Portfolio Choice with Illiquid Assets*

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PRELIMINARY AND INCOMPLETE

Abstract

We investigate how the inability to continuously trade an asset affects portfolio choice. We develop a tractable model that extends the Merton framework to include an illiquid asset that can only be traded at infrequent, stochastic intervals. Because consumption is financed through liquid wealth only, the presence of illiquidity leads to increased and state-dependent risk aversion. Illiquidity leads to under-investment in both the liquid and illiquid risky asset, relative to the standard Merton (1969) case. We find that the welfare cost of illiquidity is quantitatively important, and is mostly driven by the uncertainty regarding the length of the illiquidity period. We extend the model to allow for state-dependent probabilities of trade and derive implications about the pricing of risk associated with a rise in illiquidity – a liquidity crisis.

JEL Classification: G11, G12

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1 Introduction

Investors seeking to buy or sell assets that are not traded on centralized exchanges can face substantial difficulty in finding a counterparty or an opportunity to trade. From the investor's perspective, this inability to trade continuously represents an additional source of risk that cannot be hedged. This inability to find a willing counter-party every instant can arise for several reasons. First, trading the asset may require specialized knowledge that is in limited supply, as is the case for certain securitized fixed income and structured credit products. Second, the asset may have unique characteristics, so it may be time-consuming to find an investor willing to trade in this particular asset, for example as in real estate or certain private equity investments. Third, private equity and venture capital limited partner investments have uncertain exit and re-investment timing because the timing of the exit from the underlying investments is uncertain. Finally, markets sometimes shut down, as was the case for many fixed income markets that froze during the financial crisis in 2008/09.[1] Thus, even if an investor so desires, certain assets cannot be traded or liquidated for significant periods of time.

We view the inability to trade frequently as one of the defining characteristics of liquidity, and we investigate its effects on asset allocation. We develop a tractable model of illiquidity by extending the Merton (1973) portfolio choice framework to allow for an illiquid asset that can only be traded at infrequent and stochastically occurring times. We interpret these stochastically occurring trading times as the outcome of an un-modeled search process: the investor must find an appropriate counter-party, and such counter-parties are either not freely identifiable or are difficult to locate. We model the arrival of trading opportunities as an i.i.d. Poisson process, and so the waiting time before a counter-party is found is random. As a result, the investor is exposed to an additional source of risk that cannot be hedged.

Illiquidity affects the portfolio choice problem in two important ways. First, the in-

[1] This is not simply a question of a seller reducing prices to a level where a buyer is willing to step in. As Tirole (2011) and Krishnamurthy, Nagel, and Orlov (2011) comment, there were no bids, at any price, representing "buyers' strikes" in certain markets where whole classes of investors simply exited markets.

vestor's immediate obligations (consumption or payout) can only be financed through liquid wealth, implying that liquid and illiquid wealth are imperfect substitutes. If the investor's liquid wealth drops to zero, these obligations cannot be met until after the next rebalancing opportunity. As a result, the investor is willing to reduce her allocation to both the liquid and illiquid risky assets in order to minimize the probability that a state with zero liquid wealth (as opposed to zero total wealth) is reached. This concern corresponds to real-world situations where investors or investment funds are insolvent, not because their assets under management have hit zero, but because they cannot fund their immediate obligations.

Second, the additional uncertainty associated with the next opportunity to trade reduces the allocation to the illiquid asset. In particular, the distribution of waiting times until the next opportunity to trade is unbounded. Compared to the case when the length of the illiquidity period is deterministic, the investor needs to allocate a substantially higher fraction of her wealth into the risk-free asset in order to ensure the same minimum level of consumption until the next opportunity to trade. Hence, the fact that the ability to trade arrives at random intervals dampens the investor's ability to smooth consumption, leading to substantial welfare costs.

We find that the effect of illiquidity on welfare can be economically large. We compare the investor's optimal allocation in the presence of illiquidity and with a two-risky-asset Merton (1969, 1971) economy, where all assets can be traded continuously. We find that for realistic parameter values, an investor would be willing to trade $10\% - 15\%$ of her wealth in order to make the illiquid asset fully liquid.

Next, we explore the determinants of the welfare cost of illiquidity. First, we disentangle the effect of risk aversion and intertemporal substitution (EIS) by extending the baseline model to allow for preferences that are not separable over time. We find that the welfare cost of illiquidity is highest for agents that unwilling to substitute across time (low EIS) but are willing to substitute across states (low risk aversion). We find that the amount of investment in the illiquid asset is primarily dictated by risk aversion. Then, since a very risk averse agent is unlikely to invest in the illiquid risky asset anyway, she faces lower welfare costs of illiquidity. In contrast, a low EIS investor dislikes states of the world where illiquid wealth, and hence current consumption, is low relative to total wealth and future consumption. Since the EIS has a quantitatively minor impact on the allocation to the illiquid asset, lowering the EIS while holding risk aversion constant increases the welfare cost to the investor.

Second, we explore whether the cost of illiquidity is mostly due to the inability to trade continuously for a certain period or instead due to the fact that the length of this period is uncertain. To do so, we solve a version of the model where the rebalancing interval is deterministic, and we compare welfare and portfolio policies between the stochastic and deterministic cases. We find that both the allocation to the risky assets and investor welfare are lower in the stochastic case. Furthermore, varying the length of the illiquidity period has a major impact on welfare and optimal policies in the stochastic case, but only a minor effect in the deterministic case. Hence, our results suggest that a major component of the cost of illiquidity is the uncertainty associated with the next opportunity to trade.

Our model has rich implications for asset allocation. First, the investor should invest a substantially lower fraction of their wealth in illiquid assets. A standard calibration indicates that if turnover averages once a year, the investor should cut her investment in the illiquid asset by 33% relative to an otherwise identical but fully liquid asset. Second, the investor should reduce her holdings of *liquid* assets. In contrast to standard Merton-style portfolio models, our model implies that the optimal level of investment in the liquid risky asset is affected by the illiquid asset holdings, even when the two assets are uncorrelated or the investor has logarithmic preferences. Third, the investor should be prepared for large, skewed changes in the relative value of illiquid to liquid holdings in her portfolio. Illiquid wealth grows on average faster than liquid wealth, even if the liquidity premium is zero, because investors consume out of liquid wealth. As a result, the investor should re-balance to a mixture with fewer illiquid assets than her long run average target when given the opportunity.

In the second part of the paper, we use our model to study aggregate liquidity events. We extend the model to allow for the level of market liquidity – the probability of meeting a counterparty – to vary stochastically over time. We consider the extreme case where there are two regimes. The first regime represents 'normal' times, in which case the second asset is fully liquid. The second regime represents a liquidity event, where now one of the two assets is illiquid and can be traded only at infrequent intervals. Hence, our this model captures a broader spectrum of assets, including some that are liquid most of the time but not during a crisis. We find that the possibility that the second asset becomes illiquid has an effect on the investor's optimal portfolio policies in normal times. The agent underinvests in the potentially illiquid security, in anticipation of a liquidity event.

The possibility of a liquidity event leads to limited arbitrage. The investor is wary in entering 'arbitrage' trades, defined as trading in two perfectly correlated (positive or negative) securities with different Sharpe ratios. Even if the two securities are currently fully liquid, a shift in the level of market liquidity – a liquidity crisis – can leave the investor able to finance consumption only from one leg of the arbitrage trade, hence introducing risk. As a result, the investor will underinvest in arbitrage opportunities, even when realizing them involves no short positions.

Last, we use the model with time varying liquidity to infer the risk premium associated with a systematic liquidity event. We do so by introducing a derivative security in zero net supply that pays off on the arrival of a liquidity event – a liquidity swap – and compute the premium on this security such that the investor's optimal demand is zero. We find that the investor would be willing to pay an annual premium 5 to 12% higher than the likelihood of a liquidity event to receive a dollar in the onset of a crisis.

Our analysis falls into a large literature dealing with portfolio choice with frictions.[2] First,

[2] This literature considers transaction costs (Amihud and Mendelson, 1986; Constantinides, 1986; Vayanos, 1998; Huang, 2003; Lo, Mamaysky and Wang, 2004), the inability to trade arbitrarily large amounts (Longstaff, 2001), market shutdowns (Rogers and Zane, 2002; Kahl, Liu, and Longstaff, 2003; Dai, Li, Liu,

our work is related to papers that restrict the investor's ability to trade continuously (see e.g. Dai, Li, Liu, and Wang, 2010; Longstaff, 2009; and de Roon, Guo and ter Horst,2009). These papers consider deterministic periods of illiquidity. In contrast, the illiquid period in our model is recurring and of stochastic duration, introducing an additional, unhedgeable, source of risk from the investor's perspective. In our numerical solution, we find that the cost of illiquidity is substantially greater if the length of the illiquidity period is unknown. Longstaff (2001) allows investors to trade continuously, but with only bounded variation, which makes illiquid assets partially marketable at all times and the model closer to the literature on transaction costs. Further, in contrast to previous authors studying the effect of discrete trading (e.g. Lo, Mamaysky and Wang, 2004; Gârleanu, 2009), our model features CRRA rather than exponential utility and a choice between liquid and illiquid risky assets. The existence of wealth effects amplifies the cost of illiquidity. Further, by featuring two risky assets our model allows us to study the effect of illiquidity on the portfolio decisions over liquid assets other than cash.

Second, our work is related to the literature on transaction costs (e.g. Constantinides, 1986; Vayanos, 1998), since illiquidity is often viewed as an implicit transaction cost which investors pay when rebalancing. Our work is similar in the sense that in the presence of fixed transaction costs the investor is *unwilling* to rebalance continuously. However, in our setting the investor is *unable* to trade continuously, even at a cost.[3] In the transaction costs model, the shadow cost of illiquidity is bounded by the level of transaction costs. In contrast, in our paper the shadow cost of illiquidity is significant because it is unbounded; liquidity cannot be generated, e.g. a counter-party found, simply by paying a cost.

Third, there are some economic similarities between our setting and one in which there is a jump component in prices, as in Liu, Longstaff and Pan (2003). In particular, in our

and Wang, 2010; Longstaff, 2009; de Roon, Guo and ter Horst, 2009), search frictions associated with finding counterparties to trade (Duffie, Gârleanu and Pedersen, 2005, 2007; Vayanos and Weill, 2008; Lagos and Rocheteau, 2009), and unhedgeable labor income or business risk (Heaton and Lucas, 1996; Koo, 1998).

[3] Our setup corresponds to a situation without a centralized market, where investors need to search for suitable counter-parties.

setting total wealth can drop sharply between rebalancing times since the illiquid asset can be traded at infrequent intervals. However, a key difference between our setting and the jump-diffusion setting of Liu, Longstaff and Pan (2003) is that in our model of illiquidity, risk aversion is time-varying and portfolios drift away from optimal diversification leading to time variation in investment and consumption policies even when returns are i.i.d.

Fourth, our work is related to the literature on unhedgeable human capital risk (see e.g. Heaton and Lucas, 1996; Koo, 1998) in that part of the investor's total wealth cannot be traded, which introduces a motive to hedge using the set of tradeable securities. We differ in that our illiquid asset can be traded, though not frequently.

Last, our paper is related to the "endowment model" of asset allocation for institutional long-term investors made popular by David Swensen's work, *Pioneering Portfolio Management*, in 2000. Swensen's thesis is that highly illiquid markets, such as private equity and venture capital, have large potential payoffs to research and management skill, which are not competed away because most managers have short horizons. Leaving aside whether there are superior risk-adjusted returns in alternative investments, the endowment model does not consider the illiquidity of these investments. Recently, Siegel (2008) and Leibowitz and Bova (2009) recognize that the inability to trade illiquid assets should be taken into account in determining optimal asset allocation weights, but only investigate scenario or simulation-based procedures and do not solve for optimal asset holdings. In addition to economically characterizing the impact of illiquidity risk on portfolio choice, our certainty equivalent calculations are quantitatively useful for investors to take into account the effect of illiquidity on risk-return trade-offs.

2 Baseline Model

Here, we describe the setup of our baseline model.

2.1 Information

The information structure obeys standard technical assumptions. Specifically, there exists a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ supporting the vector of two independent Brownian motions $Z_t = (Z_t^1, Z_t^2)$ and an independent Poisson process (N_t). $\mathcal{P}$ is the corresponding measure and $\mathcal{F}$ is a right-continuous increasing filtration generated by $Z \times N$.

2.2 Assets

There are three assets in the economy: a risk-free bond B, a liquid risky asset S, and an illiquid risky asset P. The riskless bond B appreciates at a constant rate r:

$$dB_t = r\,B_t\,dt \tag{1}$$

The second asset S is a liquid risky asset whose price follows a geometric Brownian motion with drift μ and volatility σ:

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dZ_t^1. \tag{2}$$

The first two assets are liquid and holdings can be rebalanced continuously.

The third asset P is an illiquid risky asset, for which the price process evolves according to a geometric Brownian motion with drift ν and volatility ψ:

$$\frac{dP_t}{P_t} = \nu\,dt + \psi\rho\,dZ_t^1 + \psi\sqrt{1-\rho^2}\,dZ_t^2, \tag{3}$$

where ρ captures the correlation between the returns on the two risky assets.

The illiquid asset P differs from the first two assets B and S in two important ways. The first distinction is that asset P can only be rebalanced at infrequent, stochastic intervals. In particular, the illiquid asset P can only be traded at stochastic times τ, which coincide with the arrival of a Poisson process with intensity λ. Thus, the expected time between rebalancing events is $1/\lambda$. When a trading opportunity arrives, the investor is able to rebalance her holdings of the illiquid asset up to any amount. Note that P_t reflects the fundamental value of the illiquid asset, which varies randomly irrespective of whether trading in the asset is possible.

Our specification of illiquidity is motivated by the literature on search and asset prices, e.g. Duffie, Gârleanu and Pedersen, (2005, 2007). We interpret the illiquid asset P as an asset which is not traded on a centralized exchange. In this case, investors who are willing to trade in this asset need to search for a counterparty. This search process might be time-consuming, since in many cases the number of market participants with the required expertise, capital, and interest in these illiquid assets is small. Hence, the average waiting time $1/\lambda$ captures the expected period needed to find a suitable counterparty to trade the illiquid asset. For instance, investors in private equity, venture capital, and other limited partnerships face an uncertain timing of exit. In addition, if investors wish to trade their partnerships in a secondary market, they must find one of a small number of willing counterparties. Investors in real estate, hedge funds and structured credit products face similar problem when wishing to sell their investments in the secondary market.[4] As a result, investors often need to wait for an indeterminate period before rebalancing their portfolio between liquid and illiquid investments.

The second way in which the illiquid asset P differs from the liquid assets B and S is that it cannot be pledged as collateral for borrowing. Investors can issue non-state contingent debt by taking a short position in the riskless bond B, but they cannot issue risky debt using the illiquid asset as collateral. If investors were allowed to do so, they could convert the illiquid asset into liquid wealth and thus implicitly circumvent the illiquidity friction.[5]

[4] Hedge fund investors may also face problems exiting their investments directly. For instance, Ang and Bollen (2010) illustrate that investors redeeming directly from hedge funds after lockup provisions have expired may face gates, which restrict their withdrawal of capital.

[5] Alternatively, we could re-interpret P as the fraction of illiquid wealth that cannot be collateralized. In the case of real estate, we could interpret the illiquid asset P as the fraction of the value of the property that cannot be used as collateral against a mortgage or a home equity line. This interpretation assumes that the amount that the asset can be collateralized does not vary over time and that the constraint is

Our assumption is motivated by the difficulty of finding a counterparty who is willing to lend cash using illiquid assets as collateral. For instance, Krishnamurthy, Nagel, and Orlov (2011) find evidence suggesting that money market mutual funds, which are the main providers of repo financing, were unwilling to accept private asset-backed securities as collateral between the third quarter of 2008 and the third quarter of 2009.

In summary, we parsimoniously introduce illiquidity risk into a standard Merton (1969, 1971) setting by the addition of one parameter, λ, which controls how often, on average, the illiquid asset can be rebalanced. The illiquidity friction introduces a difference between the investor's liquid and illiquid wealth, since only the former can be used to finance intermediate obligations such as consumption or payout to investors.

Finally, we will assume the standard discount rate restriction from the Merton one-asset model

$$\beta > (1-\gamma)r + \frac{1-\gamma}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2. \tag{4}$$

and that the illiquid asset has at least as high a Sharpe ratio as the liquid asset

$$\frac{\nu - r}{\psi} \geq \frac{\mu - r}{\sigma}. \tag{5}$$

In contrast to the Merton problem, here the investor's problem is well defined even in the case where $|\rho| = 1$ and the illiquid security has a higher Sharpe ratio than the liquid risky asset.

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always binding. We could extend the model to allow the investor to endogenously choose the amount of collateralized borrowing every period, up to a limit. This model is equivalent to a hybrid model of infrequent trading and transaction costs, with similar qualitative effects.

2.3 Investor

The investor has CRRA utility over sequences of consumption, C_t, given by:

$$\mathrm{E}\left[\int_0^\infty e^{-\beta t} U(C_t) dt\right], \tag{6}$$

where β is the subjective discount factor and $U(C)$ is given by

$$U(C) = \begin{cases} \dfrac{C^{1-\gamma}}{1-\gamma} & \text{if} \quad \gamma > 1 \\ \ln(C) & \text{if} \quad \gamma = 1. \end{cases} \tag{7}$$

We focus on the case $\gamma > 1$ and present the results for log utility, $\gamma = 1$, in the appendix. Despite our investor having preferences that exhibit constant relative risk aversion with respect to consumption, we show that her relative risk aversion with respect to wealth is time varying.

The agent's wealth is comprised of two components, liquid and illiquid wealth. The first includes the amount invested in the liquid risky asset and the risk-free asset. The second, which includes the amount invested in the illiquid asset, cannot be immediately consumed or converted into liquid wealth. The joint evolution of the investor's liquid, W_t, and illiquid wealth, X_t, is given by:

$$\frac{dW_t}{W_t} = \left(r + (\mu - r)\,\theta_t - c_t\right) dt + \theta_t \sigma dZ_t^1 - \frac{dI_t}{W_t} \tag{8}$$

$$\frac{dX_t}{X_t} = \nu dt + \psi \rho dZ_t^1 + \psi \sqrt{1-\rho^2} dZ_t^2 + \frac{dI_t}{X_t}. \tag{9}$$

The agent invests a fraction θ of her liquid wealth into the liquid risky asset, while the remainder $(1-\theta)$ is invested in the bond. The agent consumes (C_t) out of liquid wealth, so $c_t = C_t/W_t$ is the ratio of consumption to liquid wealth. When a trading opportunity arrives, the agent can transfer an amount dI_τ from her liquid wealth to the illiquid asset.

Following Dybvig and Huang (1988) and Cox and Huang (1989), we restrict the set of admissible trading strategies, θ, to those that satisfy the standard integrability conditions.

Our first result is that trading risk eliminates any willingness by the investor either to short the illiquid asset or to net borrow in liquid wealth to fund long purchases of the illiquid asset. Thus, without loss of generality, we restrict our attention to solutions with $W_t > 0$ and $X_t \geq 0$:

Proposition 1 *Any optimal policies will have $W > 0$ and $X \geq 0$ a.s.*

Proof. Consumption is out of liquid wealth only and the illiquid asset cannot be pledged, so $W_t \leq 0$ implies zero consumption before the next trading day, leaving the objective function (6) at $-\infty$. For $|\rho| < 1$, $X_t < 0$ implies that under any admissible investment and consumption policy, there is a positive probability that at the next trading time $W_\tau + X_\tau \leq 0$, violating limited liability, implying zero consumption, and leaving the objective function (6) at $-\infty$. For $\rho = 1$, $X_t < 0$ is ruled out by assuming that the illiquid asset has a weakly higher Sharpe ratio than the liquid asset (5). For $\rho = -1$ the investor would invest positive amounts in the illiquid asset X_t and the liquid risky asset. ■

3 Solution

Because markets are not dynamically complete, we use dynamic programming techniques to solve the investor's problem. First, we establish some basic properties of the solution. Then, we compute the investor's value function and optimal portfolio and consumption policies.

3.1 Value function

The agent's value function is equal to the discounted present value of her utility flow,

$$F(W_t, X_t) = \max_{\{\theta, I, c\}} \mathrm{E}_t \left[\int_t^\infty e^{-\beta(s-t)} U(C_s) ds \right]. \tag{10}$$

Our first step is to establish bounds on (10). The trader's value function must be bounded below by the problem in which the illiquid asset does not exist, and the value function must be bounded above by the problem in which the entire portfolio can be continuously rebalanced. We refer to these as the Merton (1969, 1971) one-stock and two-stock problems, respectively. Hence, there exist constants K_{M1} and K_{M2} such that

$$K_{M1}W^{1-\gamma} \leq F(W,X) \leq K_{M2}\,(W+X)^{1-\gamma} \leq 0. \tag{11}$$

Since the Merton one-asset value function exists (4), our value function is bounded between the one-asset solution and the two-asset solution.

Since the utility function is homothetic and the return processes have constant moments, it must be the case that F is homogeneous of degree $1-\gamma$. Thus, there exists a function H with $H(\xi) = F(1-\xi,\xi)$ such that

$$F(W,X) = (W+X)^{1-\gamma}H\left(\xi\right), \quad \text{where} \quad \xi = \frac{X}{X+W}. \tag{12}$$

Hence, the investor's value function can be represented as a power function of total wealth times a function, $H(\xi)$, of her portfolio composition.

The next step involves characterizing the value function at the instant when the agent can rebalance between her liquid and illiquid wealth. When the Poisson process hits and the agent rebalances her portfolio, the value function may discretely jump. Denote the new, higher, value function after rebalancing occurs as F^*, so that the total amount of the jump is $F^* - F$. At the Poisson arrival, the agent is free to make changes to her entire portfolio, and thus we have that

$$F^*(W_t,X_t) = \max_{I\in[-X_t,W_t)} F(W_t - I, X_t + I). \tag{13}$$

Since F^* must also be homogeneous of degree $1-\gamma$, there exists a function H^* such that

$F^* = (W+X)^{1-\gamma} H^* \left(\frac{X}{X+W}\right)$. In addition, since rebalancing the illiquid asset is costless when possible, H^* is a constant function. The homogeneity of the value function implies that when a trading opportunity arrives, the investor rebalances her portfolio so that the fraction of illiquid to total wealth equals $\xi^* = \arg\max H(\xi)$, and $H^* = H(\xi^*)$.

The investor's portfolio and consumption problem can be defined as

Problem 1 (Baseline) *The investor performs the maximization in (10), subject to the two intertemporal budget constraints (8) and (9), with re-balancing ($dI_t \neq 0$) only when the Poisson process N_t^λ jumps.*

The following proposition characterizes the solution to the investor's problem

Proposition 2 (Baseline) *The solution to Problem 1 is characterized the function $H(\xi)$ and constants H^* and ξ^* that satisfy*

$$
\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{1}{1-\gamma} c^{1-\gamma} - \beta\, H(\xi) + \lambda\,(H^* - H(\xi)) + H(\xi)\, A(\xi, c, \theta) + \frac{\partial H(\xi)}{\partial \xi}\, B(\xi, c, \theta) \\
& + \frac{1}{2} \frac{\partial^2 H(\xi)}{\partial \xi^2}\, C(\xi, c, \theta) \Bigg] . \qquad (14)
\end{aligned}
$$

where the functions A, B, and C are given by

$$
\begin{aligned}
A(\xi, c, \theta) \equiv & (1-\gamma)\Big(r + (1-\xi)((\mu - r)\theta - c) + \xi(\nu - r) \\
& - \frac{1}{2}\gamma\left(\xi^2\psi^2 + (1-\xi)^2\sigma^2\theta^2 + 2\xi(1-\xi)\psi\rho\sigma\theta\right)\Big) \qquad (15)
\end{aligned}
$$

$$
B(\xi, c, \theta) \equiv \xi(1-\xi)\left(\nu - (r + (\mu - r)\theta - c) + \gamma\psi\theta\rho\sigma(2\xi - 1) + \gamma\theta^2\sigma^2(1-\xi) + \gamma\psi^2\xi\right) \qquad (16)
$$

$$
C(\xi, c, \theta) \equiv \xi^2(1-\xi)^2\left(\theta^2\sigma^2 + \psi^2 - 2\psi\theta\rho\sigma\right) \qquad (17)
$$

and

$$
H^* = \max_{\xi} H(\xi) \qquad (18)
$$

$$
\xi^* = \arg\max_{\xi} H(\xi) \qquad (19)
$$

When a trading opportunity occurs at time τ, the trader selects I_τ so that $\frac{X_\tau}{X_\tau + W_\tau} = \xi^*$.

Lemma 3 *$H(\xi)$ is concave.*

Proof. Define $Q = X + W$ to be total wealth, and let $\{Q_0, X_0^1\}$ and $\{Q_0, X_0^3\}$ be two initial values with the associated optimal policies $\{C^1, \pi^1\}$ and $\{C^3, \pi^3\}$ where $\pi = \theta W$. For $\kappa \in (0, 1)$, we consider a middle initial value $\{Q_0, X_0^2 = \kappa X_0^1 + (1-\kappa) X_0^3\}$ and the associated (possibly optimal) policies $\{C^2 = \kappa C^1 + (1-\kappa) C^3, \pi^2 = \kappa \pi^1 + (1-\kappa)\pi^3\}$, which are feasible because of the linearity of the budget constraint. From (8) and (9), we have

$$dQ_t = [rQ_t + (\mu - r)\pi_s + (\nu - r)X_t - C_t]\, dt + [\pi_s \sigma + \psi \rho X_t]\, dZ_t^1 + \psi X_t \sqrt{1-\rho^2} dZ_t^2 \tag{20}$$

for any time t. Thus, from the construction of our initial values and optimal policies, we have $Q_t^2 = \kappa Q_t^1 + (1-\kappa) Q_t^3$. Next, consider the objective function

$$\mathrm{E}\left[\int_0^\infty e^{-\beta t} U(C_t)\right] = \mathrm{E}\left[\int_0^\tau e^{-\beta t} U(C_t) dt + e^{-\beta \tau} Q_\tau^{1-\gamma} H^*\right] \tag{21}$$

Because $U(C)$ is increasing and concave, we have $U(C_t^2) > \kappa U(C_t^1) + (1-\kappa) U(C_t^3)$. From Jensen's inequality and $H^* < 0$, we have ${Q_\tau^2}^{1-\gamma} H^* > \kappa {Q_\tau^1}^{1-\gamma} H^* + (1-\kappa) {Q_\tau^3}^{1-\gamma} H^*$. Thus, $\mathrm{E}^2\left[\int_0^\infty e^{-\beta t} U(C_t)\right] > \kappa \mathrm{E}^1\left[\int_0^\infty e^{-\beta t} U(C_t)\right] + (1-\kappa)\mathrm{E}^3\left[\int_0^\infty e^{-\beta t} U(C_t)\right]$, and so the value function is concave in X for fixed Q. This is sufficient to show that the value function is concave in ξ for fixed Q, so H is concave. ■

The investor's value function is comprised of two parts. The first part $(W + X)^{1-\gamma}$ captures the effect of total wealth on the continuation utility. The second component $H(\xi)$ captures the effect of wealth composition between liquid and illiquid wealth. The function H is concave and maximized at H^*; deviations from this allocation reduce welfare for two reasons. First, there is the standard effect from lack of optimal diversification. Second, there is an asymmetric effect arising from the fact that consumption is funded by liquid wealth only. In the section below, we explore the second effect in more detail.

3.2 Imperfect Substitutability of Liquid and Illiquid Wealth

Here, we discuss some properties of the solution to provide intuition for the results. We will begin by emphasizing the way our model changes the basic Merton continuous trading intuitions and conclude by describing how those changes depend on the consumption smoothing properties of CRRA utility, as opposed to the CARA and risk-neutral utilities mostly used in prior work.

The solution to our problem differs from the solution to the Merton setup because liquid and illiquid wealth are imperfect substitutes. This non-substitutability is particularly acute when the investor's portfolio has more illiquid than liquid wealth. To understand the implication of this difference for the solution to our problem, we first examine the behavior of the solution at the limits $X \to \infty$ ($\xi \approx 1$) and $X \to 0$ ($\xi \approx 0$); the behavior of the solution at the extremes sheds light into the effect of illiquid holdings on the investor's optimal policies in the interior values of ξ.

Lemma 4 *At the boundaries, the value function satisfies*

$$\lim_{X \to \infty} F(W, X) = K_\infty W^{1-\gamma} \tag{22}$$

$$\lim_{W \to \infty} F(W, X) = 0 \tag{23}$$

and

$$F(W, X = 0) = K_0 \tag{24}$$

$$\lim_{W \to 0} F(W = 0, X) = -\infty \tag{25}$$

where the constants $K_0 < K_\infty < K_{M2} < 0$ *solve*

$$0 = -\beta + \gamma((1-\gamma)K_0)^{-\frac{1}{\gamma}} + (1-\gamma)r + \frac{1}{2}(1-\gamma)\frac{(\mu - r)^2}{\gamma\sigma^2} + \lambda\left(\frac{H^*}{K_0} - 1\right) \tag{26}$$

and

$$K_\infty = \frac{1}{1-\gamma}\left[\frac{1}{\gamma}\left(\beta+\lambda+(\gamma-1)r+\frac{1}{2}(\gamma-1)\gamma\left(\frac{\mu-r}{\gamma\sigma}\right)^2\right)\right]^{-\gamma}. \tag{27}$$

Lemma 4 demonstrates the imperfect substitutability in two ways. First, the investor cannot achieve bliss even with an unboundedly large endowment of illiquid wealth $\lim_{X\to\infty} F(X,W) < 0$. By contrast, the investor can achieve bliss with sufficiently high liquid wealth $\lim_{W\to\infty} F(X,W) = 0$. This result arises because, in contrast to liquid wealth, illiquid wealth cannot be immediately transformed into consumption. The investor needs to wait for an opportunity to trade the illiquid security, and this random delay bounds her welfare away from bliss. Second, the investor can reach states with negative infinite utility, if her liquid wealth is low enough. However, the value function is finite for investors with zero illiquid wealth. This also results follows from the fact that only liquid wealth can be transformed into consumption immediately. The trading delay prevents the investor from immediately accessing her illiquid wealth for consumption.

Next, to obtain intuition about the cost of illiquidity, consider the introduction of a fictitious market, that allows the investor to exchange 1 unit for illiquid wealth for q units of liquid wealth. Between normal rebalancing dates, the investor would be indifferent in participating in this fictitious market as long as

$$q = \frac{F_X}{F_W}. \tag{28}$$

Whenever the investor has the opportunity to rebalance, then $q = 1$. Between rebalancing dates, the relative price q differs from one, depending on whether the investor has too much, or to little illiquid wealth X relative to his desired allocation. The following lemma characterizes the behavior of the relative values of the investor's portfolio as the value of illiquid holdings becomes large:

Lemma 5 *When illiquid holdings are large, the ratio of the value of illiquid to liquid holdings* *tends to zero*

$$\lim_{X\to\infty} \frac{F_X X}{F_W W} = 0. \tag{29}$$

Lemma 5 shows that the relative value of illiquid to liquid wealth becomes arbitrarily small as the investor's allocation to illiquid wealth increases. Specifically, as the investor's illiquid holdings become large $X \to \infty$, the price the investor attaches to his illiquid wealth q tends to zero sufficiently fast, so that the relative value of his entire illiquid portfolio (29) tends to zero.[6] A direct consequence of Lemma 5 is that, as illiquid holdings become large, the investor's marginal utility of consumption is mainly affected by changes in liquid, rather than illiquid wealth:

Lemma 6 *The elasticity of substitution of the marginal utility of consumption between liquid and illiquid wealth tends to zero*

$$\lim_{X\to\infty} \frac{F_{WX} X}{F_{WW} W} = \lim_{X\to\infty} \frac{X \frac{\partial}{\partial X} U'(C)}{W \frac{\partial}{\partial W} U'(C)} = 0. \tag{30}$$

Lemma 6 has strong implications for risk preferences. In particular, even if changes in illiquid and liquid wealth are correlated, the investor treats them as separate gambles. Hence, the investor cannot use risks taken with illiquid wealth to offset risks taken with liquid wealth. As we show in Section **??** below, this imperfect substitutability implies that hedging demands will be zero when illiquid wealth becomes large.

The results of this section illustrate that, as a direct consequence of the inability to trade continuously, the investor treats his liquid and illiquid holdings as imperfect substitutes. In the limit where his illiquid wealth becomes large, the elasticity of substitution between them tends to zero. To gain further intuition about why liquid and illiquid wealth are imperfect

[6]This behavior is in contrast with the standard Merton problem in which the investor can freely rebalance. In this case, the relative price is constant, implying that $\lim_{X\to\infty} \frac{F_X X}{F_W W} = \infty$.

substitutes, consider the following approximation to the value function

$$
\begin{aligned}
F(X_t, W_t) &= \mathrm{E}_t \left[\int_t^\tau e^{-\beta(s-t)} U(C_s) ds + e^{-\beta(\tau - t)} F^*(W_\tau, X_\tau) \right] \\
&\approx K_\infty W_t^{1-\gamma} + (K_0 - K_\infty)(W_t + X_t)^{1-\gamma} \qquad (31)
\end{aligned}
$$

The investor's continuation value can be decomposed into two parts: i) the utility she derives from consumption until the next rebalancing date τ; and ii) her continuation value $F^*(W_\tau, X_\tau)$ thereafter. These two parts correspond approximately to the two components of equation (31).[7] The first component in equation (31) corresponds to the part of the value function capturing the utility of consumption until the next trading day. This part depends only on liquid wealth, W, because the investor can only instantaneously consume out of her liquid holdings. The second term in equation (31) corresponds to the investor's continuation value immediately after the next trading time. At that instant, the investor can freely convert her illiquid holdings into liquid assets and vice versa.

Equation (31) sheds light into the non-substitution results in Lemmas 4, 5, and 6. In particular, illiquid wealth affects the level of the value function only through the continuation value F^* at the next trading time $t = \tau$. In contrast, liquid wealth can fund consumption both before and after τ. Hence, liquid and illiquid wealth are not perfect substitutes. When the illiquid endowment is large $X \gg W$, this non-substitutability is particularly acute, since variation in liquid wealth becomes unimportant for long-run consumption, and the value function becomes separable in X and W:

$$
F(X, W) \underset{X \gg W}{\approx} K_\infty W^{1-\gamma} + (K_0 - K_\infty) X^{1-\gamma}.
$$

[7] In general, the weights of the exact solution on these two components are not constant but depend on ξ. The approximation is exact for $X = 0$ and $X = \infty$ and reasonably accurate for intermediate values using our parameters. This approximation generates a mean squared relative error – weighted by the invariant distribution of $X/(X + W)$ – of less that 1%. While this is a good approximation for the level of the value function, it cannot necessarily be used to generate good approximations of the optimal policies.

In this case, liquid wealth W is only used to fund immediate consumption, while illiquid wealth is used to fund future consumption. Since consumption preferences are time separable, so is the value function. As a consequence, when X is large, the hedging demand disappears and the correlation between the liquid and illiquid asset returns does not matter for portfolio allocation.

The approximation (31) also makes it clear why the agent cannot achieve bliss through an increasing allocation of the illiquid asset:

$$\lim_{X\to\infty} F(X,W) < 0 = \lim_{W\to\infty} F(X,W).$$

The first term in equation (31) bounds the value function away from zero for large values of X: the illiquid asset cannot be used to fund immediate consumption and illiquid wealth is inaccessible until after the first trading time. In contrast, the value function is not bounded away from zero for large values of W because liquid wealth can be used for consumption today.

The approximation demonstrates how illiquidity creates additional high-marginal-utility states. In contrast to the standard Merton model, the investor's marginal value of wealth in our model is high in two types of states: states where *total* wealth is low and states where *liquid* wealth is low. Even if the investor has substantial total wealth, if her liquid wealth is low, she cannot fund immediate consumption, leading to high marginal utility. As a result, the investor is concerned with smoothing not only her total wealth $W + X$, but also her liquid wealth W. These concerns lead to underinvestment in the illiquid *and* the liquid risky assets. Last, the results of this section illustrate the importance of wealth effects, which are absent from most of the model with search frictions.[8]

[8]Most of these models feature investors that are either risk neutral or have CARA preferences. Risk neutral investors have no need to smooth, and so the effects we outline do not exist. CARA investors do not have Inada conditions at zero and hence do not face the second set of high marginal utility states – states with low liquid wealth.

3.3 Numerical solution and calibration

We solve the problem numerically using value function iteration. See the appendix for more details.

3.4 Optimal Portfolio Policies

In this section we characterize the investor's optimal asset allocation and consumption policies. Even though the investment opportunity set is constant, the optimal policies vary over time and depend on the amount of illiquid assets held in the investor's portfolio.

Participation

Before characterizing the optimal allocation, we first find sufficient conditions for the investor to have a non-zero holding of the illiquid asset:

Corollary 7 *An investor prefers holding a small amount of the illiquid asset to holding a zero position if and only if*

$$\frac{\nu - r}{\psi} \geq \rho \frac{\mu - r}{\sigma}. \tag{32}$$

The condition for participation is identical to the Merton two-asset case and depends only on the mean-variance properties of the two securities. Somewhat surprisingly, the degree of illiquidity λ does not affect the decision to invest a small amount in the illiquid asset because of the infinite horizon of the agent: a trading opportunity will eventually arrive where the illiquid asset can be converted to liquid wealth and eventual consumption. However, even though the conditions for participation are the same as the standard Merton case, the optimal holdings of the illiquid and liquid assets are quite different, as we discuss next.

Illiquid Asset Holdings

Illiquidity induces underinvestment in the illiquid asset. In Table 2, we present the investor's optimal rebalancing point ξ^* along with summary statistics for the distribution of portfolio holdings for different values of λ. The optimal holding of illiquid assets at $\lambda = 1$ when rebalancing is possible is 0.37, which is lower than the optimal two-asset Merton holding at 0.60.

In addition to underinvestment in the illiquid asset, the inability to trade means that the investor can be away from optimal diversification for a long time. Figure 6 plots the stationary distribution of an investor's holding of the illiquid asset, ξ. For most of the time – the 20% to 80% range – the share of wealth allocated in illiquid securities is 0.36 to 0.45, while the 1% to 99% range is 0.30 to 0.65. Furthermore, the distribution in Figure 6 is positively skewed, since illiquid wealth grows faster on average than liquid wealth. Even though both risky assets have the same mean return, $\mu = \nu = 0.12$, liquid wealth is only partially allocated to the risky asset and consumption is taken out only from liquid wealth. As a result of this skewness, the investor chooses a rebalancing point *lower* than the mean of the steady-state distribution of portfolio holdings. When $\lambda = 1$, the mean holding is 0.41, compared to a rebalancing value of 0.37, while the distribution of portfolio holdings has a normalized skewness coefficient of 1.9.

Liquid Asset Holdings

In addition to underinvestment in the illiquid asset, illiquidity affects the investment in the liquid asset. Taking the first order condition from the investor's value function, the allocation to the liquid risk asset as a fraction of the investor's liquid holdings is equal to

$$\theta_t = \frac{\mu - r}{\sigma^2}\left(-\frac{F_W}{F_{WW}W_t}\right) + \rho\frac{\psi}{\sigma}\left(-\frac{F_{WX}X_t}{F_{WW}W_t}\right). \tag{33}$$

The investor's allocation to the liquid asset as a function of her *total* wealth is equal to $\theta W/(W+X)$.

There are two aspects of the optimal policy that merit attention. First, even in the case where the liquid and illiquid asset returns are uncorrelated, $\rho = 0$, the allocation to the liquid asset differs from the Merton benchmark. In contrast to the two-asset Merton problem, W and X are not interchangeable. In Panel *a* of Figure 5, we compare the curvature of the investor's value function with respect to liquid wealth $-F_{WW}W/F_W$ (black line) to that of a Merton investor (grey line). We see that for low values of allocation to illiquid assets, the two behave in a similar fashion: as the share of liquid wealth W declines in the investor's total wealth $W+X$, so does the investor's aversion to gambles in W. However, when the investor's liquid wealth becomes sufficiently low, the two lines diverge. The investor in our problem becomes much more averse to taking gambles in terms of liquid wealth than a Merton investor. Hence, her *effective* risk aversion increases.

Second, in the case where the liquid and illiquid asset are correlated, $\rho \neq 0$, there is an additional element that influences the demand for the liquid asset, namely the desire to hedge changes in the value of the illiquid asset. The strength of this motive depends on the strength of the correlation, ρ, and the elasticity of substitution between liquid and illiquid wealth $-F_{WX}X/F_{WW}W$. In Panel *b* of figure 5 we plot the second component for the demand for the liquid risky asset $-F_{WX}X/F_{WW}W$ (black line) and contrast it to the term corresponding to a Merton investor, which reduces to $-X/W$ (grey line), for the case where $\rho = 0.6$. Again, we see that for low values of X relative to total wealth, the two lines are very similar. However, we see that the two diverge dramatically as X increases relative to W. In the Merton case, this term goes to minus infinity. In our case, the term $-F_{WX}X/F_{WW}W$ converges to zero: at the limit, the investor does not consider illiquid wealth to be a substitute for liquid wealth. This striking behavior is a direct consequence of Lemma 6: the hedging motive disappears when illiquid securities comprise the agent's portfolio $X \to \infty$. In the limit where the agent holds large amount of illiquid wealth, her main use of the liquid asset is smoothing consumption rather than hedging fluctuations in her illiquid portfolio.

A direct consequence of the results discussed above is that at the limits where either illiquid holdings are zero ($X \to 0$) or very high ($X \to \infty$), the division of liquid assets between the stock and the bond, as a fraction of liquid wealth, are the same as the Merton benchmark. This pattern occurs *irrespective* of the degree of correlation between the two risky securities.

Lemma 8 *The agent's optimal investment policy is such that for any ρ*

$$\lim_{X \to 0} \theta(W, X) = \lim_{X \to \infty} \theta(W, X) = \frac{\mu - r}{\gamma \sigma^2}. \tag{34}$$

Figure 7 plots the agent's optimal allocation to the liquid risky asset as a function of the share of wealth in illiquid assets. In general, the investor invests a higher fraction of her *liquid* wealth into the risky asset relative to the Merton benchmark. However, as a fraction of her *total* wealth, the investor usually under-allocates to the liquid risky asset relative to the Merton benchmark. The agent partially compensates for the risk of being unable to trade the illiquid asset for a long period of time by underinvesting in the liquid risky asset.

Effect of correlation on optimal asset holdings.

In the Merton case, varying the correlation coefficient has a dramatic effect on portfolio policies, especially when the two securities have different Sharpe ratios. To illustrate this effect, we set $\nu = 0.2$ and compute the optimal allocation to the liquid and illiquid asset in the Merton case. As we see in Figure 11a, the portfolio policies are very sensitive to the correlation coefficient ρ; as ρ approaches one, the investor takes large offsetting positions in the two assets. As we vary ρ from zero to one, the allocation to the two risky assets varies from @@ and @@ to XX and XX respectively.

By contrast, when one of the assets is illiquid, then the two risky assets are not perfect substitutes. As a result, the optimal portfolio policies are fairly insensitive to ρ, as we see in Figure 11b. In particular, as we vary ρ from zero to one, the allocation to the illiquid and liquid asset vary from @@ and @@ to XX and XX respectively.

3.5 Consumption

The inability to trade the illiquid asset for long periods of time also affects the agent's optimal consumption policy. Since consumption is financed by liquid wealth, the investor's optimal consumption choice satisfies

$$U'(C) = F_W(W, X). \tag{35}$$

In contrast to the Merton setting, the investor equates the marginal utility of consumption with the marginal value of *liquid*, as opposed to *total* wealth. Using the form for the value function (12), the investor's consumption to total wealth ratio equals

$$\frac{C}{W+X} = \Big((1-\gamma)H(\xi) - H'(\xi)\xi\Big)^{-\frac{1}{\gamma}}. \tag{36}$$

Figure 8 plots the agent's optimal consumption to total wealth ratio as a function of the illiquid asset's share ξ of the agent's wealth. For comparison, we also plot the one- and two-asset Merton consumption levels. As we see, the investor always consumes a lower fraction of her total wealth than the two-asset Merton benchmark. In contrast to the Merton problem, the consumption to wealth ratio is time-varying. In particular, the marginal value of liquid wealth varies with the current allocation to the illiquid security, hence, so does the investor's optimal consumption policy.

The investor's consumption policy sheds further light into the behavior of the marginal value of liquid wealth F_W, in light of the discussion of imperfect substitution between liquid and illiquid wealth in Section 3.2. When the share of illiquid assets in the portfolio is small, the share of total wealth consumed is insensitive to portfolio composition ξ. In this region, the investor smooths lifetime consumption by consuming a higher fraction of liquid welath today as ξ increases. In this case, liquid and illiquid wealth are substitutes $F_{WX}/F_{WW} \approx -1$, since the investor's valuation of liquid wealth falls as illiquid wealth increases. In contrast, as the share of illiquid assets ξ increases towards one, the fraction of total wealth consumed drops to zero. When illiquid wealth is large, her marginal value of liquid wealth increases, leading to a lower consumption to total wealth ratio. As a result, as we see in Lemma 6, when illiquid wealth is large the investor's marginal utility of consumption becomes insensitive to illiquid asset holdings X.

3.6 The welfare cost of illiquidity

Here, we quantify the cost of illiquidity on the investor's welfare. In particular, we compute the fraction of initial wealth α the investor would be willing to give up at time zero, in order to be fully able to trade the illiquid asset

$$K_{M2}\left((W_t + X_t)(1-\alpha)\right)^{1-\gamma} = (W_t + X_t)^{1-\gamma} \int_0^1 H(\xi)\pi(\xi)d\xi, \tag{37}$$

where the left hand side of equation (37) is the value function of a Merton investor able to invest in two risky securities, and $\pi(\xi)$ is the stationary distribution of ξ under the optimal policies.

4 Determinants of the cost of illiquidity

Here, we explore the two key determinants of the cost of illiquidity. First, we explore the impact of preference parameters, in particular the coefficient of risk aversion and the elasticity of intertemporal substitution. Second, we disentangle the effect of illiquidity from illiquidity risk, by comparing our setup to a model with deterministic periods of illiquidity.

4.1 The effects of risk aversion versus the elasticity of intertemporal substitution

The discussion in the previous section illustrates that the curvature of the value function is an important determinant of the cost of illiquidity. In particular, the investor fears the probability that she reaches states with very low liquid wealth for two reasons. First, states with low liquid wealth are states with low consumption, and the investor likes to smooth consumption across *states*; the coefficient of risk aversion captures the magnitude of this preference. Second, if the agent reaches states where liquid wealth is low relative to her total wealth, she faces a steeply increasing consumption profile. The agent dislikes these states because she wants to have smooth consumption paths over *time*, and the elasticity of intertemporal substitution governs this preference.

Here, we disentangle whether the cost of illiquidity is mainly driven by the desire to smooth consumption across states versus across time. A feature of power utility preferences is that these two effects are linked. Here, we investigate these two motives separately. To disentangle the effects of risk aversion from the elasticity of intertemporal substitution, we consider the case where the agent has recursive preferences, as in Duffie and Epstein (@@@@).

The agent maximizes a utility index J, which is defined recursively according to

$$J_t = E_t \int_t^{\infty} f(C_s, J_s)\, ds, \tag{38}$$

where

$$f(C, J) \equiv \frac{\beta}{1-\zeta} \left(\frac{C^{1-\zeta}}{((1-\gamma)J)^{\frac{\gamma-\zeta}{1-\gamma}}} - (1-\gamma)\, J \right). \tag{39}$$

Equation (39) represents the continuous-time equivalent of Epstein and Zin (1989), where β is the subjective discount rate, γ is the coefficient of risk aversion and ζ is the inverse of the elasticity of intertemporal substitution. The case of power utility corresponds to $\gamma = \zeta$.

Problem 2 (Epstein-Zin) *The investor maximizes*

$$F_{ez}(W_t, X_t) = \max_{\{\theta, I, c\}} \mathrm{E}_t \left[\int_t^{\infty} f(C_s, J_s) ds \right], \tag{40}$$

subject to the budget constraints (8) and (9), with re-balancing ($dI_t \neq 0$) only when the Poisson process N_t^{λ} jumps.

As in Problem 1, the value function is homothetic and so

$$F_{ez}(W, X) = (W + X)^{1-\gamma} H_{ez}(\xi), \quad \text{where} \quad \xi = \frac{X}{X + W}.$$

The following proposition characterizes the solution:

Proposition 9 (Epstein-Zin) *The solution is characterized by the function $H_{ez}(\xi)$ and constants ${H_{ez}}^*$ and ${\xi_{ez}}^*$ such that*

$$\begin{aligned} 0 \;=\; & \max_{c, \theta} \left\{ \frac{\beta}{1-\zeta} \left(c^{1-\zeta} \left((1-\gamma) H_{ez}(\xi) \right)^{\frac{\zeta-\gamma}{1-\gamma}} - (1-\gamma) H_{ez}(\xi) \right) + \lambda \left({H_{ez}}^* - H_{ez}(\xi) \right) \right. \\ & \left. + H_{ez}(\xi)\, A(\xi, c, \theta) + \frac{\partial H_{ez}(\xi)}{\partial \xi}\, B(\xi, c, \theta) + \frac{\partial^2 H_{ez}(\xi)}{\partial \xi^2}\, C(\xi, c, \theta) \right\}. \end{aligned} \tag{41}$$

and

$${H_{ez}}^* = \max_{\xi} H_{ez}(\xi) \tag{42}$$

$${\xi_{ez}}^* = \arg\max_{\xi} H_{ez}(\xi) \tag{43}$$

where the functions A, B and C are defined in (15)-(17).

When a trading opportunity occurs at time τ, the trader selects I_τ so that $\frac{X_\tau}{X_\tau + W_\tau} = {\xi_{ez}}^*$.

In Figures 3 and 4 we plot the optimal consumption and portfolio policies as a function of illiquid holdings ξ for different values of risk aversion and the EIS respectively. As we see in Figure 3, an increase in the agent's risk aversion results in a level shift in both the consumption policy c and the allocation to the liquid asset θ, regardless of her current allocation to the illiquid asset ξ. In contrast, as we see Figure 4, as the agent's desire to smooth consumption across time increases (lower EIS), her consumption policy becomes less sensitive to ξ. When her allocation to illiquid assets is high ξ, the agent consumes a *higher* fraction of her total wealth than an agent with high EIS. The low EIS agent is concerned more with smoothing consumption across time (before and after rebalancing), and worries less about the probability her liquid wealth drops to zero. Last, the EIS has a small effect on the optimal allocation to the liquid asset θ.

To quantify the impact of risk aversion and the elasticity substitution on portfolio policies, we report the rebalancing point ξ^*, the average consumption $E(c)$ and liquid risky asset allocation $E(\theta)$. In addition, we compute the welfare cost of illiquidity, defined in Section @@. We vary the coefficient of risk aversion, the EIS and the probability of trading λ. For comparison, and in an abuse of notation, we report the consumption and portfolio policies for an investor able to continuously trade one ($\lambda = 0$) and two ($\lambda = \infty$) risky assets.

First, we vary the elasticity of intertemporal substitution $1/\zeta$ and the likelihood of trading λ, holding the coefficient of risk aversion fixed, and we show the results in Table 5. We find that the elasticity of intertemporal substitution has a quantitatively small impact on portfolio policies. As we see in Panels a and d, varying the EIS from 1.5 to 1/6 has essentially no impact on the allocation to the liquid asset, and a small impact on the allocation to the illiquid asset. In contrast, as we see in Panel c, varying the EIS has an impact on the investor's optimal consumption policy. When the investor's desire to smooth consumption across states increases ($EIS = 1/6$), her optimal fraction of wealth consumed declines faster with λ than when her elasticity of substitution is high. Hence, we find that low EIS magnifies the cost of illiquidity to the investor. This increased cost is reflected to the increased premium she is willing to pay to make the second asset fully liquid, as we see in Panel b.

Second, in Table 6 we vary the coefficient of risk aversion γ and the likelihood of trading λ, holding the EIS $1/\zeta$ fixed. We find that risk aversion affects both the optimal consumption policy and portfolio policies. In particular, as risk aversion increases, the investor's policies ξ^*, c and θ become closer to a Merton investor that can trade two liquid assets. For instance, in the case where $\lambda = 10$, and investor with a risk aversion of $\gamma = 3$ would allocate 64% of her total wealth to the illiquid asset, compared to 118% for a Merton investor. In contrast, if the investor had a risk aversion of $\gamma = 15$, she would allocate only 21% of her total wealth to the illiquid asset, compared to 24% for a Merton investor. Hence, the cost of illiquidity is particularly acute for an investor with *low* risk aversion, since she otherwise would be investing more in the illiquid security than an investor with high risk aversion. As a result, we see in Panel b that the welfare cost of illiquidity *decreases* with risk aversion.

In summary, we find that the welfare cost of illiquidity increases with the desire to smooth consumption across *time* (ζ increases) and decreases with the desire to smooth consumption across *states* (γ increases). The cost of illiquidity is that it leads to non-smooth consumption paths across states and across time. Holding portfolio allocations constant, the cost would be higher for more risk averse agents that are also reluctant to substitute across time. However, the amount of investment in the illiquid asset is primarily dictated by risk aversion. Hence, holding risk aversion constant, lowering the EIS increases the welfare cost to the investor. In contrast, since a very risk averse agent is unlikely to invest in the illiquid risky asset anyway, she faces lower welfare costs of illiquidity.

4.2 Stochastic versus deterministic illiquidity

The discussion in Section 3.2 illustrates that the risk of liquid wealth dropping to zero is a substantial cost of investing in illiquid assets. Our model features an illiquid period whose length is uncertain. Here, we disentangle the cost of illiquidity into two components. The first component is the lack of continuous rebalancing over a specific period of time – the notion of illiquidity. The second component is that the length of the illiquidity period is stochastic – the notion of illiquidity *risk*. Here, we show that the second, risk component is a major determinant of the cost of illiquidity.

We consider the case where the agent is allowed to rebalance her portfolio at regular intervals spaced T apart. In this case the length of the illiquidity period is known in advance. The investor's problem is thus

Problem 3 (Deterministic) *The investor maximizes*

$$F_T(W_t, X_t) = \max_{\{\theta, I, c\}} \mathrm{E}_t \left[\int_t^\infty U(C_s) ds \right], \tag{44}$$

subject to the budget constraints (8) and (9), with re-balancing ($dI_t \neq 0$) only at $\tau = 0, T, 2T, ...$

As in Problem 1, the value function is homothetic and so

$$F_T(W, X) = (W + X)^{1-\gamma} H_T(\xi), \quad \text{where} \quad \xi = \frac{X}{X + W}.$$

The following proposition characterizes the solution:

Proposition 10 (Deterministic) *The function $H_T(t, \xi)$ satisfies*

$$\begin{aligned} 0 \;\; = \;\; & \max_{c,\theta} \left\{ \frac{1}{1-\gamma} c^{1-\gamma} - \beta H_T(t,\xi) + \frac{\partial H_T(t,\xi)}{\partial t} + H_T(t,\xi)\, A(\xi, c, \theta) + \frac{\partial H_T(t,\xi)}{\partial \xi}\, B(\xi, c, \theta) \right. \\ & \left. + \frac{1}{2} \frac{\partial^2 H_T(t,\xi)}{\partial \xi^2}\, C(\xi, c, \theta) \right\}. \end{aligned}$$

where the functions A, B and C are defined in (15)-(17).

At the repeated trading times $t = \tau = 0, T, 2T, ...,$ the investor will optimally rebalance her portfolio so that

$$H_T(\tau, \xi) = \max_{\xi} \lim_{\epsilon \downarrow 0} H_T(\tau + \epsilon, \xi) \tag{45}$$

$$H_T^* = \max_{\xi} H_T(\tau, \xi) \tag{46}$$

$$\xi_T^* = \arg\max_{\xi} H_T(\tau, \xi) \tag{47}$$

Then $\tau = 0, T, 2T, ...,$ *a trading opportunity occurs and the trader selects* I_τ *so that* $\frac{X_\tau}{X_\tau + W_\tau} = \xi_T^*$.

We plot the solution to the problem in Figure 2. In the right panel, we plot the value function $H_T(t,\xi)$ for different times $t = 0...T$. We see that the function $H(t,\xi)$ is concave in ξ. The investor has an target allocation for illiquid securities, and deviations from this allocation reduce welfare. In contrast to the case where the trading time is stochastic, the concavity of H decreases as the investor approaches the next trading time; at $t = \tau - \epsilon$, the function H is almost flat. At this point, the investor anticipates that she will be able to rebalance very soon, so her liquid and illiquid holdings become closer to perfect substitutes.

To illustrate the differences between the deterministic and the stochastic case, we compare the optimal policies across the two cases for the same expected time between trades – τ in the deterministic case and $1/\lambda$ in the stochastic case. Figure 1 compares the optimal consumption policy and allocation to the liquid risk asset as a function of the illiquid wealth holdings ξ for $E(\tau) = 1/2$ and $E(\tau) = 2$ across the two cases. We see that when the waiting time is stochastic, varying the probability of trading λ has a substantial effect on both the optimal consumption policy c and the allocation to the liquid risky asset θ. A decrease in λ leads to lower consumption and portfolio policies for almost all levels of illiquid holdings ξ. In contrast, when the time until the next opportunity to trade is deterministic, varying the length of the illiquidity period has only minimal impact on either the consumption or the portfolio allocation policy.

To quantitatively assess the effect of illiquidity risk we compare the optimal policies and the welfare cost of illiquidity across the deterministic and stochastic cases for different expected lengths of the illiquidity period. As we see in Table 4, we find that varying the length of the deterministic illiquidity period has only a small effect on optimal policies. In this case, illiquidity has a level effect on welfare that is relatively insensitive to the length of the period. In contrast, when the time until the next opportunity is stochastic, varying λ has a substantially larger effect on optimal policies relative to the deterministic case. For example, when the time until the next trade is known in advance, varying the expected time until the next rebalancing time from 0.1 to 10 years leads to a drop in the fraction of total wealth consumed per year from 8.8% to 8.4%. In contrast, when the time until the next trade is stochastic, the fraction of total wealth consumed per year declines from 8.6% to 5.9%. In addition, when the length of the illiquidity period is sufficiently uncertain, the investor consumes a lower fraction of her total wealth than an investor who is unable to trade the second asset at all (c=7%).

Varying the length of the deterministic illiquidity period has a similarly small effect on welfare relative to the stochastic case. The difference in welfare costs between the two cases is small when the expected time until next rebalancing is small (1/10 years): in the stochastic case, the investor is willing to give up 4.1% of his total wealth to make the illiquid asset fully liquid, relative to 3.7% in the deterministic case. However, in the case of 10 years until the next opportunity to trade, the investor is willing to give up 22.2% of his wealth to be able to rebalance continuously in the stochastic case relative to 7.1% in the deterministic case. The difference arises largely from welfare changes in the stochastic case.

The above comparison disentangles the effects of illiquidity versus illiquidity *risk*. To understand the main difference between the stochastic and deterministic case, note that the investor's main concern is to avoid states of the world where her liquid wealth – and therefore her consumption – drops to zero before the next opportunity to trade. If the investor can trade at deterministic intervals, this state can be avoided with probability one by investing an appropriate amount into the riskless asset and consuming a constant fraction. For instance, if the illiquidity period lasts for an interval of T, the investor can guarantee a consumption flow of c^* by investing an amount c^*T into the riskless asset. Thus, the investor can choose a policy such that her only risk is from changes in consumption across trading intervals. In contrast, if the length of the illiquidity period is stochastic, the amount the investor needs to allocate to the risk-free asset is greater. Since the distribution of waiting times until the next trade is unbounded, there is a small probability that the investor needs to wait forever. As a result, illiquidity risk dampens the investor's ability to smooth consumption within trading intervals.

5 Liquidity crises

The model we outline in Section 2 captures the spirit of illiquidity risk. In particular, investing in illiquid securities exposes the investor to trading risk. However, this model makes the rather extreme assumption that the asset is almost always illiquid. This assumption makes our model applicable to several securities that are not naturally traded in exchanges. However, sometimes liquidity dries up even in markets that are otherwise fully liquid. For instance, Krishnamurthy, Nagel, and Orlov (2011) document that in the market for money market funds, a usually liquid market, there were instances of "buyers' strikes" during the recent financial crisis, where investors were unwilling to trade at any price. Here, we extend the model to allow for infrequent liquidity crises. These liquidity crises are temporary, and they adversely affect the liquidity of otherwise liquid securities.

5.1 A model with time-varying liquidity risk

We consider a setting where the level of market liquidity varies between two states $S_t \in \{I, L\}$.[9] In the illiquid state $S_t = I$, the opportunity to trade the illiquid asset occurs with probability λ, just like our benchmark model. In contrast, in the liquid state $S_t = L$, the investor can continuously trade both assets. The state of market liquidity S_t follows a continuous-time Markov process with transition probability matrix between time t and $t + dt$ given by

$$P = \begin{pmatrix} 1 - \chi^L \, dt & \chi^L \, dt \\ \chi^I \, dt & 1 - \chi^I \, dt \end{pmatrix}. \tag{48}$$

[9] Extending the model to allow for more than two states, or for state-contingent asset return moments, is straightforward.

Hence, at any point in time, the investor can be in a state where she can trade the illiquid security at a Poisson rate λ ($S_t = I$) or in a state where she can freely rebalance ($S_t = L$). The instantaneous probability of entering each state are $\chi^I\, dt$ and $\chi^L\, dt$ respectively. The investor's problem is

Problem 4 (Liquidity Crises) *The investor maximizes (10) subject to the budget constraints (8) and (9). The state of the economy ($S_t \in \{I, L\}$) evolves as in (48). If $S_t = L$, trade in both assets is continuous; if $S_t = I$, the investor can re-balance ($dI_t \neq 0$) only when the Poisson process N_t^λ jumps.*

As in Problem 1, the value function is homothetic and so

$$F_{LC}(W, X) = (W + X)^{1-\gamma} H_{LC}(\xi, S), \quad \text{where} \quad \xi = \frac{X}{X + W}.$$

In this case, the investor's value function depends not only on her wealth composition ξ, but also on the condition of market liquidity.

The first result is that the investor will not short the illiquid asset or net-borrow in liquid wealth to fund long purchases of the illiquid asset, even in the state in which both assets can be continuously traded:

Proposition 11 *Any optimal policies will have $W > 0$ and $X \geq 0$ a.s. for both $S = I$ and $S = L$.*

Proof. The arguments in the proof of Proposition 1 are sufficient to show that for $|\rho| < 1$, the objective function is at $-\infty$ if either $W \leq 0$ or $X < 0$ for $S = I$. For $S = L$, we observe that the state will shift to $S = I$ without the possibility of re-balancing; as a result, if either $W \leq 0$ or $X < 0$, the objective function in the liquid state is also equal to $-\infty$. For $\rho = 1$, $X < 0$ is ruled out by (4). ■

The proposition shows that due to the fear of a liquidity crisis, the investor will not use leverage – a short position in either the liquid or the potentially illiquid security – even when both securities are perfectly liquid. The transition from liquid to illiquid is a surprise event and occurs without the opportunity to re-balance. Consequently, the portfolio restrictions from the illiquid state (see Proposition 1) are imported into the liquid state. The following proposition characterizes the solution to investor's problem

Proposition 12 (Liquidity Crises) *The solution to the two state problem is characterized by an optimal consumption policy $\{c_I^*, c_L^*\}$ and portfolio policies $\{\theta_I^*, \theta_L^*\}$ and $\{\xi_I^*, \xi_L^*\}$ in the illiquid and liquid state respectively, as well as constants H_I^* and H_L^* and the function $H_I(\xi)$. The investor's value function is*

$$H_{LC}(\xi, S) = \begin{cases} H_I(\xi), & S = I \\ H_L^*, & S = L \end{cases}, \tag{49}$$

where

1. *The function $H_I(\xi)$ satisfies the Hamilton-Jacobi-Bellman equation*

$$\begin{aligned} 0 = \max_{c,\theta} \Bigg\{ & \frac{1}{1-\gamma} c^{1-\gamma} - \beta H_I(\xi) + \lambda \left(H_I^* - H_I(\xi)\right) + \chi^L \left(H_L^* - H_I(\xi)\right) \\ & + H_I(\xi)(1-\gamma) A(\xi, c, \theta) + \frac{\partial H_I(\xi)}{\partial \xi} B(\xi, c, \theta) + \frac{1}{2} \frac{\partial^2 H_I(\xi)}{\partial \xi^2} C(\xi, c, \theta) \Bigg\}, \end{aligned} \tag{50}$$

   where the functions A, B and C are defined in (15)-(17).

2. *The constants H_L^* and H_I^* solve*

$$0 = \max_{c,\theta,\xi} \left\{ \frac{1}{1-\gamma} c^{1-\gamma} - \beta H_L^* + \chi^I \left(H_I(\xi) - H_L^*\right) + A\left(\xi, \frac{c}{1-\xi}, \frac{\theta}{1-\xi}\right) H_L^* \right\} \tag{51}$$

   and

$$H_I^* = \max_{\xi} H_I(\xi) \tag{52}$$

3. *The policies* $\{c_I^*, \theta_I^*\}$ *and* $\{c_L^*, \theta_L^*, \xi_L^*\}$ *maximize (50) and (51) respectively. The policy* ξ_I^* *maximizes (52).*

Conditional on being in a liquidity crisis $S = I$, the investor's problem is similar to the problem analyzed in Section 2. As before, the fraction of wealth allocated to the illiquid asset is not under her full control, hence her value function H_I depends on the ratio of illiquid to total wealth ξ.

In contrast, the investor can freely rebalance between both risky assets in the liquid state $S = L$. Hence, the function $H_L(\xi)$ is a constant. However, the fact that her value function takes the same form as the Merton investor in the liquid state does not imply that the investor follows the same portfolio policies as the Merton investor. The investor will generally have different optimal holdings of the illiquid asset when it is illiquid and when it can be continuously traded. As a result, when the market goes from the liquid to the illiquid state, the investor is is constrained to hold her current allocation to the now illiquid security, and she needs to wait for the the next opportunity to trade to rebalance to her new optimal holding.

In Table 7 we plot the solution to the problem for different values of χ^L, χ^I and λ. We vary the average arrival rate of liquidity crisis, χ^I, from once every 5 to once every 20 years; the average length of a liquidity crisis, $1/\chi^L$, from 1 to 2 years; and we vary the probability of trading during a liquidity crisis, λ, from 12 to 1/2. We keep all other parameters the same.

Overall, we see that the investor allocates a higher fraction of her wealth in the illiquid asset in normal times ($S = L$) than during a liquidity crisis ($S = I$), but less so than in the Merton benchmark. In particular, the possibility of the illiquid state occurring affects the investor's optimal policies in the liquid state. Hence, an increase the likelihood, χ^I, or the duration, $1/\chi^L$, of a liquidity crisis leads to decreased allocation to the potentially illiquid asset ξ_L^* as well as a lower consumption rate c_L^*. In contrast, the allocation to the liquid asset θ_L^* is not affected by the possibility of a liquidity crisis. Decreasing the frequency of rebalancing λ during a liquidity crisis has similar effects.

In the illiquid state, the investor behaves in a similar fashion as in the benchmark model. The investor's target allocation to the illiquid asset ξ_I^* is particularly sensitive to the frequency of trading λ as well as the duration of the liquidity crisis χ^L, but less so to the probability of entering a crisis. In contrast, since the investor aims to smooth consumption across states, her consumption policy is affected by all three parameters, and is increasing in χ^L, $1/\chi^I$ and λ. Last, the allocation to the liquid asset θ_I^* is mostly sensitive to the probability of rebalancing λ and less so to the arrival χ^I and average duration $1/\chi^L$ of the liquidity crisis.

Overall, the results of this section illustrate that the *fear* of a liquidity crisis affects the investor's behavior when the crisis has yet to occur. In particular, the agent underinvests in assets that are currently fully liquid but have the possibility of becoming illiquid during a crisis. In the next section, we illustrate that this behavior will lead to limited arbitrage: the investor will not fully realize arbitrage opportunities – the existence of two perfectly correlated assets with different Sharpe ratios – even when the two assets are currently fully liquid.

5.2 Limits to Arbitrage

The possibility of a liquidity crisis implies that the investor will not fully take advantage of arbitrage opportunities, even when both assets are currently fully liquid. In particular, the following corollary shows that the agent will not fully take advantage of situations in which asset values are perfectly correlated but have differing Sharpe Ratios:

Corollary 13 *Limits to Arbitrage: in the case where* $|\rho| = 1$ *and* $\frac{\nu-r}{\psi} \neq \frac{\mu-r}{\sigma}$ *the investor's portfolio policies* θ_L^* *and* ξ_L^* *are finite and satisfy*

$$\frac{\nu-r}{\psi} - \rho\frac{\mu-r}{\sigma} = -\frac{\chi^I}{\psi}\frac{H_I'(\xi_L^*)}{H_L^*(1-\gamma)} \tag{53}$$

and

$$\theta_L^* = \frac{\mu - r}{\gamma\sigma^2} - \rho\frac{\psi}{\sigma}\xi_L^* \tag{54}$$

Proof. In the liquid state, the investor's optimal portfolio policies θ_L^* and ξ_L^* satisfy the first order conditions

$$\begin{aligned} 0 =& H_L^*(1-\gamma)(\mu-r) - H_L^*(1-\gamma)\,\gamma\,\theta_L^*\,\sigma^2 - H_L^*\gamma\,(1-\gamma)\,\rho\,\psi\,\sigma\,\xi_L^* \\ 0 =& \chi^I H_I'(\xi_L^*) + H_L^*(1-\gamma)(\nu-r) - H_L^*\gamma\,(1-\gamma)\,\psi^2\,\xi_L^* - H_L^*(1-\gamma)\gamma\rho\psi\theta_L^*\sigma \end{aligned}$$

Setting $\rho = 1$, dividing the first equation by $\sigma H_L^*(1-\gamma)$ and the second by $\psi H_L^*(1-\gamma)$, and then subtracting the first equation from the second leads to (53) and (54). Setting, $\rho = -1$, dividing the first equation by $\sigma H_L^*(1-\gamma)$ and the second by $\psi H_L^*(1-\gamma)$, and then adding the first equation from the second leads to the analogous expressions. ■

The investor faces a situation where she can freely trade two perfectly correlated securities with different Sharpe ratios. In the absence of any trading friction, the agent would build a zero-investment portfolio that has a positive payoff, and then she would take an infinite position in this arbitrage. Here, the investor is reluctant to do so – even though both securities are fully liquid and taking advantage of this arbitrage need not involve shorting risky assets. The reason for the underinvestment is that, a liquidity crisis can arrive with instantaneous probability $\chi^I\,dt$. In this case, the investor will no longer be free to rebalance her position in order to keep the arbitrage locally riskless. As a result, equation (53) shows that the investor will increase her holdings of the illiquid asset until the marginal welfare loss in the illiquid state times the probability of that state occurring is equal to the difference in the Sharpe ratios between the liquid and illiquid assets.

5.3 The cost of insurance against liquidity crisis

Liquidity crisis are systematic events. In principle, investors could obtain some insurance against these states by purchasing assets likely to appreciate when liquidity events occur. These assets are likely to command a negative risk premium, that is they are likely to trade at higher prices than comparable assets that do not offer similar protection. Examples of similar assets with different behavior during liquidity events include the new/old bond spread. Here, we quantify the magnitude of this illiquidity risk premium.

To determine the risk premium of liquidity crises, consider a fictitious derivative security in zero net supply that allows the investor to hedge the deterioration in market liquidity. In particular, the investor pays an annual premium equal to p in order to receive a cash payment of \$1 dollar in the event of a liquidity crisis. The following lemma computes the risk premium that would induce zero demand for the derivative security

Lemma 14 *The annual premium for liquidity protection is equal to*

$$p = \chi^I \left. \frac{F_W(W,X,I)}{F_W(W,X,L)} \right|_{\xi=\xi_L^*} \tag{55}$$

Proof. Consider a derivative security Z that pays a fixed rate of return κ when the aggregate state S switches from L to I. Denoting by dN_t^I the Poisson count process that denotes the arrival of a liquidity crisis, the price of this security evolves according to

$$\frac{dZ_t}{Z_t} = (r + \mu_Z - \kappa\, \chi^I)\, dt + \kappa dN_t^I.$$

The investor is indifferent between participating in the market for security Z and her current portfolio policy as long as the excess return μ_Z is equal to

$$\begin{aligned}\mu_Z\,dt &= -cov\left(\frac{dZ_t}{Z_t},\frac{dF_W}{F_W}\right)\\ &= \kappa\,\chi^I\,\frac{H_L^*(1-\gamma)-H_I(\xi_L^*)(1-\gamma)+\xi_L^* H_I'(\xi_L^*)}{H_L^*(1-\gamma)}\,dt\end{aligned}$$

There exists a fictitious probability measure $\mathcal{Q}$ under which the security Z has an expected excess return equal to zero and the probability of a liquidity crisis is equal to

$$\hat{\chi}_I = \chi^I\,\frac{F_W(W,X,I)}{F_W(W,X,L)} = \chi^I\,\frac{H_I(\xi_L^*)(1-\gamma)-\xi_L^* H_I'(\xi_L^*)}{H_L^*(1-\gamma)},$$

Under that measure, the present value of the expected payments p has to equal the expected payoff in the event of a liquidity crisis

$$\begin{aligned}E_t^{\mathcal{Q}}\int_t^{\tau} e^{-r(s-t)}pds &= E_t^{\mathcal{Q}}\left[e^{-r(\tau-t)}\right]\\ \Rightarrow \int_t^{\infty} e^{-(r+\hat{\chi}_I)(s-t)}pds &= \int_\tau^{\infty} e^{-(r+\hat{\chi}_I)(\tau-t)}\hat{\chi}_I d\tau\\ \Rightarrow p &= \hat{\chi}_I\end{aligned}$$

■

Lemma 55 shows that if the investor's marginal value of liquid wealth increases in the illiquid state, $F_W(W,X,I) > F_W(W,X,L)$ evaluated at $\xi = \xi_L^*$, the investor is willing to pay a higher rate that the objective probability χ_I in order to obtain protection against a liquidity crisis.

We compute the magnitude p/χ_I across different parameter values and show the results in Panel a of Table 8. In panel b, we compute the welfare cost of illiquidity, defined as the fraction of initial wealth the investor would be willing to give up in order to ensure that the second asset is always liquid ($S = L$).

Table 1: Liquid and Illiquid Asset Returns

	1981Q3 – 2010Q2			1981Q3 – 2006Q4		
	Mean	Stdev	Corr	Mean	St Dev	Corr
Equity	0.103	0.182	1.000	0.125	0.157	1.000
Illiquid Assets						
Private Equity	0.103	0.229	0.629	0.110	0.231	0.605
Buyout	0.092	0.134	0.267	0.097	0.110	0.010
Venture Capital	0.133	0.278	0.557	0.143	0.286	0.548
Illiquid Investment	0.109	0.165	0.674	0.117	0.159	0.623

The table reports summary statistics on excess returns on liquid and illiquid assets. Liquid equity returns are total returns on the S&P500. Data on private equity, buyout, and venture capital funds are obtained from Venture Economics and Cambridge Associates. We construct annual horizon log returns at the quarterly frequency. We compute log excess returns using the difference between log returns on the asset and year-on-year rollover returns on one-month T-bills expressed as a continuously compounded rate. The column "Corr" reports the correlation of excess returns with equity. The illiquid investment is a portfolio invested with equal weights in private equity, buyout, and venture capital and is rebalanced quarterly.

Table 2: Asset Holdings and Wealth Composition

			Stationary Distribution		
Average Turnover	λ	Optimal Rebalance Value	Mean	St Dev	Skew
10 years	0.1	0.0483	0.1659	0.1855	2.3967
5 years	0.2	0.1053	0.1875	0.1273	2.6560
2 years	0.5	0.2423	0.2962	0.0854	2.2373
1 year	1.0	0.3729	0.4276	0.0633	1.8724
1/2 year	2.0	0.4703	0.4884	0.0422	1.5690
1/4 year	4.0	0.5063	0.5351	0.0283	1.2308

The table summarizes the effect of illiquidity on the moments of asset holdings. The optimal rebalance value is $\left(\frac{X}{X+W}\right)^*$, while the mean, standard deviation (st dev), and normalized skewness (skew) are all taken with respect to the stationary distribution of the ratio of illiquid wealth to total wealth, $\frac{X_t}{X_t+W_t}$. The table is computed using the following other parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, and $\rho = 0$.

Table 3: Illiquidity Premiums

Average Turnover	λ	Certainty Equivalent Wealth	Liquidity Premium
$\rho = 0$			
10 years	0.1	0.2866	0.0600
5 years	0.2	0.2148	0.0433
2 years	0.5	0.1140	0.0201
1 year	1.0	0.0672	0.0093
1/2 year	2.0	0.0415	0.0066
1/4 year	4.0	0.0397	0.0063
$\rho = 0.6$			
10 years	0.1	0.1235	0.0224
5 years	0.2	0.0692	0.0141
2 years	0.5	0.0197	0.0041
1 year	1.0	0.0106	0.0022
1/2 year	2.0	0.0098	0.0020
1/4 year	4.0	0.0096	0.0020

The table summarizes the effect of the trading frequency, λ, on certainty equivalents of holding the illiquid asset. The column labeled "Certainty Equivalent Wealth" reports the fraction of wealth the agent is willing to give up in order to make the illiquid asset liquid (taken as an expectation over the stationary distribution of wealth). The column labeled "Liquidity Premium" is a certainty equivalent comparison, so a liquidity premium of 0.02 means that the utility level in the economy with liquid assets with an expected returns of 12% and 10% is equal to the utility level in an economy with one liquid and one illiquid asset, both with expected returns of 12%. The numbers are computed taking expectations over the stationary distribution with the following other parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, and $\sigma = \psi = .15$.

Table 4: Risky versus deterministic illiquidity

A. Stochastic trading opportunity

Average Turnover	Optimal	Liquidity	Average policies		
$E(T) = 1/\lambda$	Rebalance	Welfare cost	$E[\xi]$	$E[c]$	$E[\theta]$
0	0.593	-	0.593	0.089	0.593
1/10 years	0.493	0.029	0.466	0.086	0.572
1/4	0.475	0.037	0.465	0.086	0.571
1/2	0.442	0.045	0.461	0.083	0.568
1	0.373	0.067	0.409	0.081	0.558
2	0.251	0.103	0.299	0.075	0.546
4	0.132	0.165	0.212	0.069	0.536
10	0.048	0.222	0.214	0.059	0.489
∞	0.593	-	0.593	0.070	0.593

B. Deterministic trading opportunity

Turnover	Optimal	Liquidity	Average policies		
T	Rebalance	Welfare cost	$E[\xi]$	$E[c]$	$E[\theta]$
0	0.593	-	0.593	0.089	0.593
1/10 years	0.555	0.010	0.467	0.088	0.572
1/4	0.532	0.016	0.467	0.088	0.572
1/2	0.516	0.019	0.466	0.087	0.571
1	0.484	0.025	0.464	0.087	0.569
2	0.478	0.031	0.458	0.086	0.566
4	0.425	0.038	0.455	0.085	0.556
10	0.348	0.045	0.414	0.084	0.528
∞	0.593	-	0.593	0.070	0.593

The table is computed using the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, and $\rho = 0$.

Table 5: Effect of the EIS on optimal policies

λ	1/EIS				1/EIS			
	2/3	1.5	3	6	2/3	1.5	3	6
	a. Optimal Rebalance				b. Welfare cost			
∞	0.593	0.593	0.593	0.593	-	-	-	-
10	0.501	0.500	0.499	0.493	0.010	0.023	0.026	0.029
4	0.495	0.490	0.489	0.475	0.019	0.027	0.029	0.037
2	0.433	0.439	0.441	0.442	0.036	0.039	0.040	0.041
1	0.391	0.398	0.383	0.373	0.040	0.047	0.052	0.067
1/2	0.244	0.245	0.248	0.251	0.077	0.093	0.100	0.103
1/4	0.114	0.121	0.129	0.131	0.139	0.154	0.161	0.165
1/10	0.026	0.038	0.045	0.048	0.177	0.203	0.216	0.222
0	-	-	-	-	-	-	-	-
	c. $E(c)$				d. $E(\theta)$			
∞	0.109	0.096	0.092	0.090	0.593	0.593	0.593	0.593
10	0.108	0.095	0.089	0.087	0.573	0.572	0.572	0.572
4	0.108	0.095	0.089	0.087	0.572	0.571	0.571	0.571
2	0.108	0.094	0.089	0.086	0.569	0.569	0.568	0.568
1	0.107	0.093	0.088	0.085	0.561	0.558	0.558	0.557
1/2	0.103	0.089	0.083	0.080	0.542	0.544	0.545	0.545
1/4	0.103	0.084	0.077	0.074	0.534	0.534	0.534	0.534
1/10	0.099	0.078	0.069	0.065	0.508	0.510	0.511	0.511
0	0.118	0.088	0.076	0.070	0.593	0.593	0.593	0.593

The table is computed using the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, and $\rho = 0$.

Table 6: Effect of risk aversion on optimal policies

λ	Risk aversion				Risk aversion			
	3	6	10	15	3	6	10	15
	a. Optimal Rebalance				b. Welfare cost			
∞	1.185	0.593	0.356	0.237	-	-	-	-
10	0.799	0.493	0.336	0.220	0.109	0.029	0.020	0.005
4	0.666	0.475	0.291	0.201	0.114	0.037	0.031	0.021
2	0.530	0.442	0.285	0.199	0.132	0.041	0.035	0.030
1	0.409	0.373	0.280	0.197	0.172	0.067	0.041	0.035
1/2	0.289	0.246	0.192	0.149	0.256	0.107	0.051	0.041
1/4	0.178	0.132	0.101	0.075	0.295	0.165	0.106	0.073
1/10	0.061	0.047	0.035	0.025	0.325	0.222	0.158	0.116
0	-	-	-	-	-	-	-	-
	c. $E(c)$				d. $E(\theta)$			
∞	0.129	0.090	0.074	0.069	1.185	0.593	0.356	0.237
10	0.117	0.086	0.072	0.068	1.122	0.572	0.349	0.255
4	0.117	0.086	0.064	0.065	1.113	0.571	0.349	0.238
2	0.114	0.082	0.068	0.065	1.097	0.561	0.343	0.234
1	0.110	0.081	0.072	0.065	1.076	0.552	0.335	0.233
1/2	0.105	0.080	0.069	0.063	1.069	0.548	0.331	0.224
1/4	0.093	0.075	0.065	0.060	1.045	0.544	0.332	0.225
1/10	0.088	0.068	0.061	0.057	1.020	0.539	0.328	0.222
0	0.090	0.070	0.062	0.058	1.185	0.593	0.356	0.237

The table is computed using the following parameter values: $\zeta = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, and $\rho = 0$.

Table 7: Portfolio policies across regimes

State	λ	Average duration of illiquid regime – $1/\chi^L$								
		1			1.5			2		
		Average arrival of liquidity crisis – χ^I								
		1/20	1/10	1/5	1/20	1/10	1/5	1/20	1/10	1/5
		Target allocation to illiquid asset – ξ^*								
L	12	0.5866	0.5866	0.5890	0.5890	0.5890	0.5890	0.5890	0.5890	0.5866
	4	0.5793	0.5793	0.5793	0.5842	0.5842	0.5793	0.5842	0.5842	0.5793
	1	0.5695	0.5695	0.5597	0.5646	0.5646	0.5474	0.5548	0.5548	0.5374
	1/2	0.5548	0.5548	0.5374	0.5250	0.5250	0.5025	0.5000	0.5000	0.4750
I	12	0.5301	0.5301	0.5301	0.5271	0.5271	0.5271	0.5251	0.5251	0.5251
	4	0.5026	0.5026	0.5026	0.5013	0.5013	0.5013	0.5026	0.5026	05026
	1	0.4354	0.4354	0.4378	0.4280	0.4280	0.4280	0.4207	0.4207	0.4207
	1/2	0.4207	0.4207	0.4207	0.3894	0.3894	0.3917	0.3659	0.3659	0.3659
		Consumption policy, average – $E(c)$								
L	12.00	0.0892	0.0892	0.0890	0.0891	0.0891	0.0889	0.0891	0.0891	0.0887
	4.00	0.0892	0.0892	0.0890	0.0891	0.0891	0.0888	0.0890	0.0890	0.0887
	1.00	0.0891	0.0891	0.0888	0.0889	0.0889	0.0885	0.0888	0.0888	0.0883
	0.50	0.0889	0.0889	0.0886	0.0885	0.0885	0.0880	0.0880	0.0880	0.0874
I	12	0.0890	0.0890	0.0888	0.0888	0.0888	0.0886	0.0887	0.0887	0.0884
	4	0.0890	0.0890	0.0888	0.0888	0.0888	0.0885	0.0886	0.0886	0.0883
	1	0.0887	0.0887	0.0885	0.0885	0.0885	0.0881	0.0882	0.0882	0.0878
	1/2	0.0851	0.0851	0.0848	0.0846	0.0846	0.0841	0.0842	0.0842	0.0838
		Allocation to liquid risky asset, average – $E(\theta)$								
L	12	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926
	4	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926
	1	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926
	1/2	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926	0.5926
I	12	0.5922	0.5922	0.5922	0.5922	0.5922	0.5922	0.5922	0.5922	0.5922
	4	0.5909	0.5909	0.5914	0.5913	0.5913	0.5913	0.5912	0.5912	0.5913
	1	0.5777	0.5777	0.5850	0.5814	0.5814	0.5816	0.5790	0.5790	0.5793
	1/2	0.5265	0.5265	0.5273	0.5235	0.5235	0.5230	0.5238	0.5238	0.5251

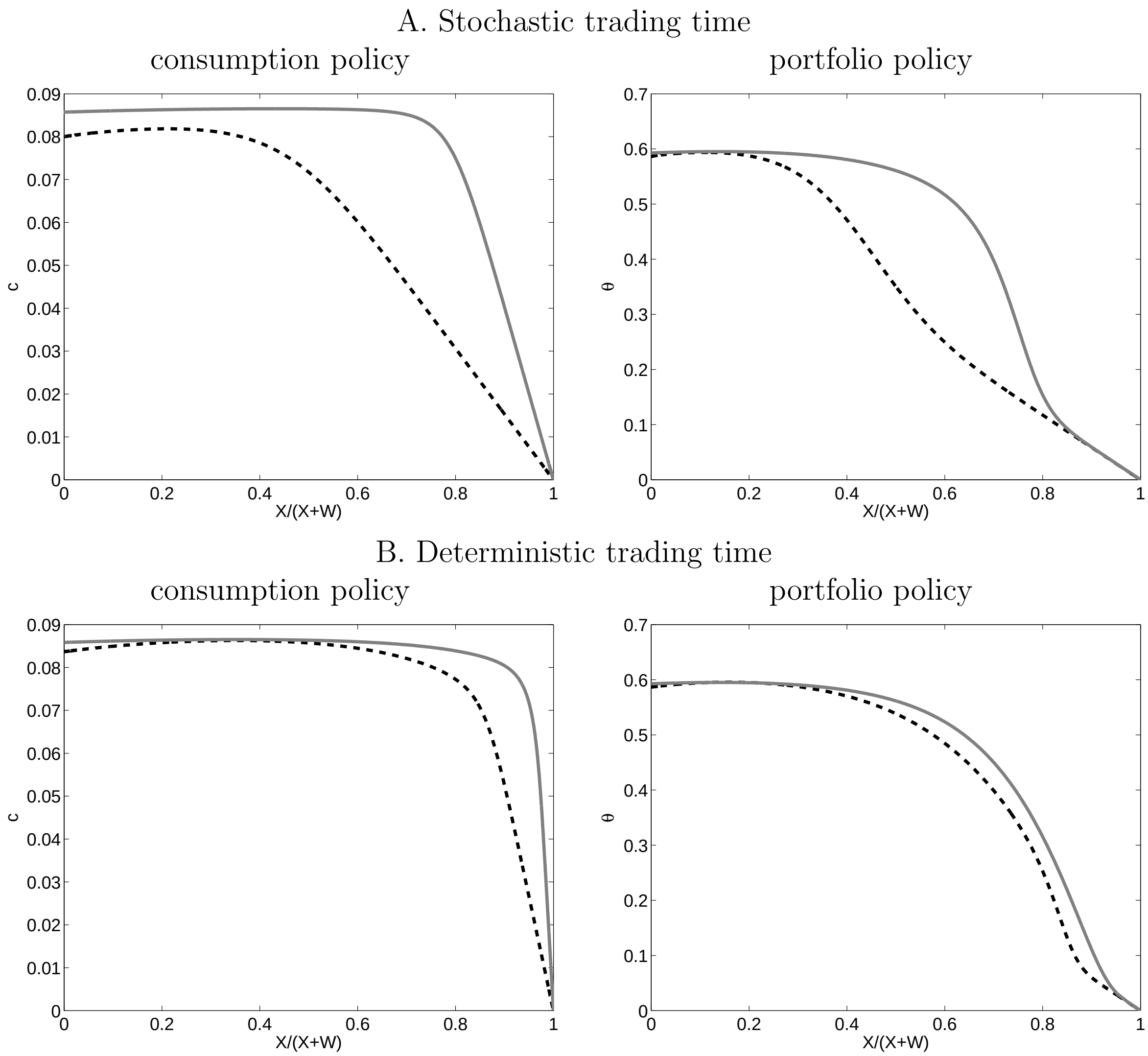
The table is computed using the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, and $\rho = 0$.

Table 8: Illiquidity risk premium versus welfare cost of illiquidity

λ	Average duration of illiquid regime – $1/\chi^L$								
	1			1.5			2		
	Average duration of liquid regime – $1/\chi^I$								
	20	10	5	20	10	5	20	10	5
	a. Illiquidity risk premium – $\hat{\chi}_I/\chi_I$								
12.0	1.0424	1.0424	1.0165	1.0239	1.0239	1.0214	1.0289	1.0289	1.0252
4.0	1.0500	1.0500	1.0209	1.0296	1.0296	1.0259	1.0347	1.0347	1.0301
1.0	1.0677	1.0677	1.0312	1.0537	1.0537	1.0419	1.0669	1.0669	1.0507
0.5	1.0814	1.0814	1.0424	1.0931	1.0931	1.0628	1.1218	1.1218	1.0772
	b. Welfare cost of illiquidity								
12	0.0039	0.0039	0.0070	0.0054	0.0054	0.0095	0.0069	0.0069	0.0117
4	0.0045	0.0045	0.0077	0.0059	0.0059	0.0103	0.0074	0.0074	0.0125
1	0.0059	0.0059	0.0099	0.0083	0.0083	0.0139	0.0109	0.0109	0.0174
1/2	0.0077	0.0077	0.0128	0.0142	0.0142	0.0212	0.0209	0.0209	0.0292

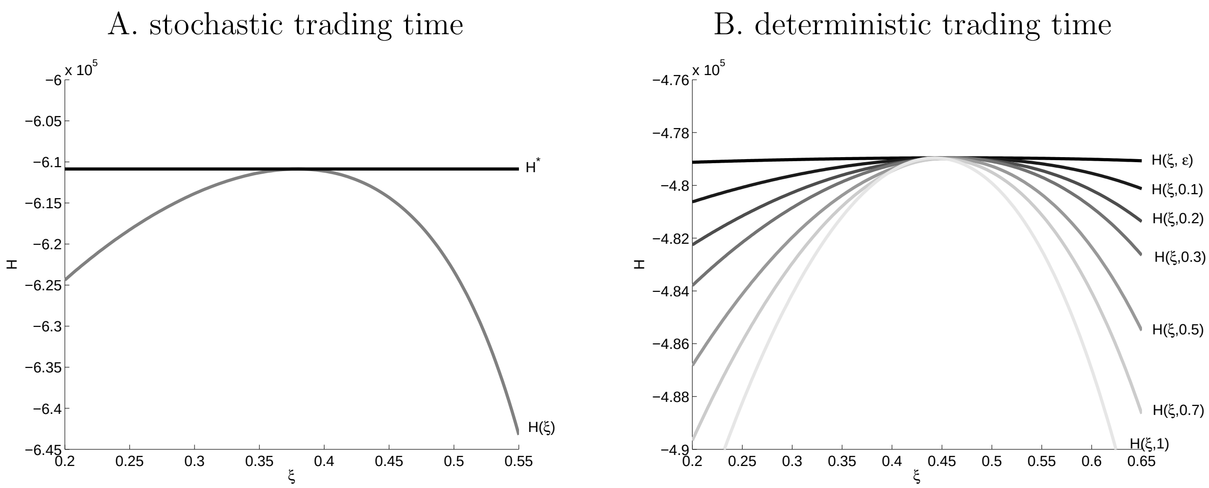
The table is computed using the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure 1: Deterministic vs stochastic trading times: effect on portfolio policies



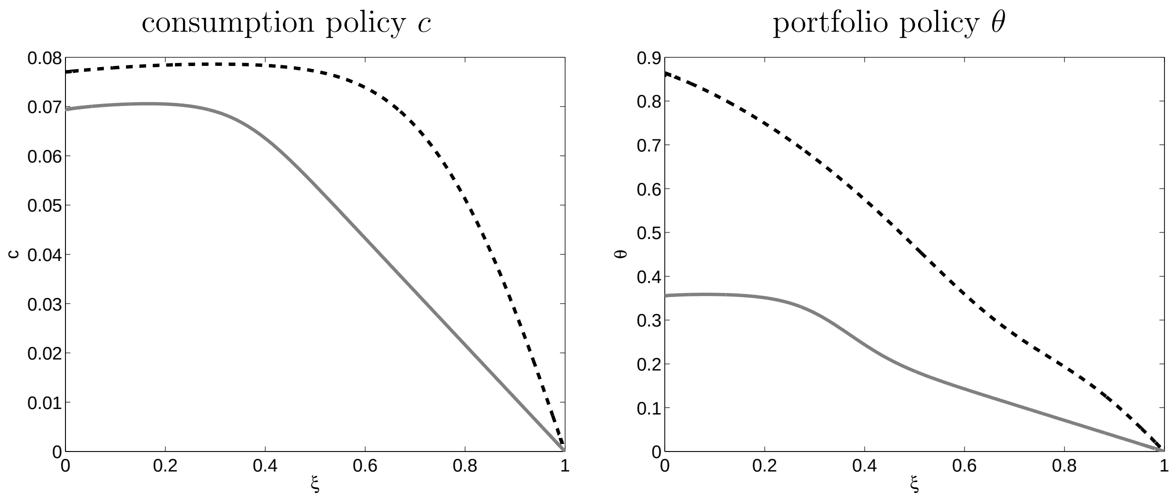
The figure compares the consumption (left figure) and portfolio allocation in the liquid asset (right figure) across different expected times until next trade: $E(T) = 1/2$ (grey line) and $E(T) = 2$ (black dotted line). In the top panel (A) the trading time is stochastic, so $E(T) = 1/\lambda$; in Panel B the time until the next trade is deterministic, so $E(T) = T$. The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure 2: Value function: comparison between deterministic and stochastic case



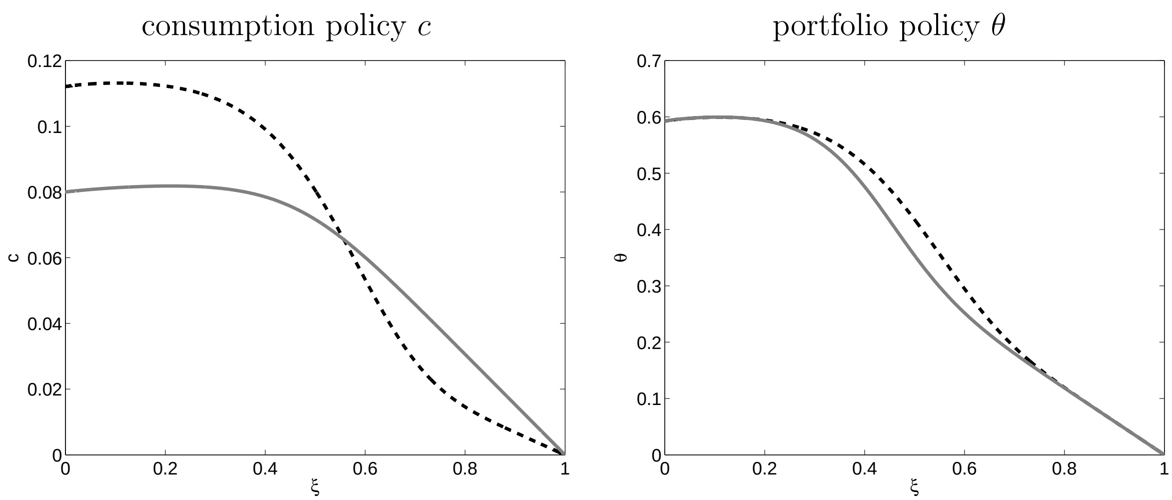
The figure compares the value function as a function of the portfolio allocation across different expected times until next trade: $E(T) = 1/2$ (grey line) and $E(T) = 2$ (black dotted line). In panel A the trading time is stochastic, so $E(T) = 1/\lambda$; in Panel B the time until the next trade is deterministic, so $E(T) = T$. The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure 3: Effect of Risk Aversion on portfolio policies



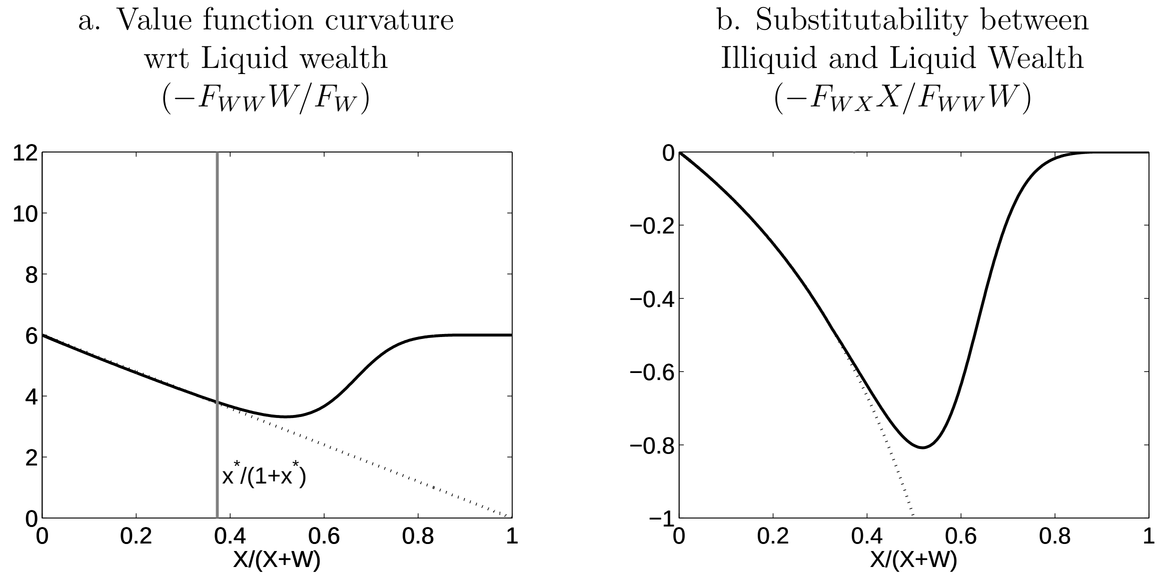
The figure compares the consumption (left figure) and portfolio allocation in the liquid asset (right figure) as a function of illiquid portfolio holdings $\xi = X/(X+W)$, for high $\gamma = 10$ (grey line) and low $\gamma = 3$ (black dotted line). The curves are plotted with the following parameter values: $\zeta = 6$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure 4: Effect of EIS on portfolio policies



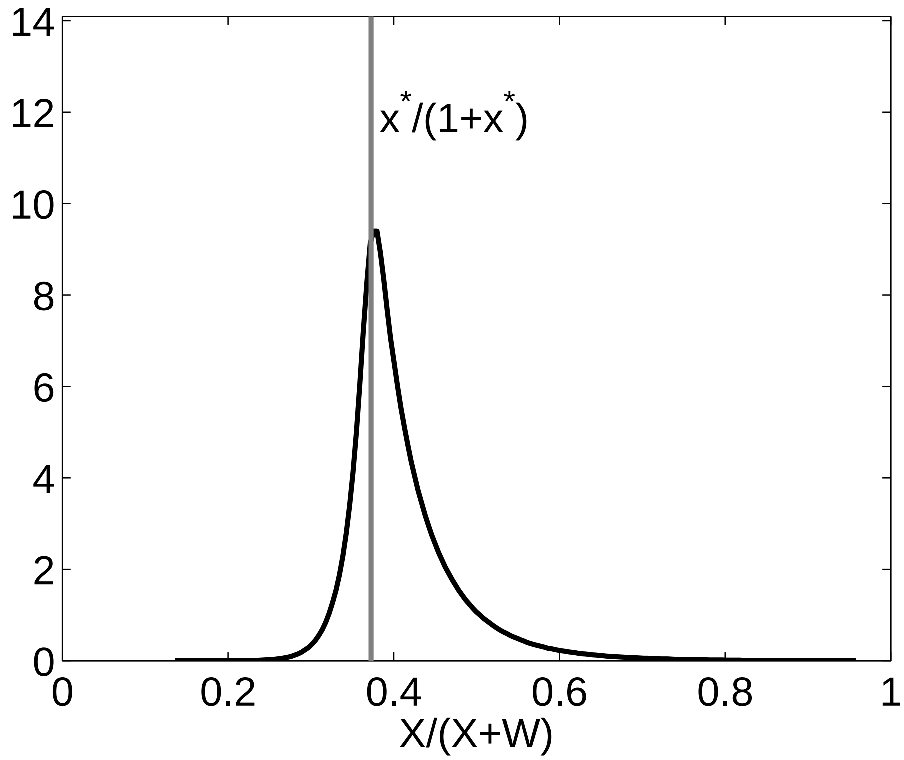
The figure compares the consumption (left figure) and portfolio allocation in the liquid asset (right figure) as a function of illiquid portfolio holdings $\xi = X/(X+W)$, for low $\zeta = 6$ (grey line) and high EIS $\zeta = 1/1.5$ (black dotted line). The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure 5: Determinants of Portfolio Choice



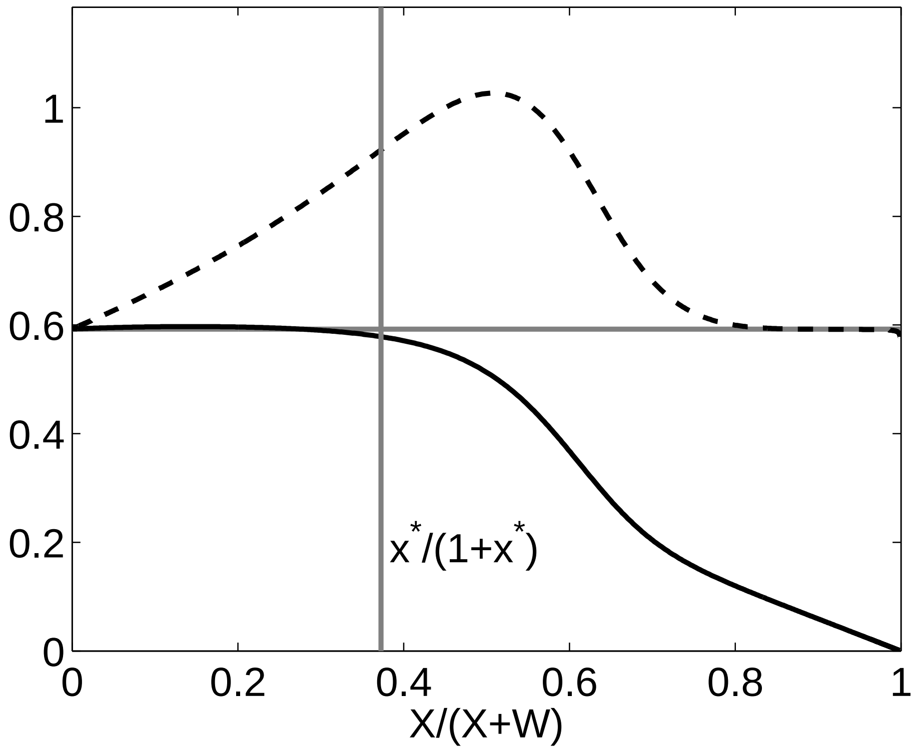
The figure in panel a plots the curvature of the value function with respect to liquid wealth, $-F_{WW}W/F_W$. The figure in panel b plots the elasticity of substitution in the value function between liquid and illiquid wealth, $F_{WX}X/F_{WW}W$. The solid lines represents the case where $\lambda = 1$. The dotted lines correspond to the Merton case $\lambda \to \infty$. The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure 6: Distribution of Illiquid Holdings



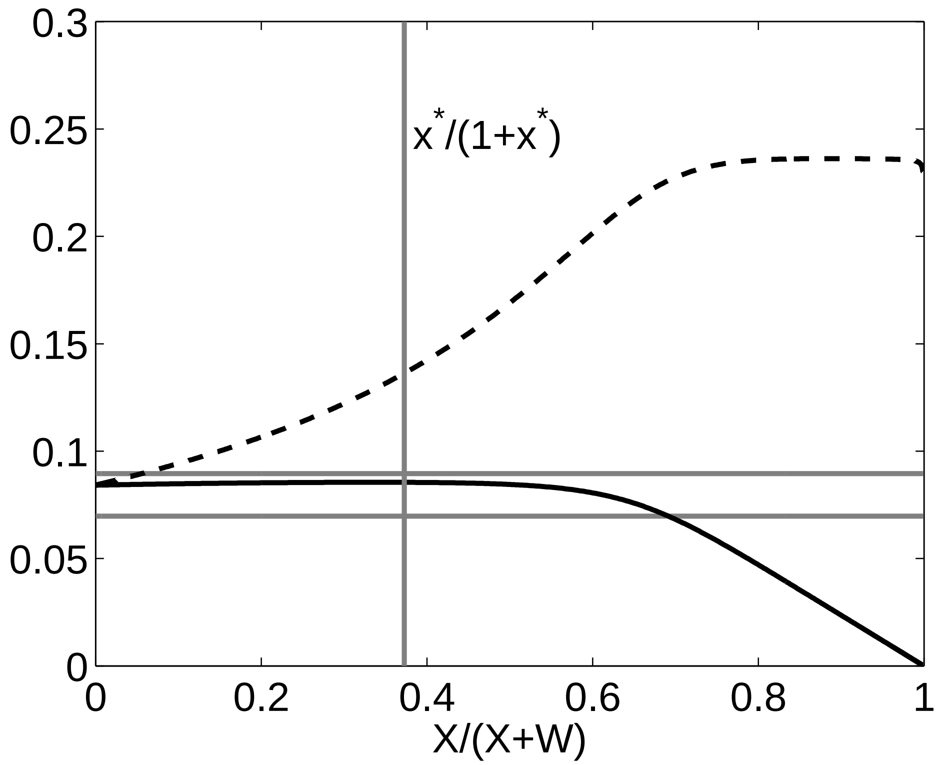
The figure plots the stationary distribution of allocation to the illiquid asset as a fraction of total wealth, $x = \frac{X}{X+W}$. The vertical solid gray line corresponds to the value of the optimal rebalancing point $x^*/(1+x^*)$, which is the desired allocation to the illiquid asset as a fraction of total wealth at the time of rebalancing. The figure uses $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure 7: Optimal Allocation to the Liquid Risky Asset



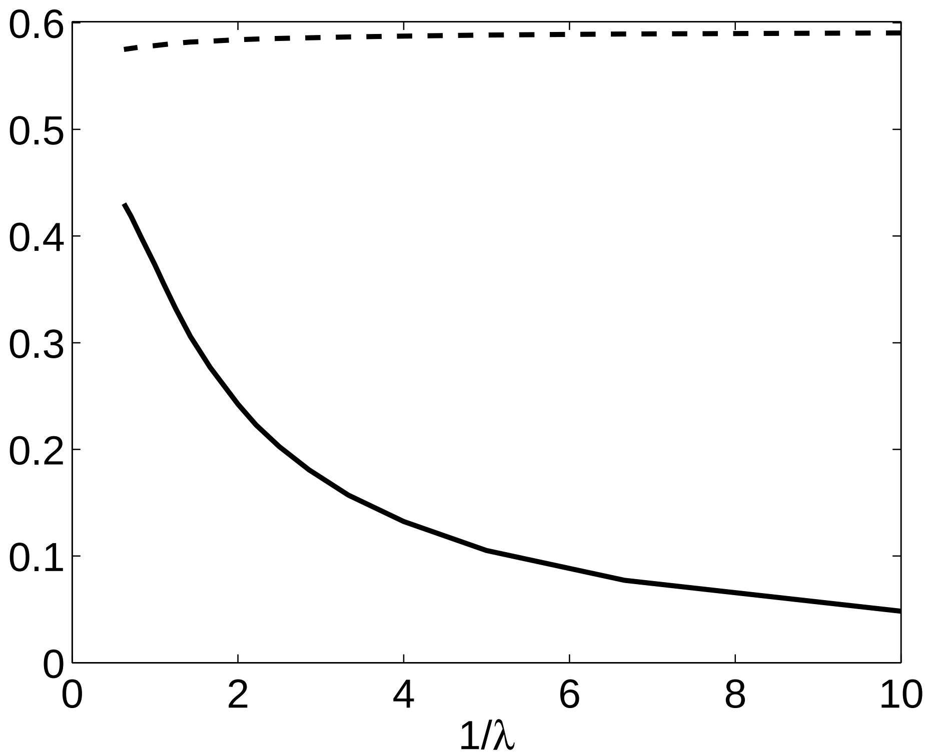
The figure displays the optimal allocation to the liquid assets. The solid black lines represent allocation to the liquid risky asset taken as a fraction of total wealth, $\theta/(1+x)$, whereas the dashed lines represent the allocation to the liquid risky asset as a fraction of liquid wealth only, θ. The gray horizontal line corresponds to the allocation to the risky asset in the one- and/or two-asset Merton economy. The vertical gray line is the point $x^*/(1+x^*)$, which is the optimal holding of illiquid assets relative to total wealth at the arrival of the trading time. The dashed line can be above one because the investor can use the liquid (but not illiquid) risky asset as collateral, as in the standard Merton problem. The solid line must remain between zero and one because the illiquid asset cannot be so used. The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure 8: Optimal Consumption



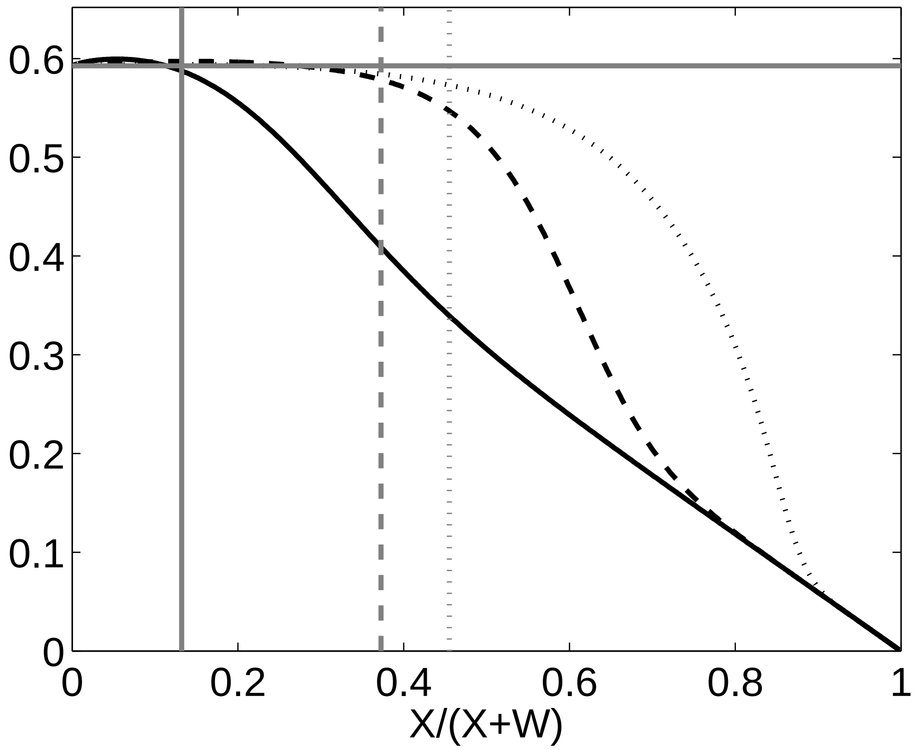
We plot the optimal consumption policy. The solid black line is the consumption policy as a fraction of total wealth, $c/(1+x)$, and the dashed line depicts consumption policy as a fraction of liquid wealth only, c. The horizontal gray lines correspond to consumption in the one- and two-asset Merton benchmarks (consumption is higher in the two-asset case). The vertical solid gray line corresponds to the value of the optimal rebalancing point, $x^*/(1+x^*)$, which is the desired allocation to the illiquid asset as a fraction of total wealth at the time of rebalancing. The curves are plotted with the following parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure 9: Effect of Illiquidity on Asset Holdings at Trading Times



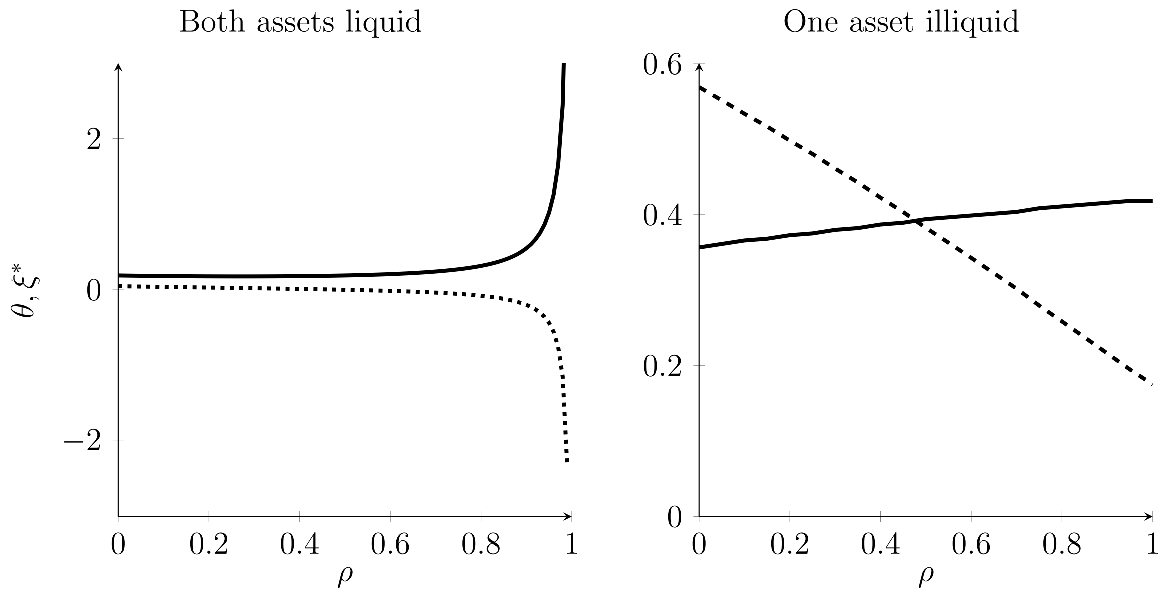
We plot the optimal allocations to the liquid risky asset as a fraction of total wealth (dashed line) and the illiquid risky asset as a fraction of total wealth (solid line) at the rebalancing time, both as a function of $1/\lambda$. The remainder is allocated to the riskless asset. The curves are plotted with the following other parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\rho = 0$, and $\sigma = \psi = .15$.

Figure 10: Effect of Illiquidity on Asset Holdings Between Trading Times



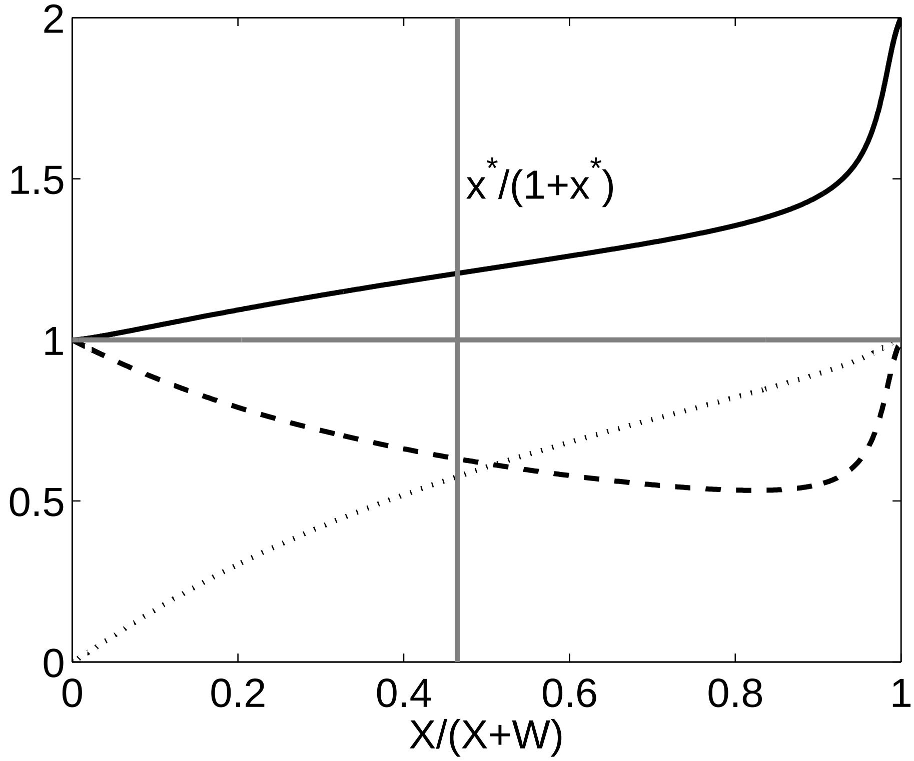
We plot the optimal allocations to the liquid risky asset as a fraction of total wealth as a function of the liquid/illiquid composition of total wealth. There are three black curves which display holdings: $\lambda = 1/4$, $\lambda = 1$, and $\lambda = 4$ correspond to the solid line, the dashed line, and the dotted line, respectively. The vertical gray lines with the same line style are the corresponding optimal rebalance levels. The curves are plotted with the following other parameter values: $\gamma = 6$, $\mu = \nu = .12$, $r = .04$, $\rho = 0$, and $\sigma = \psi = .15$.

Figure 11: Effect of Correlation on Asset Holdings



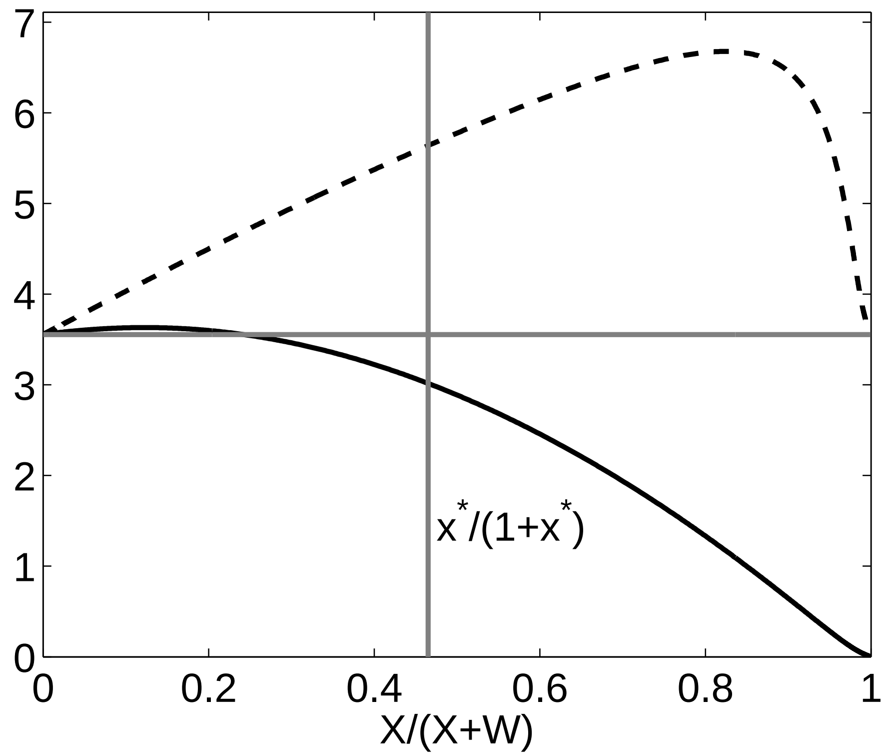
We plot the optimal allocations to the liquid risky asset (- - -) and the illiquid risky asset as a fraction of total wealth (——) at the rebalancing time, both as a function of ρ. The remainder is allocated to the riskless asset. The curves are plotted with the following other parameter values: $\gamma = 6$, $\mu = .12$, $\nu = .20$, $r = .04$, $\lambda = 1$, and $\sigma = \psi = .15$.

Figure A-1: Effective Risk Aversion: $\gamma = 1$



The figure plots the relative curvature of the value function for $\gamma = 1$. The solid line represents the total relative curvature, $\frac{F_{WW}W}{F_W} + \frac{F_{XX}X}{F_X}$, the dashed line represents the curvature with respect to W, $\frac{F_{WW}W}{F_W}$, and the dotted line represents the curvature with respect to X, $\frac{F_{XX}X}{F_X}$. The horizontal gray line is the point $x * /(1 + x^*)$, which is the optimal holding of illiquid assets relative to total wealth at the arrival of the trading time. The curves are plotted with the following parameter values: $\gamma = 1$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.

Figure A-2: Optimal Allocation to the Liquid Risky Asset: $\gamma = 1$



The figure displays the optimal allocation to the liquid assets. The solid black lines represent allocation to the liquid risky asset taken as a fraction of total wealth, $\theta/(1+x)$, whereas the dashed lines represent the allocation to the liquid risky asset as a fraction of liquid wealth only, θ. The gray lines correspond to the allocation to the risky asset in the one- and/or two-asset Merton economy. The horizontal gray line is the point $x * /(1 + x^*)$, which is the optimal holding of illiquid assets relative to total wealth at the arrival of the trading time. The curves are plotted with the following parameter values: $\gamma = 1$, $\mu = \nu = .12$, $r = .04$, $\lambda = 1$, $\sigma = \psi = .15$, and $\rho = 0$.