Kellogg School of Management, Northwestern U, Department of Finance

FINC 460
Investments

Solution to HW 3 Problems 1-3

1. You are given the following information: the variance of return on stock-1, stock-2, and the market portfolio are: $\sigma_1^2 = 0.16$, $\sigma_2^2 = 0.09$, and $\sigma_M^2 = 0.04$. The covariance between these assets are $\sigma_{12} = 0.02$, $\sigma_{1M} = 0.064$, and $\sigma_{2M} = 0.032$. Consider forming a portfolio "*p*" that has 75% invested in stock-1 and 25% invested in stock-2.

A. What is the variance of return for portfolio *p*?

Solution: Recall that:

$$\sigma_p^2 = \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\sigma_{12}$$
$$\Rightarrow \sigma_p^2 = 0.75^2 \times 0.16 + 0.25^2 \times 0.09 + 2 \times 0.75 \times 0.25 \times 0.02$$
$$\Rightarrow \sigma_p^2 = 0.103$$

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B. What are the betas of stock-1, stock-2, and *p* relative to the market (that is, what are β_{1M}, β_{2M}, and β_{pM} respectively)?

Solution: Recall:

$$\beta_{1M} = \frac{\sigma_{1M}}{\sigma_M^2} = \frac{0.064}{0.04} = 1.60$$
$$\beta_{2M} = \frac{\sigma_{2M}}{\sigma_M^2} = \frac{0.032}{0.04} = 0.80$$
$$\beta_{pM} = \omega_1\beta_{1M} + \omega_2\beta_{2M} = 0.75 \times 1.60 + 0.25 \times 0.80 = 1.40$$

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C. What are the R^2 values for regressing returns of stock-1, stock-2, and *p* on the market portfolio?

Solution: Recall that:

$$R_1^2 = \frac{\beta_{1M}^2 \sigma_M^2}{\sigma_1^2} = \frac{1.60^2 \times 0.04}{0.16} = 0.64 = 64\%$$

$$R_2^2 = \frac{\beta_{2M}^2 \sigma_M^2}{\sigma_2^2} = \frac{0.80^2 \times 0.04}{0.09} = 0.284 = 28.4\%$$

$$R_p^2 = \frac{\beta_{pM}^2 \sigma_M^2}{\sigma_p^2} = \frac{1.40^2 \times 0.04}{0.103} = 0.761 = 76.1\%$$

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2. Mr. Larson E. Rich has asked you for some financial advice. His retirement savings are currently invested as follows: \$20,000 in the riskless asset, \$40,000 in GM stock, and \$40,000 in Microsoft stock. He wants to know if this is a sensible portfolio. You decided to analyze it based on the CAPM. You want to find out if Mr. Rich's portfolio is on the Capital Market Line.

You look in a "Beta Book" and find that GM stock has a beta of 1.1 and its R^2 of the regression to market is 0.40. Microsoft stock has a beta of 0.8 and its R^2 of the regression to market is 0.30. Suppose further that the correlation between the return to GM stock and the return to Microsoft stock is 0.3.

A. If R_f is 4% and the expected excess return on the market $\left(E[R_M] - R_f\right)$ is 6%, what is the expected return on Mr. Rich's portfolio?

<u>Solution</u>: Mr. Rich's portfolio is 20% in the riskfree assets, 40% in GM stock and 40% in Microsoft stock. Hence the beta of his portfolio is:

$$\beta_P = \omega_1 \beta_1 + \omega_2 \beta_2 + \omega_3 \beta_3$$
$$\Rightarrow \beta_P = 20\% \times 0 + 40\% \times 1.1 + 40\% \times 0.8 = 0.76$$

According to the CAPM, the expected return on a portfolio is related to the market as:

$$\left(E[R_P] - R_f\right) = \beta_P \left(E[R_M] - R_f\right)$$

Hence, the expected return on Mr. Rich's portfolio is:

$$E[R_P] = R_f + \beta_P \left(E[R_M] - R_f\right) = 4\% + 0.76 \times 6\% = 8.56\%$$

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B. If market return has a volatility of 20%, compute the volatility of Mr. Rich's current portfolio.

Hint: You may use the information in the R^2 values to calculate the standard deviation of each stock's return. Then use the information about correlations between the two stock returns to calculate the portfolio standard deviation.

Solution: We first need to calculate the standard deviation of each stock return. Since the R_i^2 for stock i is given by:

$$R_i^2 = \frac{\beta_i^2 \sigma_M^2}{\sigma_i^2}$$

so that:

$$\sigma_i^2 = \frac{\beta_i^2 \sigma_M^2}{R_i^2}$$

This means that:

$$\sigma_{GM}^2 = \frac{\beta_{GM}^2 \sigma_M^2}{R_{GM}^2} = \frac{1.1^2 \times 0.2^2}{0.40} = 0.121$$

and

$$\sigma_{MICRO}^2 = \frac{\beta_{MICRO}^2 \sigma_{MICRO}^2}{R_{MICRO}^2} = \frac{0.8^2 \times 0.2^2}{0.30} = 0.0853$$

Hence, $\sigma_{GM} = 0.3479 = 34.79\%$ and $\sigma_{MICRO} = 0.2921 = 29.21\%$.

Mr. Rich invested \$100,000 in 3 assets: \$20,000 in riskless asset, \$40,000 in GM stock, and \$40,000 in Microsoft stock. Therefore, his portfolio weights are:

$$\omega_f = \frac{\$20,000}{\$100,000} = 0.2, \qquad \omega_{GM} = \frac{\$40,000}{\$100,000} = 0.4, \qquad \omega_{MICRO} = \frac{\$40,000}{\$100,000} = 0.4$$

Return on his portfolio is:

$$R_P = \omega_f R_f + \omega_{GM} R_{GM} + \omega_{MICRO} R_{MICRO}$$

Therefore:

$$\left(R_P - E[R_P]\right) = \omega_{GM}\left(R_{GM} - E[R_{GM}]\right) + \omega_{MICRO}\left(R_{MICRO} - E[R_{MICRO}]\right)$$

Portfolio variance can be obtained by squaring the above expression, and then by computing the expectation:

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\begin{aligned}
&\left(R_P - E[R_P]\right) = \omega_{GM}\left(R_{GM} - E[R_{GM}]\right) + \omega_{MICRO}\left(R_{MICRO} - E[R_{MICRO}]\right)\\
&\Rightarrow \left(R_P - E[R_P]\right)^2 = \omega_{GM}^2\left(R_{GM} - E[R_{GM}]\right)^2 + \omega_{MICRO}^2\left(R_{MICRO} - E[R_{MICRO}]\right)^2\\
&\qquad\qquad + 2\omega_{GM}\,\omega_{MICRO}\left(R_{GM} - E[R_{GM}]\right)\left(R_{MICRO} - E[R_{MICRO}]\right)\\
&\Rightarrow E\left[\left(R_P - E[R_P]\right)^2\right] = \omega_{GM}^2 E\left[\left(R_{GM} - E[R_{GM}]\right)^2\right] + \omega_{MICRO}^2 E\left[\left(R_{MICRO} - E[R_{MICRO}]\right)^2\right]\\
&\qquad\qquad + 2\omega_{GM}\,\omega_{MICRO} E\left[\left(R_{GM} - E[R_{GM}]\right)\left(R_{MICRO} - E[R_{MICRO}]\right)\right]\\
&\Rightarrow \sigma_P^2 = \omega_{GM}^2\sigma_{GM}^2 + \omega_{MICRO}^2\sigma_{MICRO}^2 + 2\omega_{GM}\,\omega_{MICRO}\operatorname{cov}\left(R_{GM}, R_{MICRO}\right)\\
&\Rightarrow \sigma_P^2 = \omega_{GM}^2\sigma_{GM}^2 + \omega_{MICRO}^2\sigma_{MICRO}^2 + 2\omega_{GM}\,\omega_{MICRO}\sigma_{GM}\sigma_{MICRO}\rho_{GM,MICRO}\\
&\Rightarrow \sigma_P^2 = 0.4^2 \times 0.121 + 0.4^2 \times 0.0853 + 2 \times 0.4 \times 0.4 \times 0.3479 \times 0.2921 \times 0.3 = 0.042764
\end{aligned}
$$

Hence, the volatility of Mr. Rich's portfolio is:

$$\sigma_P = +\sqrt{0.042764} = 0.2068 = 20.68\%$$

<u>Alternative Approach (Based on the Intuition)</u>:

The risky part of Mr Rich's portfolio is 50% in GM (as, \$40,000/\$80,000 = 0.5) and 50% in Microsoft (as, \$40,000/\$80,000 = 0.5). The variance of the return to the risky part of the portfolio is:

$$
\begin{aligned}
\sigma^2 &= 0.5^2 \times \sigma_{GM}^2 + 0.5^2 \times \sigma_{MICRO}^2 + 2 \times 0.5 \times 0.5 \times \rho_{GM,MICRO} \times \sigma_{GM} \times \sigma_{MICRO}\\
&= 0.5^2 \times 0.121 + 0.5^2 \times 0.0853 + 2 \times 0.5 \times 0.5 \times 0.3 \times 0.348 \times 0.292\\
&= 0.0668
\end{aligned}
$$

The standard deviation of this part of the portfolio is then $+\sqrt{0.0668} = 0.258$. Since the portfolio is 80% in this risky stock portfolio (40% in GM plus 40% in Microsoft), the standard deviation (that is, volatility) of the entire portfolio is:

$$\sigma_P = 0.8 \times 0.2585 = 0.2068 = 20.68\%$$

(Recall, we have seen similar analysis in 1-risky and riskfree case). Hence the variance is: $(0.2068)^2 = 0.0428$.

C. (10 Points): Assuming that the CAPM is correct, find an efficient portfolio that has the same volatility as Mr. Rich's current portfolio. What is the expected return on this portfolio? How does it compare to the expected return of his current portfolio? You may assume that that market return has a volatility of 20%, R_f is 4%, and $\left(E[R_M]-R_f\right)$ is 6%.

Solution: If the CAPM is true then the efficient portfolios are made up of an investment in the riskfree asset and the market portfolio (on CML). Let ω_M be the %-investment in the market portfolio. So that the new portfolio has the same volatility as Mr. Rich's current portfolio, it must be that:

$$\omega_M \times \sigma_M = \sigma_P$$

Or,

$$\omega_M = \frac{\sigma_P}{\sigma_M} = \frac{0.2068}{0.2} = 1.034$$

Hence the market portfolio has to be leveraged to reach Mr. Rich's target volatility on the capital market line. To invest 103.4% in the market portfolio one must borrow 3.4% in the riskfree rate.

The expected return of this efficient portfolio will be:

$$E[R_{new}] = \omega_M E[R_M] + (1-\omega_M)R_f = 1.034 \times (6\% + 4\%) + (-0.034) \times 4\% = 10.20\%$$

This is higher than the expected return of Mr. Rich's current portfolio [see part (A), it was 8.56%]. Therefore, according to the CAPM, Mr. Rich's current allocations are not rationally sensible. You should recommend him to reallocate his capital to move his portfolio to the capital market line.

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3. There are three stocks in the economy ("A", "B", and "C") that are all uncorrelated with each other. The riskfree rate for borrowing and lending is 4% over the holding period.

The table below summarizes the information for each stock regarding its expected return and variance:

Stock	Expected Return	Variance of Return
A	14%	0.004
B	12%	0.002
C	11%	0.002

A. Compute the tangency portfolio weights of these three stocks and the expected return and volatility of the tangency portfolio.

Solution:
Weights on Tangency Portfolio: To compute the tangency portfolio, we need to solve the following 3 equations:

$$\begin{aligned}
\sigma_A^2 x_A + \text{cov}(R_A, R_B)x_B + \text{cov}(R_A, R_C)x_C &= \left(E[R_A] - R_f\right)\\
\text{cov}(R_B, R_A)x_A + \sigma_B^2 x_B + \text{cov}(R_B, R_C)x_C &= \left(E[R_B] - R_f\right)\\
\text{cov}(R_C, R_A)x_A + \text{cov}(R_C, R_B)x_B + \sigma_C^2 x_C &= \left(E[R_C] - R_f\right)
\end{aligned}$$

Since the stocks are un-correlated with each other, all the covariance terms are 0. Therefore:

$$0.004 \times x_A = (0.14 - 0.04) \quad \Rightarrow \quad x_A = \frac{0.14 - 0.04}{0.004} = 25$$

$$0.002 \times x_B = (0.12 - 0.04) \quad \Rightarrow \quad x_B = \frac{0.12 - 0.04}{0.002} = 40$$

$$0.002 \times x_C = (0.11 - 0.04) \quad \Rightarrow \quad x_C = \frac{0.11 - 0.04}{0.002} = 35$$

By normalizing, the weights of the stocks on the tangency portfolio will be:

$$\omega_A = \frac{x_A}{x_A + x_B + x_C} = \frac{25}{25 + 40 + 35} = 0.25$$

$$\omega_B = \frac{x_B}{x_A + x_B + x_C} = \frac{40}{25 + 40 + 35} = 0.40$$

$$\omega_C = \frac{x_C}{x_A + x_B + x_C} = \frac{35}{25 + 40 + 35} = 0.35$$

Hence, return on tangency portfolio is:

$$R_T = 0.25R_A + 0.40R_B + 0.35R_C$$

Expected Return of Tangency Portfolio: We found:

$$R_T = 0.25R_A + 0.40R_B + 0.35R_C$$

Therefore:

$$\begin{aligned}
&E[R_T] = 0.25E[R_A] + 0.40E[R_B] + 0.35E[R_C]\\
&\Rightarrow E[R_T] = 0.25 \times 14\% + 0.40 \times 12\% + 0.35 \times 11\%\\
&\Rightarrow E[R_T] = 12.15\%
\end{aligned}$$

Volatility of Tangency Portfolio: We have:

$$R_T = 0.25R_A + 0.40R_B + 0.35R_C$$

The stock returns are un-correlated with each other.

$$\mathrm{cov}(R_j, R_k) = 0 \quad for \quad j \neq k$$

Hence:

$$\begin{aligned}
\mathrm{var}[R_T] &= \mathrm{cov}(R_T, R_T) \\
&= \mathrm{cov}(0.25R_A + 0.40R_B + 0.35R_C,\ 0.25R_A + 0.40R_B + 0.35R_C) \\
&= 0.25^2 \underbrace{\mathrm{cov}(R_A, R_A)}_{=\sigma_A^2} + 0.25 \times 0.40 \underbrace{\mathrm{cov}(R_A, R_B)}_{=0} + 0.25 \times 0.35 \underbrace{\mathrm{cov}(R_A, R_C)}_{=0} \\
&\quad + 0.40 \times 0.25 \underbrace{\mathrm{cov}(R_B, R_A)}_{=0} + 0.40^2 \underbrace{\mathrm{cov}(R_B, R_B)}_{=\sigma_B^2} + 0.40 \times 0.35 \underbrace{\mathrm{cov}(R_B, R_C)}_{=0} \\
&\quad + 0.35 \times 0.25 \underbrace{\mathrm{cov}(R_C, R_A)}_{=0} + 0.35 \times 0.40 \underbrace{\mathrm{cov}(R_C, R_B)}_{=0} + 0.35^2 \underbrace{\mathrm{cov}(R_C, R_C)}_{\sigma_C^2} \\
&= 0.25^2 \times \sigma_A^2 + 0.40^2 \times \sigma_B^2 + 0.35^2 \times \sigma_C^2 \\
&= 0.25^2 \times 0.004 + 0.40^2 \times 0.002 + 0.35^2 \times 0.002 \\
&= 0.000815
\end{aligned}$$

Therefore:

$$sd[R_T] = +\sqrt{0.000815} \approx 0.0285$$

The volatility of the tangent portfolio is (approximately) 2.85%.

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B. Suppose the CAPM holds true and these are the only three risky assets in the economy. If stock-A has a market capitalization of \$100 million, what are the market caps of stock-B and stock-C? [1-million = 1,000,000 = 10^6].

Solution: If the CAPM is true then the tangency portfolio is the value weighted market portfolio. That means, weight of any stock on the tangency portfolio is proportional to its market capitalization. We can easily calculate the proportionality constant from the given data of stock "A":

$$\omega_A \propto V_A \quad \Rightarrow \quad \omega_A = K \cdot V_A \quad \Rightarrow \quad K = \frac{\omega_A}{V_A}$$

$$\Rightarrow K = \frac{0.25}{100 - million}$$

Here K is the proportionally constant. Therefore market capitalization of stock-B is:

$$V_B = \frac{\omega_B}{K} = 0.40 \times \frac{100 - million}{0.25} = 160 - million$$

And, market capitalization of stock-C is:

$$V_C = \frac{\omega_C}{K} = 0.35 \times \frac{100 - million}{0.25} = 140 - million$$

$\blacksquare$

<u>Quick Check</u>: We know from theory:

$$\omega_j = \frac{V_j}{V_A + V_B + V_C}, \quad j = A, B, C$$

Therefore we must have:

$$K = \frac{1}{V_A + V_B + V_C}$$

We can easily verify it. In units of (million \$)$^{-1}$, we have $K = (0.25/100) = 0.0025$. Now $(V_A + V_B + V_C)$ = (\$100 + \$160 + \$140) = \$400-million. Therefore, in the units of (million \$)$^{-1}$, $1/(V_A + V_B + V_C) = 1/400 = 0.0025$.

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