

Lecture 1

History of mathematical logic in computer science

Introduction to Logic for Computer Science

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KAIST

These slides are minor variants of those made by Prof Worrell and Prof Haase for their logic course at Oxford.

Agenda

- 1 A historical perspective on logic
- 2 Recent applications of computational logic in math, CS and philosophy
- 3 Overview of the course
- 4 Practicalities

1 A historical perspective on logic

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What is logic?

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—Alfred Tarski, "Introduction to logic and to the methodology of deductive sciences".



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"To discover truths is the task of all sciences; it falls to logic to discern the laws of truth. ... I assign to logic the task of discovering the laws of truth, not of assertion or thought."

—Gottlob Frege, "Der Gedanke. Eine logische Untersuchung".



What is logic?

Logic is the study of the principles of correct reasoning.

Aristotle (384–322 BC)

Syllogism: *“...a discourse in which certain (specific) things having been supposed, something different from the things supposed results of necessity because these things are so.”*

All beings are mortal
All humans are beings
<hr/>
All humans are mortal

All B are M
All H are B
<hr/>
All H are M



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Aristotle only gave a compendium of valid arguments.



Gottfried Wilhelm Leibniz (1646–1716)

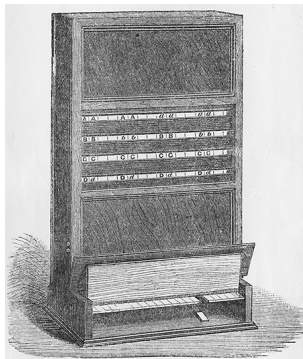
- Envisioned a calculus that allows for testing *any* argument for validity.

“It is obvious that if we could find characters or signs suited for expressing all our thoughts as clearly and as exactly as arithmetic expresses numbers or geometry expresses lines [...] all investigations which depend on reasoning would be carried out by transposing these characters and by a species of calculus.”



George Boole (1815–1864)

- Equational rules for *propositional logic* in 1854.
- Built into the logic piano by William Stanley Jevons.



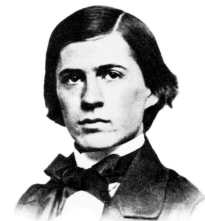
Gottlob Frege (1848–1925) and Charles Sanders Pierce (1839–1914)

- Generalisation of propositional logic to *predicate logic*.



“There is just one point where I have encountered a difficulty . . .”

— Bertrand Russell to Frege, 1902.



“Beyond doubt . . . he was one of the most original minds of the later nineteenth century, and certainly the greatest American thinker ever.”

— Bertrand Russell about Pierce, 1959.

Bertrand Russell (1872–1970) and Alfred North Whitehead (1861–1947)

- Principia Mathematica: attempt to base all of mathematics (set theory, analysis, geometry) rigorously on predicate logic.

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QUANTITY

[PART VI

*311·511. $\vdash : \text{Infin ax. } \xi \in C^{\epsilon}\Theta . Y \in C^{\epsilon}H . \supset . (\mathfrak{A}X) . X \in \xi . Y +_g X \sim \epsilon \xi$
 [*311·51. Transp]

*311·52. $\vdash : \text{Infin ax. } \xi, \eta \in C^{\epsilon}\Theta . \supset . \xi \Theta (\xi +_p \eta)$

Dem.

$\vdash . *311·511 . \supset \vdash : \text{Hp. } \supset : Y \in C^{\epsilon}H . \supset . (\mathfrak{A}X) . X \in \xi . X +_g Y \sim \epsilon \xi :$

[*311·11] $\supset : (\mathfrak{A}X, Y) . X +_g Y \in (\xi +_p \eta) - \xi :$

[*310·11, *311·27] $\supset : \xi \Theta (\xi +_p \eta) : \supset \vdash . \text{Prop}$

*311·53. $\vdash : \text{Infin ax. } \xi, \eta \in C^{\epsilon}\Theta_n . \supset . \xi \Theta_n (\xi +_p \eta)$ [*311·52·33]

*311·56. $\vdash : \text{Infin ax. } \xi \in C^{\epsilon}\Theta_g . \supset : \xi = \xi +_p \eta . \equiv . \eta = \iota^{\epsilon}0_g$ [*311·1·43·52·53]

*311·57. $\vdash : \text{Infin ax. } \supset : \xi = \xi +_p \eta . \equiv : \xi = \Lambda . \vee . \xi \in C^{\epsilon}\Theta_g . \eta = \iota^{\epsilon}0_g$
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*311·58. $\vdash : \text{Infin ax. } \mu \in C^{\epsilon}\Theta . \supset . \mu = H^{\epsilon}\mu$ [*304·3. *270·31]

*311·6. $\vdash : \text{Infin ax. } \mu \Theta \nu . X, Y \in \nu - \mu . XHY . M \in \mu . \supset . M +_g (Y -_s X) \in \nu$
Dem.

$\vdash . *310·11 . \supset \vdash : \text{Hp. } \supset . MHX .$

[*308·42·72] $\supset . \{M +_g (Y -_s X)\} HY$ (1)

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(proves that $1+1=2$ on page 379)

Kurt Gödel (1906–1978)

- No sufficiently strong logical system can be both consistent and complete.

“Kurt Gödel’s achievement in modern logic is singular and monumental - indeed it is more than a monument, it is a landmark which will remain visible far in space and time. [...] The subject of logic has certainly completely changed its nature and possibilities with Gödel’s achievement.”

— John von Neumann



Alonso Church (1903–1995) and Alan Turing (1912–1954)

- There is no algorithm that decides whether a given logical argument is valid or not.
- λ -calculus and *Turing machines* lay the foundations for theoretical computer science.

Alonzo Church, A note on the Entscheidungsproblem", *Journal of Symbolic Logic*, 1 (1936), pp 40-41.



Alan Turing, On computable numbers, with an application to the Entscheidungsproblem, *Proceedings of the London Mathematical Society*, Series 2, 42 (1936-7), pp 230-265.



Claude Shannon (1916–2001) and John Robinson (1930–2016)

Using electrical switches to compute
Boolean functions:

Claude Shannon. A Symbolic Analysis of
Relay and Switching Circuits, *MSc
Thesis*, 1937.



Resolution and *unification*: automated
reasoning and logic programming:

John Robinson. A Machine-Oriented
Logic Based on the Resolution Principle,
Journal of the ACM, 1965.



Logic is fundamental to computer science

When I was a student, even the topologists regarded mathematical logicians as living in outer space. Today the connections between logic and computers are a matter of engineering practice at every level of computer organization.

—Martin Davis, “Influences of Mathematical Logic on Computer Science”.

Logic is everywhere

- Hardware design.
- Database theory.
- Automated verification.
- Knowledge representation.
- Programming-language theory.
- Complexity theory.
- Constraint satisfaction problems.

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- Hardware design.
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- Complexity theory.
- Constraint satisfaction problems.
- ... we won't be focussing on any of the above. Our focus is on foundations of logic.
- But we study the logic from the computational perspective.

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Finding a needle amongst $1,566 \times 10^{349}$ needles

Erdős discrepancy conjecture

For any $C > 0$ and any infinite sequence $x_1 x_2 x_3 \cdots$ of $+1$'s and -1 's, there exist $d, k \in \mathbb{N}$ such that

$$\left| \sum_{1 \leq i \leq k} x_{id} \right| > C.$$

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Erdős discrepancy conjecture

For any $C > 0$ and any infinite sequence $x_1 x_2 x_3 \dots$ of $+1$'s and -1 's, there exist $d, k \in \mathbb{N}$ such that

$$\left| \sum_{1 \leq i \leq k} x_{id} \right| > C.$$

- First shown to hold when $\sum_{1 \leq i \leq k} x_{id} \leq 1$ for all d, k by A.R.D. Mathias in 1993. This implies the case $C = 1$.
- Investigated as a Polymath project in 2009-10:
"Given how long a finite sequence can be, it seems unlikely that we could answer this question [for $C = 2$] just by a clever search of all possibilities on a computer."
- B. Konev and A. Lisitsa solved the case $C = 2$ in 2014 using a SAT solver.
- 13Gb of proof that no sequence of length 1161 with discrepancy 2 exists.
- Conjecture proved for all $C > 0$ by T. Tao in 2015.

Boolean Pythagorean triples problem from Ramsey theory

Boolean Pythagorean triples problem

For which $n \in \mathbb{N}$, can we colour integers $1, 2, \dots, n$ with red or blue such that no three integers a, b, c with the same colour satisfy the Pythagorean equation $a^2 + b^2 = c^2$?

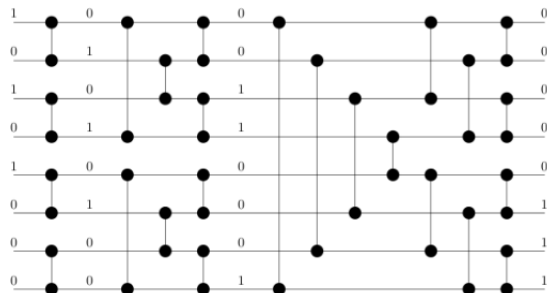
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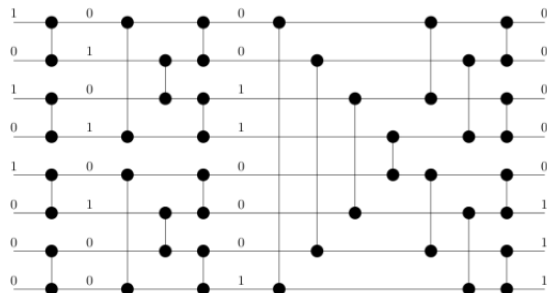
- Heule, Kullman, and Marek resolved this problem using a SAT solver in 2016.
- Their answer: precisely those n 's with $n \leq 7824$.
- The SAT solver spent 4 CPU years of computation, and generated a 200 terabyte propositional proof, which was later compressed to 68GB.

Optimal sorting networks



- Consist of wires and comparators, input flows from left to right.
- Depth is the maximum number of comparators an input can encounter.
- D. Knuth and R. Floyd found optimal depths for $n = 1, \dots, 8$ in 1960s.
- D. Bundala and J. Závodný found optimal depths for $n = 11, \dots, 16$ in 2014 using a SAT solver.

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n	5	6	7	8	9	10	11	12	13	14	15	16
d	5	5	6	6	7	7	8	8	9	9	9	9

Leibniz's ontological proof

Leibniz's proof for the existence of god

- 1 Definition: God is a being having all perfections.
- 2 Definition: A perfection is a simple and absolute property.
- 3 Existence is a perfection.
- 4 If existence is part of the essence of a thing, then it is a necessary being.
- 5 If it is possible for a necessary being to exist, then a necessary being does exist.
- 6 It is possible for a being to have all perfections.
- 7 Therefore, a necessary being (God) does exist.

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 - 6 It is possible for a being to have all perfections.
 - 7 Therefore, a necessary being (God) does exist.
- Leibniz's "algebra of concepts" formalised in the theorem prover Isabelle/HOL by Bentert, Benz Müller, Streit and Paleo in 2016.
 - Showed validity and invalidity of Leibniz's proof in the algebra of concepts depending on the interpretation of some informal statements.

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Focus

- Foundations.
- Computational aspects in particular.
- Computability and algorithms for satisfiability/validity problems.
- Based on the course by Prof Worrell and Prof Haase at Oxford.

Topics

- Propositional logic.
 - The SAT problem.
 - Polynomial-time algorithms for Horn, 2-SAT, and XOR formulas.
 - Resolution.
 - DPLL with clause learning.
 - Lower bounds for resolution proofs.
 - Compactness theorem.
- First-order logic.
 - Prenex normal form and Skolemisation.
 - Herbrand models and ground resolution.
 - Unification and resolution for predicate logic.
 - Undecidability of satisfiability.
 - Compactness theorem.
 - Logical theories and quantifier elimination.

Topics - Basics

- **Propositional logic.**
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Topics – Algorithms or semi-algorithms

- Propositional logic.
 - The SAT problem.
 - Polynomial-time algorithms for Horn, 2-SAT, and XOR formulas.
 - Resolution.
 - DPLL with clause learning.
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Topics – Computability and complexity

- Propositional logic.
 - The SAT problem.
 - Polynomial-time algorithms for Horn, 2-SAT, and XOR formulas.
 - Resolution.
 - DPLL with clause learning.
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Course Materials and Information

- The following webpage and KLMS are the main sources:

`https://github.com/hongseok-yang/logic24`

Check them regularly.

- Lecture notes, slides, and homework sheets all at the webpage.
- Announcements and Q&A at KLMS.

Teaching staffs

- Lecturer: Prof Hongseok Yang (Email: hongseok00@gmail.com).
- TA1: Mr Taeyoung Kim (Email: taeyoungkim21@kaist.ac.kr).
- TA2: Mr Jaehui Hwang (Email: hwcmi62@kaist.ac.kr).
- Hongseok's office hours: 6pm - 7pm on Monday in his office (3403, E3-1).
- TAs' office hours: To be announced.

Evaluation

- Homework (20%) – 4 problem sheets.
- Programming project and report (30%) – optimised SAT solver based on DPLL with clause learning.
- Final exam (50%).

Programming project

- Implementation of a SAT solver that uses the DPLL algorithm and other optimisations (20%).
- Report on the description and analysis of the implementation and its design decision (10%). At most 4 pages excluding bibliography and figures.
- Submission deadline: 23:59 of 20 May 2024 (Monday).
- Submit it in KLMS.
- All quoted sentences from existing literature should be marked so explicitly with pointers to their sources. If not, zero mark.

Honour code

- We adopt a very strict policy for handling the violation of the standard honour code.
- If a student is found to cheat in a test or copy answers or code from friends' or other sources, the student will get F.

Recommended textbooks

- Lecture notes in the course webpage.
- *'The Calculus of Computation: Decision Procedures with Applications to Verification'*, A. Bradley and Z. Manna.
- *'Logic in Computer Science: Modelling and Reasoning about Systems'*, M. Huth and M. Ryan.
- *'The Art of Computer Programming, 4B'*, D. Knuth.

Further Literature

- *'Logicomix: An Epic Search for Truth'*, A. Doxiadis and C. Papadimitriou.
- *'Handbook of Practical Logic and Automated Reasoning'*, J. Harrison.
- *'Mathematical Logic for Computer Science'*, M. Ben-Ari.
- *'Logic for Computer Scientists'*, U. Schöning.
- *'Gödel, Escher, Bach: an Eternal Golden Braid'*, D. Hofstadter.

Questions to check your background knowledge

- Let X be a non-negative real-valued random variable, and $c > 0$ be a positive real number. Prove that

$$P(X > c) \leq \frac{\mathbb{E}(X)}{c}.$$

- What is the time complexity of solving a system of linear equations $Ax = b$ using Gaussian elimination?
- Explain the following notions: NP, NP-complete, regular, recursive, and recursively enumerable.
- Consider a subset $L \subseteq \Sigma^*$ over a finite Σ . Show that if both L and L^c are recursively enumerable, then L is recursive.
- Assume that $L, L' \subseteq \Sigma^*$ over a finite Σ are regular. Let $f : \Sigma \rightarrow \Sigma'$ be a map to a finite Σ' . Show that $L \cap L'$ is regular. Also, prove that $f(L) = \{f(x_1) \dots f(x_n) \mid x_1 \dots x_n \in L\}$ is regular.
- Prove: the program `(y=1; while (x>0) y=y*x; x=x-1;)` computes the factorial of x .