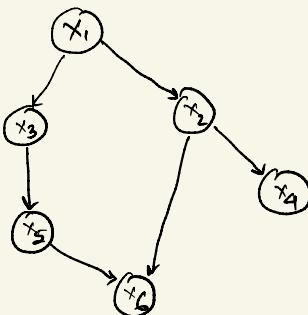
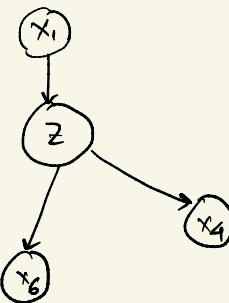


PGM Homework 3
 Shweta Subhra Ghosh



In order to apply Belief propagation method in simple BN,
 we combine node x_3 , x_2 and x_5 in node Z .



need to find
 $\text{so, } P(x_1=1 | x_4=1)$

$$\text{so, } P(x_1, z, x_6, x_4) = P(z|x_1) \cdot P(x_6|z) \cdot P(x_4|z)$$

So, forming the new CPT Table $P(x_2, x_3, x_5, x_1)$

$$= P(x_1) P(x_2|x_1) P(x_3|x_1) P(x_5|x_3)$$

$$P(z|x_1) = \frac{P(x_2, x_3, x_5, x_1)}{P(x_1)} = \frac{P(x_1) P(x_2|x_1) P(x_3|x_1) P(x_5|x_3)}{P(x_1)}$$

State	x_2	x_3	x_5	x_1	$P(z x_i)$
1	0	0	0	0	0.133
2	0	0	1	0	0.007
3	0	1	0	0	0.42
4	0	1	1	0	0.19
5	1	0	0	0	0.057
6	1	0	1	0	0.003
7	1	1	0	0	0.18
8	1	1	1	0	0.06

x_1	$P(x_1)$
0	0.8
1	0.2

x_2	x_1	$P(x_2 x_1)$
0	0	0.7
0	1	0.97
1	0	0.3
1	1	0.03

x_4	x_2	$P(x_4 x_2)$
0	0	0.98
0	1	0.4
1	0	0.02
1	1	0.6

$$P(z|x_1) = \frac{P(x_1) P(x_2|x_1) P(x_3|x_1) P(x_5|x_3)}{P(x_1)}$$

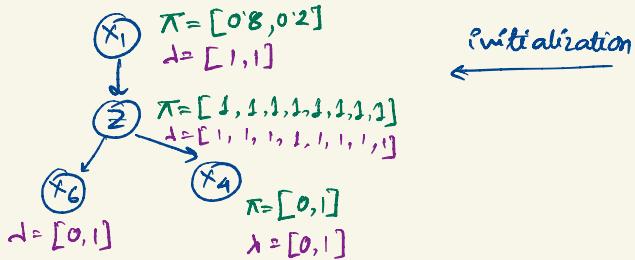
$$P(x_6|z) = P(x_6|x_5, x_2)$$

$$P(x_4|z) = P(x_4|x_2)$$

x_6	x_2	x_5	$P(x_6 x_2, x_5)$
0	0	0	0.95
0	0	1	0.5
0	1	0	0.9
0	1	1	0.3
1	0	0	0.05
1	0	1	0.5
1	1	0	0.1
1	1	1	0.7

x_3	x_1	$P(x_3 x_1)$
0	0	0.2
0	1	0.6
1	0	0.8
1	1	0.4

Now we will find the $P[x_1=1 | x_4=1]$ using belief propagation.



one important observation. — Since x_5, x_6, x_2 form a V structure and the value of the node x_5 is not given, so when $x_4=1$ is given, message cannot reach to x_1 through the node x_5 . Because the path $x_3 \rightarrow x_5 \rightarrow x_6 \leftarrow x_2$ is blocked. And hence the belief of the node x_1 will not change after the first iteration as the msg update will be following this path $x_4 \rightarrow x_2 \rightarrow x_1$. So, we only do the first iteration here in this case.

First Iteration :-

Node Z

- ① computing the π message from each of its parents in our case it's x_1 .

$$\text{So, } \pi_{x_1}(z) = \pi_{x_1} = [0.8, 0.2]$$

- ② Computing total parent msg for Z .

$$\text{So, } \pi(z) = \sum_{x_1} P[z|x_1] \pi_{x_1} = P[z, x_1=0] + P[z, x_1=1]$$

- ③ Compute λ msg from children of Z $\{x_4, x_6\}$

$$\lambda_{x_4}(z) = \sum_{x_4} \lambda(x_4) P[x_4|z] = \sum_{x_4} [0, 1] P[x_4|z] = P[x_4=1|z]$$

$$\lambda_{x_6}(z) = \sum_{x_6} \lambda(x_6) P[x_6|z] = P[x_6=0|z] + P[x_6=1|z] = 1$$

- ④ Compute total λ msg from $\{x_4, x_6\}$

$$\lambda(z) = \lambda_{x_4}(z) \cdot \lambda_{x_6}(z) = P[x_4=1|z]$$

Node x_1

- ① computing the π message from each of its parents
in our case it's {}

$$\text{So, } \pi(x_1) = \pi(x_i) = [0.8, 0.2] \quad (\text{prior})$$

- ② Computing total parent msg for x_1

$$\text{So, } \pi(x_1) = \pi(x_i) = [0.8, 0.2]$$

- ③ Compute λ msg from children of x_1 is z

$$\lambda_z(x_1) = \sum_z \lambda(z) P[z|x_1] = \sum_z P[x_4=1|z] P[z|x_1]$$

- ④ Compute total λ msg from z

$$\lambda(x_1) = \sum_z P[x_4=1|z] P[z|x_1]$$

$x_1 = 0$

State	x_2	x_3	x_5	x_4	$P(z x_1)$
1	0	0	0	0	0.133
2	0	0	1	0	0.007
3	0	1	0	0	0.42
4	0	1	1	0	0.19
5	1	0	0	0	0.057
6	1	0	1	0	0.003
7	1	1	0	0	0.18
8	1	1	1	0	0.06

$x_1 = 1$

State	x_2	x_3	x_5	x_4	$P(z x_1)$
1	0	0	0	0	0.5529
2	0	0	1	0	0.0291
3	0	1	0	0	0.291
4	0	1	1	0	0.097
5	1	0	0	0	0.0171
6	1	0	1	0	0.0009
7	1	1	0	0	0.005
8	1	1	1	0	0.003

$$\pi(z) = 0.8$$

$$\begin{bmatrix} 0.133 \\ 0.007 \\ 0.42 \\ 0.19 \\ 0.057 \\ 0.003 \\ 0.18 \\ 0.06 \end{bmatrix} + 0.2 \begin{bmatrix} 0.5529 \\ 0.0291 \\ 0.291 \\ 0.097 \\ 0.0171 \\ 0.0009 \\ 0.009 \\ 0.003 \end{bmatrix} = \begin{bmatrix} 0.21698 \\ 0.01142 \\ 0.3942 \\ 0.1314 \\ 0.04902 \\ 0.00258 \\ 0.1458 \\ 0.0486 \end{bmatrix}$$

$$\lambda(z) = P[x_4=1 | z] = P[x_4=1 | x_2] = \begin{bmatrix} 0.02 \\ 0.02 \\ 0.02 \\ 0.02 \\ 0.6 \\ 0.6 \\ 0.6 \\ 0.6 \end{bmatrix}$$

P.T.O

For node x_1 .

$$\pi(x_1) = \pi(x_i) = [0.8, 0.2]$$

$$\begin{aligned}\lambda(x_1) &= \sum_z p[x_4=1|z] p[z|x_1] \\&= p[x_4=1|z_1] p[z_1|x_1] + p[x_4=1|z_2] p[z_2|x_1] \\&\quad p[x_4=1|z_3] p[z_3|x_1] + p[x_4=1|z_4] p[z_4|x_1] \\&\quad p[x_4=1|z_5] p[z_5|x_1] + p[x_4=1|z_6] p[z_6|x_1] \\&\quad p[x_4=1|z_7] p[z_7|x_1] + p[x_4=1|z_8] p[z_8|x_1] \\&= 0.02 [p[z_1|x_1] + p[z_2|x_1] + p[z_3|x_1] + p[z_4|x_1]] + \\&\quad 0.6 [p[z_5|x_1] + p[z_6|x_1] + p[z_7|x_1] + p[z_8|x_1]]\end{aligned}$$

$$x_1=0 = 0.02 \times 0.7 + 0.6 \times 0.3 = 0.194$$

$$x_1=1 = 0.02 \times 0.97 + 0.6 \times 0.03 = 0.0374$$

$$\lambda(x_1) = [0.194, 0.0374]$$

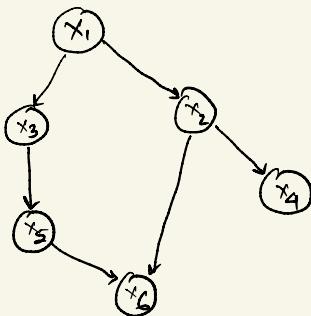
$$\begin{aligned}\text{So, Belief of } (x_1) &= \pi(x_1) \cdot \lambda(x_1) = [0.8, 0.2] \cdot [0.194, 0.0374] \\&= [0.1552, 0.00748]\end{aligned}$$

$$\text{normalized Belief of } x_1 = [0.95402016, 0.04597984)$$

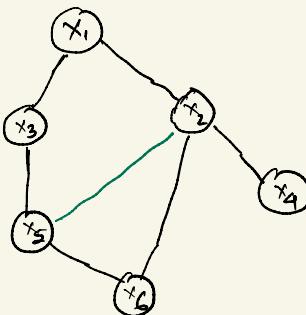
Hence $p[x_1=1 | x_4=1] = 0.0459$

Junction Tree Method

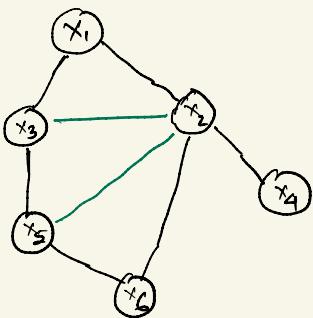
Given this BN, we will first generate the moral graph.



① Moral Graph



② Now we will triangulate the moral graph.



③ Now we will identify the maximum cliques.

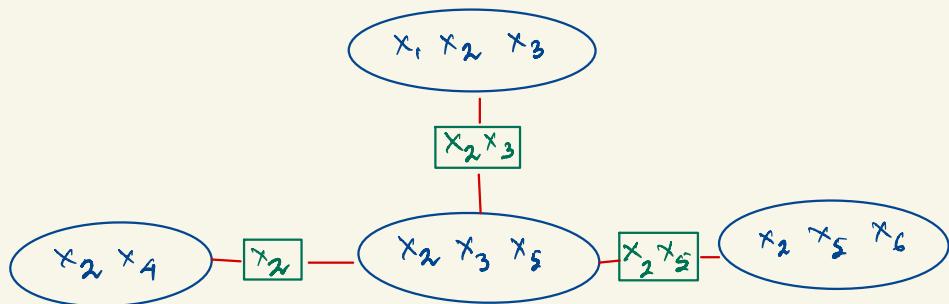
$$\{x_1, x_2, x_3\}$$

$$\{x_3, x_2, x_5\}$$

$$\{x_5, x_2, x_6\}$$

$$\{x_2, x_4\}$$

① Now we will form the junction tree.



② Now we will parameterize the cluster nodes

$$\phi_{123} = P(x_1) P(x_3|x_1) P(x_2|x_1)$$

here the evidence
is $x_4=1$.

$$\phi_{235} = P(x_5|x_3)$$

$$\phi_{256} = P(x_6|x_5, x_2)$$

$$\phi_{24} = P(x_4|x_2)$$

$$P(x_1, x_2, x_3, x_4, x_5, x_6) = \phi_{123} \phi_{235} \phi_{256} \phi_{24}$$

③ Now initializing the potential for the separators.

$$\phi(S_{23}) = 1$$

$$\phi(S_{25}) = 1$$

$$\phi(S_2) = 1$$

⑦ Belief / potential updating for node (x_2, x_3, x_5)

7.1 Obtain the initial potential for each cluster node

$$\phi_{123} = P(x_1) \cdot P(x_3|x_1) \cdot P(x_2|x_1)$$

$$\phi_{235} = P(x_5|x_3)$$

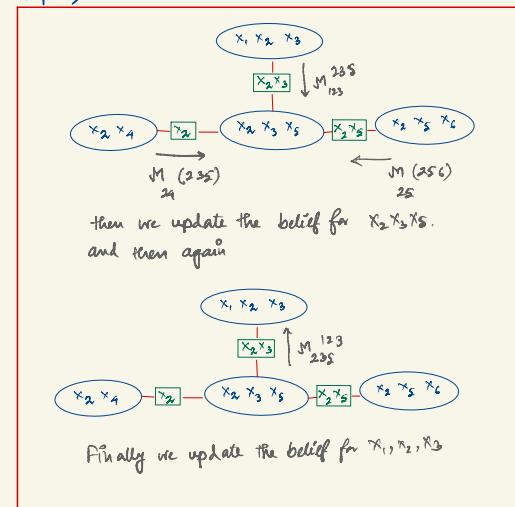
$$\phi_{256} = P(x_6|x_5, x_2)$$

$$\phi_{24} = P(x_4=1|x_2)$$

$$\phi_{23} = 1$$

$$\phi_{25} = 1$$

$$\phi_2 = 1$$



7.2 Message calculation

$$M_{24}^{(235)} = \frac{\phi_2^*}{\phi_2} = \frac{\sum_{x_4=1} \phi_{24}}{\phi_2} = \frac{P[x_4=1|x_2]}{1} = [0.02, 0.6]$$

$$M_{256}^{(235)} = \frac{\phi_{25}^*}{\phi_{25}} = \frac{\sum_{x_6} \phi_{256}}{\phi_{25}} = 1$$

$$M_{123} = \frac{\phi^*(23)}{\phi_{23}} = \frac{\sum_{x_1} \phi_{123}}{\phi_{23}} = \frac{[0.2289, 0.5256, 0.0816, 0.1944]}{1}$$

$$P(x_1) P(x_3|x_1) P(x_2|x_1)$$

x_1	x_2	x_3	$P(x_1, x_2, x_3)$
0	0	0	$0.8 \times 0.7 \times 0.2 = 0.112$
0	0	1	$0.8 \times 0.7 \times 0.8 = 0.448$
0	1	0	$0.8 \times 0.3 \times 0.2 = 0.048$
0	1	1	$0.8 \times 0.3 \times 0.8 = 0.192$
1	0	0	$0.2 \times 0.97 \times 0.6 = 0.1164$
1	0	1	$0.2 \times 0.97 \times 0.4 = 0.0776$
1	1	0	$0.2 \times 0.03 \times 0.6 = 0.0036$
1	1	1	$0.2 \times 0.03 \times 0.4 = 0.0024$

So, 7.3 Potential Updating

$$\begin{aligned}\phi_{235}^* &= \phi_{235} \frac{M_{235}}{24} \frac{M_{235}}{286} \frac{M_{235}}{123} \\ &= \phi_{235} \left[\begin{array}{cc} 0.02 & 0.6 \\ x_2=0 & 1 \end{array} \right] \left[\begin{array}{ccccc} 0.2289 & 0.5256 & 0.0816 & 0.1944 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right] P(S|B)\end{aligned}$$

x_5, x_3	$x_2=0$	$x_2=1$
0 0	0.019	0.57
0 1	0.015	0.45
1 0	0.001	0.03
1 1	0.008	0.16

S	x_3	x_2	$P(S 235)$	ϕ_{235}^*
0	0	0	$0.95 \times 0.2289 \times 0.02 = 0.00433$	-
0	1	0	$0.75 \times 0.5256 \times 0.02 = 0.00789$	-
1	0	0	$0.05 \times 0.2289 \times 0.02 = 0.002284$	-
-1	1	0	$0.25 \times 0.5256 \times 0.02 = 0.002627$	-
0	0	1	$0.95 \times 0.0816 \times 0.6 = 0.029412$	-
0	1	1	$0.75 \times 0.1944 \times 0.6 = 0.087447$	-
1	0	1	$0.05 \times 0.0816 \times 0.6 = 0.00159$	-
1	1	1	$0.25 \times 0.1944 \times 0.6 = 0.02916$	-

8. Potential updating for node x_1, x_2, x_3 .

Message passing from x_2, x_3, x_5 to x_1, x_2, x_3

$$M_{235}^{123} = \frac{\phi_{23}^*}{\phi_{23}} = \frac{\sum_{x_5} \phi_{235}}{\phi_{23}}$$

$$= \frac{[0.0048884, 0.30952, 0.01051, 0.1166]}{[0.2289, 0.5286, 0.0516, 0.1944]}$$

$$= [0.01995, 0.588, 0.20368, 0.59971]$$

So, final potential update for x_1, x_2, x_3

x_1	x_2	x_3	$P(x_1, x_2, x_3) \times M_{23}^*$
0	0	0	$0.8 \times 0.7 \times 0.2 = 0.112 \times 0.01995 = 0.0022344$
0	0	1	$0.8 \times 0.7 \times 0.8 = 0.448 \times 0.588 = 0.2634$
0	1	0	$0.8 \times 0.3 \times 0.2 = 0.048 \times 0.20368 = 0.0097769$
0	1	1	$0.8 \times 0.3 \times 0.8 = 0.192 \times 0.59971 = 0.11515$
1	0	0	$0.2 \times 0.97 \times 0.6 = 0.1164 \times 0.01995 = 0.0023$
1	0	1	$0.2 \times 0.97 \times 0.4 = 0.0776 \times 0.588 = 0.0486$
1	1	0	$0.2 \times 0.03 \times 0.6 = 0.0036 \times 0.20368 = 0.0007$
1	1	1	$0.2 \times 0.03 \times 0.4 = 0.0024 \times 0.59971 = 0.0014$

$$P[x_1 = 1 | E] = 0.05$$

→ The result is close to the junction tree method.