Lecture 3: Equilibrium Risk and Return

FE-312 Investments

NORTHWESTERN
UNIVERSITY

Introduction

- A key input in our portfolio decision is the mean (expected) return of a security
  - Last week, we saw that average returns are very hard to estimate based on historical data alone.
- More generally, we may also want to have an estimate of a security's *required rate of return*, that is, the rate of return that compensates us *appropriately* for the risk in that security.
  - This is the rate you should be using for valuation.
- It would be nice if we had an economic theory of what expected returns *should* be.
- The CAPM is an *equilibrium* model specifying a relation between expected rates of return and systematic risk (covariance) for all assets.
  - *Equilibrium* is an economic term that characterizes a situation where no investor wants to do anything differently.

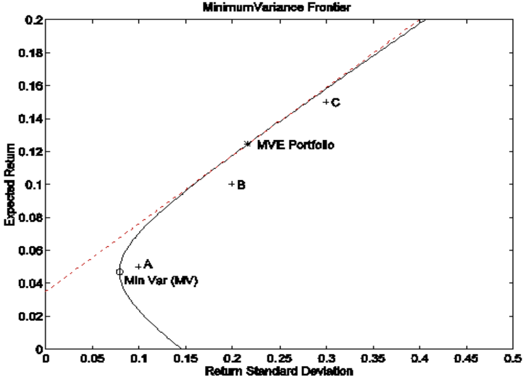
Equilibrium Pricing

- If everyone in the economy holds an efficient portfolio, then how should securities be priced so that they are actually bought 100% in equilibrium?
  - For example, if based on the prices/expected returns our model comes up with, we found that no maximizing investor would like to buy IBM, then something is not quite right.
  - IBM would be priced too high (offer too low an expected rate of return).
  - The price of IBM would have to fall to the point where, in aggregate, investors want to hold exactly the number of IBM shares outstanding.
- So, what sort of prices (risk/return relationships) are feasible in equilibrium? The CAPM will give an answer.

The CAPM Assumptions

- A number of assumptions are necessary to formally derive the CAPM:
  1. No transaction costs or taxes.
  2. Assets are all tradable and are all infinitely divisible.
  3. No individual can effect security prices (perfect competition).
  4. Investors care only about expected returns and variances.
  5. Unlimited short sales and borrowing and lending.
  6. Homogeneous expectations.
- Assumptions 4 - 6 imply everyone solves the passive portfolio problem we just finished, and they all see the same efficient frontier!
- Some of these can be relaxed without too-much effect on the results.

MV Analysis and two fund seperation

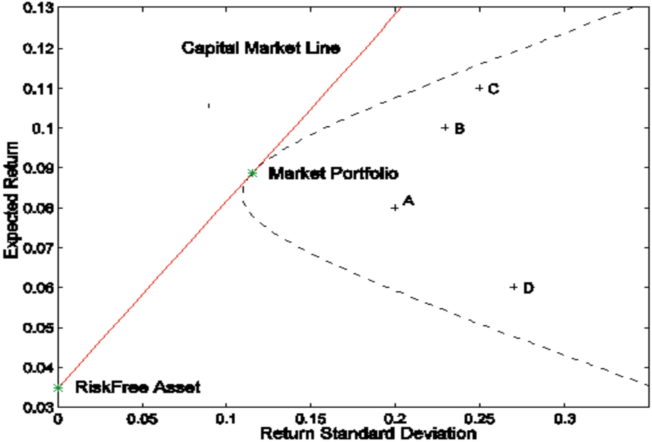


- Markowitz: everyone holds a linear combination of two portfolios
  - the risk-free security
  - the tangency portfolio
- If everyone sees the same efficient frontier and CAL, then **everyone has the same tangency portfolio**

CAPM

- What is the tangency portfolio?
  1. Markowitz: Investors *should* hold the tangency portfolio.
  2. Equilibrium Theory (market clearing):
     - The risk-free asset is in zero supply:
       Borrowing and Lending must cancel out.
     - Average investor *must* want to hold the market portfolio.
- CAPM: the tangency portfolio **must be the market portfolio**.
  - **Definition:** The 'market' or total wealth portfolio is a portfolio of **all** risky securities held in proportion to their market value. This must be the sum over all securities, i.e. stocks, bonds, real-estate, human capital, etc.
  - Here's where the assumption that all assets are tradeable comes in.

The Capital Market Line



- In equilibrium, every investor faces the same CAL.

What about Individual Assets?

- This CAL is called the *Capital Market Line* (CML). This line gives us the set of efficient or optimal risk-return combinations

  $$E(\tilde{r}_e) = r_f + \left(\frac{E(\tilde{r}_m) - r_f}{\sigma_m}\right)\sigma_e$$

  where $\tilde{r}_e$ is the return on any *efficient* portfolio (i.e. on the CML)

- Note that this says that all investors should only hold combinations of the market and the risk-free asset.
  - How does this relate to the increased popularity of index funds?

- However, the goal of the CAPM is to provide a theory for expected returns of *inefficient portfolios* (or individual assets) based on equilibrium arguments.

CAPM, a preview

- If investors want to hold the market portfolio, they should not profit by choosing any other portfolio
  - Investors only want to hold a security in their portfolio if it provides a reasonable amount of extra reward (expected return) in return for the risk (variance) it adds to the portfolio
  - Since no deviation is profitable, what each security adds to the risk of a portfolio must be exactly offset by what it adds in terms of expected return.
- Ratio of *marginal return* to *marginal variance* must be the same for all assets
  - What each asset adds in expected return is its *expected excess return*
  - What each adds in risk is proportional to its *covariance* with the portfolio we are holding (the market)
- This is the intuition for the standard form of the CAPM, which relates β (i.e. scaled covariance) to expected return.

- How does adding a small amount of security to the market portfolio affect its variance?

$$
\begin{aligned}
\sigma_m^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cov(\tilde{r}_i, \tilde{r}_j) \\
&= \sum_{i=1}^{N} w_i \left[\sum_{j=1}^{N} w_j cov(\tilde{r}_i, \tilde{r}_j)\right] \\
&= \sum_{i=1}^{N} w_i cov\left(\tilde{r}_i, \quad \underbrace{\left[\sum_{j=1}^{N} w_j \tilde{r}_j\right]}_{\text{return on the market}}\right)
\end{aligned}
$$

- The *marginal* increase in risk when you change the amount of a security in your portfolio is the **covariance with the portfolio return**.

CAPM, a derivation based on mean-variance

- Under our assumptions, all investors must hold the market portfolio
- Based on our notion of equilibrium, every investor must be content with their portfolio holdings; if this were not the case than the prices of the securities would have to change
  - This is just a supply and demand argument; if some investors want to buy IBM, and no one wants to sell, prices will have to increase
  - In equilibrium, everyone must be optimally invested
- No one can do anything to increase the Sharpe-ratio of their portfolio

Suppose you currently hold the market portfolio, but decide to borrow a small additional fraction δ_{GM} of your wealth at the risk-free rate and invest it in GM

1. The return in your new portfolio

$$\tilde{r}_c = \tilde{r}_m - \delta_{GM} \cdot r_f + \delta_{GM} \cdot \tilde{r}_{GM}$$

2. So the expected return and variance will be:

$$\begin{aligned} E(\tilde{r}_c) &= E(\tilde{r}_m) + \delta_{GM} \cdot (E(\tilde{r}_{GM}) - r_f) \\ \sigma_c^2 &= \sigma_m^2 + \delta_{GM}^2 \cdot \sigma_{GM}^2 + 2 \cdot \delta_{GM} \cdot cov(\tilde{r}_{GM}, \tilde{r}_m) \end{aligned}$$

3. The changes in each of these are:

$$\begin{aligned} \Delta E(\tilde{r}_c) &= \delta_{GM} \cdot E(\tilde{r}_{GM} - r_f) \\ \Delta \sigma_c^2 &= 2 \cdot \delta_{GM} \cdot cov(\tilde{r}_{GM}, \tilde{r}_m) \end{aligned}$$

We ignore the δ_{GM}^2 term in the variance equation because, if δ is small (say 0.001), δ^2 must be so small that we can ignore it (0.000001).

Now what if we invest δ more in GM, and invest just enough less in the IBM so that our portfolio variance stays the same.

1. The change in the variance is:

$$\Delta\sigma_c^2 = 2 \cdot (\delta_{GM} \cdot cov(\tilde{r}_{GM}, \tilde{r}_m) + \delta_{IBM} \cdot cov(\tilde{r}_{IBM}, \tilde{r}_m))$$

2. To make this zero, it must be the case that:

$$\delta_{IBM} = -\delta_{GM} \left(\frac{cov(\tilde{r}_{GM}, \tilde{r}_m)}{cov(\tilde{r}_{IBM}, \tilde{r}_m)} \right)$$

3. The change in the expected return of the portfolio will be:

$$\begin{aligned} \Delta E(\tilde{r}_c) &= \delta_{GM} \cdot E(\tilde{r}_{GM} - r_f) + \delta_{IBM} \cdot E(\tilde{r}_{IBM} - r_f) \\ &= \delta_{GM} \left[E(\tilde{r}_{GM} - r_f) - E(\tilde{r}_{IBM} - r_f) \left(\frac{cov(\tilde{r}_{GM}, \tilde{r}_m)}{cov(\tilde{r}_{IBM}, \tilde{r}_m)} \right) \right] \end{aligned}$$

CAPM, a derivation based on mean-variance

- However, we are holding the market portfolio, which is also the tangency portfolio. This portfolio has the highest Sharpe Ratio of *all* portfolios.

- Therefore, by definition, we **cannot** increase its expected return while keeping the variance constant.
  - For this to be true it must be that:
$$\frac{E(\tilde{r}_{GM})-r_f}{cov(\tilde{r}_{GM},\tilde{r}_m)} = \frac{E(\tilde{r}_{IBM})-r_f}{cov(\tilde{r}_{IBM},\tilde{r}_m)} = \lambda$$
  - λ is the ratio of the marginal benefit to the marginal cost.

- Note that this also holds for portfolios of assets as well.

- We can use the market portfolio in place of IBM:

$$\frac{E(\tilde{r}_{GM})-r_f}{cov(\tilde{r}_{GM},\tilde{r}_m)} = \frac{E(\tilde{r}_m - r_f)}{cov(\tilde{r}_m,\tilde{r}_m)} = \frac{E(\tilde{r}_m - r_f)}{\sigma_m^2} = \lambda$$

  which means that:

$$\begin{aligned} E(\tilde{r}_{GM})-r_f &= \frac{E(\tilde{r}_m - r_f)}{\sigma_m^2} cov(\tilde{r}_{GM},\tilde{r}_m) \\ &= E(\tilde{r}_m - r_f)\underbrace{\frac{cov(\tilde{r}_{GM},\tilde{r}_m)}{\sigma_m^2}}_{\beta_{GM}} \end{aligned}$$

- This characterizes the **Security Market Line(SML)**.

The equity premium

- What determines the compensation for bearing market risk, $E(\tilde{r}_m) - r_f$

- If the CAPM is true, a mean-variance investor will allocate an amount

$$w_m^* = \frac{E(R_m) - r_f}{A\sigma_m^2}$$

  to the market portfolio and the remainder to the risk-free rate

- Assume that all investors have the same risk aversion A. Since in equilibrium $w_m^* = 1$, it must be the case that

$$E(R_m) - r_f = A\sigma_m^2$$

- The compensation for market risk is increasing in risk aversion, and the amount of market risk. The same argument goes through if investors vary in risk aversion, replacing A above with a (wealth-weighted) average risk aversion coefficient for the economy

Summary

- By the definition of the tangent portfolio, investors should not be able to achieve a higher return/risk tradeoff (Sharpe Ratio) by combining the tangent portfolio with *any* other asset.
- This restriction implies a linear relationship between an asset's equilibrium return and its beta with the tangent portfolio:

$$E(\tilde{r}_i) - r_f = E(\tilde{r}_T - r_f) \times \beta_i$$

- The CAPM is the statement, that **in equilibrium**, the tangent portfolio is the market portfolio ($\tilde{r}_T = \tilde{r}_M$).
- One way to interpret this equation is as saying that the reward ($E(\tilde{r}_i) - r_f$) must equal the amount of risk that is priced (β), times its price ($E\tilde{r}_M - r_f$)

Regression

- How to estimate β?
- Regression of y on a constant and x,

$$y = a + bx + \varepsilon$$

  - a is the estimated constant
  - b is the estimated coefficient on x
  - ε is the residual
  - a and b are constants, ε differs across observations

Regression

$$y = a + bx + \varepsilon$$

- The calculation of a and b is done so that:
  - ε has zero mean
  - ε is uncorrelated with x
  - The variance of ε is as small as possible

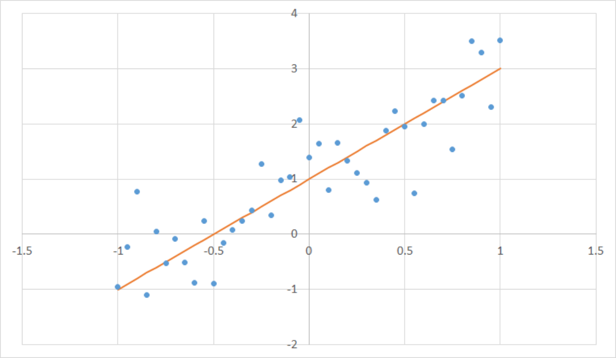
Regression

$$y = a + bx + \varepsilon$$

- Interpretation:
  - b is how much y moves for a unit change in x
  - a is the part of the *average* of y that is not explained by x
  - ε represents *fluctuations* in y not explained by x (uncorrelated with x)
- The R^2 gives you an overall measure of fit:

$$R^2 = 1 - \frac{var(\varepsilon)}{var(y)}$$

Regression



What are a and b? What is ε? What is the R^2?

The market model

- To estimate the CAPM betas, you can run the following regression:

$$R_{i,t} = a_i + \beta_i R_{m,t} + \varepsilon_{i,t}$$

  what should a_i, β_i and R^2 be according to the CAPM?

- Alternatively, you can subtract the risk-free rate from both sides

$$R_{i,t} - r_f = a_i + \beta_i (R_{m,t} - r_f) + \varepsilon_{i,t}$$

  what should a_i, β_i and R^2 be according to the CAPM?

- Both methods should give you approximately the same β

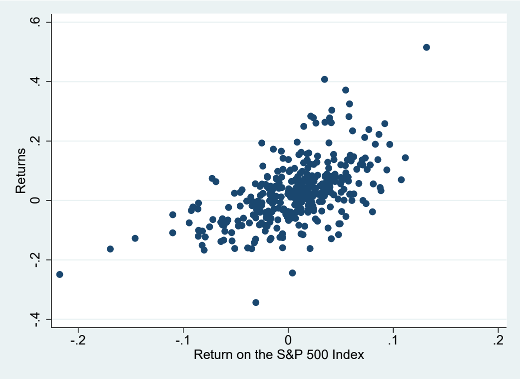
- The easiest way to run a regression is in Stata:

```
use "StockRets.dta", clear
tab ticker if permno==10107

     Ticker |
     Symbol |      Freq.     Percent        Cum.
------------+-----------------------------------
       MSFT |        357      100.00      100.00
------------+-----------------------------------
      Total |        357      100.00


reg ret sprtrn if permno==10107

      Source |       SS           df       MS      Number of obs   =       357
-------------+----------------------------------   F(1, 355)       =    166.51
       Model |  1.19477557         1  1.19477557   Prob > F        =    0.0000
    Residual |  2.54724895       355  .007175349   R-squared       =    0.3193
-------------+----------------------------------   Adj R-squared   =    0.3174
       Total |  3.74202451       356  .010511305   Root MSE        =    .08471

------------------------------------------------------------------------------
         ret |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      sprtrn |   1.320745   .1023522    12.90   0.000     1.119452    1.522038
       _cons |   .0147608   .0045403     3.25   0.001     .0058317      .02369
------------------------------------------------------------------------------
```

Running Regressions

- Stata also helps you visualize the data

```
scatter ret sprtrn if permno==10107
graph export "MSFT.pdf", as(pdf) replace
```



- Compare returns on Microsoft (10107) vs Tesla (93436):

```
tabstat ret if permno==10107| permno==93436, stat(mean sd skew k) by(permno)

Summary for variables: ret
     by categories of: permno (PERMNO)

  permno |      mean        sd  skewness  kurtosis
---------+----------------------------------------
   10107 |    .02402  .1025247  .6806583  5.414304
   93436 |  .0489231   .179453  1.676774   7.58227
---------+----------------------------------------
   Total |  .0279056  .1179383  1.409764  9.454961
--------------------------------------------------
```

- Tesla seems riskier than Microsoft

But Microsoft has a much higher beta!

```
. bysort permno: reg ret sprtrn

-----------------------------------------------------------------------------------------------------------------------
-> permno = 10107

      Source |       SS           df       MS      Number of obs   =       357
-------------+----------------------------------   F(1, 355)       =    166.51
       Model |  1.19477557         1  1.19477557   Prob > F        =    0.0000
    Residual |  2.54724895       355  .007175349   R-squared       =    0.3193
-------------+----------------------------------   Adj R-squared   =    0.3174
       Total |  3.74202451       356  .010511305   Root MSE        =    .08471

------------------------------------------------------------------------------
         ret |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      sprtrn |   1.320745   .1023522    12.90   0.000     1.119452    1.522038
       _cons |   .0147608   .0045403     3.25   0.001     .0058317      .02369
------------------------------------------------------------------------------

-----------------------------------------------------------------------------------------------------------------------
-> permno = 93436

      Source |       SS           df       MS      Number of obs   =        66
-------------+----------------------------------   F(1, 64)        =      0.48
       Model |  .015646492         1  .015646492   Prob > F        =    0.4900
    Residual |  2.07757253        64  .032462071   R-squared       =    0.0075
-------------+----------------------------------   Adj R-squared   =   -0.0080
       Total |  2.09321902        65   .03220337   Root MSE        =    .18017

------------------------------------------------------------------------------
         ret |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      sprtrn |   .4315443   .6215915     0.69   0.490    -.8102274    1.673316
       _cons |   .0441514   .0232183     1.90   0.062    -.0022325    .0905353
------------------------------------------------------------------------------
```

MV Optimization and regressions

- Suppose you've found an M–V optimal portfolio with return r_p

- For any other return, r_i, you can run the following regression

$$r_{i,t} - r_{f,t} = \alpha_i^{port} + \beta_i^{port}\left(r_{p,t} - r_{f,t}\right) + \varepsilon_{i,t}^{port}$$

  - You regress the asset's excess return on the portfolio's excess return

- **If r_p is mean-variance optimal, then $\alpha_i^{port} = 0$ for all assets!**

MV Optimization and regressions

- **If r_p is mean-variance optimal, then $\alpha_i^{port} = 0$ for all assets**

$$r_{i,t} - r_{f,t} = \alpha_i^{port} + \beta_i^{port}\left(r_{p,t} - r_{f,t}\right) + \varepsilon_{i,t}^{port}$$

- For *you*, the relevant measure of risk is β_i^{port}
  - Higher β_i^{port} implies you demand a higher return on asset i

- If $\alpha_i^{port} > 0$, then r_i has a relatively high return compared to the risk it adds to your portfolio

$$r_{i,t} - r_{f,t} = \alpha_i^{port} + \beta_i^{port}\left(r_{p,t} - r_{f,t}\right) + \varepsilon_{i,t}^{port}$$

- Any time somebody shows you a new asset to invest in, regress its returns on those of **your current portfolio**

- If $\alpha_i^{port} > 0$, then it adds value for you

- **This is the only regression that really matters** – positive α in some other regression is nice, but not directly relevant...

Example

- What happens if you have a 60/40 stock/bond portfolio and consider adding private equity?
- Call r_{6040} the return on a portfolio that is 60 percent stocks and 40 percent bonds
- r_{PE} is the return on private equity

	$E\left[r-r_f\right]$	$\text{StdDev}\left(r-r_f\right)$	Sharpe ratio
60/40	0.046	0.11	0.42
PE	0.085	0.22	0.39

$\text{Correlation}(r_{6040}, r_{PE}) = 0.40$

- Which looks more attractive? How much money would you allocate to each?
  - If you had to invest in only one, which would you prefer?

Adding private equity to 60/40 portfolio

- Regression results (using Harvard endowment forecasts)

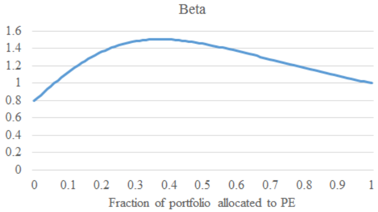
$$
\begin{aligned}
r_{PE,t} - r_{f,t} &= 0.048 + 0.80\left(r_{6040,t} - r_{f,t}\right) + \varepsilon_{PE,t} \\
R^2 &= 0.16
\end{aligned}
$$

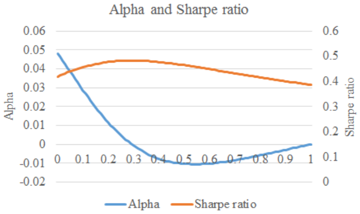
- What do those numbers mean?

What is going on?

- Suppose you hold 60/40 stocks/bonds, then private equity earns a large alpha
- Optimal response: put money into private equity
  - But then its beta (and R^2) against your portfolio rises
  - So the alpha falls
  - Add PE to the portfolio until the alpha is zero

Adding private equity to 60/40 portfolio

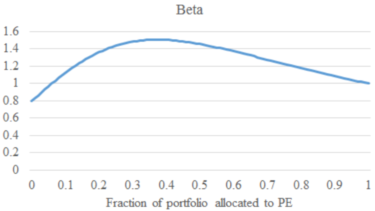


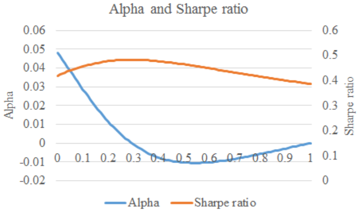


- Consider a portfolio that allocates $x\%$ to PE and $(1-x)\%$ to the 60/40 stock/bond portfolio (x is on the x-axis)

Adding private equity to 60/40 portfolio

$$r_{PE,t} - r_{f,t} = \alpha_a + \beta_a \left(a r_{PE,t} + (1-a) r_{6040,t} - r_{f,t} \right) + \varepsilon_{a,t}$$

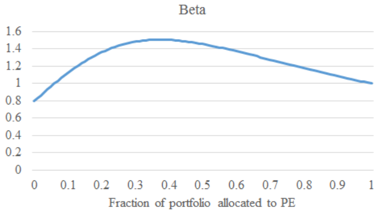


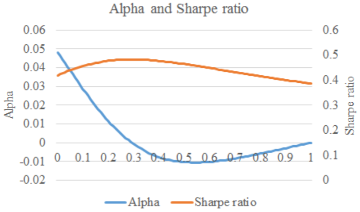


- As a rises, β_a – portfolio beta of PE – rises
- α_a falls ($a = 1$ means you're regressing PE on PE, so $\alpha_1 = 0$ always)

Adding private equity to 60/40 portfolio

$$r_{PE,t} - r_{f,t} = \alpha_a + \beta_a \left(a r_{PE,t} + (1-a) r_{6040,t} - r_{f,t} \right) + \varepsilon_{a,t}$$





- Sharpe ratio of portfolio is maximized where $\alpha_a = 0$ (around $a = 0.3$)
- That's the point where PE is no better or worse than the portfolio
  - Optimal portfolio has both PE and stocks/bonds in it

Mean–variance optimization and regressions

- Everything we have talked about so far can be thought of in terms of regressions
- New asset for portfolio: should you be long or short, by how much?
  - Look at its alpha: big positive alpha means you want to buy a lot
  - Buy it until its alpha is zero (short if alpha <0)
- Should you give money to a manager?
  - Does their portfolio earn an alpha compared to your current portfolio?
- We will run many regressions

Summary

- Perhaps because it is easy to implement, CAPM is the most widely used model for computing equilibrium, or risk-adjusted required rates of return.
- If the CAPM is true, then *all* securities should lie in the SML.

$$E(\tilde{r}_i) = r_f + \beta_i \cdot [E(\tilde{r}_m) - r_f]$$

  - The relation of expected return and β_i is linear
  - *Only* β_i explains differences in returns among securities.
  - $E(R)$ of an asset with a $\beta = 0$ is r_f.
  - $E(R)$ of an asset with a $\beta = 1$ is the expected return on the market.
- How does it work in reality?

CAPM and the data

- The first difficulty we run into is that the wealth portfolio is not really observable. At best, we observe the return to *financial* wealth, and in most cases, wealth in the stock market.

- A reasonable place to start is to assume that the 'market' portfolio is well approximated by a broad stock market index.

- How can we test the CAPM? 2 Approaches:

1. Test $\alpha_i = 0$ in

$$R_{i,t} - r_f = \alpha_i + \beta_i (R_{m,t} - r_f) + \epsilon_{i,t}$$

2. Two steps:

   - For each security, estimate $E(R_{i,t} - r_f)$ and β_i

   - test $\gamma_0 = 0$, $\gamma_1 > 0$ **and** $u_i = 0$ in

$$E(R_{i,t} - r_f) = \gamma_0 + \gamma_1 \beta_i + u_i$$

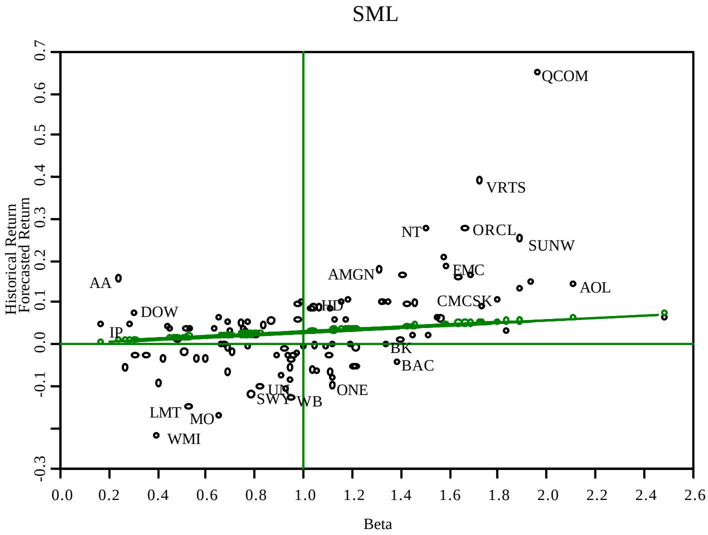
CAPM and the data: A first test

1. Collect the data.
   - We will use monthly data on 100 largest stocks
2. Estimate β_i and $E(R_{i,t} - r_f)$.
   - use a *first-pass* regression to estimate β_i
   - use historical average for $E(R_{i,t} - r_f)$
3. Set up a *second-pass* regression in Excel.
   - The dependent variable: $y_i = E(R_{i,t} - r_f)$
   - The independent variable: $x_i = \beta_i$
4. Results:

	Estimate	Standard Error	t-stat
γ_0	6.01%	1.8%	3.5
γ_1	0.17%	1.7%	0.1
R^2	2%		

What do these numbers mean?

Average returns vs market betas



CAPM and the data: A better test

- These tests suffer from a measurement error problem:
  - Market betas are measured with error
- We can get around the measurement error problem by looking at diversified portfolios.
- We can sort firms into portfolios based on characteristics that we think should explain risk premia.
- Let's try this with market beta:
  1. For every year t, use past 5 years of data to estimate market beta.
  2. At the beginning of the year sort firms into 10 portfolios based on their estimated beta.
  3. Track the performance of these portfolios over the next year.
  4. At year $t+1$ repeat.
- This test was done by Black, Jensen and Scholes.

BJS Portfolio selection technique

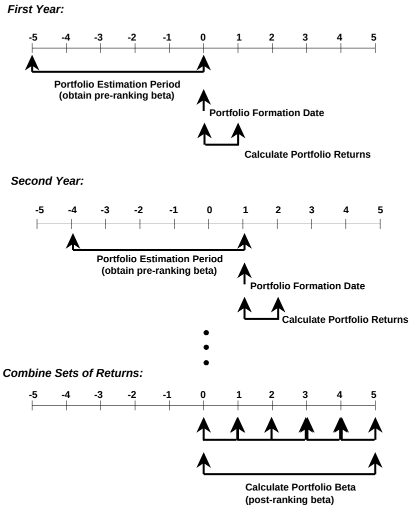
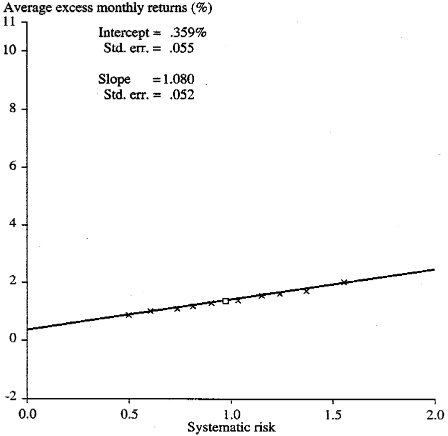


Table 2
Summary of Statistics for Time Series Tests, Entire Period (January, 1931-December, 1965)
(Sample Size for Each Regression =420)

	Portfolio Number										
Item*	1	2	3	4	5	6	7	8	9	10	$\overline{R}_M$
$\hat{\beta}$	1.5614	1.3838	1.2483	1.1625	1.0572	0.9229	0.8531	0.7534	0.6291	0.4992	1.0000
$\hat{\alpha} \bullet 10^2$	-0.0829	-0.1938	-0.0649	-0.0167	-0.0543	0.0593	0.0462	0.0812	0.1968	0.2012	
$t(\hat{\alpha})$	-0.4274	-1.9935	-0.7597	-0.2468	-0.8869	0.7878	0.7050	1.1837	2.3126	1.8684	
$r(\tilde{R}, \tilde{R}_M)$	0.9625	0.9875	0.9882	0.9914	0.9915	0.9833	0.9851	0.9793	0.9560	0.8981	
$r(\tilde{e}_t, \tilde{e}_{t-1})$	0.0549	-0.0638	0.0366	0.0073	-0.0708	-0.1248	0.1294	0.1041	0.0444	0.0992	
$\sigma(\tilde{e})$	0.0393	0.0197	0.0173	0.0137	0.0124	0.0152	0.0133	0.0139	0.0172	0.0218	
$\overline{R}$	0.0213	0.0177	0.0171	0.0163	0.0145	0.0137	0.0126	0.0115	0.0109	0.0091	0.0142
σ	0.1445	0.1248	0.1126	0.1045	0.0950	0.0836	0.0772	0.0685	0.0586	0.0495	0.0891

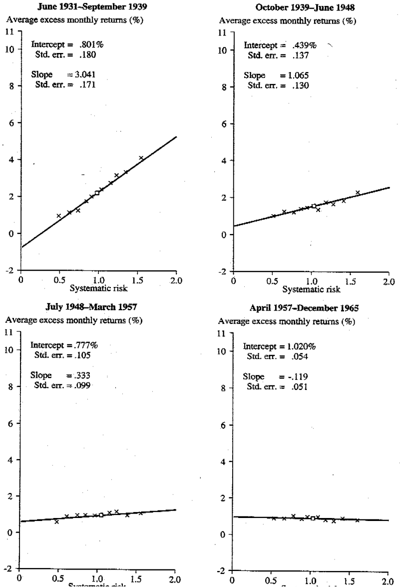
* $\overline{R}_M$ = average monthly excess returns, σ = standard deviation of the monthly excess returns, r = correlation coefficient.

BJS: Results



Average excess monthly returns (%)
Intercept = .359%
Std. err. = .055
Slope = 1.080
Std. err. = .052
11
10
8
6
4
2
0
-2
0.0
0.5
1.0
1.5
2.0
Systematic risk

BJS: Results


June 1931–September 1939
Average excess monthly returns (%)
Intercept = .801%
Std. err. = .180
Slope = 3.041
Std. err. = .171
Systematic risk
October 1939–June 1948
Average excess monthly returns (%)
Intercept = .439%
Std. err. = .137
Slope = 1.065
Std. err. = .130
Systematic risk
July 1948–March 1957
Average excess monthly returns (%)
Intercept = .777%
Std. err. = .105
Slope = .333
Std. err. = .099
April 1957–December 1965
Average excess monthly returns (%)
Intercept = 1.020%
Std. err. = .054
Slope = -.119
Std. err. = .051

CAPM and the data (1963-2016)

```
use "BetaSortedPortfolios.dta", clear

forval i = 1/10 {
qui: gen eret`i'=dec`i'-rf
qui: eststo r`i': reg  eret`i' mktrf
}
gen rHmL=dec10-dec1
qui: eststo r11: reg  rHmL mktrf

tabstat eret* rHmL, stat(mean sd) format(%9.2f)
```

stats	eret1	eret2	eret3	eret4	eret5	eret6	eret7	eret8	eret9	eret10	rHmL
mean	0.54	0.49	0.56	0.64	0.52	0.60	0.49	0.63	0.60	0.59	0.05
sd	3.46	3.80	4.06	4.59	4.75	5.07	5.43	6.01	6.65	7.89	6.55

```
esttab r1 r2 r3 r4 r5 r6 r7 r8 r9 r10 r11, r2 compress b(2) t(2) nostar
```

	(1) eret1	(2) eret2	(3) eret3	(4) eret4	(5) eret5	(6) eret6	(7) eret7	(8) eret8	(9) eret9	(10) eret10	(11) rHmL
mktrf	0.62	0.74	0.83	0.97	1.02	1.08	1.16	1.28	1.39	1.61	0.99
	(32.40)	(41.96)	(55.01)	(68.51)	(74.80)	(74.63)	(72.47)	(69.28)	(61.26)	(52.96)	(22.91)
_cons	0.23	0.13	0.14	0.15	0.01	0.06	-0.09	-0.01	-0.10	-0.22	-0.45
	(2.71)	(1.60)	(2.10)	(2.38)	(0.17)	(0.92)	(-1.27)	(-0.14)	(-0.95)	(-1.61)	(-2.32)
N	640	640	640	640	640	640	640	640	640	640	640
R-sq	0.622	0.734	0.826	0.880	0.898	0.897	0.892	0.883	0.855	0.815	0.451

t statistics in parentheses

CAPM fails ⇔ market portfolio is not MVE efficient

```
gen mktneutralp=dec1 - dec10  +  mktrf

corr mktneutralp  mktrf
(obs=640)

             | mktneu~p    mktrf
-------------+------------------
 mktneutralp |   1.0000
       mktrf |   0.0061   1.0000

. gen port=1/2*mktneutralp +1/2*mktrf

. tabstat mktneutralp mktrf port, stat(mean sd) format(%9.2f)

   stats |  mktneu~p     mktrf      port
---------+------------------------------
    mean |      0.45      0.50      0.48
      sd |      4.85      4.43      3.29
----------------------------------------
```

Equilibrium risk and return

- A more general representation of the risk-return tradeoff for each security i is the assumption that there exists a random variable m such that

$$E[m R_i] = 1, \quad \forall i$$

(here, R is a gross return, i.e. equal to $1 + r$)

- The variable m is termed the **stochastic discount factor** (SDF) and summarizes all the necessary risk-adjustment
  - This is a much more general statement that requires minimal assumptions (basically, no arbitrage)
  - The above equation can be viewed as stating that all securities have the same **risk-adjusted** returns.
- To see why this is the case, suppose that there is a finite number of states of the world tomorrow, $s = 1 \ldots S$, each occurring with probability p_s. The above can then be written as

$$\sum_{s=1}^{S} p_s m_s R_{i,s} = 1, \quad \forall i$$

You can interpret m as a risk-adjustment to the real probabilities p

Equilibrium risk and return

- In derivatives pricing, people often term the combined term $q = pm$ (normalized to sum to 1) as 'risk-neutral' probabilities. T see the connection, we can write the equation above as

$$\sum_{s=1}^{S} q_s \frac{R_{i,s}}{R_f} = 1, \quad q_s \equiv \frac{p_s m_s}{\sum_{s=1}^{S} p_s m_s}$$

- A risk-averse investor will *overweigh* 'bad' states of the world, so m will be higher in bad economic times.
- You can think of a risk averse investor as being essentially a pessimist: she behaves **as if** disasters are more likely than they actually are, and as if good things happen less frequently than they do.
  - Again, these are **as if** probabilities. Investors behave as if this is the case, even though they may know that the actual probabilities are different.
- Why bother? Under certain conditions, one can recover the market's assessment of q from the prices of financial securities

Risk-neutral probabilities

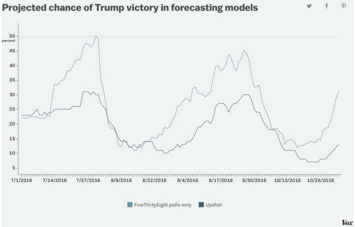
- Consider the case of a binary bet: you get \$1 if event X occurs, and 0 otherwise.
- The *market* price of that bet reveals something about the market's assessment that event X will occur, but it is conflated with the market's risk aversion.
  - Hence, these are **risk-adjusted** or risk-neutral probabilities.
- Suppose that the price of the bet is p_x. Then, given the previous expression (assume $R_f = 1$), we have that

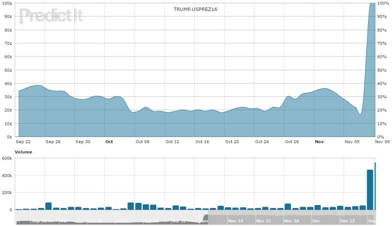
$$1 = \sum_{s=1}^{S} q_s \, R_{i,s} = q_x \frac{1}{p_x} + (1 - q_x) 0$$

$$\rightarrow q_x = p_x$$

- Since these are risk-neutral probabilities, they can be higher or lower than the actual probabilities, depending on whether the market views event X as good or bad.
- An example of this is the price of a binary (or digital) option: option pays you \$1 if the price of the underlying security moves more than x%.

Risk-neutral probabilities: 2016 Election





- If we had an assessment of what the market *actually* believed the chances of a Trump presidency was, we could back out whether it perceived it as good ($m_x < 1$) or bad ($m_x > 1$) news.

Risk-neutral probabilities can be extracted from option prices

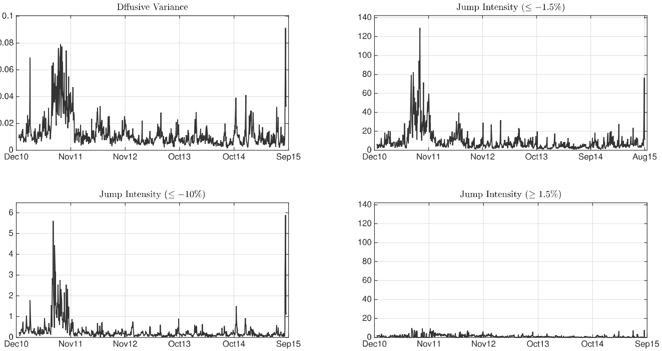


Figure 12: **Return Characteristics Extracted from the semi-nonparametric model (1**
We plot the day-by-day estimates for V_t (top left panel), $\int_{x<-0.015} \nu_t(dx)$ (top right panel), $\int_{x<-0.10} \nu_t($ (bottom left panel), and $\int_{x>0.010} \nu_t(dx)$ (bottom right panel) expressed in annualized terms, based on model (12) with $\alpha = 0.5$. The sample period is January 2011 - August 2015.

- Under certain assumptions, one can extract the market's risk-neutral probabilities from option contracts. Figure is from Andersen, Fusari and Todorov (2015). Figures 2–4 plot the implied risk-neutral probabilities (annualized) in weekly index options that the market index moves by x% over the next week.

Risk-neutral vs actual probabilities

- Yet another to understand the connection between m , p and q is to think of insurance contracts.
- Consider buying car insurance. You can view this as an investment decision that pays off if you have an automobile accident.
- Let the annual cost of car insurance be \$2,000; further, assume that, conditional on an accident occuring, the average (monetary) cost would be \$5,000 (there is clearly a lot of variation in outcomes)
- Under these assumptions, the 'return' to buying car insurance needs to satisfy

$$1 = q_A \frac{5}{2} + (1 - q_A)\, 0 \rightarrow q_A = 0.4$$

- Does this mean that the actual probability of having an accident is 40% per year? Not really. Again, this is a *risk-adjusted* probability, i.e. the product of the actual probability p_A times the risk-adjustment m_A
- Since having a car accident is a pretty bad outcome, we would expect $m_A >> 1$ and therefore $q_A >> p_A$.

Equilibrium risk and return

- To see the connection with what we did before, we can rewrite

$$
\begin{aligned}
&E[m\,R_i] = 1 \\
&\Rightarrow E[R_i] = \frac{1}{E[m]} - cov\left(\frac{m}{E[m]}, R_i\right) \\
&\Rightarrow E[R_i] = R_f \underbrace{-cov\left(\frac{m}{E[m]}, R_i\right)}_{\text{Risk adjustment}}
\end{aligned}
$$

  The last line follows from the fact that $E[m\,R_f] = 1 \rightarrow R_f = 1/E[m]$

- Recall that now m is high when times are 'bad'
- To get the CAPM as a special case, use $m = a - b\,R_m$

$$
\begin{aligned}
E[R_i] &= R_f + \lambda\, cov\,(R_m, R_i) \\
E[R_m] &= R_f + \lambda\, var\,(R_m) \\
\rightarrow E[R_i] &= R_f + \big(E[R_m] - R_f\big)\,\frac{cov\,(R_m, R_i)}{var(R_m)}
\end{aligned}
$$

Equilibrium risk and return

- Do we have complete freedom how to specify m?
  - Not exactly. Remember, it is the **same** m that prices **all** assets.
- Different models will have different specifications for m
- Equilibrium models will typically tie down m to some measure of economic fundamentals — wealth or consumption.
- Another equilibrium model that starts from first economic principles states that the stochastic discount factor (SDF) should be related to consumption growth g_c

  $$m = a - b g_c$$

  This model is based on the idea that investors' optimize their consumption-savings decisions so that consumption closely tracks wealth.
- This model, termed the *Consumption CAPM* has the advantage that consumption is much more easily measurable than wealth (think human capital).

Equilibrium risk and return

- To see the connection between m and consumption, let's bring back the utility function.
- Consider an investor who chooses how much to invest in an asset with (possibly risk) return R
- Can formulate her problem as

$$\max_x u(c_0) + \rho E[u(c_1)]$$

subject to

$$c_0 = e_0 - x$$
$$c_1 = e_1 + x R$$

- $\rho < 1$ indicates impatience; e_t is the investor's other income at time t
- The first-order condition is

$$u'(c_0) = \rho E\left[u'(c_1)R\right]$$
$$1 = E\left[\frac{\rho u'(c_1)}{u'(c_0)} R\right]$$

- That is,

$$m = \frac{\rho u'(c_1)}{u'(c_0)}$$

Equilibrium risk and return

- That is, m depends on the marginal utility tomorrow versus marginal utility today.
- Suppose utility is CRRA, so $u'(c) = c^{-\gamma}$.
- We can then approximate around c_0 to obtain

$$m = \frac{\rho u'(c_1)}{u'(c_0)} \approx \rho + \frac{c_0\, u''(c_0)}{u'(c_0)} \frac{c_1 - c_0}{c_0} = \rho - \gamma g_c$$

- The intuition is pretty general:
  - Good times: states of the world where your consumption growth is high (relative to today)
  - Bad times: states of the world where your consumption growth is low (relative to today)
- Connection to risk-neutral probabilities:
  - If you have no reason to believe that state s is related to your consumption growth — say the outcome of a sports bet — then risk-neutral and actual probabilities should coincide, i.e. $p_s = q_s$.

The CCAPM

- Perhaps the main problem with the CAPM is that we do not really observe the market portfolio.
- From that perspective, perhaps the Consumption CAPM (CCAPM)—developed by Breeden and Litzeberger (1978)—might be more useful.
- Recall, that we can view the CCAPM as a special case

$$m = a - b g_c$$

This model is based on the idea that investors' optimize their consumption-savings decisions so that consumption closely tracks wealth.

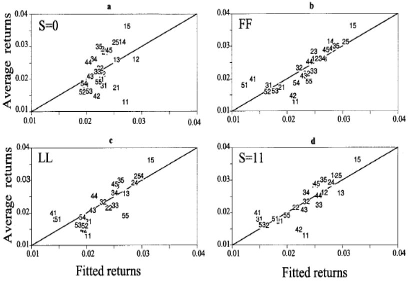
- We can write the above in expected return-covariance (or beta) form

$$E[R_i] - R_f = \lambda \, cov\,(g_c, R_i)$$

- Testing the equation above has been the focus of **much** of academic research over the last 30 years.

Parker and Julliard compare the empirical performance of four models: (1) CCAPM with only contemporaneous consumption risk, (2) the Fama-French 3-factor model, (3) Lettau and Ludvigson's conditional test of the CCAPM using their consumption/wealth ratio, cay, as a conditioning variable, and (4) ultimate consumption risk over 11 quarters. The results are quite apparent in the graphs below:

Figure 22



Source: Parker and Julliard (2005).

- In it's most basic form, the CCAPM does not work so well (panel a); it works much better if consumption growth is measured over several quarters (panel d)

The CCAPM

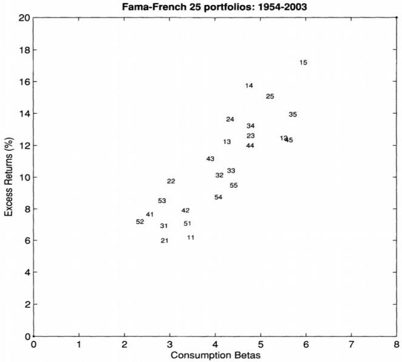
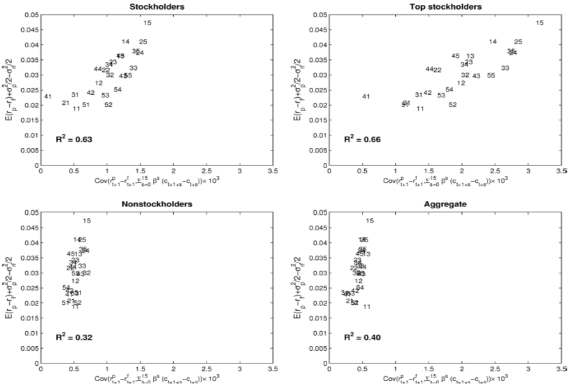


Figure 1. Annual excess returns and consumption betas. This figure plots the average annual excess returns on the 25 Fama–French portfolios and their consumption betas. Each two-digit number represents one portfolio. The first digit refers to the size quintile (1 smallest, 5 largest), and the second digit refers to the book-to-market quintile (1 lowest, 5 highest).

Source: Jagannathan and Wang (2007) , Figure 1.

- Researchers have also found that the model works better if consumption is measured over specific intervals (Q4-Q4). The idea is that households plan their future finances around the end of the year.

Panel A: Mean returns versus consumption covariances



Source: Malloy, Moskowitz and Vissing-Jorgensen (2009), Figure 1.

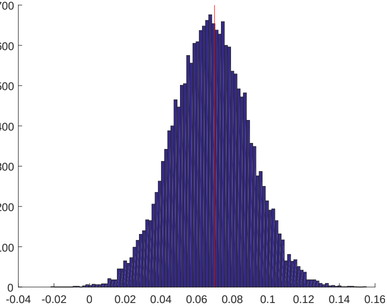
- Last, perhaps it does not make sense to use the average consumption in the economy, since most households do not participate in the stock market. Indeed, the model works much better if one focuses on the consumption growth of stockholders.

The CCAPM

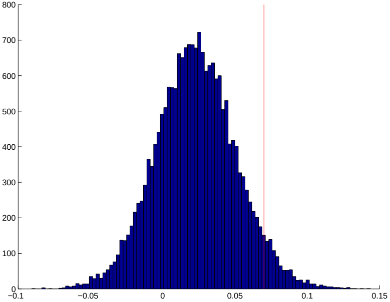
- How credible are these findings?
- One problem that had made people rather skeptical of these results is that the correlation between stock returns and consumption growth is quite low ($\approx 20\%$).
- This number is for diversified portfolios; for comparison, the correlation of these portfolios with the market proxy (SP&500) is much higher (around 80%).
- Why is this important? Because the lower the correlation with the proposed m, the higher the measurement error.
- Is this quantitatively a problem?

Testing the (C)CAPM — A simulation exercise

```
F=25; T=50;
EqPrem=0.07; Rf=0.01; sM=0.15; sE=0.75;

for isim=1:20000
    beta=normrnd(1,0.5,1,F);
    mu=Rf + beta*EqPrem;

    Rm=normrnd(EqPrem,sM,T,1);
    Ri=ones(T,1)*mu +  (Rm-EqPrem)*beta + normrnd(0,sE,T,F);

    X=[ones(T,1) Rm];    betaHAT=inv(X'*X)*X'*Ri;

    muHAT=mean(Ri-Rf,1);

    [b]=regress(muHAT',[ones(F,1) betaHAT(2,:)']);

    bsim(:,isim)=b;

end
```



Average correlation is 80%



Average correlation is 20%

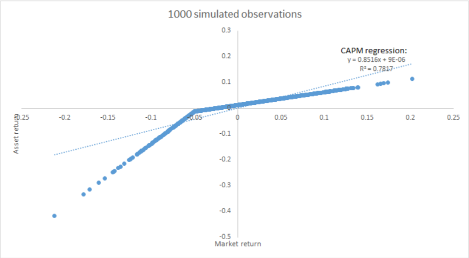
- Here, I plot the distribution of the estimated slope coefficient (mean returns versus betas) across simulations. The red line shows the true value (0.07).
- You can see that as the average correlation falls, both the mean but also the dispersion of the estimates increase
- There is no easy fix here, just a warning to take these results with a grain of salt.

Beware of non-linear returns

- The (C)CAPM is essentially a linear model
- Hence, it may perform poorly if asset returns are non-linear
- Option returns are an obvious example, but many stocks may have option-like returns
- Example: companies in distress, growth firms, . . .

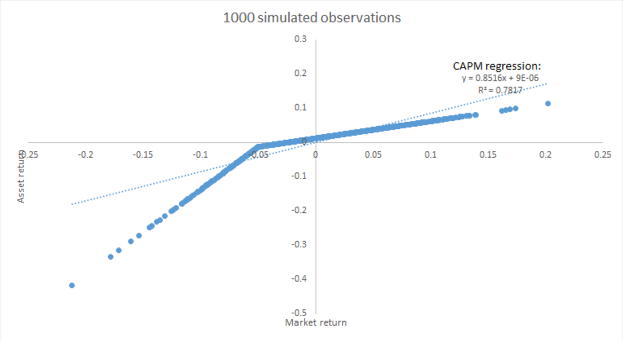
Beware of non-linear returns – Example

Figure: Hypothetical response of a stock's return to the market vs a linear fit



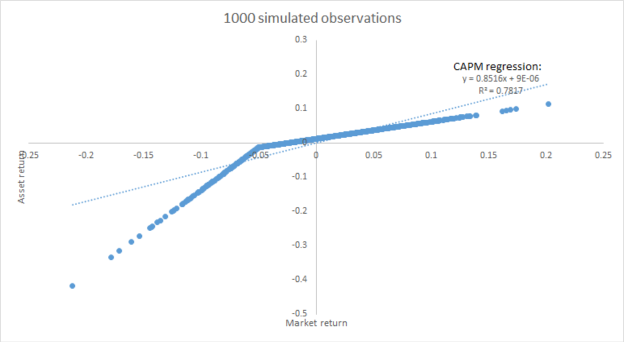
- If stocks crash periodically, measuring β can be hard
  - Especially if you're not sure you've seen a crash
- This is an option-like return
  - Hedge fund returns may look like this

Beware of non-linear returns – Example



- When the market's return is above -5%, slope $= 0.5$
- For returns below -5%, slope $= 2.5$
- If sample only includes $r_t > -5\%$, then your β estimate will be wrong

Beware of non-linear returns – Example

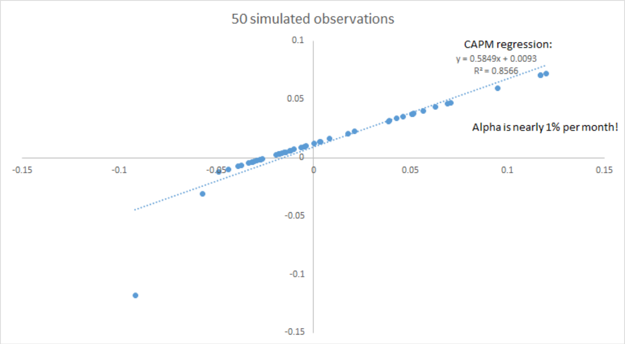


- Overall, β=0.85. If $E\left[r_m - r_f\right] = 0.07/12$,

$$E\left[r_i - r_f\right] = \beta E\left[r_m - r_f\right] = 0.85 \times 7\%/12 = 0.5\%$$

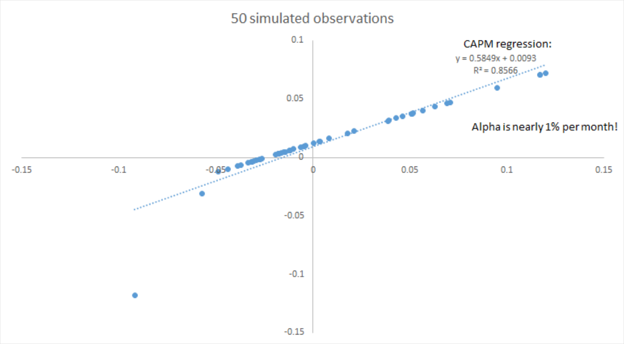
- So we should see an average return of 0.5% per month

Beware of non-linear returns – Example



- Suppose we just see 50 months of data:
- We measure $\beta = 0.59$
- Estimated α is huge – nearly 1 percent per month! Why?

Beware of non-linear returns – Example



- Measuring β too low means we think α is high
- They said they were selling you alpha, but really you got beta

Beware of non-linear returns – Example

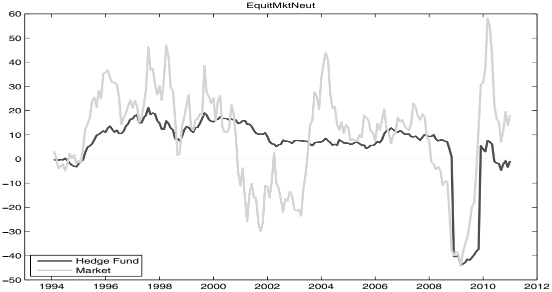


Figure 21. Hedge Fund Returns. One-year excess returns of the "equity market neutral" hedge fund index and the CRSP value-weighted portfolio. Data source: hedgeindex.com and CRSP.

Source: Cochrane (2015)

- Market-neutral hedge funds not always market neutral

Summary

- These equilibrium models have **some** empirical support in the data. In their basic form, neither do so well, but several extensions work better.
- Yet, perhaps because of its simplicity, most industry practitioners use the basic version of the CAPM:
  - In a survey of U.S. Chief Financial Officers, Graham and Harvey (2001) find that 73.5% of respondents calculate the cost of equity capital with the capital asset pricing model (CAPM).
- This is mostly due to the ease of implementation. But not just that. Many view the (modest) success of consumption-based models as somewhat suspicious: at the end of the day, consumption is not that highly correlated with stock returns.
- Over the next few lectures we will see other alternatives that are more successful, but have less robust economic foundations.