

# To quaternions and back

Paolo Bonzini

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## Abstract

Daniel White's *Mandelbulb*, with its fractal features on all axes, is possibly the closest existing three-dimensional analogue to the 2D Mandelbrot set.

The classic Mandelbrot set is based on the iteration of  $z \leftarrow z^2 + c$  in the complex plane, thus the most common 3D extension to the "Mandelbrot" set is the iteration of  $q \leftarrow q^2 + c$  on quaternions. The Mandelbulb, instead, is a three-dimensional extension of a *geometric* interpretation of complex squaring.

In this paper, I propose an alternative explanation of the Mandelbulb iteration formulas based on quaternions, with the hope of adding the necessary mathematical rigor to the discovery of the Mandelbulb.

## 1 Introduction

An extension of the Mandelbrot set to three dimension is a kind of holy grail for fractalists. The hurdles are many, because the bidimensional and three dimensional spaces are inherently very different: there is no true analogue of the complex field in three dimensions, that there are no three-dimensional conformal mappings, the *hairy ball* theorem. . . and besides all this, the expectations for such an object are incredibly high!

There are two main extensions of the Mandelbrot and Julia sets, both of them to four dimensions. One is to use quaternions instead of complex numbers. In order to reduce four dimensions to three, a "stack" of parallel 4D planes is intersected with the object and the intersecting voxels are plotted. The resulting objects possess complexity, but not the infinite level of detail shown by the 2D Mandelbrot and Julia sets—some of them are basically solids of revolution, "lathed" versions of the 2D Mandelbrot set.

Another 4D expression of the Mandelbrot set does not use quaternions, but rather composes four coordinates from the two parameters of the iteration,  $z_0$  and  $c$ . Then the standard complex iteration is used. The 2D slices of this set are very interesting, since they include both the Mandelbrot set (lying on the  $z_0 = 0$  plane) and the Julia sets (corresponding to all the constant- $c$  planes).

However, the 3D slices also fail to provide the infinite detail and sheer beauty of their 2D correspondents.

From 2007 to 2009, Daniel White experimented with a new approach to the quest, whose resulting fractals he named *Mandelbulbs*. In order to understand it, I should digress shortly and explain some properties of complex numbers. In addition to using a purely mathematical description in terms of complex numbers, the Mandelbrot set's orbits can also be studied geometrically. The real and imaginary parts  $z = a + ib$  are represented on the 2D plane by using the real axis as the  $x$  axis, and the imaginary as the  $y$ , and the arithmetic operations also have a geometric interpretation.

Addition is obvious. In order to understand multiplication, complex numbers have to be visualized in a different way using *polar* coordinates. The position of a point in a plane is identified by the distance  $\rho$  from the origin, and the angle  $\theta$  between the positive  $x$  axis and the point. This complex number can be written compactly as  $\rho e^{i\theta}$ , and the geometric interpretation of multiplication is derived as follows:

$$\rho e^{i\theta} = \rho_1 e^{i\theta_1} \cdot \rho_2 e^{i\theta_2} = \rho_1 \rho_2 e^{i(\theta_1 + \theta_2)}$$

The result of the multiplication  $z_1 z_2$  is a point that is obtained from  $z_1$  by *stretching it by a factor  $\rho_1$* , and then *rotating it by an angle  $\theta_1$* ; and the result of squaring is obtained by squaring the distance from the origin and double the angle from the positive  $x$  axis.

## 2 From 2D to 3D

Daniel White's intuition then was to invent a meaningful way of squaring and adding points in 3D space. To do the former, he had to square the distance and somehow "double" the angle between the positive  $x$  axis and the point in exam. The point is then translated by  $c$ , and the two steps are repeated until either the sequence diverges or a given number of iterations is reached.

Many things in this scheme work out easily. For example, Euclidean distance in 3D space is similarly enough to 2D that the divergence condition is the same: as soon as the distance of  $z$  from the origin exceeds 2, the iteration can be interrupted. Other things are instead noticeably different. For example, it is hard to define the effect of squaring on angles. 3D space can be described in *spherical coordinates* that couple a distance and *two* angles—corresponding to two rotations around two different axes. However, rotations in 3D space are *not* commutative. In fact, applying White's idea but choosing different angles and axes gives rise to very different fractals.

On top of this, there is some uneasiness due to the fact that 3D complex numbers were sought by mathematicians for decades and in the end they settled for four-component quaternions, despite some strange properties such as

*noncommutative multiplication.* The infinite possible ways to define the 3D rotation give the feeling of banging one's head against dead ends. Searching for the properties of three-component numbers whose multiplication is (or seems to be) commutative, feels too much like a 21st century version of squaring the circle.

Nevertheless, nothing suggested White's construction to be fundamentally flawed, and it produced (very) nice pictures, especially when the  $z^2 + c$  was tweaked to include higher exponents. This is a winning combination for fractalists, who proceeded to experiment with many different definitions of rotation. For all of them, the Mandelbulb iteration with exponent  $n$  is then obtained by the following steps:

1. compute the spherical coordinates  $(\rho, \theta, \phi)$  of the point  $c = (x_c, y_c, z_c)$  being examined<sup>1</sup>;
2. compute the rotation corresponding to a point—remember that complex squaring, by doubling the polar-coordinates angle, applies a different rotation for different points;
3. apply the rotation  $n$  times to the point  $(\rho^n, 0, 0)$ ;
4. translate the result by  $c$ .

The urge to write this in a form that resembles Julia and Mandelbrot's  $z \leftarrow z^n + c$  is hard to resist, so White defined  $z^n$  in 3D space as applying steps 1–3 of the above procedure. The actual details of exponentiation of course depend on the actual rotation chosen for step 2.

### 3 Experimenting

White himself tried many choices, but as of today, there are three main contestants to the title of “best Mandelbulb rotation”. Remember that we are defining a rotation whose details depend on the point being rotated; hence, all three compose a rotation on the  $z$  axis and a rotation on the  $y$  axis, parameterizing them by the two angles in the spherical coordinates of a point—the *azimuth*  $\theta$  and the *elevation*  $\phi$ .

$$R_z(\theta) \cdot R_y(\phi) \tag{1}$$

$$R_z(\theta) \cdot R_y(-\phi) \tag{2}$$

$$R_z(\theta) \cdot R_y(\pi/2 - \phi) \tag{3}$$

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<sup>1</sup>Unfortunately, parenthesized triples may refer throughout the remainder of this section to both cartesian and spherical coordinates. In general, the presence of Greek letters such as  $\rho$ ,  $\theta$ ,  $\phi$  or  $\pi$  will mean that a particular occurrence refers to spherical coordinates.

Each of the three has interesting properties, and the first two degenerate to  $R_z(\theta)$  (and hence to the Mandelbrot set) on the  $xy$  plane. (1) is possibly the most natural solution and produces a nice exponent-2 Mandelbulb, but it has the apparent disadvantage that  $(x, y, z)^1 = (x, y, -z)$ . Instead, (2) has  $(x, y, z)^1 = (x, y, z)$ . Even then, (3) appears to be much more weird, since  $(x, y, z)^0 = (0, 0, 1)$ .

Equation 2 was discovered by Paul Nylander, who then proceeded to define a fairly complete algebra with commutative (but nonassociative) multiplication, multiplicative inverses, and division. For example, since

$$(\rho, \theta, \phi)^n = (\rho^n, n\theta, n\phi) \quad (4)$$

the following definition of multiplication seems natural:

$$(\rho_1, \theta_1, \phi_1)(\rho_2, \theta_2, \phi_2) = (\rho_1\rho_2, \theta_1 + \theta_2, \phi_1 + \phi_2)$$

However, this attempt too seemed stuck against a dead end once the proposed algebra started to show more and more annoying differences from standard mathematical concepts. For example, properly calculating equation 4 requires a definition of  $\phi$  in the range  $[-\pi, \pi]$ , while spherical coordinates define elevation in the range  $[-\pi/2, \pi/2]$ . This in turn means that a cartesian representation of this algebra is not *power associative*<sup>2</sup>.

Leaving aside for a moment the nice pictures, the fundamental insight that White had is this: possibly, the essence of the 2D Mandelbrot does not rely on the complex field—there could be something else more fundamental to the set's appearance, and this thing could be rotations.

## 4 Reconsidering quaternions

Now, rotations are something that mathematicians have learnt to handle very well. They have two tools of choice to deal with rotations, namely matrices and... quaternions.

Matrices are an extremely general tool that can represent arbitrary linear transformations of an arbitrary vector space, including for example those that do not preserve angles. Any of the three equations in the previous section could indeed be converted to a 3x3 matrix  $R$ , and the resulting iteration would have this shape:

$$(\rho, \theta, \phi)^n = R(\theta, \phi)^n(\rho^n, 0, 0)$$

Even better, the scaling operation could be included in the transformation matrix like this:

$$(\rho, \theta, \phi)^n = R(\rho, \theta, \phi)^n(1, 0, 0)$$

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<sup>2</sup>In a power associative algebra,  $(x, y, z)^{n+m} \neq (x, y, z)^n(x, y, z)^m$

under the following conditions:

- $R$  affects areas by a factor of exactly  $\rho$ :  $\det R = \rho^3$
- $R$  is a combination of scaling and rotation:  $R^{-1} = R^T / \rho^2$

If these two conditions are imposed, however, the transformation (a scaling operation followed by a rotation) is more compactly represented by a quaternion  $q$ . Note that the operation that will be performed on the quaternion is **not**  $q \leftarrow q^2 + c$ , so this will still result in a Mandelbulb rather than the disappointing quaternion Julia sets.

Unit quaternions, that is quaternions whose norm is 1, represent the space of 3D rotations in a simple way. For the purpose of this paper, it will suffice to show how to describe a rotation by a quaternion, without explaining the theory behind this.  $\mathbf{v}$  will indicate a quaternion with a zero scalar part, that is  $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ ; the rotation by an angle  $\alpha$  around axis is then represented by the following quaternion:

$$q = \cos \frac{\alpha}{2} + \mathbf{v} \sin \frac{\alpha}{2}$$

The rotation is clockwise if the observer's line of sight points in the same direction as  $\mathbf{v}$ . A rotation  $q$  can be applied to a vector  $\mathbf{w}$  by computing two quaternion multiplications, specifically  $q\mathbf{v}q^{-1}$ .

Rotations can be composed by multiplying two quaternions; ordering is significant, since quaternion multiplication is noncommutative.  $q^n$  instead is well defined as a rotation by  $n$  times the angle around the same axis as  $q$ , since multiplication is still associative.

These formulas of course work in 2D too; in this case the rotated point will be of the form  $x\mathbf{i} + y\mathbf{j}$  and, in order to stay in the  $xy$  plane, the rotation must be performed around the  $z$  axis:

$$q = \cos \frac{\alpha}{2} + \mathbf{k} \sin \frac{\alpha}{2}$$

This is more compactly written  $e^{\mathbf{k}\alpha/2}$ . The polar representation of complex numbers also has an equivalent using quaternions on the  $xy$  plane, albeit the expression is more complicated. Let  $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$  be a point on the  $xy$  plane, and  $(\rho, \theta)$  its polar representation. Then, it's possible to write  $\mathbf{v}$  from  $\rho$  and  $\theta$  as follows:

$$\mathbf{v} = \rho e^{\mathbf{k}\theta/2} \mathbf{i} e^{-\mathbf{k}\theta/2}$$

From here it is a short step to a somewhat nontraditional formulation of the Mandelbrot set, based on quaternion rotations. It was already said repeatedly that  $\mathbf{v}^2$  corresponds to squaring  $\rho$  and doubling  $\theta$ . Then, it's possible to write  $\mathbf{v}^2 + \mathbf{c}$  as follows:

$$\mathbf{v}^2 + \mathbf{c} = \rho^2 e^{\mathbf{k}\theta} \mathbf{i} e^{-\mathbf{k}\theta} + \mathbf{c}$$

But this complicated expression can be simplified noticeably. For unit quaternions,  $q^{-1} = \bar{q}$ : inversion and conjugation are the same operation, and  $qi q^{-1}$  can also be written  $qi\bar{q}$ . When the modulus is not one, instead,  $qi\bar{q} = |q|^2 qi q^{-1}$ . Therefore we can define

$$q = \rho e^{i\mathbf{k}\theta} = x + y\mathbf{k} = -\mathbf{v}\mathbf{i}$$

and write:

$$\mathbf{v}^2 + \mathbf{c} = qi\bar{q} + \mathbf{c}$$

Changing the sign of  $q$  does not modify the effect of the rotation, and slightly simplifies the iteration. The iteration formula will then be the following for the classic Mandelbrot set:

$$\mathbf{v} \leftarrow qi\bar{q} + \mathbf{c} \text{ with } q = \mathbf{v}\mathbf{i}.$$

and, generalizing the exponent  $n$  to values other than 2:

$$\mathbf{v} \leftarrow q^{n/2} i \bar{q}^{n/2} + \mathbf{c} \text{ with } q = \mathbf{v}\mathbf{i}. \quad (5)$$

The final iteration after substitution and simplification is the following:

$$\mathbf{v} \leftarrow \mathbf{v}^{n/2} i \bar{\mathbf{v}}^{n/2} + \mathbf{c}$$

This is the *quaternion formulation of the Mandelbrot set*. However, when drawn in 3D it results simply in the “lathed” 2D Mandelbrot set. So, for the sake of generalizing to three dimensions (and rediscovering the Mandelbulb in its quaternion form), we’ll have to take a step back and look at equation 5 with 3D eyes.

## 5 Rethinking the Mandelbulb