ROBOTICS, VISION AND CONTROL

Trajectory Planning. Operational Space

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Outline

Operational Space Trajectories

Motion primitives

PROJECT

Operational Space Trajectories

Operational Space Trajectories

Operational Space Trajectories: trajectory planning in 3D, i.e. pose and orientation in the Cartesian space.

We have to take care of

- the *geometry* of the trajectory $\boldsymbol{x}_e(u)$ [Where]

  $\boldsymbol{p}(u) = \begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix}^T \in \mathbb{R}^3$, $\phi(u) = \begin{bmatrix} \varphi(u) & \theta(u) & \psi(u) \end{bmatrix}^T \in \mathbb{R}^3$, $u \in [u_i, u_f]$

  $\rightarrow$ *path*

- the *motion/timing law* $t = u(t)$, $t \in [t_i, t_F]$ [How]

  $t \in \mathbb{R}$, $\boldsymbol{p}(t)$, $\phi(t)$

  $\rightarrow$ path+motion law = *trajectory* $\tilde{\boldsymbol{p}}(t) = (\boldsymbol{p} \circ u)(t)$

Given the trajectory, it is necessary to resort to the inverse kinematics to compute the corresponding joints trajectory $\boldsymbol{q}(t)$.

Operational Space Trajectories

We will refer to *motion primitives* for the geometric features of the path and to *time primitives* for the timing law on the path itself.

n pairs $(\boldsymbol{x}_e(t_k), t_k)$: Interpolation component by component

The multi-dimensional problem can be decomposed in 6 scalar problems.

The *synchronization* among the different components is performed by imposing interpolation conditions at the *same time instants*.

Let $\boldsymbol{p} \in \mathbb{R}^3$ be a Cartesian point given by

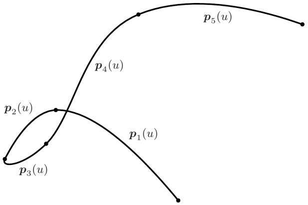
$$\boldsymbol{p} = \boldsymbol{f}(u)$$

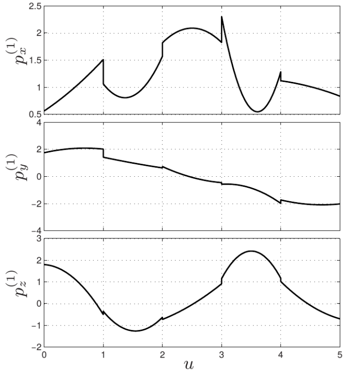
where $u \in [u_i, u_f]$ is the *parameter* of the function $\boldsymbol{f}$; $\boldsymbol{p}(u)$ is the parametric representation on the *path* Γ with

$$\begin{aligned} \boldsymbol{p}_i &:= \boldsymbol{p}(u_i) = \boldsymbol{f}(u_i) && \text{initial point} \\ \boldsymbol{p}_f &:= \boldsymbol{p}(u_f) = \boldsymbol{f}(u_f) && \text{final point} \end{aligned}$$

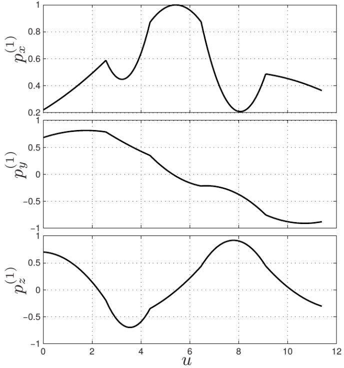
The path Γ has a direction.

$\boldsymbol{p}(u)$ $\qquad\qquad$ $\dfrac{d\boldsymbol{p}(u)}{du}$





(a)



(b)

Geometric continuity (i.e. path) $\neq$ *Parametric continuity* (vel. $\frac{d\boldsymbol{p}(u)}{du}$ and accel. $\frac{d^2\boldsymbol{p}(u)}{du^2}$)

Let $u \in [0,1]$ and $\dot{\boldsymbol{p}}(u) = \frac{d\boldsymbol{p}(u)}{du}$, $\ddot{\boldsymbol{p}}(u) = \frac{d^2\boldsymbol{p}(u)}{du^2}$, $\dots\ \boldsymbol{p}^{(i)}(u) = \frac{d^i\boldsymbol{p}(u)}{du^i}$.

Two infinitely differentiable segments $\boldsymbol{p}_k(u)$, $u \in [0,1]$, $\boldsymbol{p}_{k+1}(u)$, $u \in [0,1]$ meeting at a common point

$$\boldsymbol{p}_k(1) = \boldsymbol{p}_{k+1}(0) \tag{1}$$

satisfy the *n-order parametric continuity, C^n*, if the first n parametric derivatives match at the common point

$$\begin{aligned}
\dot{\boldsymbol{p}}_k(1) &= \dot{\boldsymbol{p}}_{k+1}(0) \\
\ddot{\boldsymbol{p}}_k(1) &= \ddot{\boldsymbol{p}}_{k+1}(0) \\
&\vdots \\
\boldsymbol{p}_k^{(n)}(1) &= \boldsymbol{p}_{k+1}^{(n)}(0)
\end{aligned}$$

Remark. The derivative vectors are not intrinsic properties of a curve.

The *tangent unit vector*

$$\boldsymbol{t} = \frac{d\boldsymbol{p}}{du} \Big/ \left\| \frac{d\boldsymbol{p}}{du} \right\|$$

and the *curvature unit vector* (also known as *normal unit vector*)

$$\boldsymbol{n} = \frac{d^2\boldsymbol{p}}{d^2u} \Big/ \left\| \frac{d^2\boldsymbol{p}}{d^2u} \right\|$$

are intrinsic properties of the curve, and they lead to the notion of geometric continuity G.

Operational Space Trajectories – Path Primitives

Two parametric curves meet with a *first order geometric continuity G^1* if and only if they have a common tangent unit vector.
The tangent direction at the joint is preserved, but the continuity of the velocity vector is not guaranteed, since the tangent vectors may have different magnitude.

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Two parametric curves meet with G^2 *continuity* if and only if they have common unit tangent and curvature vectors.

Two parametric curves $\boldsymbol{p}_k(u)$, $\boldsymbol{p}_{k+1}(u)$ meet with G^n *continuity* if and only if there exists a parametrization $\hat{u}$ equivalent to u such that $\hat{\boldsymbol{p}}_k(\hat{u})$, $\hat{\boldsymbol{p}}_{k+1}(\hat{u})$ meet with C^n continuity.

Two parameterizations u, $\hat{u}$ are equivalent if there exists a regular C^n function $f : [\hat{u}_{min}, \hat{u}_{max}] \mapsto [u_{min}, u_{max}]$ such that:

1. $\hat{\boldsymbol{p}}(\hat{u}) = \boldsymbol{p}(f(\hat{u})) = \boldsymbol{p}(u)$
2. $f([\hat{u}_{min}, \hat{u}_{max}]) = [u_{min}, u_{max}]$
3. $\frac{df}{d\hat{u}} > 0$.

The *arc length* s of the generic point $\boldsymbol{p} \in \Gamma$ is the length of the arc of Γ with extremes $\boldsymbol{p}$ and $\boldsymbol{p}_i \in \Gamma$.

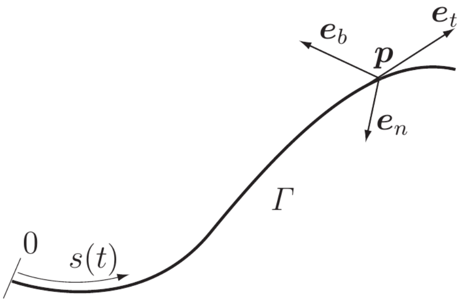
We can identify the point $\boldsymbol{p}$ using the arc length

$$\Gamma : \boldsymbol{p} = \boldsymbol{p}(s), \qquad s \in [0, L]$$

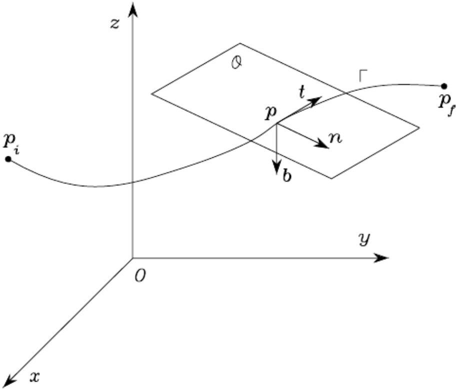
Goal: specify the orientation of the end effector on the basis of the orientation of the path at a given point

The *Frenet Frame* is a coordinate frame directly tied to the curve. It is represented by three unit vectors:

- the *tangent unit vector*
- the *normal unit vector*
- the *binormal unit vector*

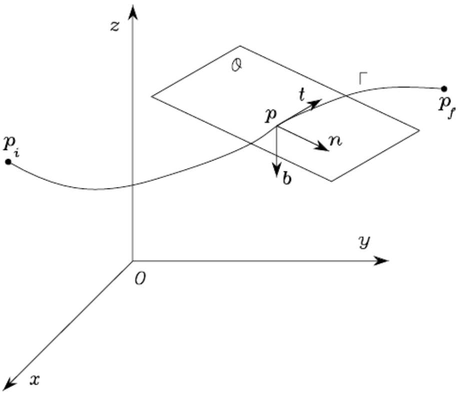


Operational Space Trajectories – Orientation



tangent unit vector $\boldsymbol{t}$: vector oriented along the positive direction induced on the path Γ by s

$$\boldsymbol{t} = \frac{d\boldsymbol{p}}{ds}$$



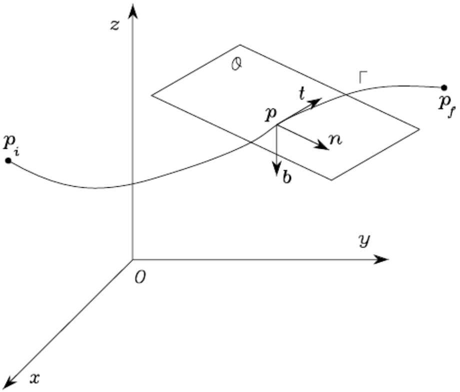
normal unit vector $\boldsymbol{n}$: vector oriented along the line intersecting $\boldsymbol{p}$ at a right angle with $\boldsymbol{t}$ and lies in the so-called osculating plane $\mathcal{O}$;

The plane $\mathcal{O}$ is the limit plane containing $\boldsymbol{t}$ and a point $\boldsymbol{p}' \in \Gamma$ when $\boldsymbol{p}'$ tends to $\boldsymbol{p}$ along the path;

The direction of $\boldsymbol{n}$ is so that the path Γ, in the neighbourhood of $\boldsymbol{p}$ with respect to the plane containing $\boldsymbol{t}$ and normal to $\boldsymbol{n}$, lies on the same side of $\boldsymbol{n}$

Another definition
The normal unit vector $\boldsymbol{n}$, lying on the line passing through the point $\boldsymbol{p}$, and orthogonal to $\boldsymbol{t}$.

The orientation of $\boldsymbol{n}$ is such that in a neighborhood of $\boldsymbol{p}$ the curve is completely on the side of $\boldsymbol{n}$ with respect to the plane passing through $\boldsymbol{t}$ and normal to $\boldsymbol{n}$.



FRENET VECTORS

tangent unit vector $\boldsymbol{t}$

$$\boldsymbol{t} = \frac{d\boldsymbol{p}}{ds}$$

normal unit vector $\boldsymbol{n}$

$$\boldsymbol{n} = \frac{d^2\boldsymbol{p}}{d^2s} \Big/ \left\| \frac{d^2\boldsymbol{p}}{d^2s} \right\|$$

binormal unit vector $\boldsymbol{b}$: vector such that the frame $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ is right-handed

$$\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$$

Remarks.

- If the curve is characterized by the arc-length parameterization s and not by a generic parameter u, the tangent vector $\boldsymbol{t}$ has unit length.
- In those applications in which the tool must have a fixed orientation with respect to the motion direction, the Frenet vectors implicitly define such an orientation
- Let $R_F(u) = \begin{bmatrix} \boldsymbol{t}(u) & \boldsymbol{n}(u) & \boldsymbol{b}(u) \end{bmatrix}$ be the Frenet frame as a function of u (the same holds for s) and R_Δ be a constant matrix rotation for the tool with respect to $R_F(u)$; then the tool orientation is

$$R_T(u) = R_\Delta R_F(u)$$

Motion primitives

Rectilinear path: linear segment connecting initial point $\boldsymbol{p}_i$ to the final point $\boldsymbol{p}_f$

- normalized parameterization

$$\boldsymbol{p}(u) = \boldsymbol{p}_i + (\boldsymbol{p}_f - \boldsymbol{p}_i)u, \qquad u \in [0, 1]$$

- arc-length

$$\boldsymbol{p}(s) = \boldsymbol{p}_i + s\frac{\boldsymbol{p}_f - \boldsymbol{p}_i}{\|\boldsymbol{p}_f - \boldsymbol{p}_i\|}, \qquad s \in [0, \|\boldsymbol{p}_f - \boldsymbol{p}_i\|]$$

The direction induced on Γ by the parametric representation s is from $\boldsymbol{p}_i$ to $\boldsymbol{p}_f$.

Tangent unit vector

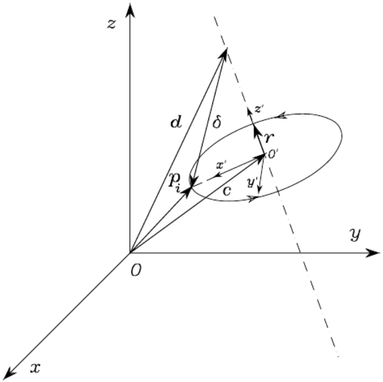
$$\boldsymbol{t} = \frac{d\boldsymbol{p}}{ds} = \frac{\boldsymbol{p}_f - \boldsymbol{p}_i}{\|\boldsymbol{p}_f - \boldsymbol{p}_i\|}$$

The normal unit vector $\boldsymbol{n}$ and the binormal unit vector $\boldsymbol{b}$ cannot be defined in a unique way since

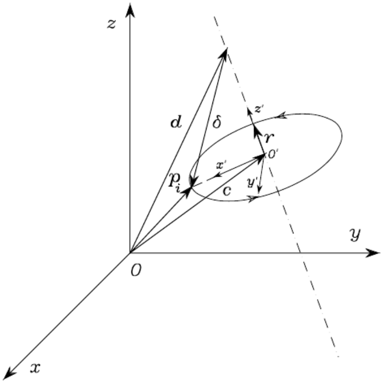
$$\frac{d^2\boldsymbol{p}}{d^2s} = 0.$$

The trajectory composed by a set of linear segments is continuous but it is characterized by discontinuous derivatives at the intermediate points

$\rightarrow$ use *blending functions* to guarantee a smooth transition between consecutive segments (e.g. ...).

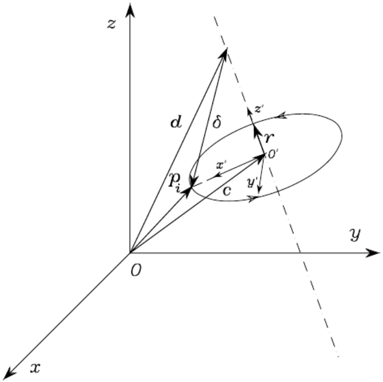


The circle Γ is specified by assigning:



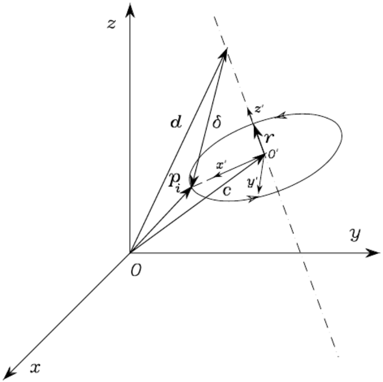
The circle Γ is specified by assigning:

- the unit vector of the circle axis $\boldsymbol{r}$,



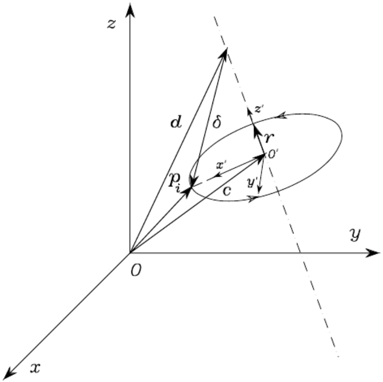
The circle Γ is specified by assigning:

- the unit vector of the circle axis $\boldsymbol{r}$,
- the position vector $\boldsymbol{d}$ of a point along the circle axis,



The circle Γ is specified by assigning:

- the unit vector of the circle axis $\boldsymbol{r}$,
- the position vector $\boldsymbol{d}$ of a point along the circle axis,
- the position vector $\boldsymbol{p}_i$ of a point on the circle.



The circle Γ is specified by assigning:

- the unit vector of the circle axis $\boldsymbol{r}$,
- the position vector $\boldsymbol{d}$ of a point along the circle axis,
- the position vector $\boldsymbol{p}_i$ of a point on the circle.

The position vector $\boldsymbol{c}$ of the centre of the circle is given by

$$\boldsymbol{c} = \boldsymbol{d} + (\delta^T \boldsymbol{r})\boldsymbol{r}$$

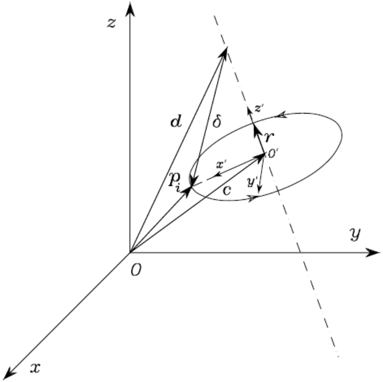
where

$$\delta = \boldsymbol{p}_i - \boldsymbol{d}$$

The radius is $\|\boldsymbol{p}_i - \boldsymbol{c}\|$

We need to find a parametric representation of the circle as a function of the arc length s.

Operational Space Trajectories – Circular Path



$$\Sigma = \{O; x, y, z\}, \qquad \Sigma' = \{O'; x', y', z'\}$$

x' oriented along the direction of the vector $\boldsymbol{p}_i - \boldsymbol{c}$
z' along $\boldsymbol{r}$
y' to have a right-handed frame

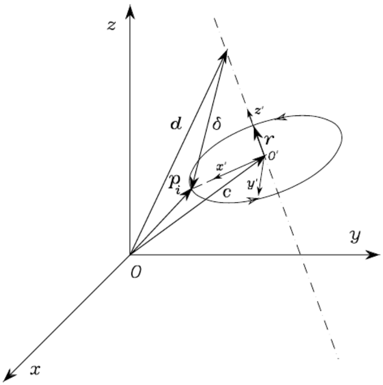
The parametric representation of the circle in Σ' is

$$\boldsymbol{p}'(s) = \begin{bmatrix} \rho\cos(s/\rho) \\ \rho\sin(s/\rho) \\ 0 \end{bmatrix}, \qquad \rho := \|\boldsymbol{p}_i - \boldsymbol{c}\|$$

The parametric representation of the circle in Σ is

$$\boldsymbol{p}(s) = \boldsymbol{c} + R\boldsymbol{p}'(s)$$

where $R = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix}$ is the rotation matrix of frame Σ' with respect to frame Σ

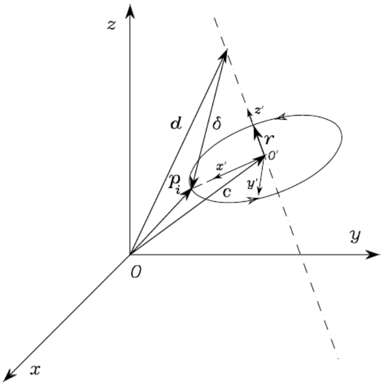


If we are interested in moving along a circular arc, then we can relay on the parametric representation

$$
\begin{aligned}
\boldsymbol{p}'(u) &= \begin{bmatrix} \rho\cos(u+\varphi) \\ \rho\sin(u+\varphi) \\ 0 \end{bmatrix}, \qquad \rho := \|\boldsymbol{p}_i - \boldsymbol{c}\| \\
\boldsymbol{p}(u) &= \boldsymbol{c} + R\boldsymbol{p}'(u)
\end{aligned}
$$

where $u \in [0, \theta]$ and

$$
\begin{aligned}
\boldsymbol{p}_0 &= \boldsymbol{p}(0) = \boldsymbol{c} + R\boldsymbol{p}'(0) \\
\boldsymbol{p}_1 &= \boldsymbol{p}(\theta) = \boldsymbol{c} + R\boldsymbol{p}'(\theta)
\end{aligned}
$$



Tangent unit vector

$$\boldsymbol{t} = \frac{d\boldsymbol{p}}{ds} = R \begin{bmatrix} -\sin(s/\rho) \\ \cos(s/\rho) \\ 0 \end{bmatrix}$$

The normal unit vector $\boldsymbol{n}$ and the binormal unit vector $\boldsymbol{b}$

$$\boldsymbol{n} = \frac{d^2\boldsymbol{p}}{d^2s} \Big/ \left\| \frac{d^2\boldsymbol{p}}{d^2s} \right\|, \qquad \boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$$

where

$$\frac{d^2\boldsymbol{p}}{ds^2} = R \begin{bmatrix} -\frac{1}{\rho}\cos(s/\rho) \\ -\frac{1}{\rho}\sin(s/\rho) \\ 0 \end{bmatrix}$$


Project

To do

- Compute the 3D trajectory (also also velocity, acceleration and jerk) in the picture as a combination of linear and circular motion primitives and compare it with the trajectory obtained using one of the multi-point methods.

  [From $(0, 0, 0)$ to $(2, 0, 2)$ and back]