Thermodynamics & Statistical Physics

Chapter 6. The most probable distribution of nearly independent particles

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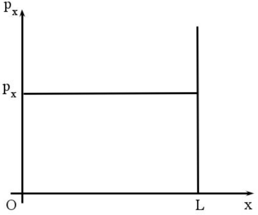

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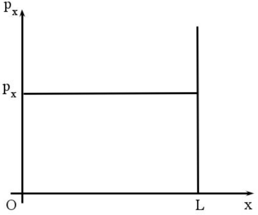

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- Also considering the spin s, the number of state enhances to $2s+1$ times.
  E.g., electrons, $s = 1/2$, degeneracy is $2s+1=2$.

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- Aim: find the number of microscopic arrangement.

Description of the system's microscopic arrangement

Classical particle system

State 1	State 2	State 3
A B		
	A B	
		A B
A	B	
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	A	B
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A	A	
	A	A
A		A

Fermion system

State 1	State 2	State 3
A	A	
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6.4 The principle of equal a priori probabilities

- Given an isolated system in equilibrium, it is found with equal probability in each of its accessible micro-states.

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- For a distribution, the number of micro-states is NOT obvious.

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4. Total number of micro-states for one distribution $\{a_l\}$:
$\boxed{\Omega_{\text{M.B.}} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l}}$.

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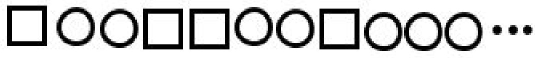

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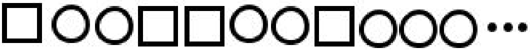

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- In the classical limit, both F.D. and B.E. reach to the same number of micro-states $\frac{\Omega_{\text{M.B.}}}{N!}$.

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phase volume	$\Delta w_1, \Delta w_2, ...\Delta w_l, ...$
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Boltzmann distribution

- $\sum_{l=1}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l$

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  or, $\boxed{a_l = \omega_l e^{-\alpha-\beta\varepsilon_l}}$.
  α, β are determined by the constraints:
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Boltzmann distribution

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  Now $\{a_l\}$ $(l = 3, 4, ...)$ are independent,
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- For any quantum state: the population is
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- Only depends on the energy for the special quantum state. Suitable for any quantum state: $f_s=e^{-\alpha-\beta\varepsilon_s}$.

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- This $\{a_l\}$ is the most probable distribution.

Deviation from the most probable distribution

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- $\frac{\Omega + \Delta\Omega}{\Omega} \sim e^{-3\times 10^{13}} \to 0.$
- The most probable distribution is very close to the whole distribution.

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  Obeys:
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  $\int\ldots\int \varepsilon(\{q_i,p_i\})e^{-\alpha-\beta\varepsilon(\{q_i,p_i\})}\frac{\mathrm{d}q_1\ldots\mathrm{d}q_r\mathrm{d}p_1\ldots\mathrm{d}p_r}{h_0^r} = E$.

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  $-\omega_l(\ln \omega_l - 1)\}$
  $= \sum [(\omega_l + a_l)\ln(\omega_l + a_l) - a_l \ln a_l - \omega_l \ln \omega_l]$.
- $\delta \ln \Omega_{\rm B} = \sum [\delta a_l \ln(\omega_l + a_l) + (\omega_l + a_l)\frac{1}{\omega_l + a_l}\delta a_l - (\delta a_l \ln a_l + a_l \frac{1}{a_l}\delta a_l) - 0]$

6.7 Bose distribution and Fermi distribution

- N, V, E conserved, require $\{a_l\}$.
- Obeys $\sum a_l = N$, $\sum a_l \varepsilon_l = E$.
- Bose system: $\Omega_{\rm B} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$.
- $\ln \Omega_{\rm B} = \sum [\ln(\omega_l + a_l - 1)! - \ln a_l! - \ln(\omega_l - 1)!]$
  $\simeq \sum [\ln(\omega_l + a_l)! - \ln a_l! - \ln \omega_l!]$
  $\simeq \sum \{(\omega_l + a_l)[\ln(\omega_l + a_l) - 1] - a_l(\ln a_l - 1)$
  $-\omega_l(\ln \omega_l - 1)\}$
  $= \sum [(\omega_l + a_l)\ln(\omega_l + a_l) - a_l \ln a_l - \omega_l \ln \omega_l]$.
- $\delta \ln \Omega_{\rm B} = \sum [\delta a_l \ln(\omega_l + a_l) + (\omega_l + a_l)\frac{1}{\omega_l + a_l}\delta a_l -$
  $(\delta a_l \ln a_l + a_l \frac{1}{a_l}\delta a_l) - 0]$
  $= \sum \ln \frac{\omega_l + a_l}{a_l} \delta a_l = 0$.

Bose distribution and Fermi distribution

- Introduce the α, β:

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l]\delta a_l=0$

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- Fermi system: $\Omega_{\mathrm{F}}=\prod\frac{\omega_l!}{a_l!(\omega_l-a_l)!}$.

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  where α, β obeys: $\sum \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1} = N$, $\sum \frac{\omega_l\varepsilon_l}{e^{\alpha+\beta\varepsilon_l}-1} = E$.
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  $=\sum[\ln(\omega_l-a_l)-\ln a_l]\delta a_l=0$.

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- Similarly, for any individual quantum state s, the average number of particles: $f_s=\frac{1}{e^{\alpha+\beta\varepsilon_s}\pm1}$, and $\sum f_s=N$, $\sum\varepsilon_s f_s=E$.

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- Remind: $\Omega_{\mathrm{M}} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l}$, $\Omega_{\mathrm{B}} = \prod \frac{(\omega_l+a_l-1)!}{a_l!(\omega_l-1)!}$, $\Omega_{\mathrm{F}} = \prod \frac{\omega_l!}{a_l!(\omega_l-a_l)!}$.

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  In the case $\frac{a_l}{\omega_l} \ll 1$, $\Omega_{\rm B} \simeq \frac{\omega_l^{a_l}}{\prod a_l!}$, $\Omega_{\rm F} \simeq \frac{\omega_l^{a_l}}{\prod a_l!}$.

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  $\therefore \Omega_\mathrm{B} \simeq \Omega_\mathrm{F} \simeq \frac{\Omega_\mathrm{M}}{N!}$.

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