Rotations for Computer Vision and Robotics

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Robotics, vision and control

Rotations

- Rotation matrix,
- Euler angles,
- Axis-angles representation,
- Quaternions,
- Exponential matrix

Matrix exponential

Matrix exponential

Remind the Taylor expansion of a generic function *f*(*x*) that is infinitesimally differentiable in a point x_0:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_o)^2}{2} + \cdots$$

$$= \sum_{n=0}^{\infty} f^{(n)}(x_0)\frac{(x - x_0)^n}{n!}$$

Matrix exponential

When we consider the exponential function $f(x) = e^x$ and we fix $x_0 = 0$ we obtain the following very known power series:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Matrix exponential

Other interesting power series are sin(x):

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Matrix exponential

Other interesting power series are cos(x):

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$$

Matrix exponential

Other interesting power series is the matrix exponential series for a square matrix M:

$$e^M = I + M + \frac{M^2}{2} + \frac{M^3}{6} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

Properties

Property 1: *the inverse matrix of the matrix exponential of M is the matrix exponential of −M.*

$$(e^{M})^{-1} = e^{-M}$$

DEMO:

It is known that if two square matrices *M* and *N* commute (i.e., MN=NM), then:

$$e^{M} \cdot e^{N} = e^{M+N}$$

Properties

Property 1: *the inverse matrix of the matrix exponential of M is the matrix exponential of –M.*

$$(e^{M})^{-1} = e^{-M}$$

DEMO:

In our case M and –M commute, therefore:

$$e^{M} \cdot e^{-M} = e^{M+(-M)} = e^{0} = I$$

Properties

Property 2: *the transpose matrix of the matrix exponential of M is the matrix exponential of M*$^\top$

$$(e^M)^\top = e^{M^\top}$$

DEMO:

$$\begin{aligned}(e^M)^\top &= (I+M+\frac{M^2}{2}+\frac{M^3}{6}+\cdots)^\top \\ &= I+(M)^\top+(\frac{M^2}{2})^\top+(\frac{M^3}{6})^\top\cdots \\ &= e^{M^\top}\end{aligned}$$

Properties

Property 3: *For a skew-symmetric matrix S the matrix exponential of the transpose matrix is the matrix exponential of the negative matrix:*

$$e^{S^\top} = e^{-S}$$

DEMO:

$$S = \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix}$$

Skey symmetric matrix

by definition $S = -S^\top$, and therefore $S^\top = -S$

Rotation matrix

From the properties of matrix exponential and the use of a skew-symmetric matrix *S* we can define a rotation matrix, i.e., an **orthogonal** matrix with determinat equal to 1:

$$(e^S)^{-1} = (e^S)^\top$$

$$\det(e^S) = 1$$

Rotation matrix

We define $R = e^S$, and considering properties 1,2 and 3 we get:

$$R^\top = (e^S)^\top = e^{S^\top} = e^{-S} = (e^S)^{-1}$$
$$= R^{-1}$$

$R = e^S$ is an **orthogonal** matrix

Rotation matrix

Moreover, the curve of orthogonal matrices e^{tS} is a path connecting the identity matrix I (when $t = 0$, that is $e^0 = I$) and the matrix R (when $t = 1$).

Therefore, R and I have the same determinant that is 1.

This shows that $\det(R) = 1$ and therefore R is a **rotation matrix**.

Rotation matrix

Now we can write a generic skew-symmetric matrix in 3D as:

$$S = \theta \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

where θ, x, y, z are real numbers with $x^2 + y^2 + z^2 = 1$. This means that x, y, z are the coordinates of a vector $\mathbf{v}$ such that $\|\mathbf{v}\| = 1$. In particular, vector $\mathbf{v}$ represents the rotation axis encoded by the rotation matrix R, and θ is the rotation angle.

Rotation matrix

The vector $\mathbf{v}' = \theta \cdot \mathbf{v}$ is such that:

$$\mathbf{v}' \times \mathbf{w} = [\mathbf{v}']_{\times} \mathbf{w}$$

for any $\mathbf{w} \in \mathcal{R}$, where $[\mathbf{v}']_{\times} = S$ is the matrix that encodes the cross product with vector $\mathbf{v}'$.

Rotation matrix

Now we can consider the Cayley-Hamilton Theorem that states:

$$-S^3 - \theta^2 S = 0$$

We can recover interesting properties of higher powers of S:

$$\begin{aligned}
S^3 &= -\theta^2 S \\
S^4 &= S^3 S = (-\theta^2 S)S = -\theta^2 S^2 \\
S^5 &= S^4 S = (-\theta^2 S^2)S = -\theta^2 S^3 = -\theta^2(-\theta^2 S) = \theta^4 S \\
S^6 &= S^5 S = \theta^4 S S = \theta^4 S^2 \\
\ldots
\end{aligned}$$

Rotation matrix

We can plug these relations to the matrix exponential power series:

$$
\begin{aligned}
R &= e^{S} \\
&= I + S + \frac{S^2}{2} + \frac{S^3}{3!} + \frac{S^4}{4!} + \frac{S^5}{5!} + \frac{S^6}{6!} + \ldots \\
&= I + S + \frac{S^2}{2} - \frac{\theta^2 S}{3!} - \frac{\theta^2 S^2}{4!} + \frac{\theta^4 S}{5!} + \frac{\theta^4 S^2}{6!} + \cdots \\
&= I + \left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \cdots\right) S + \left(\frac{1}{2} - \frac{\theta^2}{4!} + \frac{\theta^4}{6!} + \cdots\right) S^2 \\
&= I + \frac{\sin\theta}{\theta} S + \frac{1-\cos\theta}{\theta^2} S^2
\end{aligned}
$$

$$\Downarrow$$

The power series of $\sin\vartheta$ and $\cos\vartheta$ have been considered!

Rotations

Now, if we define $\hat{S} = \frac{S}{\theta}$ then we get:

$$R = I + \sin\theta \hat{S} + (1 - \cos\theta)\hat{S}^2$$

that is the well-known **Rodrigues**' formula for a rotation matrix derived by the rotation angle ϑ and the rotation axis **v**.

Recall also that $S = [\mathbf{v}']_\times$ and $\theta = \|\mathbf{v}'\|$

$$\mathbf{v} = \frac{\mathbf{v}'}{\theta} = \frac{\mathbf{v}'}{\|\mathbf{v}'\|}$$ is a unitary vector!

Lie groups

A similar definition of rotation matrix from matrix exponentials is given by the Lie algebra. Here we get the group of $SO(3)$ whose elements $R \in SO(3)$ are 3D rotation matrices. The matrix exponential is defined as $e^{[\mathbf{w}]_\times}$ where:

$$[\mathbf{w}]_\times = \begin{pmatrix} 0 & -w_3 & w2 \\ w_3 & 0 & -w1 \\ -w_2 & w_1 & 0 \end{pmatrix}$$

where $\mathbf{w}$ is the vector of non-zero components of the skew-symmetric matrix that represents the rotation vector such that $\theta = \|\mathbf{w}\|$ is the rotation angle.

Lie group

The power series is:

$$e^{[\mathbf{w}]_\times} = I + [\mathbf{w}]_\times + \frac{[\mathbf{w}]_\times^2}{2!} + \cdots + \frac{[\mathbf{w}]_\times^n}{n!}$$

Remember also that $[\mathbf{w}]_\times \mathbf{v} = \mathbf{w} \times \mathbf{v}$, $\forall \mathbf{v} \in \mathcal{R}^3$.

Lie group

We can pair the terms of power series in this way:

$$e^{[\mathbf{w}]_\times} = I + \sum_{i=0}^{\infty} \frac{[\mathbf{w}]_\times^{2i+1}}{(2i+1)!} + \frac{[\mathbf{w}]_\times^{2i+2}}{(2i+2)!}$$

Now we know that $[\mathbf{w}]_\times^2 = -(\mathbf{w}^\top \mathbf{w})[\mathbf{w}]_\times$. Therefore:

$$\begin{aligned}
\theta^2 &= \mathbf{w}^\top \mathbf{w} \\
[\mathbf{w}]_\times^{2i+1} &= (-1)^i \theta^{2i} \, [\mathbf{w}]_\times \\
[\mathbf{w}]_\times^{2i+2} &= (-1)^i \theta^{2i} \, [\mathbf{w}]_\times^2
\end{aligned}$$

Lie group

We get the same Rodrigues' formula as:

$$R = e^{[\mathbf{w}]_\times} = I + \frac{\sin\theta}{\theta}[\mathbf{w}]_\times + \left(\frac{1-\cos\theta}{\theta^2}\right)[\mathbf{w}]_\times^2$$

This is the rotation matrix that encodes a rotation (in radiants) around the axis $\mathbf{w}$ of an angle θ.

Lie group

The exponential map can be inverted using the logaritm. We get:

$$\theta = \arccos\left(\frac{Tr(R) - 1}{2}\right)$$

That is derived from: $Tr(R) = 1 + 2\cos\theta$, and

$$[\mathbf{w}]_\times = \log(R) = \frac{\theta}{2\sin\theta}(R - R^\top)$$

Note that by definition $(R - R^\top)$ is skew-symmetric. Note also that $\|vec((R - R^\top))\| = 2\sin\theta$, and therefore $\|\vec{(}[\mathbf{w}]_\times)\| = \theta$

Euler angles

The Euler angles are the three angles (*α, β, γ*) around the axis *X*, *Y* , and *Z*. We can define the following rotation matrices:

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \quad R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Euler angles

The generic rotation *R* can be represented by the products of the above three rotation matrices:

$$R = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

Euler angle

We can therefore compute the matrix product and recover the rotation matrix from the Euler angles:

$$R = \begin{pmatrix} \cos\beta\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \cos\beta\sin\gamma & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma \\ -\sin\beta & \sin\alpha\cos\beta & \cos\alpha\cos\beta \end{pmatrix}$$

We can also recover the Euler angles from the rotation matrix:

$$\begin{array}{llll} \beta: & R_{3,1} = -\sin\beta, & & i.e., \beta = \arcsin(-R_{3,1}), \\ \alpha: & \frac{R_{3,2}}{R_{3,3}} = \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} = \tan(\alpha), & & i.e., \alpha = \arctan_2(R_{3,2}, R_{3,3}), \\ \gamma: & \frac{R_{2,1}}{R_{1,1}} = \frac{\cos\beta\sin\gamma}{\cos\beta\cos\gamma} = \tan(\gamma), & & i.e., \gamma = \arctan_2(R_{2,1}, R_{1,1}). \end{array}$$

Axis angle

Every rotation R in 3D is defined by its rotation axis $\mathbf{u}$.

This rotation axis is such that $R\mathbf{u} = \mathbf{u}$, i.e., the vector $\mathbf{u}$ remains unchanged after the rotation (by definition of rotaion axis).

We can also write $R\mathbf{u} = I\mathbf{u}$ or:

$$(R - I)\mathbf{u} = 0$$

This means that $\mathbf{u}$ lies in the null space of matrix $(R - I)$, and $\mathbf{u}$ is an eigenvector of R corresponding to the eigenvalue $\lambda = 1$.

Axis angle

- Every rotation matrix must have this eigenvalue, the other two eigenvalues being complex conjugate of each other.
- It follows that a general rotation matrix in 3D has, up to a multiplicative constraint, only one real eigenvector.
- In principle the rotation axis can be computed by the eigen-decomposition of the matrix R.
- Then the rotation angle is recovered by the equation $tr(R) = 1 + 2\cos\vartheta$.

Axis angle

A more specific procedure to compute the rotation axis is given by:

$$\begin{aligned} 0 &= R^\top \mathbf{0} + \mathbf{0} \\ &= R^\top (R - I)\mathbf{u} + (R - I)\mathbf{u} \\ &= (R^\top R - R^\top + R - I)\mathbf{u} \\ &= (I - R^\top + R - I)\mathbf{u} \\ &= (R - R^\top)\mathbf{u} \end{aligned}$$

Axis angle

Since by definition ($R - R^{\top}$) is a skwe-symmetric matrix we can choose **u** such that:

$$[\mathbf{u}]_{\times} = (R - R^{\top}).$$

We get:

$$(R - R^{\top})\mathbf{u} = [\mathbf{u}]_{\times}\mathbf{u} = \mathbf{u} \times \mathbf{u} = 0$$

Axis angle

In more details, if R is defined as:

$$R = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

therefore:

$$(R - R^\top) = \begin{pmatrix} 0 & b-d & c-g \\ d-b & 0 & f-h \\ g-c & h-f & 0 \end{pmatrix}$$

from which we recover:

$$\mathbf{u} = \begin{pmatrix} h-f \\ c-g \\ d-b \end{pmatrix}$$

Note that in this representation the $\|\mathbf{u}\| = \|vec(R - R^\top\| = 2\sin\theta$, where θ is the rotation angle. Also in this case the rotation angle can be directly computed from the rotation matrix R by $tr(R) = 1 + 2\cos\theta$.

Axis angle

Now if we consider the unit vector $\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$ representing the rotation axis, and the rotation angle θ, the rotation matrix R can be recovered as:

$$\begin{pmatrix} \cos\theta + x^2(1-\cos\theta) & xy(1-\cos\theta) - z\sin\theta & xz(1-\cos\theta) + y\sin\theta \\ yx(1-\cos\theta) + z\sin\theta & \cos\theta + y^2(1-\cos\theta) & yz(1-\cos\theta) - x\sin\theta \\ zx(1-\cos\theta) - y\sin\theta & zy(1-\cos\theta) + x\sin\theta & \cos\theta + z^2(1-\cos\theta) \end{pmatrix}$$

where $\mathbf{u} = (x, y, z)$. This matrix can be written in a more concise form as:

$$R = (\cos\theta)I + (\sin\theta)[\mathbf{u}]_\times + (1-\cos\theta)(\mathbf{u}\otimes\mathbf{u}),$$

where $(\mathbf{u}\otimes\mathbf{u}) = \mathbf{u}\cdot\mathbf{u}^\top$ is the outer product. This expression is equivalent to the Rodrigue's rotation formula.