Chapter 9: Hypothesis tests: one sample

Lianming Wang

University of South Carolina

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Sections 9.1: The concepts and procedure of hypothesis testing

A motivating Example:
A city has certain specifications about breaking strengths of pipes in its sewers. The average break point must be more than 2400 pplf (pounds per linear foot). A certain manufacturer wants the contract for the city's sewer pipes, and claims that their pipes meet the specifications.
Suppose we want to test this claim. We then examine a random sample of 50 sections of pipe from this manufacturer. This sample yields an average break point of 2460 pplf with a standard deviation of 200 pplf.

The elements of a hypothesis test:

- **Null Hypothesis** H_0: A theory or claim about the values of parameters. This claim generally represents the status quo. This claim remains the "truth" unless our analysis indicates otherwise.
- **Alternative Hypothesis** H_a: Also called the research hypothesis, this is a claim that contradicts the null hypothesis. We only accept it as the "truth" if our analysis strongly indicates to do so.

The elements of a hypothesis test (continued.):

- **Test Statistic**: a sample statistic used to decide whether to reject H_0 .
- **Rejection Region**: The set of numerical values for the test statistic that will lead us to reject the null hypothesis in favor of the alternative hypothesis.
- **Conclusions**: (two possible conclusions)
  - If the numerical value of the test statistic falls in the RR, we reject H_0 in favor of H_a.
  - If the test statistic is not in the RR, we fail to reject H_0.
- Note: Check the assumptions needed for the test.

Concepts related to hypothesis tests:

- Type I error: Deciding to reject H_0 when it is in fact true. $P(\text{type I error}) = \alpha$
- Type II error: Deciding to fail to reject H_0 when it is in fact false. $P(\text{type II error}) = \beta$
- Power of a test is the probability of correctly rejecting H_0 when H_0 is false. $power = 1 - \beta$

Comments about the two types of errors:

- One can not completely avoid both type I and type II errors for a specific test with a fixed sample size. If you decrease one error, then the other will increase.
- Setting up a value of Type I error (i.e., α) determines the cutoff point and the size of rejection region.
- Increasing the sample size can decrease both types of errors (and improve the power of the test).

State the right hypotheses (putting the research statement as alternative hypothesis):

a. More than 20% of the arrivals for a certain airline are late.
   - From customers' point of view: need evidence to support their complaints.
   - From airline company's position: need evidence to defend themselves.

b. The mean lifetime of the bulbs is above 2200 hours.

c. The defective rate on a manufacturing line is less than 10%.

Section 9.2. Large-Sample Test of Hypothesis about a Population Mean μ

One-Tailed Test

$H_0 : \mu = \mu_0$

$H_a : \mu > \mu_0$ (or $H_a : \mu < \mu_0$)

Test statistic: $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \approx \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$

Rejection Region: $z > z_\alpha$ (or $z < -z_\alpha$ when $H_a : \mu < \mu_0$)

Two-Tailed Test

$H_0 : \mu = \mu_0$

$H_a : \mu \neq \mu_0$

Test statistic: $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \approx \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$

Rejection Region: $z > z_{\frac{\alpha}{2}}$ or $z < -z_{\frac{\alpha}{2}}$

Conditions required for a valid **large-sample** hypothesis test for μ

(1) A random sample from the target population is required.

(2) The sample size n is large (i.e. $n \geq 30$).

Exercise 1. A random sample of 30 quart cartons of ice cream was taken from a large production run. It was found that the sample mean fat content was 12.6 percent with the standard deviation of 1.25 percent. Based on this, do we believe that the average fat content in this type of ice cream is more than 12 percent? Use the 0.01 level of significance.

Exercise 2. A trucking firm doubts a tire manufacturer's claim that a certain type of truck tire averages at least 28,000 miles before a blowout. To check the claim, the trucking firm put 40 of these tires on its trucks. If they obtained a mean life of 27,563 miles with a standard deviation of 1,324 miles, what conclusion should they reach if they allow a type I error rate of 1%?

Exercise 3. An oceanographer wants to test, on the basis of a random sample of size 35, whether the average depth of the ocean in a certain area is 72.4 fathoms. At the 0.05 level of significance, what will the oceanographer decide if she observed a sample mean of 73.2? Assume that the population standard deviation is 2.1.

The Observed Significance Level: p-values

Definition: The **observed significance level**, or p-**value**, for a specific statistical test is the probability (assuming the H_0 is true) of observing a value of the test statistic that is at least as contradictory to the null hypothesis as the actual one computed from the sample data.

P-value is reported in all statistical packages!

Steps for calculating the p-value for a test of hypothesis

(a) If the test is one-tailed, the p-value is equal to the tail area beyond the calculated test statistic in the same direction as the alternative hypothesis.

(b) If the test is two-tailed, the p-value is equal to twice the tail area beyond the calculated test statistic in the direction of the sign of the calculated test statistic. That is, if the calculated test statistic is positive, the tail area is to its right; if the test statistic is negative, the tail area is to its left.

Formula of calculating p-value for the large-sample z test

One-Tailed Test

$H_0 : \mu = \mu_0$

$H_a : \mu > \mu_0$ (or $H_a : \mu < \mu_0$)

Test statistic: $z^* = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \approx \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$

Rejection Region: $z > z_\alpha$ (or $z < -z_\alpha$ when $H_a : \mu < \mu_0$)

p-value is $P(z > z^*)$ (or $P(z < z^*)$ when $H_a : \mu < \mu_0$)

Two-Tailed Test

$H_0 : \mu = \mu_0$

$H_a : \mu \neq \mu_0$

Test statistic: $z^* = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \approx \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$

Rejection Region: $z > z_{\frac{\alpha}{2}}$ or $z < -z_{\frac{\alpha}{2}}$

p-value is $2P(z > |z^*|)$

Reporting test results based on p-values: criteria to decide whether to reject H_0.

(a) Choose the level of significance α (type I error rate) prior to conducting the hypothesis test.

(b) If the observed significance level (p-value) of the test is less than the chosen α, reject H_0. Otherwise, fail to reject H_0.

Section 9.3 Small-Sample Test of Hypothesis about a Population Mean μ

One-Tailed Test

$H_0 : \mu = \mu_0$

$H_a : \mu > \mu_0$ (or $H_a : \mu < \mu_0$)

Test statistic: $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$

Rejection Region: $t > t_{\alpha,n-1}$ (or $t < -t_{\alpha,n-1}$ when $H_a : \mu < \mu_0$)

p-value is $P(t > t^*)$ (or $P(t < t^*)$ when $H_a : \mu < \mu_0$)

Two-Tailed Test

$H_0 : \mu = \mu_0$

$H_a : \mu \neq \mu_0$

Test statistic: $t^* = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$

Rejection Region: $t > t_{\frac{\alpha}{2},n-1}$ or $t < -t_{\frac{\alpha}{2},n-1}$

p-value$=2P(t > |t^*|)$

Conditions required for a valid **small-sample** hypothesis test for μ

(1) A random sample from the target population is required.

(2) The population from which the sample is selected has a distribution that is approximately normal.

Exercise 4. The sodium content of 300-gram boxes of organic cornflakes was determined. The data (in milligrams) from a randomly selected 20 of such boxes are displayed as follows.

131.15	130.69	130.91	129.54	129.64
128.77	130.72	128.33	128.24	129.65
130.14	129.29	128.71	129.00	129.39
130.42	129.53	130.12	129.78	130.92

Conduct a hypothesis test at a 0.05 level of significance to test whether the true mean sodium content of this brand of cornflakes differ from 130 milligram.

Section 9.5: Large-Sample Test of Hypothesis about a Population Proportion p

One-Tailed Test

$H_0 : p = p_0$ vs. $H_a : p > p_0$ (or $H_a : p < p_0$)

Test statistic: $z = \frac{\hat{p}-p_0}{\sigma_{\hat{p}}} = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$

Rejection Region: $z > z_\alpha$ (or $z < -z_\alpha$ when $H_a : p < p_0$)

p-value is $P(z > z^*)$ (or $P(z < z^*)$ when $H_a : \mu < \mu_0$)

Two-Tailed Test

$H_0 : p = p_0$ vs. $H_a : p \neq p_0$

Test statistic: $z^* = \frac{\hat{p}-p_0}{\sigma_{\hat{p}}} = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$

Rejection Region: $z > z_{\frac{\alpha}{2}}$ or $z < -z_{\frac{\alpha}{2}}$

p-value$=2P(z > |z^*|)$

Conditions required for a valid **large-sample** hypothesis test for p

(1) A random sample from the target population is required.

(2) The sample size n is large (the sample size is large enough if both $n\hat{p} \geq 15$ and $n(1 - \hat{p}) \geq 15$).

Exercise 5. A city council member claims that 45% of her constituency is "very concerned" about drug trafficking. To see whether this claim is too low, a sample of 265 citizens was taken and 135 of them indicated that they were "very concerned" about drug trafficking. Test at the .05 level of significance. Also compute the p-value of the test.

Exercise 6. The advertised claim for batteries for cell phones is set at 48 operating hours with proper charging procedures. A study of 5000 batteries is carried out and 15 stop operating prior to 48 hours. Do these experimental results support the claim that more than 0.2 percent of the company's batteries will fail during the advertised time period, with proper charging procedures? Use a hypothesis-testing procedure with $\alpha = 0.01$.