# ROBOTICS, VISION AND CONTROL

## Trajectory Planning. A quick look at geometry

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# Outline

2D space

# 2D space

# Straight line – Point and direction

Let $\boldsymbol{p}_0 = (x_0, y_0)$ be a point on the plane, and $\boldsymbol{u} = (l, m)$ a unit vector (i.e. a direction).

The unit vector $(l, m)$ is called *direction vector*.

The parametric representation of the *straight line* on the plane passing by $\boldsymbol{p}_0$ along the direction $\boldsymbol{u}$ is

$$\begin{cases} x(\sigma) & = & x_0 + l\,\sigma \\ y(\sigma) & = & y_0 + m\,\sigma \end{cases}$$

for $\sigma \in \mathbb{R}_{\geq 0}$.

According to our previous notation

$$\boldsymbol{p}(\sigma) = \boldsymbol{p}_0 + \boldsymbol{u}\,\sigma$$

# Straight line – Two points

Let $\boldsymbol{p}_1 = (x_1, y_1)$ and $\boldsymbol{p}_2 = (x_2, y_2)$ be two distinct points on the plane.

The direction vector $\boldsymbol{u} = (l, m)$ is proportional to $\boldsymbol{p}_2 - \boldsymbol{p}_1$

$$\begin{aligned} l &= x_2 - x_1 \\ m &= y_2 - y_1 \end{aligned}$$

The parametric representation of the *straight line* passing by $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ is

$$\begin{cases} x(\sigma) = x_0 + (x_2 - x_1)\,\sigma \\ y(\sigma) = y_0 + (y_2 - y_1)\,\sigma \end{cases}$$

for $\sigma \in \mathbb{R}_{\geq 0}$.

*The parametric equations are NOT unique.*

# Straight line – Implicit equation

The *implicit equation* or *Cartesian equation* of a straight line on a plane is

$$a\,x + b\,y + c = 0$$

with $a$ and $b$ not both null.

Given two points $\boldsymbol{p}_1 = (x_1, y_1)$ and $\boldsymbol{p}_2 = (x_2, y_2)$, the implicit equation is given by

$$\det \begin{bmatrix} x - x_1 & y - y_1 \\ x_2 - x_1 & y_2 - y_1 \end{bmatrix} = 0$$

or

$$\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

The direction vector is $\boldsymbol{u} = (b, -a)$

# Straight line – Distance

Given a point $\bar{\boldsymbol{p}} = (\bar{x}, \bar{y})$ and the straight line

$$L : \begin{cases} x(t) & = & x_0 + l\,\sigma \\ y(t) & = & y_0 + m\,\sigma \end{cases}$$

The *distance* from a point $\bar{\boldsymbol{p}}$ to a line $L$ is the shortest distance from $\bar{\boldsymbol{p}}$ to any point on an infinite straight line

It is the length of the segment connecting $\bar{\boldsymbol{p}}$ to its orthogonal projection $\boldsymbol{h}$ on $L$

$$d(\bar{\boldsymbol{p}}, L) = d(\bar{\boldsymbol{p}}, \boldsymbol{h}) = \|\bar{\boldsymbol{p}} - \boldsymbol{h}\|$$

How can we compute $\boldsymbol{h}$?

Given a point $\bar{\boldsymbol{p}} = (\bar{x}, \bar{y})$ and the straight line

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How can we compute $\boldsymbol{h}$?

**Proposition.** Two lines

$$\begin{aligned} L : \quad & a\,x + b\,y + c & = & \quad 0 \\ \bar{L} : \quad & \bar{a}\,x + \bar{b}\,y + \bar{c} & = & \quad 0 \end{aligned}$$

are orthogonal if and only if

$$a\bar{a} + b\bar{b} = 0.$$

□

The point $\boldsymbol{h}$ is the intersection of

$$L : \begin{cases} x(t) = x_0 + l\,\sigma \\ y(t) = y_0 + m\,\sigma \end{cases}, \bar{L} : \begin{cases} x(t) = \bar{x} - m\,\sigma \\ y(t) = \bar{y} + l\,\sigma \end{cases}$$

The distance is

$$d(\bar{\boldsymbol{p}}, L) = \frac{|a\bar{x} + b\bar{y} + c|}{\sqrt{a^2 + b^2}}$$

**Exercise 1.** Given a line $L : ax + by + c = 0$ and two points $\boldsymbol{p} \in L$ and $\boldsymbol{q} \notin L$. Find the point(s) $\boldsymbol{h}$ on $L$ such that the area of the triangle $\boldsymbol{p} - \boldsymbol{q} - \boldsymbol{h}$ is equal to 5.

**Exercise 2.** Given a line $L : ax + by + c = 0$ and two points $\boldsymbol{p} \in L$ and $\boldsymbol{q} \notin L$. Find the point $\boldsymbol{h}$ on $L$ such that the triangle $\boldsymbol{p} - \boldsymbol{q} - \boldsymbol{h}$ is equilateral.

# Circle

The *circle* with centre coordinates $(a, b)$ and radius $r$ is the set of all points $(x, y)$ such that

Cartesian coordinates $(x, y)$

$$C: \qquad (x-a)^2 + (y-b)^2 = r^2$$

Polar coordinates $(\rho, \theta)$

$$C: \qquad \rho^2 - 2\rho\rho_0 \cos(\theta - \theta_0) + \rho_0^2 = r^2$$

where $(\rho_0, \theta_0)$ is the center of the circle

Parametric form

$$C: \begin{cases} x(\sigma) & = & a + r\cos(\sigma) \\ y(\sigma) & = & b + r\sin(\sigma) \end{cases}$$

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Cartesian coordinates (3-point form)

Given three points not on a line $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, the equation of the circle is

$$\frac{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)}{(y-y_1)(x-x_2)-(y-y_2)(x-x_1)} = \frac{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_1)}{(y_3-y_1)(x_3-x_2)-(y_3-y_2)(x_3-x_1)}$$

which is the solution of

$$\det \begin{bmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{bmatrix} = 0$$

# Circle



Given the circle $C : (x - a)^2 + (y - b)^2 = r^2$ and a point $\boldsymbol{p} = (\bar{x}, \bar{y}) \in C$, the tangent line at $\boldsymbol{p}$ is

$$(\bar{x} - a)(x - a) + (\bar{y} - b)(y - b) = r^2$$

or

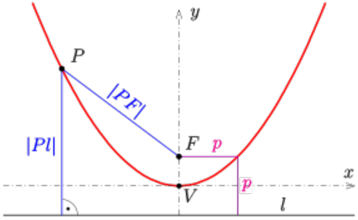
$$(\bar{x} - a)x + (\bar{y} - b)y = (\bar{x} - a)\bar{x} + (\bar{y} - b)\bar{y}$$

If $\bar{y} \neq 0$, the slope is

$$m = \frac{dy}{dx} = -\frac{\bar{x} - a}{\bar{y} - b}$$

# Parabola

*Parabola*: set of points $\boldsymbol{p}$ such that the distance $\operatorname{dist}(\boldsymbol{p}, \boldsymbol{f})$ to a fixed point $\boldsymbol{f}$, called the focus, is equal to the distance $\operatorname{dist}(\boldsymbol{p}, L)$ to a fixed line $L$, called the directrix




$|PF| = |Pl|$

The midpoint $\boldsymbol{v}$ of the perpendicular from the focus $\boldsymbol{f}$ onto the directrix $L$ is the vertex.
The line along the vector $\boldsymbol{f} - \boldsymbol{v}$ is the axis of symmetry of the parabola.

If $\boldsymbol{f} = (0, f)$ the expression for the parabola in the picture is

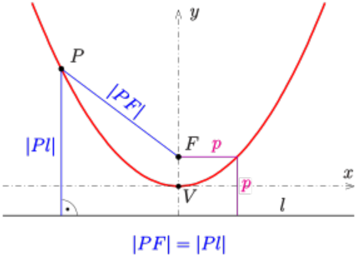
$$x^2 + (y - f)^2 = (y + f)^2$$

i.e.

$$y = \frac{1}{4f} x^2$$

If $f < 0$, the parabola has a downward opening.

If one exchanges $x$ and $y$, the parabola is rotated by 90 deg.



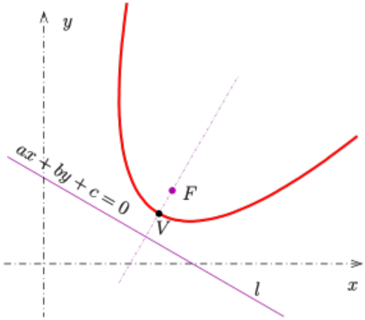

Let $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{f} = \boldsymbol{f} + (0, f) = (v_1, v_2 + f)$, the parabola has the equation

$$\begin{aligned} y &= \frac{1}{4f}(x - v_1)^2 + v_2 \\ &= \frac{1}{4f}x^2 - \frac{v_1}{2f}x + \frac{1}{4f}v_1^2 + v_2 \end{aligned}$$

The parametric representation of a normalized parabola $y = x^2$ is

$$P : \begin{cases} x(\sigma) &= \sigma \\ y(\sigma) &= \sigma^2 \end{cases}$$

What about the equation for a tilted parabola?



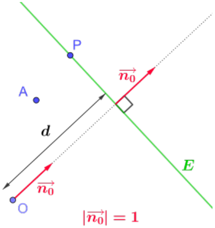

Let $\boldsymbol{f} = (f_1, f_2)$ and the directrix given by the line

$$a\,x + b\,y + c = 0$$

The parabola equation is

$$\frac{(a\,x + b\,y + c)^2}{a^2 + b^2} = (x - f_1)^2 + (y - f_2)^2$$

Hint. (Hesse normal form of a line or o a plane)



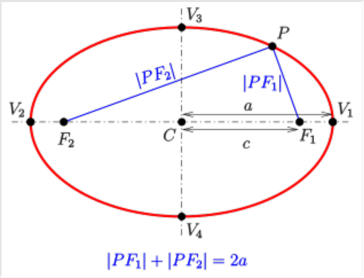

$\boldsymbol{p} \in E$ if

$$\langle \boldsymbol{p}, \boldsymbol{n}_0 \rangle - d = 0$$

where $\boldsymbol{n}_0$ is a normal unit vector to $E$ and $d \geq 0$

*Ellipse*: Given two fixed points $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ called the foci and a distance $2a$ larger than the distance between the foci, the ellipse $E$ is the set of points $\boldsymbol{p}$ such that

$$\text{dist}\,(\boldsymbol{p}, \boldsymbol{f}_1) + \text{dist}\,(\boldsymbol{p}, \boldsymbol{f}_2) = 2a$$



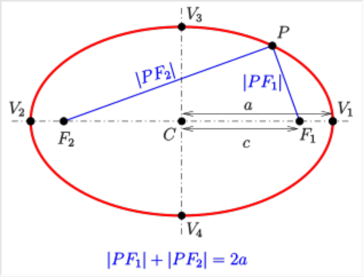

The line through the foci is called the *major axis*
The line perpendicular to it through the center is the *minor axis*

The foci are $\boldsymbol{f}_1 = (c, 0)$, $\boldsymbol{f}_1 = (-c, 0)$ where

$$c = \sqrt{a^2 - b^2}$$

Eccentricity

$$e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}$$




Standard ellipse centered at the origin with width $2a$ and height $2b$

$$\begin{aligned} \boldsymbol{f}_1 &= (c, 0), & \boldsymbol{f}_2 &= (-c, 0) \\ \boldsymbol{v}_1 &= (a, 0), & \boldsymbol{v}_2 &= (-a, 0) \\ \boldsymbol{v}_3 &= (b, 0), & \boldsymbol{v}_4 &= (-b, 0) \end{aligned}$$

Cartesian coordinates $(x, y)$

$$E: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parametric form

$$E: \begin{cases} x(\sigma) = a\cos(\sigma) \\ y(\sigma) = b\sin(\sigma) \end{cases}$$

# Ellipse

The tangent at a point $\boldsymbol{p} = (x_1, y_1)$ of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has the coordinate equation:

$$\frac{x_1}{a^2}x + \frac{y_1}{b^2}y = 1$$

A parametric equation of the tangent is:

$$\begin{cases} x(\rho) & = & x_1 - y_1 a^2 \rho \\ y(\rho) & = & y_1 + x_1 b^2 \rho \end{cases}$$

with $\rho \in \mathbb{R}$.

# Ellipse

*Translated ellipse in $(x_0, y_0)$*

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

*Rotated ellipse by $\theta$*

$$\begin{cases} x(\sigma) & = & a\cos\theta\cos\sigma - b\sin\theta\sin\sigma \\ y(\sigma) & = & a\sin\theta\cos\sigma + b\cos\theta\sin\sigma \end{cases}$$

*Translated ellipse in $(x_0, y_0)$ and rotated by $\theta$*

$$\begin{array}{rcl} X & = & (x-x_0)\cos\theta + (y-y_0)\sin\theta \\ Y & = & -(x-x_0)\sin\theta + (y-y_0)\cos\theta \end{array}$$

within

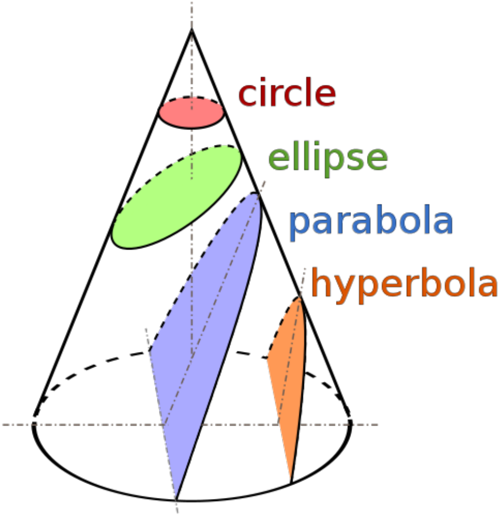
$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

gives

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where

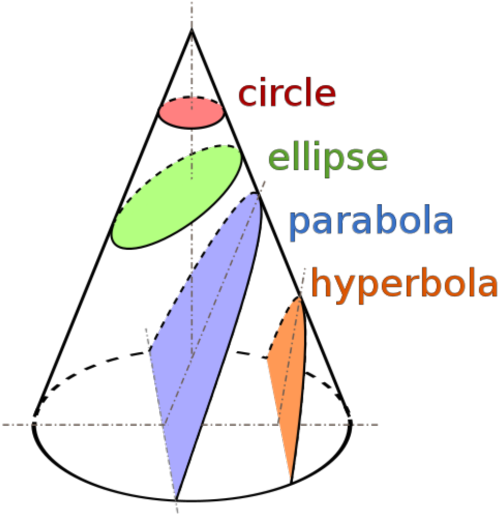
$$\begin{aligned} A &= a^2\sin^2\theta + b^2\cos^2\theta \\ B &= 2\left(b^2 - a^2\right)\sin\theta\cos\theta \\ C &= a^2\cos^2\theta + b^2\sin^2\theta \\ D &= -2Ax_0 - By_0 \\ E &= -Bx_0 - 2Cy_0 \\ F &= Ax_0^2 + Bx_0y_0 + Cy_0^2 - a^2b^2 \end{aligned}$$




The general equation of a conic section

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

with all coefficients are real numbers and $A$, $B$, $C$ not all zero.




The general equation of a conic section

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with all coefficients are real numbers and $A$, $B$, $C$ not all zero.

If the conic is non-degenerate, then

- if $B^2 - 4AC < 0$, the equation represents an ellipse;
- if $B^2 - 4AC < 0$ and $A = C$ and $B = 0$, the equation represents a circle
- if $B^2 - 4AC = 0$, the equation represents a parabola;
- if $B^2 - 4AC > 0$, the equation represents a hyperbola;

**Exercise.** Evaluate the Frenet frames attached to a circle, a parabola and an ellipse