Chapter 4: Continuous Random Variables

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Sections 4.1-4.4: Continuous probability distributions

- The graphical form of the probability distribution for a continuous random variable X is a smooth curve.
- This curve, a function of x, is denoted by $f(x)$ and is called as a probability density function, a frequency function, or a probability distribution.
- The density function can be approximated by histogram. The larger sample size, the better approximation.

The probability density function $f(x)$

- $f(x) \geq 0$ for any x.
- The area under the curve corresponds to the probability. The whole area under the curve is 1.
- For an interval (a, b), $P(a < X < b) = \int_a^b f(x)dx$, the area under the curve between a and b.
- For any x, $P(X = x) = 0$.
- $P(a \leq X \leq b) = P(a < X < b)$.

- Cumulative distribution function
  - $F(x) = P(X \leq x)$ for any real value x.
  - $F(x) = \int_{-\infty}^{x} f(y)dy$ for any real value x.
  - $F(x)$ is a continuous, non-decreasing function.
- Mean $\mu = E(X) = \int_{-\infty}^{+\infty} yf(y)dy$.
- Variance $\sigma^2 = \int_{-\infty}^{+\infty} (y-\mu)^2 f(y)dy$.
- Standard deviation $\sigma = \sqrt{\sigma^2}$.

Section 4.5: Uniform distribution

- The density function of a uniform distribution within (c, d) is given by $f(x) = \frac{1}{d-c}$ for any $x \in (c, d)$ and $f(x) = 0$ for any $x \in (c, d)$.
- This distribution is evenly distributed over the interval (c, d).
- For any two points a and b in (c, d) with $a < b$, $P(a < X < b) = \frac{b-a}{d-c}$.
- The cumulative distribution function$F(x) = P(X \leq x)$ is given by $F(x) = \frac{x-c}{d-c}$ for any $x \in (c, d)$; $F(x) = 0$ if $x \leq c$; and $F(x) = 1$ for all $x \geq d$.
- Mean: $\mu = \frac{(c+d)}{2}$; variance: $\sigma^2 = \frac{(d-c)^2}{12}$.

Exercise 1. The manager of a local soft-drink bottling company believes that when a new beverage-dispensing machine is set to dispense 7 ounces, it in fact dispenses an amount X at random anywhere between 6.5 and 7.5 ounces. Suppose X has a uniform probability distribution.

a. Find the mean and standard deviation of the amount of a beverage X dispensed by the machine.

b. Find the probability that the next bottle contains more than 7.25 ounces.

c. Find the probability that all of the next three bottles contain more than 7.25 ounces. Assuming the bottles are dispensed independently.

Sections 4.6: The Normal Distribution

Normal distribution

- The density function $f(y)$ is a bell shaped curve.

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}, \quad -\infty < y < \infty$$

- $\pi = 3.1416$ and $e = 2.7183$
- $Y \sim N(\mu, \sigma^2)$, where μ and σ^2 are the population mean and variance, respectively.
- Standard normal: $Z \sim N(0, 1)$ i.e., $\mu = 0, \ \sigma = 1$.

Calculate probability for standard normal distribution using normal probability table.

- Always in the form of $P(Z < z)$ for any z.
- Note: $P(Z \le z) = P(Z < z)$
- $P(Z < 1.25)$
- $P(Z < -2.12)$
- $P(Z > 1.25)$
- $P(-0.58 < Z < 1.25)$

Calculate probability for normal distribution

- If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$.
- Always convert to standard normal when calculating probabilities

  $P(a < X < b) = P(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma})$
- Suppose $X \sim N(\mu = 3, \sigma^2 = 4)$, calculate

  $P(X > 3)$

  $P(2 \leq X \leq 5)$

  $P(X > 5)$

Exercise 2.

Assume that the length of time, X, between charges of a cellular phone is normally distributed with a mean of 10 hours and a standard deviation of 1.5 hours.

a. Find the probability that the cell phone will last between 8 and 12 hours between charges.

b. Find the probability that the cell phone will last at least 11 hours.

Inverse probability problems

a. If $Z \sim N(0,1)$, find z_0 such that $P(Z < z_0) = .9750$.

b. If $Z \sim N(0,1)$, find z_0 such that $P(Z > z_0) = .9515$.

c. If $Z \sim N(0,1)$, find z_0 such that $P(-1 < Z < z_0) = 0.2189$.

d. If $X \sim N(2,4)$, find x_0 such that $P(X > x_0) = .025$.

e. If $X \sim N(3,0.25)$, find x_0 such that $P(2 < X < x_0) = .95$.

Exercise 3.

Suppose the score, X, on a college entrance examination is normally distributed with a mean of 550 and a standard deviation of 100. A prestigious university will consider for admission only those applicants whose scores exceed the 90th percentile of the distribution. Find the minimum score an applicant must achieve in order to receive consideration for admission to the university.

Sections 4.8: The Exponential Distribution

Exponential distribution, $X \sim Exp(\theta)$

- The density function $f(y)$ takes the following form:

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0.$$

- The CDF $F(x) = 1 - e^{-\frac{x}{\theta}}$ for any $x > 0$.
- Mean $\mu = \theta$; variance $\sigma^2 = \theta^2$.

Exercise 4.

In the National Hockey League (NHL), games that are tied at the end of three periods are sent into "sudden death" overtime. In overtime, the team to score the first goal wins. An analyst of NHL overtime games showed that the length of elapsed before the winning goal is scored has an exponential distribution with mean 9.15 minutes.

a. For a randomly selected overtime NHL game, find the probability that the winning goal is scored in 3 minutes or less.

b. In the NHL, each period including overtime last 20 minutes. If neither team scores a goal in overtime, the game is considered a tie. What is the probability of an NHL game ending in a tie?

The memoryless property of the Exponential distribution, Suppose $X \sim Exp(\theta)$

- For any $t > 0$, $s > 0$,

$$P(T > t + s | T > s) = e^{-\frac{t}{\theta}} = P(T > t).$$

- Why does it happen? The hazard function is free of time.
- Implication: "Old works as good as new" or "No wearing effect".

Continued on exercise 4

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a. If an overtime has been played 10 minutes without any goal, what is the probability that the winning goal will be scored in the next 3 minutes?

b. If an overtime has been played 10 minutes without any goal, what is the probability of this NHL game will end in a tie?