Thermodynamics & Statistical Physics

Chapter 10. Fluctuation

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  For S (or T), means $\overline{[S(N, V, E) - S(\overline{N}, \overline{V}, \overline{E})]^2}$.

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Fluctuation of grand canonical ensemble

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  $\Rightarrow p(t) = p(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi$.
- Take the average, noticing $\overline{F(\xi)} = 0$,
  $\Rightarrow \overline{p}(t) = p(0)e^{-\gamma t}$,

Diffusion and temporal correlation of Brownian particle's momentum $\boxed{\frac{\mathrm{d}p}{\mathrm{d}t} = -\gamma p + F(t)}$

- Back to the Langevin's equation:
  $e^{\gamma t}\frac{\mathrm{d}p}{\mathrm{d}t} = -e^{\gamma t}\gamma p + e^{\gamma t}F(t)$,
  $\Rightarrow \frac{\mathrm{d}(pe^{\gamma t})}{\mathrm{d}t} = e^{\gamma t}F(t)$,
  $\Rightarrow p(t) = p(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi$.
- Take the average, noticing $\overline{F(\xi)} = 0$,
  $\Rightarrow \overline{p}(t) = p(0)e^{-\gamma t}$,
  where $\frac{1}{\gamma}$ is the time scale for a notable change of momentum (larger than τ_c).

Diffusion of Brownian particle's momentum

$$\boxed{p(t) = p(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi}$$

- $\overline{(\Delta p)^2} = \overline{[p(t) - \overline{p}(t)]^2}$

Diffusion of Brownian particle's momentum

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- 1. $\tau_c \ll t \ll \frac{1}{\gamma}$, $\overline{(\Delta p)^2} = 2D_p t$;
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  In equilibrium, $\frac{\overline{p^2}}{2m} = \frac{1}{2}kT$, $\therefore D_p = m\gamma kT = \alpha kT$.

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- 2. Similarly $t > t'$, $\overline{p(t)p(t')} = \frac{D_p}{\gamma}[e^{-\gamma(t-t')} - e^{-\gamma(t+t')}]$.

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- 2. Similarly $t > t'$, $\overline{p(t)p(t')} = \frac{D_p}{\gamma}[e^{-\gamma(t-t')} - e^{-\gamma(t+t')}]$.
- The combination: $\overline{p(t)p(t')} = \frac{D_p}{\gamma}[e^{-\gamma|t-t'|} - e^{-\gamma(t+t')}]$.

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- Back to $\overline{x^2(t)}$ again. $x(t) = \int_0^t \dot{x}(\xi)\mathrm{d}\xi = \frac{1}{m}\int_0^t p(\xi)\mathrm{d}\xi$.

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  $x(t) = \int_0^t \dot{x}(\xi)\mathrm{d}\xi = \frac{1}{m}\int_0^t p(\xi)\mathrm{d}\xi$.
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Temporal correlation of the momentum

$$\boxed{\overline{p(t)p(t')} = \frac{D_p}{\gamma}[e^{-\gamma|t-t'|} - e^{-\gamma(t+t')}]}$$

- For $t, t' > \frac{1}{\gamma}$ (normal), $e^{-\gamma(t+t')} \to 0$; then $\overline{p(t)p(t')} = \frac{D_p}{\gamma}e^{-\gamma|t-t'|} = mkTe^{-\gamma|t-t'|}$.
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&= \frac{kT}{m}\int_0^t \mathrm{d}\xi \int_0^t \mathrm{d}\xi' e^{-\gamma|\xi-\xi'|} \\
&= \frac{kT}{m}\int_0^t \mathrm{d}\xi \left[\int_0^\xi \mathrm{d}\xi' e^{-\gamma(\xi-\xi')} + \int_\xi^t \mathrm{d}\xi' e^{-\gamma(\xi'-\xi)}\right]
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&= \frac{2kT}{\alpha}t.\ \text{(same as (10.5.8))}.
\end{aligned}
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- Abstracting the external voltage $-\frac{q(t)}{C}$ to $\mathcal{V}(t)$, $L\frac{\mathrm{d}i}{\mathrm{d}t} = \mathcal{V}(t) - Ri + V(t)$.
- $m\frac{\mathrm{d}v}{\mathrm{d}t} = \mathcal{F} - \alpha v + F(t)$, Langevin's equation. $L \leftrightarrow m$, $i \leftrightarrow v$, $\mathcal{V} \leftrightarrow \mathcal{F}$, $R \leftrightarrow \alpha$, and $V \leftrightarrow F$.

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  $= \frac{kTR}{\pi}\delta(\omega-\omega')$, (white noise)
  where $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega'-\omega)t}\mathrm{d}t = \delta(\omega-\omega')$.

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- When the excited atom transits to the ground state, it radiates a photon and feels a repulsive force $F_z(t)$.
- Langevin's equation: $m\frac{\mathrm{d}v_z}{\mathrm{d}t} = -\alpha v_z + F_z$.

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- Frequency of laser: ω_L; of atom: ω_A. ($\omega_L \lesssim \omega_A$.)
- For the atom at velocity v_z (1 direction), it feels the frequency of laser $\omega_L(1 \pm \frac{|v_z|}{c})$. Higher velocity, closer to ω_A; more easier to absorb the photon and lose its momentum. Effective viscous force $-\alpha v_z$.
- When the excited atom transits to the ground state, it radiates a photon and feels a repulsive force $F_z(t)$.
- Langevin's equation: $m\frac{\mathrm{d}v_z}{\mathrm{d}t} = -\alpha v_z + F_z$.
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