Mat 160 HW 2.

Problem 2: Let $A$ be an $m \times n$ matrix and $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$ vectors. Prove that the set $P = \{x : Ax \leq b\}$ is a convex set.

Solution: Pick $x_1 \in P$ , $Ax_1 \leq b$

$x_2 \in P$ , $Ax_2 \leq b$

The line segment $(x_1, x_2)$ :

$$A(\theta x_1 + (1-\theta)x_2)$$

$$= Ax_1\theta + Ax_2 - Ax_2\theta$$

$$= (Ax_1 - Ax_2)\theta + Ax_2$$

Because $Ax_1 \leq b$, $Ax_2 \leq b$ ,

$\therefore$ $Ax_1 - Ax_2 \leq 0$

Because $0 \leq \theta \leq 1$ , and $Ax_2 \leq b$

$$\therefore \underbrace{(Ax_1 - Ax_2)}_{\leq 0}\underbrace{\theta}_{\leq 1} + \underbrace{Ax_2}_{\leq b} \leq b$$

Thus $(x_1, x_2) \in P$

Therefore, $P = \{x \mid Ax \leq b\}$ is a convex set.

Problem 3: Let $C$ be a nonempty subset of $\mathbb{R}^n$, and let $\lambda_1$ and $\lambda_2$ be positive scalars. Show that if $C$ is a convex set, then $(\lambda_1+\lambda_2)C = \lambda_1 C + \lambda_2 C$. Show by example that this need not be true when $C$ is not convex.

Solution: ① If $C$ is convex,

Let $x_1, x_2$ be vectors in $C$ and $x$ is in $\lambda_1 C + \lambda_2 C$.

Then $x = \lambda_1 x_1 + \lambda_2 x_2$

Because $C$ is a convex set

$\therefore \frac{\lambda_1}{\lambda_1+\lambda_2} x_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} x_2$ is also in $C$

Thus, $(\lambda_1 x_1 + \lambda_2 x_2) \cdot \frac{1}{\lambda_1+\lambda_2} \in C$

$\therefore \lambda_1 x_1 + \lambda_2 x_2 \in \lambda_1+\lambda_2 C$

Because $x \in \lambda_1 C + \lambda_2 C$

Therefore, $\lambda_1 C + \lambda_2 C \subset (\lambda_1+\lambda_2) C$

On the other hand, we know that $(\lambda_1+\lambda_2) C$ is always a subset of $\lambda_1 C + \lambda_2 C$,

therefore, $(\lambda_1+\lambda_2) C = \lambda_1 C + \lambda_2 C$

② When $C$ is not convex, Let $\lambda_1 = \lambda_2 = 1$

Suppose $C$ consists of two vectors $0$ and $x$.

then $(\lambda_1+\lambda_2) C = 2C = (0, 2x)$

~~$2\lambda_1+\lambda_2$~~ $\lambda_1 C + \lambda_2 C = C + C = (0, x, 2x)$

$\therefore (\lambda_1+\lambda_2) C \neq \lambda_1 C + \lambda_2 C$.

Problem 4: Show that the image and inverse image of a cone under a linear transformation is a cone. Show that a subset $C$ is a convex cone iff. it is closed under addition and positive scalar multiplication.

Solution: ① Let $C$ be a cone, and $A \cdot C$ be the image of $C$ under linear transformation.

Then, for some $x$ in $C$, there must exist a $z$, s.t. $z = Ax$, $z \in A \cdot C$

Because $C$ is a cone, after scalar multiplication, $a$, $ax \in C$.

Thus, $(Ax)a = (z)a$ is also true.

$\therefore A(ax) = az$

Thus, $az$ is in $AC$

Because $z \in AC$

Therefore, $AC$ is a cone

Let $C$ be a cone and $A^{-1}C$ be the inverse image of $C$ under linear transformation.

Then for some $x$ in $A^{-1}C$, $Ax \in C$

Because $C$ is a cone, Scaling it be $a$,

$aAx \in C$ is also true.

Therefore, $A(ax)$ is in $C$

$\therefore ax \in A^{-1}C$

Because $x \in A^{-1}C$

Thus, $A^{-1}C$ must be a cone.

Problem 4. Continued.

(2) Assume that $C$ is a convex cone.

Then because a convex cone includes all conic combinations.

$$\lambda C \subset C$$

Let $x, y \in C$, Because $C$ is convex, there must exist a $z$ in $C$, s.t.

$$z = \frac{1}{2}(x+y)$$

$\therefore$ $x+y \in C$

Because $x, y \in C$

Thus, $C+C \subset C$.

Therefore, if $C$ is a convex cone, then $\lambda C \subset C$ and $C+C \subset C$.

Assume that $\lambda C \subset C$ and $C+C \subset C$

Then $C$ includes all conic combinations of points

Thus, $C$ is a cone.

Let $x \in C$, $y \in C$. Because $C+C \subset C$,

$\therefore$ $\theta x+(1-\theta)y \in C$

$\therefore$ $C$ is a convex set

Therefore, if $\lambda C \subset C$ and $C+C \subset C$, $C$ is a convex cone.

Problem 5: ① Show that for $x, y$ positive scalar real numbers $ye^{\frac{x}{y}} = \max_{a>0} a(x+y) - y\,a\ln(a)$. ② Use this to prove that the function $ye^{\frac{x}{y}}$ is convex inside the positive orthant. ③ Let $f(x) = \ln(e^{x_1} + \cdots + e^{x_n})$, is this convex.

Solution:

① Prove: $ye^{\frac{x}{y}} = \max_{a>0} a(x+y) - y\,a\ln(a)$

First derivative of rhs $= (x+y) - (y\ln a + y \cdot a \frac{1}{a})$

Second derivative of rhs $= (x - y\ln a)' = -\frac{y}{a}$

$\because a>0 \quad \therefore -\frac{y}{a} < 0$

$\therefore$ When $1^{st}$ derivate $=0$, there exist a local maximum extrema

Let $rhs' = 0$

$(x+y) - (y\ln a + y) = 0$.

$$x = y\ln a$$
$$\frac{x}{y} = \ln a$$
$$a = e^{\frac{x}{y}}.$$

$\therefore rhs = e^{\frac{x}{y}}(x+y) - ya\ln(a)$

$= xe^{\frac{x}{y}} + ye^{\frac{x}{y}} - xe^{\frac{x}{y}}$.

$= ye^{\frac{x}{y}}$

$\therefore lhs = rhs$ when at local max

Problem 5 / ②

$$\text{rhs} = \max_{a>0} \; a(x+y) - y \cdot a \cdot \ln(a)$$

$\because$ $a(x+y)$ is a linear (convex) function

and $\underbrace{-\underbrace{y\,a\ln(a)}_{\text{concave}}}_{\text{convex.}}$ is a convex function.

By lemma: $g(x) = \max\{f_1(x), f_2(x), \cdots f_n(x)\}$

is convex if $f_i(x)$ is convex.

$\therefore$ rhs is convex.

Therefore, lhs $ye^{\frac{x}{y}}$ is also convex

Problem 5 / ③.

Prove: $f(x) = \ln(e^{x_1} + \cdots + e^{x_n})$ is convex

$$\frac{\partial f}{\partial x_1} = \left[\ln(e^{x_1} + a)\right]' ; \; a = e^{x_2} + \cdots + e^{x_n}$$

$$= \frac{e^{x_1}}{e^{x_1} + a}$$

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{0}{[e^{x_1}+a]^2} = 0$$

Similarly: $\frac{\partial^2 f}{\partial x_i^2} = 0$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \left[\frac{e^{x_1}}{e^{x_2} + e^{x_1} + \cdots + e^{x_n}}\right]' = \frac{0 - e^{x_1} e^{x_2}}{(e^{x_1} + \cdots + e^{x_n})^2} < 0.$$

Similarly, $\frac{\partial^2 f}{\partial x_i \partial x_j} < 0$

Therefore

$$\nabla^2 f(x) =$$

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

All terms on the diagnal are zero.
All other terms are negative

Therefore, clearly, all principal minors of this matrix are in the form of $0 - (\text{negative term})$ ~~constant~~;
thus, are all non-negative.
Thus, the matrix is PSD.

Therefore, $f(x) = \ln(e^{x_1} + \cdots + e^{x_n})$ is convex.

Problem 6: In this problem you need to test whether the following functions are convex or not:

1. The function $S_k : R^n \to R$ which is defined as $S_k(x) = \sum_{i=1}^{k} x_{[i]}$ where $x_{[i]}$ is the $i$th largest component of the vector $x$.

2. For $n = 2k-1$ odd consider the function $\phi : R^n \to R$ with $\phi(x) = \frac{1}{n} \sum_{i=1}^{n} |x_i - \mathrm{med}(x)|$ where $\mathrm{med}(x)$ is the median of the components of $x$.

Solution:

① When $n=3$, $k=2$ suppose $x = [2, 8, 5]$

$$S_2(x) = \sum_{i=1}^{2} x_{[i]}$$

$$= 8+5 = 13$$

$$S_k(x) = \max \underbrace{\sum^{k} x_i}$$ ; where $i$ could be any number from 1 to $n$.

Pick the maximum (sum) combination of $k$ elements in vector $x$

Sum of (any) $k$ elements in $x$

$\downarrow$

Linear function $\to$ Convex

By lemma: $g(x)$ is convex if $f_i(x)$ is convex where $g(x) = \max\{f_1(x), f_2(x), \dots, f_n(x)\}$.

Therefore, $S_k(x)$ is convex.

Problem 6 / ⓑ.

$$\phi(x) = \frac{1}{n}\left[\sum_{i=1}^{k-1}(x_{[i]} - x_{[k]}) + \sum_{i=k+1}^{n}(x_{[k]} - x_{[i]})\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{k-1} x_{[i]} - \sum_{i=k+1}^{n} x_{[i]}\right]$$

$$= \underbrace{S_{k-1}(x)}_{\text{convex}} + \underbrace{S_k(x)}_{\text{convex}} - \sum_{i=1}^{n} x_i$$

Therefore, $\phi$ is convex.