Kellogg School of Management

Investments

Solution to HW 1 Problems 1-4

1. You bought 100 shares of ABC Inc. common stock at \$100 per share today at the opening of the market. ABC Inc. just announced a dividend of \$2.00 per share payable in exactly one year from today. It is widely believed in the market that in one year from now the economy will either be in a 'recession', a state of 'normal growth', or a 'boom' with probabilities of 30%, 40%, and 30% respectively. After analyzing ABC Inc. you are convinced that the price of ABC stock a year from now in these various states of the economy will be:

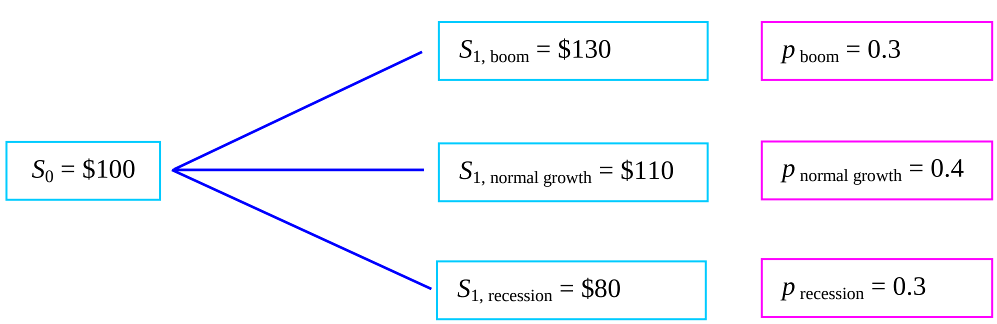
State of Economy	Price of ABC Share
Recession	\$80
Normal Growth	\$110
Boom	\$130

What are your estimated expected return and volatility over the next year to your investment in ABC stock?

Solution: You created a portfolio of 100 shares of ABC stock today. We denote "today" by time $t = 0$. You will hold this portfolio for one year. The end of one year is denoted by the time $t = 1$. At $t = 1$, the stock pays a dividend of $D_1 = \$2.00$ per share. Therefore values of your portfolio at $t = 0$ and $t = 1$ are, respectively:

$$P_0 = 100 \times S_0 \qquad \textit{and} \qquad P_1 = 100 \times (S_1 + D_1)$$

According to the given data, the state diagram for the stock price movement is:



Therefore the return on your portfolio in the various states will be:

$$R_{1,boom} = \frac{P_{1,boom}}{P_0} - 1 = \frac{100 \times (S_{1,boom} + D_1)}{100 \times S_0} - 1 = \frac{S_{1,boom} + D_1}{S_0} - 1 = \frac{\$130 + \$2}{\$100} - 1 = 0.32$$

$$R_{1,normal\ growth} = \frac{P_{1,normal\ growth}}{P_0} - 1 = \frac{100 \times (S_{1,normal\ growth} + D_1)}{100 \times S_0} - 1 = \frac{S_{1,normal\ growth} + D_1}{S_0} - 1 = \frac{\$110 + \$2}{\$100} - 1 = 0.12$$

$$R_{1,recession} = \frac{P_{1,recession}}{P_0} - 1 = \frac{100 \times (S_{1,recession} + D_1)}{100 \times S_0} - 1 = \frac{S_{1,recession} + D_1}{S_0} - 1 = \frac{\$80 + \$2}{\$100} - 1 = -0.18$$

Statistics of portfolio return is summarized in the following table:

State of Economy	Return on Portfolio (R_1)	Probability (p)
Boom	0.32	0.3
Normal Growth	0.12	0.4
Recession	-0.18	0.3

Therefore, the expected return is:

$$\begin{aligned} E[R_1] &= p_{boom} \times R_{1,boom} + p_{normal\ growth} \times R_{1,normal\ growth} + p_{recession} \times R_{1,recession} \\ &= 0.3 \times 0.32 + 0.4 \times 0.12 + 0.3 \times (-0.18) \\ &= 0.09 = 9\% \end{aligned}$$

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The variance of return is:

$$\begin{aligned} \operatorname{var}[R_1] &= p_{boom} \times (R_{1,boom} - E[R_1])^2 + p_{normal\ growth} \times (R_{1,normal\ growth} - E[R_1])^2 + p_{recession} \times (R_{1,recession} - E[R_1])^2 \\ &= 0.3 \times (0.32 - 0.09)^2 + 0.4 \times (0.12 - 0.09)^2 + 0.3 \times (-0.18 - 0.09)^2 \\ &= 0.0381 \end{aligned}$$

Volatility is measured by the standard deviation of the return. Therefore the volatility is:

$$sd[R_1] = +\sqrt{\operatorname{var}[R_1]} = +\sqrt{0.0381} \approx 0.1952 = 19.52\%$$

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2. ABC fund invests 25% of their assets in IBM stock, 50% in GE stock, and 25% in T-Bills. You invested 50% of your wealth in ABC fund and rest in the T-Bills. What % of your wealth is been invested in each stock and in the T-Bills?

<u>Solution</u>: ABC fund's portfolio is:

$$R_{ABC} = 0.25 \times R_f + 0.25 \times R_{IBM} + 0.5 \times R_{GE}$$

Return on your investment portfolio 'P' is:

$$R_P = 0.5 \times R_f + 0.5 \times R_{ABC}$$

By substituting ABC fund's return in your portfolio we find:

$$\begin{aligned} R_P &= 0.5 \times R_f + 0.5 \times \left(0.25 \times R_f + 0.25 \times R_{IBM} + 0.5 \times R_{GE}\right) \\ &= \left(0.5 + 0.5 \times 0.25\right) \times R_f + 0.5 \times 0.25 \times R_{IBM} + 0.5 \times 0.5 \times R_{GE} \\ &= 0.625 \times R_f + 0.125 \times R_{IBM} + 0.25 \times R_{GE} \end{aligned}$$

Hence, 62.5% of your wealth is invested in riskfree T-Bills, 12.5% in IBM stock, and 25% in GE stock (note that, total = 100%, as it always should be).

$\blacksquare$

3. Based on your examination of the historic records, you estimate that the expected return on the S&P-500 index over the next year will be 6% over the riskfree T-bills with a standard deviation of 15%. Currently a T-bill with one year to maturity and face value of $10,000 is selling for $9,615. You have $1 million to invest and you will put all of your money in some combination of the S&P-500 index and the 1-year T-bills. Calculate the expected return and the volatility for the following 3 different portfolios:

A. (5 Points): 'Portfolio-1' is invested in 100% in the S&P-500 index.

B. (5 Points): 'Portfolio-2' is invested 50% in S&P-500 index.

C. (5 Points): 'Portfolio-3' is invested 10% in the S&P-500 index.

<u>Solution</u>: The one year risk-free return is the return on the T-bill. It is:

$$R_f = \frac{\$10{,}000}{\$9{,}615} - 1 = 4\%$$

Hence the estimated expected return of the S&P-500 over the coming year is (6% + 4%) = 10% with a standard deviation of 15%.

Let ω be the weight put on the S&P-500 index in each portfolio. Then the return on portfolio ‘P’ is:

$$R_P = (1-\omega)R_f + \omega R_{S\&P-500}$$

The expected portfolio return is:

$$E[R_P] = (1-\omega)\cdot R_f + \omega \cdot E[R_{S\&P-500}]$$
$$\Rightarrow E[R_P] = (1-\omega)\times 4\% + \omega \times 10\%$$

with a standard deviation of:

$$sd[R_p] = \omega \cdot sd[R_{S\&P-500}]$$
$$sd[R_P] = \omega \times 15\%, \quad \omega \geq 0$$

In this problem, different portfolios are created by assuming different levels of ω. Expected returns and Volatilities for those different levels of ω can be easily computed from the above formulas, and they are given in the table below:

ω	$E[R_P]$	$sd[R_P]$
100%	10.0%	15.0%
50%	7.0%	7.5%
10%	4.6%	1.5%

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4. You are considering investing in two stocks. There are two possible states for the economy over the next year: “Good” and “Bad”. Each state is equally likely (that is, probability for each state is ½). Stock returns in each possible state are estimated as follows:

State	Return to Stock 1	Return to Stock 2
Good	30%	5%
Bad	10%	10%

Pre-Solution: This problem is pretty straightforward when you recall the following definitions from statistics:

$$E[X] = \sum_{j=1}^{N} p_j X_j$$

$$\operatorname{var}[X] = \sum_{j=1}^{N} p_j \left[\left(X_j - E[X] \right)^2 \right], \qquad sd[X] = +\sqrt{\operatorname{var}[X]}$$

$$\operatorname{cov}(X,Y) = E\left[\left(X - E[X] \right)\left(Y - E[Y] \right) \right], \qquad corr(X,Y) = \rho_{XY} = \frac{\operatorname{cov}(X,Y)}{sd[X] \cdot sd[Y]}$$

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A. What are the expected return and volatility of each stock?

Solution: It is given that the probability of each state is ½. We find the expected return on stock-1 by applying the first formula.

$$E[R_1] = 0.5 \times 0.3 + 0.5 \times 0.1 = 0.2 = 20\%$$

The variance of return of stock-1 is obtained from the 2nd formula.

$$\operatorname{var}[R_1] = 0.5 \times (0.3 - 0.2)^2 + 0.5 \times (0.1 - 0.2)^2 = 0.01$$

Volatility is defined as the standard deviation of return. Therefore the volatility of stock-1 is:

$$sd[R_1] = +\sqrt{\operatorname{var}[R_1]} = +\sqrt{0.01} = 0.1 = 10\%$$

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Similarly, the expected return on stock-2 is:

$$E[R_2] = 0.5 \times 0.05 + 0.5 \times 0.1 = 0.075 = 7.5\%$$

The variance of return of stock-2 is:

$$\text{var}[R_2] = 0.5 \times (0.05 - 0.075)^2 + 0.5 \times (0.1 - 0.075)^2 = 0.000625$$

Therefore the volatility of stock-2 is:

$$sd[R_2] = +\sqrt{\text{var}[R_2]} = +\sqrt{0.000625} = 0.025 = 2.5\%$$

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B. What are the covariance and the correlation between their returns?

Solution: By applying the third formula from the pre-solution we can find the covariance between the two stock returns.

$$\text{cov}(R_1, R_2) = 0.5 \times (0.3 - 0.2) \times (0.05 - 0.075) + 0.5 \times (0.1 - 0.2) \times (0.1 - 0.075) = -0.0025$$

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Hence, the correlation between the stocks is:

$$corr(R_1, R_2) = \rho_{12} = \frac{\text{cov}(R_1, R_2)}{sd[R_1] \cdot sd[R_2]} = \frac{-0.0025}{0.1 \times 0.025} = -1$$

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C. Suppose that a riskfree investment of 5% is also available. Does this present a profit opportunity to you? Why or why not? Explain.

Solution: Since the two stock returns are perfectly negatively correlated, it is possible to create a riskfree portfolio by holding these two stocks in appropriate weights (see the diagram in part (D) – this portfolio is called "perfectly hedged" portfolio). Suppose ω is the weight on stock-1 for this riskfree portfolio, then:

$$\operatorname{var}[R_P] = (\omega \cdot sd[R_1] - (1-\omega) \cdot sd[R_2])^2 = 0$$

$$\Rightarrow \omega \cdot sd[R_1] - (1-\omega) \cdot sd[R_2] = 0$$

$$\Rightarrow \omega = \frac{sd[R_2]}{sd[R_1] + sd[R_2]} = \frac{0.025}{0.1 + 0.025} = 0.2 = 20\%$$

Hence a riskfree return can be obtained by holding a portfolio invested 20% of its capital in stock-1 and 80% of its capital in stock-2. According to the following formula, this portfolio has a return of 10%:

$$R_{riskfree} = E[R] = \omega \cdot E[R_1] + (1-\omega) \cdot E[R_2] = 0.2 \times 0.2 + 0.8 \times 0.075 = 0.1 = 10\%$$

Clearly this return is riskfree and it is different from 5%. In case there is a riskfree investment in the market returning 5% over the holding period, then there is a significant arbitrage profit opportunity in this market when you are allowed to short-sell this instrument costlessly. The arbitrage trading strategy will be to borrow at the riskfree rate of 5% from the market (that is, to short-sell the riskfree investment) and to invest the entire capital in the stock portfolio with 20% in stock-1 and 80% in stock-2.

It is very important to note that the above arbitrage trading strategy is possible if and only if you can costlessly borrow at the riskfree rate of at 5%, or alternatively, there is no restriction or cost involved in short-selling this asset. Otherwise, the analysis will not work, and there will be no available arbitrage opportunity among these two stocks and the other riskfree investment. It is called ‘limits-to-arbitrage’ when market restrictions, transaction costs, or other market frictions eliminate the otherwise possible arbitrage opportunities.

You may have noticed that there is always a spread between various riskfree interest rates in the real world. Often those rates do not create arbitrage opportunities. Either those rates are one way rates (for either borrowing or depositing, but not both), or there are high transaction costs involved to enter into those rates. Typical transaction costs are big enough to wash out the spread arbitrage. Hence limit-to-arbitrage!

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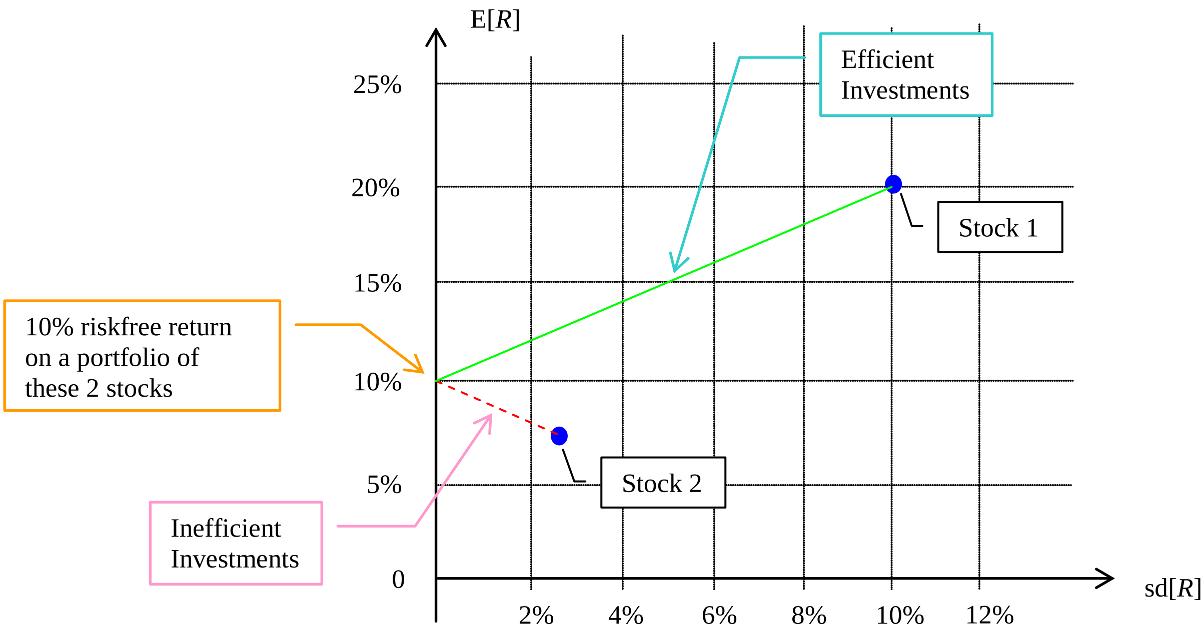
D. Draw a diagram to illustrate the tradeoff between risk and return (that is available portfolios or funds) by investing in these two stocks (assume no short selling).

Solution: The possibilities are given in the following diagram:



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5. Consider a risky portfolio that offers a rate of return of 15% per year with a standard deviation of 20% per year. Suppose an investor is indifferent between investing in the risky portfolio and investing in a risk free asset earning 8% per year.

   a) What is the investor's risk aversion coefficient?

      To find the implied risk aversion coefficient A of the investor, we equate the risk-free (certainty) equivalent rate of return for the risky asset to the risk free rate (Why?). Risk-free equivalent return for the risky asset

      $$r_{CE} = 0.15 - 0.5 \times A \times (0.22) = 0.15 - 0.02 \times A = 0.08(\text{the risk free rate}).$$

      Hence, $A = (0.15 - 0.08)/0.02 = 0.07/0.02 = 3.5$.

   b) If allowed to invest in a combination of the risky portfolio and the risk free asset, what proportion would the investor hold in the risky portfolio?

      The optimal fraction invested in the risky asset, x^*, is given by the formula,

      $$x^* = \frac{E(R_i) - r_f}{A\sigma^2}$$

      Substituting the values for the variables involved gives:

      $$x^* = (0.15 - 0.08)/[3.5(0.22)] = 0.5$$

   c) What is the expected rate of return and the standard deviation of the rate of return on the optimally chosen combination?

      The expected return on the optimally chosen combination,

      $$E(r) = 0.5(.15) + 0.5(0.08) = 0.115.$$

      The standard deviation of the portfolio is $0.5(0.20) = 0.10$.

   d) What would be the investor's certainty equivalent return for the optimally chosen combination?

The certainty equivalent of the optimally chosen combination equals, $0.115-0.5(3.5)(0.12) = 0.0975$. The investor gains $0.0975 - 0.08 = 0.0175$ or 1.75% by choosing the optimal combination instead of investing in the risk-free asset alone or the risky asset alone.

6. Consider an investor who has an asset allocation of 50% in equities and the rest in T-Bills. Suppose the expected rate of return on equities is 10%/year and the standard deviation of the return on equities is 15%/year. T-Bills earn 6%/year.

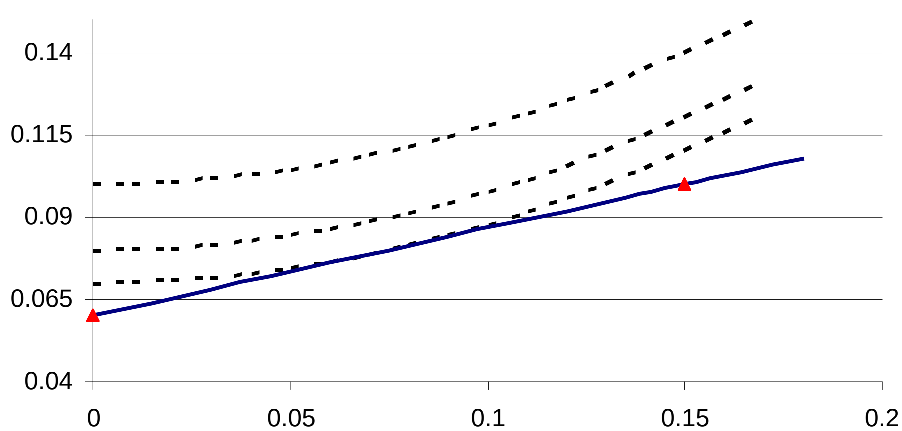
    a) What is the implied risk aversion coefficient of the investor?

    The optimal fraction invested in the risky asset, x*, is 0.5. Substituting this value for x* and the values for the expected return on the risky asset, the risk free rate and the standard deviation of the risky asset return we get:

$$0.5 = \frac{0.1 - 0.06}{A \times 0.15^2} = \frac{0.04}{0.0225A}$$

$$A = \frac{0.04}{0.0225 \times 0.5} = 3.556$$

    b) Plot the CAL along with a couple of indifference curves for the investor type identified above.



    c) Use Excel's solver to maximize the investor's utility and confirm that you get a 50% allocation in stocks.

7. You can invest in a risky asset with an expected rate of return of 20% per year and a standard deviation of 40% per year or a risk free asset earning 5% per year or a

combination of the two. The borrowing rate is 6% per year.

a) Draw the Capital Allocation Line. Indicate the points corresponding to (a) 50% in the risk-less asset and 50% in the risky asset; and (b) -50% in the riskless asset and 150% in the risky asset.

   The Capital Allocation Line will be kinked. The first part of the CAL that joins the risk free rate and the risky asset will have a slope of (20% - 5%)/40% = 0.375. The second part of the CAL that extends beyond the risky asset will have a smaller slope since the borrowing rate is 6%. The slope for that part will be (20%-6%)/40% = 0.35.

b) Compute the expected rate of return and standard deviation for (a) and (b).

   The portfolio with 50% in the risk free asset and 50% in the risky asset will plot in the first part of the CAL. The expected return on the portfolio is 0.5(5%) + 0.5(20%) = 12.5%. The standard deviation is 0.5(40%) = 20%. The portfolio with 150% in the risky asset and -50% in the riskless asset will have an expected return of 1.50(20%) - 0.5(6%) = 27%. Its standard deviation will be 1.5(40%) = 60%. It will lie on the second part of the CAL.

c) Suppose you have a target risk level of 50% per year. How would you construct a portfolio of the risky and the riskless asset to attain this target level of risk? What is the expected rate of return on the portfolio so constructed?

   Suppose the fraction invested in the risky asset is y. The standard deviation of the portfolio will be y 40%. We want to choose y such that the standard deviation equals 50%. Hence choose y = 50%/40% = 1.25. The expected return on the portfolio will be 1.25(20%) - 0.25(6%) = 23.5%