Instructions

- *The duration of the exam is 2 hours. You can refer to the prescribed textbook.*
- *To get full credit, you should justify your answers with valid arguments. It is not necessary to simplify expressions involving permutations P, combinations C or factorials - for instance, writing $7P_2$ suffices.*
- *No doubts/clarifications would be entertained during the examination. Make appropriate assumptions.*

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1. In how many ways can 5 drinks be chosen from a refrigerator containing 8 different types of drinks. Multiple units of the same drink can be chosen. Assume that the refrigerator contains at least 5 drinks of each type. **(2 marks)**

2. How many ways are there to travel in $xyzw$ space from $(0, 0, 0, 0)$ to $(5, 3, 2, 6)$ by taking one unit in the positive x, positive y, positive z or positive w direction? Moving in the negative x, y, z or w direction is not allowed. **(2 marks)**

3. What is the coefficient of x^4y^5 in $(2x - y)^9 + (x + 4y)^9$? **(2 marks)**

4. Ten identical tokens have to be distributed among four (different) persons such that each person gets at least one token. In how many ways can this be done? **(2 marks)**

5. How many ways are there to distribute 15 distinguishable objects into 5 distinguishable boxes when:
   (a) there is no restriction on the number of objects in a box. A box could even be empty.
   (b) the boxes have $1, 2, 3, 4, 5$ objects in them respectively. **(4 marks)**

6. (a) Give a sequence of 10 positive integers (where repetition of numbers is not allowed) that contains no increasing subsequence of 4 terms and no decreasing subsequence of 6 terms.
   (b) What is the minimum length of a sequence of positive integers (again, where repetition of numbers is not allowed) that guarantees the presence of either an increasing subsequence of 4 terms or a decreasing subsequence of 6 terms? **(4 marks)**

7. Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r and k are nonnegative integers with $r \leq n$ and $k \leq r$ using a combinatorial argument. **(4 marks)**

8. In an exam containing 2 problems, 70% of them solved the first problem and 80% solved the second problem. Show that at least 50% of them solved both the problems. **(5 marks)**

9. Let D_n denote the number of derangements of n distinct objects. Prove the equation: **(5 marks)**

$$D_n = n! - \binom{n}{1}D_{n-1} - \binom{n}{2}D_{n-2} - \cdots - \binom{n}{n-1}D_1 - 1$$

10. There are 200 pigeons placed in 101 pigeonholes such that each pigeonhole contains at least one pigeon. Show that there is some subset of pigeonholes containing exactly 100 pigeons. **(5 marks)**

    Hint: You can get some idea from Example 10 in Section 6.2 of the book. You can also choose to do it in a different way if you wish.