# Seminar 13

$\boxed{32.43}$ Să se determine rangul listelor de vectori din $\mathbb{R}^4$:

(1) $\left[[0,1,3,2],[1,0,5,1],[-1,0,1,1],[3,-1,-3,-4],[2,0,1,-1]\right]^t$

$$A=\begin{pmatrix} 0 & 1 & -1 & 3 & 2 \\ 1 & 0 & 0 & -1 & 0 \\ 3 & 5 & 1 & -3 & 1 \\ 2 & 1 & 1 & -4 & -1 \end{pmatrix} \overset{C_1 \leftrightarrow C_2}{\sim} \begin{pmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 0 & -1 & 0 \\ 5 & 3 & 1 & -3 & 1 \\ 1 & 2 & 1 & -4 & -1 \end{pmatrix}$$

$$\underset{L_4-L_1}{\overset{L_3-5L_1}{\sim}} \begin{pmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 3 & 6 & -18 & -9 \\ 0 & 2 & 2 & -7 & -3 \end{pmatrix} \underset{L_4-2L_2}{\overset{L_3-3L_2}{\sim}} \begin{pmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 6 & -15 & -9 \\ 0 & 0 & 2 & -5 & -3 \end{pmatrix}$$

$$\overset{L_4-\frac{1}{3}L_3}{\sim} \begin{pmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 6 & -15 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \operatorname{rang} A = 3$$

(pe o pseudo-diagonală avem doar 3 valori $\neq 0$)

(2) $\left[[1,2,3,0],[0,1,-1,1],[3,7,8,1],[1,3,2,1]\right]^t$

$$A=\begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 1 & 7 & 3 \\ 3 & -1 & 8 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \underset{L_3-3L_1}{\overset{L_2-2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \underset{L_4-L_2}{\overset{L_3+L_2}{\sim}}$$

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \operatorname{rang} A = 2$$

(3) $\left[ [1,2,-1,2], [2,3,0,-1], [2,4,0,6], [1,2,1,4], [3,6,-1,-1], [1,3,-1,0] \right]^t$

$$A = \begin{pmatrix} 1 & 2 & 2 & 1 & 3 & 1 \\ 2 & 3 & 4 & 2 & 6 & 3 \\ -1 & 0 & 0 & 1 & -1 & -1 \\ 2 & 1 & 6 & 4 & -1 & 0 \end{pmatrix} \underset{\substack{L_3+L_1 \\ L_4-2L_1}}{\overset{L_2-2L_1}{\sim}} \begin{pmatrix} 1 & 2 & 2 & 1 & 3 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & 2 & 0 \\ 0 & -5 & -2 & 0 & -13 & -6 \end{pmatrix}$$

$$\underset{L_4-4L_2}{\overset{L_3+2L_2}{\sim}} \begin{pmatrix} 1 & 2 & 2 & 1 & 3 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & -2 & 0 & -13 & -10 \end{pmatrix} \overset{L_3+L_4}{\sim} \begin{pmatrix} 1 & 2 & 2 & 1 & 3 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & -11 & -8 \end{pmatrix}$$

$\Rightarrow$ rang $A = 4$

$\boxed{3.2.45}$ Să se calculeze dim. și câte o bază a subspațiilor Kerf și Imf, în cazurile:

(4) $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f[x_1, x_2] = [x_1 - 3x_2, 2x_1, -x_1 + x_2]$

$Kerf = \{x \in \mathbb{R}^2 \mid f(x) = 0\}$

$f(x_1, x_2) = (0,0,0) \iff (x_1 - 3x_2, 2x_1, -x_1 + x_2) = (0,0,0)$

$$\Rightarrow \begin{cases} x_1 - 3x_2 = 0 \\ 2x_1 = 0 \\ -x_1 + x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 = 0$$

$Kerf = \{(0,0)\} \Rightarrow \dim Kerf = 0$

$Imf = \{y \in \mathbb{R}^3 \mid f(x) = y\}$

$f(x_1, x_2) = (y_1, y_2, y_3) \iff (x_1 - 3x_2, 2x_1, -x_1 + x_2) = (y_1, y_2, y_3)$

$$\Rightarrow (5) \begin{cases} x_1 - 3x_2 = y_1 \\ 2x_1 = y_2 \\ -x_1 + x_2 = y_3 \end{cases} \qquad \bar{A} = \left(\begin{array}{cc|c} 1 & -3 & y_1 \\ 2 & 0 & y_2 \\ -1 & 1 & y_3 \end{array}\right)$$

$y \in Imf \iff$ (5) compatibil $\iff rang A = rang \bar{A}$

$rang A = 2 \Rightarrow rang \bar{A} \neq 3$

$\Rightarrow \quad \det B = 0$

$$\begin{vmatrix} 1 & -3 & y_1 \\ 2 & 0 & y_2 \\ -1 & 1 & y_3 \end{vmatrix} = 0 \Rightarrow 0 + 2y_1 + 3y_2 + 0 - y_2 + 6y_3 = 0$$

$2y_1 + 2y_2 + 6y_3 = 0 \mid :2$

$y_1 + y_2 + 3y_3 = 0$

$y_1 = -3y_3 - y_2$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = y_1 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + y_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + y_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow \quad b = [(-1, 1, 0), (-3, 0, 1)\}$ bază

- $f : V \to W$ <u>aplicație liniară</u>

$\dim V = \dim Ker f + \dim Im f$

$\dim Im f = \dim V - \dim Ker f$

$x = (x_1, x_2) \qquad \alpha x = (\alpha x_1, \alpha x_2)$

$y = (y_1, y_2) \qquad \beta y = (\beta y_1, \beta y_2)$

$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = (\underline{\alpha x_1} + \underline{\beta y_1} \underline{-3\alpha x_2} \underline{-3\beta y_2},$

$\underline{2\alpha x_1} + \underline{2\beta y_1}, \underline{-\alpha x_1} - \underline{\beta y_1} \underline{+\alpha x_2} + \underline{\beta y_2}) =$

$= (\alpha x_1 - 3\alpha x_2, 2\alpha x_1, -\alpha x_1 + \alpha x_2) +$

$+ (\beta y_1 - 3\beta y_2, 2\beta y_1, -\beta y_1 + \beta y_2)$

$= \alpha(x_1 - 3x_2, 2x_1, -x_1 + x_2) + \beta(y_1 - 3y_2, 2y_1, -y_1 + y_2)$

$= \alpha f(x) + \beta f(y)$

$\Rightarrow f$ este apl. liniară $\Rightarrow \dim Im f = 2 - 0 = 2$

3.3.10 Fie $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x_1, x_2, x_3) = (x_2, -x_1)$ și $v = [[1,1,0], [0,1,1], [1,0,1]]^t$ și $w = [[1,1], [1,-2]]^t$

a.) Să se arate că $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$

b.) Să se arate că $v$ și $w$ sunt baze în $\mathbb{R}^3$, resp. $\mathbb{R}^2$ și să se determine matricile $[f]_{v,e}$ și $[f]_{v,w}$ unde $e$ este baza canonică din $\mathbb{R}^2$.

<u>Soluție:</u> a.) $f(\alpha x + \beta y) \overset{?}{=} \alpha f(x) + \beta f(y)$

$x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$

$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$

$f(\alpha x + \beta y) = (\alpha x_2 + \beta y_2, -\alpha x_1 - \beta y_1)$

$\alpha f(x) + \beta f(y) = \alpha(x_2, -x_1) + \beta(y_2, -y_1) =$

$= (\alpha x_2 + \beta y_2, -\alpha x_1 - \beta y_1)$

$\Rightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ ✓

b.) • $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ (liniar ind.)

$$\begin{cases} \alpha_1 + 0\alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_2 + 0\alpha_3 = 0 \\ 0 \cdot \alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases}$$

$\det A \neq 0 \Rightarrow \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 + 1 + 0 - 0 - 0 - 0 = 2 \neq 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow v$ este bază în $\mathbb{R}^3$

• $\alpha_1 w_1 + \alpha_2 w_2 = 0$

$$\begin{cases} \alpha_1 + \alpha_2 = 0 \\ \alpha_1 - 2\alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0 \Rightarrow w \text{ este bază în } \mathbb{R}^2$$

$$[f]_{v,e} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \end{pmatrix} \in \mathcal{M}_{2,3}$$

$$\begin{cases} f(v_1) = \alpha_{11} e_1 + \alpha_{21} e_2 \\ f(v_2) = \alpha_{12} e_1 + \alpha_{22} e_2 \\ f(v_3) = \alpha_{13} e_1 + \alpha_{23} e_2 \end{cases} \Rightarrow \begin{cases} (1,-1) = \alpha_{11}(1,0) + \alpha_{21}(0,1) \\ (1,0) = \alpha_{12}(1,0) + \alpha_{22}(0,1) \\ (0,-1) = \alpha_{13}(1,0) + \alpha_{23}(0,1) \end{cases}$$

$$\Rightarrow \begin{cases} (1,-1) = (\alpha_{11}, \alpha_{21}) \\ (1,0) = (\alpha_{12}, \alpha_{22}) \\ (0,-1) = (\alpha_{13}, \alpha_{23}) \end{cases} \Rightarrow \begin{cases} \alpha_{11} = 1, \ \alpha_{21} = -1 \\ \alpha_{12} = 1, \ \alpha_{22} = 0 \\ \alpha_{13} = 0, \ \alpha_{23} = -1 \end{cases}$$

$$\Rightarrow [f]_{v,e} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$[f]_{v,w} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{33} \end{pmatrix}$$

$$\begin{cases} f(v_1) = \alpha_{11} \cdot w_1 + \alpha_{21} \cdot w_2 \\ f(v_2) = \alpha_{12} \cdot w_1 + \alpha_{22} \cdot w_2 \\ f(v_3) = \alpha_{13} \cdot w_1 + \alpha_{33} \cdot w_2 \end{cases} \Leftrightarrow \begin{cases} (1,-1) = \alpha_{11} \cdot (1,1) + \alpha_{21}(1,-2) \\ (1,0) = \alpha_{12}(1,1) + \alpha_{22}(1,-2) \\ (0,-1) = \alpha_{13}(1,1) + \alpha_{33}(1,-2) \end{cases}$$

$$\Leftrightarrow \begin{cases} (1,-1) = (\alpha_{11} + \alpha_{21}, \alpha_{11} - 2\alpha_{21}) \\ (1,0) = (\alpha_{12} + \alpha_{22}, \alpha_{12} - 2\alpha_{22}) \\ (0,-1) = (\alpha_{13} + \alpha_{33}, \alpha_{13} - 2\alpha_{33}) \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_{11} + \alpha_{21} = 1 \\ \alpha_{11} - 2\alpha_{21} = -1 \end{cases} \ominus$$

$$3\alpha_{21} = 2 \Rightarrow \boxed{\alpha_{21} = \frac{2}{3}}$$

$$\alpha_{11} = 1 - \frac{2}{3}$$

$$\boxed{\alpha_{11} = \frac{1}{3}}$$

$$\begin{cases} \alpha_{12} + \alpha_{22} = 1 \\ \alpha_{12} - 2\alpha_{22} = 0 \end{cases} \ominus$$

$$3\alpha_{22} = 1$$

$$\boxed{\alpha_{22} = \frac{1}{3}}$$

$$\alpha_{12} = 1 - \frac{1}{3}$$

$$\boxed{\alpha_{12} = \frac{2}{3}}$$

$$\begin{cases} \alpha_{13} + \alpha_{33} = 0 \\ \alpha_{13} - 2\alpha_{33} = -1 \end{cases} \ominus$$

$$3\alpha_{33} = +1$$

$$\boxed{\alpha_{33} = \frac{1}{3}} \Rightarrow \boxed{\alpha_{13} = -\frac{1}{3}}$$

$$\Rightarrow [f]_{v,w} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$