Instructions

- *The duration of the exam is 1 hour.*

- *To get full credit, you should justify your answers with valid arguments.*

- *No doubts/clarifications would be entertained during the examination. Make appropriate assumptions.*

- *If you are caught copying, you would directly get a fail grade.*

Part A ($5 \times 2 = 10$ marks)

1. Let C_n be the number of strings w consisting of n X's and n Y's such that no initial segment of w has more Y's than X's. Now consider the following problem. A person stands on the edge of a swimming pool holding a bag of n red and n blue balls. He draws a ball out one at a time and discards it. If he draws a blue ball, he takes one step back, if he draws a red ball, he moves one step forward. What is the probability that the person remains dry?

   (a) $\dfrac{C_n}{2^{2n}}$ (b) $\dfrac{C_n}{\binom{2n}{n}}$ (c) $\dfrac{n \cdot C_n}{(2n)!}$ (d) $\dfrac{n \cdot C_n}{\binom{2n}{n}}$

2. How many paths are there in the plane from $(0,0)$ to $(m,n) \in \mathbb{N} \times \mathbb{N}$, if the possible steps from (i,j) are either $(i+1,j)$ or $(i,j+1)$?

   (a) $\binom{2m}{n}$ (b) $\binom{m}{n}$ (c) $\binom{m+n}{n}$ (d) m^n

3. For every integer n, there exist $i, j \geq 0$ with $i \neq j$ such that

   (a) $2^i - 2^j$ is a multiple of n

   (b) $2^i + 2^j$ is a multiple of n

   (c) $2^i \cdot 2^j$ is a multiple of n

   (d) none of the above

4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them. What is the minimum number of balls to be selected so that she is guaranteed to get 3 blue balls?

   (a) 4   (b) 7   (c) 13   (d) none of the above

5. Four siblings go shopping with their father. If Abhay gets shoes, then Asha does not get a necklace. If Arun gets a T-shirt, then Aditi gets bangles. If Abhay does not get shoes or Aditi gets bangles, the mother will be happy. Which of the following is true?

   (a) If the mother is happy, then Aditi got bangles.

   (b) If Aditi got bangles, then Abhay got shoes.

   (c) If the mother is not happy, then Asha did not get a necklace and Arun did not get a T-shirt.

   (d) None of the above.

Part B (30 marks)

6. Let A be a set of n elements. What is the total number of distinct functions from A to A? How many of them are onto?

7. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all the five balls. In how many ways can we place the balls so that no box remains empty?

8. (i) In how many ways can a pack of 52 cards be divided equally amongst four players in order.

   (ii) In how many ways can you divide these cards in four sets, three of them having 17 cards each and a the fourth just 1 card?

9. Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

10. Prove the validity of the following sequents:

   (a) $p \rightarrow q \vdash \neg q \rightarrow \neg p$

   (b) $p \rightarrow q, r \rightarrow s \vdash p \wedge r \rightarrow q \wedge s$

   (c) $\vdash \neg p \vee q \rightarrow (p \rightarrow q)$