ECSE 4810/6810 PGM Homework 1
Due 11:59 pm, Sept. 18, 2022 via Email

1. [20 points] Suppose we wish to calculate P(H,E1,E2), which of the following sets of numbers are sufficient for the calculation
   - P(E1,E2), P(H), P(E1|H), and P(E2|H)
   - P(E1,E2), P(H), and P(E1,E2|H)
   - P(E1|H), P(E2|H), and P(H)

   For each case, justify your response either by showing how to calculate the desired answer from the numbers given or by explaining why this is not possible.
   Suppose we know E1 and E2 are independent given H. Now which of the preceding three sets is sufficient? Justify your answer as before.

2. [20 points] Prove or disapprove each of the following independence
   a) $X \perp Y,W|Z$ implies $X \perp Y|Z$
   b) $X \perp Y|Z$ and $X,Y \perp W|Z$ implies $X \perp W|Z$
   c) $X \perp Y,W|Z$ and $Y \perp W|Z$ implies $X,W \perp Y|Z$
   d) $X \perp Y|Z$ and $X \perp Y|W$ implies $X \perp Y|Z,W$

   Note $X \perp Y,W|Z$ means X is independent Y and W given Z. $X,Y \perp W|Z$ means X and Y are independent of W given Z. Hint: use sum and chain rules.

3. [20 points] Given a binary random RV $X \in \{0,1\}$. Follow the procedure discussed in class to generate N (N=50,100,150, 200) random samples of X, with p(X=1)=0.4. Given the samples, obtain the maximum likelihood estimate (MLE) of p(X=1) by counting the number of times (n) X=1 and dividing it by N. Compute the 95% confidence interval of the estimate, using the Normal method and the exact method.

   1) Discuss the procedure used to generate the random samples.
   2) Plot the 95% interval as a function of N. What can you observe and why ?
   2) Compare the tightness (the difference between the upper and lower bounds) of the the intervals computed by the two methods. What can you observe and why ?

4. [40 points] For a Bernoulli trial involving coin toss, assume we toss the coin N times, resulting in data $D=[x_1, x_2, \ldots, x_N]$, out of which we observe head up (x=1) K times. Let θ be the probability of head up. i.e., $\theta=p(x=1)$

   1) Given D, obtain the MLE estimate (θ_{MLE}), MAP estimate (θ_{MAP}), and Bayesian estimate (θ_{Bay}) of θ respectively. Show the derivations. For MAP and Bayesian estimation, assume θ has a uniform prior with beta distribution $\alpha=\beta=1$.
   2) Compute the probability that head is up for the N+1's toss respectively using θ_{MLE}, θ_{MAP}, and θ_{Bay}. Justify your answers.
   3) Perform a Bayesian prediction of the probability that head is up for the N+1 toss, i.e., $p(x_{N+1}=1|D)$. Show the derivation details.