Lecture 7: Multifactor Models

Investments

Introduction

- The empirical failure of the CAPM is not that surprising:
  - $\hookrightarrow$ We had to make a number of pretty unrealistic assumptions to prove the CAPM.
  - $\hookrightarrow$ For example, assumed that all investors were rational, had identical preferences, had the same information, and hold the same portfolio (the market).
  - $\hookrightarrow$ Also, identifying and measuring the market return is difficult, if not impossible (the Roll Critique)
- In this lecture we will study a different approach to asset pricing called the **Arbitrage Pricing Theory** or **APT**.
- The APT specifies a pricing relationship with a number of "systematic" factors.

Returns Are Multi-Dimensional

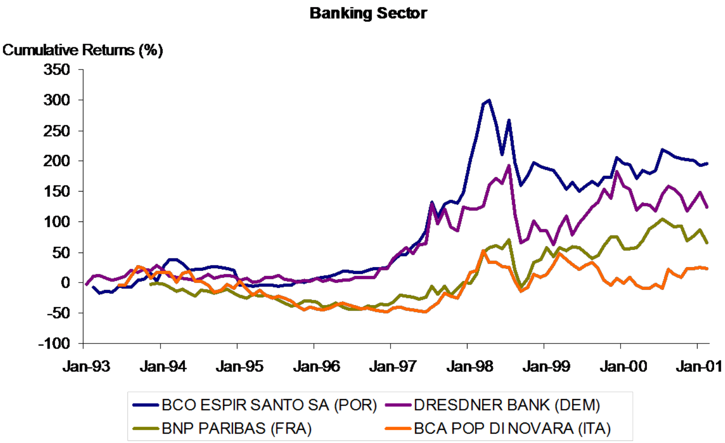
Barr Rosenberg, “Extra Market Components of Covariance in Security Markets,” Journal of Financial and Quantitative Analysis, 1974:

> *“Companies possessing similar characteristics may, in a given month, show returns that are different from the other companies. The pattern of differing shows up as the factor relation.”*

- A set of common factors - not just a monolithic market - influence returns

Comovement

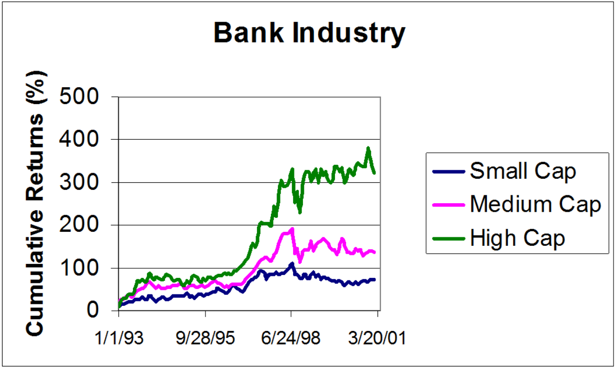
- Stocks in the same industry tend to move together.
- Example: European Banks



Source: BARRA

Comovement

- However, there are other common factors that simultaneously affect returns.
- Within the banking industry, the size factor is at work



Source: BARRA

Multiple factors and the CAPM

- The presence of multiple factors makes the CAPM a much more restrictive theory.
- Consider two sources of risk, “technology” and “monetary policy”.
  - $\hookrightarrow$ Do all stocks respond the same way to technological innovation?
  - $\hookrightarrow$ Do all stocks respond the same way to changes in interest rates?
- Assume that $R_M = R_T + R_I$,
  - $\hookrightarrow$ CAPM then implies that

$$E[R_i] = r_f + (E(R_m) - r_f)\,\beta_i$$

where

$$\beta_i \equiv \frac{cov(R_i, R_m)}{var(R_m)} = \frac{cov(R_i, R_T) + cov(R_i, R_I)}{var(R_m)}$$

- CAPM says that both covariances are priced exactly the same.

Arbitrage Pricing Theory

- The APT is an approach to determining asset values based on law of one price and no arbitrage.
- It is a multi-factor model of asset pricing.
- The APT is derived from a *statistical* model whereas the CAPM is an *equilibrium* asset pricing model.
  - ↪ We don't have to assume that *everyone* is optimizing.
  - ↪ Recall that we did need this assumption to get the CAPM.
  - ↪ This makes the APT a much more "reasonable" theory.
- Unlike the CAPM, we need very few assumptions to get the APT.

APT Assumptions

- The assumptions necessary for the APT are:
  a) All securities have finite expected values and variances.
  b) *Some* agents can form well diversified portfolios
  c) There are no taxes
  d) There are no transaction costs
- Notice that we have considerably fewer than with the CAPM!
- The central idea behind the APT will be that we can price some assets *relative* to other assets.
  a) We will derive restrictions on the price of assets based on no-arbitrage.
  b) We will be able to say something stronger if we exclude "near-arbitrage" or extremely good deals.

No Arbitrage

- Pricing restrictions in the APT come from the *Absence of Arbitrage*.
- Absence of Arbitrage in financial markets means that *NO SECURITY EXISTS WHICH HAS A NEGATIVE PRICE AND A NON-NEGATIVE PAYOFF.*
  - $\hookrightarrow$ Also, no security can be created which has this property.
- This rule also implies that:
  a) Two securities that always have the same payoff must have the same price.
  b) No security exists which has a zero price and a strictly positive payoff.
- This is the same idea that has been applied to option pricing!

No Arbitrage

1. In an efficiently functioning financial market arbitrage opportunities should not exist (for very long).
2. Unlike *equilibrium* rules such as the CAPM, arbitrage rules require only that there just one intelligent investor in the economy.
   - ↪ This is why derivatives security pricing models do a better job of predicting prices than equilibrium based models.
   - ↪ In this lecture, we will see how to apply the same concept to pricing portfolios of assets.
3. If arbitrage rules are violated, then unlimited risk-free profits are possible.

Specifying Uncertainty

- We will assume that investors know exactly which *states of nature* can occur, and exactly what will happen in each of the states.
- We know which states are possible, but not which will actually occur:
  - $\hookrightarrow$ We only know what the probabilities of each of the states are.
  - $\hookrightarrow$ You can think of this as meaning that we can always draw a multi-branched *tree* for the world.
- One implication is that investors know what the price of each and every security will be if the economy evolves in a particular way.
- With standard factor models, there are an infinite number of states of nature, since the factors can take on *any* values.

Arbitrage - Example

- Suppose that there are only two possible **states** for Inflation and Interest Rates: high or low
- We know *exactly* how four securities will perform in each of the possible states:

State/ Stock	High Real Int. Rates		Low Real Int. Rates	
	High Infl.	Low Infl.	High Infl.	Low Infl.
Int. Rate	5%	5%	0%	0%
Inflation	10%	0%	10%	0%
Prob.	0.25	0.25	0.25	0.25
Apex (A)	-20	20	40	60
Bull (B)	0	70	30	-20
Crush (C)	90	-20	-10	70
Dreck (D)	15	23	15	36

Arbitrage - Example

- Let's assume the prices of each of the four securities are \$100 and calculate expected returns, standard deviations and correlations:
- Everything looks normal here, but there is a simple **arbitrage opportunity** lurking in these numbers!

Stock	Current Price	Expect. Return(%)	Standard Dev. (%)	Correlation Matrix			
				A	B	C	D
A	100	25.00	29.58	1.00	-0.15	-0.29	0.68
B	100	20.00	33.91	-0.15	1.00	-0.87	-0.38
C	100	32.50	48.15	-0.29	-0.87	1.00	0.22
D	100	22.25	8.58	0.68	-0.38	0.22	1.00

Arbitrage - Example

- Consider the return of an Equal-Weighted portfolio of A, B and C, and compare this with the return of D:

State/ Port.	High Real Int. Rates		Low Real Int. Rates	
	High Infl.	Low Infl.	High Infl.	Low Infl.
EW Port. of A,B,C	23.33	23.33	20.00	36.67
Dreck (D)	15	23	15	36

- This table shows that the return of the EW portfolio is higher in all states, this means that there is an *arbitrage opportunity*.
- What would happen to the price of D in a well functioning market?

Arbitrage - Example

- Based on the arguments presented earlier, we will take the position that arbitrage opportunities cannot exist (for very long!). That means that there is something wrong with these prices.
- Our goal in this lecture is to come up with a model of security prices where:
  a) If prices/returns obey this model, there is no arbitrage.
  b) If prices/returns *fail to* obey this model, there *is* arbitrage.

Arbitrage Pricing Theory

- The previous example here is too restrictive. In reality:
  1. An infinite number of states are possible
  2. There are generally a large number of *factors*, and a *continuum* of possible factor realizations.
- The factor model framework gives us a systematic way of describing how expected security returns must relate to their comovement with the economy.
- If there is to no arbitrage in the economy, the we can also price assets relative to one another based on their comovement with these factors. *This is the basis of the APT*.

- First we need a **Factor Model** (or *Return Generating Process (RGP)*), which is a mathematical expression for how the security returns move with economic factors:
  - $\hookrightarrow$ We will call the sources of movement **factors**
  - $\hookrightarrow$ We will call the stock sensitivities to the factors **factor loadings** (or, equivalently, *factor betas* or *factor sensitivities*).
- We have already seen an example of such a model. It is the single-index model that we used to simplify the correlation structure between securities.

The **Factor Model** (or RGP) is:

$$\tilde{r}_i - E(\tilde{r}_i) = b_{i,1}\tilde{f}_1 + \ldots + b_{i,K}\tilde{f}_K + \tilde{e}_i$$

- The f_i's are common factors that affect most securities. Examples are economic growth, interest rates, and inflation.
- We require that, for each of the factors, $E(\tilde{f}_i) = 0$. This means that, instead of defining an f directly as economic growth, we would have to define it as the deviation of economic growth from what was expected.
- Often we assume $cov(\tilde{f}_i, \tilde{f}_j) = 0$ for $i \neq j$.

Factor Model

The **Factor Model** (or RGP) can be written as:

$$\tilde{r}_i = E(\tilde{r}_i) + b_{i,1}\tilde{f}_1 + ... + b_{i,K}\tilde{f}_K + \tilde{e}_i$$

- $b_{i,j}$ denotes the **loading** of the i'th asset on the j'th factor. This tells you how much the asset's return goes up when the factor is one unit higher than expected.
- The $\tilde{e}_i$ in this equation is *idiosyncratic* risk.
  - ↪ For example, $\tilde{e}_i$ will be negative when a firm's president dies, or a firm loses a big contract.
  - ↪ We will assume that $cov(\tilde{e}_i, \tilde{e}_j) = 0$ for all securities i and j.

Factor Model - Example

An example using this return generating process (*BKM*, chapter 10):

> *Suppose that two factors have been identified for the U.S. economy: the growth rate of industrial production, IP, and the inflation rate, IR. IP is expected to be 4%, and IR 6%. A stock with a beta of 1.0 on IP and 0.4 on IR currently is expected to provide a rate of return of 14%. If industrial production actually grows by 5%, while the inflation rate turns out to be 7%, what is your revised estimate of the realized return on the stock?*

Factor Model - Example

- We know $E(IP) = 4\%$ and $b_{IP} = 1$, $E(IR) = 6\%$, $b_{IR} = .4$, and $E(r_i) = 14\%$
- The two factors are therefore:

$$\begin{aligned} f_{IP} &= (0.05 - 0.04) = 0.01 \\ f_{IR} &= (0.07 - 0.06) = 0.01 \end{aligned}$$

- Plug these into the return generating process gives the expected return **conditional on** these realization of the industrial production growth rate (IP) and the inflation rate (IR):

$$\begin{aligned} E(\tilde{r}_i | f_{IP}, f_{IR}) &= 0.14 + 1 \cdot 0.01 + 0.4 \cdot 0.01 \\ &= 15.4\% \end{aligned}$$

$\hookrightarrow$ What about the idiosyncratic risk (e_i)?

Factor Model

- Using the return generating process we can calculate the variance of this portfolio as (for two factors):

$$\begin{aligned} var(R_i) &= var\left(b_{i,1}\tilde{f}_1 + b_{i,2}\tilde{f}_2 + \tilde{e}_i\right) \\ &= b_{i,1}^2 var(f_1) + b_{i,2}^2 var(f_2) + 2 \cdot b_{i,1} \cdot b_{i,2} \cdot cov(\tilde{f}_1, \tilde{f}_2) + \sigma_{e,i}^2 \\ &= b_{i,1}^2 var(f_1) + b_{i,2}^2 var(f_2) + \sigma_{e,i}^2 \qquad \text{(if factors are uncorrelated)} \end{aligned}$$

the general formula for n factors:

$$\sigma_i^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} b_{i,j} \cdot b_{i,k} \cdot \sigma_{j,k} + \sigma_{e,i}^2$$

where $\sigma_{j,k}$ denotes the covariance between the j'th and k'th factors. (What is $\sigma_{2,2}$?)

1. The *systematic* variance is $\sum_{j=1}^{n} \sum_{k=1}^{n} b_{i,j} \cdot b_{i,k} \cdot \sigma_{j,k}$.
2. The *idiosyncratic* variance is $\sigma_{e,i}^2$.

Factor Model

- Under the factor model, the covariance between two stocks, i and j (for two factors)

$$
\begin{aligned}
cov(R_i, R_j) &= cov(b_{i,1} f_1 + b_{i,2} f_2 + e_i, b_{j,1} f_1 + b_{j,2} f_2 + e_j) \\
&= b_{i,1} b_{j,1} var(f_1) + b_{i,2} b_{j,2} var(f_2) + \\
&\quad + (b_{i,1} b_{j,2} + b_{j,1} b_{i,2}) cov(f_1, f_2) \\
&= b_{i,1} b_{j,1} var(f_1) + b_{i,2} b_{j,2} var(f_2) \\
&\quad \text{(if factors are uncorrelated)}
\end{aligned}
$$

Definitions

- In the context of the APT, a diversified portfolio is a portfolio that carries no idiosyncratic risk:

$$\tilde{r}_p - E(\tilde{r}_p) = b_{p,1}\tilde{f}_1 + \ldots + b_{p,K}\tilde{f}_K$$

- This is defined *relative to a specific factor model*.
- We will assume that investors can form such diversified portfolios.

Arbitrage Pricing Theory

- The APT Pricing Equation is:

$$E(\tilde{r}_i) = \lambda_0 + \lambda_1 b_{i,1} + \ldots + \lambda_K b_{i,K}$$

$\hookrightarrow$ λ_i is the *factor risk premia*, that is the extra return for each extra unit of risk b your portfolio has.

  - There is one λ for each factor in the economy, plus one extra: λ_0.
  - If there is a risk-free asset, then it must be the case that portfolios with no risk (with all bs equal to zero) have a return of the risk-free rate, *i.e.*, that $\lambda_0 = r_f$.

- How does it compare to the CAPM?

APT - Example 1: Harvard management

- The facts:
  - $\hookrightarrow$ In January 1997, the US Treasury started issuing inflation-protected securities (TIPS), whose principal and coupon grow with inflation.
  - $\hookrightarrow$ In February 2000, the Harvard Management Company (HMC) considers investing a significant fraction of the university's endowment in TIPS.
  - $\hookrightarrow$ Proposal also suggests the inclusion of TIPS as a seperate asset class.

How do TIPS work?

- Principal and coupons grow at the rate of inflation:
  - $\hookrightarrow$ coupon at t: $c \times CPI_t \times P$
  - $\hookrightarrow$ principal at T: $P \times CPI_T$
- TIPS protect against inflation.
- TIPS only sensitive to real interest rate variation.
- A portfolio long TIPS and short T-Bonds is a bet on inflation.
  - $\hookrightarrow$ It will do well if inflation is higher than expected.

HMC's estimates

		Expected Real Return (%)	S.D. (%)	Correlations											
				Domestic Equity	Foreign Equity	Emerging Markets	Private Equity	Absolute Return	High Yield	Commodities	Real Estate	Domestic Bonds	Foreign Bonds	Infl-Indexed Bonds	Cash
1	Domestic Equity	6.5	16.0	1.00	0.50	0.40	0.40	0.60	0.55	(0.05)	0.20	0.40	0.15	0.10	0.10
2	Foreign Equity	6.5	17.0	0.50	1.00	0.35	0.30	0.50	0.35	(0.05)	0.15	0.25	0.40	(0.05)	0.05
3	Emerging Markets	8.5	20.0	0.40	0.35	1.00	0.25	0.30	0.35	0.00	0.15	0.15	0.10	0.00	0.00
4	Private Equity	9.5	22.0	0.40	0.30	0.25	1.00	0.30	0.20	(0.10)	0.15	0.20	0.10	0.10	0.05
5	Absolute Return	5.5	12.0	0.60	0.50	0.30	0.30	1.00	0.40	0.00	0.15	0.30	0.20	0.20	0.10
6	High Yield	5.5	12.0	0.55	0.35	0.35	0.20	0.40	1.00	0.10	0.10	0.45	0.15	0.30	0.10
7	Commodities	4.5	12.0	(0.05)	(0.05)	0.00	(0.10)	0.00	0.10	1.00	0.00	(0.15)	(0.10)	0.20	(0.05)
8	Real Estate	5.5	12.0	0.20	0.15	0.15	0.15	0.15	0.10	0.00	1.00	0.20	0.10	0.20	0.15
9	Domestic Bonds	4.3	7.0	0.40	0.25	0.15	0.20	0.30	0.45	(0.15)	0.20	1.00	0.40	0.50	0.15
10	Foreign Bonds	4.3	8.0	0.15	0.40	0.10	0.10	0.20	0.15	(0.10)	0.10	0.40	1.00	0.10	0.10
11	Infl-Indexed Bonds	4.0	3.0	0.10	(0.05)	0.00	0.10	0.20	0.30	0.20	0.20	0.50	0.10	1.00	(0.10)
12	Cash	3.5	1.0	0.10	0.05	0.00	0.05	0.10	0.10	(0.05)	0.15	0.15	0.10	(0.10)	1.00

- TIPS have been issued for only three years.
- As such, using historical data to obtain estimates of expected returns, standard deviations and correlations is difficult.
- HMC has formed its own, forward looking estimates.

- How are HMC's estimates formed? TIPS (T) and T-Bonds (B) have exposure only to real rates (r) and inflation (p)

$$R_T - E[R_T] = b_{T,p}\tilde{f}_p + b_{T,r}\tilde{f}_r$$
$$R_B - E[R_B] = b_{B,p}\tilde{f}_p + b_{B,r}\tilde{f}_r$$

  - ↪ Assuming they have the same maturity (duration), TIPS and Bonds will have roughly the same exposure to changes in real rates, so $b_{T,r} = b_{B,r}$
  - ↪ TIPS have inflation protection, so $b_{T,p} = 0$

- So, a portfolio TIPS minus Bonds is a pure inflation bet

$$R_T - R_B = E[R_T] - E[R_B] + (0 - b_{B,p})\tilde{f}_p + \underbrace{(b_{T,r} - b_{B,r})}_{=0}\tilde{f}_r$$

TIPS, Treasuries and the factor model

- Then, the APT implies that

$$E[R_T] - E[R_B] = -\lambda_p b_{B,p}$$

- HMC's estimates imply that

$$E[R_T] - E[R_B] = -0.30\%$$

- What does HMC's estimates imply about the price of inflation risk λ_p ?
- If we knew Bonds' sensitivity to inflation (which should be approximately equal to their duration), we could compute λ_p as equal to $\lambda_p = 0.30\%/b_{B,p}$
- Often it is not easy to find securities (or portfolios) that are pure factor bets, so we need to do a bit more work...

APT - Example 2

- Suppose that IBM is going to pay a liquidating dividend in exactly one year, and this is the only payment that it will make.
- However, that dividend that IBM pays is uncertain – it depends on how well the economy is doing:
  - ↪ If the economy is in an expansion, then its dividend will be **140.**
  - ↪ If the economy is in a recession, then its dividend will be only **100.**
  - ↪ If the two states are equally likely, the expected cash flow from IBM is $E(CF_1^{\text{IBM}}) = 120$.
- Note that, consistent with our definition of uncertainty, we know exactly what will happen to IBM *in each scenario*, but we don't know which scenario will occur.

APT - Example 2

	IBM
Boom Payoff (Pr=0.5)	140
Bust Payoff (Pr=0.5)	100
$E(CF_1)$	120
Time 0 Price	100
Discount Rate	20%

- Assuming that the price of IBM is $100, we see that investors in this economy are applying a discount rate of 20% to IBM's expected cash-flows.
- The way they come up with the price of IBM is to take the expected cash flow at time 1 from IBM, $E(CF_1^{\text{IBM}}) = \$120$, and discount this cash-flow back to the present at the "appropriate" discount rate, which is apparently 20%.
- Equivalently, we can say that the expected return of IBM is 20%.

APT - Example 2

- Now let's consider a second security, DELL which, like IBM, will pay a liquidating dividend in one year, and which has only two possible cash flows, depending on whether the economy booms or goes into a recession over the next year.

	IBM	DELL
Boom Payoff (Pr=0.5)	140	160
Bust Payoff (Pr=0.5)	100	80
$E(CF_1)$	120	120
Time 0 Price	100	?
Discount Rate	20%	?

- Now let's consider how investors will "price" DELL.
- Note the IBM and DELL have the same expected cash-flow, so one might guess that a reasonable price for DELL might also be $100.

APT - Example 2

- However, even though the expected cash-flows for DELL are the same, we see that the pattern of cash flows across the two states are probably worse for DELL:
- DELL's payoff is lower in the bust/recession state, which is when we are more likely to need the money. It only does better when things are good.
  - ↪ The recession is when the rest of our portfolio is more likely to do poorly, and when we are more likely to lose our job, and our consulting income is likely to be lower. We "need" the cash more in a recession
  - ↪ In the boom/expansion, our portfolio will probably do better; we're more likely to have a good job; we'll probably have more consulting income.
- This means that if DELL were the same price as IBM, we would buy IBM.

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APT - Example 2

- Therefore, to induce investors to buy all of the outstanding DELL shares, it will have to be the case that DELL's price is lower than \$100.
- Also, note that the three are statements are equivalent:
  - ↪ The price investors will pay for DELL will be less than \$100, even though DELL's expected cash flows are the same as IBM's.
  - ↪ The discount rate investors will apply to DELL's cash flows will be higher than the 20% applied to IBM's cash flows.
  - ↪ The expected return investors will require from DELL will be higher than the IBM's expected return of 20%.
- Let's assume that investors are only willing to buy up all of DELL's shares if the price of DELL is \$90, or, equivalently, that the discount rate that they will apply to DELL is 33.33%:

$$E(R_{DELL}) = \frac{E(CF_1^{\text{DELL}}) - P_{DELL}}{P_{DELL}} = \frac{120-90}{90} = \frac{30}{90} = 0.3333$$

APT - Example 2

	IBM	DELL
Boom Payoff (Pr=0.5)	140	160
Bust Payoff (Pr=0.5)	100	80
$E(CF_1)$	120	120
Time 0 Price	\$100	\$90
Discount Rate	20%	33.33%

- The way to think about expected returns is that this is something that investors determine.
- After looking at the pattern of cash-flows from any investment, and deciding whether they *like* or *dislike* this pattern, investors determine what rate they will discount these cash flows at (to determine the price.)
- This rate is what we then call the "expected return."
- Since investors accurately calculate the expected cash-flows, the average return they realize on the investment *will be* the discount rate.

APT - Example 2

Now let's see how all of this relates to the APT equations:

1. Calculating the business cycle factor f_{BC} in the two states:
   - $\hookrightarrow$ First, assume that we use the NBER (National Bureau of Economic Research) business cycle indicator to construct our factor.
     - The indicator is one (at the end of the next year) if the economy is in an expansion, and zero if the economy is in a recession.
   - $\hookrightarrow$ However, remember that for the factor, we need the *unexpected component* of the business cycle.
   - $\hookrightarrow$ Assuming there is a 50%/50% chance that we will be in an expansion/recession, the expected value of the indicator is 0.5.
   - $\hookrightarrow$ This means that the business-cycle factor has a value of $0.5 = 1 - 0.5$ if the economy booms, and $-0.5 = 0 - 0.5$ if the economy goes bust.

APT - Example 2

2. Calculating the factor loadings for the two securities:

↪ We run a time-series regression of the returns of IBM on the factor:

$$r_{IBM,t} = E(r_{IBM}) + b_{IBM,BC} \cdot f_{BC,t} + e_{IBM,t}$$

where $E(r_{IBM})$ is the intercept and $b_{IBM,BC}$ is the slope coefficient.

↪ Here, we have only two points, so we can fit a line exactly:

$$\begin{aligned} 0.40 &= E(r_{IBM}) + b_{IBM,BC} \cdot 0.5 \qquad \text{(boom)} \\ 0.00 &= E(r_{IBM}) + b_{IBM,BC} \cdot -0.5 \qquad \text{(bust)} \end{aligned}$$

↪ Solving these gives $E(r_{IBM}) = 0.20$ and $b_{IBM,BC} = 0.4$. Similarly, $E(r_{DELL}) = 0.33$ and $b_{DELL,BC} = 0.89$.

APT - Example 2

3. Interpreting the factor loadings.

   ↪ The factor loadings $b_{IBM,BC}$ and $b_{DELL,BC}$ tell us how much *risk* IBM and DELL have.

   ↪ Risk means that the security moves up or down when the factor (the business cycle) moves up and down.

   ↪ The expected return and the factor loading $b_{i,BC}$ of each security are sufficient to calculate payoffs, per dollar invested, in every state of nature.

      a) This is true for every well-diversified portfolio, in every factor model.
      b) The expected return tells us what the reward is, and the b's tell us what the risk of the security is.
      c) The APT (like the CAPM) tells us that risk and reward are linked.

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APT - Example 2

4. Calculating the factor risk premia (λ's)
   - $\hookrightarrow$ The Factor Model tells us nothing about why investors are discounting the cash flows from the different securities at different rates.
   - $\hookrightarrow$ To determine how investors view the risks associated with each of the risks in the economy, we have to evaluate the APT **Pricing Equation**.
   - $\hookrightarrow$ Given the factor loadings, we can determine the factor risk-premia (λ), by regressing expected returns Er_i on factor loadings b_{ik}.
   - $\hookrightarrow$ Since there is only one factor and two stocks, we are fitting a line

$$\begin{aligned} E(r_{IBM}) &= \lambda_0 + \lambda_{BC} \cdot b_{IBM,BC} \\ E(r_{DELL}) &= \lambda_0 + \lambda_{BC} \cdot b_{DELL,BC} \end{aligned}$$

to which the solutions are $\lambda_0 = 0.0909$ and $\lambda_{BC} = 0.2727$.

5. Interpreting the risk-premia (λs):

   $\hookrightarrow$ The λ_{BC} is a measure of how much more investors discount a stock as a result of having one extra unit of risk relating to the BC factor. Here, $\lambda_{BC} > 0$ since

   - DELL is considered to be a riskier security than IBM, which is reflected in DELL's higher loading on the BC factor.
   - Investors discount DELL's cashflows at a higher rate than IBM.

   $\hookrightarrow$ λ_0 is the return investors require if a security has no risk.

APT - Example 2

- Using the set of securities that we used to calculated the λ's for the pricing equations, we can always construct a portfolio p with *any* set of factor loadings ($b_{p,k}$).
- The pricing equation then tells us what the expected return (or discount rate) for the portfolio of securities is.
- Alternatively, this equation tells us, if we find a (well-diversified) portfolio with certain factor loadings, what discount-rate that portfolio must have for there to be no arbitrage.
- Finally, if we add a third security to the mix and calculate its factor loadings, the APT will tell us what its expected return **must be** in order to avoid arbitrage opportunities.
- Essentially what we doing is pricing securities *relative* to other securities, in much the same way we are pricing derivatives relative to the underlying security.

Finding Arbitrages

Arbitrage arises when the price of risk differs across securities.

1. If investors are pricing risk inconsistently across securities, then – assuming these securities are well diversified – arbitrage will be possible.
2. To see this, let's extend the example to include a risk-free asset which we can buy or sell at a rate of 5%.
3. Since $\lambda_0 = 0.0909$, we know that we can combine DELL and IBM in such a way that we can create a *synthetic* risk-free asset with a return of 9.09%.
4. So we borrow money at 5% (by selling the risk-free asset) and lend money at 9.09%

Building the Arbitrage Portfolio

- To determine how much of IBM and DELL we buy/sell, we solve the equation for the weights on IBM and DELL in a portfolio which is risk-free:

$$w_{IBM} \cdot b_{IBM,BC} + (1 - w_{IBM}) \cdot b_{DELL,BC} = b_{p,BC} = 0$$

  or

$$w_{IBM} = \frac{b_{DELL,BC}}{b_{ELL,BC} - b_{IBM,BC}} = 1.8182$$

  and $w_{DELL} = (1 - w_{IBM}) = -0.8182$.
- This means that, to create an arbitrage portfolio, we can invest \$1.8182 in IBM, short \$0.8182 worth of DELL, and short \$1 worth of the risk-free asset.
- This portfolio requires *zero* initial investment and has a positive payoff in *all* states.

Building the Arbitrage Portfolio

- To verify that this works, lets look at the payoff to the IBM/DELL risk-free portfolio in the two states:

$$\text{Payoff(boom)} = w_{IBM} \cdot (140/100) + (1 - w_{IBM}) \cdot (160/90) = 1.0909$$

$$\text{Payoff(bust)} = w_{IBM} \cdot (100/100) + (1 - w_{IBM}) \cdot (80/90) = 1.0909$$

  This means that the payoff from the zero-investment portfolio is **$0.0409** $= 1.0909 - 1.05$ in both the boom and bust states. So this portfolio (in this simple economy) is risk free.

- Of course, we can scale this up as much as we like. For $1 million investment in the long and short portfolios, we would get a risk-free payoff of $40,900 (assuming prices did not move with our trades)

Interpreting the Arbitrage

- What is going on here is that investors are pricing risk inconsistently across securities.
- With any pair of securities, we can calculate λ_0 and λ_{BC}, however, each pair of securities will give you a different set of λ's:

Security 1	Security 2	λ_0	λ_{BC}
IBM	DELL	0.0909	0.2727
IBM	RF	0.05	0.3750 $(=(0.2-0.05)/0.4)$
DELL	RF	0.05	0.3187 $(=(0.3333-0.05)/0.8889)$

- This means that, to find the arbitrage, we could have
  1. calculated the λ's using any pair of securities, and then
  2. calculated the expected return (or discount rate) for the third security
  3. bought the high return and sold the low return.

Interpreting the Arbitrage

- Whenever this happens, there will be an arbitrage.
- **APT:** If there are arbitrageurs in the economy, they will move prices until the arbitrage disappears, and risk is priced consistently.
- Note, that this assumes that the arbitrageurs have unlimited capital and patience. We will see later that in some cases that this may be an erroneous assumption and in fact, there may be *limits to arbitrage*.

Summary

What have we learned?

1. Here, we have used a simple example to understand how investors price risky securities.
2. The idea behind the APT is that investors require different rates of return from different securities, depending on the riskiness of the securities.
3. If, however, risk is priced inconsistently across securities, then there will be arbitrage opportunities.
4. Arbitrageurs will take advantage of these arbitrage opportunities until prices are pushed back into line, risk is priced consistently across securities, and arbitrages disappear.

Glossary

We have defined the following terms:

- **Factors** ($\tilde{f}$'s) move up and down with the economy, and affect the future cash flows of securities.
- **Factor loadings** (b's) *for each security, for each factor*, tell you how much the security moves (on average, in percent) when the factor moves by 1%.
- **Factor risk-premia:** (λ's) *for each factor*, tell you how much higher a discount-rate investors apply to a security if its factor loading on a particular factor is higher by one.
- **Arbitrage** arises if risk is priced inconsistently across (well-diversified) securities.
  - $\hookrightarrow$ One way of thinking about this is that we can find two well diversified securities/portfolios, with exactly the same risk/factor-loadings, which have their cash flows discounted at different discount rates.

Summary

- The APT is an alternative model to the CAPM.
- It makes fewer assumptions, and as a result gets weaker predictions.
- In particular, the APT *does not say* what the systematic factors are, whereas the CAPM says that the market portfolio is the only systematic source of risk.
- The APT can be used in place of the CAPM for
  a) pricing assets
  b) performance evaluation
  c) risk management

Implementing the APT

- The APT makes fewer assumptions and this comes at a cost:
  - $\hookrightarrow$ unlike the CAPM, the APT does not tell us *which* are the systematic factors driving returns.
- The APT *relies* on a statistical model for returns.
  - $\hookrightarrow$ Its ability to price asset will depend on getting the “right” factors.
- Next: How to specify the factors?
  a) Factors can be specified a priori: they could be macroeconomic variables (ex inflation, output) that capture the systematic risk in the economy or portfolios proxying for these risks.
  b) Factors can be extracted via Principal Components or Factor Analysis.

Selecting the factors: Three Approaches

1) We treat the factors as observable and specify the $\tilde{f}_j$ directly. *(Macroeconomic Approach)*

   $\hookrightarrow$ The factors can be macroeconomic variables like inflation, output. We think that these variables will be sufficient to capture the systematic risk in the economy.

   $\hookrightarrow$ An example of this approach is the Chen, Roll and Ross model.

2) We could treat the *loadings* ($b_{i,j}$'s) as observable and infer the security sensitivities from fundamental information about the securities. *(Fundamental Approach)*

   $\hookrightarrow$ We construct indices of some firm characteristics (such as B/M), and treat these risk indices as sensitivities to the factors associated with those characteristics (such as a B/M factor).

   $\hookrightarrow$ An example of this approach is the Fama-French 3-factor model.

Macroeconomic Variables as Factors

- This approach requires us to specify the factors a priori.
- You want to consider types of risk which you think are
  - ↪ Systematic
  - ↪ Rewarded by the market
- Each investment firm has its own preferred set of factors and puts a lot of effort into identifying factors and factor sensitivities.
- APT and factor models are very popular in practice, with BARRA the leading provider of factor models.

Many factor models

- Salomon Smith Barney's Factor Model
  - ↪ Market trend (past market returns)
  - ↪ Economic growth
  - ↪ Credit quality
  - ↪ Interest rates
  - ↪ Inflation shocks
  - ↪ Small cap premium
- Morgan Stanley's Macro Proxy Model
  1. GDP growth
  2. Long-term interest rates
  3. Foreign exchange (Yen, Euro, Pound basket)
  4. Market Factor
  5. Commodities or oil price index

Macroeconomic Variables as Factors

- Use leading indicators

A. Leading indicators

1. Average weekly hours of production workers (manufacturing)
2. Initial claims for unemployment insurance
3. Manufacturers' new orders (consumer goods and materials industrie
4. Vendor performance—slower deliveries diffusion index
5. New orders for nondefense capital goods
6. New private housing units authorized by local building permits
7. Yield curve slope: 10-year Treasury minus federal funds rate
8. Stock prices, 500 common stocks
9. Money supply (M2)
10. Index of consumer expectations

B. Coincident indicators

1. Employees on nonagricultural payrolls
2. Personal income less transfer payments
3. Industrial production
4. Manufacturing and trade sales

C. Lagging indicators

1. Average duration of unemployment
2. Ratio of trade inventories to sales
3. Change in index of labor cost per unit of output
4. Average prime rate charged by banks
5. Commercial and industrial loans outstanding
6. Ratio of consumer installment credit outstanding to personal income
7. Change in consumer price index for services

Macroeconomic Variables as Factors

In order to obtain the factor 'surprises',$(\tilde{f})$ we need to subtract the market expectations from the realized values, $(f - Ef)$. How to get market expectations?

1. Historical Averages / Statistical model
2. Analyst surveys

This Week's Calendar

Date	ET	Release	For	Actual	Briefing.com	Consensus	Prior	Revised From
Feb 21	10:00	Leading Indicators	Jan	1.1%	0.6%	0.5%	0.3%	0.1%
Feb 21	14:00	FOMC Minutes	Jan 31					
Feb 22	08:30	Core CPI	Jan	0.2%	0.2%	0.2%	0.1%	0.2%
Feb 22	08:30	CPI	Jan	0.7%	0.4%	0.5%	-0.1%	
Feb 23	08:30	Initial Claims	02/18	278K	285K	300K	298K	297K
Feb 23	10:00	Help-Wanted Index	Jan	37	40	40	38	39
Feb 23	10:30	Crude Inventories	02/17	1121K	NA	NA	4853K	
Feb 24	08:30	Durable Orders	Jan	-10.2%	-4.0%	-2.0%	2.5%	1.3%

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Macroeconomic Variables as Factors

- Choose factors that market reacts to:

Updated: 24-Feb-06

Market Snapshot

Dow		-32.57	(%)
Nasdaq		+1.38	(%)
SP 500		+0.08	(%)
10-yr Note			
NYSE	**Adv** 1731	**Dec** 1424	**Vol** 910.1 mln
Nasdaq	**Adv** 1441	**Dec** 1476	**Vol** 997.2 mln

Industry Watch

Strong: oil & gas refiners, equipment & services, explorers, and drillers; gold, diversified metals & minerals

Weak: specialty consulting services, fertilizer & agricultural chemicals, apparel retail, auto parts & equipment, drug retail, airlines

Moving the Market

- 10-year -03/32 at 4.57%
- Durable Orders fell 10.2% in Jan. (consensus -2.0%), caused by a sharp drop in transportation orders. Excluding transportation, orders were up 0.6%.
- Geopolitical concerns related to Nigeria militants and a foiled attack on a Saudi oil center are affecting energy prices.

Macroeconomic Variables as Factors

Once the factors $k = 1..K$ have been specified

1. Construct the factor surprises $\tilde{f} = f - E(f)$.

2. Estimate the loadings b_{ik} for each security $i = 1..N$. You can use individual stocks or portfolios.

$$R_{i,t} = a_i + b_{i1}\tilde{f}_{1,t} + b_{i2}\tilde{f}_{2,t} + ... + b_{iK}\tilde{f}_{K,t} + \varepsilon_{i,t}$$

$\hookrightarrow$ the $\varepsilon_{i,t}$ is the idiosyncratic news in each security. The APT makes no assumptions on $\varepsilon_{i,t}$ except that it is uncorrelated with $\varepsilon_{j,t}$.

$\hookrightarrow$ $\hat{b}_{ik}$ is going to be your estimate of the loading of security i on factor k.

Macroeconomic Variables as Factors

3. Estimate the average return of each security in the sample, $\hat{\mu}_i$. That is your estimate of this security's expected return.

4. Estimate the market prices of risk for each factor, $\lambda_1 \ldots \lambda_K$, by a multivariate regression of the N average returns $\hat{\mu}_i$ for the N assets on the $N \times K$ loadings including a constant.

$$\hat{\mu}_i = \lambda_0 + \lambda_1 \hat{b}_{i1} + \lambda_2 \hat{b}_{i2} + \ldots + \lambda_K \hat{b}_{iK} + u_i$$

- $\hookrightarrow$ The u_i's are deviations from the APT. If the APT holds then all the u_i's should be zero. This procedure is similar to what we used to test the CAPM.
- $\hookrightarrow$ The estimated prices of risk, $\hat{\lambda}_k$, are the reward for bearing factor-k risk. They provide the answer to the question: What is the difference in average returns (cost of capital, discount rates) between two otherwise identical securities that have differential exposure to factor k.

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Interpreting the λs

- How should we interpret the sign of the λ's?
- One easy way to think about what the sign of the λ's should be is as follows:
  - $\hookrightarrow$ As long as the factors are normally distributed, $Pr(\tilde{f} > 0) = Pr(\tilde{f} < 0) = 1/2$.
  - $\hookrightarrow$ Consider two derivative contracts, trade at prices P_H and P_L:
    1. C_H pays $\$1m$ if $\tilde{f} > 0$ and zero otherwise.
    2. C_L pays $\$1m$ if $\tilde{f} < 0$ and zero otherwise.

    - If $P_L > P_H$ then $\lambda > 0$. Examples: Market portfolio, GDP.
    - If $P_H > P_L$ then $\lambda < 0$. Examples: Inflation, Oil Prices.
    - If $P_L = P_H$ then $\lambda = 0$. Factor is not priced because we are indifferent.

Macroeconomic Variables as Factors - Example

Chen, Roll and Ross (1986) specify the factors to be

1. Monthly and annual unanticipated growth in industrial production (MP)
2. Changes in expected inflation, as measured by the change in r_{T-Bill} (DEI).
3. Unexpected inflation (UI)
4. Unanticipated changes in risk premiums, as measured by $r_{Baa} - r_{AAA}$ (UPR)
   $\hookrightarrow$ This is often called the "Default Spread"
5. Unanticipated changes in the slope of the term structure, as measured by $r_{T-Bond} - r_{T-Bill}$. (UTS)
   $\hookrightarrow$ This is often called the "Term Spread"
6. In their pricing equation they also include the return on the equal-weighted (EWNY) and value-weighted (VWNY) NYSE market portfolio.

Macroeconomic Variables as Factors - Example

- Chen, Roll and Ross (1986) find the factor prices λ_k (t-stats in parenthesis).

Table 11.4 Economic Variables and Pricing (percent per month × 10), Multivariate Approach

A	Years	YP	MP	DEI	UI	UPR	UTS	Constant
	1958–84	4.341	13.984	−.111	−.672	7.941	−5.8	4.112
		(.538)	(3.727)	(−1.499)	(−2.052)	(2.807)	(−1.844)	(1.334)
	1958–67	.417	15.760	.014	−.133	5.584	.535	4.868
		(.032)	(2.270)	(.191)	(−.259)	(1.923)	(.240)	(1.156)
	1968–77	1.819	15.645	−.264	−1.420	14.352	−14.329	−2.544
		(.145)	(2.504)	(−3.397)	(−3.470)	(3.161)	(−2.672)	(−.464)
	1978–84	13.549	8.937	−.070	−.373	2.150	−2.941	12.541
		(.774)	(1.602)	(−.289)	(−.442)	(.279)	(−.327)	(1.911)

B	Years	MP	DEI	UI	UPR	UTS	Constant
	1958–84	13.589	−.125	−6.29	7.205	−5.211	4.124
		(3.561)	(−1.640)	(−1.979)	(2.590)	(−1.690)	(1.361)
	1958–67	13.155	.006	−.191	5.560	−.008	4.989
		(1.897)	(.092)	(−.382)	(1.935)	(−.004)	(1.271)
	1968−77	16.966	−.245	−1.353	12.717	−13.142	−1.889
		(2.638)	(−3.215)	(−3.320)	(2.852)	(−2.554)	(−.334)
	1978–84	9.383	−.140	−.221	1.679	−1.312	11.477
		(1.588)	(−.552)	(−.274)	(.221)	(−.149)	(1.747)

Macroeconomic Variables as Factors - Example

- Interestingly, they find that proxies for the market portfolio are not priced.

C	Years	EWNY	MP	DEI	UI	UPR	UTS	Constant
	1958–84	5.021	14.009	−.128	.848	.130	−5.017	6.409
		(1.218)	(3.774)	(−1.666)	(−2.541)	(2.855)	(−1.576)	(1.848)
	1958–67	6.575	14.936	−.005	−.279	5.747	−.146	7.349
		(1.199)	(2.336)	(−.060)	(−.558)	(2.070)	(−.067)	(1.591)
	1968–77	2.334	17.593	−.248	−1.501	12.512	−9.904	3.542
		(.283)	(2.715)	(−3.039)	(−3.366)	(2.758)	(−2.015)	(.558)
	1978–84	6.638	7.563	−.132	−.729	5.273	−4.993	9.164
		(.906)	(1.253)	(−.529)	(−.847)	(.663)	(−.520)	(1.245)

D	Years	VWNY	MP	DEI	UI	UPR	UTS	Constant
	1958–84	−2.403	11.756	−.123	.795	8.274	−5.905	10.713
		(−.633)	(3.054)	(−1.600)	(−2.376)	(2.972)	(−1.879)	(2.755)
	1958–67	1.359	12.394	.005	−.209	5.204	−.086	9.527
		(.277)	(1.789)	(.064)	(−.415)	(1.815)	(−.040)	(1.984)
	1968–77	−5.269	13.466	−.255	−1.421	12.897	−11.708	8.582
		(−.717)	(2.038)	(−3.237)	(−3.106)	(2.955)	(−2.299)	(1.167)
	1978–84	−3.683	8.402	−.116	.739	6.056	−5.928	15.452
		(−.491)	(1.432)	(−.458)	(−.869)	(.782)	(−.644)	(1.867)

Fundamental Approach

- Identify specific firm-characteristics that we a priori think are proxies for differential sensitivity to systematic risk. Example:
  a) membership in specific industries (e.g. durable goods)
  b) book to market (value-growth)
  c) liquidity
  d) leverage
- Form portfolios of stocks sorted based on these characteristics.
  - $\hookrightarrow$ Example: Buy stocks in top quintile of leverage, short stocks in bottom quintile.
- When we want to use portfolio r_k as a factor, we should
  - $\hookrightarrow$ estimate the price of risk by $\lambda_k = Er_k$
  - $\hookrightarrow$ construct the factor "surprise" by $\tilde{f}_k = r_k - Er_k$

Fundamental Approach

- How to select firm characteristics?
  a) differences in the characteristic should be associated with differences in expected returns.
     - Example: Value firms have higher returns than growth firms.
  b) firms that have the same characteristics should also move together:
     - Example: Value firms co-move more with other value firms and less with growth firms
- These portfolios should typically satisfy the following criteria:
  - ↪ They will capture systematic movements in stock returns, i.e. a lot of stocks will have non-zero b_is with these portfolios.
  - ↪ These portfolios will typically be mispriced by the CAPM.

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Example: The Fama-French 3 factor Model

- When we looked at the performance of the CAPM in real data, we found that when it failed, it failed in a *systematic* way.
  - $\hookrightarrow$ We saw that value firms and small firms had higher returns than those implied by the CAPM.
- One possible explanation is that value firms and small firms are exposed to sources of systematic risk that are not captured by the existing proxies of the market portfolio, e.g. the S&P 500.
- We saw some possible explanations:
  1. value firms may proxy for “distress”.
  2. small firms may proxy for “liquidity” risk.

Example: The Fama-French 3 factor Model

- Fama and French in their 1993 *Journal of Financial Economics* paper, specified a version of the APT referred to as the *Fama-French 3 factor model*

$$E(R_i) - r_f = \beta_{mkt}(E(R_m) - r_f) + \beta_{smb}E(R_{smb}) + \beta_{hml}E(R_{hml})$$

- A stock's systematic risk is now summarized by three betas: the market, size and value beta.
- SMB and HML are the two Fama-French factors
  - $\hookrightarrow$ SMB: small minus big
  - $\hookrightarrow$ HML: high minus low book-to-market

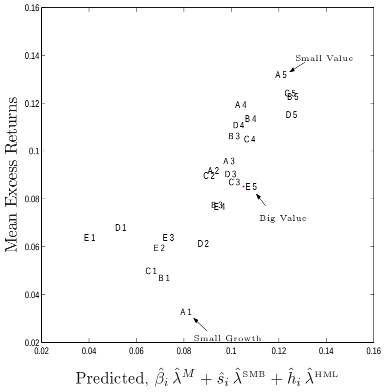
The Fama-French 3 factor Model

- Fama and French split the universe of stocks into 6 portfolios, based on size (Market Capitalization) and value/growth (Book to Market):

	Book-to-market		
Market Cap	**Low**	**Medium**	**High**
Small	Portfolio 1: Small growth	Portfolio 2: Small core	Portfolio 3: Small value
Large	Portfolio 4: Large growth	Portfolio 5: Large core	Portfolio 6: Large value

- They then constructed 2 factors
  1. Small minus Big (SMB): $(1/2SG + 1/2SV) - (1/2LG + 1/2LV)$
  2. High minus Low (HML): $(1/2SV + 1/2LV) - (1/2SG + 1/2LG)$
- Their version of the APT includes as factors the market portfolio, SMB and HML.

The Fama-French 3 factor Model



- The Fama-French model does a good job explaining the value and size anomalies.
- This is not purely mechanical: it works because there is comovement among value and size stocks.

The Fama-French 3 factor Model

	Deciles									
	1	2	3	4	5	6	7	8	9	10
BE/ME	**Low**									**High**
Mean	0.42	0.50	0.53	0.58	0.65	0.72	0.81	0.84	1.03	1.22
Std. Dev.	5.81	5.56	5.57	5.52	5.23	5.03	4.96	5.06	5.52	6.82
t(Mean)	1.39	1.72	1.82	2.02	2.38	2.74	3.10	3.17	3.55	3.43
Ave. ME	2256	1390	1125	1037	1001	864	838	730	572	362
E/P	**Low**									**High**
Mean	0.55	0.45	0.54	0.63	0.67	0.77	0.82	0.90	0.99	1.03
Std. Dev.	6.09	5.62	5.51	5.35	5.14	5.18	4.94	4.88	5.05	5.87
t(Mean)	1.72	1.52	1.89	2.24	2.49	2.84	3.16	3.51	3.74	3.37
Ave. ME	1294	1367	1211	1209	1411	1029	1022	909	862	661
C/P	**Low**									**High**
Mean	0.43	0.45	0.60	0.67	0.70	0.76	0.77	0.86	0.97	1.16
Std. Dev.	5.80	5.67	5.57	5.39	5.39	5.19	5.00	4.88	4.96	6.36
t (Mean)	1.41	1.52	2.06	2.37	2.47	2.78	2.93	3.36	3.75	3.47
Ave. ME	1491	1266	1112	1198	990	994	974	951	990	652
5-Yr SR	**High**									**Low**
Mean	0.47	0.63	0.70	0.68	0.67	0.74	0.70	0.78	0.89	1.03
Std. Dev.	6.39	5.66	5.46	5.15	5.22	5.10	5.00	5.10	5.25	6.13
t (Mean)	1.42	2.14	2.45	2.52	2.46	2.78	2.68	2.91	3.23	3.21
Ave. ME	937	1233	1075	1182	1265	1186	1075	884	744	434

- The Fama-French model also does a good job explaining related patterns

The Fama-French 3 factor Model

	Deciles									
	1	2	3	4	5	6	7	8	9	10
BE/ME	**Low**									**High**
a	0.08	−0.02	−0.09	−0.11	−0.08	−0.03	0.01	−0.04	0.03	−0.00
$t(a)$	1.19	−0.26	−1.25	−1.39	−1.16	−0.40	0.15	−0.61	0.43	−0.02
R^2	0.95	0.95	0.94	0.93	0.94	0.94	0.94	0.94	0.95	0.89
E/P	**Low**									**High**
a	−0.00	−0.07	−0.07	−0.04	−0.03	0.02	0.06	0.09	0.12	0.00
$t(a)$	−0.07	−1.07	−0.94	−0.52	−0.43	0.24	1.01	1.46	1.49	0.05
R^2	0.91	0.95	0.94	0.94	0.94	0.94	0.94	0.94	0.92	0.92
C/P	**Low**									**High**
a	0.02	−0.08	−0.07	−0.00	−0.04	0.00	0.00	0.05	0.06	0.01
b	1.04	1.06	1.08	1.06	1.05	1.04	0.99	1.00	0.98	1.14
s	0.45	0.50	0.54	0.51	0.55	0.50	0.53	0.48	0.57	0.92
h	−0.39	−0.18	0.07	0.11	0.23	0.31	0.36	0.50	0.67	0.79
$t(a)$	0.22	−1.14	−1.00	−0.04	−0.51	0.00	0.06	0.72	0.92	0.14
$t(b)$	51.45	61.16	62.49	64.15	59.04	61.28	60.02	63.36	58.92	46.49
$t(s)$	15.56	20.32	22.11	21.57	21.49	20.72	22.19	21.17	24.13	26.18
$t(h)$	−12.03	−6.52	2.56	4.28	7.85	11.40	13.52	19.46	24.88	19.74
R^2	0.93	0.95	0.95	0.95	0.94	0.94	0.94	0.94	0.94	0.92
5-Yr SR	**High**									**Low**
a	−0.21	−0.06	−0.03	−0.01	−0.04	−0.02	−0.04	0.00	0.04	0.07
b	1.16	1.10	1.09	1.03	1.03	1.03	1.00	0.99	0.99	1.02
s	0.72	0.56	0.52	0.49	0.52	0.51	0.50	0.57	0.67	0.95
h	−0.09	0.09	0.21	0.20	0.24	0.33	0.33	0.36	0.47	0.50
$t(a)$	−2.60	−0.97	−0.49	−0.20	−0.61	−0.25	−0.66	0.07	0.47	0.60
$t(b)$	59.01	70.59	67.65	65.34	56.68	68.89	62.49	54.12	50.08	34.54
$t(s)$	25.69	25.11	22.59	21.65	20.15	23.64	21.89	21.65	23.65	22.34
$t(h)$	−2.88	3.55	8.05	7.98	8.07	13.63	12.80	12.13	14.78	10.32
R^2	0.95	0.96	0.95	0.95	0.93	0.95	0.94	0.93	0.92	0.87

The Carhart 4 factor model

- The Fama-French model does not help explain the alphas in the momentum trading strategy
- Carhart (1997) proposed as an extension the Carhart four-factor model, that includes a fourth factor, MOM, constructed as the ‘winners’ minus ‘losers’
- Carhart’s four factor model is the industry-standard in performance evaluation of mutual fund managers

Factor-model Diagnostics

- The APT, as an asset pricing model, is only as good as the factor model it assumes. A good factor model should
  1. Explain the common movements in returns. That is, if $\tilde{f}_k$ are the factors, a regression of

     $$R_{i,t} = a_i + \sum b_{i,k} \tilde{f}_k + \varepsilon_i$$

     should yield ε_i's that are more or less uncorrelated amongst securities and $b_{i,k}$'s that are statistically significant.
  2. Differences in factor loadings between assets have to be associated with differences in expected returns.
  3. It helps if there is an economic reason why a portfolio is a factor!

APT vs the CAPM

- If the CAPM holds exactly, the market portfolio is the mean-variance efficient portfolio
- If the APT holds exactly, the mean-variance efficient portfolio is a linear combination **only** of the factors f
  - $\hookrightarrow$ Desire for diversification implies that it is never optimal to take on non-factor risk (ε)
  - $\hookrightarrow$ Reward for investing in factor i is the risk premium of factor i
  - $\hookrightarrow$ When you are making an investment decision you are choosing among a portfolio of risks. The APT captures this idea.

Example

- Suppose that there are three diversified portfolios: A,B and C,

$$\begin{aligned}\tilde{r}_A &= E[\tilde{r}_A] + 1.2\tilde{f}_1 + 0.5\tilde{f}_2 \\ \tilde{r}_B &= E[\tilde{r}_B] + 0.5\tilde{f}_1 + 0.7\tilde{f}_2 \\ \tilde{r}_C &= E[\tilde{r}_C] - 0.5\tilde{f}_1 + 1.2\tilde{f}_2\end{aligned}$$

- $\sigma_1 = 0.1$ and $\sigma_2 = 0.2$. and $\lambda_0 = 0.016$, $\lambda_1 = 0.02$ and $\lambda_2 = 0.12$

- Factors are uncorrelated

- The market capitalization of A, B and C is the same.

- What is the portfolio that has the maximum Sharpe Ratio?

Example

- Approach 1: Plug in returns for the three portfolios, A, B and C into mean-variance optimizer.
- Approach 2: Realize that because the portfolios are diversified and there are two factors, there are *only two risky assets* and a synthetic riskless asset:
  - $\hookrightarrow$ the factor 1 mimicking portfolio with expected excess return $\lambda_1 = 0.02$ and $\sigma_1 = 0.1$
  - $\hookrightarrow$ the factor 2 mimicking portfolio with expected excess return $\lambda_2 = 0.12$ and $\sigma_2 = 0.2$
  - $\hookrightarrow$ a risk-free asset with return $\lambda_0 = 0.016$

- We know that there is an analytic formula for the two risky-asset case:

$$x_A = \frac{E(\tilde{r}^e_A)\sigma^2_B - E(\tilde{r}^e_B)cov(\tilde{r}^e_A,\tilde{r}^e_B)}{E(\tilde{r}^e_A)\sigma^2_B + E(\tilde{r}^e_B)\sigma^2_A - [E(\tilde{r}^e_A)+E(\tilde{r}^e_B)]\,cov(r^e_A,r^e_B)}$$

- Here, the assumption that $cov(\tilde{f}_1,\tilde{f}_2)=0$ simplifies the algebra
- Plugging things into the formula, we find that

$$w_1 = \frac{\lambda_1\sigma^2_2}{\lambda_1\sigma^2_2+\lambda_2\sigma^2_1} = \frac{0.02\times 0.2^2}{0.02\times 0.2^2+0.12\times 0.1^2} = 0.4$$

- and $w_2 = 1 - w_1 = 0.6$.

Example

- If we want to find the weights in portfolios A, B and C, we first need to figure out which combination of A, B, and C mimics factors 1 and 2, by solving the systems

$$\begin{aligned} w_A^1 + w_B^1 + w_C^1 &= 1 \\ 1.2w_A^1 + 0.5w_B^1 - 0.5w_C^1 &= 1 \\ 0.5w_A^1 + 0.7w_B^1 + 1.2w_C^1 &= 0 \end{aligned}$$

- whose solution is $w^1 = [-3, 6.6, -2.6]$ and

$$\begin{aligned} w_A^2 + w_B^2 + w_C^2 &= 1 \\ 1.2w_A^2 + 0.5w_B^2 - 0.5w_C^2 &= 0 \\ 0.5w_A^2 + 0.7w_B^2 + 1.2w_C^2 &= 1 \end{aligned}$$

- whose solution is solution is $w^2 = [-1/3, -0.0667, 0.7333]$

Example

- Then, we can find the weights of the optimal portfolio on the three assets, A, B and C:

$$\begin{aligned}
w_A^* = 0.4 \times w_A^1 + 0.6 \times w_A^2 &= 0.4 \times (-3) + 0.6 \times (-1/3) = -1.4 \\
w_B^* = 0.4 \times w_B^1 + 0.6 \times w_B^2 &= 0.4 \times (6.6) + 0.6 \times (-0.0667) = 2.6 \\
w_C^* = 0.4 \times w_C^1 + 0.6 \times w_C^2 &= 0.4 \times (-2.6) + 0.6 \times (0.7333) = -0.6
\end{aligned}$$

- And the final step is to multiply all the weights by $1/0.6$ so that they sum to 1.

Example

Q. Is it possible that the CAPM holds as well?

A. Only if the market portfolio is MVE.

- Examine the loadings of the market portfolio on the two factors and compare them to $x_1 = 0.4$ and $x_2 = 0.6$ we found earlier. Since the market is an equally weighted average of A, B, and Cs, it has loadings

$$\begin{aligned} w_1^M &= 1/3 \times (1.2 + 0.5 - 0.5) = 0.4 \\ w_2^M &= 1/3 \times (0.5 + 0.7 + 1.2) = 0.8 \end{aligned}$$

From the above, it can be seen that the market portfolio is not a scaled up version of the MVE portfolio found earlier, since the ratio of $w_1^M / w_2^M \neq x_1 / x_2$.

Variations

- **Factor Tilting:** We have superior information to the market, that is we can forecast f better than the market, so $E[\tilde{f}_{1,t}] = a \neq 0$
  - $\hookrightarrow$ We adjust the expected return on the factor to $\lambda_1 + a$ in order to take advantage of our view
- **Hedging:** We want to target a particular risk exposure on a particular factor
  - $\hookrightarrow$ We can impose constraints on Markowitz to target a particular exposure to a factor
- **Take views:** Even if you think the APT may not hold exactly, it is still useful as an alternative baseline in place of the CAPM
  - $\hookrightarrow$ Modify the Black-Litterman model to use the APT instead of the CAPM as a baseline

Kellogg School of Management

Factor-Mimicking Portfolios

- Sometimes common sense dictates what the appropriate asset is that mimics the factor.
  - $\hookrightarrow$ Use oil futures to mimic an oil price factor.
  - $\hookrightarrow$ Use TIPS-Treasuries to mimic an inflation factor.
  - $\hookrightarrow$ Use REITs to mimic a real-estate price factor.
  - $\hookrightarrow$ Use gold futures to mimic a gold price factor.
- In other cases, we have to do a bit more work ...

How to construct a factor mimicking portfolio

1. Construct factor surprises $\tilde{f}_1$
2. Regress individual stocks on macro factors – save the beta with the factor
3. Sort firms into quintiles each year based on historical estimated factor beta
4. Create your factor mimicking portfolio as the long-short portfolios of most and least sensitive stocks [top 20% minus bottom 20%]
5. Verify that return on macro proxy is correlated with the factor

An example of this approach is the Morgan Stanley “Macro-proxy” model

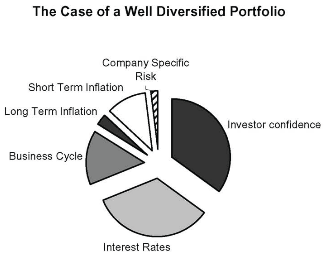
Factor-Mimicking Portfolios

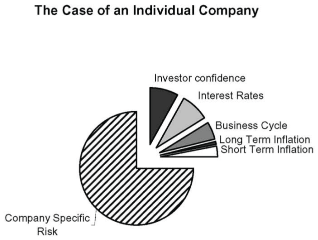
- Factor mimicking portfolios are useful
  - ↪ For estimating the risk-premia of the factor. The expected return on a portfolio that mimics factor f (and is uncorrelated with the other APT factors) equals

$$ER_{f,t} = \lambda_f$$

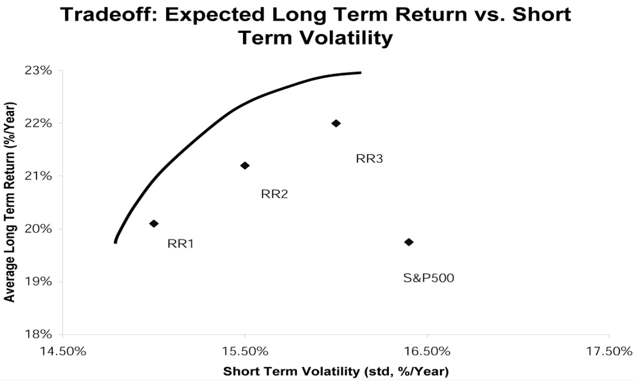
  - ↪ Remember: factor-mimicking portfolios are constructed from excess returns, so they are already zero-investment portfolios.
- For hedging factor risks.
- For expressing particular macro-views
- For pricing individual securities
- When you use the APT, you are effectively pricing securities **relative to the factor-mimicking portfolios**

- An example of a firm using the APT to manage money is the *Roll and Ross Asset Management Corporation*.
- A portfolio's long-term return and its volatility are completely determined by its factor loadings.

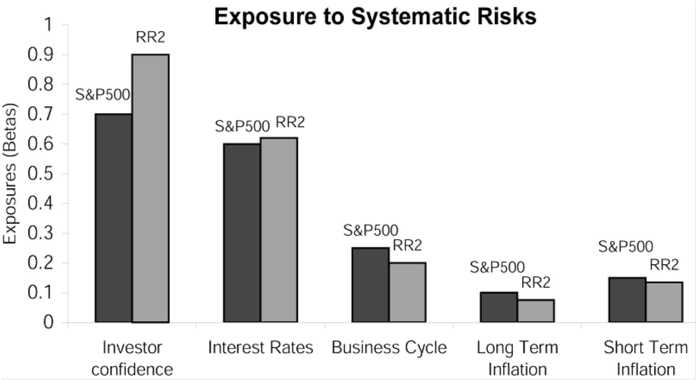




- The benchmark is in general not an “optimal” portfolio

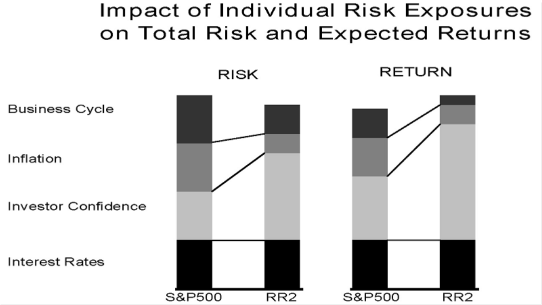


- The factors contribute differently to the aggregate risk and return of the portfolio



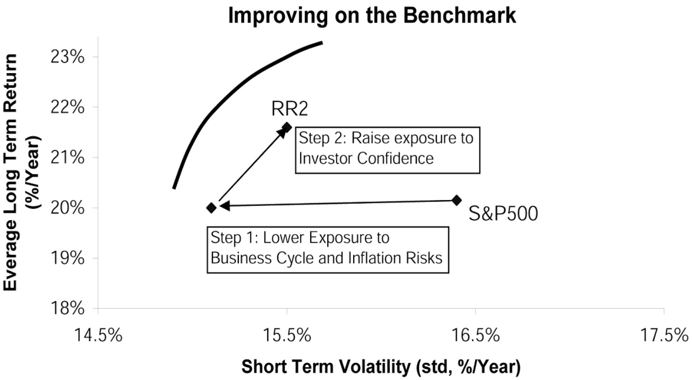
- Assuming that the factors are uncorrelated, the Sharpe ratio of the portfolio is:

$$SR = \frac{\lambda_1 b_1 + \lambda_2 b_2 + \ldots \lambda_k b_k}{\sqrt{\sigma_1^2 b_1^2 + \sigma_2^2 b_2^2 + \ldots \sigma_K^2 b_K^2}}$$

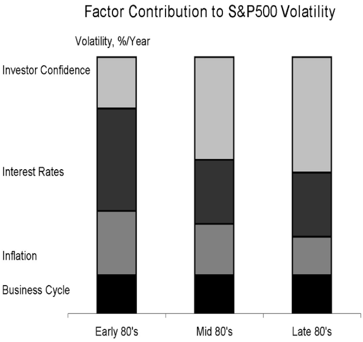


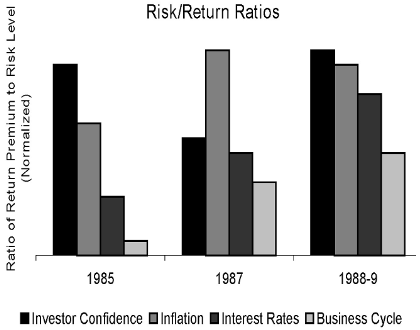
Asset Allocation using the APT (Roll and Ross)

- By measuring and controlling a portfolio's relative systematic risk exposures, the firm of Roll and Ross can produce the highest possible return for a given level of risk.



- Factor volatilities and rewards change over time, so we need to rebalance often.





Summary

- The APT *does not* specify what the systematic factors are
- As many versions of the APT as factor models
- In order to implement APT in practice, we need to specify factors.
    - $\hookrightarrow$ Factors can be specified a priori: they could be macroeconomic variables (ex inflation, output) that capture the systematic risk in the economy or portfolios proxying for these risks.
    - $\hookrightarrow$ Factors can be extracted via statistical techniques
- Once the factors are identified, the APT can be used as an alternative to the CAPM