z}^i}{(\mathbf{w}^i)^T\mathbf{z}^i}, \qquad \mathbf{w}^{i+1} = -\mathbf{r}^i + \beta^i\mathbf{w}^i, \qquad \mathbf{z}^{i+1} = A\mathbf{w}^{i+1}$$
  and
  $$\alpha^{i+1} = \frac{(\mathbf{r}^i)^T\mathbf{w}^{i+1}}{(\mathbf{w}^{i+1})^T\mathbf{z}^{i+1}}, \qquad \mathbf{x}^{i+1} = \mathbf{x}^i + \alpha^{i+1}\mathbf{w}^{i+1}$$
- *Outputs:* $\mathbf{x}^i$ and $\|\mathbf{r}^i\|$

This exercise is about solving the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

1. Check that A is symmetric and positive definite.
2. Perform two iterations of the conjugate gradient method starting with

$$\mathbf{x}^0 = \begin{pmatrix} \frac{3}{2} \\ 2 \end{pmatrix}$$

   and verify that $\mathbf{x}^2 = \mathbf{x}$ is the solution of the linear system.