Lecture 1: Asset Allocation

FE-312 Investments

NORTHWESTERN
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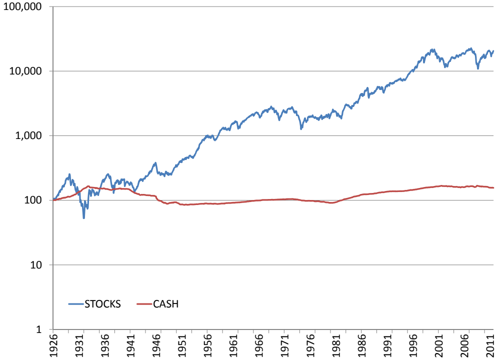
Introduction

- One of the first decisions we need to make is to decide how much to invest in risky assets, and how much to leave in the safe asset.

- Example: If you had \$ 1,000,000 to invest, how would you allocate your money between stocks and bonds?

- Stocks are riskier than bonds, but they also yield higher returns.

Returns of Stocks vs Cash, adjusted for inflation



	mean	volatility
Stocks	8.0%	19.0%
T-Bills	1.0%	0.9%

- What do these numbers mean?

Returns of Stocks vs Cash, adjusted for inflation

- IF returns were normally distributed, over a one-year period:
  1. If you buy stocks:
     - You are 90% confident that your return will range between $8\% \pm 1.65 \times 19\% = [-23\%, 39\%]$
     - The probability that you have a negative return is $\approx 33\%$.
  2. If you buy bonds:
     - You are 90% confident that your return will range between $1\% \pm 1.65 \times 0.9\% = [-0.5\%, 2.5\%]$
     - The probability that you have a negative return is $\approx 14\%$.

Stocks vs Cash

How can we make sense of these numbers?

- Should we invest in the stock market, or T-Bills?

- Which factors influence this decision?

- Should this decision be the same for everyone?

Preferences

- We need a way to describe the tradeoff between risk and expected return, i.e. we need a way to allow investors to rank (ex-ante) different portfolios (different patterns of $\tilde{r}$)
- As a starting point, we assume people are *risk averse*, i.e.
  1. like high expected returns $E(\tilde{r})$
  2. dislike high variance $\sigma^2(\tilde{r})$
- Keep in mind that $\tilde{r}$ is a random variable; we have assumed for now that all investors care about is the mean and variance of $\tilde{r}$.
- So, we have reduced a complicated, multidimensional problem down to two dimensions, risk (variance) and return (mean)

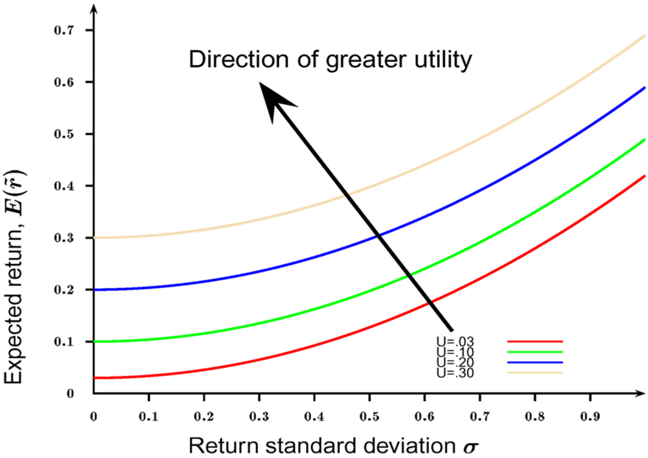
Preferences

- One way to quantify this tradeoff is to assume that investors make choices to maximize their *utility* or happiness they derive from a pattern of returns $\tilde{r}$:

$$U(\tilde{r}) = E(\tilde{r}) - \frac{1}{2}A\sigma^2(\tilde{r})$$

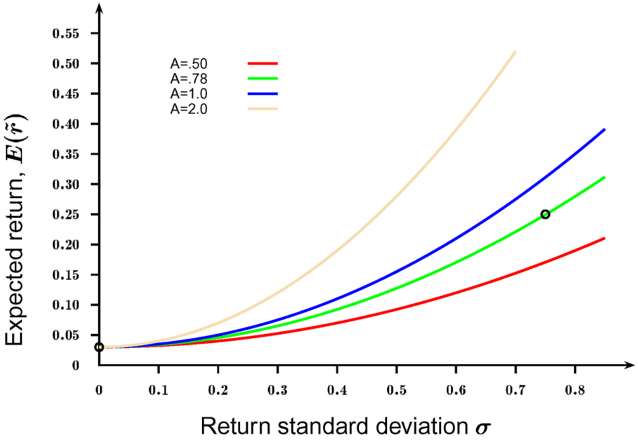
  - A measures the investor's level of *risk-aversion*, i.e. their attitude towards trading off risk (variance) versus higher expected return (mean)
  - The higher A, the higher an investor's dislike of risk

Indifference Curves



- Investor preferences can be depicted as indifference curves.
- Each curve represents different utility levels for fixed risk aversion A.
- Each curve traces out the combinations of $E(r)$, $\sigma(r)$ yielding the same level of utility U.

Indifference Curves



- Each curve plots the same utility level for different risk aversion A.
- Higher A implies that for a given σ, investors require higher mean return to achieve same level of utility.

Which Asset to Choose?

- Recall the choice versus stocks and bonds. Let's ignore inflation risk and pretend that investing in T-bill is riskless.
- Following table compares the utility from investing in stocks versus bonds for different levels of risk aversion

A	U(stocks)	U(T-bill)
1	0.0620	0.010
2	0.0439	0.010
3	0.0259	0.010
4	0.0078	0.010
5	- 0.0103	0.010

- If you have $A = 2$ would you hold bonds or stocks?
- What level of risk-aversion do you have to have to be indifferent between stocks and bonds?

Other utility functions

- In sum, the utility function is an index that measures bad times
  - High utility: good times
  - Low utility: bad times
- Mean/Variance utility seems to be the industry standard, but other utility functions are possible.
- More generally, utility maps outcomes of asset returns which occur with some probability to a numerical index (expected utility). This index
  - Takes into account how large the returns are: what can happen
  - And how often they happen: probability

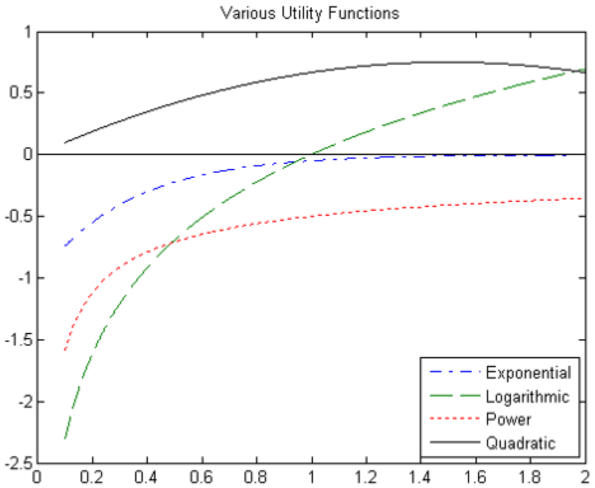
Other utility functions

- In general, utility theory allows us to represent investor preferences as
$$U = E[u(W)]$$
where $u(W)$ is a concave function of wealth.

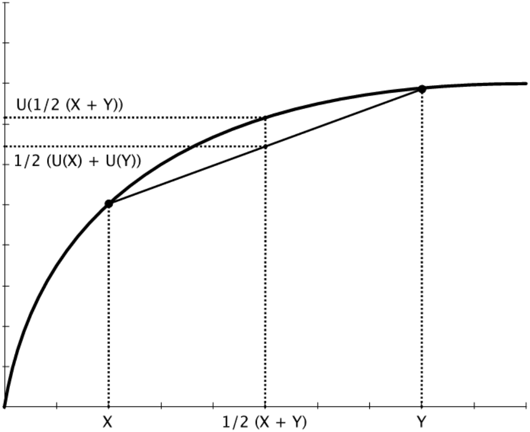
- Common examples are:
  - Exponential Utility: $u(W) = -e^{-\gamma W}$
  - Power Utility: $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$
  (same choices as MV utility if returns are normally distributed)
  - Logarithmic Utility: $u(W) = \log W$
  (special case of power utility as $\gamma = 1$

- Concavity is a desirable property as it implies that investor's are **risk averse**.

Other utility functions



Other utility functions

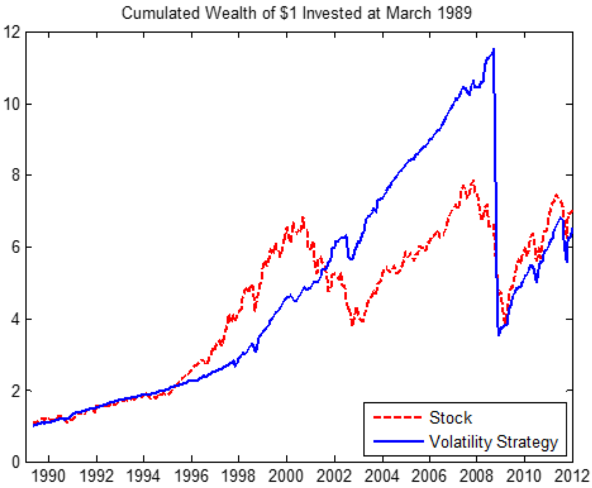


- A sure value of $(x+y)/2$ is preferred to a 50-50 chance of x or y. The degree of concavity is a measure of risk aversion, which trades off risk and return.

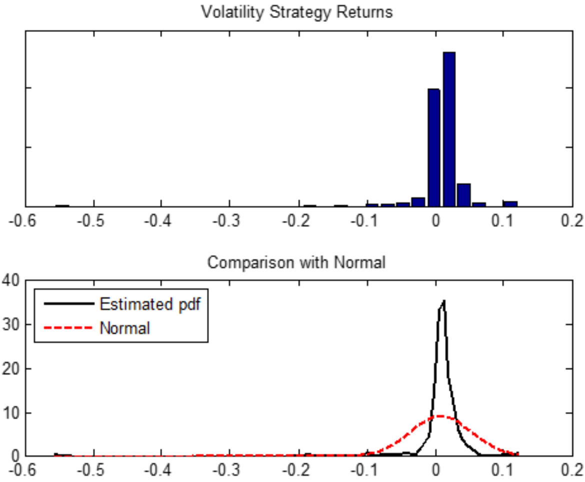
Other utility functions

- Why bother?
- Sometimes, variance may not be a sufficient statistic for risk.
- Compare
  - Volatility strategy: investment strategy which nets a premium during stable periods (period after 2009), but has large losses during volatile times (financial crisis 2008-2009)
    - Technically, we can implement this by selling out-of-the-money put options
  - S&P 500 equities

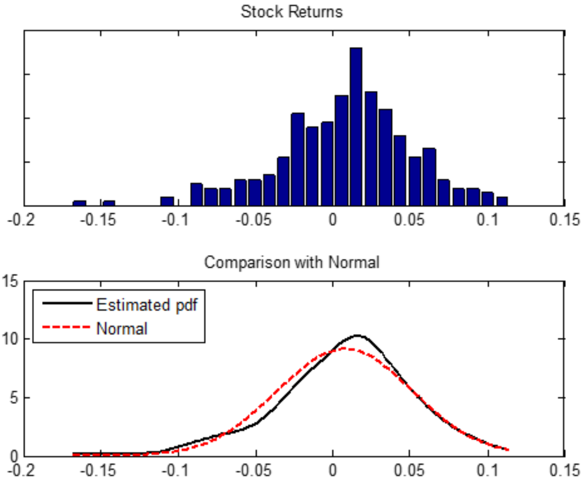
Other utility functions



Other utility functions



Other utility functions



Other utility functions

- Average return is approximately the same, at around 10%
- Dispersion, as measured by standard deviation = 15%, is also approximately the same
- But the volatility strategy exhibits large negative skewness, that is prone to occasional frightful losses

	Vol Strategy	S&P 500
Mean	9.9%	9.7%
Stdev	15.2%	15.1%
Skewness	-8.3	-0.6
Kurtosis	104.4	4.0

Utility and Portfolio Choice

- In sum, the investor's problem can be described as her making portfolio and consumption/savings choices that maximize her *expected utility*, subject to constraints, determined by
  - Her available wealth
  - The assets she can invest in
  - Limits on the positions she can undertake
- We will start with the simplest problem: the investor has already decided how much she wants to save for tomorrow, and her only choice is how to invest the money between today and tomorrow.

Choosing a Portfolio of Risky and Risk-free Assets

- Let's first determine how to build an optimal portfolio of risky and risk-free assets.

- Calculating the return on a portfolio p consisting of of one risky asset and a risk-free asset.

$$
\begin{array}{rll}
\tilde{r}_A = & \text{return on stocks} & = \tilde{r}_A \\
E(\tilde{r}_A) = & \text{expected risky rate of return} & = 8\% \\
\sigma_A = & \text{standard deviation} & = 19\% \\
r_f = & \text{risk-free rate} & = 1\% \\
w = & \text{fraction of portfolio } p \text{ invested in asset A} & = ??
\end{array}
$$

Choosing a Portfolio of Risky and Risk-free Assets

- The return and expected return on a portfolio with weight w on the risky security and $1-w$ on the risk-free asset is:

$$\begin{aligned}
\tilde{r}_p &= w\tilde{r}_A + (1-w)\cdot r_f \\
\tilde{r}_p &= r_f + w(\underbrace{\tilde{r}_A - r_f}_{\tilde{r}_A^e}) \\
E(\tilde{r}_p) &= r_f + wE(\tilde{r}_A^e) \qquad (1)
\end{aligned}$$

- The risk (variance) of this combined portfolio is:

$$\begin{aligned}
\sigma_p^2 &= E[(\tilde{r}_p - \overline{r_p})^2] \\
&= E((w\tilde{r}_A - w\overline{r_A})^2] \\
&= w^2 E[(\tilde{r}_A - \overline{r_A})^2] \\
&= w^2\sigma_A^2 \qquad (2)
\end{aligned}$$

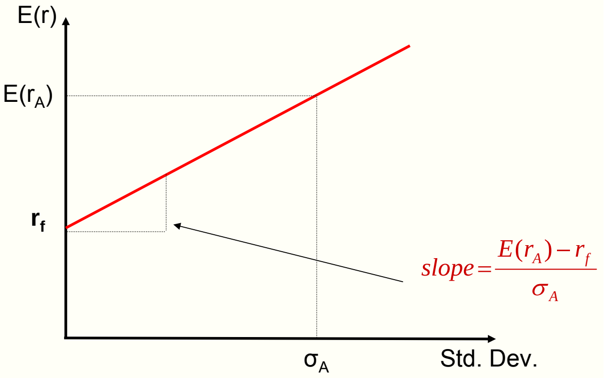
Capital Allocation Line

- We can derive the Capital Allocation Line, i.e. the set of investment possibilities created by all combinations of the risky and riskless asset.
- Combining (1) and (2), we can characterize the expected return on a portfolio with σ_p:

$$E(\tilde{r}_p) = r_f + \underbrace{\left[\frac{E(\tilde{r}_A) - r_f}{\sigma_A}\right]}_{\text{price of risk}} \underbrace{\sigma_p}_{\text{amount of risk}}$$

- The price of risk is the return premium per unit of portfolio risk (standard deviation) and depends **only** the prices of available securities.
- The standard term for this ratio is the *Sharpe Ratio*.

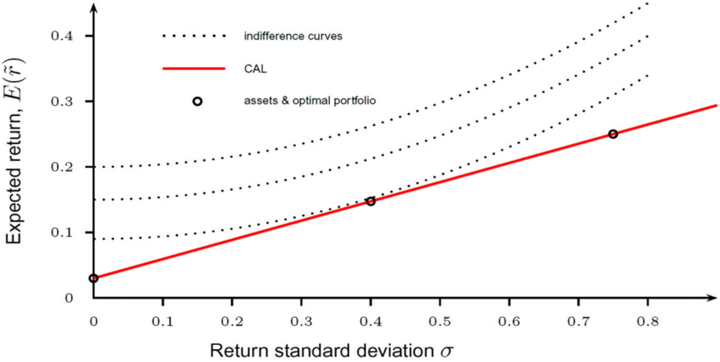
Capital Allocation Line



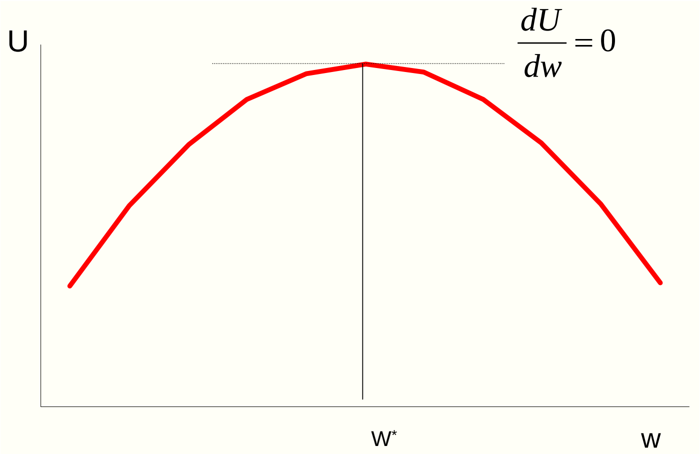
- The CAL shows all risk-return combinations possible from a portfolio of one risky-asset and the risk-free return.
- The slope of the CAL is the *Sharpe Ratio*.

Which Portfolio?

- Which risk-return combination along the CAL do we want?
- To answer this we need the utility function!



Utility as a function of portfolio weight



- At the optimum, investors are indifferent between small changes in w.

Which portfolio?

- The optimal portfolio is the solution to the following problem:

$$U^* = \max_w U(\tilde{r}_p) = \max_w E(\tilde{r}_p) - \frac{1}{2} A\sigma_p^2$$

where, we know,

$$E(\tilde{r}_p) = r_f + wE(\tilde{r}_A - r_f) \qquad \sigma_p^2 = w^2\sigma_A^2$$

Combining these two equations we get:

$$\max_w U(\tilde{r}_p) = \max_w \left(r_f + wE(\tilde{r}_A - r_f) - \frac{1}{2} Aw^2\sigma_A^2 \right)$$

Solution

$$\left.\frac{dU}{dw}\right|_{w=w^*} = 0 \Rightarrow w^* = \frac{E(\tilde{r}_A - r_f)}{A\sigma_A^2}$$

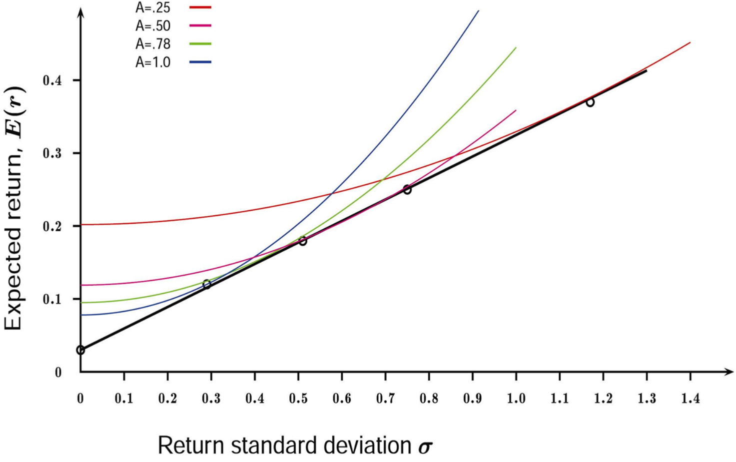
Which portfolio?

For the choice between stocks and bonds:

A	w^*	$E(r_p)$	σ_p
1	1.939	0.146	0.368
2	0.970	0.078	0.184
3	0.646	0.055	0.123
4	0.485	0.044	0.092
5	0.388	0.037	0.074

- What is the meaning of 1.939 in the above table?
- Can you ever get a negative w^*?
- How do changes in A affect the optimal portfolio?
- How do changes in the Sharpe ratio affect the optimal portfolio?

Which Portfolio?



- Different level of risk aversion leads to different choices.

The Investor's Risk Tolerance

Asset **Allocation Planner** Step 1

Your Personal Risk Tolerance

Your prior investment experience can help determine your attitude toward investment risk.

10 Have you ever invested in individual bonds or bond mutual funds? (Aside from U.S. savings bonds.)

- No, and I would be uncomfortable with the risk if I did.
- No, but I would be comfortable with the risk if I did.
- Yes, but I was uncomfortable with the risk.
- Yes, and I felt comfortable with the risk.

11 Have you ever invested in individual stocks or stock mutual funds?

- No, and I would be uncomfortable with the risk if I did.
- No, but I would be comfortable with the risk if I did.
- Yes, but I was uncomfortable with the risk.
- Yes, and I felt comfortable with the risk.

Your comfort level with investment risk influences how aggressively or conservatively you choose to invest. It should be balanced with the potential of achieving your investment goals.

12 Which ONE of the following statements best describes your feelings about investment risk?

- I would only select investments that have a low degree of risk associated with them (i.e., it is unlikely I will lose my original investment).
- I prefer to select a mix of investments with emphasis on those with a low degree of risk and a small portion in others that have a higher degree of risk that may yield greater returns.
- I prefer to select a balanced mix of investments -- some that have a low degree of risk, others that have a higher degree of risk that may yield greater returns.
- I prefer to select an aggressive mix of investments -- some that have a low degree of risk, but with emphasis on others that have a higher degree of risk that may yield greater returns.
- I would select an investment that has only a higher degree of risk and a greater potential for higher returns.

13 If you could increase your chances of improving your returns by taking more risk, would you:

- Be willing to take *a lot* more risk with *all* your money.
- Be willing to take *a lot* more risk with *some* of your money.
- Be willing to take *a little* more risk with *all* your money.
- Be willing to take *a little* more risk with *some* of your money.
- Be unlikely to take much more risk.

The Investor's Risk Tolerance, example

- What are reasonable values for A?

- This is a very tricky question and the subject of much debate among financial economists.
  - At most, we can infer A from investor's choices, *conditional on a particular model.*

- A common investment advice is to invest 60% of your portfolio in stocks and 40% in bonds. Using this framework, and the moments of stocks and T-bills we saw earlier, what is the investor risk tolerance implicit in this advice?

Two Risky Assets and no Risk-Free Asset.

- Now that we understand how to allocate capital between the risky and riskfree asset, we need to show that it is really true that there is a single optimal risky portfolio.

- To start, we'll ask the question:

  How should you combine two risky securities in your portfolio?

- We will plot out possible set of expected returns and standard deviations for different combinations of the assets.

- **Definition:** *Minimum Variance Frontier*, is the set of portfolios with the lowest variance for a given expected return.

A Portfolio of Two Risky Assets

1. The expected return for the portfolio is

$$E(\tilde{r}_p) = w \cdot E(\tilde{r}_A) + (1 - w) \cdot E(\tilde{r}_B)$$

   - $w \equiv w_A^p$ is the fraction that is invested in asset A. Hence, $w_B^p = (1 - w)$

2. The variance of the portfolio is:

$$\begin{aligned} \sigma_p^2 &= E[(\tilde{r}_p - \overline{r_p})^2] \\ &= w^2 \sigma_A^2 + (1-w)^2 \sigma_B^2 + 2w(1-w) cov(\tilde{r}_A, \tilde{r}_B) \end{aligned}$$

   or, since $\rho_{AB} = cov(\tilde{r}_A, \tilde{r}_B)/(\sigma_A \cdot \sigma_B)$,

$$\sigma_p^2 = w^2 \sigma_A^2 + (1-w)^2 \sigma_B^2 + 2w(1-w) \rho_{AB} \sigma_A \sigma_B$$

Importantly, the variance of the portfolio depends on the correlation between the two securities.

A Portfolio of Two Risky Assets: Example

- As an example let's assume that the investor can trade not only in US stocks, but also invest in the Japanese stock market (the Nikkei 225 index).
- Over the last 30 years, the Nikkei 225 index has delivered mean returns of 3.65% per year, with a standard deviation of 25%.
- The correlation between stock returns in the US vs Japan is approximately 50%.
- Let's derive the minimum variance frontier using Excel (hard) and Matlab (easy).

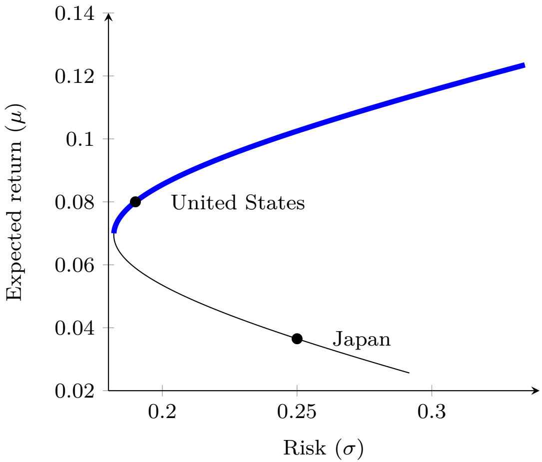
The minimum variance frontier, the hard way

Create three columns in Excel. The first column includes a list of possible w's. The second and third column will contain the mean and standard deviation associated with these weights:

$$E(\tilde{r}_p) = 0.08w + 0.0365 \cdot (1-w)$$
$$\sigma_p = \sqrt{0.19^2 w^2 + 0.25^2 \cdot (1-w)^2 + 2\,w(1-w)0.5 \times 0.19 \times 0.25}$$

w	$E(\tilde{r}_p)$	σ_p
-0.25	0.0256	0.2917
0	0.0365	0.2500
0.25	0.0474	0.2152
0.5	0.0583	0.1911
0.75	0.0691	0.1820
1	0.0800	0.1900
1.25	0.0909	0.2132
1.5	0.1018	0.2474
1.75	0.1126	0.2887

The minimum variance frontier



- The upper part of the frontier (blue) is termed the 'efficient frontier'

The minimum variance frontier, the easier way

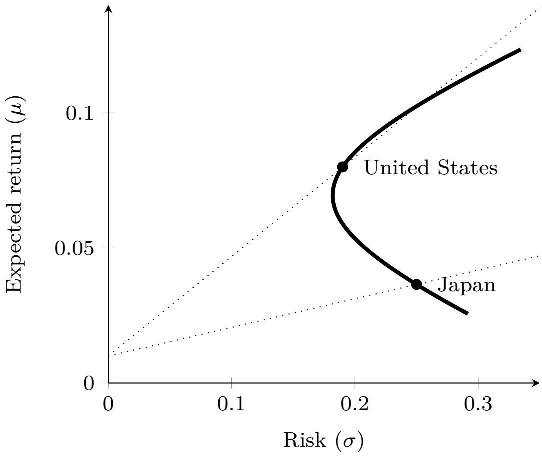
With just two assets, the minimum variance frontier can also easily be computed in MATLAB:

```
w=[-0.25:0.01:2];
mu=0.08*w+(1-w)*0.0365;
sigma=sqrt(0.19^2*w.^2+0.25^2*(1-w).^2+2*w.*(1-w)*0.19*0.25*0.5);
p=plot(sigma,mu);
```

How about multiple assets? If you have access to the Financial Toolbox, you can also do

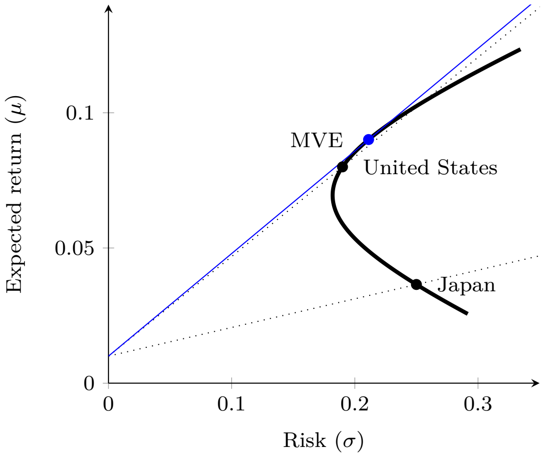
```
AssetBoundsNC=[-100 -100;100 100] ;
mu1=0.08;
mu2=0.0365;
Sigma1=0.19;
Sigma2=0.25;
rho12=0.5;
PortReturn=[0.069:0.005:0.15];
V=[Sigma1^2 rho12*Sigma1*Sigma2;  rho12*Sigma1*Sigma2  Sigma2^2];
mu=[mu1;mu2]';
[PortRiskTNC, PortReturnTNC] = frontcon(mu,V,[],PortReturn,AssetBoundsNC);
```

The minimum variance frontier



- Now, let's also allow the investor to allocate some money into T-bills
- The dotted lines represent the investment possibilities of Tbills +

The minimum variance frontier



- There exists a unique combination of the US and Japan index that has the *higherst* Sharpe Ratio
- Any portfolio along the blue line (the Capital Allocation Line)

The Mean-Variance Efficient Portfolio

- We call the optimal portfolio combining investment in the US and Japan the **mean-variance efficient portfolio**
- It is the portfolio that, in combination with the risk-free asset, it provides the steepest Capital Allocation Line
- Why is this the portfolio we want? Recall that when we chose between 'stocks' and 'bonds' we were trading off risk and return according to

$$E(\tilde{r}_p) = r_f + \underbrace{\left[\frac{E(\tilde{r}_A) - r_f}{\sigma_A}\right]}_{\text{Sharpe Ratio}} \underbrace{\sigma_p}_{\text{amount of risk}}$$

- The MVE portfolio has the highest Sharpe Ratio and therefore provides the best tradeoff between risk and return.

MVE portfolio

- How do we find the *MVE* portfolio?
- Mathematically, we need to find the portfolio with the highest Sharpe Ratio:

$$\max_{w} \frac{E(\tilde{r}_p) - r_f}{\sigma_p}$$

where

$$\begin{aligned} E(\tilde{r}_p) &= wE(r_B) + (1-w)E(\tilde{r}_C) \\ \sigma_p &= \left[w^2\sigma_B^2 + (1-w)^2\sigma_C^2 + 2w(1-w)\rho_{BC}\sigma_B\sigma_C\right]^{1/2} \end{aligned}$$

- Unfortunately, the solution is pretty complicated:

$$w_B^p = \frac{E(\tilde{r}_B^e)\sigma_C^2 - E(\tilde{r}_C^e)cov(\tilde{r}_B, \tilde{r}_C)}{E(\tilde{r}_B^e)\sigma_C^2 + E(\tilde{r}_C^e)\sigma_B^2 - \left[E(\tilde{r}_B^e) + E(\tilde{r}_C^e)\right]cov(r_B, r_C)}$$

MVE portfolio

Life is easier using Matlab (need the Financial Toolbox)

```
AssetBoundsNC=[-100 -100;100 100] ;
mu1=0.08;
mu2=0.0365;
Sigma1=0.19;
Sigma2=0.25;
rho12=0.5;
PortReturn=[0.069:0.005:0.15];
V=[Sigma1^2 rho12*Sigma1*Sigma2;  rho12*Sigma1*Sigma2  Sigma2^2];
mu=[mu1;mu2]';
RiskAversion=4;

[PortRiskTNC, PortReturnTNC, PortWtsTNC] = ...
    frontcon(mu,Sigma,[],PortReturn,AssetBoundsNC);
[OPTRiskTNC, OPTReturnTNC, RiskyWtsTNC] = ...
    portalloc(PortRiskTNC, PortReturnTNC, PortWtsTNC, RisklessRate);
```

MVE portfolio

- Here, the MVE portfolio allocates $w = 1.23$ in the United States and $1 - w = -0.23$ in Japan.
- The *Sharpe Ratio* of the MVE portfolio is:

$$SR_{MVE} = \frac{E(r^e_{MVE})}{\sigma_{MVE}} = \frac{0.09 - 0.01}{0.2112} = 0.3793$$

- The United States and Japan have Sharpe Ratios of 0.368 and 0.146
- The Sharpe Ratio of the MVE portfolio is **always** higher than those of each individual assets.
- Have we determined the optimal allocation between risky and riskless assets?
- Almost. What remains is to decide how much to invest between the MVE portfolio and the risk-free asset. That fraction will depend on the investor's risk aversion, as we saw earlier.

Optimal portfolio

- Earlier, we saw that the allocation to the risky asset (now, the tangency or MVE portfolio) should be

$$w_T^* = \frac{\bar{r}_T - r_f}{A\sigma_T^2}$$

That solution for w_T^* implies U is (just by plugging in)

$$U^* = \frac{1}{2}\frac{1}{A}\left(\underbrace{\frac{\bar{r}_T - r_f}{\sigma_T}}_{\text{Sharpe Ratio}}\right)^2$$

- Utility depends on the *Sharpe ratio* of the risky portfolio
- *Utility is maximized when your risky portfolio has the maximum Sharpe ratio*

MVF with many risky assets.

- Now let's look at the optimal portfolio problem when there are three assets.

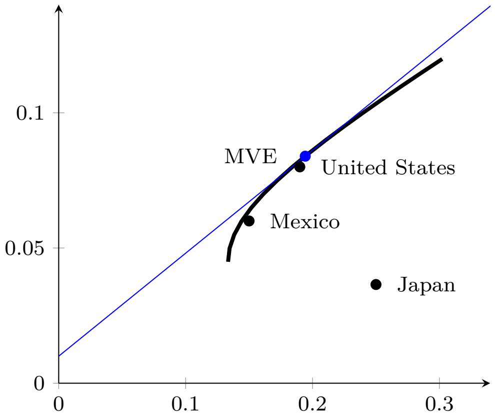
- Expected returns, standard deviations, and correlation matrix are again:

Asset	$E(r)$	σ
US	8.0%	19%
Japan	3.7%	25%
Mexico	6.0%	15%

Correlations			
Assets	US	J	M
US	1.0	0.5	0.8
J	0.5	1.0	0.2
M	0.8	0.2	1.0

- What does the minimum variance frontier look like now?

The minimum variance frontier



- The MVE portfolio allocates 97%, -17% and 20% in the US, Japan and Mexico, respectively.
- The Sharpe Ratio of the MVE portfolio is now 0.381 compared to 0.379 when we were investing in the US and Japan alone. Not a

MVF with many risky assets.

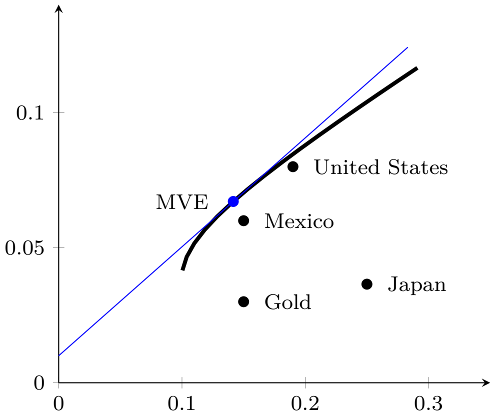
- Let us now add a fourth security, Gold

Asset	$E(r)$	σ
US	8.0%	19%
Japan	3.7%	25%
Mexico	6.0%	15%
Gold	3.0%	15%

Correlations				
Assets	US	J	M	G
US	1.0	0.5	0.8	0.0
J	0.5	1.0	0.2	0.0
M	0.8	0.2	1.0	0.0
G	0.0	0.0	0.0	1.0

- What does the minimum variance frontier look like now?

The minimum variance frontier



- The MVE portfolio allocates 67%, -12%, 13%, and 32% in the US, Japan, Mexico and Gold, respectively.
- The Sharpe Ratio of the MVE portfolio is now 0.41 compared to 0.38 relative to before. What happened?

MVF with many risky assets.

- To drive the point home, suppose now that the volatility of Gold increases from 0.15 to 0.25

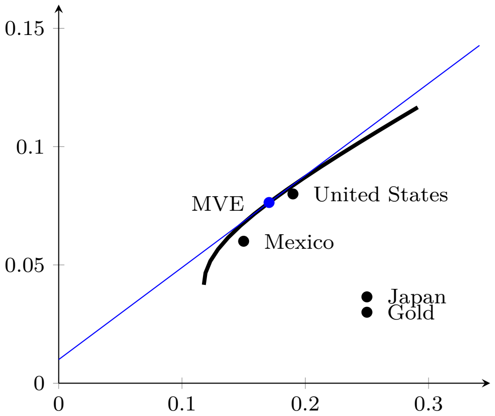
Asset	$E(r)$	σ
US	8.0%	19%
Japan	3.7%	25%
Mexico	6.0%	15%
Gold	3.0%	25%

Correlations				
Assets	US	J	M	G
US	1.0	0.5	0.8	0.0
J	0.5	1.0	0.2	0.0
M	0.8	0.2	1.0	0.0
G	0.0	0.0	0.0	1.0

- Gold has now the **lowest** Sharpe Ratio of any security.

- What does the minimum variance frontier look like now?

The minimum variance frontier



- The MVE portfolio allocates 85%, -16%, 17%, and 14% in the US, Japan, Mexico and Gold, respectively.
- investing in Gold looks worse than investing in Japan, yet we allocate more money to it. Why?

MVF with many risky assets.

- The answer is Diversification
- We are not investing in Gold alone. We are investing in Gold **as part of a portfolio**
- When choosing which securities to invest in, what matters is not just the mean and variance of each individual security, but also their **correlation** with the other securities in our basket.
- We would like to choose the portfolio that has the highest Sharpe Ratio. But that does **not** involve choosing securities that have high Sharpe Ratios themselves!

Computing the Mean Variance Frontier

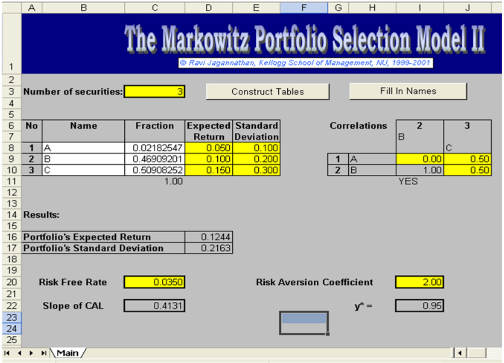
- The mathematics of the problem quickly become complicated as we add more risky assets.
- We are going to need a general purpose method for solving the multiple asset problem.
- Fortunately, there are several alternatives.
- MATLAB (or Octave, a free alternative) are a good option
- Alternatively, Excel's "solver" function can solve the problem using the spreadsheet called MarkowitzII.xls which can be found on the course homepage.

MVE portfolio with multiple risky assets

If you don't have the Financial Toolbox, you can use this code with MATLAB

```
mu1=0.08; mu2=0.0365; Sigma1=0.19; Sigma2=0.25; rho12=0.5;

% S is matrix of security covariances
S = [Sigma1^2 rho12*Sigma1*Sigma2;rho12*Sigma1*Sigma2 Sigma2^2]; stdevs = sqrt(diag(S));
% Vector of security expected returns
zbar = [mu1; mu2];

unity = ones(length(zbar),1);
A = unity'*S^-1*unity; B = unity'*S^-1*zbar; C = zbar'*S^-1*zbar; D = A*C-B^2;

% Plot Efficient Frontier (portfolio with least risk associated with each mu)
minvar = ((A*mu.^2)-2*B*mu+C)/D; minstd = sqrt(minvar); mu = [0.035:0.001:0.12];
plot(minstd,mu,stdevs,zbar,'*')

% Weights for Tangency Portfolio,
B1 = unity'*S^-1*ezbar; w_mve = (S^-1*ezbar)/B1;

% Expected Return of Tangency Portfolio
mu_mve = w_mve'*zbar;

% Variance and Standard Deviation of Tangency Portfolio
var_mve = w_mve'*S*w_mve; std_mve = sqrt(var_mve)
```

MVF with many risky assets in Excel



- Fill in the input data (yellow cells).
- Given a set of weights w, the spreadsheet computes the slope of the CAL, $E(r_{MVE})$, σ_{MVE}
- To find the MVE, ask solver to choose weights to maximize the slope of the CAL (subject to the constraint that weights sum to one).

Understanding Diversification

Basic Message: Your risk/return tradeoff is improved by holding many assets with less than perfect correlation.

1. Start with our equation for variance:

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} w_i w_j cov(\tilde{r}_i, \tilde{r}_j)$$

2. Then make the simplifying assumption that $w_i = 1/N$ for all assets:

$$\sigma_p^2 = \left(\frac{1}{N^2}\right) \sum_{i=1}^{N} \sigma_i^2 + \sum_{i=1}^{N} \left(\frac{1}{N^2}\right) \sum_{\substack{j=1 \\ i \neq j}}^{N} cov(\tilde{r}_i, \tilde{r}_j)$$

3. The average variance and covariance of the securities are:

$$\overline{\sigma^2} = \left(\frac{1}{N}\right) \sum_{i=1}^{N} \sigma_i^2 \qquad \overline{cov} = \frac{1}{N(N-1)} \sum_{\substack{j=1 \\ i \neq j}}^{N} cov(\tilde{r}_i, \tilde{r}_j)$$

Understanding Diversification

1. Plugging these into our equation gives:

$$\sigma_p^2 = \left(\frac{1}{N}\right)\overline{\sigma^2} + \left(\frac{N-1}{N}\right)\overline{cov}$$

2. What happens as N becomes large?

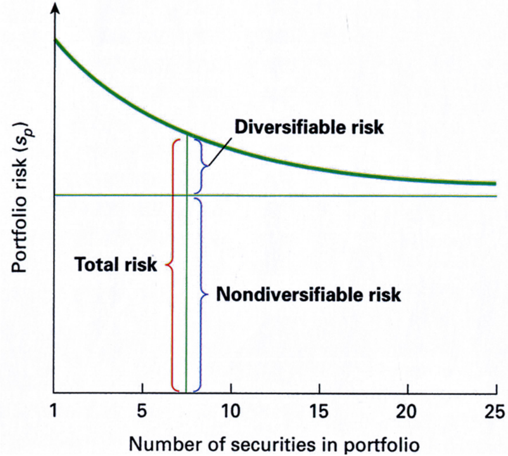
$$\sigma_p^2 \rightarrow \overline{cov}$$

3. **Only the average covariance matters for large portfolios.**

4. If the average covariance is zero, then the portfolio variance is close to zero for large portfolios.

Understanding Diversification

- This plot shows how the standard deviation of a portfolio of average NYSE stocks changes as we change the number of assets in the portfolio.



Understanding Diversification

- The component of risk that can be diversified away we call the *diversifiable* or *non-systematic* risk.
- Empirical Facts
  - The average (annual) return standard deviation is 49%
  - The average (annual) covariance between stocks is 0.037, and the average correlation is about 39%.
- Since the average covariance is positive, even a very large portfolio of stocks will be risky. We call the risk that cannot be diversified away the *systematic risk.*

What happens in practice?

- In a survey of retail investors, Hung et. al. (2008) found that 73% of all individuals surveyed consult a financial adviser before purchasing shares or mutual funds.
- Mullainathan, Noeth, Schoar (2012) examined the advice given to clients by financial advisers.
  - Advisers often reinforced investor biases. Most strikingly, they were unsupportive of the (efficient) index portfolio and suggested a change to actively managed funds.
  - Most advisers did ask clients about their demographic characteristics to determine risk preferences. However, in many cases, the information did not get used in the way predicted by portfolio theory: advisers did not seem to tailor portfolio advice with the age of the client at hand.
  - In general, most advisers encouraged clients to invest in high-fee investments, downplaying the magnitude of fees: "This fund has 2% fee but that is not much above industry average."

"MY FIRST PIECE OF ADVICE IS NOT TO PUT ALL YOUR EGGS IN ONE BASKET."

Conclusions

In this lecture we have developed modern portfolio theory

1. In it's basic form, we often call it mean-variance analysis because we assume that all that matters to investors is the average return and the return variance of their portfolio.
   - This is appropriate if returns are normally distributed.
2. There are a couple of key lessons from mean-variance analysis:
   - You should hold the same portfolio of risky assets no matter what your tolerance for risk.
     - If you want less risk, combine this portfolio with investment in the risk-free asset.
     - If you want more risk, buy the portfolio on margin.
   - In large portfolios, covariance is important, not variance.