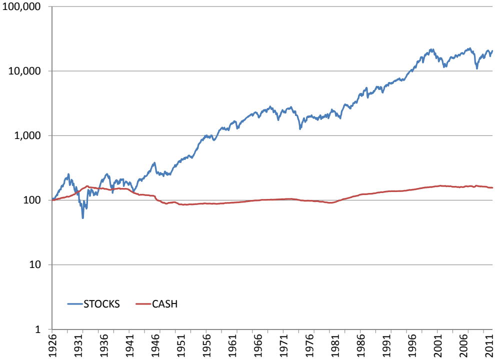
Lecture 1: Asset Allocation

Investments

Introduction

- One of the first decisions we need to make is to decide how much to invest in risky assets, and how much to leave in the safe asset.
- Example: If you had \$ 1,000,000 to invest, how would you allocate your money between stocks and bonds?
- Stocks are riskier than bonds, but they also yield higher returns.

Returns of Stocks vs Cash, adjusted for inflation.



	mean	volatility
Stocks	8.02%	18.94%
T-Bills	0.52%	1.85%

Stocks vs Cash

- Should we invest in the stock market, or T-Bills?
- Which factors influence this decision?
- Should this decision be the same for everyone?

Introduction

- The goal of this lecture is to understand *asset-allocation.* theory.
- Modern portfolio theory has its roots in mean-variance portfolio analysis.
  - $\hookrightarrow$ Developed by Harry Markowitz in the early 1960's.
  - $\hookrightarrow$ His work was the first step in the development of modern finance.
- The asset allocation problem answers the question:
  How much of your wealth should you invest in each security?
- This is an area that we have come to understand much better in the last forty years!

Plan

1. Investor's Risk Tolerance
2. Allocating Capital Between a Risky and Riskless asset
3. Allocating Capital among Multiple Risky Securities
4. Conclusions

Mean-Variance Analysis

- Up until the mean-variance analysis of Markowitz became known, an investment advisor would have given you advice like:

  ↪ *If you are young you should be putting money into a couple of good growth stocks, maybe even into a few small stocks. Now is the time to take risks.*

  ↪ *If you're close to retirement, you should be putting all of your money into bonds and safe stocks, and nothing into the risky stocks – don't take risks with your portfolio at this stage in your life.*

- This advice is intuitively compelling. However, it is not entirely correct.

Mean-Variance Analysis: Preview

- The main insight from MV-Analysis is that the optimal portfolio of risky assets is exactly the same for everyone, **no matter what their tolerance for risk.**

1. Investors should control the risk of their portfolio not by reallocating among risky assets, but through the split between risky and risk-free assets.
2. The portfolio of risky assets should contain a large number of assets – it should be a **well diversified portfolio.**
3. The optimal portfolio of risky assets has the highest Sharpe ratio, i.e. it gives the highest expected return for a given level of risk.

Mean-Variance Analysis: Assumptions

- **Note:** These results are derived under the assumptions that:

  a) Either,

     1) all returns are normally distributed
     2) investors care only about mean return and variance.

  b) All assets are tradable.

  c) There are no transaction costs.

- We will discuss the implications of relaxing these assumptions later in the class.

Allocating Capital Between Risky and Risk-free Assets

- *Two-Fund separation* is a key result in Modern Portfolio theory. It implies that the investment problem can be decomposed into two steps:

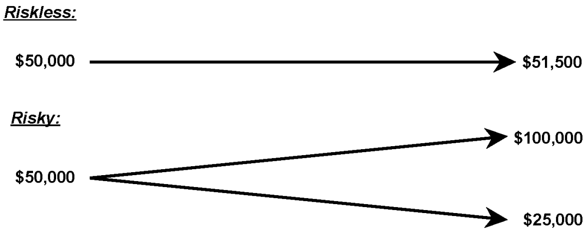
   1. Find the optimal portfolio of risky securities
   2. Find the best combination of the risk-free asset and this optimal risky portfolio

- We'll consider part (2) first. Afterwards, we will show that, if we have many risky assets, there is an optimal portfolio of these risky assets that all investors prefer.

- But first, we need a theoretical framework for understanding the tradeoff between risk and return.

Framework

- An investor has the choice of investing \$50,000 in a risk-free investment or a risky investment.
  - ↪ The risky investment will either halve or double, with equal probability
  - ↪ The riskless investment will yield a certain return of \$51,500.



- How should he decide which of these investments to take?

Framework

1. Calculate the expected return for each investment

   $\hookrightarrow$ The (simple) return on the risk free investment is:

$$r_f = \frac{51,500}{50,000} - 1 = 3\%$$

   $\hookrightarrow$ The expected return on the risky investment is:

$$E(\tilde{r}) = \frac{1}{2} \cdot \underbrace{\left(\frac{100}{50} - 1\right)}_{100\%} + \frac{1}{2} \cdot \underbrace{\left(\frac{25}{50} - 1\right)}_{-50\%} = 25\%$$

2. Calculate the *risk premium* on the risky investment.

   $\hookrightarrow$ **Definition:** The *excess return* is the return net of the risk-free rate

$$\tilde{r}^e = \tilde{r} - r_f$$

   $\hookrightarrow$ **Definition:** The risk premium is the expected excess return

$$E(\tilde{r}^e) = E(\tilde{r} - r_f) = \frac{1}{2} \cdot 97\% + \frac{1}{2} \cdot (-53\%) = E(\tilde{r}) - r_f = 22\%$$

Framework

3. Calculate the riskiness of the investments

↪ To answer this we need a measure of risk. The measure we will use for now, is the return *variance* or *standard deviation*;

- For the risk-free asset, the variance is zero.
- For the risky investment the return variance is:

$$\sigma^2(\tilde{r}) = \frac{1}{2} \cdot \left[(1.00 - 0.25)^2 + (-0.50 - 0.25)^2\right] = 0.56$$

and the return standard-deviation is the positive square-root of σ^2:

$$\sigma(r) = \sqrt{0.56} = 0.75 = 75\%$$

↪ If asset returns are normally distributed, this is a perfect measure of risk (why?).

↪ If returns are not normal (as is the case here), you need other assumptions to make variance a perfect *proxy* for riskiness.

4. Finally, we need to determine if this is a reasonable amount of risk for the extra expected return.

   ↪ We need to *quantify* our attitudes or preferences over risk and return.

   ↪ For a starting point, we assume people

   4.1 like high expected returns $E(\tilde{r})$

   4.2 dislike high variance $\sigma^2(\tilde{r})$

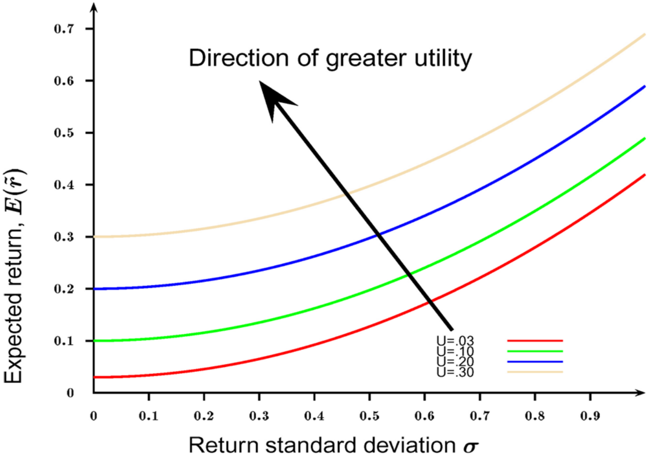
   ↪ that is, investors are *risk averse.*

   ↪ Their *utility* or happiness from a pattern of returns $\tilde{r}$ is:

$$U(\tilde{r}) = E(\tilde{r}) - \frac{1}{2}A\sigma^2(\tilde{r})$$

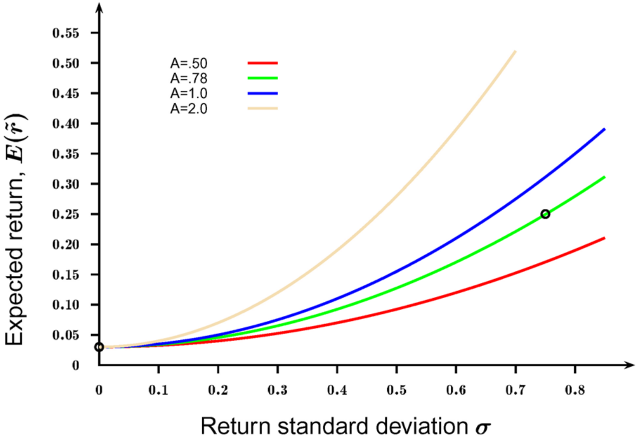
   - A measures the investor's level of *risk-aversion*
   - The higher A, the higher an investor's dislike of risk

Indifference Curves



- Investor preferences can be depicted as indifference curves.
- Each curve represents different utility levels for fixed risk aversion A.
- Each curve traces out the combinations of $E(r)$, $\sigma(r)$ yielding the same level of utility

Indifference Curves



- Each curve plots the same utility level for different risk aversion A.
- Higher A implies that for a given σ, investors require higher mean return to achieve same level of utility.

Certainty Equivalent return (CER)

- **Definition:** The *certainty equivalent rate of return* r_{CE} for a risky portfolio is the return such that you are *indifferent* between that portfolio and earning a *certain* return r_{CE}.

- r_{CE} depends on the characteristics of the portfolio ($E(\tilde{r})$, $\sigma(\tilde{r}$ and the investor's risk tolerance. With our specification for the Utility function,

$$r_{CE} = U(\tilde{r}) = E(\tilde{r}) - \frac{1}{2}A\sigma^2(\tilde{r})$$

Which Asset to Choose?

- For the risky asset in our example, where $E(\tilde{r}) = 0.25$ and $\sigma_r^2 = 0.56$, let's determine r_{CE} for different levels of risk-aversion:

A	0.04	0.50	0.78	1.00
r_{CE}	24%	11%	3%	-3%

- If you have $A = 0.50$ would you hold the risky or risk-free asset?
- What level of risk-aversion do you have to have to be indifferent between the risky and the risk-free asset?
- If you are more risk-averse will r_{CE} be larger or smaller?

"MY FIRST PIECE OF ADVICE IS NOT TO PUT ALL YOUR EGGS IN ONE BASKET."

- Of course, we don't usually have a *binary* choice like this: we can hold a *portfolio* of risky and risk-free assets.
- Let's determine how to build an optimal portfolio of risky and risk-free assets.
- Calculating the return on a portfolio p consisting of of one risky asset and a risk-free asset.

$$
\begin{aligned}
\tilde{r}_A &= \text{return on (risky) asset A} && = \tilde{r}_A \\
E(\tilde{r}_A) &= \text{expected risky rate of return} && = 25\% \\
\sigma_A &= \text{standard deviation} && = 75\% \\
r_f &= \text{risk-free rate} && = 3\% \\
w &= \text{fraction of portfolio } p \text{ invested in asset A} && = ??
\end{aligned}
$$

- The return and expected return on a portfolio with weight w on the risky security and $1-w$ on the risk-free asset is:

$$\begin{aligned} \tilde{r}_p &= w\tilde{r}_A + (1-w)\cdot r_f \\ \tilde{r}_p &= r_f + w(\underbrace{\tilde{r}_A - r_f}_{\tilde{r}^e_A}) \\ E(\tilde{r}_p) &= r_f + wE(\tilde{r}^e_A) \end{aligned} \tag{1}$$

- The risk (variance) of this combined portfolio is:

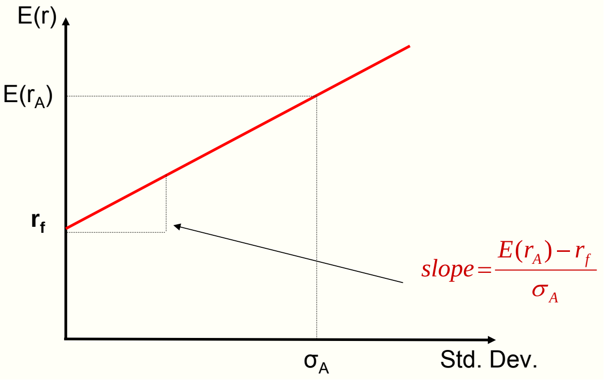
$$\begin{aligned} \sigma^2_p &= E[(\tilde{r}_p - \overline{r_p})^2] \\ &= E((w\tilde{r}_A - w\overline{r_A})^2] \\ &= w^2E[(\tilde{r}_A - \overline{r_A})^2] \\ &= w^2\sigma^2_A \end{aligned} \tag{2}$$

- We can derive the Capital Allocation Line, i.e. the set of investment possibilities created by all combinations of the risky and riskless asset.
- Combining (1) and (2), we can characterize the expected return on a portfolio with σ_p:

$$E(\tilde{r}_p) = r_f + \underbrace{\left[\frac{E(\tilde{r}_A) - r_f}{\sigma_A}\right]}_{\text{price of risk}} \underbrace{\sigma_p}_{\text{amount of risk}}$$

- The price of risk is the return premium per unit of portfolio risk (standard deviation) and depends **only** the prices of available securities.
- The standard term for this ratio is the *Sharpe Ratio*.

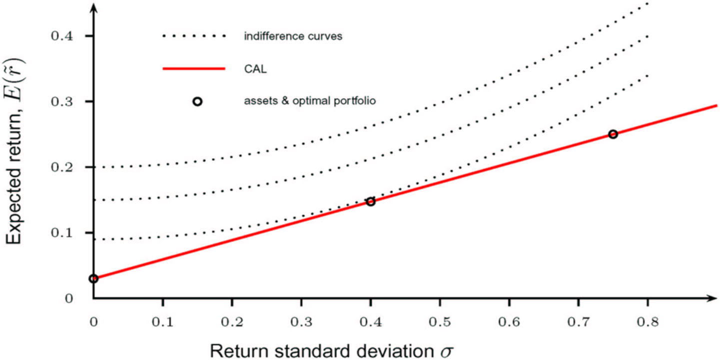
Capital Allocation Line



- The CAL shows all risk-return combinations possible from a portfolio of one risky-asset and the risk-free return.
- The slope of the CAL is the *Sharpe Ratio*.

Which Portfolio?

- Which risk-return combination along the CAL do we want?
- To answer this we need the utility function!



Which Portfolio?

- Mathematically, the optimal portfolio is the solution to the following problem:

$$U^* = \max_w U(\tilde{r}_p) = \max_w E(\tilde{r}_p) - \frac{1}{2}A\sigma_p^2$$

where, we know,

$$E(\tilde{r}_p) = r_f + wE(\tilde{r}_A - r_f) \qquad \sigma_p^2 = w^2\sigma_A^2$$

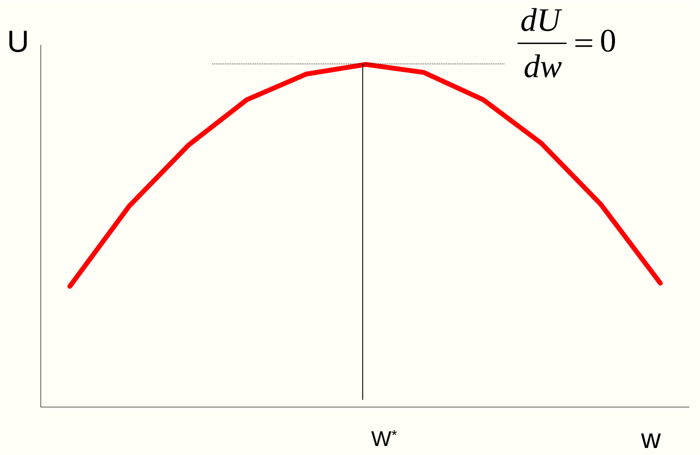
Combining these two equations we get:

$$\max_w U(\tilde{r}_p) = \max_w \left(r_f + wE(\tilde{r}_A - r_f) - \frac{1}{2}Aw^2\sigma_A^2 \right)$$

Solution

$$\frac{dU}{dw}\Big|_{w=w^*} = 0 \Rightarrow w^* = \frac{E(\tilde{r}_A - r_f)}{A\sigma_A^2}$$

Utility as a function of portfolio weight



- At the optimum, investors are indifferent between small changes in w.

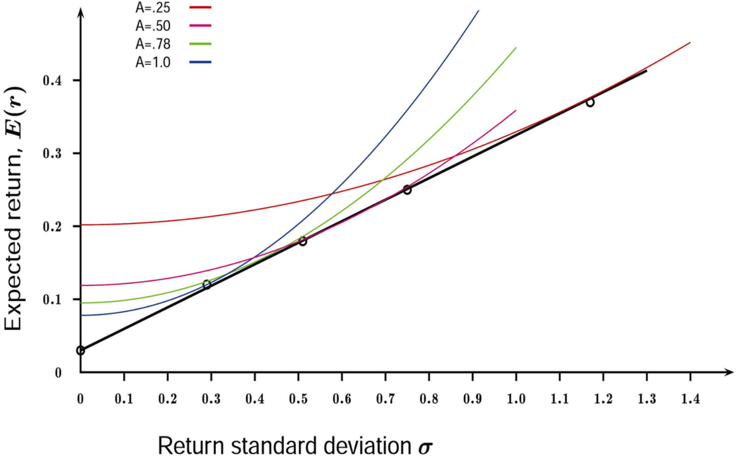
Which Portfolio?

- For the example we are considering:

A	w^*	$E(r_p)$	σ_p
0.25	1.56	0.37	1.17
0.50	0.78	0.20	0.51
0.78	0.49	0.14	0.37
1.00	0.39	0.12	0.29

- What is the meaning of 1.56 in the above table?
- Can you ever get a negative w^*?
- How do changes in A affect the optimal portfolio?
- How do changes in the Sharpe ratio affect the optimal portfolio?

Which Portfolio?



- Different level of risk aversion leads to different choices.

The Investor's Risk Tolerance

Asset **Allocation Planner** Step 1

Your Personal Risk Tolerance

Your prior investment experience can help determine your attitude toward investment risk.

10 Have you ever invested in individual bonds or bond mutual funds? (Aside from U.S. savings bonds.)

- ☐ No, and I would be uncomfortable with the risk if I did.
- ☐ No, but I would be comfortable with the risk if I did.
- ☐ Yes, but I was uncomfortable with the risk.
- ☐ Yes, and I felt comfortable with the risk.

11 Have you ever invested in individual stocks or stock mutual funds?

- ☐ No, and I would be uncomfortable with the risk if I did.
- ☐ No, but I would be comfortable with the risk if I did.
- ☐ Yes, but I was uncomfortable with the risk.
- ☐ Yes, and I felt comfortable with the risk.

Your comfort level with investment risk influences how aggressively or conservatively you choose to invest. It should be balanced with the potential of achieving your investment goals.

12 Which ONE of the following statements best describes your feelings about investment risk?

- ☐ I would only select investments that have a low degree of risk associated with them (i.e., it is unlikely I will lose my original investment).
- ☐ I prefer to select a mix of investments with emphasis on those with a low degree of risk and a small portion in others that have a higher degree of risk that may yield greater returns.
- ☐ I prefer to select a balanced mix of investments -- some that have a low degree of risk, others that have a higher degree of risk that may yield greater returns.
- ☐ I prefer to select an aggressive mix of investments -- some that have a low degree of risk, but with emphasis on others that have a higher degree of risk that may yield greater returns.
- ☐ I would select an investment that has only a higher degree of risk and a greater potential for higher returns.

13 If you could increase your chances of improving your returns by taking more risk, would you:

- ☐ Be willing to take *a lot* more risk with *all* your money.
- ☐ Be willing to take *a lot* more risk with *some* of your money.
- ☐ Be willing to take *a little* more risk with *all* your money.
- ☐ Be willing to take *a little* more risk with *some* of your money.
- ☐ Be unlikely to take much more risk.

The Investor's Risk Tolerance, example

- What are reasonable values for A?
- This is a very tricky question and the subject of much debate among financial economists.
  - $\hookrightarrow$ At most, we can infer A from investor's choices, *conditional on a particular model*.
- A common investment advice is to invest 60% of your portfolio in stocks and 40% in bonds. Using this framework, and the moments of stocks and T-bills we saw earlier, what is the investor risk tolerance implicit in this advice?

Two Risky Assets and no Risk-Free Asset.

- Now that we understand how to allocate capital between the risky and riskfree asset, we need to show that it is really true that there is a single optimal risky portfolio.

- To start, we'll ask the question:

  How should you combine two risky securities in your portfolio?

- We will plot out possible set of expected returns and standard deviations for different combinations of the assets.

- **Definition:** *Minimum Variance Frontier*, is the set of portfolios with the lowest variance for a given expected return.

A Portfolio of Two Risky Assets

1. The expected return for the portfolio is

$$E(\tilde{r}_p) = w \cdot E(\tilde{r}_A) + (1-w) \cdot E(\tilde{r}_B)$$

   ↪ $w \equiv w_A^p$ is the fraction that is invested in asset A.
   ↪ Note that, based on this, $w_B^p = (1-w)$

2. The variance of the portfolio is:

$$\begin{aligned} \sigma_p^2 &= E[(\tilde{r}_p - \overline{r_p})^2] \\ &= w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)cov(\tilde{r}_A, \tilde{r}_B) \end{aligned}$$

   or, since $\rho_{AB} = cov(\tilde{r}_A, \tilde{r}_B)/(\sigma_A \cdot \sigma_B)$,

$$\sigma_p^2 = w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\rho_{AB}\sigma_A\sigma_B$$

3. Notice that the variance of the portfolio depends on the correlation between the two securities.

A Portfolio of Two Risky Assets: Example

- As an example let's assume that we can trade in asset A of the previous example, and in an asset B:

Asset	$E(\tilde{r})$	σ
A	25%	75%
B	10%	25%

- To develop intuition for how correlation affects the risk of the possible portfolios, we will derive the minimum variance frontier under 3 different assumptions:
  1. $\rho_{AB} = 1$
  2. $\rho_{AB} = -1$
  3. $\rho_{AB} = 0$

Case $\rho_{AB} = 1$

- Plug these numbers into these two equations:

$$\begin{aligned} E(\tilde{r}_p) &= 0.25w + 0.10 \cdot (1-w) \\ &= 0.15w + 0.10 \\ \sigma_p &= 0.75w + 0.25 \cdot (1-w) \\ &= 0.50w + 0.25 \end{aligned}$$

- In *Excel*, we can build a table with various possible w's:

w	$E(\tilde{r}_p)$	σ_p
-.5	2.5%	0.0%
0	10%	25%
0.5	17.5%	50.0%
1	25.0%	75.0%
1.5	32.5%	100.0%

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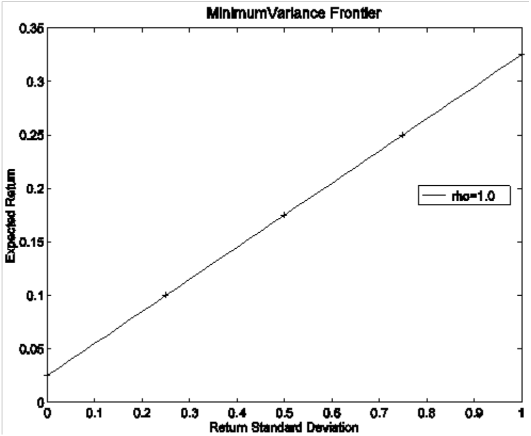
Two Risky assets, $\rho_{AB} = 1$



- The picture looks very similar to the case where there one risky and one riskless asset.
- Because the two assets are perfectly correlated, we can build a 'synthetic' riskless asset.

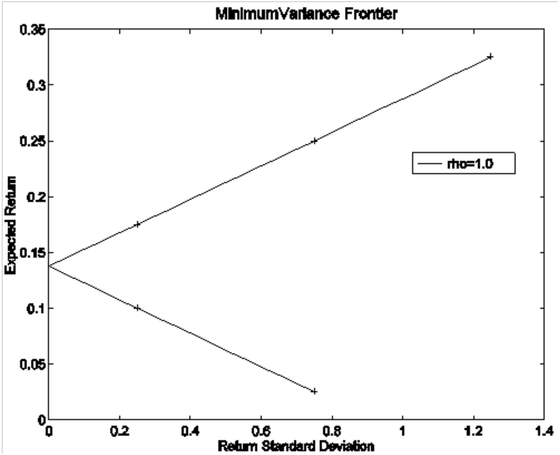
Case $\rho_{AB} = -1$

- When $\rho = -1$ we can again simplify the variance equation:

$$
\begin{aligned}
\sigma_p^2 &= w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\rho_{AB}\sigma_A\sigma_B \\
\sigma_p^2 &= w^2\sigma_A^2 + (1-w)^2\sigma_B^2 - 2w(1-w)\sigma_A\sigma_B \\
&= (w\sigma_A - (1-w)\sigma_B)^2 \\
\sigma_p &= |w\sigma_A - (1-w)\sigma_B|
\end{aligned}
$$

- Again, if we create a table of the expected returns and variances for different weights and plot these, we get: (here for $-0.5 \le w \le 1.5$):

Two Risky assets, $\rho_{AB} = -1$



- Because the two assets are perfectly correlated, we can build a 'synthetic' riskless asset.
- Some combinations are 'dominated' in this case. Which ones?

- In the cases where $|\rho| = 1$, it is possible to find a perfect hedge with these 2 securities.
- **Definition:** *Perfect Hedge* is a hedge that gives a portfolio with zero risk. ($\sigma_p = 0$)
- To solve out for this zero risk portfolio in the case $\rho = -1$, set the risk to zero and solve for w:

$$\begin{aligned} \sigma_p &= w\sigma_A - (1-w)\sigma_B \\ 0 &= w\sigma_A - (1-w)\sigma_B \\ \Rightarrow w &= \sigma_B/(\sigma_A + \sigma_B) = 0.25 \end{aligned}$$

Plugging this into the expected return equation we get:

$$\begin{aligned} E(r_p) &= 0.25w + 0.10(1-w) \\ &= (0.25)(0.25) + (0.10)(0.75) = 13.75\% \end{aligned}$$

- We have created a 'synthetic' risk-free security!

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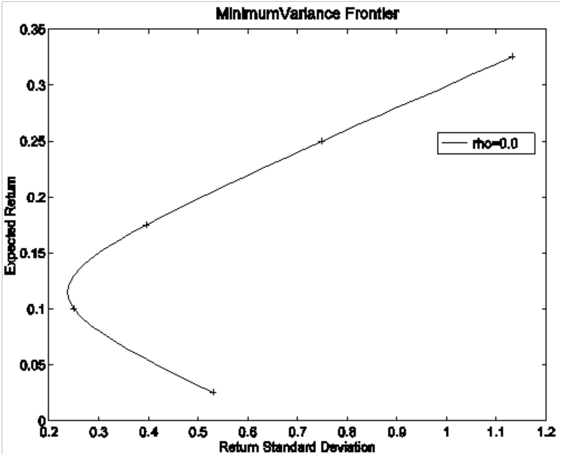
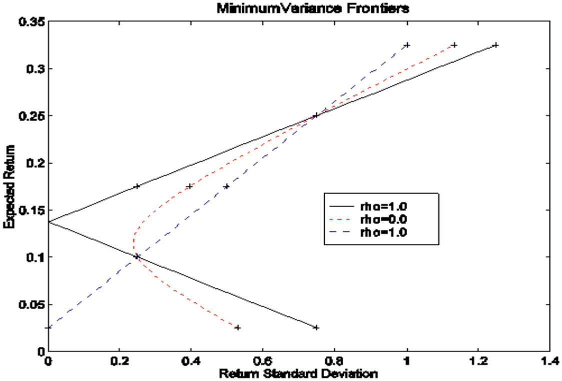
Two Risky assets, $\rho_{AB} = 0$



- The plot shows that there is now some hedging effect (though not as much as when $\rho = 1$ or $\rho = -1$).

All cases together



- To calculate the minimum variance frontier in this 2 asset world you the following:
  $\hookrightarrow$ $E(r_A)$, $E(r_B)$, σ_A, σ_B, and ρ_{AB}
- Where do you get these in the real world?

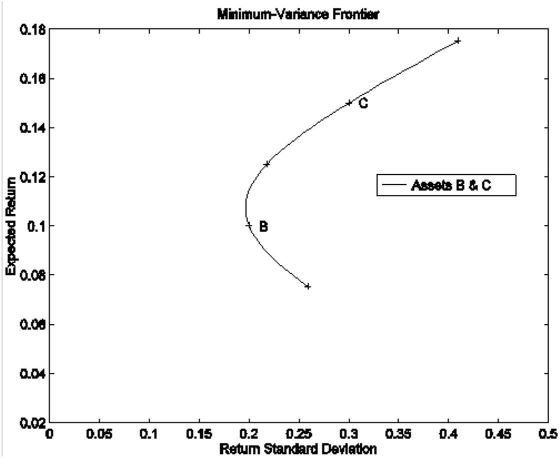
- For this section, let's assume we can only trade in the risk-free asset ($r_f = 0.03$) and risky assets B and C, where,

Asset	$E(r)$	σ
B	10%	20%
C	15%	30%

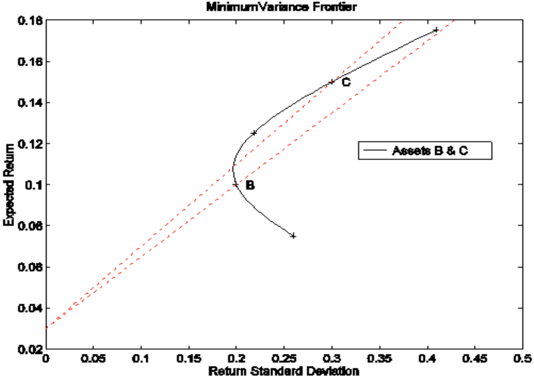
and $\rho_{BC} = 0.5$.

- We can compute the Minimum Variance frontier created by combinations of the B and C

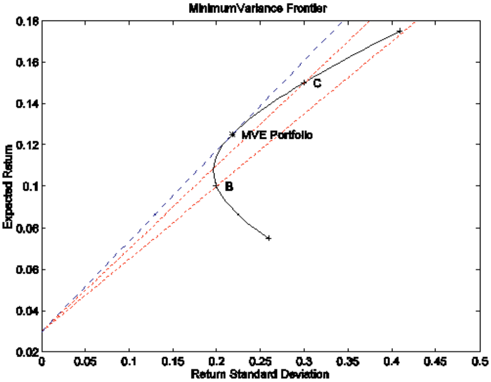
The CAL with Two Risky Assets



- Now, lets look at the situation where we can include either B, C or the risk-free asset in our portfolio.
- If we use either the risk-free asset plus asset B, or the risk-free asset plus asset C, the two possible CALs are:



- What is the optimal combination?



- We call this portfolio the tangency portfolio (why?). In combination with the risk-free asset, it provides the "steepest" CAL (the one with the highest slope)
- It is sometimes called the *Mean-Variance Efficient* or *MVE* portfolio.
- Why is this the portfolio we want?

MVE portfolio

- How do we find *MVE* portfolio mathematically?
- Find the portfolio with the highest Sharpe Ratio (Why?):

$$\max_{w} \frac{E(\tilde{r}_p) - r_f}{\sigma_p}$$

where

$$\begin{aligned} E(\tilde{r}_p) &= wE(r_B) + (1-w)E(\tilde{r}_C) \\ \sigma_p &= \left[w^2\sigma_B^2 + (1-w)^2\sigma_C^2 + 2w(1-w)\rho_{BC}\sigma_B\sigma_C\right]^{1/2} \end{aligned}$$

- Unfortunately, the solution is pretty complicated:

$$w_B^p = \frac{E(\tilde{r}_B^e)\sigma_C^2 - E(\tilde{r}_C^e)cov(\tilde{r}_B, \tilde{r}_C)}{E(\tilde{r}_B^e)\sigma_C^2 + E(\tilde{r}_C^e)\sigma_B^2 - \left[E(\tilde{r}_B^e) + E(\tilde{r}_C^e)\right] cov(r_B, r_C)}$$

MVE portfolio

- Here, $w_B^p = 0.5$, which gives a mean and a variance for this portfolio of $E(r_{MVE}) = 0.1250$ and $\sigma_{MVE} = 0.2179$.
- Also, the *Sharpe Ratio* of the MVE portfolio is:

$$SR_{MVE} = \frac{E(r^e_{MVE})}{\sigma_{MVE}} = \frac{0.095}{0.2179} = 0.4359$$

- Assets B and C have Sharpe Ratios of 0.35 and 0.40.
- The Sharpe Ratio of the MVE portfolio is higher than those of assets B and C. Will this always be the case?
- Have we determined the optimal allocation between risky and riskless assets?

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MVF with many risky assets.

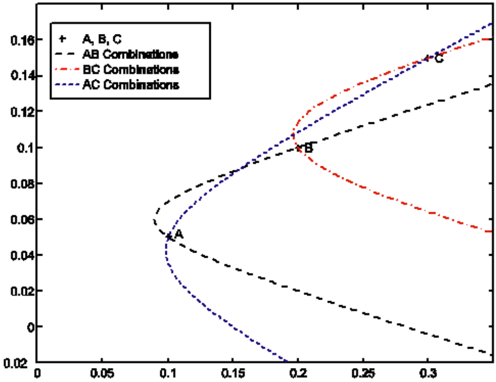
- Now let's look at the optimal portfolio problem when there are three assets.
- The expected returns, standard deviations, and correlation matrix are again:

Asset	$E(r)$	σ
A	5%	10%
B	10%	20%
C	15%	30%

Correlations			
Assets	A	B	C
A	1.0	0.0	0.5
B	0.0	1.0	0.5
C	0.5	0.5	1.0

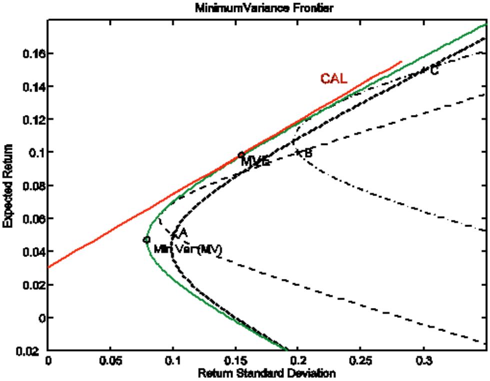
- What does the minimum variance frontier look like now?

MVF with many risky assets.



- If we combine A and B, B and C, or A and C, we get the above possible portfolio combinations.
- However, we can do better.

MVF with many risky assets.



- This plot adds the mean-variance frontier and the CAL to the two-asset portfolios.
- This shows that the MVE portfolio will be a portfolio of portfolios, or a combination of all of the assets.

Allocating Capital among Multiple Risky Securities

- The mathematics of the problem quickly become complicated as we add more risky assets.
- We are going to need a general purpose method for solving the multiple asset problem.
- Fortunately Excel's "solver" function can solve the problem using the spreadsheet called MarkowitzII.xls which can be found on the course homepage.
- Today, we'll go through a brief tutorial on how to use the program.

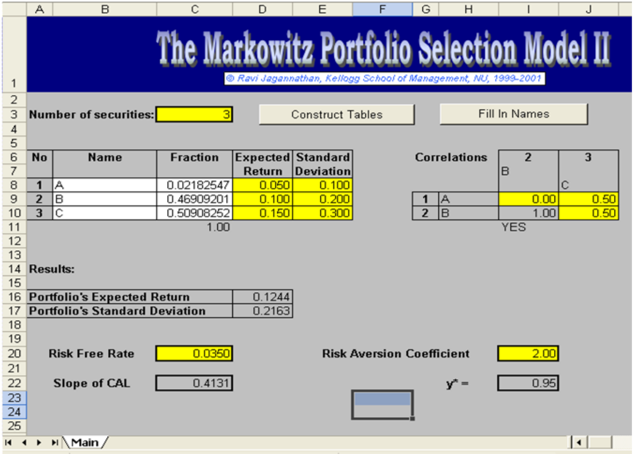
MVF with many risky assets.

- Take the three risky assets A,B,C from before with $r_f = 3.5\%$.
- The relevant expected returns, standard deviation, and correlation data are:

Asset	$E(r)$	σ
A	5%	10%
B	10%	20%
C	15%	30%

Correlations			
Assets	A	B	C
A	1.0	0.0	0.5
B	0.0	1.0	0.5
C	0.5	0.5	1.0

- Just fill in the input data (yellow cells).
- The slope of the CAL, optimal weights w and $E(r_{MVE})$, σ_{MVE} can be calculated for you!



The Markowitz Portfolio Selection Model II

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Number of securities: 3 | Construct Tables | Fill In Names

No	Name	Fraction	Expected Return	Standard Deviation
1	A	0.02182547	0.050	0.100
2	B	0.46909201	0.100	0.200
3	C	0.50908252	0.150	0.300
		1.00		

Correlations		2 B	3 C
1	A	0.00	0.50
2	B	1.00	0.50
		YES	

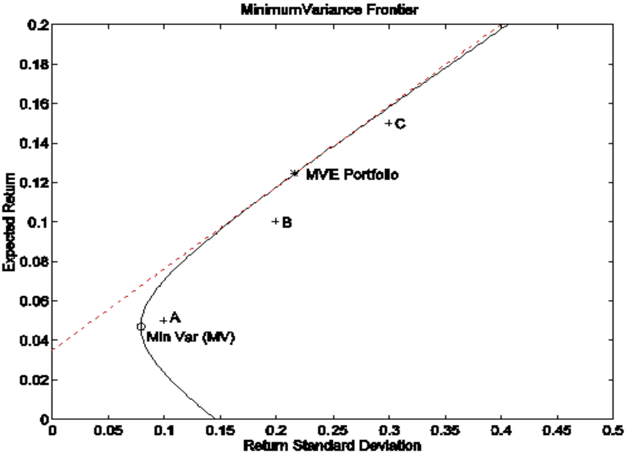
Results:

Portfolio's Expected Return	0.1244
Portfolio's Standard Deviation	0.2163

Risk Free Rate	0.0350	Risk Aversion Coefficient	2.00
Slope of CAL	0.4131	y* =	0.95

Main

MVF with many risky assets

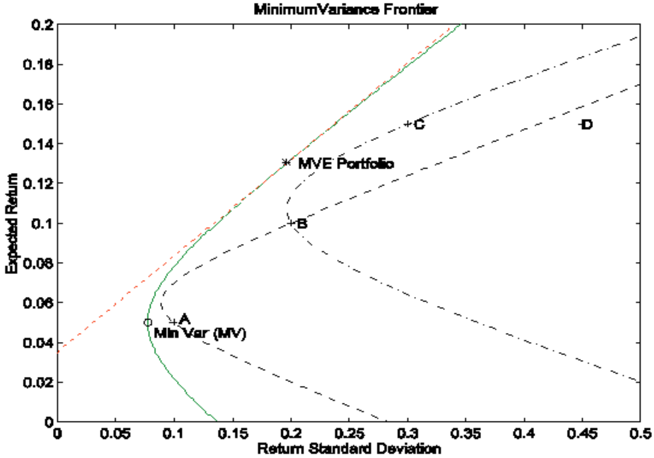


$$\mathbf{w}_{MVE} = \begin{bmatrix} 0.0218 \\ 0.4619 \\ 0.5091 \end{bmatrix}$$

$$\bar{r}_{MVE} = 0.1244$$
$$\sigma_{MVE} = 0.2163$$
$$SR_{MVE} = 0.4131$$

MVF with many risky assets

- Let's add a fourth security, say, D with $E(r_D) = 15\%$, $\sigma_D = 45\%$.
- Assume that it is a zero correlation with all of the other securities.
- Will anyone want to hold this security?



MVF with many risky assets

- The optimal allocation looks like this

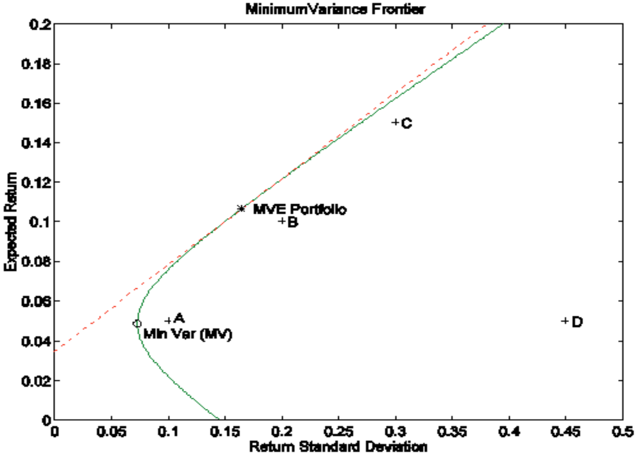
$$\mathbf{w}_{MVE} = \begin{bmatrix} 0.0168 \\ 0.3616 \\ 0.3924 \\ 0.2292 \end{bmatrix}$$

$$\begin{aligned} \bar{r}_{MVE} &= 0.1302 \\ \sigma_{MVE} &= 0.1961 \\ SR_{MVE} &= 0.4858 \end{aligned}$$

- What explains this?
  - ↪ D is apparently strictly dominated by C
  - ↪ D is uncorrelated with A,B,C
  - ↪ How can D contribute to the overall portfolio?
  - ↪ Wouldn't increasing the share of C by 23% dominate the allocation above?

MVF with many risky assets

- Let's change the fourth security. Suppose D has $E(r_D) = 5\%$, $\sigma_D = 45\%$.
- Assume that it is a correlation of -0.2 with all of the other securities.
- Will anyone want to D now?



MVF with many risky assets

- The new optimal allocation looks like this

$$\mathbf{w}_{MVE} = \begin{bmatrix} 0.1215 \\ 0.3924 \\ 0.3685 \\ 0.1175 \end{bmatrix}$$

$$\bar{r}_{MVE} = 0.1065$$
$$\sigma_{MVE} = 0.1646$$
$$SR_{MVE} = 0.4342$$

- Compare to previous ($E(r_D) = 15\%$, zero correlation):

$$\mathbf{w}_{MVE} = \begin{bmatrix} 0.0168 \\ 0.3616 \\ 0.3924 \\ 0.2292 \end{bmatrix}$$

$$\bar{r}_{MVE} = 0.1302$$
$$\sigma_{MVE} = 0.1961$$
$$SR_{MVE} = 0.4858$$

- Why are we still holding D?
- Are we worse off with the new D?

MVF with many risky assets

- Basic Message: Your risk/return tradeoff is improved by holding many assets with less than perfect correlation.
- Far from everybody agrees:

"To suppose that safety-first consists in having a small gamble in a large number of different companies where I have no information to reach good judgment, as compared with a substantial stake in a company where one's information is adequate, strikes me as a travesty of investment policy"

- J.M. Keynes 1939

"Diversification is an admission of not knowing what to do and striking an average"

- G. Loeb, 1935

"…and is the business understandable? Despite high regard for Microsoft, Mr. Buffet avoids its stock because the field puzzles him. Ignorance, he says, increases danger. This belief is a departure from the common wisdom of stock diversification. Owning many different stocks – good, bad and mediocre – depresses the returns a more selective portfolio would achieve, he believes, and makes it impossible to understand all that you own. Thus in 1987 his $2 billion portfolio had just three companies. Today nearly $15 billion is spread among just 10."

- W. Buffet, 1996

Kellogg School of Management

Understanding Diversification

1. Start with our equation for variance:

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} w_i w_j cov(\tilde{r}_i, \tilde{r}_j)$$

2. Then make the simplifying assumption that $w_i = 1/N$ for all assets:

$$\sigma_p^2 = \left(\frac{1}{N^2}\right) \sum_{i=1}^{N} \sigma_i^2 + \sum_{i=1}^{N} \left(\frac{1}{N^2}\right) \sum_{\substack{j=1 \\ i \neq j}}^{N} cov(\tilde{r}_i, \tilde{r}_j)$$

3. The average variance and covariance of the securities are:

$$\overline{\sigma^2} = \left(\frac{1}{N}\right) \sum_{i=1}^{N} \sigma_i^2 \qquad \overline{cov} = \frac{1}{N(N-1)} \sum_{\substack{j=1 \\ i \neq j}}^{N} cov(\tilde{r}_i, \tilde{r}_j)$$

Understanding Diversification

1. Plugging these into our equation gives:

$$\sigma_p^2 = \left(\frac{1}{N}\right)\overline{\sigma^2} + \left(\frac{N-1}{N}\right)\overline{cov}$$

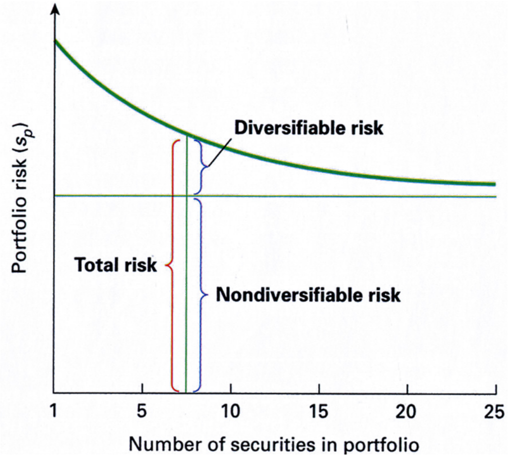
2. What happens as N becomes large?

$$\left(\frac{1}{N}\right) \to 0 \text{ and } \left(\frac{N-1}{N}\right) \to 1$$

3. *Only the average covariance matters for large portfolios.*
4. If the average covariance is zero, then the portfolio variance is close to zero for large portfolios.

Understanding Diversification

- This plot shows how the standard deviation of a portfolio of average NYSE stocks changes as we change the number of assets in the portfolio.



Understanding Diversification

- The component of risk that can be diversified away we call the *diversifiable* or *non-systematic* risk.

- Empirical Facts

  ↪ The average (annual) return standard deviation is 49%

  ↪ The average (annual) covariance between stocks is 0.037, and the average correlation is about 39%.

- Since the average covariance is positive, even a very large portfolio of stocks will be risky. We call the risk that cannot be diversified away the *systematic risk.*

Conclusions

In this lecture we have developed mean-variance portfolio analysis.

1. We call it mean-variance analysis because we assume that all that matters to investors is the average return and the return variance of their portfolio.

   $\hookrightarrow$ This is appropriate if returns are normally distributed.

2. There are a couple of key lessons from mean-variance analysis:

   $\hookrightarrow$ You should hold the same portfolio of risky assets no matter what your tolerance for risk.

   - If you want less risk, combine this portfolio with investment in the risk-free asset.
   - If you want more risk, buy the portfolio on margin.

   $\hookrightarrow$ In large portfolios, covariance is important, not variance.

What is wrong with mean-variance analysis?

- Not much! This is one of the few things in finance about which there is complete agreement.
  - $\hookrightarrow$ *Caveat:* remember that you have to include every asset you have in the analysis; including human capital, real estate, *etc.*
- Finally, Markowitz's theory tells us nothing about where the prices, returns, variances or covariances come from.
  - $\hookrightarrow$ This is what we will spend much of the rest of the course on!
- We will visit some of this issues next week.