數值分析Numerical Analysis

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聽眾Audience

- 碩一first year graduate
- 基礎課程 Basic curriculum
- 19 students

Objective

- 引導您了解如何自學。Guide you to know how to learn by yourself.
- 數值分析（NA）的歷史是什麼？
- What is the history of the Numerical Analysis (NA)?
  - Who are the important people NA? Newton, Lagrange, Gauss and Euler.
  - What are the important events of NA?
  - Where is the location of NA?
  - Which objects are related to NA?
  - When is NA popular?

材料Materials

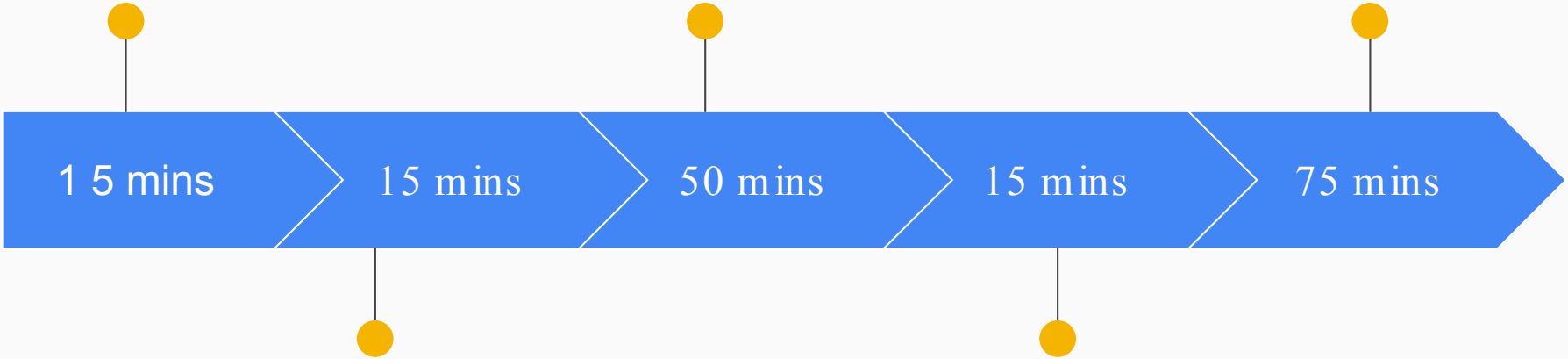
- What is NA
- Python Language
- Tensor flow
- Project and report

程序Procedure

Introduction

Step by step to learn small concepts by doing

A project and report should be done by yourself



Demo:How to solve problem?

Conclusions

家庭作業1/Homework1

Learning efficiency depends on motivation

1. Why do you choose this course?
2. What kind of jobs for NA?
3. What kind of jobs you want?

How to learn knowledge of NA efficiently?

1. Python programming
2. Mathematics
3. Tensorflow, scikit learn and, karas

Python programming

Learning topics

1. Using Python as a Calculator
    a. Arithmetic operations
2. Variables
3. if Statements
4. for Statements
5. The range() Function
6. Executing modules as scripts
7. Mathematics
8. Python Functions - W3Schools

Learning Resources

1. Welcome to Python.org
2. Download
3. Python For Beginners
4. The Python Tutorial
5. IntroductoryBooks
6. Python Functions - W3Schools
7. A Visual Introduction to Python
8. Try Jupyter

Python programming

Learning topics

1. Python Functions

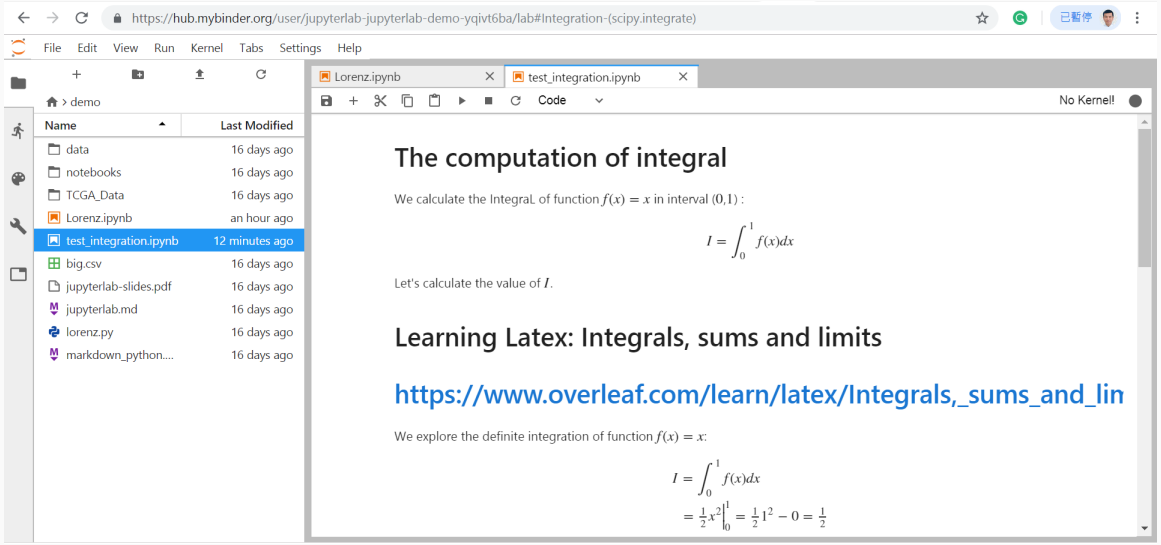
```
def my_function():
  print("Hello from a function")

my_function()
```

Learning Resources

1. Python Functions - W3Schools
2. Run web python or Anaconda
3. http://jupyter.org/try
4. https://hub.mybinder.org/user/jupyterlab-jupyterlab-demo-yqivt6ba/lab#Integration-(scipy.integrate)

Python programming



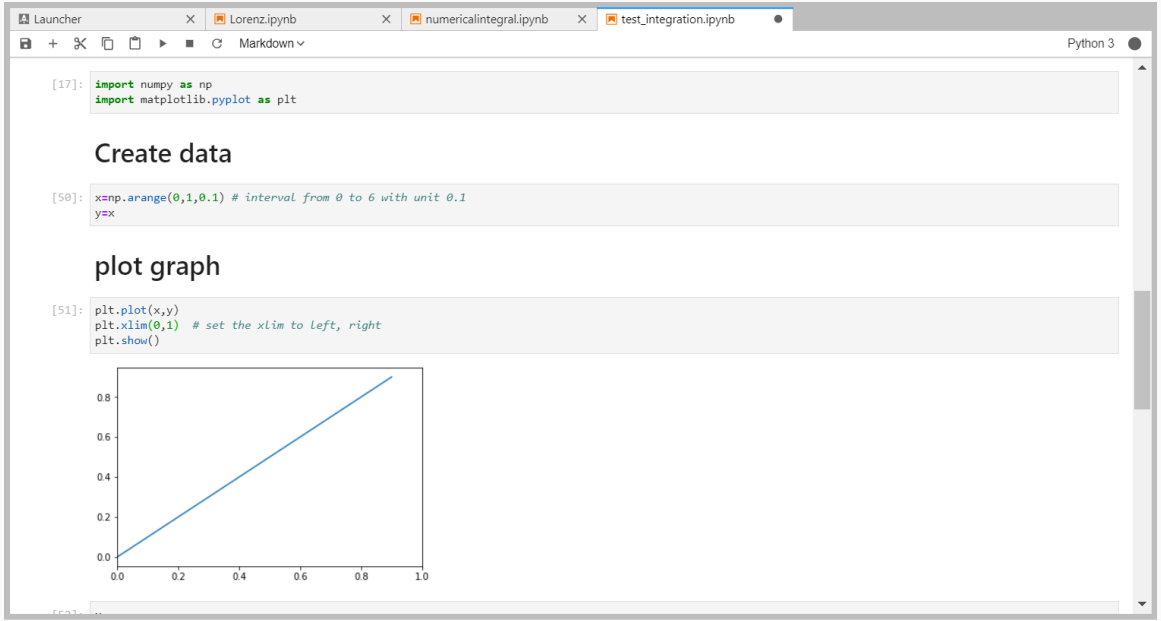
1. http://jupyter.org/try

Python programming

Plot graph curve y=f(x)=x

```
import numpy as np
import matplotlib.pyplot as plt

# Create data
x=np.arange(0,1,0.1) # interval from 0 to 6 with unit 0.1
y=x
# plot graph
plt.plot(x,y)
plt.xlim(0,1)  # set the xlim to left, right
plt.show()
```

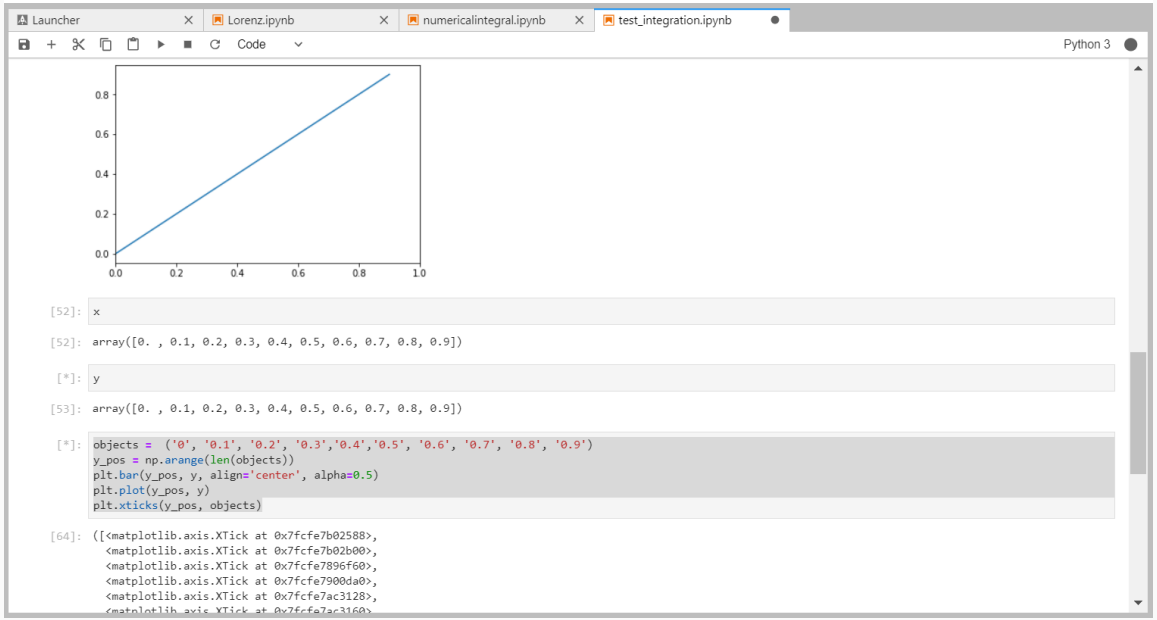


http://jupyter.org/try

程式設計 Python programming

Data
X
y

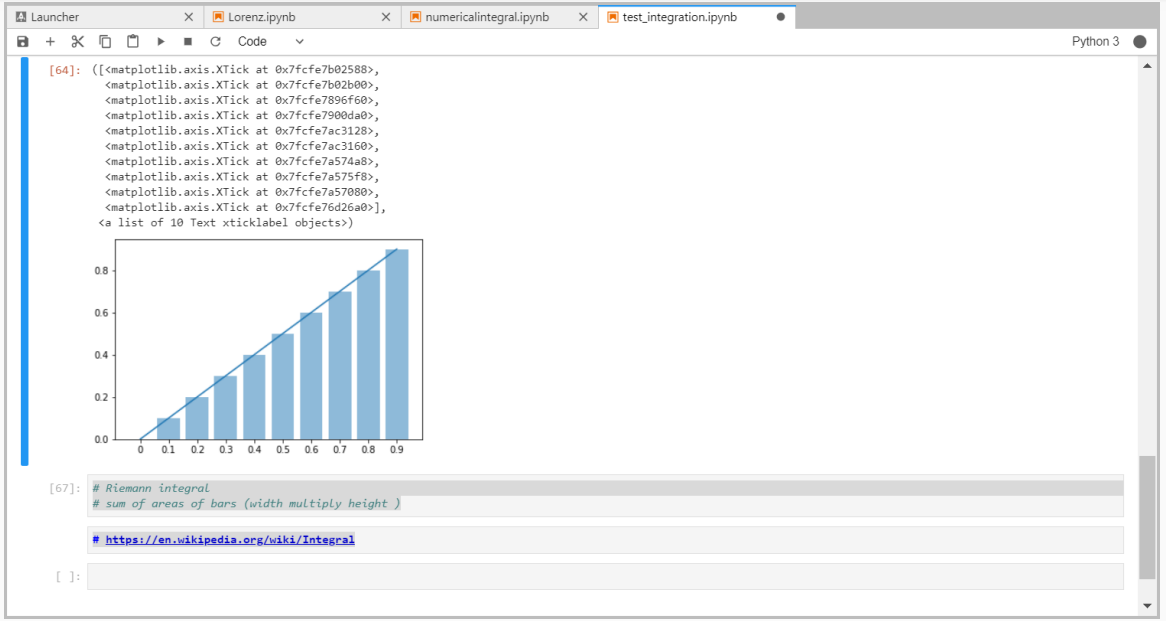
1. http://jupyter.org/try



程式設計 Python programming

Bar Plot
Riemann integral
sum of areas of bars (width multiply height)
https://en.wikipedia.org/wiki/Integral

```
objects =  ('0', '0.1', '0.2', '0.3','0.4','0.5', '0.6', '0.7', '0.8', '0.9')
y_pos = np.arange(len(objects))
plt.bar(y_pos, y, align='center', alpha=0.5)
plt.plot(y_pos, y)
plt.xticks(y_pos, objects)
```

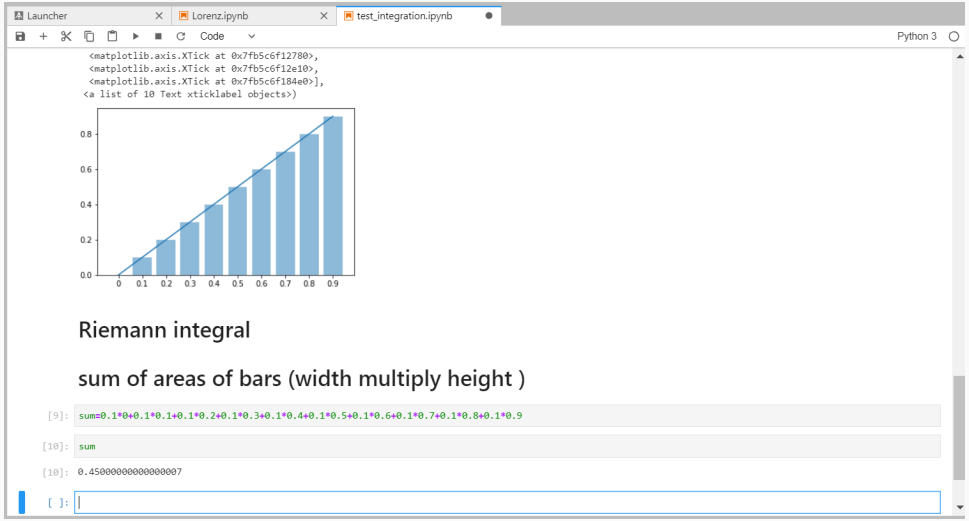


http://jupyter.org/try

Python programming

 # Riemann integral
 # sum of areas of bars (width multiply height)

 # https://en.wikipedia.org/wiki/Integral



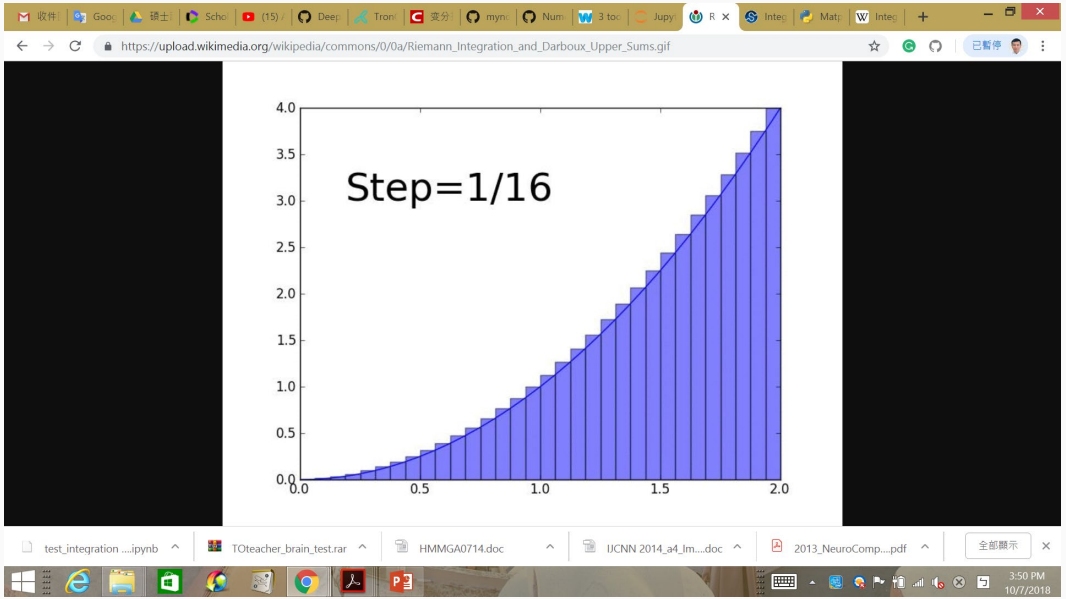
https://en.wikipedia.org/wiki/Numerical_integration

Python programming

Question:
Y=x**2 [0 1]
Summing the areas of these rectangles, we get a better approximation for the sought integral, namely

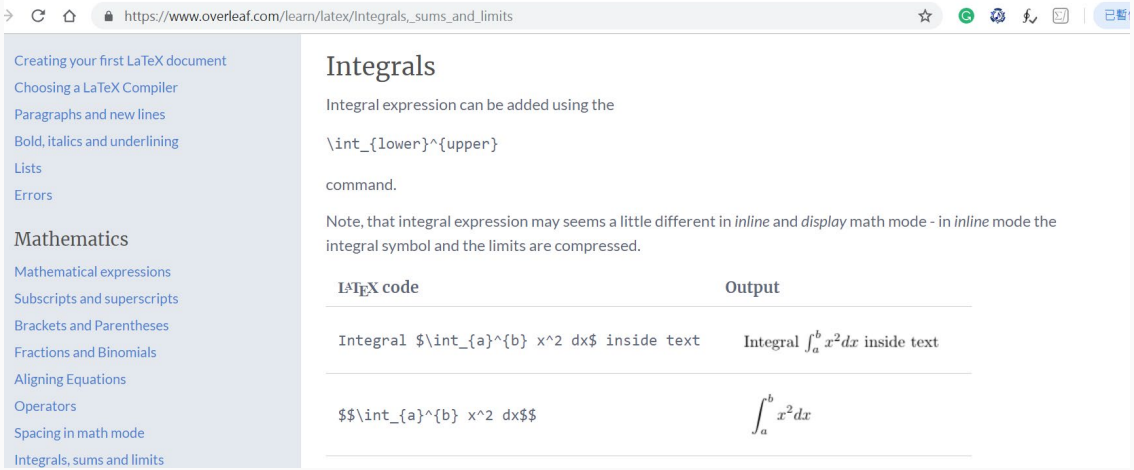
https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.quad.html

https://en.wikipedia.org/wiki/Integral



Latex programming

Learning to Latex to Integrals, sums and limits

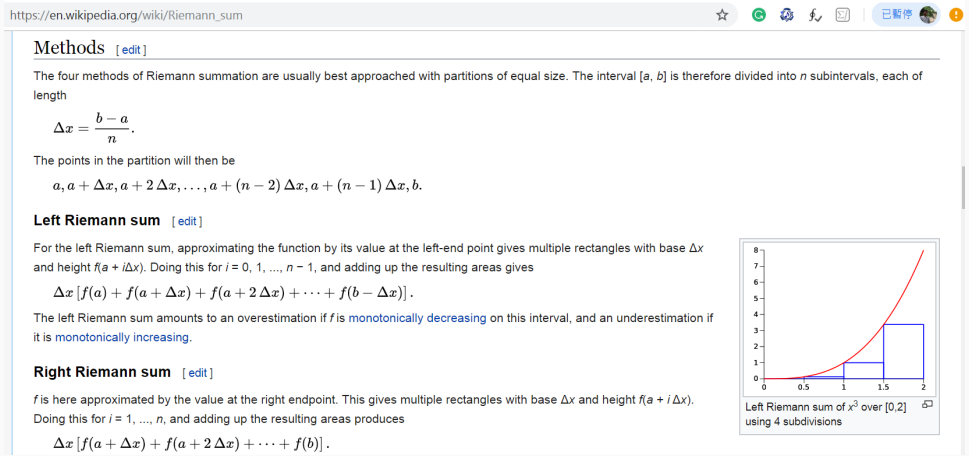


https://www.overleaf.com/learn/latex/Integrals,_sums_and_limits

Python programming

Learning to implement the algorithm on wiki

https://en.wikipedia.org/wiki/Riemann_sum



https://en.wikipedia.org/wiki/Riemann_sum

Methods [edit]

The four methods of Riemann summation are usually best approached with partitions of equal size. The interval [*a*, *b*] is therefore divided into *n* subintervals, each of length

$$\Delta x = \frac{b-a}{n}.$$

The points in the partition will then be

$$a, a+\Delta x, a+2\,\Delta x, \ldots, a+(n-2)\,\Delta x, a+(n-1)\,\Delta x, b.$$

Left Riemann sum [edit]

For the left Riemann sum, approximating the function by its value at the left-end point gives multiple rectangles with base Δx and height $f(a + i\Delta x)$. Doing this for $i = 0, 1, \ldots, n-1$, and adding up the resulting areas gives

$$\Delta x\left[f(a)+f(a+\Delta x)+f(a+2\,\Delta x)+\cdots+f(b-\Delta x)\right].$$

The left Riemann sum amounts to an overestimation if *f* is monotonically decreasing on this interval, and an underestimation if it is monotonically increasing.

Left Riemann sum of x^3 over [0,2] using 4 subdivisions

Right Riemann sum [edit]

f is here approximated by the value at the right endpoint. This gives multiple rectangles with base Δx and height $f(a + i\,\Delta x)$. Doing this for $i = 1, \ldots, n$, and adding up the resulting areas produces

$$\Delta x\left[f(a+\Delta x)+f(a+2\,\Delta x)+\cdots+f(b)\right].$$

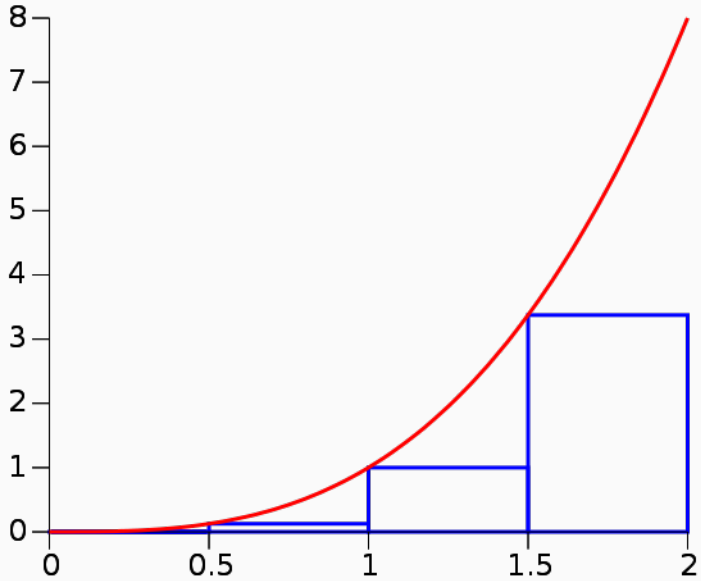
Python programming

Question:

Refer to:
https://github.com/tccnchsu/Numerical_Analysis/blob/master/NA_W4_Integration.ipynb

F(x)=x^2#

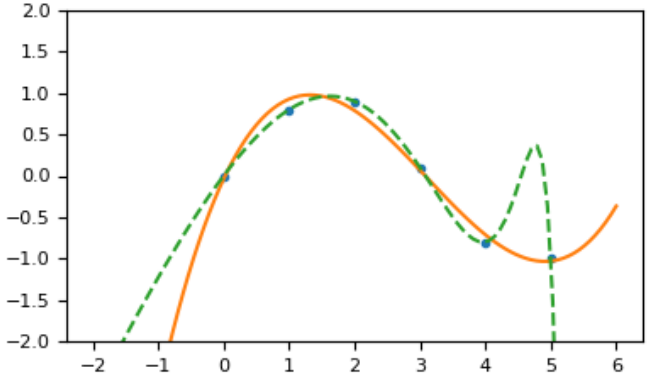
https://en.wikipedia.org/wiki/Riemann_sum



Python programming

polynomial interpolation

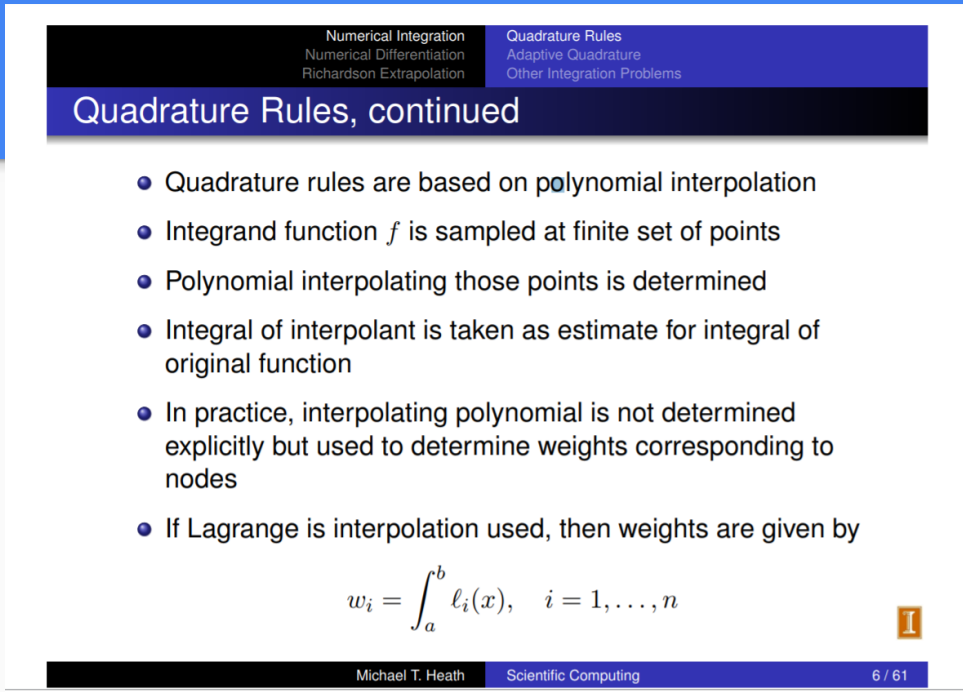
Two points <-> 2,3,4 degree polynomial
Three points <-> 2,3,4 degree polynomial
Four points <-> 2,3,4 degree polynomial



https://docs.scipy.org/doc/numpy-1.15.0/reference/generated/numpy.polyfit.html

Python programming

We will see two types of global (interpolatory) rules:
– Newton-Cotes — interpolatory on uniformly spaced nodes.
– Gauss rules — interpolatory on optimally chosen point sets.



https://relate.cs.illinois.edu/course/cs450-f17/file-version/f753d2d1a50ecbf2ca66202aa894e6d67c24efdb/lecture-notes/chap08_lec2.pdf

Python programming

5-point Gaussian quadrature rule

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_{-1}^{1} f\left(\frac{b-a}{2}t+\frac{a+b}{2}\right)\frac{b-a}{2}\,dt\\ &\approx \sum_{i=1}^{5} f\left(\frac{b-a}{2}r_{5,i}+\frac{a+b}{2}\right)\frac{b-a}{2}c_{5,i}\\ &\approx \sum_{i=1}^{5} f(x_i)w_i,\end{aligned}$$

where

$$x_i = \frac{b-a}{2}r_{5,i}+\frac{a+b}{2}, \quad w_i = \frac{b-a}{2}c_{5,i}, \quad i=1,\ldots,5.$$

In this example, $a=0$ and $b=1$, so the nodes and weights for a 5-point Gaussian quadrature rule for integrating over $[0,1]$ are given by

$$x_i = \frac{1}{2}r_{5,i}+\frac{1}{2}, \quad w_i = \frac{1}{2}c_{5,i}, \quad i=1,\ldots,5,$$

which yields

i	Nodes x_i	Weights w_i
1	0.95308992295	0.11846344250
2	0.76923465505	0.23931433525
3	0.50000000000	0.28444444444
4	0.23076534495	0.23931433525
5	0.04691007705	0.11846344250

It follows that

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &\approx \sum_{i=1}^{5} e^{-x_i^2}w_i\\ &\approx 0.11846344250e^{-0.95308992295^2}+0.23931433525e^{-0.76923465505^2}+\\ &\quad 0.28444444444e^{-0.5^2}+0.23931433525e^{-0.23076534495^2}+\\ &\quad 0.11846344250e^{-0.04691007705^2}\\ &\approx 0.74682412673352.\end{aligned}$$

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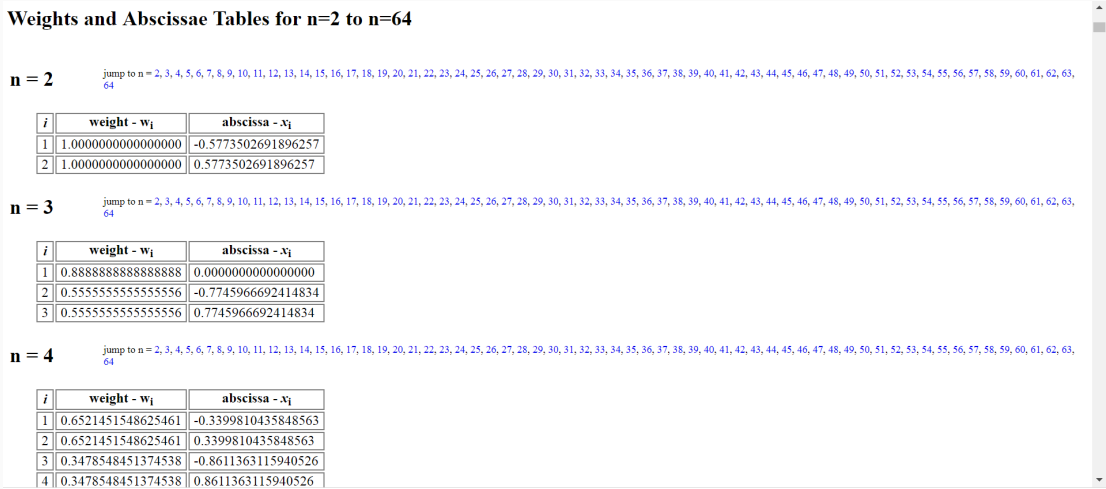
http://www.math.usm.edu/lambers/mat460/fall09/lecture31.pdf

Python programming

Gaussian Quadrature Weights and Abscissae



Weights and Abscissae Tables for n=2 to n=64

n = 2 jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	1.0000000000000000	-0.5773502691896257
2	1.0000000000000000	0.5773502691896257

n = 3 jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.8888888888888888	0.0000000000000000
2	0.5555555555555556	-0.7745966692414834
3	0.5555555555555556	0.7745966692414834

n = 4 jump to n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64

i	weight - w_i	abscissa - x_i
1	0.6521451548625461	-0.3399810435848563
2	0.6521451548625461	0.3399810435848563
3	0.3478548451374538	-0.8611363115940526
4	0.3478548451374538	0.8611363115940526

https://pomax.github.io/bezierinfo/legendre-gauss.html

Python programming

Quadrature rules based on interpolating functions

https://en.wikipedia.org/wiki/Numerical_integration

Quadrature rules based on interpolating functions [edit]

A large class of quadrature rules can be derived by constructing interpolating functions that are easy to integrate. Typically these interpolating functions are polynomials. In practice, since polynomials of very high degree tend to oscillate wildly, only polynomials of low degree are used, typically linear and quadratic.

The simplest method of this type is to let the interpolating function be a constant function (a polynomial of degree zero) that passes through the point $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$. This is called the *midpoint rule* or *rectangle rule*

$$\int_a^b f(x)\,dx \approx (b-a) f\left(\frac{a+b}{2}\right).$$

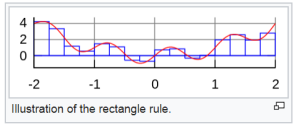


Illustration of the rectangle rule.

The interpolating function may be a straight line (an affine function, i.e. a polynomial of degree 1) passing through the points $(a, f(a))$ and $(b, f(b))$. This is called the *trapezoidal rule*

$$\int_a^b f(x)\,dx \approx (b-a)\left(\frac{f(a)+f(b)}{2}\right).$$

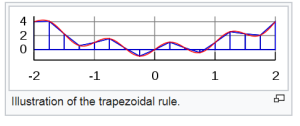


Illustration of the trapezoidal rule.

For either one of these rules, we can make a more accurate approximation by breaking up the interval $[a, b]$ into some number n of subintervals, computing an approximation for each subinterval, then adding up all the results. This is called a *composite rule*, *extended rule*, or *iterated rule*. For example, the composite trapezoidal rule can be stated as

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left(\frac{f(a)}{2} + \sum_{k=1}^{n-1}\left(f\left(a+k\frac{b-a}{n}\right)\right) + \frac{f(b)}{2}\right),$$

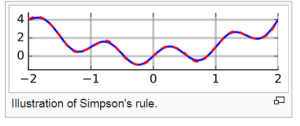


Illustration of Simpson's rule.

where the subintervals have the form $[a+kh, a+(k+1)h] \subset [a, b]$, with $h = \frac{b-a}{n}$ and $k = 0, \ldots, n-1$. Here we used subintervals of the same length h but one could also use intervals of varying length $(h_k)_k$.

https://en.wikipedia.org/wiki/Numerical_integration