Math 160 (De Loera) **Homework 4**

Due date: May 27 2016

INSTRUCTIONS

All homeworks will have many problems, both theoretical and practical. Programming exercises need to be submitted via SMARTSITE using the assignment boxes NOT DROPBOX. Other methods of submission without prior approval will receive zero points.

Write legibly preferably using word processing if your hand-writing is unclear. Be organized and use the notation appropriately. Show your work on every problem. Correct answers with no support work will not receive full credit.

1. **PROJECT 4:** Automatically solving SUDOKU puzzles, predicting difficulty of Sudoku's:

   You probably know the famous sudoku puzzles, but just in case A 9×9 matrix A is partitioned into nine 3×3 denoted $A_1, A_2, \ldots, A_9$. A few entries are filled in advanced, then a solution to the Sudoku game is an assignment of integers from 1 to 9 to each (unassigned) entry of the matrix such that each row of A, each column of A and each A_1 contains every number from 1 to 9 exactly once.

   - Formulate the problem of finding a solution for Sudoku as a discrete model. (HINT: Think of the assignment model, but use variables with 3 indices $x_{i,j,k}$ that takes value 1 or 0, depending on whether entry $A_{i,j}$ is assigned the number k.)
   - Implement the model in ZIMPL/SCIP and use it to solve the following three SUDOKU puzzles, which one took longer?

			1	5	6			3
6	7	2		8		1		
				2	9	4	8	
9	3	4						1
			9	1			7	4
8			6		5			
		1		9	8		4	
5	4	9	2			3		
		6	4		1	2		9

9			8			2	7	3
			4					
	3	2						6
1		8						
		5		8		3		
	9				5		6	
				1		6	2	
					2	8		
5		3						9

8					7			
	3		1	5				7
							3	
9						1	6	8
		6			4			
1	7		5			4		
						2		1
2							7	
6			8				4	

   - Not all SUDOKU puzzles have a unique solution. If one is not careful when building one, there may be many solutions. Demonstrate how to you use your model to decide whether the following SUDOKU has a single solution. Come up with a "bad" SUDOKU that has more than one solution (i.e., at least two ways to complete the clues given to fill the SUDOKU).

2. Consider the linear program $\min\{c^T x : Ax \leq b\}$ compute the Lagrangian and the Lagrangian dual problem. Verify that the optimal value of the first is bounded below by the second.

3. A vertex-cover of a graph G is a set S of vertices of G such that each edge of G is incident with at least one vertex of S. Formulate a discrete model that given a graph finds a vertex cover with smallest number of vertices. Explain the reasoning on your variables and constraints. Show that if M is a matching and S is some vertex cover $|S| \geq |M|$. In particular show that

$$\max\{|M| : M \text{ is a matching of } G\} \leq \min\{|S| : S \text{ is a vertex-cover of } G\}.$$

   HINT: How many vertices do you need to cover just the edges of M?

4. Give an example of an Integer Linear program which has no feasible integer solutions, but its LP relaxation has a feasible set in R^2 of area at least 100.

5. Given a graph $G = (V, E)$ representing say the cities of California connected by highways, we wish to find out whether there is a tour of the vertices of G (a cycle visiting each vertex exactly once) which only uses the existing edges in E. Suppose you have a powerful software that solves the Traveling salesman problem. How would you use that algorithm for the TSP to answer that question? What costs should you pick on arcs?

6. Directed graphs are important for decision models.

   We saw they can be used to decide the "most important node on a network". Pagerank uses the idea that a stationary distribution on Markov chain, which simulates a random walk on a directed graph with probabilities as weights of transition (Perron-Frobenius theorem).

   A very small link matrix is available at

   `http://www.math.ucdavis.edu/~deloera/TEACHING/MATH160/link-matrix.mat`

   Draw the corresponding link graph (this is a 10 node network only!!). Compute, using the MATLAB command eig, and plot the pagerank vector. Who is the most important node?

7. A directed graph $G = (V, A)$ can be used to represent the order of actions to be take in a project (task a needs to be done before b, if there is an arc from a to b. A topological ordering of the vertices is assignment of the value y_i to each vertex such that for every arc ij then $y_i \geq y_j + 1$. Show that a directed graph has a topological ordering, then there is no directed cycle. Write a simple discrete model to detect whether a directed graph has a cycle.