ADVANCED CONTROL SYSTEMS

Motion Control - Intro & Joint Space -

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Outline

Problem statement

Joint Space PD Control with Gravity Compensation

PROJECT

Joint Space Inverse Dynamics Control

PROJECT

Problem statement

Control Problem

Goal: execute a pre-determined task involving specified motions for the manipulator operating in free space (*motion control*), or specified interactions of the manipulator's end-effector with the environment (*force control*).

General observations:

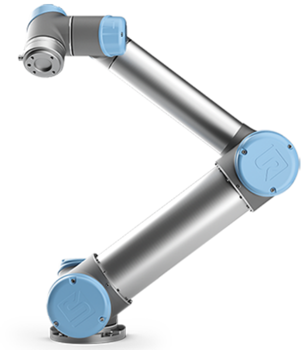
- The control scheme choice depends on the manipulator mechanics, the joints actuators, the knowledge we have about the robot, the control library the vendor makes available;
- Control techniques can be divided into several categories:
  - *Joint space* or *Operational space* control schemes;
  - *Model-based* or not
  - *Decentralized* control (when the single manipulator joint is controlled independently of the others) or *Centralized* (when the dynamic interaction effects between the joints are taken into account) control.

desired trajectory

$q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$
or
$x_d(t), \dot{x}_d(t), \ddot{x}_d(t)$

$t \in [0, T]$

$\rightarrow$



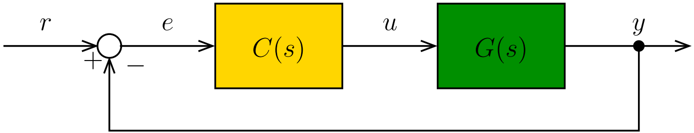
$\rightarrow$

actual pose / velocity

$q(t), \dot{q}(t), \ddot{q}(t)$
or
$x(t), \dot{x}(t), \ddot{x}(t)$

model

$$\begin{aligned} B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) &= \tau - J^T(q)h_e \\ B_A(x)\ddot{x} + C_A(x,\dot{x})\dot{x} + g_A(x) &= u - u_e \end{aligned}$$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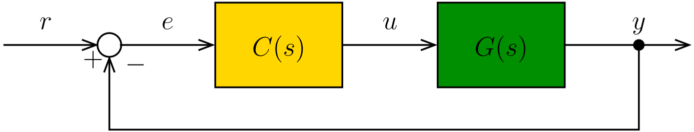
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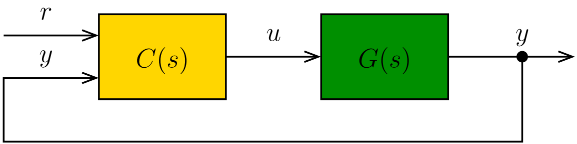
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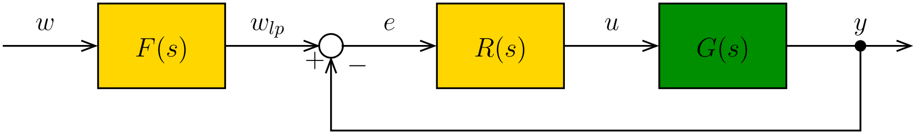
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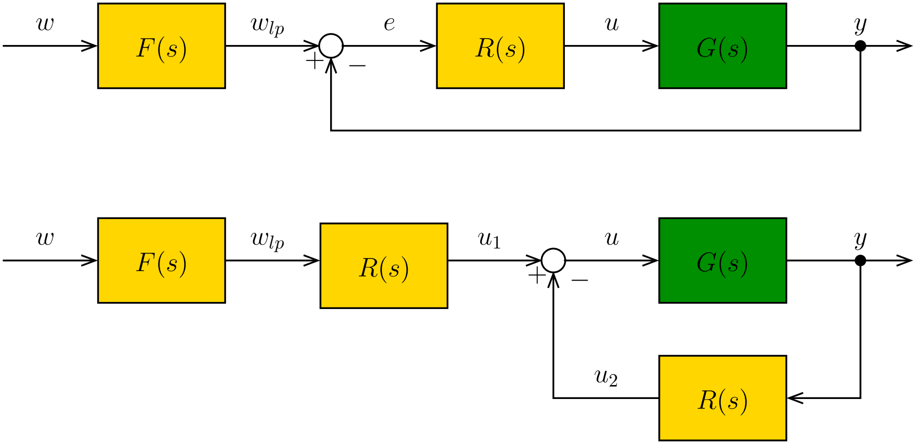

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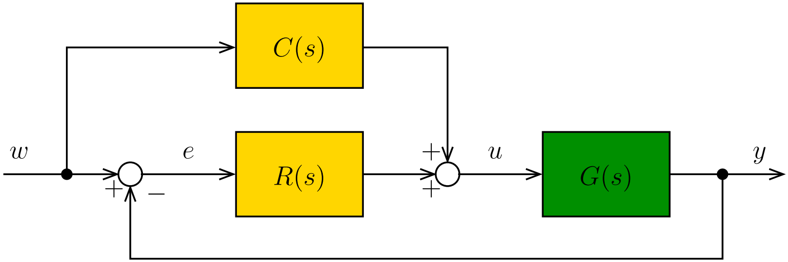
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Static specification: accuracy and repeatability in steady state

poor accuracy
poor repeatability

poor accuracy
good repeatability

good accuracy
poor repeatability

good accuracy
good repeatability

Specifications on $C(s)$:

- static specification: steady state error
- dynamic specifications:
  - setting time
  - rise time
  - overshoot
  - bandwidth
- robustness (uncertain parameters, nonlinear effects not taken into account as backlash, flexibility, friction, saturations)
- sensitivity to measurements noise
- sensitivity to external disturbances

- discretization $C(z)$ of $C(s)$; choice of the sample time T_s
- estimation of not measurable signals, e.g. $\dot{q}$, $\ddot{q}$
- real-time constraints

Motion Control

Different kinds of 'standard' control architectures:

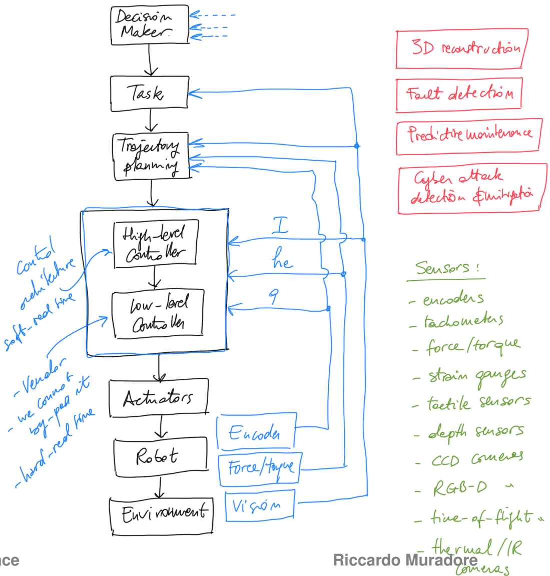
★ *feedback control*: insensitive to mild disturbances and small variations of parameters

★ *robust control*: tolerate relatively large uncertainties of *known* range

★ *adaptive control*: improve performance on line by adapting the control law to *unknown* range of uncertainties and/or large (but not too fast) parameter variations

The research (i.e. money €, $) is now focusing on *embodied AI*, i.e., robotics + AI

Modern Control Unit



With $B(q)$ is nonsingular, the dynamical model

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = u$$

can be written in the *state-space representation*

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} x_2 \\ -B^{-1}(x_1)\left(C(x_1,x_2)x_2 + Fx_2 + g(x_1)\right) \end{bmatrix} + \begin{bmatrix} 0 \\ B^{-1}(x_1)u \end{bmatrix} \\
\dot{x} &= F(x) + G(x)u
\end{aligned}$$

where the state is

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleq \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

When the *input is identically zero, $u = 0$*, the equilibrium point(s) are the solutions of

$$F(x_e) = 0, \qquad \Rightarrow \qquad \begin{cases} x_{e,2} = 0 \\ g(x_{e,1}) = 0 \end{cases} \qquad \text{i.e.} \qquad \begin{cases} \dot{q}_e = 0 \\ g(q_e) = 0 \end{cases}$$

In any equilibrium point x_e, the velocity must be equal to zero: $x_{e,2} = 0$ $(\dot{q} = 0)$

When the *input is constant, $u = u(x_e)$*, the equilibrium point is

$$F(x_e) + G(x_e)u(x_e) = 0, \qquad \Rightarrow \qquad \begin{cases} x_{e,2} = 0 \\ g(x_{e,1}) = u(x_e) \end{cases} \qquad \text{i.e.} \qquad \begin{cases} \dot{q}_e = 0 \\ g(q_e) = u(q_e) \end{cases}$$

The command torque u should be equal to the gravity term in x_e.

Check the definitions of stability, asymptotic stability, Lyapunov method,...

Exponential stability

$$\exists \delta > 0, c > 0, \lambda > 0 \; : \; \|x(t_0) - x_e\| < \delta \qquad \Rightarrow \qquad \|x(t) - x_e\| \leq c e^{-\lambda(t-t_0)} \|x(t_0) - x_e\|$$

Ultimately uniformly bounded stability. Given a set S

$$\exists T = T(x(t_0), S) \; : \qquad x(t) \in S, \quad \forall t \geq t_0 + T(x(t_0), S)$$

Theorem

The system $\dot{x} = f(x)$ is ultimately uniformly bounded in S, if there exists a function V such that

(i) *S is a level set of V for a given a, i.e.*

$$S = \{x \in \mathbb{R}^n \,|\, V(x) \leq a\}\,;$$

(ii) *$\dot{V}(x) < 0$ along $\dot{x} = f(x)$, $x \notin S$.*

Stability

In the general case of a system

$$\dot{x}(t) = F(x(t)) + G(x(t))u(t)$$

we have to check the stability of an equivalent time-varying system

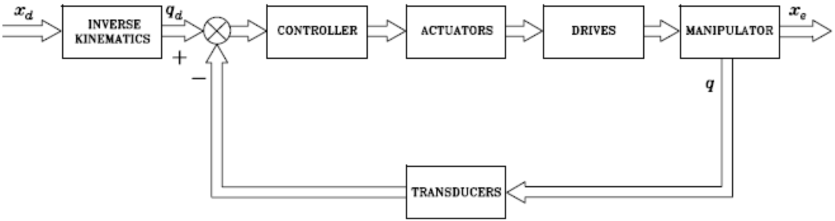
$$\dot{x}(t) = f(x(t), t).$$

This is the case of a robot controlled to track a time-varying trajectory.

The *task specification* (end-effector motion) is usually carried out in the operational space, whereas *control actions* (joint actuator generalized forces) are performed in the joint space.

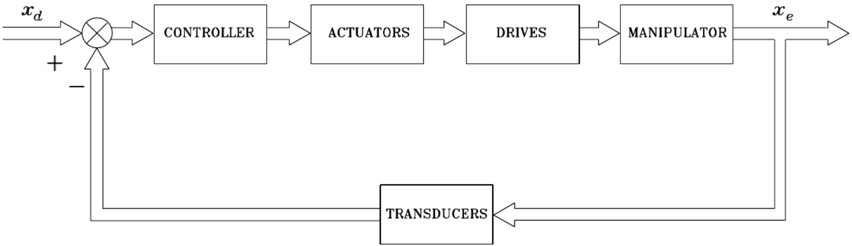
The joint space control problem consists of two subproblems:

- The inverse kinematics problem transforms the motion requirements x_d in the operational space into the desired motion q_d in the joint space.
- The joint space control scheme allows the actual motion q to track the desired trajectory q_d.



A joint space control scheme does not control *directly* the operational space variables x.

Any uncertainty of the structure (construction tolerance, lack of calibration, gear backlash, elasticity) or any imprecision in the knowledge of the end-effector pose will increase the error $x_d - x$ and so the motion accuracy of the end effector.



The *operational space control problem* requires a greater algorithmic complexity.

Joint space

The manipulator *dynamic model equation* in the joint space is

$$B(q)\ddot{q} + \underbrace{C(q,\dot{q})\dot{q} + F\dot{q} + g(q)}_{\triangleq n(q,\dot{q})} = \tau \qquad (1)$$

where

$$n(q,\dot{q}) = C(q,\dot{q})\dot{q} + F\dot{q} + g(q). \qquad (2)$$

To control the motion of the manipulator in free space means to determine the n components of generalized forces τ that allows $q(t)$ to track $q_d(t)$.

Ideally

$$q(t) \equiv q_d(t).$$

Joint space

The manipulator *dynamic model equation* in the joint space is

$$B(q)\ddot{q} + \underbrace{C(q,\dot{q})\dot{q} + F\dot{q} + g(q)}_{\triangleq n(q,\dot{q})} = \tau \tag{1}$$

where

$$n(q,\dot{q}) = C(q,\dot{q})\dot{q} + F\dot{q} + g(q). \tag{2}$$

To control the motion of the manipulator in free space means to determine the n components of generalized forces τ that allows $q(t)$ to track $q_d(t)$.

Ideally

$$q(t) \equiv q_d(t).$$

In real robots we cannot force the joint torque/force τ_i but we have to command a "corresponding" voltage to the motor actuator which will apply a torque to the joint.
The torque mismatch is due to the motor dynamics and to the control driver.

Joint Space PD Control with Gravity Compensation

Goal: Design a controller which ensures global asymptotic stability of the desired posture q_d (*constant equilibrium posture*).

System: nonlinear multivariable systems with state $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleq \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$

$$\begin{bmatrix} \frac{dq}{dt} \\ \frac{d\dot{q}}{dt} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -B^{-1}(q)\left(C(q,\dot{q})\dot{q} + F\dot{q} + g(q)\right) \end{bmatrix} + \begin{bmatrix} 0 \\ B^{-1}(q)\tau \end{bmatrix}$$

How: Lyapunov direct method for nonlinear multivariable systems.

Assumption: partial or complete knowledge of the manipulator dynamic model.

Since the equilibrium point is $\begin{bmatrix} q_d \\ 0 \end{bmatrix}$, we consider as error

$$\begin{bmatrix} \tilde{q}(t) \\ \dot{\tilde{q}}(t) \end{bmatrix} = \begin{bmatrix} q_d \\ 0 \end{bmatrix} - \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} q_d - q(t) \\ -\dot{q}(t) \end{bmatrix}$$

Then, we need to study the system state $\begin{bmatrix} \tilde{q}(t) \\ -\dot{q}(t) \end{bmatrix}$ around the equilibrium point $\begin{bmatrix} q_d \\ 0 \end{bmatrix}$

Let $K_P \in \mathbb{R}^{n \times n}$ be a symmetric and positive definite matrix; we choose as *Lyapunov candidate function*

$$V(\dot{q}, \tilde{q}) = \frac{1}{2}\dot{q}^T B(q)\dot{q} + \frac{1}{2}\tilde{q}^T K_P \tilde{q} \tag{3}$$

$V(\dot{q}, \tilde{q})$ is positive definite (always non-negative and equal to zero only for $\dot{q} = 0$, $\tilde{q} = 0$)

$$V(\dot{q}, \tilde{q}) \succ 0$$

Energetic interpretation

$$V(\dot{q}, \tilde{q}) = \underbrace{\frac{1}{2}\dot{q}^T B(q)\dot{q}}_{\text{Kinetic energy of the manipulator}} + \underbrace{\frac{1}{2}\tilde{q}^T K_P \tilde{q}}_{\text{Spring potential energy}}$$

Potential energy of a multidimentional spring with stiffness matrix $K_P \succ 0$ and rest position q_d

$$\frac{1}{2}\tilde{q}^T K_P \tilde{q} = \frac{1}{2}(q_d - q)^T K_P (q_d - q)$$

The term $K_P(q_d - q)$ can be seen as a set of *n position feedback loops*: one for each degree of freedom.

Joint Space PD Control with Gravity Compensation

We compute the time derivative of V with respect of time

$$\dot{V}(\dot{q},\tilde{q}) = \frac{d}{dt}\left(\frac{1}{2}\dot{q}^T B(q)\dot{q} + \frac{1}{2}\tilde{q}^T K_P \tilde{q}\right) = \dot{q}^T B(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{B}(q)\dot{q} - \dot{q}^T K_P \tilde{q},$$

where we use the fact that q_d is constant ($\frac{d}{dt}(\tilde{q}) = \frac{d}{dt}(q_d - q) = -\dot{q}$).

Using the dynamic model of the robot for $B(q)\ddot{q}$

$$\begin{aligned} B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) &= \tau \\ B(q)\ddot{q} &= \tau - C(q,\dot{q})\dot{q} - F\dot{q} - g(q) \end{aligned}$$

we end up with

$$\dot{V}(\dot{q},\tilde{q}) = \dot{q}^T\left(\dot{B}(q) - 2C(q,\dot{q})\right)\dot{q} - \dot{q}^T F\dot{q} + \dot{q}^T\left(\tau - g(q) - K_P\tilde{q}\right).$$

Since $N \triangleq \dot{B} - 2C$ is anti-symmetric, $\dot{q}^T N\dot{q} = 0$ we finally get

$$\dot{V}(\dot{q},\tilde{q}) = -\dot{q}^T F\dot{q} + \dot{q}^T\left(\tau - g(q) - K_P\tilde{q}\right).$$

Joint Space PD Control with Gravity Compensation

The expression for $\dot{V}$ gives us some hint on how to choose the control law

$$\dot{V}(\dot{q}, \tilde{q}) = -\dot{q}^T F \dot{q} + \dot{q}^T \left(\tau - g(q) - K_P \tilde{q}\right).$$

By choosing

$$\tau = g(q) + K_P \tilde{q} \tag{4}$$

we have

$$\dot{V}(\dot{q}, \tilde{q}) = -\dot{q}^T F \dot{q} \tag{5}$$

which is negative semi-definite $\dot{V}(\dot{q}, \tilde{q}) \preceq 0$

By the Lyapunov theorem, the equilibrium point $\begin{bmatrix} q_d \\ 0 \end{bmatrix}$ is at least stable for the closed-loop system composed of the robotic manipulator and the controller $\tau = g(q) + K_P \tilde{q}$.

The controller $\tau = g(q) + K_P \tilde{q}$ is a controller with a *proportional action* and compensation of *gravitational terms*.

It is also possible to define

$$u = g(q) + K_P\tilde{q} - K_D\dot{q}, \tag{6}$$

which consists of *a linear proportional-derivative (PD) action* with compensation of *gravitational terms*.

Also with this control law, $\dot{V}$ is negative semi-definite

$$\dot{V}(\dot{q}, \tilde{q}) = -\dot{q}^T \left(F + K_D\right) \dot{q} \tag{7}$$

Since both $F \succ 0$ and $K_D \succ 0$, the derivative action improves the performance, i.e. the system reaches the equilibrium point faster.

Both control laws are non linear due to the term $g(q)$.

Is the equilibrium just stable or asymptotically stable?

We observe that

$$\dot{V}(\dot{q},\tilde{q}) = -\dot{q}^T F \dot{q} \equiv 0$$

for $\dot{q} = 0$.

At the equilibrium, besides $\dot{q} = 0$, we also have $\ddot{q} = 0$.

Substituting such values in the robotic dynamics with the PD control law with gravity compensation

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = g(q) + K_P\tilde{q} - K_D\dot{q}$$

we end up with

$$0 = K_P\tilde{q}$$

Since K_P is non-singular, we have

$$\tilde{q} = 0, \qquad \Longleftrightarrow \qquad q = q_d$$

We have proven the following theorem

Theorem

A PD controller with gravity compensation

$$u = g(q) + K_P(q_d - q) - K_D\dot{q}, \qquad K_P \succ 0, K_D \succ 0$$

guarantees that the equilibrium point $(q_d, 0)$ *of the system*

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = u$$

is globally asymptotically stable.

Joint Space PD Control with Gravity Compensation

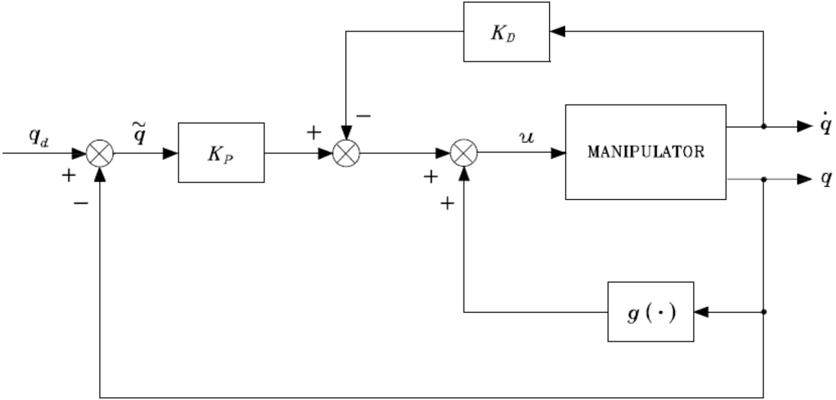


Figure: Joint Space PD control with gravity compensation block scheme

Final observations:

- A PD Control with Gravity Compensation guarantees the *global asymptotic stability* for any choice of $K_P \succ 0$ and $K_D \succ 0$
- The control law requires the on-line computation of the term $g(q)$.
- The previous proof of asymptotic stability is an application of the *La Salle theorem*: the only invariant set for $\dot{V} = 0$ is $(q, \dot{q}) = (q_d, 0)$.
- Besides the common gravity compensation term $g(q)$, if the matrices $K_P \succ 0$ and $K_D \succ 0$ are chosen diagonal, we have *n decentralized PD controllers*: one for each degree of freedom of the robotic manipulator

  PD controller $\sim$ *software* spring-damper system

  The equilibrium point for the i-th spring is $q_{d,i}$.

- If $F \neq 0$, then $K_D \succ 0$ is not strictly necessary, compare (5) and (7).

  However, the choice of $K_D \succ 0$ can be exploited to have the desired dynamics for the controlled robot.
- In real applications, it is necessary to discretize the control law since the controller will be implemented on digital hardware and the measurements provided by encoders are available on discrete time kT_s, where $k \in \mathbb{Z}$ and T_s is the sample time.
- If the gravity term is uncertain then the above results may not hold.

- If the gravity term is not implemented or $\hat{g}(q) \neq g(q)$, then

$$\begin{aligned} \dot{q}(t) &\rightarrow 0 \\ q(t) &\rightarrow q_e \neq q_d \end{aligned}$$

With a pure PD controller

$$u = K_P(q_d - q) - K_D\dot{q}$$

the equilibrium $(q_e, 0)$ for

$$B(q)\ddot{q} + C(q, \dot{q})\dot{q} + F\dot{q} + g(q) = u$$

is given by

$$g(q_e) = K_P(q_d - q_e)$$

and $q_d - q_e \neq 0$.

Joint Space PD Control with Gravity Compensation

- The steady-state error can be corrected if an integral action is included in the controller

$$u = K_P(q_d - q) - K_D\dot{q} + K_I \int_0^t \Big(q_d - q(\tau)\Big) d\tau, \qquad K_P \succ 0, K_D \succ 0, K_I \succ 0$$

  In steady state, the final value for the integral action will converge to the gravity term at q_d

$$K_I \int_0^\infty \Big(q_d - q(\tau)\Big) d\tau = g(q_d).$$

- To prove the stability of the robot with a PID controller is not trivial.

  [Ref. Kelly, Santibanez, Loria, “Control of Robot Manipulators in Joint Space”]

Joint Space PD Control with Gravity Compensation

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Theorem

A PD controller with constant gravity compensation

$$u = g(q_d) + K_P(q_d - q) - K_D\dot{q}, \qquad K_P \succ 0, K_D \succ 0$$

such that $K_{P,i} > \alpha$, $i = 1, \ldots, n$ *with*

$$\left\| \frac{\partial g(q)}{\partial q} \right\| \leq \alpha, \qquad \forall q$$

guarantees that the equilibrium point $(q_d, 0)$ *is globally asymptotically stable.*

Proof.

Exercise

Hint. Use as Lyapunov candidate

$$V(\dot{q}, \tilde{q}) = \frac{1}{2}\dot{q}^T B(q)\dot{q} + \frac{1}{2}\tilde{q}^T K_P \tilde{q} + U(q) - U(q_d) + \tilde{q}^T g(q_d)$$

where $g(q) = \frac{\partial U(q)}{\partial q}$. □

Exercise. Given the desired pose q_d, try to estimate the term $g(q_d)$ using the iterative scheme for the control law

$$u = K_P(q_d - q) - K_D\dot{q} + u_{i-1}$$

with initial value for the constant term $u_0 = 0$, i.e. pure PD controller.

The i-th control law should be used till the robot reaches the equilibrium position $q_{e,i}$

$$g(q_{e,i}) = K_P(q_d - q_{e,i}) + u_{i-1}$$

and so the steady-state error $e_i = q_d - q_{e,i}$.

The i-th constant term will be

$$u_i := K_P(q_d - q_{e,i}) + u_{i-1}$$

Under mild conditions, it can be proved that $u_i \to g(q_d)$, for $i \to \infty$ (and so $q_{e,i} \to q_d$).


Project

PROJECT – Assignment # 6

To do

- Design the Joint Space PD control law with gravity compensation
- What happens if $g(q)$ is not taken into account?
- What happens if the gravity term is set constant and equal to $g(q_d)$ within the control law?
- What happens if q_d is not constant (e.g. $q_d(t) = \bar{q}_d + \Delta \sin(\omega t)$)?

Joint Space Inverse Dynamics Control

The 'Joint Space PD Control with Gravity Compensation' scheme:

- was meant for steering the robot toward an equilibrium point $\begin{bmatrix} q_d \\ 0 \end{bmatrix}$, even though it can be also used for the tracking problem;
- was "partially" model-based (only $g(q)$ was needed in the control law).

We now introduce a better control law for handling the *tracking problem* that makes use of the whole dynamic model of the manipulator.

Let's recall the mathematical model

$$B(q)\ddot{q} + \underbrace{C(q,\dot{q})\dot{q} + F\dot{q} + g(q)}_{\triangleq n(q,\dot{q})} = \tau$$

with

$$n(q,\dot{q}) = C(q,\dot{q})\dot{q} + F\dot{q} + g(q).$$

Open loop solution: If the mathematical model of the manipulator

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = \tau$$

is perfectly known, and the desired trajectory $q_d(t)$ is twice-differentiable, then the *feedforward torque* (open-loop !)

$$\tau_d = B(q_d)\ddot{q}_d + C(q_d,\dot{q}_d)\dot{q}_d + F\dot{q}_d + g(q_d)$$

guarantees that $q(t) \equiv q_d(t)$, $\forall t$, if $q_d(0) = q(0)$ and $\dot{q}_d(0) = \dot{q}(0)$.

Problems:

- model mismatching, $\hat{B} \neq B$, $\hat{C} \neq C$, ...
- external disturbances
- $q_d(0) \neq q(0)$ and $\dot{q}_d(0) \neq \dot{q}(0)$
- unmodeled or time-varying parameters (e.g. friction, payload, ...), measurement noise, ...

Joint Space Inverse Dynamics Control

The *Joint Space Inverse Dynamics Control* consists of two parts:

1. a *nonlinear state feedback* able to make an *exact linearization* of the nonlinear system dynamics
2. a *stabilizing linear controller*

Assuming to perfectly know $B(q)$ and $n(q, \dot{q})$, it is possible to define the control τ as a function of the manipulator state $(q, \dot{q})$

$$\tau = B(q)y + n(q, \dot{q}). \tag{8}$$

Then we have

$$B(q)\ddot{q} + n(q, \dot{q}) = B(q)y + n(q, \dot{q}). \tag{9}$$

Since $B(q)$ is nonsingular, it leads to the double integrator dynamics

$$\ddot{q} = y \tag{10}$$

where y the *new input vector*. (nominal conditions)

Joint Space Inverse Dynamics Control

The control law

$$\tau = B(q)y + n(q, \dot{q}).$$

is the *inverse dynamics control* whereas, y is the control input of a set of n second order *linear and decoupled* systems

$$\ddot{q}_i = y_i, \qquad i = 1, \ldots, n$$

The dependency of each joint variable q_i on the other q_j has been *canceled* by the inverse dynamics control.

The *stabilizing linear control law* y can be chosen as

$$y = -K_P q - K_D \dot{q} + r \tag{11}$$

where K_P and K_D are positive definite matrices. Then we end up with to the second-order MIMO system

$$\ddot{q} + K_D \dot{q} + K_P q = r. \tag{12}$$

where r is related to the reference input.

Choosing

$$\begin{aligned} K_P &= \text{diag}\{\omega_{n1}^2, ..., \omega_{nn}^2\} \\ K_D &= \text{diag}\{2\xi_1\omega_{n1}, ..., 2\xi_n\omega_{nn}\}, \end{aligned}$$

with $\omega_{ni} > 0, \xi_i > 0$ the natural frequency and the damping ratio of i-th joint

$$\ddot{q}_i + 2\xi_i\omega_{ni}\dot{q}_i + \omega_{ni}^2 q_i = r_i. \qquad G_i(s) = \frac{1}{s^2 + 2\xi_i\omega_{ni}s + \omega_{ni}^2}$$

where the component r_i only influences the joint variable q_i thanks to the decoupled systems.

Remarks:

- n time-invariant, linear and decoupled second-order systems
- each q_i evolves independently of the other components
- each subsystem has a unitary mass (the coefficient in front of $\ddot{q}_i$ is 1) in the joint space

Joint Space Inverse Dynamics Control

Given a desired trajectory $q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$, the input r can be chosen as

$$r = \ddot{q}_d + K_D \dot{q}_d + K_P q_d \tag{13}$$

Substituting this expression in

$$\begin{aligned} \ddot{q} + K_D \dot{q} + K_P q &= r \\ &= \ddot{q}_d + K_D \dot{q}_d + K_P q_d \end{aligned}$$

we end up with a set of n *homogeneous second-order differential equation* on the position error $\tilde{q}(t) = q_d(t) - q(t)$

$$\ddot{\tilde{q}} + K_D \dot{\tilde{q}} + K_P \tilde{q} = 0 \tag{14}$$

which converges to zero with our choice for K_D, K_P for any initial condition $\tilde{q}(t_0), \dot{\tilde{q}}(t_0)$.

Since

$$\ddot{\tilde{q}}_i + 2\xi_i\omega_{ni}\dot{\tilde{q}}_i + \omega_{ni}^2\tilde{q}_i = 0$$

it is possible to analytically compute the error settling time, rise time and overshoot as a function of ξ_i and ω_{ni}.

$\tilde{q}_i(t) \to 0$ for $t \to \infty$ without the integral action thanks to the compensation of the gravity term.

The Joint Space Inverse Dynamics Control ensures the *trajectory tracking*.

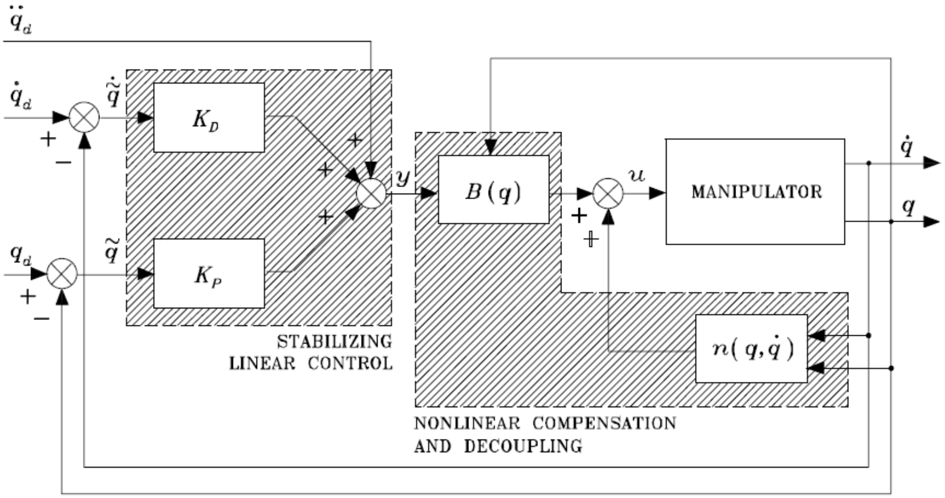

Figure: Joint Space Inverse Dynamics block scheme.

- *Inner feedback loop* based on the manipulator dynamic model; it computes the inverse dynamics.

  This loop is nonlinear and it is meant to linearize and decoupled each joint variable.
- *Outer feedback loop* controlling the joint tracking error and stabilize the overall system

$$\tau = B(q)\left[\ddot{q}_d + K_D(\dot{q}_d - \dot{q}) + K_P(q_d - q)\right] + n(q, \dot{q}). \tag{15}$$

The inertia matrix $B(q)$ and the vector of Coriolis, centrifugal, gravitational, and damping terms $n(q, \dot{q})$ must be known and computed at run-time.

This control law is based on the assumption of perfect cancellation of dynamic terms, and then it is quite natural to raise questions about sensitivity and robustness.

Uncertainty due to, e.g., imperfect knowledge of manipulator mechanical parameters, unmodelled dynamics, end-effector payloads, etc

Instead of the Inverse Dynamics Control based on state feedback linearization

$$\tau = B(q)\left[\ddot{q}_d + K_D(\dot{q}_d - \dot{q}) + K_P(q_d - q)\right] + C(q, \dot{q})\dot{q} + F\dot{q}, \tag{16}$$

it is possible to implement a different control command which consists of

- an inverse dynamics feedforward term τ_{ff}, and
- a linear PD control law

$$\tau = \underbrace{B(q)\ddot{q}_d + C(q, \dot{q})\dot{q}_d + g(q) + F\dot{q}_d}_{\tau_{ff}} + \underbrace{K_D(\dot{q}_d - \dot{q}) + K_P(q_d - q)}_{\text{PD}} \tag{17}$$

Remark. We can think of pre-computing the feedforward term as

$$\tau_{ff} = B(q_d)\ddot{q}_d + C(q_d, \dot{q}_d)\dot{q}_d + g(q) + F\dot{q}_d.$$

In this case the proof of the following theorem does not hold any longer.

Joint Space Inverse Dynamics Control II

Theorem

The control law (17) applied to the robotic arm

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = \tau \tag{18}$$

guarantees the global asymptotic stability of $(e(t), \dot{e}(t)) = (0, 0)$, *where*

$$e(t) := q_d(t) - q(t).$$

Remark. If q_d is constant, then (17) is the PD control law with gravity compensation.

Proof

Let

$$V = \frac{1}{2}\dot{e}^T B(q)\dot{e} + \frac{1}{2}e^T K_P e \geq 0$$

be the Lyapunov candidate function. Its time derivative is

$$\dot{V} = \frac{1}{2}\dot{e}^T \dot{B}(q)\dot{e} + \dot{e}^T B(q)\ddot{e} + e^T K_P \dot{e}.$$

Substituting (17) into (18) we have the closed-loop system

$$B(q)\ddot{e} + C(q,\dot{q})\dot{e} + F\dot{e} + K_D\dot{e} + K_P e = 0$$

i.e.

$$B(q)\ddot{e} = -C(q,\dot{q})\dot{e} - (F + K_D)\dot{e} - K_P e \qquad (19)$$

Substituting (19) into (19), and exploiting the skew-symmetry of $\dot{B}(q) - 2C(q,\dot{q})$, we end up with

$$\dot{V} = -\dot{e}^T(F + K_D)\dot{e} \leq 0 \qquad (20)$$

We cannot apply the La Salle theorem because the dynamic equation for e is not time-invariant. Since

$$q = q_d - e, \qquad \dot{q} = \dot{q}_d - \dot{e}$$

the equation (19) is equivalent to

$$B(q_d - e)\ddot{e} = -C(q_d - e, \dot{q}_d - \dot{e})\dot{e} - (F + K_D)\dot{e} - K_P e$$

which means that $B(q_d - e)$ and $C(q_d - e, \dot{q}_d - \dot{e})$ are time-varying due to $q_d(t)$, $\dot{q}_d(t)$.

It is worth remarking that the Lyapunov function depends on t, i.e.

$$V = V(e, \dot{e}, t), \qquad \dot{V} = \dot{V}(e, \dot{e}, t).$$

We have to use the Barbalat lemma!

Lemma [Barbalat lemma] If the double-integrable function $V(x, t)$ satisfies

(i) $V(x, t)$ is lower bounded

(ii) $\dot{V}(x, t) \leq 0$

Then $\exists \lim_{t\to\infty} V(x, t)$.
If in addition

(iii) $\ddot{V}(x, t)$ is bounded

Then $\lim_{t\to\infty} \dot{V}(x, t) = 0$.

Corollary If the Lyapunov candidate function $V(x, t)$ satisfies the Barbalat lemma along the trajectory of $\dot{x} = f(x, t)$, then the system trajectories converge asymptotically to the largest invariant set

$$\mathcal{N} \subseteq \left\{ x \in \mathbb{R}^n : \dot{V}(x, t) = 0 \right\}.$$

This is an extension of the La Salle theorem.

It is easy to verify that our Lyapunov candidate function $V(e, \dot{e}, t)$, with $x = (e, \dot{e})$ satisfies (i) and (ii).

Since $\exists \lim_{t\to\infty} V(x, t)$, i.e. $V(x, t)$ is bounded, we can conclude that e and $\dot{e}$ are bounded.

Assuming that also the desired trajectory has bounded velocity $\dot{q}_d$, we can conclude that also q and $\dot{q}$ must be bounded.

To check the third condition, we need to compute $\ddot{V}$:

$$\ddot{V} = -2\dot{e}^T (F + K_D)\ddot{e}$$

From (19) we can derive $\ddot{e}$ as

$$\ddot{e} = -B^{-1}(q) \left[C(q, \dot{q})\dot{e} + (F + K_D)\dot{e} + K_P e \right]$$

Exploiting the two properties

- $0 < B_m \leq \|B(q)\| < B_M < \infty$, which implies
  $0 < \bar{B}_m \leq \|B^{-1}(q)\| < \bar{B}_M < \infty$
- $\|C(q, \dot{q})\| \leq \alpha \|\dot{q}\|$

it turns out that all the matrices and signals defining $\ddot{e}$ are bounded.

Since $\ddot{e}$ and $\dot{e}$ are bounded, $\ddot{V}$ is bounded as well; for the Barbalat lemma we have

$$\lim_{t \to \infty} \dot{V}(x, t) = 0.$$

Remembering that

$$\dot{V} = -\dot{e}^T (F + K_D) \dot{e} \leq 0$$

we can say that

$$\dot{V} = 0 \qquad \Leftrightarrow \qquad \dot{e} = 0$$

The closed loop dynamics (19) becomes

$$B(q)\ddot{e} = -K_P e$$

which implies that

$$\ddot{e} = 0 \qquad \Leftrightarrow \qquad e = 0.$$

The point $(e, \dot{e}) = (0, 0)$ is the largest invariant set in $\dot{V} = 0$.

We have then proved the global asymptotic tracking. $\square$

Exact linearization via (state) feedback

Given the model

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau,$$

the design of the *nonlinear compensation and decoupling* inner loop

$$\tau = B(q)y + n(q,\dot{q}).$$

which makes the closed-loop system equivalent to

$$\ddot{q} = y, \qquad \text{i.e. } \frac{d^2 q_i}{dt^2} = y_i, \;\; i = 1, \dots, n$$

is a (straightforward in our case) application of a very important tool in nonlinear control systems called *exact linearization via state feedback*.

Exact linearization via (state) feedback

Let
$$\dot{x} = f(x) + g(x)u$$
be a generic nonlinear system. If x is available and $g(x)$ is a nonsingular matrix function, then the control u can be chosen as
$$u = g^{-1}(x)\left(-f(x) + v\right).$$
We end up with
$$\dot{x} = v.$$

What are x, $f(x)$, $g(x)$ in the robotic arm case?

Exact linearization via (state) feedback

Given a robotic arm with elastic joints

$$\begin{aligned} B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + K(q-\theta) &= 0 \\ B_m\ddot{\theta} + K(\theta - q) &= u \end{aligned}$$

it is possible to find a control law

$$u = \alpha(q,\theta,\dot{q},\dot{\theta}) + \beta(q,\theta,\dot{q},\dot{\theta})y$$

such that the controlled system is equivalent to n chains of four integrators

$$\frac{d^4 q_i}{dt^4} = y_i, \;\; i = 1,\ldots,n$$

[In the rigid case there were n chains of two integrators]


Project

PROJECT – Assignment # 7

To do

- Design the Joint Space Inverse Dynamics Control law
- Check that in the nominal case the dynamic behaviour is equivalent to the one of a set of stabilized double integrators
- Check the behavior of the control law when the $\hat{B}$, $\hat{C}$, $\hat{g}$ used within the controller are different than the “true ones” B, C, g (e.g. slightly modify the masses, the frictions, ...)
- What happens to the torque values when the settling time of the equivalent second order systems is chosen very small?