Chapter 2: Probability

Lianming Wang

University of South Carolina

January 4, 2023

Basic concepts:

- An **experiment** is an act or process of observation that leads to a single outcome that can not be predicted with certainty.
- A **sample point** is the most basic outcome of an experiment.
- The **sample space** of an experiment is the collection of all its sample points. Denoted by S.

A definition of probability

- The **probability** of a sample point is a number between 0 and 1 that measures the likelihood that the outcome will occur in an experiment.
- Let p_i denote the probability of sample point i.
  - $0 \leq p_i \leq 1$.
  - $\sum_i p_i = 1$.
- This definition is simple and general and applies only to discrete space in which there are only a countable number of outcomes.

- An **event** is a specific collection of sample points.
- The probability of an event A is calculated by summing the probability of the sample points contained in A.
- In the case that all sample points have the same probability,

$$P(A) = \frac{\#\text{ in A}}{\#\text{ in S}}.$$

Counting techniques

- Multiplication rule for counting

  Assume an operation can be described as a sequence of k steps, and the number ways of completing step i is n_i for $i = 1, \cdots, k$. Then the total number of ways of completing the operation is

$$n_1 \cdot n_2 \cdot \cdots \cdot n_k.$$

Permutation rule:

- Then the number of orderings of r different objects selected from a set of n different element is

$$P_r^n = n \times (n-1) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}$$

where $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

Combination rule

- Suppose a sample of k elements is to be drawn from a set of n elements. Then the number of different sample possible is

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$$

- Note: we define $0! = 1$ for mathematical convenience.

- The **union** of two events A and B is the event that occurs if either A or B (or both) occurs. The union of A and B, denoted by $A \cup B$, consists of all the sample points that belong to A or B or both.
- The **intersection** of two events A and B is the event that occurs if both A and B occur. The intersection of A and B, denoted by $A \cap B$, consists of all the sample points that belong to A and B.

- Event A and event B are **mutually exclusive** if the intersection of A and B contains no sample points. In this case, $A \cap B = \phi$, an empty set.
- For mutually exclusive events A and B, $P(A \cap B) = 0$.
- **Additive Rule of Probability**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- For mutually exclusive events A and B, $P(A \cup B) = P(A) + P(B)$.

- The **complement** of an event A is the event that A does not occur - that is, the event consisting of all sample points that are not in event A. We denote the complement of A by A^c.
- For complementary events A and A^c, $A \cup A^c = S$ and $A \cap A^c = \phi$.
- Complementary rule of probability: $P(A^c) = 1 - P(A)$.

Exercise 1

Four bits are transmitted over a digital communications channel. Each bit is either distorted or received without distortion. Let A_i denote the event that the ith bit is distorted, $i = 1, 2, 3, 4$.

a. Describe the sample space for this experiment.

b. List all elements in A_1.

c. Are these A_i's mutually exclusive?

d. Describe the outcomes in the following events:
   (i) $A_1 \cap A_2 \cap A_3 \cap A_4$
   (ii) $A_1 \cup A_2 \cup A_3 \cup A_4$

N means bit is not distorted,
D means bit is distorted

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a. $S = \{$ NNNN, NNND, NNDN, NNDD,
NDNN, NDND, NDDN, NDDD,
DNNN, DNND, DNDN, DNDD,
DDNN, DDND, DDDN, DDDD $\}$

b. $A_1 = \{$ DNNN, DNND, DNDN, DNDD,
DDNN, DDND, DDDN, DDDD $\}$

c. No, then there would only be five points in the sample space: $\{$ NNNN, NNND, NNDN, NDNN, DNNN $\}$
For these events to be mutually exclusive,
$A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset$ must be true.

d. (i). $\{DDDD\}$

(ii). $\{NNND, NNDN, NNDD,$
$NDNN, NDND, NDDN, NDDD,$
$DNNN, DNND, DNDN, DNDD,$
$DDNN, DDND, DDDN, DDDD\}$

Exercise 2: Disks of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized as follows:

	Shock Resistance	
Scratch Resistance	High	Low
High	70	9
Low	16	5

If one disk is randomly selected from those 100 disks,

a. What is the probability that the selected disk has high scratch resistance?

b. What is the probability that the selected disk has high scratch resistance and high shock resistance?

c. What is the probability that the selected disk has high scratch resistance but low shock resistance?

a. There are 79 disks with High Scratch Resistance out of 100 total, so the probability of getting one randomly is $\frac{79}{100}$.

b. There are only 70 disks with both High Scratch Resistance and High Shock Resistance, so the probability of getting one randomly is $\frac{70}{100} = \frac{7}{10}$.

c. There are only 5 disks that meet the condition, so the probability of getting one randomly is $\frac{5}{100} = \frac{1}{20}$.

Sections 2.4-2.6: Conditional probability, multiplicative rule, and independence

- Consider the probability that event A occurs given that event B occurs. This probability is called **conditional probability** of event A given event B and denoted by $P(A|B)$.
- Formula for conditional probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$.
- **Multiplicative Rule of Probability**:

$$P(A \cap B) = P(A)P(B|A)$$

$$P(A \cap B) = P(B)P(A|B)$$

- Events A and B are said to be **independent** if $P(A|B) = P(A)$. Otherwise they are dependent.

- Multiplicative Rule of Probability for independent events:

$$P(A \cap B) = P(A)P(B)$$

- Additive rule for independent events:

$$P(A \cup B) = P(A) + P(B) - P(A)P(B)$$

Exercise 3. A box of 100 semiconductor chips contains 20 that are defective. Two are selected randomly, without replacement from the box.

a. What is the probability that the first one selected is defective?

b. What is the probability that the second one selected is defective given that the first one was defective?

c. What is the probability that both are defective?

d. How does the answer to part (b) change if chips were replaced prior to the next selection?

a. $\frac{20}{100} = \frac{1}{5}$

b. If we know that the first chip is defective, then there are 19 defective chips left out of 99, So the probability is $\frac{19}{99}$.

c. $P(A \cap B) = P(A)P(A|B) = \frac{20}{100} \cdot \frac{19}{99} = \frac{19}{495}$

d. If the first chip is replaced, then there is another, potentially defective chip, so the probability that the second is defective is also $\frac{20}{100} = \frac{1}{5}$.

Exercise 4. Suppose that instructors of junior-level math courses give 25% of students As and 35% of students Bs.

a. If two students are selected at random, what is the probability they both got Bs or better?

b. If two students are selected at random, what is the probability that at least one of them got a B or better?

Exercise 5.

For undergraduate students, the probability of having high blood pressure is 0.03. For male undergraduates, the probability of having high blood pressure is 0.04. For undergraduate students, are “being male” and “having high blood pressure” independent? Show why or why not?

Exercise 6. Workers at a company were cross-classified according to their job location (plant, office, or sales) and their sex (male or female).

	Male	Female	Total
Plant	1,600	200	1,800
Office	20	140	160
Sales	40	0	40
Total	1,660	340	2,000

A person is selected at random from these 2,000 workers.

a. What is the probability he or she does not work in the office?
b. What is the probability that the person is female and works in the plant?
c. If the selected person is male, what is the probability that he works in sales?
d. Are the events "selected person is female" and "selected person works in the plant" independent? Show your reason.

Exercise 7. The following table summarizes the analysis of samples of galvanized steel for coating weight and surface roughness.

	Coating Weight	
Surface Roughness	High	Low
High	12	16
Low	88	34

a. If the coating weight of a sample is high, what is the probability that the surface roughness is high?

b. If the surface roughness of a sample is high, what is the probability that the coating weight is high?

c. Are the status of coating weight and surface roughness of a sample independent or not?

- Law of total probability

$$P(A) = P(B)P(A|B) + P(B^c)P(A|B^c)$$

- Bayes' Theorem

$$P(B|A) = \frac{P(B)P(A|B)}{P(B)P(A|B) + P(B^c)P(A|B^c)}$$

In the case that (1) $\cup_{i=1}^{k} B_i = S$ and (2) $B_i \cap B_j = \phi$ for all $i \neq j$,

- Law of total probability

$$P(A) = P(B_1)P(A|B_1) + \cdots + P(B_k)P(A|B_k)$$

- Bayes' Theorem

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(B_1)P(A|B_1) + \cdots + P(B_k)P(A|B_k)}$$

for $i = 1, \cdots, k$.

Exercise 8. An inspector working for a manufacturing company has 99% chance of correctly identifying good items and a 5% chance of incorrectly classifying a defective item as good. The company has evidence that its line produces 3% defective items.

a. What is the probability that an item seleced for inspection is classified as defective?

b. If an item selected at random is classified as good, what is the probability that it is truely good?

Section 2.8: Random sampling

- A random sample: If n elements are selected from a population in such a way that every set of n elements in the population has an equal probability of being selected, then the n elements are said to be a random sample.
- We can use statistical software to draw a random sample from a finite population. For example, use the function "sample" in R.
- We assume that by default the observed data is a random sample of the population that we want to study.