Thermodynamics & Statistical Physics

Chapter 10. Fluctuation

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Table of contents

§10.1 Quasi-thermodynamics of fluctuation

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  For S (or T), means $\overline{[S(N, V, E) - S(\overline{N}, \overline{V}, \overline{E})]^2}$.

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Table of contents

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- As $t = 0$, $x = 0$. $\Rightarrow C_2 = -C_1$.

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  $\Rightarrow p(t) = p(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi$.
- Take the average, noticing $\overline{F(\xi)} = 0$,
  $\Rightarrow \overline{p}(t) = p(0)e^{-\gamma t}$,

Diffusion and temporal correlation of Brownian particle's momentum $\boxed{\frac{\mathrm{d}p}{\mathrm{d}t} = -\gamma p + F(t)}$

- Back to the Langevin's equation:
  $e^{\gamma t}\frac{\mathrm{d}p}{\mathrm{d}t} = -e^{\gamma t}\gamma p + e^{\gamma t}F(t)$,
  $\Rightarrow \frac{\mathrm{d}(pe^{\gamma t})}{\mathrm{d}t} = e^{\gamma t}F(t)$,
  $\Rightarrow p(t) = p(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi$.
- Take the average, noticing $\overline{F(\xi)} = 0$,
  $\Rightarrow \overline{p}(t) = p(0)e^{-\gamma t}$,
  where $\frac{1}{\gamma}$ is the time scale for a notable change of momentum (larger than τ_c).

Diffusion of Brownian particle's momentum

$$\boxed{p(t) = p(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi}$$

- $\overline{(\Delta p)^2} = \overline{[p(t) - \overline{p}(t)]^2}$

Diffusion of Brownian particle's momentum

$$\boxed{p(t) = p(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi}$$

- $$\begin{aligned}\overline{(\Delta p)^2} &= \overline{[p(t) - \overline{p}(t)]^2} \\ &= \overline{[e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi]^2}\end{aligned}$$

Diffusion of Brownian particle's momentum

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- $\overline{(\Delta p)^2} = \overline{[p(t) - \overline{p}(t)]^2}$
  $= \overline{[e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi]^2} = \overline{[\int_0^t F(\xi)e^{-\gamma(t-\xi)}\mathrm{d}\xi]^2}$

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Diffusion of Brownian particle's momentum

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- $$\begin{aligned}\overline{(\Delta p)^2} &= \overline{[p(t) - \overline{p}(t)]^2} \\ &= \overline{[e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi]^2} = \overline{[\int_0^t F(\xi)e^{-\gamma(t-\xi)}\mathrm{d}\xi]^2} \\ &= \overline{\int_0^t \mathrm{d}\xi \int_0^t \mathrm{d}\xi' F(\xi)F(\xi')e^{-\gamma(t-\xi)}e^{-\gamma(t-\xi')}} \\ &= \int_0^t \mathrm{d}\xi \int_0^t \mathrm{d}\xi' \overline{F(\xi)F(\xi')}e^{-\gamma(t-\xi)}e^{-\gamma(t-\xi')} \\ &= \int_0^t \mathrm{d}\xi \int_0^t \mathrm{d}\xi' 2D_p\delta(\xi - \xi')e^{-\gamma(t-\xi)}e^{-\gamma(t-\xi')} \\ &= \int_0^t \mathrm{d}\xi 2D_p e^{-2\gamma(t-\xi)} = \frac{D_p}{\gamma}(1 - e^{-2\gamma t}).\end{aligned}$$
- 1. $\tau_c \ll t \ll \frac{1}{\gamma}$, $\overline{(\Delta p)^2} = 2D_p t$;

Diffusion of Brownian particle's momentum

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- 1. $\tau_c \ll t \ll \frac{1}{\gamma}$, $\overline{(\Delta p)^2} = 2D_p t$;
- 2. $t \gg \frac{1}{\gamma}$, $\overline{p}(t) = 0$, so $\overline{p^2} = \overline{(\Delta p)^2} = \frac{D_p}{\gamma}$.

Diffusion of Brownian particle's momentum

$$\boxed{p(t)=p(0)e^{-\gamma t}+e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi}$$

- $\overline{(\Delta p)^2}=\overline{[p(t)-\overline{p}(t)]^2}$
$$=\overline{[e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi]^2}=\overline{[\int_0^t F(\xi)e^{-\gamma(t-\xi)}\mathrm{d}\xi]^2}$$
$$=\overline{\int_0^t \mathrm{d}\xi\int_0^t \mathrm{d}\xi' F(\xi)F(\xi')e^{-\gamma(t-\xi)}e^{-\gamma(t-\xi')}}$$
$$=\int_0^t \mathrm{d}\xi\int_0^t \mathrm{d}\xi' \overline{F(\xi)F(\xi')}e^{-\gamma(t-\xi)}e^{-\gamma(t-\xi')}$$
$$=\int_0^t \mathrm{d}\xi\int_0^t \mathrm{d}\xi' 2D_p\delta(\xi-\xi')e^{-\gamma(t-\xi)}e^{-\gamma(t-\xi')}$$
$$=\int_0^t \mathrm{d}\xi 2D_p e^{-2\gamma(t-\xi)}=\frac{D_p}{\gamma}(1-e^{-2\gamma t}).$$
- 1. $\tau_c\ll t\ll\frac{1}{\gamma}$, $\overline{(\Delta p)^2}=2D_p t$;
- 2. $t\gg\frac{1}{\gamma}$, $\overline{p}(t)=0$, so $\overline{p^2}=\overline{(\Delta p)^2}=\frac{D_p}{\gamma}$.

  In equilibrium, $\frac{\overline{p^2}}{2m}=\frac{1}{2}kT$, $\therefore D_p=m\gamma kT=\alpha kT$.

Temporal correlation of the momentum

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$\overline{p(t)p(t')}$

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- 1. If $t < t'$, $\overline{p(t)p(t')} = e^{-\gamma(t+t')}\int_0^t \mathrm{d}\xi 2D_p e^{2\gamma\xi}$

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- 1. If $t < t'$, $\overline{p(t)p(t')} = e^{-\gamma(t+t')}\int_0^t \mathrm{d}\xi 2D_p e^{2\gamma\xi}$

$= 2D_p e^{-\gamma(t+t')}\frac{1}{2\gamma}(e^{2\gamma t} - 1)$

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Temporal correlation of the momentum

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- 1. If $t < t'$, $\overline{p(t)p(t')} = e^{-\gamma(t+t')}\int_0^t \mathrm{d}\xi 2D_p e^{2\gamma\xi}$ $= 2D_p e^{-\gamma(t+t')}\frac{1}{2\gamma}(e^{2\gamma t} - 1) = \frac{D_p}{\gamma}[e^{-\gamma(t'-t)} - e^{-\gamma(t+t')}]$.
- 2. Similarly $t > t'$, $\overline{p(t)p(t')} = \frac{D_p}{\gamma}[e^{-\gamma(t-t')} - e^{-\gamma(t+t')}]$.

Temporal correlation of the momentum

$$\boxed{p(t) = p(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t F(\xi)e^{\gamma\xi}\mathrm{d}\xi}$$

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$$= e^{-\gamma(t+t')}\int_0^t \mathrm{d}\xi \int_0^{t'} \mathrm{d}\xi' \overline{F(\xi)F(\xi')}e^{\gamma(\xi+\xi')}$$
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$= 2D_p e^{-\gamma(t+t')}\frac{1}{2\gamma}(e^{2\gamma t} - 1) = \frac{D_p}{\gamma}[e^{-\gamma(t'-t)} - e^{-\gamma(t+t')}]$.
- 2. Similarly $t > t'$, $\overline{p(t)p(t')} = \frac{D_p}{\gamma}[e^{-\gamma(t-t')} - e^{-\gamma(t+t')}]$.
- The combination: $\overline{p(t)p(t')} = \frac{D_p}{\gamma}[e^{-\gamma|t-t'|} - e^{-\gamma(t+t')}]$.

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- Back to $\overline{x^2(t)}$ again. $x(t) = \int_0^t \dot{x}(\xi)\mathrm{d}\xi = \frac{1}{m}\int_0^t p(\xi)\mathrm{d}\xi$.

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- Back to $\overline{x^2(t)}$ again.
  $x(t) = \int_0^t \dot{x}(\xi)\mathrm{d}\xi = \frac{1}{m}\int_0^t p(\xi)\mathrm{d}\xi$.
  $\overline{x^2(t)} = \overline{\frac{1}{m}\int_0^t p(\xi)\mathrm{d}\xi \cdot \frac{1}{m}\int_0^t p(\xi')\mathrm{d}\xi'}$

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  $$\overline{x^2(t)} = \overline{\frac{1}{m}\int_0^t p(\xi)\mathrm{d}\xi \cdot \frac{1}{m}\int_0^t p(\xi')\mathrm{d}\xi'}$$
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$$\boxed{\overline{p(t)p(t')} = \frac{D_p}{\gamma}[e^{-\gamma|t-t'|} - e^{-\gamma(t+t')}]}$$

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&= \frac{kT}{m}\int_0^t \mathrm{d}\xi \int_0^t \mathrm{d}\xi' e^{-\gamma|\xi-\xi'|} \\
&= \frac{kT}{m}\int_0^t \mathrm{d}\xi \left[\int_0^\xi \mathrm{d}\xi' e^{-\gamma(\xi-\xi')} + \int_\xi^t \mathrm{d}\xi' e^{-\gamma(\xi'-\xi)}\right]
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\end{aligned}
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Table of contents

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- $m\frac{\mathrm{d}v}{\mathrm{d}t} = \mathcal{F} - \alpha v + F(t)$, Langevin's equation. $L \leftrightarrow m$, $i \leftrightarrow v$, $\mathcal{V} \leftrightarrow \mathcal{F}$, $R \leftrightarrow \alpha$, and $V \leftrightarrow F$.

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  $= \frac{kTR}{\pi}\delta(\omega-\omega')$, (white noise)
  where $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega'-\omega)t}\mathrm{d}t = \delta(\omega-\omega')$.

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- When the excited atom transits to the ground state, it radiates a photon and feels a repulsive force $F_z(t)$.
- Langevin's equation: $m\frac{\mathrm{d}v_z}{\mathrm{d}t} = -\alpha v_z + F_z$.

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- Frequency of laser: ω_L; of atom: ω_A. ($\omega_L \lesssim \omega_A$.)
- For the atom at velocity v_z (1 direction), it feels the frequency of laser $\omega_L(1 \pm \frac{|v_z|}{c})$. Higher velocity, closer to ω_A; more easier to absorb the photon and lose its momentum. Effective viscous force $-\alpha v_z$.
- When the excited atom transits to the ground state, it radiates a photon and feels a repulsive force $F_z(t)$.
- Langevin's equation: $m\frac{\mathrm{d}v_z}{\mathrm{d}t} = -\alpha v_z + F_z$.
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Table of contents