PGM Homework- 3

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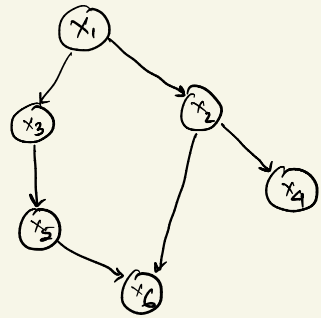
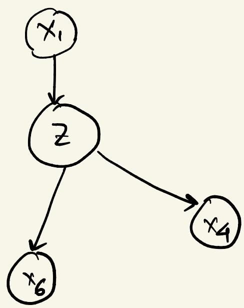


In order to apply Belief propagation method in simple BN, we combine node x_3, x_2 and x_5 in node Z.



need to find

So, $P(X_1=1 \mid X_4=1)$

So, $P(x_1, Z, x_6, x_4) = P(Z|x_1) \cdot P(x_6|Z) \cdot P(x_4|Z)$

So, forming the new CPT Table $P(x_2, x_3, x_5, x_1)$

$$= P(x_1)\, P(x_2|x_1)\, P(x_3|x_1)\, P(x_5|x_3)$$

$$P(z_1|x_1) = \frac{P(x_2, x_3, x_5, x_1)}{P(x_1)} = \frac{P(x_1)\, P(x_2|x_1)\, P(x_3|x_1)\, P(x_5|x_3)}{P(x_1)}$$

State	x_2	x_3	x_5	x_1	$P(Z\|x_1)$
1	0	0	0	0	0.133
2	0	0	1	0	0.007
3	0	1	0	0	0.42
4	0	1	1	0	0.14
5	1	0	0	0	0.057
6	1	0	1	0	0.003
7	1	1	0	0	0.18
8	1	1	1	0	0.06

x_1	$P(x_1)$
0	0.8
1	0.2

x_2	x_1	$P(x_2\|x_1)$
0	0	0.7
0	1	0.97
1	0	0.3
1	1	0.03

x_4	x_2	$P(x_4\|x_2)$
0	0	0.98
0	1	0.4
1	0	0.02
1	1	0.6

$$P(Z|x_1) = \frac{P(x_1)\, P(x_2|x_1)\, P(x_3|x_1)\, P(x_5|x_3)}{P(x_1)}$$

$$P(x_6|Z) = P(x_6|x_5, x_2)$$

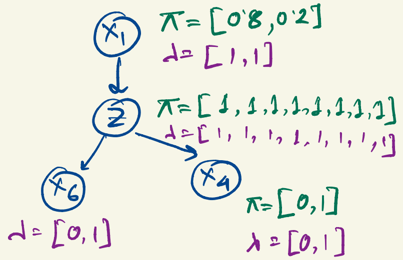
$$P(x_4|Z) = P(x_4|x_2)$$

x_5	x_3	$P(x_5\|x_3)$
0	0	0.95
0	1	0.75
1	0	0.05
1	1	0.25

x_6	x_2	x_5	$P(x_6\|x_2, x_5)$
0	0	0	0.95
0	0	1	0.5
0	1	0	0.9
0	1	1	0.3
1	0	0	0.05
1	0	1	0.5
1	1	0	0.1
1	1	1	0.7

x_3	x_1	$P(x_3\|x_1)$
0	0	0.2
0	1	0.6
1	0	0.8
1	1	0.4

Now we will find the $P[x_1=1 \mid x_4=1]$ using belief propagation.



initialization

One important observation. — Since x_5, x_6, x_2 form a V structure and the value of the node x_6 is not given, so when $x_4=1$ is given, message cannot reach to x_1 through the node x_6 Because the path $x_3 \rightarrow x_5 \rightarrow x_6 \leftarrow x_2$ is blocked. And hence the belief of the node x_1 will not change after the first iteration as the msg update will be following this path $x_4 \rightarrow x_2 \rightarrow x_1$. So, we only do the first iteration here in this case.

First Iteration :-

$\boxed{\text{Node } Z}$

① computing the π message from each of it's parents in our case its x_1.

$$\text{So,} \quad \pi_{x_1}(z) = \pi_{x_1} = [0.8, 0.2]$$

② Computing total parent msg for Z.

$$\text{So,} \quad \pi(z) = \sum_{x_1} P[z \mid x_1]\, \pi_{x_1} = P[z, x_1=0] + P[z, x_1=1]$$

③ compute λ msg from children of z $\{x_4, x_6\}$

$$\lambda_{x_4}(z) = \sum_{x_4} \lambda(x_4)\, P[x_4 \mid z] = \sum_{x_4} [0,1]\, P[x_4 \mid z] = P[x_4=1 \mid z]$$

$$\lambda_{x_6}(z) = \sum_{x_6} \lambda(x_6)\, P[x_6 \mid z] = P[x_6=0 \mid z] + P[x_6=1 \mid z] = 1$$

④ compute total λ msg from $\{x_4, x_6\}$

$$\lambda(z) = \lambda_{x_4}(z) \cdot \lambda_{x_6}(z) = P[x_4=1 \mid z]$$

Node x_1

① computing the π message from each of it's parents in our case its { }

So, $\pi(x_1) = \pi(x_1) = [0.8, 0.2]$ (prior)

② computing total parent msg for x_1

So, $\pi(x_1) = \pi(x_1) = [0.8, 0.2]$

③ compute λ msg from children of x_1 is Z

$$\lambda_Z(x_1) = \sum_Z \lambda(Z)\, P[Z \mid x_1] = \sum_Z P[x_4 = 1 \mid Z]\, P[Z \mid x_1]$$

④ compute total λ msg from Z

$$\lambda(x_1) = \sum_Z P[x_4 = 1 \mid Z]\, P[Z \mid x_1]$$

$x_1 = 0$

State	x_2	x_3	x_5	x_1	$P(Z \mid x_1)$
1	0	0	0	0	0.133
2	0	0	1	0	0.007
3	0	1	0	0	0.42
4	0	1	1	0	0.14
5	1	0	0	0	0.057
6	1	0	1	0	0.003
7	1	1	0	0	0.18
8	1	1	1	0	0.06

$x_1 = 1$

State	x_2	x_3	x_5	x_1	$P(Z \mid x_1)$
1	0	0	0	0	0.5529
2	0	0	1	0	0.0291
3	0	1	0	0	0.291
4	0	1	1	0	0.097
5	1	0	0	0	0.0171
6	1	0	1	0	0.0009
7	1	1	0	0	0.009
8	1	1	1	0	0.003

$$\pi(Z) = 0.8 \begin{bmatrix} 0.133 \\ 0.007 \\ 0.42 \\ 0.19 \\ 0.057 \\ 0.003 \\ 0.18 \\ 0.06 \end{bmatrix} + 0.2 \begin{bmatrix} 0.5529 \\ 0.0291 \\ 0.291 \\ 0.097 \\ 0.0171 \\ 0.0009 \\ 0.009 \\ 0.003 \end{bmatrix} = \begin{bmatrix} 0.21698 \\ 0.01142 \\ 0.3942 \\ 0.1314 \\ 0.04902 \\ 0.00258 \\ 0.1458 \\ 0.0486 \end{bmatrix}$$

$$\lambda(Z) = P[x_4 = 1 \mid Z] = P[x_4 = 1 \mid x_2] = \begin{bmatrix} 0.02 \\ 0.02 \\ 0.02 \\ 0.02 \\ 0.6 \\ 0.6 \\ 0.6 \\ 0.6 \end{bmatrix}$$

P.T.O

For node x_1.

$$\pi(x_1) = \pi(x_1) = [0.8, 0.2]$$

$$\lambda(x_1) = \sum_z P[x_4=1 \mid z]\, P[z \mid x_1]$$

$$= P[x_4=1 \mid z_1]\, P[z_1 \mid x_1] + P[x_4=1 \mid z_2]\, P[z_2 \mid x_1]$$
$$P[x_4=1 \mid z_3]\, P[z_3 \mid x_1] + P[x_4=1 \mid z_4]\, P[z_4 \mid x_1]$$
$$P[x_4=1 \mid z_5]\, P[z_5 \mid x_1] + P[x_4=1 \mid z_6]\, P[z_6 \mid x_1]$$
$$P[x_4=1 \mid z_7]\, P[z_7 \mid x_1] + P[x_4=1 \mid z_8]\, P[z_8 \mid x_1]$$

$$= 0.02\left[P[z_1 \mid x_1] + P[z_2 \mid x_1] + P[z_3 \mid x_1] + P[z_4 \mid x_1]\right] +$$
$$0.6\left[P[z_5 \mid x_1] + P[z_6 \mid x_1] + P[z_7 \mid x_1] + P[z_8 \mid x_1]\right]$$

$$x_1 = 0 \quad = 0.02 \times 0.7 + 0.6 \times 0.3 = 0.194$$

$$x_1 = 1 \quad = 0.02 \times 0.97 + 0.6 \times 0.03 = 0.0374$$

$$\lambda(x_1) = [0.194, 0.0374]$$

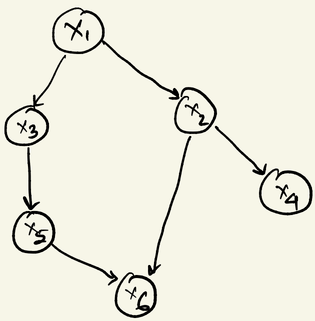
So, Belief of $(x_1) = \pi(x_1).\,\lambda(x_1) = [0.8, 0.2] .* [0.194, 0.0374]$

$$= [0.1552, 0.00748]$$

normalized Belief of x_1 = $[0.95402016, 0.04597984]$

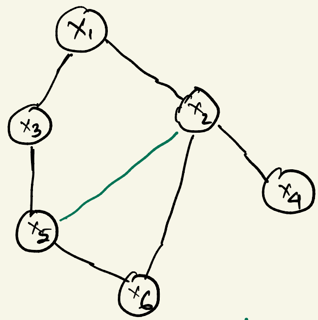
Hence $\boxed{P[x_1=1 \mid x_4=1] = 0.0459}$

Junction Tree Method

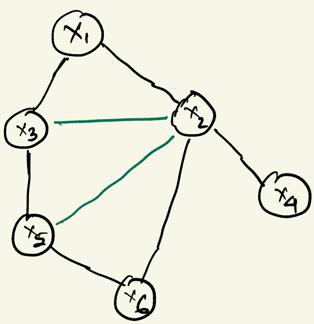


Given this BN, we will first generate the moral graph.

① Moral Graph



② Now we will triangularize the moral graph.



③ Now we will identify the maximum cliques.

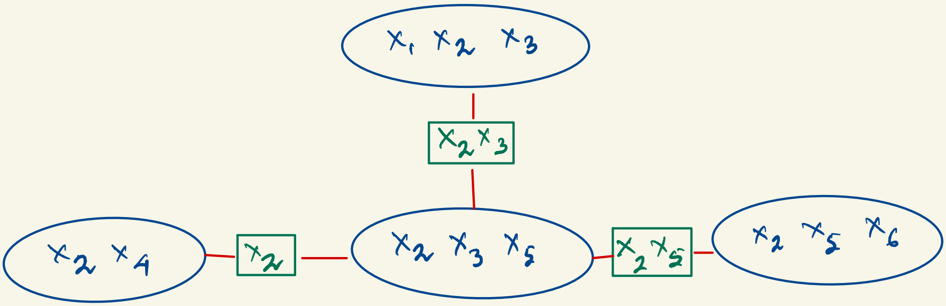
$\{x_1, x_2, x_3\}$

$\{x_3, x_2, x_5\}$

$\{x_5, x_2, x_6\}$

$\{x_2, x_4\}$

④ Now we will form the junction tree.



⑤ Now we will parameterize the cluster nodes

$$\phi_{123} = P(x_1)\ P(x_3|x_1)\ P(x_2|x_1)$$

here the evidence is $x_4 = 1$.

$$\phi_{235} = P(x_5|x_3)$$

$$\phi_{256} = P(x_6|x_5, x_2)$$

$$\phi_{24} = P(x_4|x_2)$$

$$P(x_1, x_2, x_3, x_4, x_5, x_6) = \phi_{123}\ \phi_{235}\ \phi_{256}\ \phi_{24}$$

⑥ Now initializing the potential for the separators.

$$\phi(S_{23}) = 1$$

$$\phi(S_{25}) = 1$$

$$\phi(S_2) = 1$$

⑦ Belief / potential updating for node (x_2, x_3, x_5)

$\boxed{7.1}$ Obtain the initial potential for each cluster node

$$\phi_{123} = P(x_1)\ P(x_3|x_1)\ P(x_2|x_1)$$

$$\phi_{235} = P(x_5|x_3)$$

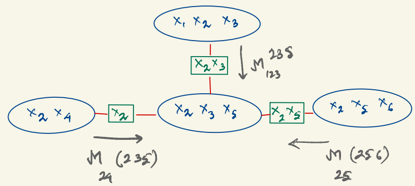
$$\phi_{256} = P(x_6|x_5, x_2)$$

$$\phi_{24} = P(x_4=1|x_2)$$

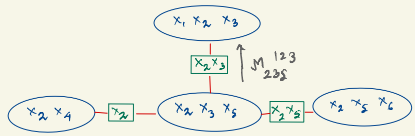
$$\phi_{23} = 1$$

$$\phi_{25} = 1$$

$$\phi_{2} = 1$$



then we update the belief for $x_2 x_3 x_5$.
and then again



Finally we update the belief for x_1, x_2, x_3

$\boxed{7.2}$ Message calculation

$$M_{24}(235) = \frac{\phi^*_2}{\phi_2} = \frac{\sum_{x_4=1} \phi_{24}}{\phi_2} = \frac{P[X_4=1 \mid X_2]}{1}$$

$$= [0'02, 0'6]$$

$$M_{256}(235) = \frac{\phi^*_{25}}{\phi_{25}} = \frac{\sum_{x_6} \phi_{256}}{\phi_{25}} = 1$$

$$\mu^{235}_{123} = \frac{\phi^*(23)}{\phi_{23}} = \frac{\sum_{x_1} \phi_{123}}{\phi_{23}} = \frac{[0.2284, 0.5256, 0.0516, 0.1944]}{1}$$

$P(x_1)$ $P(x_3|x_1)$ $P(x_2|x_1)$

x_1	x_2	x_3	$P(x_1,x_2,x_3)$
0	0	0	$0.8 \times 0.7 \times 0.2 = 0.112$
0	0	1	$0.8 \times 0.7 \times 0.8 = 0.448$
0	1	0	$0.8 \times 0.3 \times 0.2 = 0.048$
0	1	1	$0.8 \times 0.3 \times 0.8 = 0.192$
1	0	0	$0.2 \times 0.97 \times 0.6 = 0.1164$
1	0	1	$0.2 \times 0.97 \times 0.4 = 0.0776$
1	1	0	$0.2 \times 0.03 \times 0.6 = 0.0036$
1	1	1	$0.2 \times 0.03 \times 0.4 = 0.0024$

So, $\boxed{7.3}$ Potential Updating

$$\phi^*_{235} = \phi_{235}\, \mu^{235}_{24}\, \mu^{235}_{256}\, \mu^{235}_{123}$$

$$= \phi_{235} \begin{array}{c} \begin{bmatrix} 0.02 & 0.6 \end{bmatrix} \\ x_2 = 0 \quad\quad 1 \end{array} \begin{array}{c} [0.2284, 0.5256, 0.0516, 0.1944] \\ x_2x_3 \quad 00 \quad\quad 01 \quad\quad 10 \quad\quad 11 \end{array}$$

$P[5|3]$

5	3	2	$P(5\|2,3)$	ϕ^*_{235} d
0	0	0	$0.95 \times 0.2284 \times 0.02$	$= 0.00433$
0	1	0	$0.75 \times 0.5256 \times 0.02$	$= 0.007884$
1	0	0	$0.05 \times 0.2284 \times 0.02$	$= 0.0002284$
1	1	0	$0.25 \times 0.5256 \times 0.02$	$= 0.002627$
0	0	1	$0.95 \times 0.0516 \times 0.6$	$= 0.029412$
0	1	1	$0.75 \times 0.1944 \times 0.6$	$= 0.08747$
1	0	1	$0.05 \times 0.0516 \times 0.6$	$= 0.00154$
1	1	1	$0.25 \times 0.1944 \times 0.6$	$= 0.02916$

8. Potential updating for node x_1, x_2, x_3.

message passing from x_2, x_3, x_5 to x_1, x_2, x_3

$$M_{235}^{123} = \frac{\phi^*_{23}}{\phi_{23}} = \frac{\sum_{x_5} \phi_{235}}{\phi_{23}}$$

$$= \frac{[0.0045584,\ 0.30952,\ 0.01051,\ 0.1166]}{[0.2284,\ 0.5256,\ 0.0516,\ 0.1944]}$$

$$= [0.02,\ 0.6,\ 0.02,\ 0.6]$$

So, final potential update for x_1, x_2, x_3

x_1	x_2	x_3	$P(x_1,x_2,x_3) \times M^*_{23}$
0	0	0	$0.8 \times 0.7 \times 0.2 = 0.112 \times 0.02 = 0.00224$
0	0	1	$0.8 \times 0.7 \times 0.8 = 0.448 \times 0.6 = 0.0288$
0	1	0	$0.8 \times 0.3 \times 0.2 = 0.048 \times 0.02 = 0.00896$
0	1	1	$0.8 \times 0.3 \times 0.8 = 0.192 \times 0.06 = 0.1152$
1	0	0	$0.2 \times 0.97 \times 0.6 = 0.1164 \times 0.02 = 0.002328$
1	0	1	$0.2 \times 0.97 \times 0.4 = 0.0776 \times 0.6 = 0.00216$
1	1	0	$0.2 \times 0.03 \times 0.6 = 0.0036 \times 0.02 = 0.001552$
1	1	1	$0.2 \times 0.03 \times 0.4 = 0.0024 \times 0.06 = 0.00144$

$P[x_1 = 1 \mid E] = 0.0459$ (after normalization)

→ The result matches the junction tree results.