ROBOTICS, VISION AND CONTROL

Trajectory Planning. Operational Space

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Outline

Introduction

End-Effector Position

End-Effector Orientation

Introduction

From Geometric Paths to Trajectories

Given a geometric path Γ represented by a parametric curve $\boldsymbol{p} = \boldsymbol{p}(u)$, the *trajectory* is completely defined only when the motion law $u = u(t)$ is provided

Particular motion law is adopted to guarantee that the trajectory is *compliant* with velocity and acceleration *constraints*

> *the motion law $u(t)$ is a re-parameterization of the curve, which modifies the velocity and acceleration vectors*

$$\begin{aligned} \boldsymbol{p}(u) \quad &: \quad [u_{min}, u_{max}] \mapsto \mathbb{R}^3 \\ u(t) \quad &: \quad [t_{min}, t_{max}] \mapsto [u_{min}, u_{max}] \end{aligned}$$

$$\tilde{\boldsymbol{p}}(t) = (\boldsymbol{p} \circ u)(t) \quad : \quad [t_{min}, t_{max}] \mapsto \mathbb{R}^3$$

From Geometric Paths to Trajectories

Computing the time derivation exploiting the chain rule, we have

- *velocity*

$$\dot{\tilde{\boldsymbol{p}}}(t) = \frac{d\boldsymbol{p}}{du}\dot{u}(t)$$

the velocity vector of is equal to the first derivative of the parametric curve modulated by the velocity of the motion law

the velocity is always oriented along the direction tangent to the curve (i.e. $\boldsymbol{t}$)

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- *acceleration*

$$\ddot{\tilde{\boldsymbol{p}}}(t) = \frac{d\boldsymbol{p}}{du}\ddot{u}(t) + \frac{d^2\boldsymbol{p}}{du^2}\dot{u}^2(t)$$

the acceleration depends on both the acceleration and the square speed of $u(t)$.

The acceleration is composed by two components, oriented along the directions tangent (i.e. $\boldsymbol{t}$, *tangential acceleration*) and normal (i.e. $\boldsymbol{n}$, *centripetal acceleration*) to the curve

Constant scaling

When the motion law is a linear scaling of time

$$u(t) = \lambda\, t, \qquad \lambda > 0$$

the k-th derivative of the parametric curve is simply scaled by a factor λ^k

$$\begin{aligned}
\dot{\tilde{\boldsymbol{p}}}(t) &= \frac{d\boldsymbol{p}}{du}\lambda \\
\ddot{\tilde{\boldsymbol{p}}}(t) &= \frac{d^2\boldsymbol{p}}{du^2}\lambda^2 \\
\dddot{\tilde{\boldsymbol{p}}}(t) &= \frac{d^3\boldsymbol{p}}{du^2}\lambda^3 \\
&\vdots
\end{aligned}$$

Constraints on velocity, acceleration and jerk are satisfied by choosing

$$\lambda = \min\left\{ \frac{\dot{\tilde{\boldsymbol{p}}}^{max}}{\max_u \|\frac{d\boldsymbol{p}}{du}\|}, \sqrt{\frac{\ddot{\tilde{\boldsymbol{p}}}^{max}}{\max_u \|\frac{d^2\boldsymbol{p}}{du^2}\|}}, \left(\frac{\dddot{\tilde{\boldsymbol{p}}}^{max}}{\max_u \|\frac{d^3\boldsymbol{p}}{du^3}\|}\right)^{1/3} \right\}$$

Remarks.

- A constant scaling cannot guarantee a smooth starting/ending (with initial and final velocities and accelerations equal to zero).

- Let consider a multi-segment trajectory. By considering only a constant scaling λ for the overall trajectory, we have

$$u(t) = \lambda\, t, \qquad t \in \left[\frac{u_k}{\lambda}, \frac{u_{k+1}}{\lambda}\right]$$

  where $[u_k, u_{k+1}]$ is the parametric interval for the k-th segment. The value of λ is obtained as described in the previous slide.

- When a generic motion law is assumed to describe the relation between t and u, it is not simple to find a motion law which satisfies prescribed values of acceleration or jerk, since the derivatives of $u(t)$ are mixed in the expression of $\ddot{\tilde{\boldsymbol{p}}}(t)$ and $\dddot{\tilde{\boldsymbol{p}}}(t)$

Segment connecting point $\boldsymbol{p}_i$ with point $\boldsymbol{p}_f$: $\boldsymbol{p}(u) = \boldsymbol{p}_i + (\boldsymbol{p}_f - \boldsymbol{p}_i)u, \qquad u \in [0, 1]$

Since

$$\begin{aligned} \frac{d\boldsymbol{p}}{du} &= \boldsymbol{p}_f - \boldsymbol{p}_i \\ \frac{d^2\boldsymbol{p}}{du^2} &= 0 \\ \frac{d^3\boldsymbol{p}}{du^3} &= 0 \\ &\vdots \end{aligned}$$

then

$$\begin{aligned} \dot{\tilde{\boldsymbol{p}}}(t) &= (\boldsymbol{p}_f - \boldsymbol{p}_i)\dot{u}(t) \\ \ddot{\tilde{\boldsymbol{p}}}(t) &= (\boldsymbol{p}_f - \boldsymbol{p}_i)\ddot{u}(t) \\ \dddot{\tilde{\boldsymbol{p}}}(t) &= (\boldsymbol{p}_f - \boldsymbol{p}_i)\dddot{u}(t) \\ &\vdots \end{aligned}$$

The constraints on $\dot{\tilde{\boldsymbol{p}}}(t)$, $\ddot{\tilde{\boldsymbol{p}}}(t)$, $\dddot{\tilde{\boldsymbol{p}}}(t)$ can be easily translated into constraints on the derivative of the motion law

$$\begin{aligned} |\dot{u}(t)| &\leq \frac{\dot{\tilde{\boldsymbol{p}}}^{max}}{\|\boldsymbol{p}_f - \boldsymbol{p}_i\|} \\ |\ddot{u}(t)| &\leq \frac{\ddot{\tilde{\boldsymbol{p}}}^{max}}{\|\boldsymbol{p}_f - \boldsymbol{p}_i\|} \\ |\dddot{u}(t)| &\leq \frac{\dddot{\tilde{\boldsymbol{p}}}^{max}}{\|\boldsymbol{p}_f - \boldsymbol{p}_i\|} \end{aligned}$$

It is also possible to refine the inequalities for the three components.

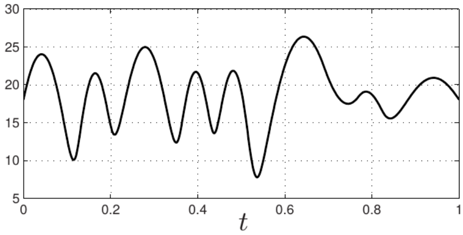
Constant Feed Rate

In several applications, it is necessary to plan the motion of the tool in the workspace with a constant speed (e.g. milling machines, welding robots)

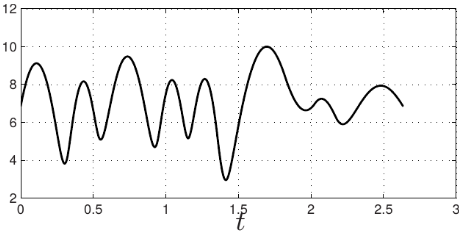
$$u(t) \text{ such that } \|\dot{\tilde{\boldsymbol{p}}}(t)\| = v = \text{constant}, \forall t$$

Using one of the standard techniques discussed so far, we could have the following $\|\dot{\tilde{\boldsymbol{p}}}(t)\|$ with $u(t) = t$ (left plot) that we can 'modulate' by using $u(t) = \lambda t$ (right plot)

$\|\dot{\tilde{\boldsymbol{p}}}(t)\|$ with $u(t) = t$



$\|\dot{\tilde{\boldsymbol{p}}}(t)\|$ with $u(t) = \lambda t$



Discrete approximation: let numerically compute $u(t)$ in real-time at each sampling time $t_k = kT_s$ using the Taylor approximation

$$u_{k+1} := u(t_{k+1}) = u_k + T_s \dot{u}_k + \frac{T_s^2}{2} \ddot{u}_k + O\left(T_s^3 \dddot{u}_k\right)$$

First-order approximation

Since

$$\dot{\boldsymbol{p}}(t) = \frac{d\boldsymbol{p}}{du} \dot{u}(t)$$

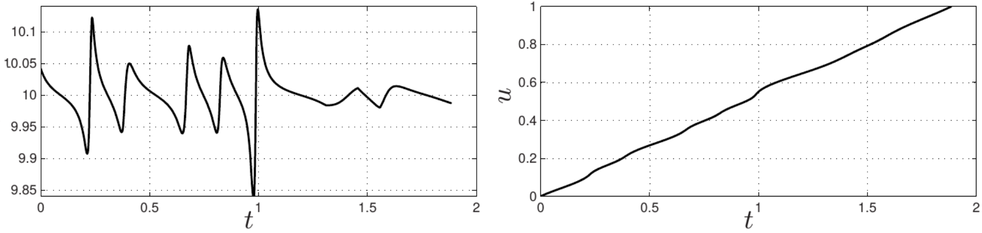
the constant feed rate v requires

$$\dot{u}(t) = \frac{v}{\left\| \frac{d\boldsymbol{p}}{du} \right\|}, \qquad \forall t$$

and finally

$$u_{k+1} = u_k + T_s \frac{v}{\left\| \left. \frac{d\boldsymbol{p}}{du} \right|_{u=u_k} \right\|}, \qquad \forall t = t_k$$

$$\|\dot{\tilde{\boldsymbol{p}}}(t)\| \text{ with } u_{k+1} = u_k + T_s \frac{v}{\left\| \frac{d\boldsymbol{p}}{du}\Big|_{u=u_k} \right\|}, \qquad \forall t = t_k$$



Second-order approximation
Taking into account also the second derivative

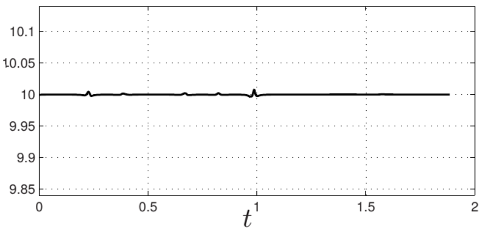
$$\ddot{u}(t) \quad = \quad -v^2 \frac{\frac{d\mathbf{p}}{du}^T \frac{d^2\mathbf{p}}{du^2}}{\left\| \frac{d\mathbf{p}}{du} \right\|^4},$$

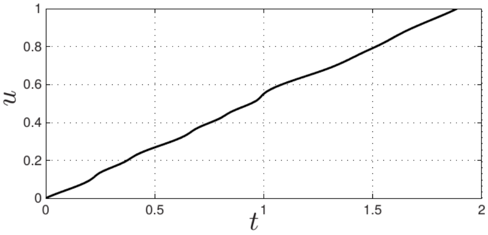
we get

$$u_{k+1} = u_k + \frac{T_s\, v}{\left\| \left. \frac{d\mathbf{p}}{du} \right|_{u=u_k} \right\|} - \frac{(T_s\, v)^2}{2} \left. \frac{\frac{d\mathbf{p}}{du}^T \frac{d^2\mathbf{p}}{du^2}}{\left\| \frac{d\mathbf{p}}{du} \right\|^4} \right|_{u=u_k}, \qquad \forall t = t_k$$

Constant Feed Rate

$$\|\dot{\tilde{\boldsymbol{p}}}(t)\| \text{ with } u_{k+1} = u_k + \frac{T_s\, v}{\left\| \left.\frac{d\boldsymbol{p}}{du}\right|_{u=u_k} \right\|} - \frac{(T_s\, v)^2}{2} \left. \frac{\frac{d\boldsymbol{p}}{du}^T \frac{d^2\boldsymbol{p}}{du^2}}{\| \frac{d\boldsymbol{p}}{du} \|^4} \right|_{u=u_k}, \qquad \forall t = t_k$$





Drawback. Computation burden.

Constant Feed Rate

Arc-length parameterization

Using the arc-length parameterization (*uniform parameterization*), we have by construction a tangent vector with unitary magnitude over the entire curve

$$\|\boldsymbol{t}(s)\| = \|\frac{d\boldsymbol{p}}{ds}\|$$

The re-parameterization of the curve which produces a constant feed rate v is then

$$u(t) = v\,t$$

Its sampled version is

$$s_{k+1} = s_k + T_s\,v, \qquad \forall t = t_k$$

Generic Feed Rate Profile

Suppose that we have given profiles for the velocity $v(t)$ and the acceleration $a(t)$ over the time horizon.

The expressions for the first derivative $\dot{u}(t)$ and second derivative $\ddot{u}(t)$ will be a function of time as well

$$\dot{\tilde{\boldsymbol{p}}}(t) = \frac{d\boldsymbol{p}}{du}\dot{u}(t) \quad \text{end} \quad \|\dot{\tilde{\boldsymbol{p}}}(t)\| = v(t) \quad \Rightarrow \quad \dot{u}(t) = \frac{v(t)}{\|\frac{d\boldsymbol{p}}{du}\|}$$

$$\|\ddot{\tilde{\boldsymbol{p}}}(t)\| = a(t) \quad \Rightarrow \quad \ddot{u}(t) = \frac{a(t)}{\|\frac{d\boldsymbol{p}}{du}\|} - v(t)^2 \frac{\frac{d\boldsymbol{p}}{du}^T \frac{d^2\boldsymbol{p}}{du^2}}{\|\frac{d\boldsymbol{p}}{du}\|^4},$$

Let $v_k = v(t_k)$ and $a_k = a(t_k)$, then

$$u_{k+1} = u_k + \frac{T_s\, v_k}{\left\| \left.\frac{d\boldsymbol{p}}{du}\right|_{u=u_k} \right\|} + \frac{T_s^2}{2}\left(\frac{a_k}{\left\| \left.\frac{d\boldsymbol{p}}{du}\right|_{u=u_k} \right\|} - v_k^2 \left. \frac{\frac{d\boldsymbol{p}}{du}^T \frac{d^2\boldsymbol{p}}{du^2}}{\|\frac{d\boldsymbol{p}}{du}\|^4} \right|_{u=u_k} \right), \qquad \forall t = t_k$$

Generic Feed Rate Profile

Using the arc-length parameterization, we have the simpler expression

$$s_{k+1} = s_k + T_s\, v_k + \frac{T_s^2}{2}\left(a_k - v_k^2 \left. \frac{d\boldsymbol{p}}{du}^T \frac{d^2\boldsymbol{p}}{du^2} \right|_{u=u_k} \right), \qquad \forall t = t_k$$

End-Effector Position

Operational Space Trajectories – EE Position

Let $\boldsymbol{x}_e(t) = \begin{bmatrix} \boldsymbol{p}_e \\ \phi_e \end{bmatrix}$ be the pose of the manipulator's end-effector at time t.

Let $\boldsymbol{p}_e(s) = \boldsymbol{f}(s) \in \mathbb{R}^3$ be the position vector of the parametric representation of the path Γ as a function of the arc length s with initial position $\boldsymbol{p}_i$ at $s = 0$ (at $t = 0$) and final position $\boldsymbol{p}_f$ at $s = s_f$ (at $t = t_f$).

The analytic expression for $s(t)$ can be derived using the techniques explained in the generation of joint trajectory (e.g. cubic polynomials, trapezoidal shape, ...).

Goal: Compute the time evolution of $\boldsymbol{p}_e$ on Γ as a function of the timing law $s(t)$

$$\dot{\boldsymbol{p}}_e = \frac{d\boldsymbol{p}_e}{dt} = \frac{d\boldsymbol{p}_e}{ds}\frac{ds}{dt} = \dot{s}\frac{d\boldsymbol{p}_e}{ds} = \dot{s}\boldsymbol{t}$$

where $\boldsymbol{t}$ is evaluated at $\boldsymbol{p}_e$ on Γ.

The magnitude of $\dot{\boldsymbol{p}}_e$ starts from zero at $t = 0$, then it varies with a parabolic or trapezoidal profile, and finally it returns to zero at $t = t_f$.

Operational Space Trajectories – EE Position

Segment connecting point $\boldsymbol{p}_i$ with point $\boldsymbol{p}_f$: $\boldsymbol{p}_e(s) = \boldsymbol{p}_i + s\frac{\boldsymbol{p}_f - \boldsymbol{p}_i}{\|\boldsymbol{p}_f - \boldsymbol{p}_i\|}, \qquad s \in [0, \|\boldsymbol{p}_f - \boldsymbol{p}_i\|]$

$$
\begin{aligned}
\dot{\boldsymbol{p}}_e &= \dot{s}\frac{\boldsymbol{p}_f - \boldsymbol{p}_i}{\|\boldsymbol{p}_f - \boldsymbol{p}_i\|} = \dot{s}\boldsymbol{t} \\
\ddot{\boldsymbol{p}}_e &= \ddot{s}\frac{\boldsymbol{p}_f - \boldsymbol{p}_i}{\|\boldsymbol{p}_f - \boldsymbol{p}_i\|} = \ddot{s}\boldsymbol{t}
\end{aligned}
$$

Circle: $\boldsymbol{p}_e(s) = \boldsymbol{c} + R \begin{bmatrix} \rho\cos(s/\rho) \\ \rho\sin(s/\rho) \\ 0 \end{bmatrix}$

$$\dot{\boldsymbol{p}}_e = R \begin{bmatrix} -\dot{s}\sin(s/\rho) \\ \dot{s}\cos(s/\rho) \\ 0 \end{bmatrix} = R\boldsymbol{t}$$

$$\ddot{\boldsymbol{p}}_e = R \begin{bmatrix} -\dot{s}^2\frac{1}{\rho}\cos(s/\rho) - \ddot{s}\sin(s/\rho) \\ -\dot{s}^2\frac{1}{\rho}\sin(s/\rho) + \ddot{s}\cos(s/\rho) \\ 0 \end{bmatrix} = \underbrace{R\dot{s}^2 \left\| \begin{bmatrix} -\frac{1}{\rho}\cos(s/\rho) \\ -\frac{1}{\rho}\sin(s/\rho) \\ 0 \end{bmatrix} \right\| \boldsymbol{n}}_{\text{centripetal acceleration}} + \underbrace{R\ddot{s}\boldsymbol{t}}_{\text{tangential acceleration}}$$

End-Effector Orientation

Operational Space Trajectories – EE Orientation

The end-effector orientation is specified in terms of the rotation matrix of the (time-varying) end-effector frame Σ_e with respect to the base frame Σ_b. Let $R = \begin{bmatrix} \boldsymbol{n}_e & \boldsymbol{s}_e & \boldsymbol{a}_e \end{bmatrix}$, a linear interpolation on the unit vectors $\boldsymbol{n}_e, \boldsymbol{s}_e, \boldsymbol{a}_e$ describing the initial and final orientation does not guarantee orthonormality of the above vectors at each instant of time.

Possible solutions:

- Euler angles
- Angle and axis

Euler angles

$\phi_e = (\varphi, \theta, \psi)$ moves along the segment connecting the initial value ϕ_i to its final value ϕ_f with, e.g., a cubic polynomial timing law.

$$\phi_e = \phi_i + s\frac{\phi_f - \phi_i}{\|\phi_f - \phi_i\|}, \qquad \dot{\phi}_e = \dot{s}\frac{\phi_f - \phi_i}{\|\phi_f - \phi_i\|}, \qquad \ddot{\phi}_e = \ddot{s}\frac{\phi_f - \phi_i}{\|\phi_f - \phi_i\|}$$

where the timing law for $s(t)$ has to be specified.

Angle and axis

Given an initial frame and a final frame, it is possible to find a unit vector so that the second frame can be obtained from the first frame by a rotation of a proper angle about the axis of such unit vector.

R_i the rotation matrix of the initial frame $\Sigma_i = \{O_i; x_i, y_i, z_i\}$ with respect to the base frame
R_f the rotation matrix of the final frame $\Sigma_f = \{O_f; x_f, y_f, z_f\}$ with respect to the base frame

Rotation from Σ_f to Σ_i

$$R_f^i = R_i^T R_f = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \qquad \theta_f = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right), \qquad \boldsymbol{r} = \frac{1}{2\sin\theta_f}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$R^i(t) = R^i(\theta(t), \boldsymbol{r})$ describes the transition from R_i to R_f ($\boldsymbol{r}$ is in the reference frame Σ_i);

$$R^i(0) = R^i(\theta(0), \boldsymbol{r}) = I, \qquad R^i(t_f) = R^i(\theta(t_f), \boldsymbol{r}) = R_f^i$$

where $\theta(t)$ is a timing law of the type of those presented for the single joint with $\theta(0) = 0$ and $\theta(t_f) = \theta_f$.

Since the rotation unit vector $\boldsymbol{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$ is constant (it depends only on R_i to R_f), the only time-varying element is $\theta(t)$. Then

$$R^i(t) = R^i(\theta(t), \boldsymbol{r}) = \begin{bmatrix} r_x^2(1-\cos\theta)+\cos\theta & r_x r_y(1-\cos\theta)-r_z\sin\theta & r_x r_z(1-\cos\theta)+r_y\sin\theta \\ r_x r_y(1-\cos\theta)+r_z\sin\theta & r_y^2(1-\cos\theta)+\cos\theta & r_y r_z(1-\cos\theta)-r_x\sin\theta \\ r_x r_z(1-\cos\theta)-r_y\sin\theta & r_y r_z(1-\cos\theta)+r_x\sin\theta & r_z^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

Moreover

$$\begin{aligned} \boldsymbol{\omega}^i(t) &= \dot{\theta}(t)\boldsymbol{r} \\ \dot{\boldsymbol{\omega}}^i(t) &= \ddot{\theta}(t)\boldsymbol{r} \end{aligned}$$

and finally

$$\begin{aligned} R_e(t) &= R_i R^i(t), && \rightarrow \phi_e(t) \\ \boldsymbol{\omega}_e(t) &= R_i \boldsymbol{\omega}^i(t), && \rightarrow \dot{\phi}_e(t) \\ \dot{\boldsymbol{\omega}}_e(t) &= R_i \dot{\boldsymbol{\omega}}^i(t), && \rightarrow \ddot{\phi}_e(t) \end{aligned}$$