CSCI 6968 Weekly Participation 4

Shiuli Subhra Ghosh

February 17, 2023

1 Question

Consider the ordinary least square problem with L-1 norm.

$$\arg\min_{x\in\mathbb{R}} \frac{1}{2}(x-a)^2 + \lambda \mid x \mid$$

1. Argue that this is a convex optimization problem and has a unique solution given any a.
Solution:
We know that x^2 is a strictly convex function. $x - a$ is an affine function which is convex in nature, but $(x-a)^2$ is the composition of a strictly convex function and a convex function. So, it is also strictly convex. $|X|$ is a convex function. The sum of two convex functions is convex. Also, we can say this is strictly convex as we have a quadratic function. Alternatively, if we find the hessian of the function,

$$f(x) = \frac{1}{2}(x-a)^2 + \lambda \mid x \mid$$
$$\nabla^2 f(x) = 1 > 0$$

Hence, As the loss function is strictly convex, it can have a unique solution.
2. Let $S_\lambda(a)$ be the unique solution to this optimization problem, given an a. State Fermat's optimality condition as concisely as you can, using our rules for subdifferential manipulations.
Solution:

First, computing the subgradient of the loss function.

$$\partial f(x) = \begin{cases} \{x-a+\lambda\} & x > 0 \\ \{x-a-\lambda\} & x < 0 \\ [x-a-\lambda, x-a+\lambda] & x = 0 \end{cases}$$

So, according to Fermat's optimality condition, $x^* \in \arg\min_{x\in\mathbb{R}} f(x)$ iff $0 \in \partial f(x^*)$.

$$0 \in \partial f(x^*) = \begin{cases} \{x^*-a+\lambda\} & x > 0 \\ \{x^*-a-\lambda\} & x < 0 \\ [x^*-a-\lambda, x^*-a+\lambda] & x = 0 \end{cases}$$

3. Use Fermat's optimality condition to find an expression for $s_\lambda(a)$, and draw a plot of s_λ.
Solution:

$$s_\lambda(a) = \begin{cases} a - \lambda & x > \lambda \\ a + \lambda & x < -\lambda \\ 0 & -\lambda \leq x \leq \lambda \end{cases}$$