Due on 20 June 2021

ECSE 6810 Assignment 2

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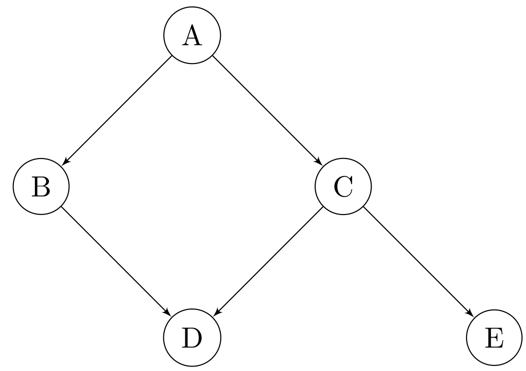
1. Given The BN below,



All the nodes are binary and their CPTs are given below,

P(A = 1)	P(A = 2)
0.5	0.5

A	P(B = 1 | A)	P(B = 2 | A)
A = 1	0.5741	0.4259
A=2	0.1379	0.8621

A	P(C = 1 | A)	P(C = 2 | A)
A = 1	0.8321	0.1679
A=2	0.0346	0.9654

C	P(E = 1 | C)	P(E = 2 | C)
C = 1	0.0496	0.9504
C=2	0.8243	0.1757

B C	P(D = 1 | B,C)	P(D = 2 | B,C)
1 1	0.8	0.2
2 1	0.3	0.7
1 2	0.6	0.4
2 2	0.1	0.9

(a)

$$P(A \mid B=1, C=2, D=1, E=2) = \frac{P(A, B=1, C=2, D=1, E=2)}{P(B=1, C=2, D=1, E=2)}$$

$$= \frac{P(A=1)P(B=1 \mid A=1)P(C=2 \mid A=1)P(D=1 \mid B=1, C=2)P(E=2 \mid C=2)}{\sum_a P(A=a, B=1, C=2, D=1, E=2)}$$

$$\begin{aligned}
P(A=1) &= 0.5 \\
P(B=1 \mid A=1) &= 0.5741 \\
P(C=2 \mid A=1) &= 0.1679 \\
P(D=1 \mid B=1, C=2) &= 0.6 \\
P(E=2 \mid C=2) &= 0.1757
\end{aligned}$$

Also,

$$\begin{aligned}
P(A=2) &= 0.5 \\
P(B=1 \mid A=2) &= 0.1379 \\
P(C=2 \mid A=2) &= 0.9654 \\
P(D=1 \mid B=1, C=2) &= 0.6 \\
P(E=2 \mid C=2) &= 0.1757
\end{aligned}$$

$P(A=1 \mid B=1, C=2, D=1, E=2)$

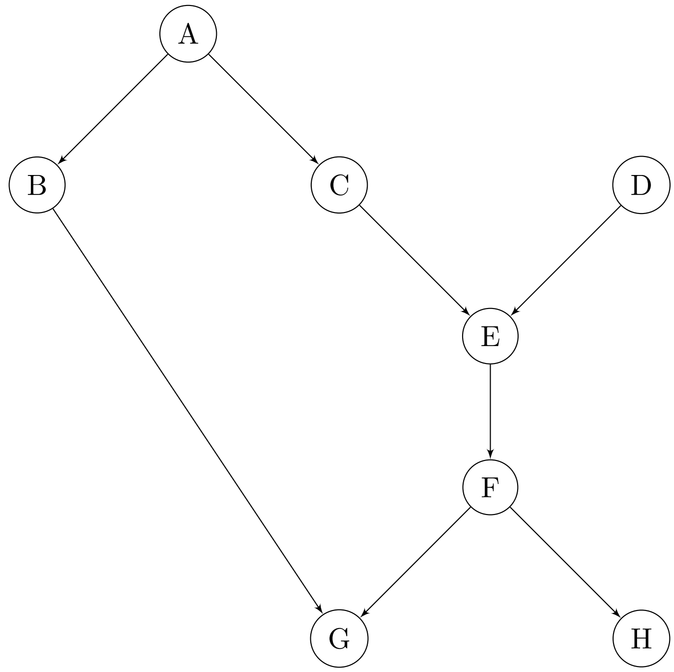
$$\begin{aligned}
&= \frac{0.5 \times 0.5741 \times 0.1679 \times 0.6 \times 0.1757}{(0.5 \times 0.5741 \times 0.1679 \times 0.6 \times 0.1757) + (0.5 \times 0.1379 \times 0.9654 \times 0.6 \times 0.1757)} \\
&= 0.4199
\end{aligned}$$

$$\begin{aligned}
P(A=2 \mid B=1, C=2, D=1, E=2) &= 1 - P(A=1 \mid B=1, C=2, D=1, E=2) \\
&= 1 - 0.4199 \\
&= 0.5801
\end{aligned}$$

(b)

$$\begin{aligned}
P(D=1 \mid A=1, B=2) &= \sum_c P(D=1, C=c \mid A=1, B=2) \\
&= \sum_c P(C=c \mid A=1, B=2) P(D=1 \mid A=1, B=2, C=c) \\
&= \sum_c P(C=c \mid A=1) P(D=1 \mid B=2, C=c) \\
&= 0.8231 \times 0.3 + 0.1679 \times 0.1 \\
&= 0.26372
\end{aligned}$$

2. For the BN below,



(a) Give the factorized joint probability distribution.
So, the joint probability distribution is,
$P(A, B, C, D, E, F, G, H) = P(A)P(D)P(B \mid A)P(C \mid A)P(E \mid C, D)P(F \mid E)P(G \mid F, B)P(H \mid F)$

(b) $B \perp C$ - False.
There are two paths from B to C.

First path - $B \leftarrow A \rightarrow C$. So, A is the confounder in the path and hence B is independent on C given A. But in this path, the given set is Φ. Hence the path is active.

Second path - Another path from B to C is $B \rightarrow G \leftarrow F \leftarrow E \leftarrow C$. So, there is one collider in the path that is G. So, This path is blocked given the set Φ.

(c) $B \perp C \mid A$ - True.
There are two paths from B to C.

First path - $B \leftarrow A \rightarrow C$. So, A is the confounder in the path and hence B is independent on C given A.

Second path - Another path from B to C is $B \rightarrow G \leftarrow F \leftarrow E \leftarrow C$. So, there is one collider in the path that is G. So, This path is blocked given the set Φ hence, $B \perp C \mid A$.

(d) $B \perp C \mid A, G$ - False. There are two paths from B to C.

First path - $B \leftarrow A \rightarrow C$. So, A is the confounder in the path and hence B is independent on C given A.

Second path - Another path from B to C is $B \to G \leftarrow F \leftarrow E \leftarrow C$. So, there is one collider in the path that is G. So, This path is active given the set G. Hence, according to the d-separation criteria, B and C are not deseperated because, second path from B to C is not blocked given the nodes A and G because, in this path there is a common effect G in the given set.

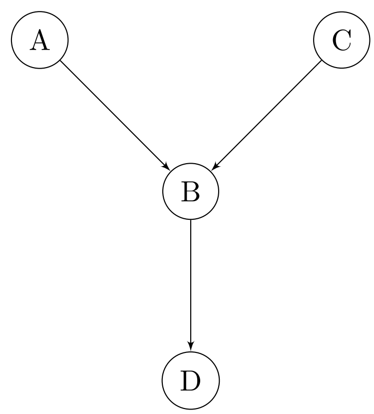
(e) $A \perp D \mid H$ - False

There is a path from A to D, $A \to C \to E \leftarrow D$. But H is a descendent of the common effect(collider) E in the path. Hence, the path is active given H.

(f) List all variables that are independent of A given evidence on G.

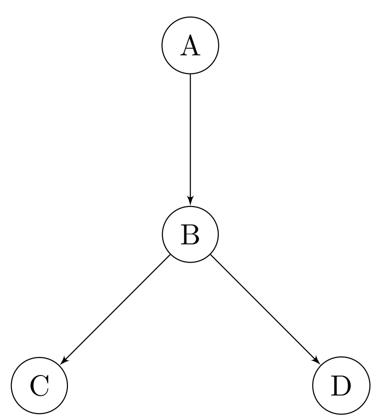
I didn't find any set. All off the variables are dependent on A given evidence on G. In any path from A to any other node G appears to be either a collider or a descendent of a collider. Hence, There are no such variables which is independent of A given evidence on G.

3. For two BNs below, consider if there are any other BNs that are I-equivalent to either of them.



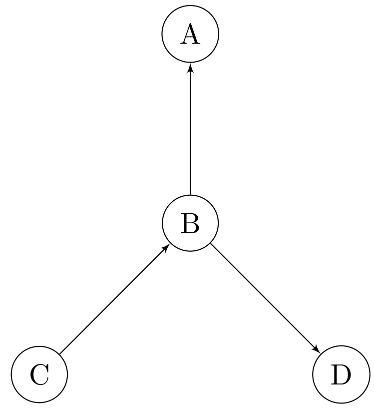
In this graph, we cannot find any I-equivalent because. there is a very specific V-structure. $A \to B \leftarrow C$. If we reverse the arrow direction in between B and D, it will create another v structure which is not desired.

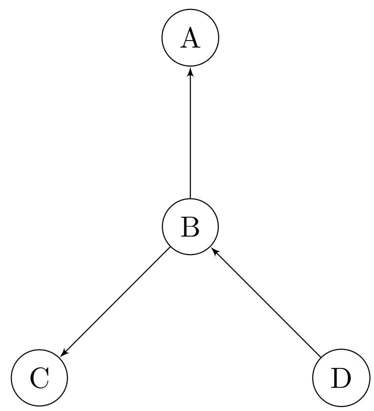
But for this image,

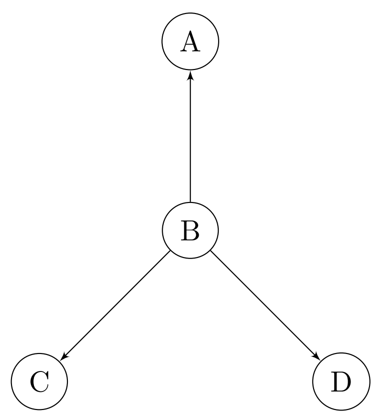


We can find another I-equivalent by changing the direction of the arrows. In this way, it won't create any extra v structure or impart ant structural change.

So, the I-equivalent is -







4. Given the joint probability distribution P(X,Y,Z) for the three binary random variables X,Y,Z as follows -

X	Y	Z	P(X,Y,Z)
0	0	0	1/12
0	0	1	1/6
0	1	0	1/6
0	1	1	1/12
1	0	0	1/6
1	0	1	1/12
1	1	0	1/12
1	1	1	1/6

(a) Determine I(P), i.e., all independences and conditional independences in P.
To approach this problem we will start trying to find marginal distributions.

X	Y	P(X,Y)
0	0	1/4
0	1	1/4
1	0	1/4
1	1	1/4

Y	Z	P(Y,Z)
0	0	1/4
0	1	1/4
1	0	1/4
1	1	1/4

X	Z	P(X,Z)
0	0	1/4
0	1	1/4
1	0	1/4
1	1	1/4

X	P(X)
0	1/2
1	1/2

Y	P(Y)
0	1/2
1	1/2

Z	P(Z)
0	1/2
1	1/2

Finding Independencies: Basically from the probability distributions, it is very clear that X,Y and Z are pairwise independent because, $P(X,Y) = P(X)P(Y)$, $P(Y,Z) = P(Y)P(Z)$, $P(Z,X) = P(Z)P(Z)$. It is evident from the probability distributions. But $P(X,Y,Z) \neq P(X)P(Y)P(Z)$.

Finding Conditional Independencies: Again we will try to find the distributions from the table. So, $P(X,Y \mid Z) = \frac{P(X,Y,Z)}{P(Z)}$, $P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$ and following this equation for all the 3 cases.

X	Y	Z	$P(X,Y \mid Z)$
0	0	0	2/12
0	0	1	2/6
0	1	0	2/6
0	1	1	2/12
1	0	0	2/6
1	0	1	2/12
1	1	0	2/12
1	1	1	2/6

X	Z	$P(X \mid Z)$
0	0	1/2
0	1	1/2
1	0	1/2
1	1	1/2

Y	Z	$P(Y \mid Z)$
0	0	1/2
0	1	1/2
1	0	1/2
1	1	1/2

X	Y	Z	$P(Y, Z \mid X)$
0	0	0	2/12
0	0	1	2/6
0	1	0	2/6
0	1	1	2/12
1	0	0	2/6
1	0	1	2/12
1	1	0	2/12
1	1	1	2/6

Y	X	$P(Y \mid X)$
0	0	1/2
0	1	1/2
1	0	1/2
1	1	1/2

Z	X	$P(Z \mid X)$
0	0	1/2
0	1	1/2
1	0	1/2
1	1	1/2

X	Y	Z	$P(Z, X \mid Y)$
0	0	0	2/12
0	0	1	2/6
0	1	0	2/6
0	1	1	2/12
1	0	0	2/6
1	0	1	2/12
1	1	0	2/12
1	1	1	2/6

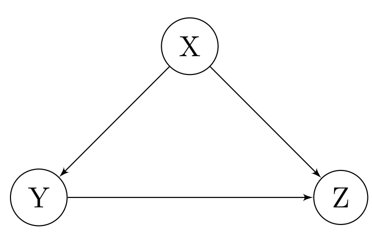
X	Y	$P(X \mid Y)$
0	0	1/2
0	1	1/2
1	0	1/2
1	1	1/2

Z	Y	$P(Z \mid Y)$
0	0	1/2
0	1	1/2
1	0	1/2
1	1	1/2

So, from these tables it is also clear that, X, Y and Z are not conditionally dependent.

(b) Based on I(P), determine a BN G, that is an I-map of P.

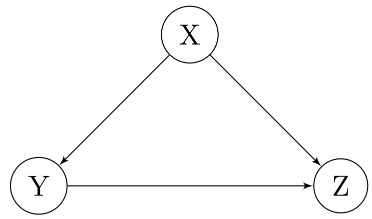
(G,P) is a BN representing P. If $I(G) \subseteq I(P)$ holds, we say G is an I-map for P which means, the structural independencies in G is a subset of statistical indenpendencies in P. So, independencies shown in a DAG should be a subset of statistical independencies. In our case $I(P) = \{X \perp Y, Y \perp Z, X \perp Z\}$ So, the I-map of P can be,



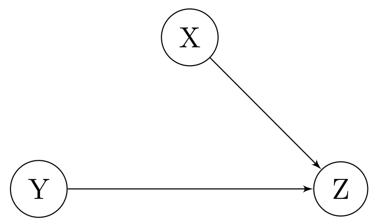
This is an I-map and there is not independencies of conditional independencies. So, $\Phi \subset I(P)$.

(c) Determine a G that is the minimum I-map of P.

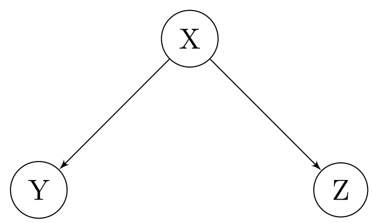
The minimal I-map of P is -



Because, if we remove the edge between X and Y, then it is sill an I-map because, as Z is the collider then $X \perp Y \mid \Phi$.

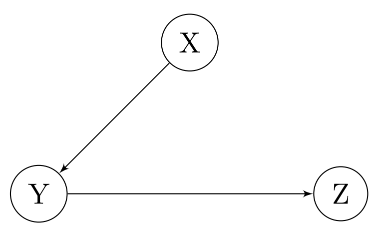


But if we remove any other edge, it will induce another conditional independencies.



Here. X is a confounder. Hence, $Y \perp Z \mid X$, which is not present in I(P).

Also for,

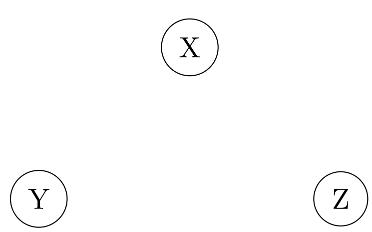


Y is in the path from X to Z. Then, $X \perp Z \mid Y,$, which is not present in I(P).

Hence, the I-map we produced earlier is the minimal I-map.

(d) Determine a G that is a perfect map of P if it exists.

A BN structure is called Perfect map for distribution P if I(G) = I(P). But in our case, there is no way, we can create a graph which has same independencies equal to P. Also for this case,



there are more additional independencies like $\{X \perp Y \mid Z, Y \perp Z \mid X, Z \perp X \mid Y\} \not\subseteq I(P)$.