To quaternions and back

Paolo Bonzini

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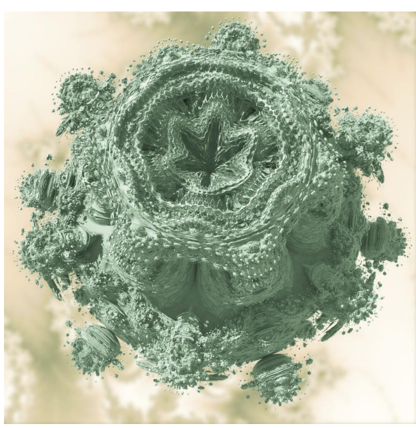
Abstract

Daniel White's *Mandelbulb*, with its fractal features on all axes, is possibly the closest existing three-dimensional analogue to the 2D Mandelbrot set.

The classic Mandelbrot set is based on the iteration of $z \leftarrow z^2 + c$ in the complex plane, thus the most common 3D extension to the "Mandelbrot" set is the iteration of $q \leftarrow q^2 + c$ on quaternions. The Mandelbulb, instead, extends to three dimension the *geometric* interpretation of complex squaring.

In this paper, I propose an alternative explanation of the Mandelbulb formulas using standard quaternion arithmetic, with the hope of adding the necessary mathematical rigor to the discovery of the Mandelbulb. A novel quaternion formulation is derived for the Mandelbrot iteration formulas; by parameterizing this formula, and appropriately choosing the parameters, the various Mandelbulb fractals are produced without resorting to non-standard algebras.



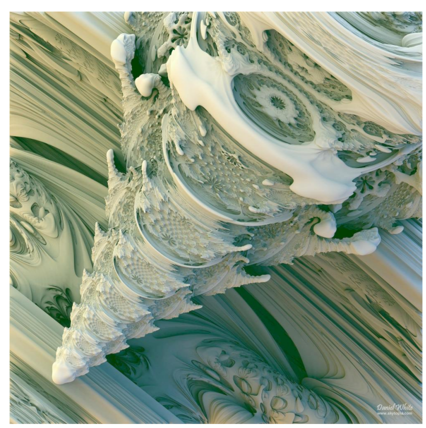


Figure 1: The entire exponent-8 Mandelbulb and a zoom on a 3D structure. Pictures by Krzysztof Marczak and Daniel White.

1 Introduction

An extension of the Mandelbrot set to three dimension is a kind of holy grail for fractalists. The hurdles are many and stem from the inherent difference between the two- and three-dimensional spaces: no true analogue of the complex field in three dimensions, no three-dimensional conformal mappings, the *hairy ball* theorem… and besides all this, the expectations for such an object are incredibly high!

There are two main extensions of the Mandelbrot and Julia sets, both of them to four dimensions. One is to use quaternions instead of complex numbers. In order to reduce four dimensions to three, a "stack" of parallel 4D planes is intersected with the object and the intersecting voxels are plotted. The resulting objects possess complexity, but not the infinite level of detail shown by the 2D Mandelbrot and Julia sets—some of them are basically solids of revolution, "lathed" versions of the 2D Mandelbrot set.

Another 4D expression of the Mandelbrot set does not use quaternions, but rather composes four coordinates from the two parameters of the iteration, z_0 and c. Then the standard complex iteration is used. The 2D slices of this set are very interesting, since they include both the Mandelbrot set (lying on the $z_0 = 0$ plane) and the Julia sets (corresponding to all the constant-c planes). However, the 3D slices also fail to provide the infinite detail and sheer beauty of their 2D correspondents.

From 2007 to 2009, Daniel White experimented with a new approach to the quest, whose resulting fractals he named *Mandelbulbs*. In order to understand it, I should digress shortly and explain some properties of complex numbers. In addition to using a purely mathematical description in terms of complex numbers, the Mandelbrot set's orbits can also be studied geometrically. The real and imaginary parts $z = a+ib$ are represented on the 2D plane by using the real axis as the x axis, and the imaginary as the y, and the arithmetic operations also have a geometric interpretation.

Addition is obvious. In order to understand multiplication, complex numbers have to be visualized in a different way using *polar* coordinates. The position of a point in a plane is identified by the distance ρ from the origin, and the angle θ between the positive x axis and the point. This complex number can be written compactly as $\rho e^{i\theta}$, and the geometric interpretation of multiplication is derived as follows:

$$\rho e^{i\theta} = \rho_1 e^{i\theta_1} \cdot \rho_2 e^{i\theta_2} = \rho_1 \rho_2 e^{i(\theta_1+\theta_2)}$$

The result of the multiplication $z_1 z_2$ is a point that is obtained from z_1 by *stretching it by a factor* ρ_2, and then *rotating it by an angle* θ_2; and the result of squaring is obtained by squaring the distance from the origin and doubling the angle from the positive x axis.

2 From 2D to 3D

Daniel White's intuition then was to invent a meaningful way of squaring and adding points in 3D space. To do the former, he had to square the distance and somehow "double" the angle between the positive x axis and the point in exam. The point is then translated by c, and the two steps are repeated until either the sequence diverges or a given number of iterations is reached. This idea had actually been proposed before by Rudy Rucker, but his formulas had a small error that caused them to produce less interesting results.

Many things in this scheme work out easily. For example, Euclidean distance in 3D space is similarly enough to 2D that the divergence condition is the same: as soon as the distance of z from the origin exceeds 2, the iteration can be interrupted. Other things are instead noticeably different. For example, it is hard to define the effect of squaring on angles. 3D space can be described in *spherical coordinates* that couple a distance and *two* angles—corresponding to two rotations around two different axes. However, rotations in 3D space are *not* commutative. In fact, applying White's idea but choosing different angles and axes gives rise to very different fractals.

On top of this, there is some uneasiness due to the fact that 3D complex numbers were sought by mathematicians for decades and in the end they settled for four-component quaternions $w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, despite these possess strange properties such as *noncommutative multiplication*. The infinite possible ways to define the 3D rotation give the feeling of banging one's head against dead ends. Searching for the properties of three-component numbers whose multiplication is (or seems to be) commutative, feels too much like a 21st century version of squaring the circle.

Nevertheless, nothing suggested White's construction to be fundamentally flawed, and it produced (very) nice pictures, especially when the $z^2 + c$ was tweaked to include higher exponents[1]. This is a winning combination for fractalists, who proceeded to experiment with many different definitions of rotation. For all of them, the Mandelbulb iteration with exponent n is then obtained by the following steps:

1. compute the spherical coordinates (ρ, θ, ϕ) of the point $c = (x_c, y_c, z_c)$ being examined[2];

2. compute the rotation corresponding to a point—remember that complex squaring, by doubling the polar-coordinates angle, applies a different rotation for different points;

3. apply the rotation n times to the point $(\rho^n, 0, 0)$;

[1]This was first tried by Paul Nylander.

[2]Unfortunately, parenthesized triples may refer throughout the remainder of this section to both cartesian and spherical coordinates. In general, the presence of Greek letters such as ρ, θ, ϕ or π will mean that a particular occurrence refers to spherical coordinates.

4. translate the result by c.

The urge to write this in a form that resembles Julia and Mandelbrot's $z \leftarrow z^n + c$ is hard to resist, so White defined z^n in 3D space as applying steps 1– 3 of the above procedure. The actual details of exponentiation of course depend on the actual rotation chosen for step 2.

3 Experimenting

White himself tried many choices, but as of today, there are three main contestants to the title of "best Mandelbulb rotation". Remember that we are defining a rotation whose details depend on the point being rotated; hence, all three compose a rotation on the z axis and a rotation on the y axis, parameterizing them by the two angles in the spherical coordinates of a point—the *azimuth* θ and the *elevation* ϕ.

$$R_z(\theta) \cdot R_y(\phi) \tag{1}$$

$$R_z(\theta) \cdot R_y(-\phi) \tag{2}$$

$$R_z(\theta) \cdot R_y(\pi/2 - \phi) \tag{3}$$

Each of the three has interesting properties, and the first two degenerate to $R_z(\theta)$ (and hence to the Mandelbrot set) on the xy plane. (1) is possibly the most natural solution and produces a nice exponent-2 Mandelbulb, but it has the apparent disadvantage that $(x, y, z)^1 = (x, y, -z)$. Instead, (2) has $(x, y, z)^1 = (x, y, z)$. Even then, (3) appears to be much more weird, since $(x, y, z)^0 = (0, 0, 1)$.

Equation (2) was discovered by Paul Nylander, who then proceeded to define a fairly complete algebra with commutative (but nonassociative) multiplication, multiplicative inverses, and division. For example, since

$$(\rho, \theta, \phi)^n = (\rho^n, n\theta, n\phi) \tag{4}$$

the following definition of multiplication seems natural:

$$(\rho_1, \theta_1, \phi_1)(\rho_2, \theta_2, \phi_2) = (\rho_1\rho_2, \theta_1 + \theta_2, \phi_1 + \phi_2)$$

However, this attempt too seemed stuck against a dead end once the proposed algebra started to show more and more annoying differences from standard mathematical concepts. For example, properly calculating equation (4) requires a definition of ϕ in the range $[-\pi, \pi]$, while spherical coordinates define elevation in the range $[-\pi/2, \pi/2]$. This in turn means that a cartesian representation of this algebra is not *power associative*[3].

[3]In a power associative algebra, $(x, y, z)^{n+m} \neq (x, y, z)^n(x, y, z)^m$

Leaving aside for a moment the nice pictures, the fundamental insight that White had is this: possibly, the essence of the 2D Mandelbrot does not rely on the complex field—there could be something else more fundamental to the set's appearance, and this thing could be rotations.

4 Reconsidering quaternions

Now, rotations are something that mathematicians have learnt to handle very well. They have two tools of choice to deal with rotations, namely matrices and… quaternions.

Matrices are an extremely general tool that can represent arbitrary linear transformations of an arbitrary vector space, including for example those that do not preserve angles. Any of the three equations in the previous section could indeed be converted to a 3x3 matrix R, and the resulting iteration would have this shape:

$$(\rho, \theta, \phi)^n = R(\theta, \phi)^n (\rho^n, 0, 0)$$

Even better, the scaling operation could be included in the transformation matrix like this:

$$(\rho, \theta, \phi)^n = R(\rho, \theta, \phi)^n (1, 0, 0)$$

under the following conditions:

- R affects areas by a factor of exactly ρ: $\det R = \rho^3$
- R is a combination of scaling and rotation: $R^{-1} = R^T/\rho^2$

If these two conditions are imposed, however, the transformation is more compactly represented by a quaternion q. *Note that the operation that will be performed on the quaternion is* ***not*** $q \leftarrow q^2 + c$*, so this will still result in a Mandelbulb* rather than the disappointing quaternion Julia sets. Only, the 3-tuples of Section 2 will be replaced a well-known mathematical object, the quaternion.

Unit quaternions, that is quaternions whose norm is 1, represent the space of 3D rotations in a simple way. For the purpose of this paper, it will suffice to show how to describe a rotation by a quaternion, without explaining the theory behind this. $\mathbf{v}$ will indicate a quaternion with a zero scalar part, that is $\mathbf{i}x + \mathbf{j}y + \mathbf{k}z$; the rotation by an angle α around axis $\mathbf{v}$ is then represented by the following quaternion:

$$q = \cos\frac{\alpha}{2} + \mathbf{v}\sin\frac{\alpha}{2}$$

The rotation is clockwise if the observer's line of sight points in the same direction as $\mathbf{v}$. A rotation q can be applied to a vector $\mathbf{w}$ by computing two quaternion multiplications, specifically $q\mathbf{w}q^{-1}$.

Rotations can be composed by multiplying two quaternions; ordering is significant, since quaternion multiplication is noncommutative. q^n instead is well defined as a rotation by n times the angle around the same axis as q, since multiplication is still associative.

These formulas of course work in 2D too; in this case the rotated point will be of the form $\mathbf{i}x + \mathbf{j}y$ and, in order to stay in the xy plane, the rotation must be performed around the z axis:

$$q = \cos\frac{\alpha}{2} + \mathbf{k}\sin\frac{\alpha}{2}$$

This is more compactly written $e^{\mathbf{k}\alpha/2}$. The polar representation of complex numbers also has an equivalent using quaternions on the xy plane, albeit the expression is more complicated. Let $\mathbf{v} = \mathbf{i}x + \mathbf{j}y$ be a point on the xy plane, and (ρ, θ) its polar representation. Then, it's possible to write $\mathbf{v}$ from ρ and θ as follows:

$$\mathbf{v} = \rho e^{\mathbf{k}\theta/2}\mathbf{i}e^{-\mathbf{k}\theta/2}$$

From here it is a short step to a somewhat nontraditional formulation of the Mandelbrot set, based on quaternion rotations. It was already said repeatedly that $\mathbf{v}^2$ corresponds to squaring ρ and doubling θ. Then, it's possible to write $\mathbf{v}^2 + c$ as follows:

$$\mathbf{v}^2 + \mathbf{c} = \rho^2 e^{\mathbf{k}\theta}\mathbf{i}e^{-\mathbf{k}\theta} + \mathbf{c}$$

But this complicated expression can be simplified noticeably. For unit quaternions, $q^{-1} = \overline{q}$: inversion and conjugation are the same operation, and $q\mathbf{i}q^{-1}$ can also be written $q\mathbf{i}\overline{q}$. When the modulus is not one, instead, $q\mathbf{i}\overline{q} = |q|^2 q\mathbf{i}q^{-1}$. Therefore we can define

$$q = \rho e^{\mathbf{k}\theta} = x + y\mathbf{k} = -\mathbf{v}\mathbf{i}$$

and write:

$$\mathbf{v}^2 + \mathbf{c} = q\mathbf{i}\overline{q} + \mathbf{c}$$

Changing the sign of q does not modify the effect of the rotation[4], and it slightly simplifies the iteration. The iteration formula will then be the following for the classic Mandelbrot set:

$$\mathbf{v} \leftarrow q\mathbf{i}\overline{q} + \mathbf{c} \text{ with } q = \mathbf{v}\mathbf{i}.$$

and, generalizing the exponent n to values other than 2:

$$\mathbf{v} \leftarrow q^{n/2}\mathbf{i}\overline{q}^{n/2} + \mathbf{c} \text{ with } q = \mathbf{v}\mathbf{i}. \tag{5}$$

[4]This is true for all quaternions, since quaternions are actually a *double cover* of the space of rotations.

For $n = 2$ it is possible to substitute and simplify q, giving the following[5]:

$$\mathbf{v} \leftarrow \mathbf{v}\mathbf{i}\overline{\mathbf{v}} + \mathbf{c} \tag{6}$$

This elegant iteration is the *quaternion formulation of the Mandelbrot set*. However, equation (6), when drawn in 3D, it results simply in the "lathed" 2D Mandelbrot set. So, for the sake of generalizing to three dimensions (and rediscovering the Mandelbulb in its quaternion form), we'll have to take a step back to equation (5), and see if it can be actually generalized to 3D.

5 Rethinking the Mandelbulb

Since there is no single candidate to the role of 3D Mandelbrot, it makes sense to define a family of iterations that can be used to produce these fractals. Some of them will be more interestings and other more boring, but all of them will share the property of degenerating to a same-order Mandelbrot set on a plane—for simplicity, the choice will be the xy plane.

Each different fractal will be described by a function $Q(\mathbf{v})$ associating a quaternion to a position in 3D space v. To satisfy the first condition, the function should satisfy $|Q(\mathbf{v})| = |\mathbf{v}|$; the second condition can be imposed by checking that

$$Q(\mathbf{i}x + \mathbf{j}y) = x + \mathbf{k}y$$

whenever the $\mathbf{k}$ component of $\mathbf{v}$ is zero (alternatively, if $\mathbf{v}$'s spherical coordinates are used, whenever $\phi = 0$).

Note that this would exclude equation (3), which indeed has a very distorted appearance for $n = 2$. Still, under this definition there are infinite possible definitions of $Q(\mathbf{v})$ that make sense and many produce interesting drawings. The main advantage is that the function $Q(\mathbf{v})$ is the same for all exponents: its definition automatically includes the generalization of squaring to exponentiation. It is also includes possibilities such as adding phase shifts to ϕ or multiplying by arbitrary factors.

Let's consider for example equation (2) for $n = 2$. This gives the following value for $Q(\mathbf{v})$[6]:

$$\begin{aligned} Q(\mathbf{v}) &= \rho(\cos\theta + \mathbf{k}\sin\theta)(\cos\phi - \mathbf{j}\sin\phi) \\ &= \rho\cos\theta\cos\phi + \mathbf{i}\rho\sin\theta\sin\phi - \mathbf{j}\rho\cos\theta\sin\phi + \mathbf{k}\rho\sin\theta\cos\phi \\ &= x + \mathbf{i}\frac{yz}{\sqrt{x^2+y^2}} - \mathbf{j}\frac{xz}{\sqrt{x^2+y^2}} + \mathbf{k}y \end{aligned} \tag{7}$$

[5] An interesting point in this formula is that $\mathbf{v}\mathbf{i}\overline{\mathbf{v}}$ is also a quaternion rotation, more precisely a 180 degree rotation around axis $\mathbf{v}$. In 2D this is a reflection, so the result will still lie on the xy plane even though the axis is not $\mathbf{k}$. Multiplying on the right by $-\mathbf{i}$, alternatively, corresponds to adding a 180 degree rotation around the x axis, which is irrelevant because the transformed point always lies on that axis.

[6] Since $n = 2$, these functions use rotations of 2θ and 2ϕ, and this avoids the introduction of fractional terms $\theta/2$ and $\phi/2$ as in equation (4).

For $x = y = 0$, the function is not continuous, and any value of θ can be used. The simplest possibilities are $Q(\mathbf{v}) = -\mathbf{i}z$ or $Q(\mathbf{v}) = \mathbf{j}z$.

From this formula it is trivial to verify the condition expressed above for embedding the Mandelbrot set. It is also possible, if needed, to easily convert the quaternion q back to (ρ, θ, ϕ):

$$\rho = |q|, \theta = \arctan -\frac{q_x}{q_y} = \arctan \frac{q_z}{q_w}, \phi = \arctan \frac{q_x}{q_z} = \arctan -\frac{q_y}{q_w}$$

One important property of the values of equation (7) is that $Q(\mathbf{v})$ takes a different value for each θ and ϕ in the range $(-\pi, \pi)$. In other words, unlike when using spherical coordinates, rotation of the elevation is expressed using its full 2π range; you may recall this to be a problem in the 3-tuple representation, and we can now see why this problem is present. In Nylander's multiplication formula two objects with two different meanings (a rotation and a vector) appear in a commutative way, but since rotations are "twice as many" as vectors, the rotation cannot be described exactly. While it happens to suffice in the case of squaring, this does not extend to higher powers. Therefore, unlike exponentiation and due to its non-associativity, *"triplex multiplication" does not have a geometrical meaning* when expressed in cartesian coordinates. This is also related to the lack of power-associativity in White and Nylander's 3-tuples; instead, the quaternion formulation intrinsically avoid the problem by representing rotations and vectors differently.

Instead, "triplex exponentiation" does have a meaning since the cartesian formulas for it include the conversion to rotation quaternions as well as applying the rotation to $\mathbf{i}$. This is advantageous, because the exponentiation formulas are faster than using quaternion multiplication.

As another example of defining $Q(\mathbf{v})$, equation (1) can be written as follows for $n = 2$:

$$\begin{aligned} Q(\mathbf{v}) &= \rho(\cos\theta + \mathbf{k}\sin\theta)(\cos\phi + \mathbf{j}\sin\phi) \\ &= \rho\cos\theta\cos\phi - \mathbf{i}\rho\sin\theta\sin\phi + \mathbf{j}\rho\cos\theta\sin\phi + \mathbf{k}\rho\sin\theta\cos\phi \\ &= x - \mathbf{i}\tfrac{yz}{\sqrt{x^2+y^2}} + \mathbf{j}\tfrac{xz}{\sqrt{x^2+y^2}} + \mathbf{k}y \end{aligned}$$

Unfortunately, the definition of $Q(v)$ proposed so far will match the corresponding Mandelbulb only for $n = 2$. For higher orders, raising equations (7) and (5) to $n/2$ will alternate rotations around the z and y axes, instead of making a single rotation around the z axis followed by one around the y axis.

So, instead of using exponentiation as in equation (5), $Q(\mathbf{v})$ has to be generalized a family of functions $Q_\nu(\mathbf{v})$ satisfying $Q_\nu(\mathbf{i}x + \mathbf{j}y) = (x + \mathbf{k}y)^\nu$. The iteration will be defined as follows:

$$\mathbf{v} \leftarrow q\mathbf{i}\overline{q} + \mathbf{c} \text{ with } q = Q_{n/2}(\mathbf{v}) \tag{8}$$

The family of functions corresponding to equation (2) becomes:

$$Q_\nu(\mathbf{v}) = \rho(\cos\nu\theta + \mathbf{k}\sin\nu\theta)(\cos\nu\phi - \mathbf{j}\sin\nu\phi) \tag{9}$$

This of course degenerates to equation (7) when $\nu = 1$.

6 Conclusion

In this paper I proposed to describe the family of Mandelbulb fractals using quaternions, so that each possible definition of exponentiation can be written as a functions from a vector to a quaternion. This formulation makes it easy to distinguish members which embed the Mandelbrot fractal, and eliminates the need to define new non-standard algebras.

An alternative method using transformation matrices was also presented briefly. This method is more useful for extensions beyond the third dimension.

However, this paper barely scratched the surface in terms of analyzing the properties of the fractals itself. There are many questions that can be analyzed. For example, what are the boundaries of the set? Or, why do some fractals have a flat appearance for $n = 2$?

Since there may be no true 3D equivalent of the Mandelbrot fractal, having defined a space of $Q_\nu(\mathbf{v})$ functions may at least help steering the search for new fractal creations of this family. It is my hope that this paper will provide a tool to analyze their characteristics rigorously and, maybe, will contribute to the discovery of new beautiful mathematical objects.

A Expansion of quaternion formulas

Rotation of i by q

$$q\mathbf{i}\overline{q} = \mathbf{i}(q_w^2 + q_x^2 - q_y^2 - q_z^2) + \mathbf{j}(2q_xq_y + 2q_wq_z) + \mathbf{k}(2q_xq_z - 2q_wq_y)$$

This is the main step in the computation of the Mandelbulb for $n = 2$.

Quaternion squaring and multiplication

$$\begin{aligned} q^2 &= (q_w^2 - q_x^2 - q_y^2 - q_z^2) + 2\mathbf{i}q_wq_x + 2\mathbf{j}q_wq_y + 2\mathbf{k}q_wq_z \\ pq &= (p_wq_w - p_xq_x - p_yq_y - p_zq_z) + \mathbf{i}(p_wq_x + p_xq_w + p_yq_z - p_zq_y) \\ &\quad + \mathbf{j}(p_wq_y - p_xq_z + p_yq_w + p_zq_x) + \mathbf{k}(p_wq_z + p_xq_y - p_yq_x + p_zq_w) \end{aligned}$$

$Q(v)^2$ for the definition of equation (7)

$$Q(v)^2 = (x^2 - y^2 - z^2) + \frac{2xz}{\sqrt{x^2 + y^2}}(-\mathbf{i}y + \mathbf{j}x) + 2\mathbf{k}xy$$

This is slightly faster than computing $Q(v)$ and then squaring, due to the simplifications in the q_w term. It can be used to compute the Mandelbulb for $n = 4$.

Real part of $Q(v)^4$ for the definition of equation (7)

$$\operatorname{Re} Q(v)^4 = (x^2 - y^2 - z^2)^2 - 4x^2y^2 - 4x^2z^2$$

For $n = 8$ one more squaring is necessary. However, the real term can be simplified considerably using the above expression. Note that this formula does *not* give the order-8 Mandelbulb represented in figure 1.