Thermodynamics & Statistical Physics

Chapter 8. Bose statistics and Fermi statistics

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$$= -\mathrm{d}(\beta \frac{\partial \ln \mathcal{Z}}{\partial \beta}) - \mathrm{d}(\alpha \frac{\partial \ln \mathcal{Z}}{\partial \alpha}) + \mathrm{d} \ln \mathcal{Z}$$
$$= \mathrm{d}(\ln \mathcal{Z} - \alpha \frac{\partial \ln \mathcal{Z}}{\partial \alpha} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta}).$$

$$\boxed{\ln \mathcal{Z} = -\sum \omega_l \ln(1 - e^{-\alpha-\beta\varepsilon_l})}$$

- Eq. (4.1.12), $\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V + \sum \mu_i \mathrm{d}n_i$.

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- $$\begin{aligned}U&=g\frac{2\pi V}{h^3}(2mkT)^{\frac{3}{2}}kT\int_0^\infty\frac{x^{\frac{3}{2}}\mathrm{d}x}{e^{\alpha+x}\pm 1}\\&\simeq g\frac{2\pi V}{h^3}(2mkT)^{\frac{3}{2}}kT\int_0^\infty e^{-\alpha-x}(1\mp e^{-\alpha-x})x^{\frac{3}{2}}\mathrm{d}x\\&=g\frac{2\pi V}{h^3}(2mkT)^{\frac{3}{2}}kTe^{-\alpha}[\Gamma(\frac{5}{2})\mp e^{-\alpha}2^{-\frac{5}{2}}\Gamma(\frac{5}{2})]\\&=g\frac{2\pi V}{h^3}(2mkT)^{\frac{3}{2}}kTe^{-\alpha}\frac{3}{2}\frac{\sqrt{\pi}}{2}(1\mp e^{-\alpha}2^{-\frac{5}{2}})\\&=g(\frac{2\pi mkT}{h^2})^{\frac{3}{2}}V\frac{3}{2}kTe^{-\alpha}(1\mp 2^{-\frac{5}{2}}e^{-\alpha}).\end{aligned}$$
- $$\frac{U}{N}=\frac{g(\frac{2\pi mkT}{h^2})^{\frac{3}{2}}V\frac{3}{2}kTe^{-\alpha}(1\mp 2^{-\frac{5}{2}}e^{-\alpha})}{g(\frac{2\pi mkT}{h^2})^{\frac{3}{2}}Ve^{-\alpha}(1\mp 2^{-\frac{3}{2}}e^{-\alpha})}$$

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  1st term, same as the Boltzmann distribution; 2nd term, modification by the identical principle of particles (Fermions, repel; Bosons, attract).

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  $= \frac{2\pi V}{h^3}(2mkT)^{\frac{3}{2}}(kT) \cdot \frac{3}{4}\sqrt{\pi} \cdot 1.341 = 0.77NkT(\frac{T}{T_c})^{\frac{3}{2}}$.

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- Magneto optical trap – 3-D harmonic oscillator.

BEC – realization

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- Number of photons at $(\omega, \omega + \mathrm{d}\omega)$: $a(\omega)\mathrm{d}\omega = \frac{V}{\pi^2 c^3} \frac{\omega^2 \mathrm{d}\omega}{e^{\frac{\hbar\omega}{kT}} - 1}$.
- Internal energy: $U(\omega, T)\mathrm{d}\omega = \frac{V}{\pi^2 c^3} \frac{\hbar\omega^3 \mathrm{d}\omega}{e^{\frac{\hbar\omega}{kT}} - 1}$.
- Total internal energy: $U = \int_0^\infty U(\omega, T)\mathrm{d}\omega$
$= \int_0^\infty \frac{V}{\pi^2 c^3} \frac{\hbar\omega^3 \mathrm{d}\omega}{e^{\frac{\hbar\omega}{kT}} - 1} = \int_0^\infty \frac{V\hbar}{\pi^2 c^3} (\frac{\hbar}{kT})^{-4} \frac{(\frac{\hbar\omega}{kT})^3 \mathrm{d}(\frac{\hbar\omega}{kT})}{e^{\frac{\hbar\omega}{kT}} - 1}$
$= \frac{V(kT)^4}{\pi^2 c^3 \hbar^3} \int_0^\infty \frac{x^3 \mathrm{d}x}{e^x - 1}$

Photon gas

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  $= \frac{V(kT)^4}{\pi^2 c^3 \hbar^3} \int_0^\infty \frac{x^3 \mathrm{d}x}{e^x - 1} = \frac{\pi^2 k^4}{15 c^3 \hbar^3} T^4 V$.

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Photon gas $\boxed{D(\omega)\mathrm{d}\omega = \frac{V}{\pi^2 c^3}\omega^2 \mathrm{d}\omega}$

- Maximum for $U(\omega, T)\mathrm{d}\omega = \frac{V}{\pi^2 c^3}\frac{\hbar\omega^3 \mathrm{d}\omega}{e^{\frac{\hbar\omega}{kT}} - 1}$:

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  $= -\frac{V}{\pi^2 c^3}(\beta\hbar)^{-3}\int_0^\infty (\beta\hbar\omega)^2 \ln(1 - e^{-\beta\hbar\omega})\mathrm{d}(\beta\hbar\omega)$

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&= -\frac{V}{\pi^2 c^3}\frac{1}{(\beta\hbar)^3}[\frac{x^3}{3}\ln(1 - e^{-x})|_0^\infty - \int_0^\infty \frac{x^3}{3}\cdot\frac{e^{-x}}{1-e^{-x}}\mathrm{d}x]
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&= \frac{V}{\pi^2 c^3}\frac{1}{(\beta\hbar)^3}\frac{1}{3}\int_0^\infty \frac{x^3}{e^x-1}\mathrm{d}x = \frac{\pi^2 V}{45 c^3}\frac{1}{(\beta\hbar)^3}.
\end{aligned}$$

Photon gas $\boxed{U = \frac{\pi^2 k^4}{15 c^3 \hbar^3} T^4 V}$, $\boxed{\ln \mathcal{Z} = \frac{\pi^2 V}{45 c^3} \frac{1}{(\beta\hbar)^3}}$.

- Internal energy:
  $U = -\frac{\partial}{\partial\beta} \ln \mathcal{Z}$

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- Pressure:
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- Entropy:
$$S = k(\ln \mathcal{Z} - \alpha \frac{\partial \ln \mathcal{Z}}{\partial \alpha} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta})$$

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$$= k\left[\frac{\pi^2 V}{45 c^3} \frac{1}{(\beta\hbar)^3} + \beta \frac{\pi^2 k^4}{15 c^3 \hbar^3} T^4 V\right]$$

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$$= k\left[\frac{\pi^2 V}{45 c^3} \frac{1}{(\beta\hbar)^3} + \beta \frac{\pi^2 k^4}{15 c^3 \hbar^3} T^4 V\right]$$
$$= k\left(\frac{\pi^2 V}{45 c^3} \frac{k^3 T^3}{\hbar^3} + \frac{\pi^2 V}{15 c^3} \frac{k^3 T^3}{\hbar^3}\right)$$

Photon gas $\boxed{U = \frac{\pi^2 k^4}{15 c^3 \hbar^3} T^4 V}$, $\boxed{\ln \mathcal{Z} = \frac{\pi^2 V}{45 c^3} \frac{1}{(\beta\hbar)^3}}$.

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$$U = -\frac{\partial}{\partial \beta} \ln \mathcal{Z} = -\frac{\pi^2 V}{45 c^3} \frac{1}{\hbar^3} \frac{-3}{\beta^4} = \frac{\pi^2 k^4}{15 c^3 \hbar^3} T^4 V.$$
- Pressure:
$$p = \frac{1}{\beta} \frac{\partial}{\partial V} \ln \mathcal{Z} = \frac{1}{\beta} \frac{\pi^2}{45 c^3} \frac{1}{(\beta\hbar)^3} = \frac{\pi^2 k^4}{45 c^3 \hbar^3} T^4 = \frac{1}{3} \frac{U}{V}.$$
- Entropy:
$$S = k(\ln \mathcal{Z} - \alpha \frac{\partial \ln \mathcal{Z}}{\partial \alpha} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta}) = k(\ln \mathcal{Z} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta})$$
$$= k \left[\frac{\pi^2 V}{45 c^3} \frac{1}{(\beta\hbar)^3} + \beta \frac{\pi^2 k^4}{15 c^3 \hbar^3} T^4 V \right]$$
$$= k \left(\frac{\pi^2 V}{45 c^3} \frac{k^3 T^3}{\hbar^3} + \frac{\pi^2 V}{15 c^3} \frac{k^3 T^3}{\hbar^3} \right)$$
$$= \frac{4}{45} \frac{\pi^2 k^4 V}{c^3 \hbar^3} T^3.$$ (same as (2.6.4))

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  $T_{\mathrm{F}} = \frac{\mu(0)}{k} = \frac{1}{k}\frac{\hbar^2}{2m}(3\pi^2\frac{N}{V})^{\frac{2}{3}}$
  $= \frac{1}{k}\frac{\hbar^2}{2m}(3\pi^2 \cdot 8.5 \times 10^{28}\mathrm{m}^{-3})^{\frac{2}{3}} \simeq 8.2 \times 10^4\mathrm{K}.$
- Internal energy: $U(0) = \frac{4\pi V}{h^3}(2m)^{\frac{3}{2}} \int_0^\infty \frac{\varepsilon^{\frac{3}{2}}}{e^{\frac{\varepsilon-\mu}{kT}}+1}\mathrm{d}\varepsilon$
  $= \frac{4\pi V}{h^3}(2m)^{\frac{3}{2}} \int_0^{\mu(0)} \varepsilon^{\frac{3}{2}}\mathrm{d}\varepsilon = \frac{3}{5}N\mu(0).$
- Pressure: $p = \frac{2}{3}\frac{U}{V} = \frac{2}{5}n\mu(0).$
  $p_{\mathrm{Cu}} = \frac{2}{5} \cdot 8.5 \times 10^{28}\mathrm{m}^{-3} \cdot 1.38 \times 10^{-23}\mathrm{J} \cdot \mathrm{K}^{-1} \cdot 8.2 \times 10^4\mathrm{K}$

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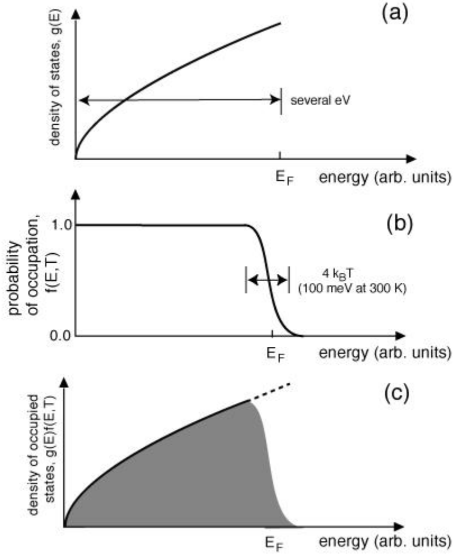
- $f \begin{cases} > \frac{1}{2}, & \varepsilon < \mu; \\ = \frac{1}{2}, & \varepsilon = \mu; \\ < \frac{1}{2}, & \varepsilon > \mu. \end{cases}$
- Number of states: $\frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}\mathrm{d}\varepsilon$.
- Energy density: $\frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}\frac{\varepsilon^{\frac{3}{2}}}{e^{\frac{\varepsilon-\mu}{kT}}+1}\mathrm{d}\varepsilon$.



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  $E = \frac{2}{5} C \mu^{\frac{5}{2}} [1 + \frac{5\pi^2}{8} (\frac{kT}{\mu})^2]$, where $C = \frac{4\pi V}{h^3}(2m)^{\frac{3}{2}}$.

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- $\therefore C_V = (\frac{\partial U}{\partial T})_V = Nk\frac{\pi^2 k}{2\mu(0)}T = \frac{\pi^2}{2}Nk\frac{T}{T_F} \propto T$.

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