Portfolio Choice with Illiquid Assets

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A Motivating Example: Harvard in 2009

- Harvard's endowment is large ($43.0B net assets in June 2008) and is used to fund 34% of university operations. Spending of out of the endowment is smooth and has stayed around 4.5%.
- Investment is in a leveraged, diversified portfolio with many "alternative investments":
  - $\hookrightarrow$ Real Assets (e.g. Timber) make up 29% of assets. Hedge Funds and Private Equity are an additional 32%.
  - $\hookrightarrow$ Foreign and Domestic Equity make up 19% of assets.
  - $\hookrightarrow$ Derivatives, fixed income, and emerging markets are 30%.
  - $\hookrightarrow$ Cash position is -10% of assets.
- From June 2008 through June 2009, the endowment lost 27% of its value.
  - $\hookrightarrow$ For comparison, the S&P 500 was down 30%.

A Motivating Example: Harvard in 2009, Continued

Harvard could not reasonably get out of its illiquid positions, even in marquee partnership names.

> *Money Manager: "Hey, look, I'll buy it back from you. I'll buy my interest back."*
>
> *HMC President: "Great."*
>
> *Money Manager: "Here, I think its worth, you know, today the value is a dollar, so I'll pay you 50 cents."*
>
> *HMC President: "Then why would I sell it?"*
>
> *Money Manager: "Well, why are you? I don't know. You're the one who wants to sell, not me. If you guys want to sell, I'm happy to rip your lungs out. If you are desperate, I'm a buyer."*
>
> *– Reported in "Rich Harvard, Poor Harvard", Nina Munk, Vanity Fair.*

Harvard did not sell its illiquid positions. Instead it borrowed money and cut programs.

A Motivating Example: Harvard in 2009, Continued

What happened?

- Harvard's theory: Endowments are long-term investors and so they can better absorb liquidity shocks.
- In reality, Harvard has immediate institutional cash flow obligations (e.g. salaries, maintenance).
- Faced with the choice of selling endowment assets at a large loss or cutting university funding, Harvard chose to reduce its operating budget by 20%.

Harvard's problem was the result of the *interaction* between the need for smooth cash flows and risk in trading conditions.

Liquidity Literature

Broad categories of theoretical literature on non-tradability:

1. Market Closures
   - $\hookrightarrow$ Investors cannot trade for known, fixed durations (e.g. blackout periods).
   - $\hookrightarrow$ Examples include Dai, Li, Liu, and Wang (2010) and Longstaff (2009).

2. Partially Marketable Securities
   - $\hookrightarrow$ Investors can trade at any time at posted prices, but only in limited quantities or with transactions costs.
   - $\hookrightarrow$ Examples include Longstaff (2001) and Constantinides (1986).

3. Search Models
   - $\hookrightarrow$ Aim to understand market-makers. Constructed to avoid wealth effects, avoid liquid/illiquid asset interactions.
   - $\hookrightarrow$ Example include Garleanu (2009) and Duffie, Garleanu, and Pedersen (2005).

Liquidity

Missing: The interaction between infrequent, **random** trading opportunities and the need to smooth consumption/payout.

Many assets cannot be traded for significant lengths of time:

- Real Assets, particularly Real Estate.
- Private Equity and Venture Capital.
- Many securitized fixed income and structured credit products.

$\Rightarrow$ Periodic, random liquidity (reduced form for search).

Most individuals and institutions have intermediate funding needs:

- Outside investors, operational needs, consumption

$\Rightarrow$ Payout needs to be smoothed.

Needed: A portfolio choice model

- Emphasis is on portfolio decisions, not price formation.

Contribution

Our Questions: How should an investor allocate funds to assets that do not trade continuously?

1. How do changes in the value of an illiquid asset affect allocations to liquid *and* illiquid assets?
2. How does one hedge the risk of illiquid assets?
3. How much does the inability to trade impact welfare?

Contribution

Preview of Answers:

1. There is disinvestment is *both* the liquid and illiquid asset. Numerically large effects for calibrated parameters.

2. Liquid assets are not effective for hedging illiquid assets. Correlation is much less important as a driver of investment than in the all-liquid case.

3. Welfare effects are large. Illiquid assets are much less valuable than they would be if liquid.

Model

Three Assets:

1. Riskless bond. Interest rate r. Freely tradeable.
2. Liquid risky asset. Freely tradeable.

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dZ_t^1$$

3. Illiquid risky asset. Tradeable only at random times $\tau \sim \mathrm{Poisson}(\lambda)$. Expected waiting time is $\frac{1}{\lambda}$.

$$\frac{dP_t}{P_t} = \nu\, dt + \psi\rho\, dZ_t^1 + \psi\sqrt{1-\rho^2}\, dZ_t^2$$

Model

Investor has CRRA utility over consumption:

$$\mathrm{E}\left[\int_0^\infty e^{-\beta t}\frac{C_t^{1-\gamma}}{1-\gamma}dt\right]$$

- Wealth effect play an important role in our setup
- Extend to recursive preferences that separate risk aversion (desire to smooth over states) from the elasticity of intertemporal substitution (desire to smooth over time).

Calibration: Liquid and Illiquid Asset Returns

	1981Q3 – 2010Q2			1981Q3 – 2006Q4		
	Mean	Stdev	Corr	Mean	St Dev	Corr
S&P 500	0.103	0.182	1.000	0.125	0.157	1.000
Illiquid Assets						
Private Equity	0.103	0.229	0.629	0.110	0.231	0.605
Buyout	0.092	0.134	0.267	0.097	0.110	0.010
Venture Capital	0.133	0.278	0.557	0.143	0.286	0.548
Illiquid Investment	0.109	0.165	0.674	0.117	0.159	0.623

We base our liquid asset on the S&P 500 and our illiquid asset on the Composite Illiquid Investment. $\gamma = 6$.

Model (continued)

We have to keep track of two types of wealth:

- Liquid Wealth W_t: invested in the liquid stock and bond.
- Illiquid Wealth X_t: invested in the illiquid asset.
- Consumption is out of liquid wealth.

Wealth can only be transferred at trading times.

- Because the investor is CRRA, and the problem is stationary, the agent will always re-balance so that a set fraction of his wealth is illiquid.
- The investor can move far from the optimal portfolio between re-balancing times.
- The relevant state variable is $\frac{X_t}{X_t+W_t}$: fraction of wealth that is illiquid.

Solution

- The value function takes the form

$$F(W_t, X_t) = \max \mathrm{E}\left[\int_0^\infty e^{-\beta t}\frac{C_t^{1-\gamma}}{1-\gamma}dt\right] \Rightarrow W_t^{1-\gamma} H\left(\frac{X_t}{X_t + W_t}\right)$$

- When a trading time arrives, there is a jump in the agent's continuation value as he re-balances. $F \to F^*$.

$$F^*(W_t, X_t) = \max_{I \in [-X_t, W_t]} F(W_t - I, X_t + I) = K(W_t + X_t)^{1-\gamma}$$

High-Marginal-Utility States

Intuition: Illiquidity creates time separation in the value function:

$$F(W,X) = \max \mathrm{E}\left[\int_0^\tau e^{-\beta t}\frac{C_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}K(W_\tau + X_\tau)^{1-\gamma}\right]$$

$$\approx \underbrace{K_{short}W^{1-\gamma}}_{\text{Welfare before next trade}} + \underbrace{K_{long}(W+X)^{1-\gamma}}_{\text{Welfare after next trade}}$$

Result: Two types of high marginal utility states:

- low total wealth
- low **liquid** wealth

Illiquid wealth cannot be used to fund consumption:

Illiquidity and risk-taking

Implication 1: Illiquidity affects the portfolio allocation to *all* risky assets

- The investor behaves as if she were more risk averse, and shifts allocation towards cash

Implication 2: Liquid and illiquid assets are not perfect substitutes

- If the two risky assets are correlated, the investor could use the liquid risky asset to hedge changes in illiquid wealth.
- However, since the investor also cares about liquid wealth, the liquid asset is *not* a good hedge

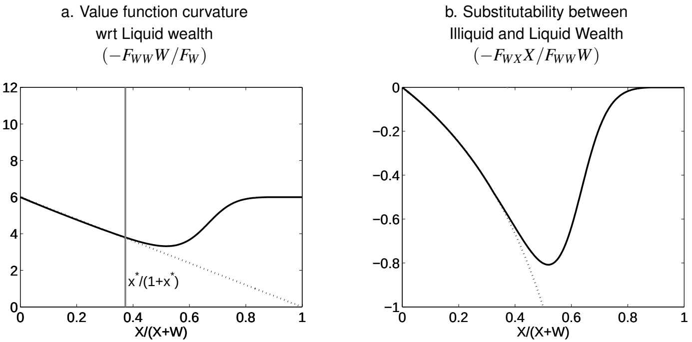
Illiquidity and risk-taking

The allocation to the liquid risky asset is

$$\theta_t = \frac{\mu - r}{\sigma^2}\left(-\frac{F_W}{F_{WW}W_t}\right) + \rho\frac{\psi}{\sigma}\left(-\frac{F_{WX}X_t}{F_{WW}W_t}\right)$$

- term $-\frac{F_W}{F_{WW}W_t}$ captures investor's effective risk aversion
- term $-\frac{F_{WX}X_t}{F_{WW}W_t}$ captures the degree to which the investor views the liquid and illiquid asset as substitutes

Determinants of Portfolio Choice



Compared to the Merton investor (dotted line), our investor

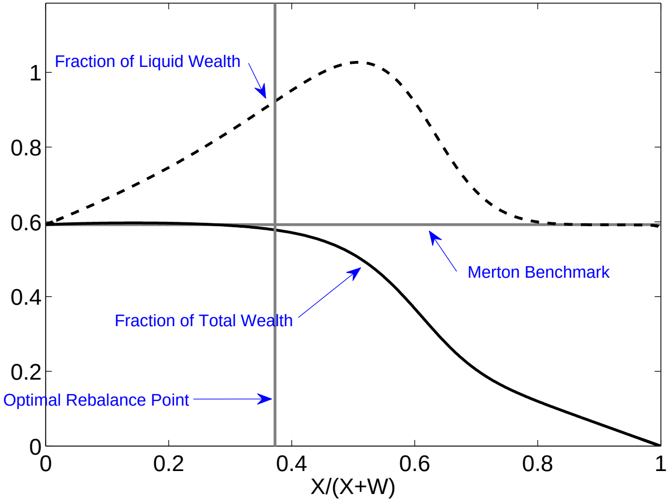
- is more risk averse towards gambles in W
- views liquid and illiquid assets as imperfect substitutes

Asset Holdings and Wealth Composition

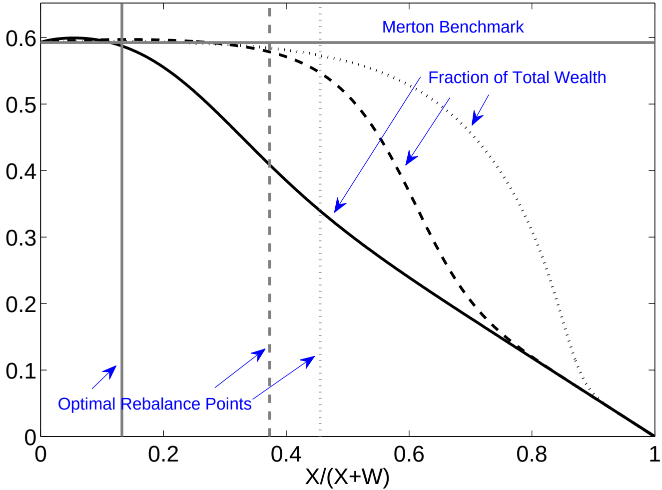
			Stationary Distribution		
Average Turnover	λ	Optimal Rebalance Value	Mean	St Dev	Skew
10 years	0.1	0.0483	0.1659	0.1855	2.3967
5 years	0.2	0.1053	0.1875	0.1273	2.6560
2 years	0.5	0.2423	0.2962	0.0854	2.2373
1 year	1.0	0.3729	0.4076	0.0633	1.8724
1/2 year	2.0	0.4403	0.4584	0.0422	1.5690
1/4 year	4.0	0.4963	0.5051	0.0283	1.2308

- High skewness
- Mean Allocation is greater than the Rebalance Value
- Variation is high

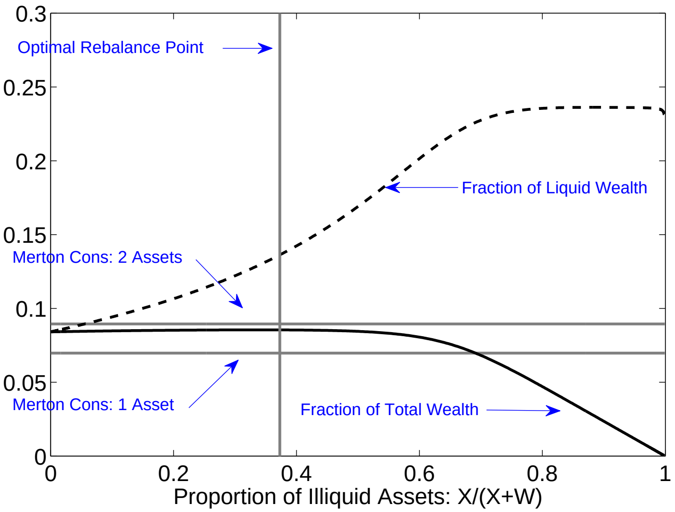
Liquid Asset Allocation



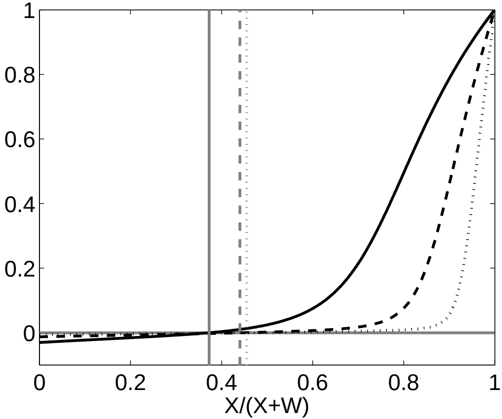
Effect of Liquidity: Illiquid Asset Allocation: $\lambda = 1/4,\ 1,\ 4$



Consumption Rules



How Large a Haircut? $\lambda = 1,\, 2,\, 4$



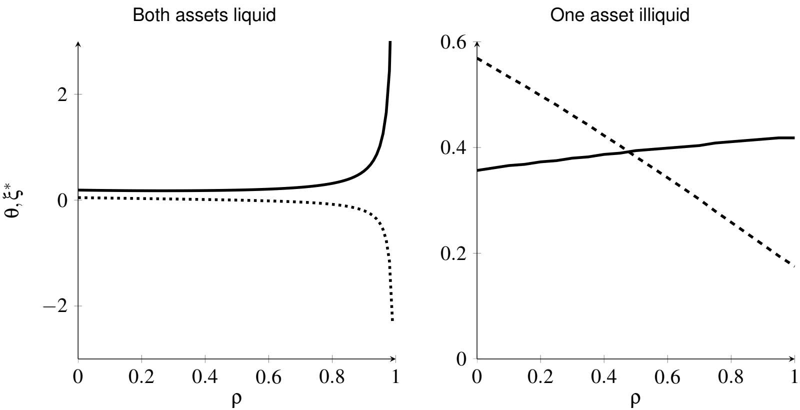
Hedging the Illiquid Asset

Question Can the investor hedge fluctuations in illiquid wealth?

Yes, if the two assets are correlated. However, she does not want to.

- Liquid and illiquid asset imperfect substitutes
- Effect of correlation muted in our setting

Hedging the Illiquid Asset



- Allocation to illiquid asset (——) and liquid asset (- - -) versus correlation

Welfare

Average Turnover	λ	Certainty Equivalent Wealth	Liquidity Premium
$\rho = 0$			
10 years	0.1	0.2866	0.0600
5 years	0.2	0.2148	0.0433
2 years	0.5	0.1140	0.0201
1 year	1.0	0.0572	0.0093
1/2 year	2.0	0.0415	0.0066
1/4 year	4.0	0.0397	0.0063
$\rho = 0.6$			
10 years	0.1	0.1235	0.0224
5 years	0.2	0.0692	0.0141
2 years	0.5	0.0197	0.0041
1 year	1.0	0.0106	0.0022
1/2 year	2.0	0.0098	0.0020
1/4 year	4.0	0.0096	0.0020

Welfare cost of illiquidity

- To better understand the determinants of the utility cost of illiquidity, we extend the model to recursive preferences, that separate

  ↪ Risk aversion: the desire to smooth consumption across *states*

  ↪ 1/EIS: the desire to smooth consumption across *time*.

  Welfare costs are large for investors with *low* risk aversion and *high* 1/EIS.

  ↪ High risk averse investors do not want to invest in risky assets anyway

  ↪ Low EIS guys dislike the fact that they cannot consume today out of their illiquid wealth

Liquidity versus Liquidity Risk

Question: How important is liquidity *risk* as opposed to the simple inability to trade for a finite time?

Approach: Compare the effects of average trading delays to those of deterministic trading delays.

- Agent can trade every $1/\lambda$ periods
- Compare optimal policies and welfare cost

Risky versus deterministic illiquidity

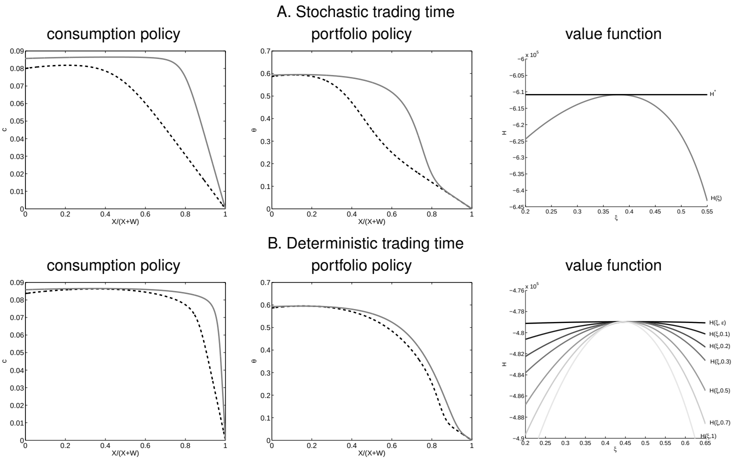
A. Stochastic trading opportunity

Average Turnover $E(T) = 1/\lambda$	Optimal Rebalance	Liquidity Welfare cost	Average policies $E[\xi]$	$E[c]$	$E[\theta]$
0	0.593	-	0.593	0.089	0.593
1/10 years	0.493	0.029	0.466	0.086	0.572
1/4	0.475	0.037	0.465	0.086	0.571
1	0.373	0.067	0.409	0.081	0.558
4	0.132	0.165	0.212	0.069	0.536
10	0.048	0.222	0.214	0.059	0.489

B. Deterministic trading opportunity

Turnover T	Optimal Rebalance	Liquidity Welfare cost	Average policies $E[\xi]$	$E[c]$	$E[\theta]$
0	0.593	-	0.593	0.089	0.593
1/10 years	0.555	0.010	0.467	0.088	0.572
1/4	0.532	0.016	0.467	0.088	0.572
1	0.484	0.025	0.464	0.087	0.569
4	0.425	0.038	0.455	0.085	0.556
10	0.348	0.045	0.414	0.084	0.528

Effect of liquidity risk on portfolio policies



Illiquidity risk

- The real cost of illiquidity is that the waiting time until the next opportunity to trade is **uncertain**.
  - ↪ If the illiquidity period had a known duration T, the investor can guarantee herself a flow period payoff $c\,dt$ by investing cT into the bond
  - ↪ If the duration of the illiquidity period is unknown, then this strategy is not feasible; the distribution of waiting times is unbounded

Liquidity crises

Approach: Introduce a regime shifting process such that

1. All assets can be traded continuously $(S = L)$
2. One asset traded infrequently $(S = I)$

- Aggregate liquidity state S switches between I and L with transition probabilities χ_I and χ_L
  - $\hookrightarrow$ χ_I probability of a liquidity crisis
  - $\hookrightarrow$ $1/\chi_L$ average duration of a crisis

- The possibility of a liquidity crisis occuring affects outcomes in the non-crisis state

Liquidity crises

State	λ	Average duration of illiquid regime – $1/\chi^L$								
		1			1.5			2		
		Average arrival of liquidity crisis – χ^I								
		$1/20$	$1/10$	$1/5$	$1/20$	$1/10$	$1/5$	$1/20$	$1/10$	$1/5$
		Target allocation to illiquid asset – ξ^*								
L	12	0.587	0.587	0.589	0.589	0.589	0.589	0.589	0.589	0.587
	4	0.579	0.579	0.579	0.584	0.584	0.579	0.584	0.584	0.579
	1	0.570	0.570	0.560	0.565	0.565	0.547	0.555	0.555	0.537
	1/2	0.555	0.555	0.537	0.525	0.525	0.503	0.500	0.500	0.475
I	12	0.530	0.530	0.530	0.527	0.527	0.527	0.525	0.525	0.525
	4	0.503	0.503	0.503	0.501	0.501	0.501	0.503	0.503	5026.000
	1	0.435	0.435	0.438	0.428	0.428	0.428	0.421	0.421	0.421
	1/2	0.421	0.421	0.421	0.389	0.389	0.392	0.366	0.366	0.366
		Allocation to liquid risky asset, average – $E(\theta)$								
L	12	0.593	0.593	0.593	0.593	0.593	0.593	0.593	0.593	0.593
	4	0.593	0.593	0.593	0.593	0.593	0.593	0.593	0.593	0.593
	1	0.593	0.593	0.593	0.593	0.593	0.593	0.593	0.593	0.593
	1/2	0.593	0.593	0.593	0.593	0.593	0.593	0.593	0.593	0.593
I	12	0.592	0.592	0.592	0.592	0.592	0.592	0.592	0.592	0.592
	4	0.591	0.591	0.591	0.591	0.591	0.591	0.591	0.591	0.591
	1	0.578	0.578	0.585	0.581	0.581	0.582	0.579	0.579	0.579
	1/2	0.527	0.527	0.527	0.524	0.524	0.523	0.524	0.524	0.525

Liquidity crises

Result: Limits to arbitrage

- Consider an arbitrage opportunity: $|\rho| = 1$ and $\frac{\nu - r}{\psi} \neq \frac{\mu - r}{\sigma}$
- The investor will not fully take advantage of the arbitrage opportunity, **even when both assets are currently liquid**.

Risk of a liquidity crisis leads to underinvestment in the arbitrage

- With probability χ_I the state will switch to $S = I$ and the investor will be stuck with very high leverage (short W or short X) until the next opportunity to rebalance; in the meantime, her liquid wealth can drop to zero.
- Even though the arbitrage is locally riskless in terms of total wealth, during a crisis proceeds from the liquid part very important.

Liquidity risk premium

Our model allows us to derive an analytic expression for the risk premium of a liquidity crisis.

- Liquidity insurance: the investor pays a premium $p\,dt$ every period; receives 1 if the state switches to $S = I$
- Risk premium determined by the 'jump' in marginal value of wealth from L to I

$$p = \chi^I \left. \frac{F_W(W,X,I)}{F_W(W,X,L)} \right|_{\xi=\xi_L^*}$$

Liquidity risk premium

λ	Average duration of illiquid regime – $1/\chi^L$								
	1			1.5			2		
	Average duration of liquid regime – $1/\chi^I$								
	$1/20$	$1/10$	$1/5$	$1/20$	$1/10$	$1/5$	$1/20$	$1/10$	$1/5$
	a. Illiquidity risk premium – p/χ_I								
12	1.042	1.042	1.017	1.024	1.024	1.021	1.029	1.029	1.025
4	1.050	1.050	1.021	1.030	1.030	1.026	1.035	1.035	1.030
1	1.068	1.068	1.031	1.054	1.054	1.042	1.067	1.067	1.051
1/2	1.081	1.081	1.042	1.093	1.093	1.063	1.122	1.122	1.077
	b. Welfare cost of illiquidity								
12	0.004	0.004	0.007	0.005	0.005	0.010	0.007	0.007	0.012
4	0.005	0.005	0.008	0.006	0.006	0.010	0.007	0.007	0.013
1	0.006	0.006	0.010	0.008	0.008	0.014	0.011	0.011	0.017
1/2	0.008	0.008	0.013	0.014	0.014	0.021	0.021	0.021	0.029

Conclusion

- New definition of liquidity that focuses on periodic, random inability to trade.
- Tractable model allows us to understand the interaction between non-marketability and immediate funding needs.
- Investor becomes more risk averse over gambles in *liquid* wealth. Underinvestment in all risky assets.
- Even if illiquidity spells are periodic, the possibility of a liquidity crisis induces limits to arbitrage.