Optimal portfolio liquidation

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3. Naive strategies
4. Optimal strategies

Portfolio liquidation

Financial problem

- We want to sell a large quantity of a stock (or of several stocks) in one day.
- How to choose the transaction times ?

Strategies (1)

Naive strategies

2 extreme strategies :

- Sell everything right now$\rightarrow$ huge transaction cost since we need to "eat" a lot in the order book. However this cost is known.
- Sell regularly in the day small amounts of assets$\rightarrow$ small transaction costs (volumes are much smaller) but the final profit is unknown because of the daily price fluctuations : Volatility risk.

Strategies (2)

Optimization

- We need to optimize between transaction costs and volatility risk.
- To do so, we use the Almgren and Chriss framework which takes into account the market impact phenomenon and emphasizes the importance of having good statistical estimators of market parameters.

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Trading strategy

Setup

- We consider we are selling one asset. We have X shares of this assets at $t_0 = 0$.
- We want everything to be sold at $t = T$.
- We split $[0, T]$ into N intervals of length $\tau = T/N$ and set $t_k = k\tau$, $k = 0, \dots, N$.
- A trading strategy is a vector $(x_0, \dots, x_N)$, with x_k the number of shares we still have at time t_k.
- $x_0 = X$, $x_N = 0$ and $n_k = x_{k-1} - x_k$ is the number of assets sold between t_{k-1} and t_k, decided at time t_{k-1}.

Price decomposition

Price components

The price we have access to moves because of :

- The drift $\rightarrow$ negligible at the intraday level.
- The volatility.
- The market impact.

Permanent market impact

Permanent impact component

- Market participants see us selling large quantities.
- Thus they revise their prices down.
- Therefore, the "equilibrium price" of the asset is modified in a permanent way.
- Let S_k be the equilibrium price at time t_k :

$$S_k = S_{k-1} + \sigma\tau^{1/2}\xi_k - \tau g(n_k/\tau),$$

with ξ_k iid standard Gaussian and n_k/τ the average trading rate between t_{k-1} and t_k.

Temporary market impact

Temporary impact component

- It is due to the transaction costs : we are liquidity taker since we "eat" the order book.
- If we sell a large amount of shares, our price per share is significantly worse than when selling only one share.
- We assume this effect is temporary and the liquidity comes back after each period.
- Let $\tilde{S}_k = (\sum n_{k,i} p_i)/n_k$, with $n_{k,i}$ the number of shares sold at price p_i between t_{k-1} and t_k. We set

$$\tilde{S}_k = S_{k-1} - h(n_k/\tau).$$

- The term $h(n_k/\tau)$ does not influence the next equilibrium price S_k.

Profit and Loss

Cost of trading

- The result of the sell of the asset is

$$\sum_{k=1}^{N} n_k \tilde{S}_k$$

$$= XS_0 + \sum_{k=1}^{N} \big(\sigma\tau^{1/2}\xi_k - \tau g(n_k/\tau)\big)x_k - \sum_{k=1}^{N} n_k h(n_k/\tau).$$

- The trading cost $\mathcal{C} = XS_0 - \sum_{k=1}^{N} n_k \tilde{S}_k$ is equal to

$$\text{Vol. cost} + \text{Perm. Impact cost} + \text{Temp. Impact cost.}$$

Mean-Variance analysis

Moments

- Consider a static strategy (fully known in t_0), which is in fact optimal in this framework. We have

$$\mathbb{E}[\mathcal{C}] = \sum_{k=1}^{N} \tau x_k g(n_k/\tau) + \sum_{k=1}^{N} n_k h(n_k/\tau), \ \ \mathrm{Var}[\mathcal{C}] = \sigma^2 \sum_{k=1}^{N} \tau x_k^2.$$

- In order to build optimal trading trajectories, we will look for strategies minimizing

$$\mathbb{E}[\mathcal{C}] + \lambda \mathrm{Var}[\mathcal{C}],$$

with λ a risk aversion parameter.

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Assumptions (1)

Permanent impact

- Linear permanent impact : $g(v) = \gamma v$.
- If we sell n shares, the price per share decreases by γn. Thus

$$S_k = S_0 + \sigma \sum_{j=1}^{k} \tau^{1/2} \xi_j - \gamma(X - x_k).$$

and in $\mathbb{E}[\mathcal{C}]$, the permanent impact component satisfies

$$\sum_{k=1}^{N} \tau x_k g(n_k/\tau) = \gamma \sum_{k=1}^{N} x_k(x_{k-1} - x_k) = \frac{1}{2}\gamma X^2 - \frac{1}{2}\gamma \sum_{k=1}^{N} n_k^2.$$

Assumptions (2)

Temporary impact

- Affine temporary impact : $h(n_k/\tau) = \varepsilon + \eta(n_k/\tau)$.
- ε represents a fixed cost : fees + bid ask spread.
- Let $\tilde{\eta} = \eta - \frac{1}{2}\gamma\tau$, we get

$$\mathbb{E}[\mathcal{C}] = \frac{1}{2}\gamma X^2 + \varepsilon X + \frac{\tilde{\eta}}{\tau}\sum_{k=1}^{N} n_k^2.$$

Regular liquidation

Regular strategy

- Take $n_k = X/N$, $x_k = (N-k)X/N$, $k = 1, \dots, N$.
- We easily get

$$\mathbb{E}[\mathcal{C}] = \frac{1}{2}\gamma X^2 + \varepsilon X + \tilde{\eta}\frac{X^2}{T},$$
$$\mathrm{Var}[\mathcal{C}] = \frac{\sigma^2}{3} X^2 T \Big(1 - \frac{1}{N}\Big)\Big(1 - \frac{1}{2N}\Big).$$

- We can show this strategy has the smallest expectation. However the variance can be very big if T is large.

Immediate selling

Selling everything at t_0

- Take $n_1 = X$, $n_2 = \ldots = n_N = 0$, $x_1 = \ldots = x_N = 0$.
- We get

$$\mathbb{E}[\mathcal{C}] = \varepsilon X + \frac{\eta X^2}{\tau},$$
$$\mathrm{Var}[\mathcal{C}] = 0.$$

- This strategy has the smallest variance. However, if τ is small, the expectation can be very large.

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Optimization (1)

Optimization program

- The trader wants to minimize

$$U(\mathcal{C}) = \mathbb{E}[\mathcal{C}] + \lambda \mathrm{Var}[\mathcal{C}].$$

- $U(\mathcal{C})$ is equal to

$$\frac{1}{2}\gamma X^2 + \varepsilon X + \frac{\tilde{\eta}}{\tau}\sum_{k=1}^{N}(x_{k-1} - x_k)^2 + \lambda\sigma^2\sum_{k=1}^{N}\tau x_k^2.$$

Optimization (2)

Derivation

- For $j = 1, \ldots, N-1$,

$$\frac{\partial U}{\partial x_j} = 2\tau\big(\lambda\sigma^2 x_j - \tilde{\eta}\frac{(x_{j-1} - 2x_j + x_{j+1})}{\tau^2}\big).$$

- Therefore

$$\frac{\partial U}{\partial x_j} = 0 \Leftrightarrow \frac{(x_{j-1} - 2x_j + x_{j+1})}{\tau^2} = \tilde{K} x_j,$$

with $\tilde{K} = \lambda\sigma^2/\tilde{\eta}$.

Optimization (3)

Solution

- It is shown that the solution can be written $x_0 = X$ and for $j = 1, \dots, N$:

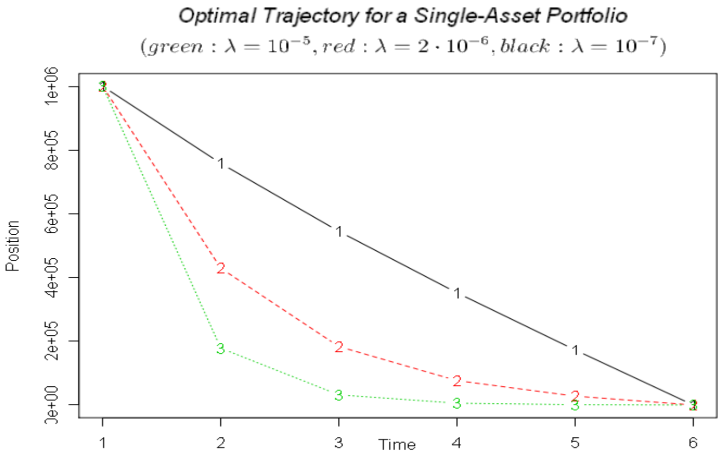
$$x_j = \frac{\sinh\big(K(T - t_j)\big)}{\sinh(KT)} X,$$

$$n_j = \frac{2\sinh(K\tau/2)}{\sinh(KT)} \cosh\big(K(T - j\tau + \tau/2)\big),$$

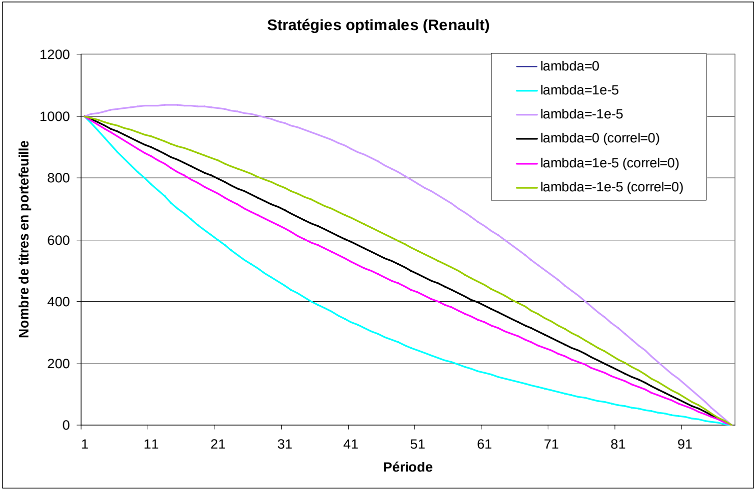
where K satisfies $\frac{2}{\tau^2}\big(\cosh(K\tau) - 1\big) = \tilde{K}$.

- If $\lambda = 0$, then $\tilde{K} = K = 0$ and so $n_j = \tau/T = X/N$. We retrieve the strategy with minimal expected cost.

Optimal strategies



Optimal strategies on real data



Remarks on this approach

Remarks

- It is easy to show that the solution is time homogenous : if we compute the optimal strategy in t_k, we obtain the value between t_k and T of the optimal strategy computed in t_0.
- In this approach, we obtain an efficient frontier of trading.
- The optimal trajectories are very sensitive to the volatility parameter. It is therefore important to obtain accurate volatility estimates.
- The Almgren and Chriss framework can be extended in dimension n (if we sell several assets). In that case, correlation parameters come into the picture.

Optimal strategies in dimension 2