ROBOTICS, VISION AND CONTROL

Trajectory Planning. Multipoint

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Outline

Complements

Complements

Choice of the time instants

In several cases, only the interpolation points q_k, $k = 0, \ldots, n$, the initial time t_0 and the final time t_n are known.

How to choose the *intermediate time instants* $t_1, t_2, \ldots, t_{n-1}$?

Let's focus on a normalized interval, i.e. $t_0 = 0$ and $t_n = 1$

The distribution of the intermediate time instants is

$$t_k = t_{k-1} + \frac{d_k}{d}, \qquad d = \sum_{k=0}^{n-1} d_k$$

The problem is now: how to choose d_k?

[Since t_n is fixed, different distributions in time of the interpolation points do non change the duration of the trajectory]

Choice of the time instants

Three main options:

- equally spaced points

$$d_k = \frac{1}{n-1}$$

- cord length distribution

$$d_k = |q_{k+1} - q_k|$$

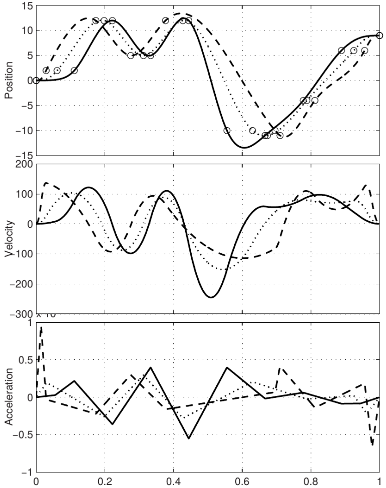
- centripetal distribution

$$d_k = \sqrt{|q_{k+1} - q_k|}$$

- or

$$d_k = |q_{k+1} - q_k|^{\mu}$$

Choice of the time instants



$$q_0 = 0, q_1 = 2, q_2 = 12, q_3 = 5,$$
$$q_4 = 12, q_5 = -10, q_6 = -11, q_7 = -4,$$
$$q_8 = 6, q_9 = 9$$

uniformly spaced (solid),
cord length (dashed),
centripetal (dotted)

Choice of the time instants

Exercise. Scale the trajectory duration to a generic t_f

Optimization of cubic splines

The total duration of a spline trajectory $s(t)$ interpolating the points (t_k, q_k), $k = 0, \ldots, n$ is

$$T = \sum_{k=0}^{n-1} T_k, \qquad T_k = t_{k+1} - t_k$$

and so

$$T = t_n - t_0$$

The goal is to minimize T such that the constraints on min/max velocity and min/max accelerations are satisfied

$$\{T_k^\circ\} = \arg\min_{T_k} T \quad = \quad \sum_{k=0}^{n-1} T_k$$

$$\text{s. to} \qquad \begin{aligned} |\dot{s}(t; T_0, T_1, \ldots, T_{n-1})| &< \dot{q}^{max} \\ |\ddot{s}(t; T_0, T_1, \ldots, T_{n-1})| &< \ddot{q}^{max} \end{aligned}$$

It is a *nonlinear optimum problem* with a linear objective function, solvable with classical techniques of operational research.

Suboptimal solution
Since the coefficients which determine the spline (and as a consequence the value of the velocity and of the acceleration along the trajectory) are computed as a function of the intervals T_k, the optimization problem can be solved in a iterative way, by *scaling in time the segments* which compose the spline

If the time interval T_k is replaced by

$$T_k \quad \rightarrow \quad \lambda T_k$$

then the velocity, acceleration and jerk are scaled by

$$\begin{aligned}
\dot{\Pi}_k(t) \quad &\rightarrow \quad \frac{1}{\lambda}\dot{\Pi}_k(t) \\
\ddot{\Pi}_k(t) \quad &\rightarrow \quad \frac{1}{\lambda^2}\ddot{\Pi}_k(t) \\
\dddot{\Pi}_k(t) \quad &\rightarrow \quad \frac{1}{\lambda^3}\dddot{\Pi}_k(t)
\end{aligned}$$

Then the optimal λ is

$$\lambda^{\circ} = \max\{\lambda_v, \lambda_a, \lambda_j\}$$

where

$$\lambda_v = \max_k \lambda_{v,k}, \qquad \lambda_{v,k} = \max_{t\in[t_k,t_{k+1})} \frac{|\dot{\Pi}_k(t)|}{\dot{q}^{max}}$$

$$\lambda_a = \max_k \lambda_{a,k}, \qquad \lambda_{a,k} = \max_{t\in[t_k,t_{k+1})} \sqrt{\frac{|\ddot{\Pi}_k(t)|}{\ddot{q}^{max}}}$$

$$\lambda_j = \max_k \lambda_{j,k}, \qquad \lambda_{j,k} = \max_{t\in[t_k,t_{k+1})} \left(\frac{|\dddot{\Pi}_k(t)|}{\dddot{q}^{max}}\right)^{1/3}$$

The spline $s(t)$ will reach the maximum speed or the maximum acceleration or the maximum jerk, in at least a point of the interval $[t_0, t_n]$

Geometric Modification of a Trajectory

- *space-translation* of a trajectory $q(t)$ from $(0, 0)$ to (t_1, q_1)

$$\bar{q}(t) = q(t) + q_0$$

$\bar{q}(t) : (0, q_0) \mapsto (t_1, q_0 + q_1)$

- *time-translation* of a trajectory $q(t)$ from $(0, 0)$ to (t_1, q_1)

$$\bar{q}(t) = q(t - t_0)$$

$\bar{q}(t) : (t_0, 0) \mapsto (t_0 + t_1, q_1)$

- *space-reflection* of a trajectory $q(t)$ from $(0, 0)$ to (t_1, q_1)

$$\bar{q}(t) = -q(t)$$

$\bar{q}(t) : (0, 0) \mapsto (t_1, -q_1)$

- *scaling in space* of a 'unitary' trajectory $q(t)$ from $(0,0)$ to $(1,1)$

$$\bar{q}(t) = q_0 + h\,q(t), \qquad h = q_1 - q_0$$

  $\bar{q}(t) : (0, q_0) \mapsto (1, q_1)$

- *scaling in time* of a 'unitary' trajectory $q(t)$ from $(0,0)$ to $(1,1)$

$$\bar{q}(t) = q(t/t_1)$$

  $\bar{q}(t) : (0, 0) \mapsto (t_1, 1)$

Geometric modifications

Exercise. Starting from the 'unitary' cubic polynomial $q(t)$ from $(0, 0)$ to $(1, 1)$ with zero initial and final velocities, design the multi-point trajectory passing through $(0, 0)$, $(2, 1)$, $(3, -1)$, $(5, 2)$ using the previous properties.

Scaling in time

The scaling in time is useful when we have to satisfied the following constraints

- *Kinematic saturation*: limits on velocity and acceleration

$$|\dot{q}(t)| \leq \dot{q}^{max}, \qquad |\ddot{q}(t)| \leq \ddot{q}^{max}$$

- *Dynamic saturation*: limits on the torques requested to the motors

$$|\tau(t)| \leq \tau^{max}$$

and the profile is already planned.

Scaling in time

Given the original trajectory $q(t)$, define a strictly increasing function σ

$$t = \sigma(\bar{t})$$

such that the scaled trajectory

$$\bar{q}(\bar{t}) = (q \circ \sigma)(\bar{t}) = q(\sigma(\bar{t}))$$

has its velocity and acceleration profiles

$$\begin{aligned}
\dot{\bar{q}}(\bar{t}) &= \frac{dq(\sigma)}{d\sigma} \frac{d\sigma(\bar{t})}{d\bar{t}} \\
\ddot{\bar{q}}(\bar{t}) &= \frac{dq(\sigma)}{d\sigma} \frac{d^2\sigma(\bar{t})}{d\bar{t}^2} + \frac{d^2q(\sigma)}{d\sigma^2} \left(\frac{d\sigma(\bar{t})}{d\bar{t}}\right)^2
\end{aligned}$$

that satisfied the kinematic contraints

A common σ function is the linear one

$$t = \lambda \bar{t}$$

with $\lambda > 0$. Then we have

$$\begin{aligned}
\dot{\bar{q}}(\bar{t}) &= \frac{dq(\sigma)}{d\sigma}\lambda &= \lambda\,\dot{q}(t) \\
\ddot{\bar{q}}(\bar{t}) &= \frac{d^2q(\sigma)}{d\sigma^2}\lambda^2 &= \lambda^2\,\ddot{q}(t) \\
\dddot{\bar{q}}(\bar{t}) &= \frac{d^3q(\sigma)}{d\sigma^3}\lambda^3 &= \lambda^3\,\dddot{q}(t) \\
&\vdots
\end{aligned}$$

The choice

$$\lambda = \min\left\{\frac{\dot{q}^{max}}{\max_t|\dot{q}(t)|}, \sqrt{\frac{\ddot{q}^{max}}{\max_t|\ddot{q}(t)|}}, \left(\frac{\dddot{q}^{max}}{\max_t|\dddot{q}(t)|}\right)^{1/3}\right\}$$

guarantees that the maximum values of speed, acceleration and jerk are never exceeded.

Scaling in time – Kinematics constraint

Let $\tilde{q}(\tau)$ be the *normalized trajectory*

$$0 \leq \tilde{q}(\tau) \leq 1, \qquad 0 \leq \tau \leq 1$$

A generic trajectory from (t_0, q_0) to (t_1, q_1) can be written as

$$\begin{aligned} q(t) &= q_0 + (q_1 - q_0)\tilde{q}\left(\frac{t - t_0}{t_1 - t_0}\right) \\ &= q_0 + \Delta q\, \tilde{q}\left(\frac{t - t_0}{\Delta T}\right) \end{aligned}$$

It is then possible to compute the time derivative of q as a function of the time derivative of $\tilde{q}$

$$\dot{q}(t) = \frac{\Delta q}{\Delta T}\dot{\tilde{q}}(\tau), \qquad \ddot{q}(t) = \frac{\Delta q}{(\Delta T)^2}\ddot{\tilde{q}}(\tau), \qquad \dddot{q}(t) = \frac{\Delta q}{(\Delta T)^3}\dddot{\tilde{q}}(\tau), \qquad \dots$$

Since the maximum values of velocity, acceleration, jerk of $q(t)$ are obtained in correspondence of the maximum values of the velocity, acceleration, jerk of $\tilde{q}(\tau)$, it is easy to compute both these values and the corresponding time instants τ from the a given parameterization $\tilde{q}$.

Scaling in time – Kinematics constraint

By scaling ΔT we obtain motion profiles with maximum velocity/acceleration values equal to the saturation limits.

Example. Normalized trajectory = polynomial of order 3 with zero initial and final velocities

$$\begin{aligned}\tilde{q}(\tau) &= a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 \\ &= 3\tau^2 - 2\tau^3\end{aligned}$$

Then

$$\dot{\tilde{q}}(\tau) = 6\tau - 6\tau^2, \qquad \ddot{\tilde{q}}(\tau) = 6 - 12\tau, \qquad \dddot{\tilde{q}}(\tau) = -12$$

and the maximum velocity and acceleration are

$$\dot{\tilde{q}}^{max} = \max_\tau \dot{\tilde{q}}(\tau) = \dot{\tilde{q}}(0.5) = \frac{3}{2}, \qquad \ddot{\tilde{q}}^{max} = \max_\tau \ddot{\tilde{q}}(\tau) = \ddot{\tilde{q}}(0) = 6,$$

According to the expression in the previous slide, we end up with

$$\dot{q}^{max} = \frac{3}{2}\frac{\Delta q}{\Delta T}, \qquad \ddot{q}^{max} = 6\frac{\Delta q}{(\Delta T)^2}$$

Exercise. Compute the expression for the maximum velocity, acceleration and jerk when the nominal trajectory is a polynomial of order 5 with zero initial and final velocities, and zero initial and final accelerations.

The equations of motion of a robotic manipulator with n DoF are

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\dot{\boldsymbol{q}}, \boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{1}$$

The i-th row is

$$\boldsymbol{b}_i^T(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{L}_i(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{g}_i(\boldsymbol{q}) = \boldsymbol{\tau}_i \tag{2}$$

where $\frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{L}_i(\boldsymbol{q})\dot{\boldsymbol{q}}$ is an equivalent expression of the i-th component of $\boldsymbol{C}(\dot{\boldsymbol{q}}, \boldsymbol{q})\dot{\boldsymbol{q}}$.

Given the trajectory $\boldsymbol{q}(t)$, $t \in [0, T]$, we have

$$\underbrace{\boldsymbol{b}_i^T(\boldsymbol{q}(t))\ddot{\boldsymbol{q}}(t) + \frac{1}{2}\dot{\boldsymbol{q}}^T(t)\boldsymbol{L}_i(\boldsymbol{q}(t))\dot{\boldsymbol{q}}(t)}_{\boldsymbol{\tau}_{s,i}(t)} + \underbrace{\boldsymbol{g}_i(\boldsymbol{q}(t))}_{\boldsymbol{\tau}_{p,i}(t)} = \boldsymbol{\tau}_i(t) \tag{3}$$

where $\boldsymbol{\tau}_{p,i}(t)$ depends only on the position.

Scaling in time – Dynamics constraint

We now consider a scaled version $\bar{\boldsymbol{q}}(\bar{t})$, $\bar{t} \in [0, \bar{T}]$ of $\boldsymbol{q}(t)$, $t \in [0, T]$ with

$$t = \sigma(\bar{t}) \tag{4}$$

The corresponding torques $\bar{\tau}_i(t)$, $i = 1, \ldots, n$ are

$$\bar{\tau}_i(\bar{t}) = \boldsymbol{b}_i^T(\bar{\boldsymbol{q}}(\bar{t}))\ddot{\bar{\boldsymbol{q}}}(\bar{t}) + \frac{1}{2}\dot{\bar{\boldsymbol{q}}}^T(\bar{t})\boldsymbol{L}_i(\bar{\boldsymbol{q}}(\bar{t}))\dot{\bar{\boldsymbol{q}}}(\bar{t}) + \boldsymbol{g}_i(\bar{\boldsymbol{q}}(\bar{t})); \tag{5}$$

exploiting the relationships

$$\bar{\boldsymbol{q}}(\bar{t}) = (\bar{\boldsymbol{q}} \circ \sigma)(\bar{t}) \tag{6}$$

$$\dot{\bar{\boldsymbol{q}}}(\bar{t}) = \dot{\boldsymbol{q}}(t)\dot{\sigma}, \qquad \dot{\sigma} = \frac{d\sigma}{d\bar{t}} \tag{7}$$

$$\ddot{\bar{\boldsymbol{q}}}(\bar{t}) = \ddot{\boldsymbol{q}}(t)\dot{\sigma}^2 + \dot{\boldsymbol{q}}(t)\ddot{\sigma}, \qquad \ddot{\sigma} = \frac{d^2\sigma}{d\bar{t}^2} \tag{8}$$

we have

$$\bar{\tau}_i(\bar{t}) = \boldsymbol{b}_i^T(\boldsymbol{q}(t))\dot{\boldsymbol{q}}(t)\ddot{\sigma} + \left[\boldsymbol{b}_i^T(\boldsymbol{q}(t))\ddot{\boldsymbol{q}}(t) + \frac{1}{2}\dot{\boldsymbol{q}}^T(t)\boldsymbol{L}_i(\boldsymbol{q}(t))\dot{\boldsymbol{q}}(t)\right]\dot{\sigma}^2 + \boldsymbol{g}_i(\boldsymbol{q}(t)) \tag{9}$$

The contribution of the gravity term is independent of the time scaling.

Focusing on $\bar{\boldsymbol{\tau}}_{s,i}$

$$\begin{aligned} \bar{\boldsymbol{\tau}}_{s,i}(\bar{t}) &= \boldsymbol{b}_i^T(\boldsymbol{q}(t))\dot{\boldsymbol{q}}(t)\ddot{\sigma} + \left[\boldsymbol{b}_i^T(\boldsymbol{q}(t))\ddot{\boldsymbol{q}}(t) + \frac{1}{2}\dot{\boldsymbol{q}}^T(t)\boldsymbol{L}_i(\boldsymbol{q}(t))\dot{\boldsymbol{q}}(t)\right]\dot{\sigma}^2 && (10) \\ &= \boldsymbol{b}_i^T(\boldsymbol{q}(t))\dot{\boldsymbol{q}}(t)\ddot{\sigma} + \boldsymbol{\tau}_{s,i}(t)\dot{\sigma}^2 && (11) \end{aligned}$$

In the linear scaling case $t = \sigma(\bar{t}) = \lambda\,\bar{t}$

$$\dot{\sigma}(\bar{t}) = \lambda, \qquad \ddot{\sigma}(\bar{t}) = 0 \tag{12}$$

Then

$$\bar{\boldsymbol{\tau}}_{s,i}(\bar{t}) = \boldsymbol{\tau}_{s,i}(t)\lambda^2 \tag{13}$$

and finally

$$\bar{\tau}_i(\bar{t}) - \boldsymbol{g}_i(\bar{\boldsymbol{q}}(\bar{t})) = \lambda^2\left[\tau_i(t) - \boldsymbol{g}_i(\boldsymbol{q}(t))\right] \tag{14}$$

The duration of the scaled trajectory is $\bar{T} = T/\lambda$

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Scaling in time – Dynamics constraint

Let $\boldsymbol{q}(t)$ be the planned trajectory. The scaling factor can be used to increase ($\lambda < 1$) or decrease ($\lambda > 1$) the duration in order to allow for at least to one torque to be equal to the maximum value

$$\boldsymbol{q}(t) \quad \rightarrow \quad \boldsymbol{\tau}(t) \quad \rightarrow \quad \boldsymbol{\tau}_i^{max} = \max_t \boldsymbol{\tau}_i \tag{15}$$

Then

$$\lambda^2 = \min \left\{ \frac{\boldsymbol{\tau}_1^{max}}{|\max_t \boldsymbol{\tau}_1(t)|}, \frac{\boldsymbol{\tau}_2^{max}}{|\max_t \boldsymbol{\tau}_2(t)|}, \ldots, \frac{\boldsymbol{\tau}_n^{max}}{|\max_t \boldsymbol{\tau}_n(t)|} \right\} \tag{16}$$

and

$$\bar{t} = \frac{t}{\lambda} \tag{17}$$

$$\bar{T} = \frac{T}{\lambda} \tag{18}$$

Scaling in time – Dynamics constraint

Remark 1. The scaling factor λ is the same for the whole trajectory.

A more efficient approach could be to scale down the time only in the interval(s) where one torque is larger than the corresponding maximum.

$\rightarrow$ *variable scaling*

Remark 2. the minimum time motion along a given path saturates the torque or the acceleration or the velocity of one of the actuators in at least a point of each segment.