Hand-eye calibration

Umberto Castellani

Robotics, vision and control

Overall aim

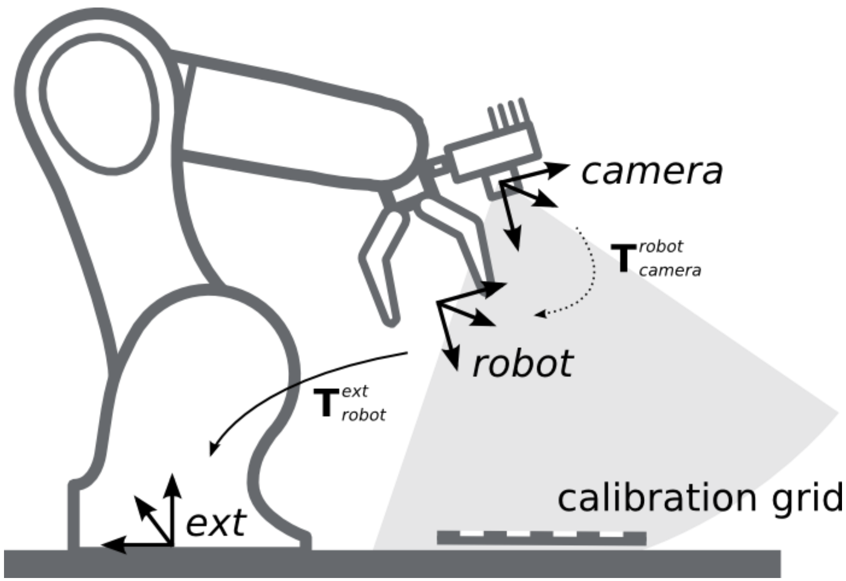
The overall aim of **hand-eye calibration** procedure is to find the **camera-to-robot** transformation.

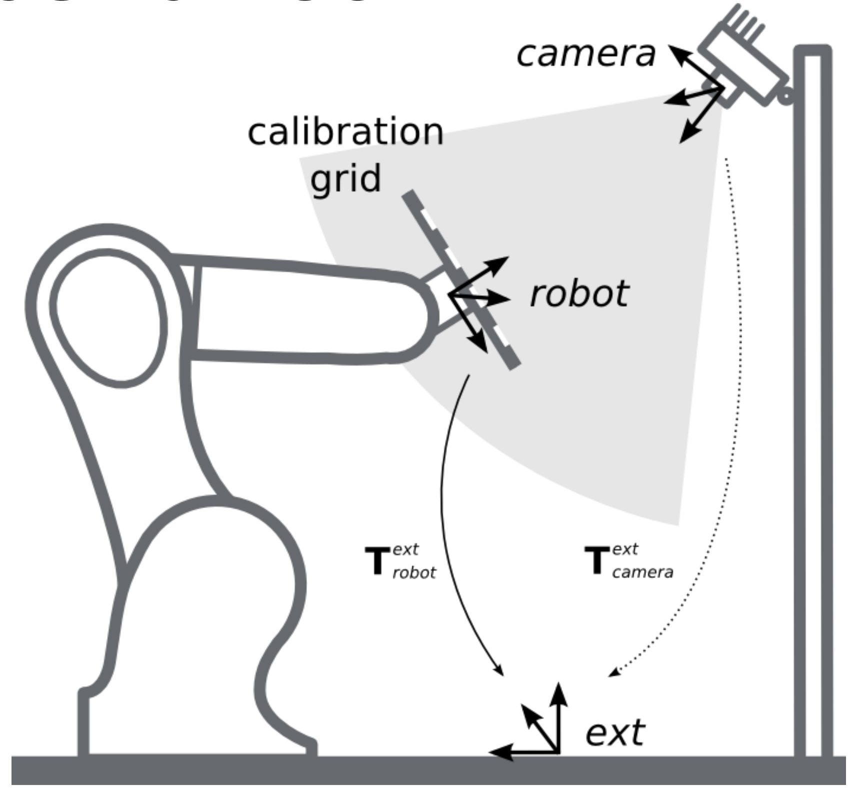
In the basic scenarios:

- The scene is composted by one object, one camera (e.g., RGBD device), and a robot arm with a gripper,
- The camera observes the object in the scene, detect the object position and tell the robot where to pick it.

The camera has to know the robot positon to transfrom the object coordinates from the camera to the robot reference system !

Different scenarios





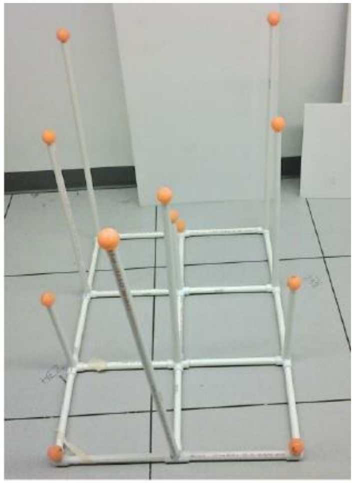
camera attached to the end-effector

static camera

https://answers.opencv.org/answers/204935/revisions/

System calibration

A calibration object is used



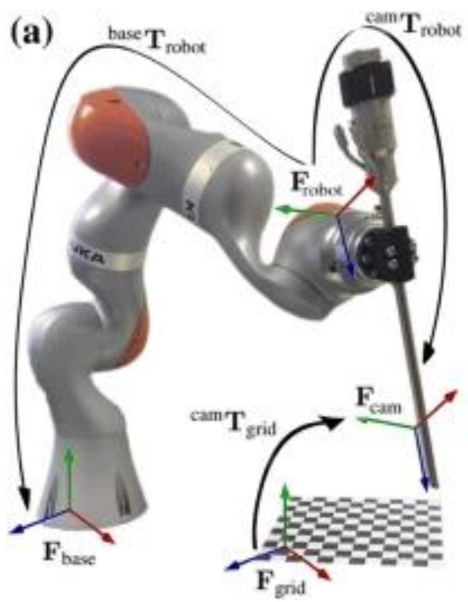
- The aim of calibration object is to easily detect 3D-2D correspondences,
- Sometimes we ask the end-effector to touch the interest points of the calibration object,
- The calibration object may define the world reference system.

Calibration System

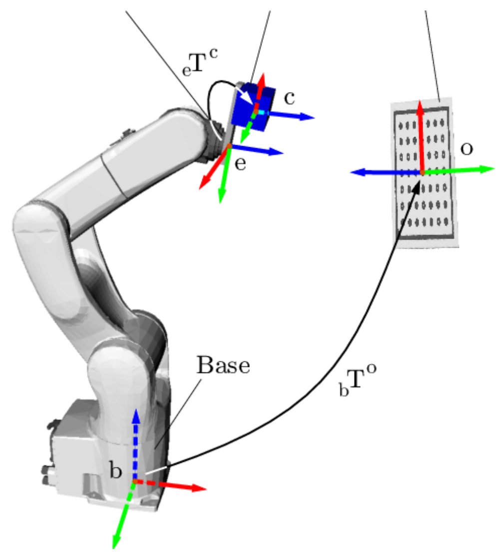
Different calibration scenarios:

- Chessboard mounted on the end-effector (we need to estimate also the transformation between the end-effector and the chessboard).
- Fixed Chessboard with camera mounted to the end effector (we need to estimate also the transformation betwen end-effector and camera)

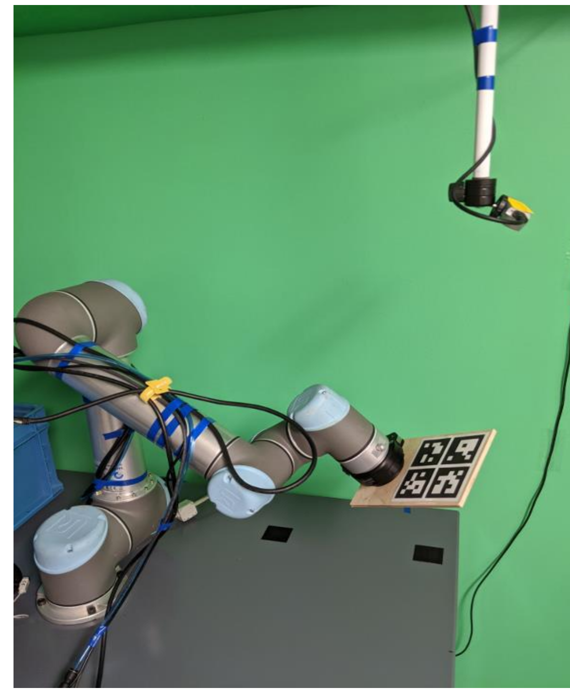
Calibration system



End-effector touches the calibration object



Fixed calibration object and camera mounted to the end-effector



Fixed camera and calibration object mounted to the end-effector

Transformations

- Each system involves different reference systems:
  - Calibration object (i.e., chessboard),
  - Camera,
  - Robot (base),
  - Gripper

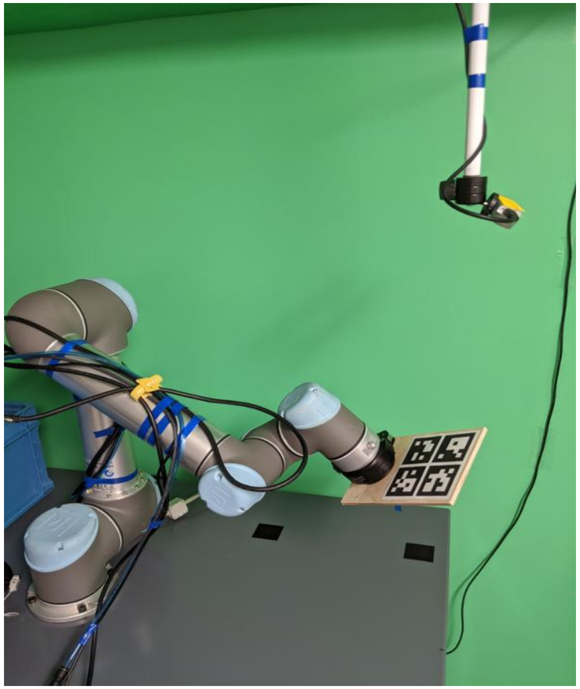
We can compute matrix transformations to move among the reference systems

Probem solution

- We need to combine the various trasformations in order to obtain reasonable equations to estimate our unknowns,
- We need to define a suitable optimization problem,
  - We should solve the optimization problem in closed-form, or...
  - We should employ an iterative (numerical) method.

The litterature on hand-eye calibration differs on the definition of the optimization problem according to the involved Transformations (and the respective scenarios)

Case 1



Fixed camera and calibration object mounted to the end-effector

Involved transformation

- For this scenario we define the respective **transformation chain,** here we have a fixed camera scenario**:**

$$p\_cam=K*BaseT*armPose*gripT*P\_ch$$

where

- p_cam= pixel coordinates,
- K =camera projection matrix,
- BaseT= robot to camera transform,
- armPose=end-effector to robot transform,
- gripT=chessboard to end-effector transform,
- P_ch= 3D coordinates of points on the chessboard

Involved transformation

- For this scenario we define the respective **transformation chain,** here we have a fixed camera scenario**:**

p_cam=K*BaseT*armPose*gripT*P_ch

where

- p_cam= pixel coordinates (observed by image processing),
- K =camera projection matrix (estimates as pre-processing),
- BaseT= robot to camera transform (unknown),
- armPose=end-effector to robot transform (given),
- gripT=chessboard to end-effector transform (unknown),
- P_ch= 3D coordinates of points on the chessboard (given).

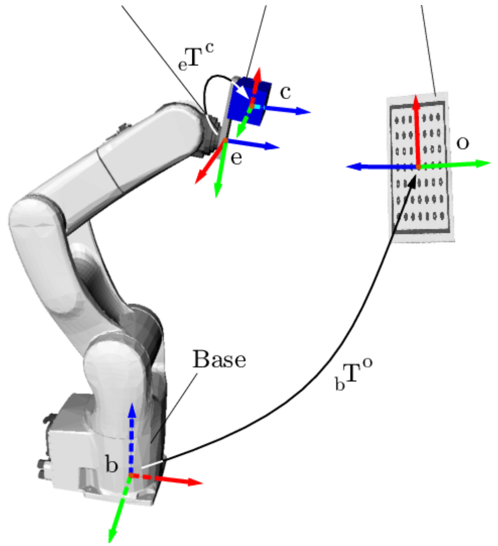
Estimated solution

- For this example we solve a generic optimization problem by collecting a set of equations derived by the transformation chain,
- We can refer to this code:

https://github.com/ZacharyTaylor/Camera-to-Arm-Calibration

- The most important part is the 'ProjectError' function.

Case 2



Fixed calibration object and camera mounted to the end-effector

Calibration setup

- The robot carrying a camera makes a series of motions with the camera acquiring a picture of a calibration object at each motion.
- The involved reference systems are:
  - G_i: the gripper coordinate system, i.e., the coordinate frame fixed on the robot gripper.
  - C_i: the camera coordinate system, i.e., the coordinate frame fixed on the camera.
  - R_w: robot world coordinate system, i.e., the coordinate frame fixed on the robot.
  - C_w: the calibration world coordinate system. This frame is located on the calibration object used to calibrate the camera.

Calibration setup

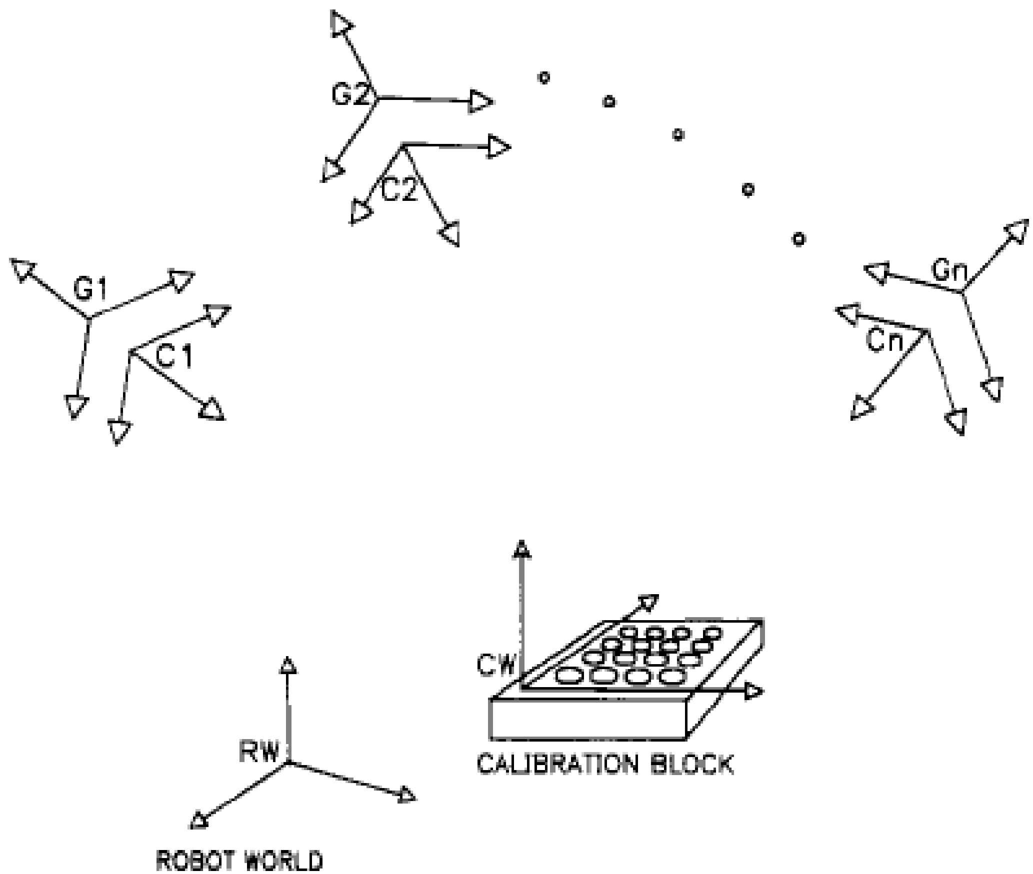


Figure 1. Basic setup for robot hand/eye calibration. C_i and G_i are coordinate frames for the camera and gripper respectively.

Transformations

- We involve the following transformations:
  - H_g^i: from G_i to R_w, available from robot control.
  - H_c^i:from C_w to C_i, available from camera calibration as extrinsic parameters.
  - $H_g^{(i,j)}$: from gripper position G_i to gripper position G_j.
  - $H_c^{(i,j)}$: from camera position C_i to camera position C_j.
  - H_{cg}: from camera frame to gripper frame. This transformation is fixed for all positions since the gripper and the camera are moving together. This transformation is the unknown.

Transformation

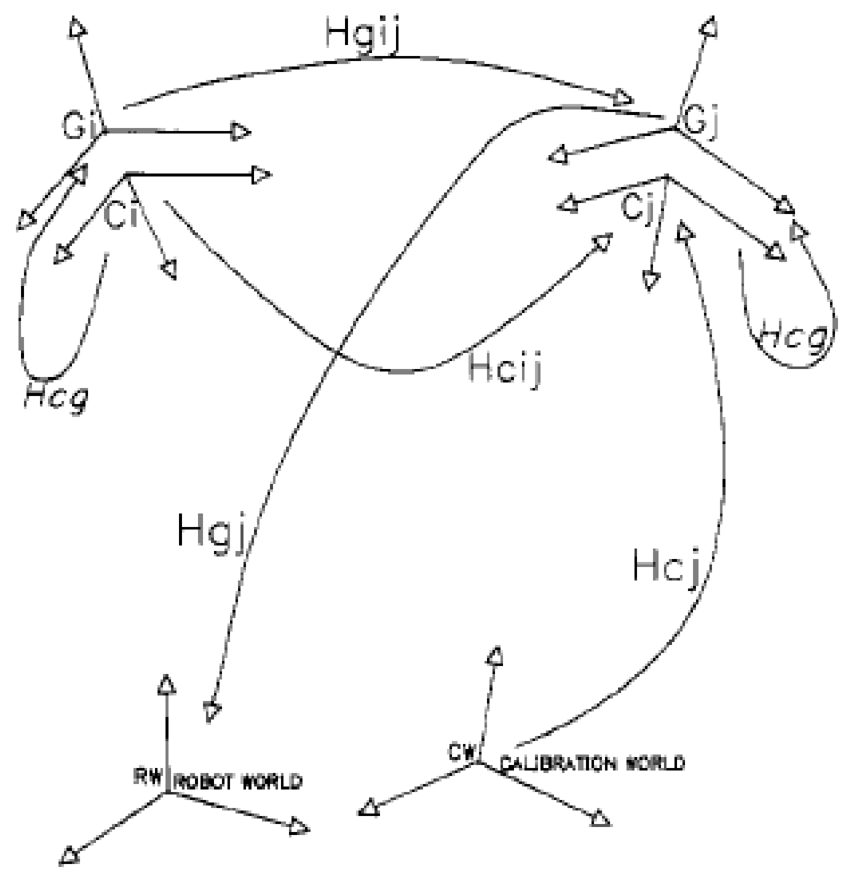


Figure 4. Relationship between the homogeneous matrices and the coordinate frames.

Interesting equations

- We can recover the following relations:

$$H_g^{(i,j)} = (H_g^j)^{-1} H_g^i$$

$$H_c^{(i,j)} = H_c^j (H_c^i)^{-1}$$

More interesting: $\longrightarrow$ $$H_g^{(i,j)} = H_{cg} H_c^{(i,j)} (H_{cg})^{-1}$$

$$\Downarrow$$

$$H_g^{(i,j)} H_{cg} = H_{cg} H_c^{(i,j)}$$

Unknown

End eye calibration equation

- This class of equations are of the form:

$$AX = XB$$

- In our case we can decompose the problem in rotation and translation components as:

$$R_A R_X = R_X R_B \qquad (R_A - I)t_x = R_X t_b - t_a$$

Tsai's metod

- Tsai proposed a further variation of the rotation matrix representation from the rotation axis $\boldsymbol{v}$:

$$R = (1 - \frac{\|\mathbf{v}\|^2}{2})I + \frac{1}{2}\left(\mathbf{v}\mathbf{v}^\top + \sqrt{4 - \|\mathbf{v}\|^2}\,[\mathbf{v}]_\times\right)$$

Where: $\|\mathbf{v}\| = 2\sin\left(\frac{\theta}{2}\right)$

This representation is particularly convenient because it is without any trigonometric function

Tsai's method

The main procedure is summarized to the following main steps:

1. For each pair of station (*i, j*) set up a system of linear equations with $\mathbf{v}'_{cg}$ as unknown, with $\mathbf{v}'_{cg}$ is a properly rescaled version of $\mathbf{v}_{cg}$ to satisfy certain properties (see later):

$$[\mathbf{v}_g^{(i,j)} + \mathbf{v}_c^{(i,j)}]_\times \mathbf{v}'_{cg} = (\mathbf{v}_c^{(i,j)} - \mathbf{v}_g^{(i,j)})$$

2. Compute $\vartheta_{cg} = 2 \tan^{-1} \|\mathbf{v}'_{cg}\|$ (this step is not necessary).

Tsai's method

The main procedure is summarized to the following main steps:

3. Compute: $$\mathbf{v}_{cg} = \frac{2\mathbf{v}'_{cg}}{\sqrt{1+\|\mathbf{v}'_{cg}\|^2}}$$

Then from $\mathbf{v}_{cg}$ we can recover R_{cg} using:

$$R = (1 - \frac{\|\mathbf{v}\|^2}{2})I + \frac{1}{2}\left(\mathbf{v}\mathbf{v}^\top + \sqrt{4 - \|\mathbf{v}\|^2}\,[\mathbf{v}]_\times\right)$$

Finally, the translation vector $\mathbf{t}_{cg}$ is recovered by solving:

$$(R_g^{(i,j)} - 1)\mathbf{t}_{cg} = R_{cg}\mathbf{t}_c^{(i,j)} - \mathbf{t}_g^{(i,j)}$$

Tsai's method

- **DEMO**:

Working with rotation first: $R_g^{(i,j)} = R_{cg} R_c^{(i,j)} R_{cg}^\top$

Property 1: *The eigenvector matrix of $R_g^{(i,j)}$ can be transformed from that of $R_c^{(i,j)}$ using R_{cg}*

$$R_g^{(i,j)} R_{cg} = R_{cg} R_c^{(i,j)}$$

$$\Downarrow$$

$$\begin{aligned} R_g^{(i,j)} R_{cg} \mathbf{v}_c^{(i,j)} &= R_{cg} R_c^{(i,j)} \mathbf{v}_c^{(i,j)} \\ R_g^{(i,j)} \boxed{R_{cg} \mathbf{v}_c^{(i,j)}} &= \boxed{R_{cg} \mathbf{v}_c^{(i,j)}} \\ R_g^{(i,j)} \mathbf{v}_g^{(i,j)} &= \mathbf{v}_g^{(i,j)} \end{aligned}$$

Rotation axis (by definition)

Tsai's method

- **DEMO**:

Working with rotation first: $R_g^{(i,j)} = R_{cg} R_c^{(i,j)} R_{cg}^\top$

Property 1: *The eigenvector matrix of* $R_g^{(i,j)}$ *can be transformed from that of* $R_c^{(i,j)}$ *using* R_{cg}

$$R_g^{(i,j)} R_{cg} = R_{cg} R_c^{(i,j)}$$

$$\Downarrow$$

$$\mathbf{v}_g^{(i,j)} = R_{cg} \mathbf{v}_c^{(i,j)}$$

Tsai's method

- **DEMO**:

Property 2 :*The rotation axis of R_{cg} is perpendicular to the vector joining the ends of the rotation axis for $R_g^{(i,j)}$ end $R_c^{(i,j)}$*

$$\mathbf{v}_{cg} \perp \left(\mathbf{v}_g^{(i,j)} - \mathbf{v}_c^{(i,j)}\right)$$

Tsai's method

- **DEMO:** $\mathbf{v}_{cg} \perp (\mathbf{v}_g^{(i,j)} - \mathbf{v}_c^{(i,j)})$

$$
\begin{aligned}
(\mathbf{v}_g - \mathbf{v}_c)^\top \mathbf{v}_{cg} &= (\mathbf{v}_g - \mathbf{v}_c)^\top R_{cg}^\top R_{cg} \mathbf{v}_{cg} \\
&= (R_{cg}(\mathbf{v}_g - \mathbf{v}_c)^\top)^\top R_{cg} \mathbf{v}_{cg} \\
&= (R_{cg}\mathbf{v}_g - R_{cg}\mathbf{v}_c)^\top \mathbf{v}_{cg} \\
&= (R_{cg}\mathbf{v}_g - \mathbf{v}_g)^\top \mathbf{v}_{cg} \\
&= ((R_{cg} - I)\mathbf{v}_g)^\top \mathbf{v}_{cg} \\
&= \mathbf{v}_g^\top (R_{cg} - I)^\top \mathbf{v}_{cg} \\
&= \mathbf{v}_g^\top (R_{cg}^\top - I)\mathbf{v}_{cg} \\
&= \mathbf{v}_g^\top (R_{cg}^\top \mathbf{v}_{cg} - \mathbf{v}_{cg}) \\
&= \mathbf{v}_g^\top (\mathbf{v}_{cg} - \mathbf{v}_{cg}) \\
&= 0 \quad \blacksquare
\end{aligned}
$$

* Here we omit (i,j) to clarify notation

Tsai's method

- **DEMO**:

Property 3: $(\mathbf{v}_g - \mathbf{v}_c)$*is collinear with* $(\mathbf{v}_g + \mathbf{v}_c) \times \mathbf{v}_{cg}$.

This means that $(\mathbf{v}_g - \mathbf{v}_c)$ is simultaneously orthogonal to $\mathbf{v}_{cg}$ (according to Property 2) and also to $(\mathbf{v}_g + \mathbf{v}_c)$. This can be shown considering that $(\mathbf{v}_g + \mathbf{v}_c)$ and $(\mathbf{v}_g - \mathbf{v}_c)$ are the minor and major diagonals of the parallelogram defined by the cross product (see picture). This provides this important constraint:

$$(\mathbf{v}_g - \mathbf{v}_c) = s \cdot (\mathbf{v}_g + \mathbf{v}_c) \times \mathbf{v}_{cg}$$

Tsai's method

- **DEMO**:

Property 4: *we can define* $\mathbf{v}'_{cg}$ *such that* $s = 1$.

This means that vectors $(\mathbf{v}_g - \mathbf{v}_c)$ and $(\mathbf{v}_g + \mathbf{v}_c) \times \mathbf{v}'_{cg}$ have the same length (other than being collinear). Let be α the angle between $\mathbf{v}'_{cg}$ and $(\mathbf{v}_g + \mathbf{v}_c)$. Then by definition we have:

$$\|(\mathbf{v}_g + \mathbf{v}_c) \times \mathbf{v}'_{cg}\| = \|(\mathbf{v}_g + \mathbf{v}_c)\| \cdot \|\mathbf{v}'_{cg}\| \cdot \cos\alpha$$

Tsai's method

- **DEMO**:

Property 4: *we can define* $\mathbf{v}'_{cg}$ *such that* $s = 1$.

Here we focuses on the term $\|\mathbf{v}'_{cg}\|$, from step 3 of the main Tsai's procedure we can derive that:

$$\mathbf{v}'_{cg} = \frac{\mathbf{v}_{cg}}{\sqrt{4 - \|\mathbf{v}_{cg}\|^2}}$$

Tsai's method

- **DEMO**:

Property 4: *we can define $\mathbf{v}'_{cg}$ such that $s = 1$.*

Here we focuses on the term $\|\mathbf{v}'_{cg}\|$, from step 3 of the main Tsai's procedure we can derive that:

$$\mathbf{v}'_{cg} = \frac{\mathbf{v}_{cg}}{\sqrt{4 - \|\mathbf{v}_{cg}\|^2}} \Rightarrow \begin{aligned} \|\mathbf{v}'_{cg}\| &= \left\| \frac{\mathbf{v}_{cg}}{\sqrt{4 - \|\mathbf{v}_{cg}\|^2}} \right\| \\ &= \frac{\|\mathbf{v}_{cg}\|}{\sqrt{4 - 4\sin^2 \frac{\theta_{cg}}{2}}} \\ &= \frac{2 \sin \frac{\theta_{cg}}{2}}{\sqrt{4 - 4\sin^2 \frac{\theta_{cg}}{2}}} \\ &= 2 \sin \frac{\theta_{cg}}{2} \cdot \frac{1}{2 \cos \frac{\theta_{cg}}{2}} \\ &= \tan \frac{\theta_{cg}}{2} \end{aligned}$$

$$\boxed{\|\mathbf{v}\| = 2 \sin\left(\frac{\theta}{2}\right)}$$

$$\boxed{\begin{gathered} 4\sin^2\alpha + 4\cos^2\alpha = 4, \\ \text{therefore } 4\cos^2\alpha = 4 - 4\sin^2\alpha \\ \text{and } \sqrt{4 - 4\sin^2\alpha} = 2\cos\alpha \end{gathered}}$$

Tsai's method

- **DEMO**:

Property 4: *we can define* $\mathbf{v'}_{cg}$ *such that* $s = 1$.

$$
\begin{aligned}
\|(\mathbf{v}_g + \mathbf{v}_c) \times \mathbf{v'}_{cg}\| &= \|(\mathbf{v}_g + \mathbf{v}_c)\| \cdot \tan\frac{\theta_{cg}}{2} \cdot \sin\alpha \\
&= 2\|OB\| \cdot \sin\alpha \cdot tan\frac{\theta_{cg}}{2} \\
&= 2\|AB\| \cdot tan\frac{\theta_{cg}}{2} \\
&= 2\|CB\| \\
&= \|CG\| \\
&= \|(\mathbf{v}_g - \mathbf{v}_c)\| \\
&\blacksquare
\end{aligned}
$$

Tsai's method

- **DEMO**:

Property 4:

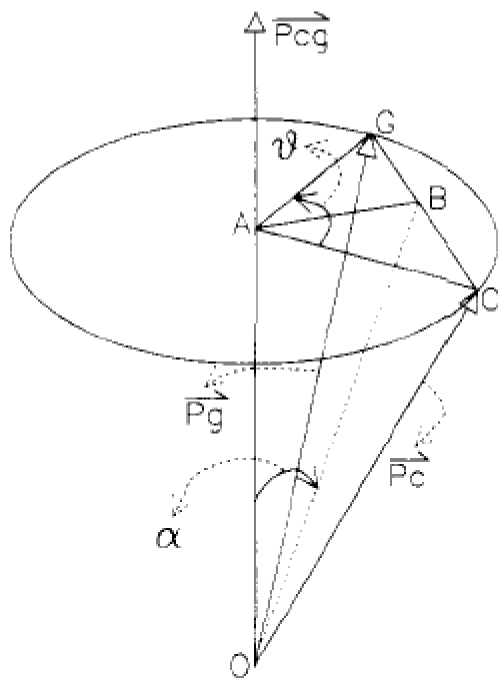


Figure 5. Geometrical relationships between P_{cg}, P_c and P_g. P_{cg} rotates P_c into P_g. The plane containing the circle is perpendicular to P_{cg}, and point B is the midpoint of point C and G.

Tsai's method

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Property 4: *we can define* $\mathbf{v'}_{cg}$ *such that* $s = 1$.

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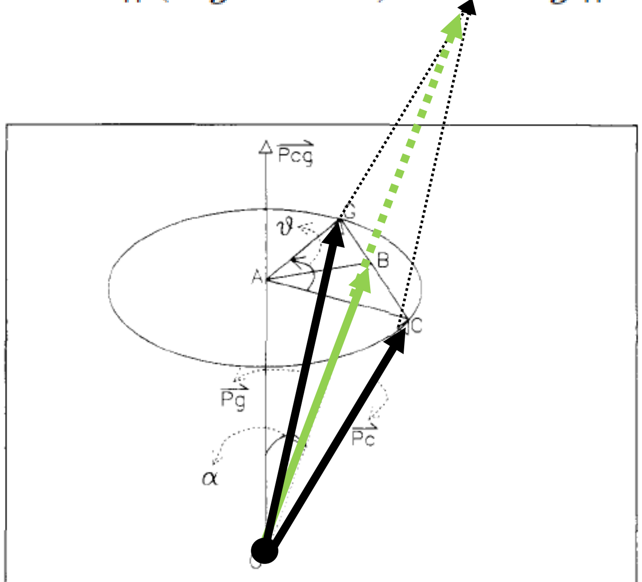

Figure 5. Geometrical relationships between P_{cg}, P_c and P_g. P_{cg} rotates P_c into P_g. The plane containing the circle is perpendicular to P_{cg}, and point B is the midpoint of point C and G.

Tsai's method

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\begin{aligned}
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&= 2\underline{\|AB\|} \cdot tan\frac{\theta_{cg}}{2} \\
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&\blacksquare
\end{aligned}
$$

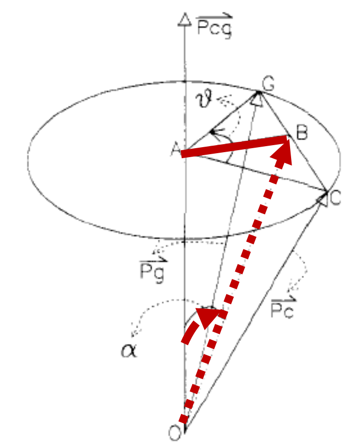
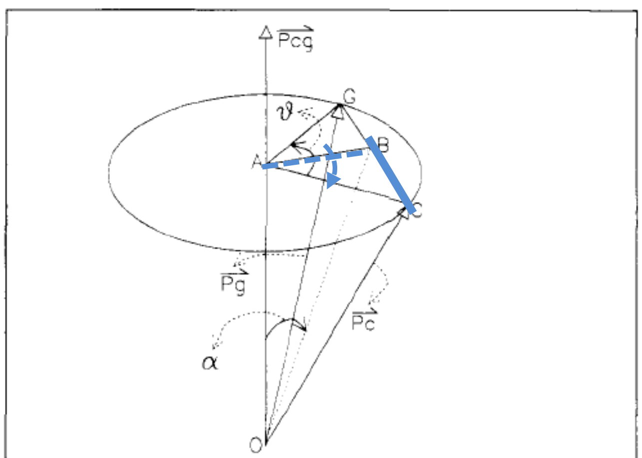


Figure 5. Geometrical relationships between P_{cg}, P_g and P_c. P_{cg} rotates P_g into P_c. The plane containing the circle is perpendicular to P_{cg}, and point B is the midpoint of point C and G.

Tsai’s method

- **DEMO**:

Property 4: *we can define* $\mathbf{v}'_{cg}$ *such that* $s = 1$.



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$$
\begin{aligned}
\|(\mathbf{v}_g + \mathbf{v}_c) \times \mathbf{v}'_{cg}\| &= \|(\mathbf{v}_g + \mathbf{v}_c)\| \cdot \tan\frac{\theta_{cg}}{2} \cdot \sin\alpha \\
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Tsai’s method

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&= 2\|AB\| \cdot tan\frac{\theta_{cg}}{2} \\
&= \underline{2\|CB\|} \\
&= \underline{\|CG\|} \\
&= \|(\mathbf{v}_g - \mathbf{v}_c)\| \\
&\blacksquare
\end{aligned}
$$

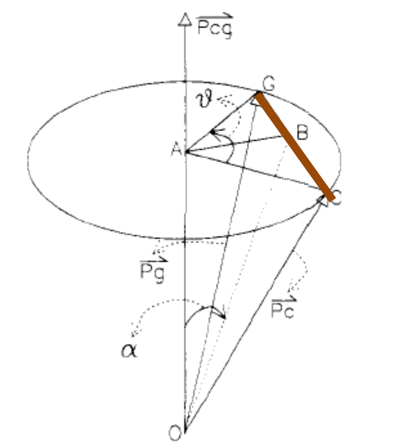

Figure 5. Geometrical relationships between P_{cg}, P_c and P_g. P_{cg} rotates P_c into P_g. The plane containing the circle is perpendicular to P_{cg}, and point B is the midpoint of point C and G.

Tsai's method

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$$

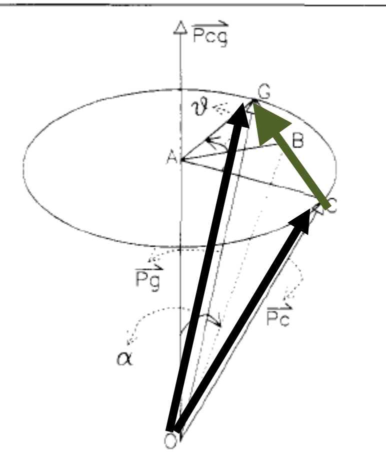

Figure 5. Geometrical relationships between P_{cg}, P_c and P_g. P_{cg} rotates P_c into P_g. The plane containing the circle is perpendicular to P_{cg}, and point B is the midpoint of point C and G.

Tsai's method

Moreover, we can recover the rotation angle θ_{cg} from the rotation angle θ'_{cg} considering that $\|\mathbf{v}'_{cg}\| = \tan\frac{\theta_{cg}}{2}$ but also $\|\mathbf{v}'_{cg}\| = 2\sin\frac{\theta'_{cg}}{2}$. We can combine these relations to get:

$$
\begin{aligned}
\tan\frac{\theta_{cg}}{2} &= 2\sin\frac{\theta'_{cg}}{2} \\
\frac{\theta_{cg}}{2} &= \tan^{-1}\left(2\sin\frac{\theta'_{cg}}{2}\right) \\
\theta_{cg} &= 2\tan^{-1}\left(2\sin\frac{\theta'_{cg}}{2}\right) \\
\theta_{cg} &= 2\tan^{-1}\|\mathbf{v}'_{cg}\|
\end{aligned}
$$

that was introduced on the optional step 2 of the Tsai's procedure.