ROBOTICS, VISION AND CONTROL

Point-to-Point. Trapezoidal

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Outline

Composition of Elementary Trajectories
Trapezoidal velocity

PROJECT

Composition of Elementary Trajectories Trapezoidal velocity

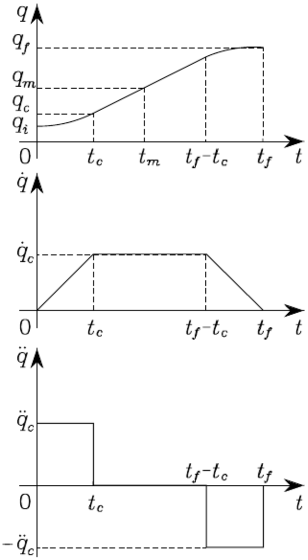
Trapezoidal velocity

To obtain trajectories with a continuous velocity profile is to use linear motions with parabolic blends, characterized therefore by the typical *trapezoidal velocity profiles*.

Popular solution with industrial robots.

Trajectories are divided into three parts:

1. constant acceleration in the start phase
2. constant cruise velocity
3. constant deceleration in the arrival phase



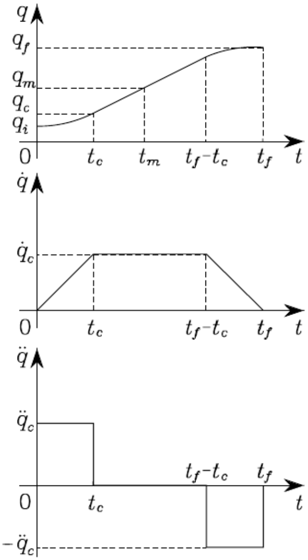
The duration of the acceleration phase is usually assumed equal to the duration of the deceleration phase $\rightarrow t_c$

Constraints:

- t_i, t_f
- q_i, q_f

Assumptions:

- $t_i = 0$
- $\dot{q}_i(t_i) = 0$, $\dot{q}_f(t_f) = 0$
- acc time = decel time = t_c ($\Rightarrow$ symmetric profile)



Trapezoidal velocity

1. constant acceleration in the start phase

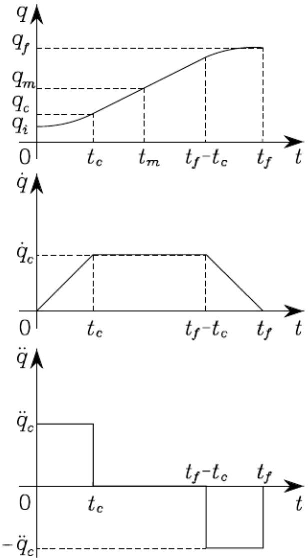
   Assuming a positive displacement, $q_f > q_i$, in the first part the acceleration is positive and constant, and therefore the velocity is a linear function of time and the position is a parabolic curve.

$$
\begin{aligned}
q(t) &= a_0 + a_1 t + a_2 t^2 \\
\dot{q}(t) &= a_1 + 2 a_2 t \qquad\qquad t \in [0, t_c] \\
\ddot{q}(t) &= 2 a_2
\end{aligned}
$$

If $\dot{q}_i = 0$

$$
\begin{cases}
a_0 = q_i \\
a_1 = 0 \\
a_2 = \frac{\dot{q}_c}{2 t_c}
\end{cases}
$$

The acceleration is $\ddot{q}_c = \frac{\dot{q}_c}{t_c}$



Trapezoidal velocity

2. constant cruise velocity

   In the second part the acceleration is null, the velocity is constant and the position is a linear function of time.

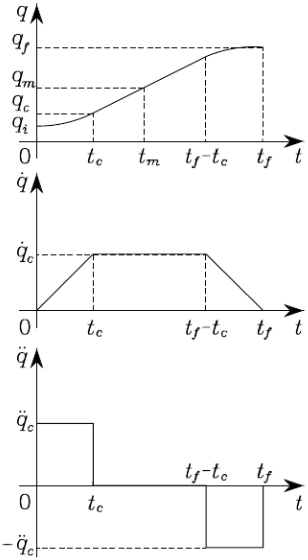
$$\begin{aligned} q(t) &= b_0 + b_1 t \\ \dot{q}(t) &= b_1 \qquad\qquad t \in [t_c, t_f - t_c] \\ \ddot{q}(t) &= 0 \end{aligned}$$

   Then

$$\begin{cases} b_1 = \dot{q}_c \\ b_0 = q_i - \frac{\dot{q}_c t_c}{2} \end{cases}$$

   because

$$q(t_c) = \underbrace{q_i + \frac{\dot{q}_c}{2} t_c}_{\text{final point phase 1}} = b_0 + \dot{q}_c t_c$$



3. constant deceleration in the arrival phase

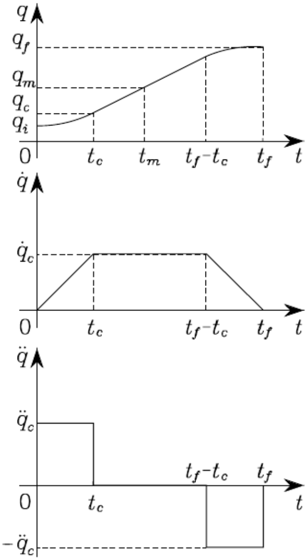
   In the last part, a constant negative acceleration is present, the velocity decreases linearly and the position is again a polynomial function of degree two.

$$\begin{aligned} q(t) &= c_0 + c_1 t + c_2 t^2 \\ \dot{q}(t) &= c_1 + 2c_2 t \qquad\qquad t \in [t_f - t_c, t_f] \\ \ddot{q}(t) &= 2c_2 \end{aligned}$$

If $\dot{q}_f = 0$

$$\begin{cases} c_0 = q_f - \dfrac{\dot{q}_c t_f^2}{2t_c} \\ c_1 = \dfrac{\dot{q}_c t_f}{t_c} \\ c_2 = -\dfrac{\dot{q}_c}{2t_c} \end{cases}$$

The deceleration is $\ddot{q}_c = -\frac{\dot{q}_c}{t_c}$



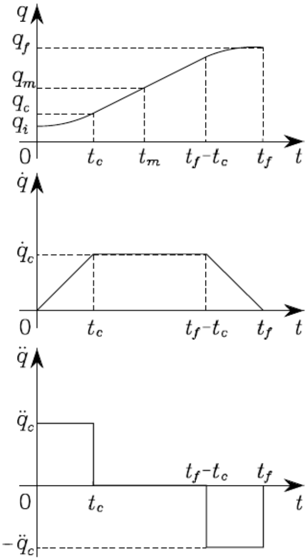
Trapezoidal velocity

Finally

$$q(t) = \begin{cases} q_i + \frac{\dot{q}_c}{2t_c} t^2, & 0 \le t \le t_c \\ q_i + \dot{q}_c \left(t - \frac{t_c}{2}\right), & t_c < t \le t_f - t_c \\ q_f - \frac{\dot{q}_c}{2t_c} (t_f - t)^2, & t_f - t_c < t \le t_f \end{cases}$$

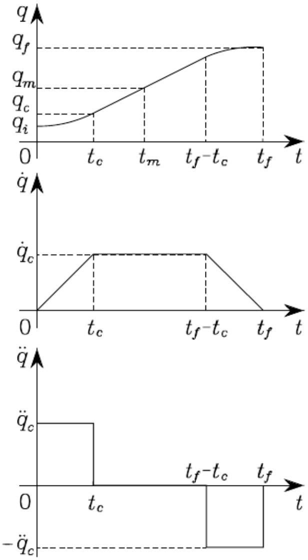
If $t_i \neq 0$

$$q(t) = \begin{cases} q_i + \frac{\dot{q}_c}{2t_c} (t - t_i)^2, & t_i \le t \le t_i + t_c \\ q_i + \dot{q}_c \left(t - t_i - \frac{t_c}{2}\right), & t_i + t_c < t \le t_f - t_c \\ q_f - \frac{\dot{q}_c}{2t_c} (t_f - t)^2, & t_f - t_c < t \le t_f \end{cases}$$



Trapezoidal velocity

$$q(t) = \begin{cases} q_i + \frac{\dot{q}_c}{2t_c} t^2, & 0 \le t \le t_c \\ q_i + \dot{q}_c \left(t - \frac{t_c}{2}\right), & t_c < t \le t_f - t_c \\ q_f - \frac{\dot{q}_c}{2t_c}(t_f - t)^2, & t_f - t_c < t \le t_f \end{cases}$$



We need further constraints on $t_c, \dot{q}_c$ to compute the trajectory. Not all values provide feasible solutions.

First of all, let's state the following condition

$$t_c \le \frac{t_f - t_i}{2}$$

and conditions on the maximum velocity and acceleration

Conditions on the maximum velocity and acceleration. Let

$$\begin{aligned} q_m &= \frac{q_i + q_f}{2} \\ t_m &= \frac{t_f - t_i}{2} \\ q_c &= q(t_i + t_c) \end{aligned}$$

from the velocity continuity condition at $t = t_i + t_c$, we get

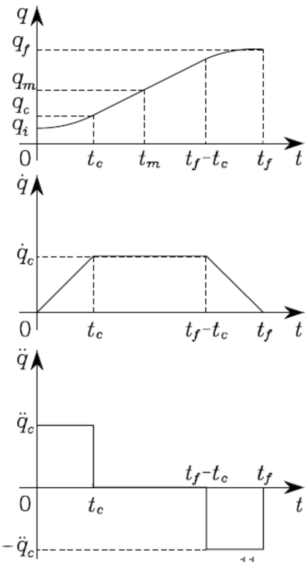
$$\ddot{q}_c t_c = \frac{q_m - q_c}{t_m - t_c}$$

and

$$q_c = q_i + \frac{1}{2}\ddot{q}_c t_c^2$$

Finally, we end up with the equation

$$\ddot{q}_c t_c^2 - \ddot{q}_c(t_f - t_i)t_c + (q_f - q_i) = 0$$



Trapezoidal velocity

$$q(t) = \begin{cases} q_i + \frac{\dot{q}_c}{2t_c} t^2, & 0 \leq t \leq t_c \\ q_i + \dot{q}_c \left(t - \frac{t_c}{2}\right), & t_c < t \leq t_f - t_c \\ q_f - \frac{\dot{q}_c}{2t_c} (t_f - t)^2, & t_f - t_c < t \leq t_f \end{cases}$$

Given t_c $(t_i = 0)$	Given $\ddot{q}_c$ $(t_i = 0)$	Given $\dot{q}_c$ $(t_i = 0)$
$\ddot{q}_c = \frac{q_f - q_i}{t_c\, t_f - t_c^2}$ $\dot{q}_c = \ddot{q}_c t_c$ If $t_c = \alpha t_f$ with $0 < \alpha \leq \frac{1}{2}$ $\ddot{q}_c = \frac{q_f - q_i}{\alpha(1-\alpha) t_f^2}$ $\dot{q}_c = \frac{q_f - q_i}{(1-\alpha) t_f}$	$t_c = \frac{t_f}{2} - \frac{1}{2}\sqrt{\frac{t_f^2 \ddot{q}_c - 4(q_f - q_i)}{\ddot{q}_c}}$ N.B. $\Delta > 0$	$t_c = \frac{q_i - q_f + \dot{q}_c t_f}{\dot{q}_c}$ $\ddot{q}_c = \frac{\dot{q}_c^2}{q_i - q_f + \dot{q}_c t_f}$ N.B. $t_c > 0$, $t_c < t_f - t_c$

Preassigned acceleration $\ddot{q}_c$ **and** velocity $\dot{q}_c$.

Conditions ($t_i = 0$):

$$
\begin{aligned}
t_c &= \frac{\dot{q}_c}{\ddot{q}_c} \\
t_f &= \frac{\dot{q}_c^2 + \ddot{q}_c(q_f - q_i)}{\dot{q}_c \ddot{q}_c}
\end{aligned}
$$

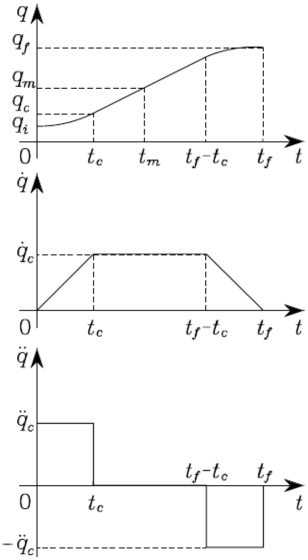
Since $t_c \leq \frac{t_f}{2}$, the linear segment exists if and only if

$$
q_f - q_i \geq \frac{\dot{q}_c^2}{\ddot{q}_c}
$$

If not, the velocity profile has a triangular shape.

Instead of using $(t_c, \dot{q}_c)$, it is possible to derive the coefficients of the polynomials as a function of $(t_c, \ddot{q}_c)$

$$q(t) = \begin{cases} q_i + \frac{1}{2}\ddot{q}_c t^2, & 0 \leq t \leq t_c \\ q_i + \ddot{q}_c t_c (t - t_c/2), & t_c < t \leq t_f - t_c \\ q_f - \frac{1}{2}\ddot{q}_c (t_f - t)^2, & t_c < t \leq t_f - t_c \end{cases}$$



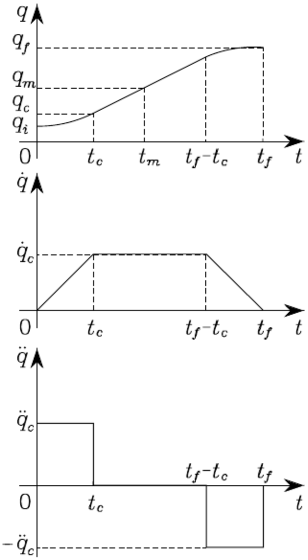
Exercise. Compute the above expressions for the position, and derive velocity and acceleration in the three phases.

Exercise. Include in the above expressions the initial time t_i.

Trapezoidal velocity

$$
q(t) = \begin{cases} q_i + \frac{\dot{q}_c}{2t_c} t^2, & 0 \le t \le t_c \\ q_i + \dot{q}_c \left(t - \frac{t_c}{2} \right), & t_c < t \le t_f - t_c \\ q_f - \frac{\dot{q}_c}{2t_c} (t_f - t)^2, & t_f - t_c < t \le t_f \end{cases}
$$

$$
q(t) = \begin{cases} q_i + \frac{1}{2} \ddot{q}_c t^2, & 0 \le t \le t_c \\ q_i + \ddot{q}_c t_c (t - t_c/2), & t_c < t \le t_f - t_c \\ q_f - \frac{1}{2} \ddot{q}_c (t_f - t)^2, & t_c < t \le t_f - t_c \end{cases}
$$



Trapezoidal velocity

Observation. If several actuators must be coordinated, all the movements must be the defined according to the slowest one, or the one with the largest displacement.

Any problems in the Operational coordinates?!?

Observation. If a trajectory through a sequence of points is planned with trapezoidal velocity, the resulting motion will present null velocities in the intermediate points.
Since this may be unacceptable, the generic intermediate motion between the points q_k and q_{k+1} may be anticipated in such a way that it starts before the motion between the points q_{k-1} and q_k is concluded.

Trajectory with non-null initial velocity $\dot{q}_i \neq 0$ and final velocity $\dot{q}_f \neq 0$, and null initial and final accelerations $\ddot{q}_i = 0$, $\ddot{q}_f = 0$. ($q_f > q_i$)
t_a: Acceleration time
t_d: Deceleration time

Acceleration phase

$$\begin{aligned} q(t) &= q_i + \dot{q}_i(t - t_i) + \frac{\dot{q}_c - \dot{q}_i}{2t_a}(t - t_i)^2 \\ \dot{q}(t) &= \dot{q}_i + \frac{\dot{q}_c - \dot{q}_i}{t_a}(t - t_i) \qquad\qquad t \in [t_i, t_i + t_a] \\ \ddot{q}(t) &= \frac{\dot{q}_c - \dot{q}_i}{t_a} =: \ddot{q}_c \end{aligned}$$

Trapezoidal velocity

Constant velocity phase

$$
\begin{aligned}
q(t) &= q_i + \dot{q}_i \frac{t_a}{2} + \dot{q}_c \left(t - t_i - \frac{t_a}{2} \right) \\
\dot{q}(t) &= \dot{q}_c & t \in [t_i + t_a, t_f - t_d] \\
\ddot{q}(t) &= 0
\end{aligned}
$$

Deceleration phase

$$
\begin{aligned}
q(t) &= q_f - \dot{q}_f(t_f - t) - \frac{\dot{q}_c - \dot{q}_f}{2t_d}(t_f - t)^2 \\
\dot{q}(t) &= \dot{q}_f + \frac{\dot{q}_c - \dot{q}_f}{t_d}(t_f - t) \qquad\qquad t \in [t_f - t_d, t_f] \\
\ddot{q}(t) &= -\frac{\dot{q}_c - \dot{q}_f}{t_d} =: -\ddot{q}_c
\end{aligned}
$$

Remark. As in the case with null initial and final velocities, there are a few choices to be made before defining the trajectory.

Trapezoidal velocity

Trajectory with preassigned duration $\Delta T = t_f - t_i$ and maximum acceleration $\ddot{q}_c^{max}$

- Check feasibility, i.e. $\ddot{q}(t) \leq \ddot{q}_c^{max}$

$$\ddot{q}_c^{max} \Delta q > \frac{|\dot{q}_i^2 - \dot{q}_f^2|}{2}$$

  If '$<$' it is not possible to find a trapezoidal trajectory compliant with the given constraints on initial and final velocities and maximum acceleration, since the displacement $q_f - q_i$ is too small with respect to $\dot{q}_i$ or $\dot{q}_f$.

  If the initial and final speeds are both null, the trajectory is always feasible.

- If the trapezoidal trajectory exists, it is possible to compute the constant velocity

$$\dot{q}_c = \frac{1}{2}\left(\dot{q}_i + \dot{q}_f + \ddot{q}_c^{max}\Delta T - \sqrt{(\ddot{q}_c^{max})^2 \Delta T^2 - 4\ddot{q}_c^{max}\Delta q + 2\ddot{q}_c^{max}(\dot{q}_i + \dot{q}_f)\Delta T - (\dot{q}_i - \dot{q}_f)^2}\right)$$

  where $\ddot{q}_c^{max}$ must satisfy

$$(\ddot{q}_c^{max})^2 \Delta T^2 - 4\ddot{q}_c^{max}\Delta q + 2\ddot{q}_c^{max}(\dot{q}_i + \dot{q}_f)\Delta T - (\dot{q}_i - \dot{q}_f)^2 > 0$$

- The max acceleration must be larger than a limit value

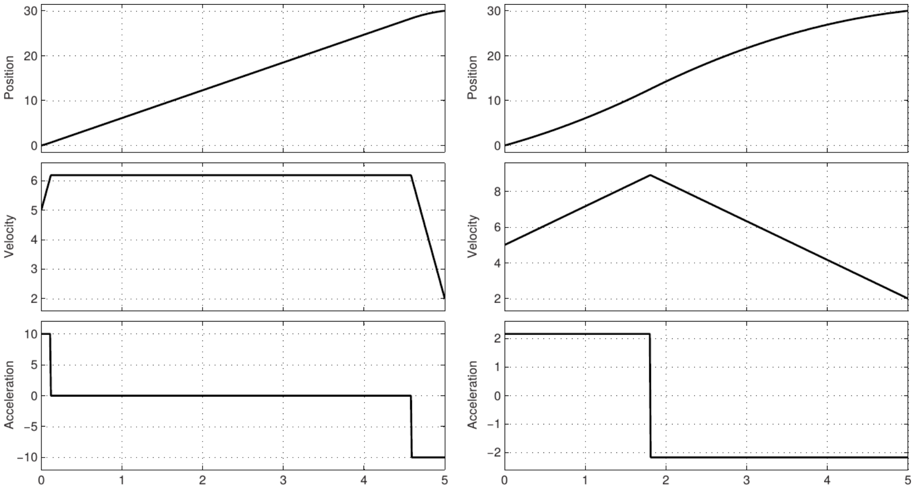
$$\ddot{q}_c^{max} \geq \ddot{q}_c^{lim} = \frac{2\Delta q - (\dot{q}_i + \dot{q}_f)\Delta T + \sqrt{4\Delta q^2 - 4\Delta q(\dot{q}_i + \dot{q}_f)\Delta T + 2(\dot{q}_i^2 + \dot{q}_f^2)\Delta T^2}}{\Delta T^2}$$

  If $\ddot{q}_c^{max} = \ddot{q}_c^{lim}$ there is no constant velocity phase.

- Computation of the acceleration/deceleration periods

$$t_a = \frac{\dot{q}_c - \dot{q}_i}{\ddot{q}_c^{max}}, \qquad t_d = \frac{\dot{q}_c - \dot{q}_f}{\ddot{q}_c^{max}}$$

Trapezoidal velocity



Trapezoidal velocity

Trajectory with preassigned maximum acceleration $\ddot{q}_c^{max}$ and maximum velocity $\dot{q}_c^{max}$ (ΔT is one of the output of the planning.)

- Check feasibility, i.e. $\ddot{q}(t) \leq \ddot{q}_c^{max}$

$$\ddot{q}_c^{max} \Delta q > \frac{|\dot{q}_i^2 - \dot{q}_f^2|}{2} \tag{1}$$

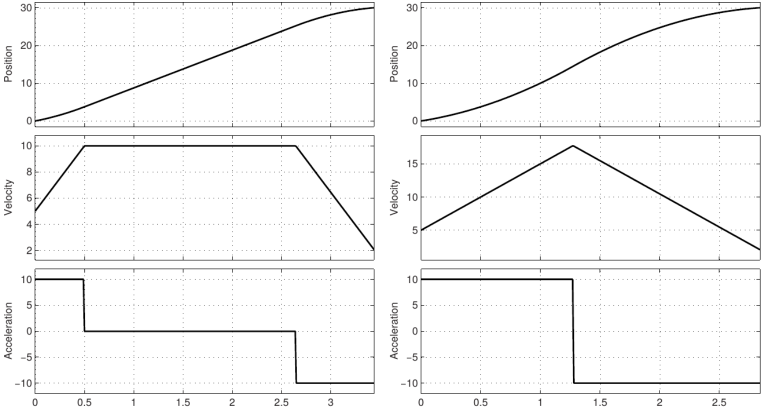
- If the trajectory exists, two cases are possible according to the fact that the maximum velocity $\dot{q}_c^{max}$ is reached ($>$) or not ($<$)

$$\ddot{q}_c^{max} \Delta q \quad \gtreqless \quad (\dot{q}_c^{max})^2 - \frac{\dot{q}_i^2 + \dot{q}_f^2}{2} \tag{2}$$

  - If $\dot{q}_c^{max}$ is reached, then during the constant velocity phase $\dot{q}_c = \dot{q}_c^{max}$
  - If $\dot{q}_c^{max}$ is not reached, then the constant velocity phase is not present

Trapezoidal velocity

$\dot{q}_c^{max}$ is reached	$\dot{q}_c^{max}$ is not reached
$\dot{q}_c = \dot{q}_c^{max}$	The maximum velocity of the trajectory is $\dot{q}_c = \dot{q}_c^{lim} = \sqrt{\ddot{q}_c^{max}\Delta q + \frac{\dot{q}_i^2 + \dot{q}_f^2}{2}} < \dot{q}_c^{max}$
Acceleration / Deceleration periods	Acceleration / Deceleration periods
$t_a = \frac{\dot{q}_c^{max} - \dot{q}_i}{\ddot{q}_c^{max}}, \qquad t_d = \frac{\dot{q}_c^{max} - \dot{q}_f}{\ddot{q}_c^{max}}$	$t_a = \frac{\dot{q}_c^{lim} - \dot{q}_i}{\ddot{q}_c^{max}}, \qquad t_d = \frac{\dot{q}_c^{lim} - \dot{q}_f}{\ddot{q}_c^{max}}$
Trajectory duration	
$\Delta T = \frac{\Delta q}{\Delta T} + \frac{\dot{q}_c^{max}}{2\ddot{q}_c^{max}}\left(1 - \frac{\dot{q}_i}{\dot{q}_c^{max}}\right)^2 + \frac{\dot{q}_c^{max}}{2\ddot{q}_c^{max}}\left(1 - \frac{\dot{q}_f}{\dot{q}_c^{max}}\right)^2$	Trajectory duration



Same initial and final positions and velocity with $\ddot{q}_c^{max} = 10$;
Left $\dot{q}_c^{max} = 10$, Right $\dot{q}_c^{max} = 20$. $\Delta T \neq$

Trapezoidal velocity

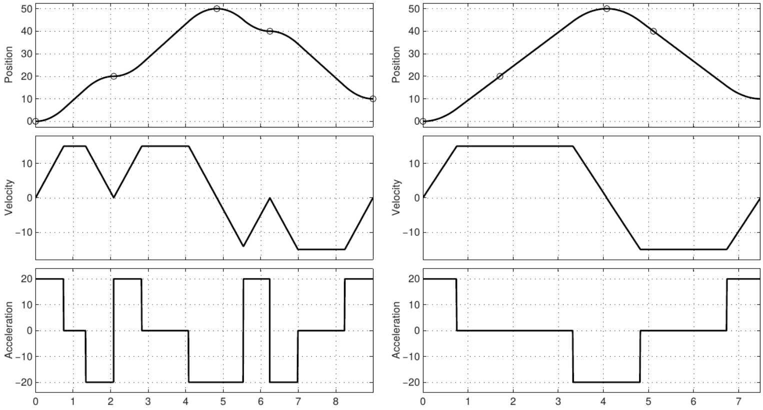
If the goal is to design a trajectory through a sequence of points, the velocity at each point can be defined as

$$
\begin{aligned}
\dot{q}(t_i) &= \dot{q}_i \\
\dot{q}(t_k) &= \begin{cases} 0, & \text{if } \operatorname{sign}(\Delta Q_k) \neq \operatorname{sign}(\Delta Q_{k+1}) \\ \operatorname{sign}(\Delta Q_k)\dot{q}^{max}, & \text{if } \operatorname{sign}(\Delta Q_k) = \operatorname{sign}(\Delta Q_{k+1}) \end{cases} \\
\dot{q}(t_f) &= \dot{q}_f
\end{aligned}
$$

where

$$\Delta Q_k = q_k - q_{k-1}.$$

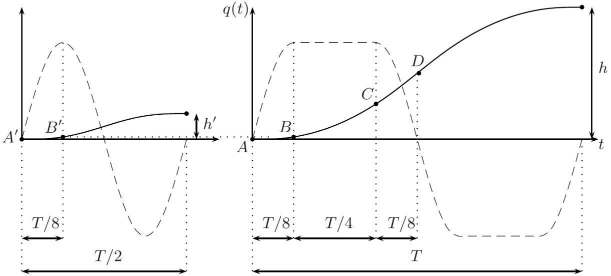
Trapezoidal velocity

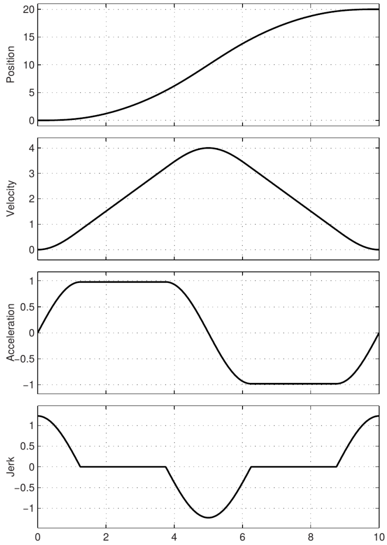


What is the difference?

Modified Trapezoidal trajectory

Modified Trapezoidal trajectory: the trajectory is subdivided into six parts: the second and the fifth segments are defined by second degree polynomials, while the remaining ones are expressed by cycloidal functions ($\Rightarrow$ continuity of the acceleration profile).





With trajectory composed by constant velocity/acceleration segments connected by trigonometric blends

Minimum-time trajectories: Given the initial and final position values q_i and q_f, compute the trajectory in order to minimize its duration.

This is equivalent to impose in each segment of the trajectory the maximum value allowed for the velocity or the acceleration


Project

PROJECT – Assignment # 2

To do

- Implement in Matlab the Trapezoidal trajectory taking into account the different constraints.