ROBOTICS, VISION AND CONTROL

Trajectory Planning. Point-to-Point Polynomials

Nicola Piccinelli

UNIVERSITÀ di VERONA
Dipartimento di INGEGNERIA PER LA MEDICINA DI INNOVAZIONE

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Outline

Problem statement

Joint Space Trajectories
Point-to-Point

PROJECT

Problem statement

Trajectory planning

Goal: compute the *reference inputs* for the motion control system to move the manipulator's end-effector to accomplish a specific task.

The trajectory planning problem consists in finding a relationship between two elements belonging to different domains: *time* and *space*.

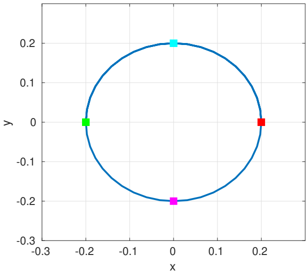
Planned trajectory is a *time sequence* of values, i.e. a parametric function of time

Geometric Path: the locus of points in the joint space ($\boldsymbol{q}$) or in the operational space ($\boldsymbol{x}$)

Trajectory: is a path on which a timing law (or motion law) is specified, for instance in terms of velocities and/or accelerations at each point, e.g. $\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), \ddot{\boldsymbol{q}}(t)$, or $\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \ddot{\boldsymbol{x}}(t)$

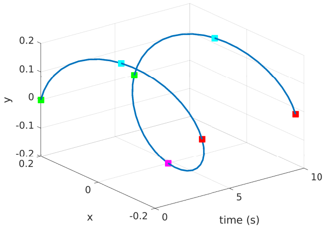
path

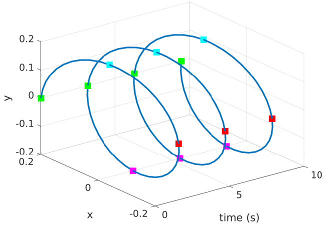
$$\boldsymbol{p} = \boldsymbol{p}(u)$$



trajectories

$$\boldsymbol{p}(t) = \boldsymbol{p}(u(t)) = \boldsymbol{p}(u) \circ u(t)$$





Trajectory planning

Inputs:

- path description
- path constraints (e.g. positions and velocities should be continuous functions of time)
- trajectory time (duration)
- dynamics constraints (manipulator)
- obstacles

$\rightarrow$

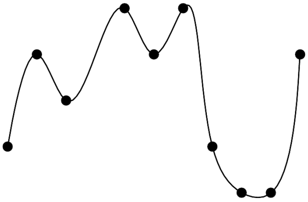
More often:

- extremal points
- intermediate points
- geometric primitives interpolating the points,
- velocity constraints
- acceleration constraints
- velocity and acceleration at particular points of interest

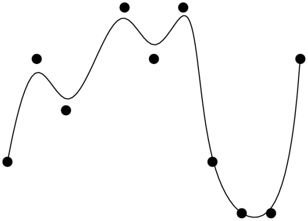
Outputs:

- time sequences of the positions (end-effector poses), velocities and accelerations which satisfy the constraints (*if a trajectory exists...*).

Interpolation: the curve crosses the given points for some values of the time



Approximation: the curve does not pass exactly through the points, but there is an error that may be assigned by specifying a prescribed tolerance



For given boundary conditions (initial and final positions, velocities, accelerations, etc.) and duration, the *typology of the trajectory* has a strong influence on the peak values of the velocity and acceleration in the intermediate points ($\rightarrow$ frequency aspects, vibrations)

	point-to-point motion (point-to-point trajectory)	motion through a sequence of points (multi-point trajectory)
joint space	$(q_i, t_i), (q_f, t_f)$	$(q_i, t_i), (q_1, t_1), \ldots, (q_f, t_f)$
operational space	$(x_i, t_i), (x_f, t_f)$	$(x_i, t_i), (x_1, t_1), \ldots, (x_f, t_f)$

Joint Space Trajectories
Point-to-Point

Polynomials

Assumption 1: *scalar case*, i.e. single actuator, or single axis of motion
Assumption 2: *no dynamic model is taken into account*

Data: initial joint configuration $q_i \in \mathbb{R}$ at t_i, final joint configuration $q_f \in \mathbb{R}$ at t_f ($t_f = t_i +$traveling time), no constraints

Trajectories described by polynomial functions

$$q(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_2 t^2 + a_1 t + a_0, \qquad t \in [t_i, t_f]$$

where the $n+1$ coefficients a_i are determined so that the initial and final constraints are satisfied.

- The degree n of the polynomial depends on the number of conditions to be satisfied (e.g. velocities and accelerations in specific time instant $t_j \in [t_i, t_f]$) and on the desired “smoothness” of the resulting motion.
- Since the number of boundary conditions is usually even, the degree n of the polynomial function is odd.

Linear trajectory

The *linear trajectory* (constant velocity) requires a first order polynomial

$$q(t) = a_0 + a_1(t - t_i), \qquad t \in [t_i, t_f]$$

Conditions on the coefficients:

$$\begin{cases} q(t_i) = q_i = a_0 \\ q(t_f) = q_f = a_0 + a_1(t_f - t_i) \end{cases}$$

We need to solve

$$\begin{bmatrix} 1 & 0 \\ 1 & \Delta T \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} q_i \\ q_f \end{bmatrix}$$

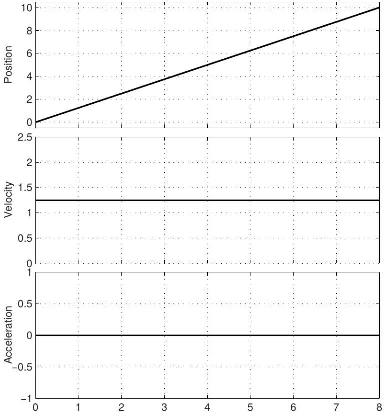
where $\Delta T = t_f - t_i$. Then, considering $\Delta q = q_f - q_i$ we have

$$\begin{cases} a_0 = q_i \\ a_1 = \frac{\Delta q}{\Delta T} \end{cases}$$

Linear trajectory

Remarks:

- The velocity $\dot{q}(t) = a_1$ is constant
- The acceleration is null in the interior of the trajectory and has an impulsive behaviour at the extremities



Parabolic trajectory

The *parabolic trajectory* (constant acceleration) requires the composition of two second degree polynomials, one from t_i to t_m (the flex point) and the second from t_m to t_f

$$t_m := \frac{t_i + t_f}{2}, \qquad q(t_m) = q_m := \frac{q_i + q_f}{2}$$

In the case of trajectory symmetric with respect to its middle point.

Acceleration period

$$q_a(t) = a_0 + a_1(t - t_i) + a_2(t - t_i)^2, \qquad t \in [t_i, t_m]$$

Conditions on the coefficients:

$$\begin{cases} q_a(t_i) = q_i = a_0 \\ q_a(t_m) = q_m = a_0 + a_1(t_m - t_i) + a_2(t_m - t_i)^2 \\ \dot{q}_a(t_i) = \dot{q}_i = a_1 \end{cases} \quad \Rightarrow \quad \begin{cases} a_0 = q_i \\ a_1 = \dot{q}_i \\ a_2 = \frac{2}{\Delta T^2}(\Delta q - \dot{q}_i \Delta T) \end{cases}$$

Parabolic trajectory

Deceleration period

$$q_d(t) = a_3 + a_4(t - t_m) + a_5(t - t_m)^2, \qquad t \in [t_m, t_f]$$

Conditions on the coefficients:

$$\begin{cases} q_d(t_m) = q_m = a_3 \\ q_d(t_f) = q_f = a_3 + a_4(t_f - t_m) + a_5(t_f - t_m)^2 \\ \dot{q}_d(t_f) = \dot{q}_f = a_4 + 2a_5(t_f - t_m) \end{cases} \Rightarrow \begin{cases} a_3 = q_m \\ a_4 = 2\frac{\Delta q}{\Delta T} - \dot{q}_f \\ a_5 = \frac{2}{\Delta T^2}(\dot{q}_f \Delta T - \Delta q) \end{cases}$$

Remarks:

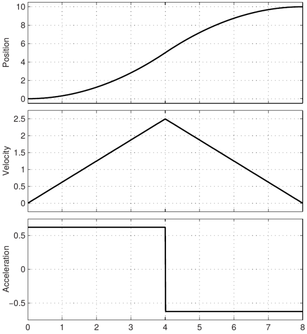
- if $\dot{q}_i \neq \dot{q}_f$ the velocity profile is discontinuous at t_m
- the maximum velocity in obtained at the flex point

$$\dot{q}_{max} = \dot{q}_a(t_m) = 2\frac{\Delta q}{\Delta T} - \dot{q}_i$$

- acceleration profile is piecewise constant with opposite sign in the acceleration/deceleration periods

$$\ddot{q}_a = \frac{4}{\Delta T^2}(\Delta q - \dot{q}_i \Delta T), \quad \ddot{q}_d = \frac{4}{\Delta T^2}(\dot{q}_f \Delta T - \Delta q)$$

- The jerk is always null except at the flex point, when the acceleration changes its sign and it assumes an infinite value.



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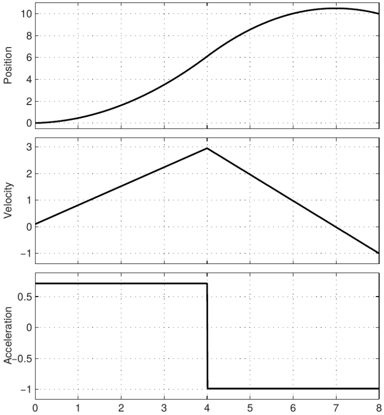
If $\dot{q}_i \neq \dot{q}_f$ (i.e. the velocity profile is discontinuous at t_m with the previous approach) and the constraint on the position at t_m (i.e. $q_m = \frac{q_i+g_f}{2}$) is not assigned, to have a continuous velocity profile ($\dot{q}_a(t_m) = \dot{q}_d(t_m)$) we can solve the following system of six equations in six unknowns

$$\begin{cases} q_a(t_i) = q_i = a_0 \\ \dot{q}_a(t_i) = \dot{q}_i = a_1 \\ q_d(t_f) = q_f = a_3 + a_4\frac{\Delta T}{2} + a_5\left(\frac{\Delta T}{2}\right)^2 \\ \dot{q}_d(t_f) = \dot{q}_f = a_4 + 2a_5\frac{\Delta T}{2} \\ q_a(t_m) = a_0 + a_1\frac{\Delta T}{2} + a_2\left(\frac{\Delta T}{2}\right)^2 = a_3 = q_d(t_m) \\ \dot{q}_a(t_m) = a_1 + 2a_2\frac{\Delta T}{2} = a_4 = \dot{q}_d(t_m) \end{cases} \Rightarrow \begin{cases} a_0 = q_i \\ a_1 = \dot{q}_i \\ a_2 = \dots \\ a_3 = \\ a_4 = \\ a_5 = \end{cases}$$

where $\frac{\Delta T}{2} = t_m - t_i = t_f - t_m$. It is possible to generate trajectory with asymmetric constant acceleration, i.e. $t_m \neq \frac{t_i+t_f}{2}$, but t_m is given.

Parabolic trajectory



Cubic Polynomials

$\Rightarrow$ Infinite solutions! We need a criterion / metric / performance index

$\Rightarrow$ optimization problem

We know from “Robotics” that a *cubic polynomial* (third-order polynomial function) minimizes the energy associated to the motion. Then

$$q(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

with

$$\dot{q}(t) = 3a_3 t^2 + 2a_2 t + a_1$$
$$\ddot{q}(t) = 6a_3 t + 2a_2$$

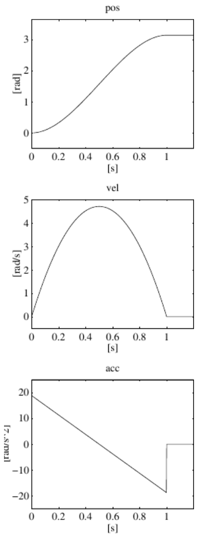
Cubic Polynomials

Since we have 4 unkowns, we need 4 constraints: $q_i, \dot{q}_i$ at t_i, $q_f, \dot{q}_f$ at t_f

$$
\begin{aligned}
a_3 t_i^3 + a_2 t_i^2 + a_1 t_i + a_0 &= q_i \\
3a_3 t_i^2 + 2a_2 t_i + a_1 &= \dot{q}_i \\
a_3 t_f^3 + a_2 t_f^2 + a_1 t_f + a_0 &= q_f \\
3a_3 t_f^2 + 2a_2 t_f + a_1 &= \dot{q}_f
\end{aligned}
$$

If $t_i = 0$

$$
\begin{aligned}
a_0 &= q_i \\
a_1 &= \dot{q}_i \\
a_3 t_f^3 + a_2 t_f^2 + a_1 t_f + a_0 &= q_f \\
3a_3 t_f^2 + 2a_2 t_f + a_1 &= \dot{q}_f
\end{aligned}
$$



Cubic Polynomials

Exercise: Derive the four conditions using the cubic polynomial

$$q(t) = a_3(t - t_i)^3 + a_2(t - t_i)^2 + a_1(t - t_i) + a_0, \quad t \in [t_i, t_f]$$

as a function of Δq and ΔT.

Solution:

$$\begin{cases} a_0 = q_i \\ a_1 = \dot{q}_i \\ a_2 = \frac{3\Delta q - (2\dot{q}_i + \dot{q}_f)\Delta T}{\Delta T^2} \\ a_3 = \frac{-2\Delta q + (\dot{q}_i + \dot{q}_f)\Delta T}{\Delta T^3} \end{cases}$$

By exploiting this result, it is very simple to compute a trajectory with continuous velocity through a sequence of n points by connecting with the previous equations pairs of points. (SEE LATER)

Remark: Discontinuities in the desired trajectory may generate vibrations in the robotic manipulator due to the induced discontinuities in the applied forces and the elastic effects of the mechanical system.

Solution 1: If we would like to assign also the initial and final values of acceleration, six constraints have to be satisfied and then a polynomial of at least *fifth order* is needed

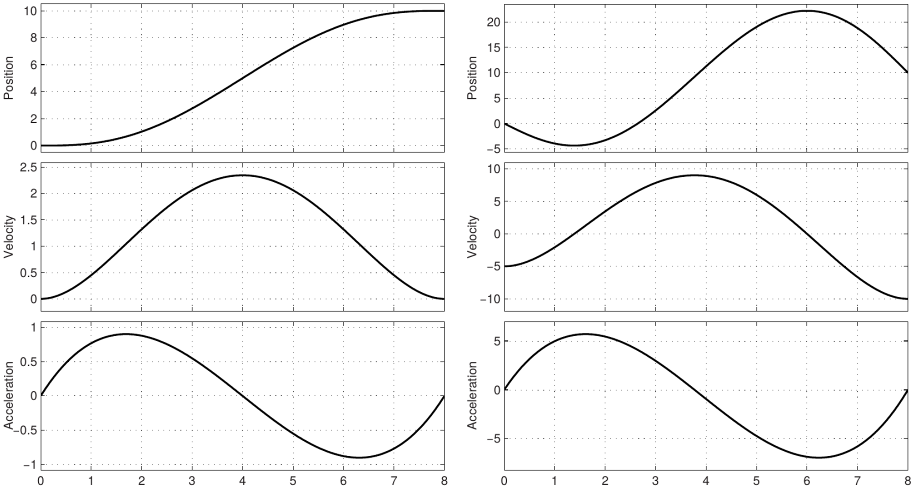
$$q(t) = a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

Then we also have the equations

$$\begin{aligned} \dots &= \ddot{q}_i \\ \dots &= \ddot{q}_f \end{aligned}$$

$\Rightarrow$ *trajectories with continuous acceleration, but discontinuous jerk*

5th-order Polynomials



Solution 2: If we would like to assign also the initial and final values of jerk, eight constraints have to be satisfied and then a polynomial of at least *seventh order* is needed

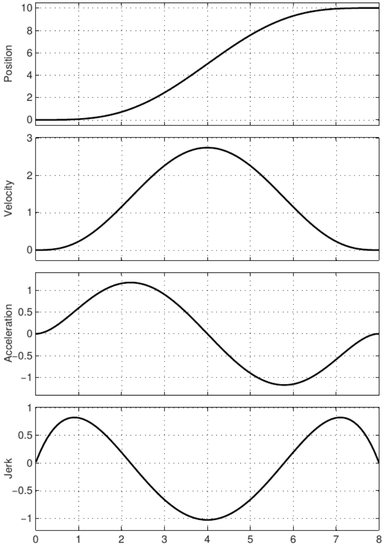
$$q(t) = a_7 t^7 + a_6 t^6 + a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

Then we also have the equations

$$\begin{aligned} \dots &= \dddot{q}_i \\ \dots &= \dddot{q}_f \end{aligned}$$

$\Rightarrow$ *trajectories with continuous jerk*

7th-order Polynomials




Project

TODO:

- Implement in Matlab 3rd– (cubic), 5th–, 7th–order polynomials for $q_i > q_f$ and $q_i < q_f$, and in both formulations

$$q(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0, \qquad t \in [t_i, t_f]$$

and

$$q(t) = a_3 (t - t_i)^3 + a_2 (t - t_i)^2 + a_1 (t - t_i) + a_0, \qquad t \in [0, \Delta T]$$