

Thermodynamics & Statistical Physics

Chapter 8. Bose statistics and Fermi statistics

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 $\Rightarrow \beta = \frac{1}{kT}$, $\alpha = -\beta\mu = -\frac{\mu}{kT}$.

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Comparing with eq.(6.7.4),

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Thermal parameters in statistics for Fermion system

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- $$N = -\frac{\partial}{\partial \alpha} \ln \mathcal{Z}.$$

- $$U = -\frac{\partial}{\partial \beta} \ln \mathcal{Z}.$$

- $$Y = -\frac{1}{\beta} \frac{\partial}{\partial y} \ln \mathcal{Z}.$$

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- 8.1 Thermal parameters in statistics
- 8.2 U of weak degenerated ideal Fermion(Boson) gas
- 8.3 Bose-Einstein Condensate (BEC)
- 8.4 Photon gas
- 8.5 Free electron gas in metal

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$$U = \sum a_l \varepsilon_l$$

$$D(\varepsilon)d\varepsilon = g \frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \varepsilon^{\frac{1}{2}} d\varepsilon$$

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$$N = \sum a_l = \sum \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l} \pm 1} = g \frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \int_0^\infty \frac{\varepsilon^{\frac{1}{2}} d\varepsilon}{e^{\alpha+\beta\varepsilon} \pm 1}.$$

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 &= g \frac{2\pi V}{h^3} (2mkT)^{\frac{3}{2}} e^{-\alpha} \\
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 &= g \frac{2\pi V}{h^3} (2mkT)^{\frac{3}{2}} e^{-\alpha} \frac{\sqrt{\pi}}{2} (1 \mp e^{-\alpha} 2^{-\frac{3}{2}}) \dots
 \end{aligned}$$

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- $$\frac{U}{N} = \frac{g \left(\frac{2\pi mkT}{h^2} \right)^{\frac{3}{2}} V \frac{3}{2} kT e^{-\alpha} (1 \mp 2^{-\frac{5}{2}} e^{-\alpha})}{g \left(\frac{2\pi mkT}{h^2} \right)^{\frac{3}{2}} V e^{-\alpha} (1 \mp 2^{-\frac{3}{2}} e^{-\alpha})}$$

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1st term, same as the Boltzmann distribution; 2nd term, modification by the identical principle of particles (Fermions, repel; Bosons, attract).

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- 8.1 Thermal parameters in statistics
- 8.2 U of weak degenerated ideal Fermion(Boson) gas
- 8.3 Bose-Einstein Condensate (BEC)
- 8.4 Photon gas
- 8.5 Free electron gas in metal

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$T \downarrow$, $\mu \uparrow$, until 0. Corresponds to a critical temperature: T_c .

BEC – idea gas

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- Magneto optical trap – 3-D harmonic oscillator.

BEC – realization

- $\varepsilon_{n_x, n_y, n_z} = \hbar\omega_x\left(n_x + \frac{1}{2}\right) + \hbar\omega_y\left(n_y + \frac{1}{2}\right) + \hbar\omega_z\left(n_z + \frac{1}{2}\right).$

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&= (\frac{kT_c}{\hbar\bar{\omega}})^3 \iiint \sum_{l=1}^{\infty} e^{-l(x+y+z)} dx dy dz \dots
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- 8.5 Free electron gas in metal

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$$= \int_0^\infty \frac{V}{\pi^2 c^3} \frac{\hbar\omega^3 d\omega}{e^{\frac{\hbar\omega}{kT}} - 1} = \int_0^\infty \frac{V\hbar}{\pi^2 c^3} \left(\frac{\hbar}{kT}\right)^{-4} \frac{(\frac{\hbar\omega}{kT})^3 d(\frac{\hbar\omega}{kT})}{e^{\frac{\hbar\omega}{kT}} - 1}$$

$$= \frac{V(kT)^4}{\pi^2 c^3 \hbar^3} \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

Photon gas

- Number of quantum states: $2 \cdot \frac{V4\pi p^2 dp}{h^3} = \frac{8\pi V p^2 dp}{h^3}$,
where 2 is because of the polarization.
- Convert to frequency $\hbar\omega = \varepsilon = pc, \Rightarrow p = \frac{\hbar\omega}{c}$,

$$D(\omega)d\omega = \frac{8\pi V(\frac{\hbar\omega}{c})^2 d(\frac{\hbar\omega}{c})}{h^3} = \frac{V}{\pi^2 c^3} \omega^2 d\omega.$$
- Number of photons at $(\omega, \omega + d\omega)$:

$$a(\omega)d\omega = \frac{V}{\pi^2 c^3} \frac{\omega^2 d\omega}{e^{\frac{\hbar\omega}{kT}} - 1}.$$
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Photon gas $D(\omega)d\omega = \frac{V}{\pi^2 c^3} \omega^2 d\omega$

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$$D(\omega)d\omega = \frac{V}{\pi^2 c^3} \omega^2 d\omega$$

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Photon gas

$$U = \frac{\pi^2 k^4}{15c^3 \hbar^3} T^4 V,$$

$$\ln \mathcal{Z} = \frac{\pi^2 V}{45c^3} \frac{1}{(\beta \hbar)^3}.$$

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$$U = -\frac{\partial}{\partial \beta} \ln \mathcal{Z}$$

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$$p = \frac{1}{\beta} \frac{\partial}{\partial V} \ln \mathcal{Z}$$

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$$U = \frac{\pi^2 k^4}{15c^3 \hbar^3} T^4 V, \quad \ln \mathcal{Z} = \frac{\pi^2 V}{45c^3} \frac{1}{(\beta \hbar)^3}.$$

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- Entropy:

$$S = k(\ln \mathcal{Z} - \alpha \frac{\partial \ln \mathcal{Z}}{\partial \alpha} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta})$$

Photon gas

$$U = \frac{\pi^2 k^4}{15c^3 \hbar^3} T^4 V, \quad \ln \mathcal{Z} = \frac{\pi^2 V}{45c^3} \frac{1}{(\beta \hbar)^3}.$$

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- Entropy:

$$S = k(\ln \mathcal{Z} - \alpha \frac{\partial \ln \mathcal{Z}}{\partial \alpha} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta}) = k(\ln \mathcal{Z} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta})$$

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- Pressure:

$$p = \frac{1}{\beta} \frac{\partial}{\partial V} \ln \mathcal{Z} = \frac{1}{\beta} \frac{\pi^2}{45c^3} \frac{1}{(\beta \hbar)^3} = \frac{\pi^2 k^4}{45c^3 \hbar^3} T^4 = \frac{1}{3} \frac{U}{V}.$$

- Entropy:

$$\begin{aligned} S &= k \left(\ln \mathcal{Z} - \alpha \frac{\partial \ln \mathcal{Z}}{\partial \alpha} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta} \right) = k \left(\ln \mathcal{Z} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta} \right) \\ &= k \left[\frac{\pi^2 V}{45c^3} \frac{1}{(\beta \hbar)^3} + \beta \frac{\pi^2 k^4}{15c^3 \hbar^3} T^4 V \right] \end{aligned}$$

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$$U = \frac{\pi^2 k^4}{15c^3 \hbar^3} T^4 V,$$

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- Entropy:

$$\begin{aligned} S &= k \left(\ln \mathcal{Z} - \alpha \frac{\partial \ln \mathcal{Z}}{\partial \alpha} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta} \right) = k \left(\ln \mathcal{Z} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta} \right) \\ &= k \left[\frac{\pi^2 V}{45c^3} \frac{1}{(\beta \hbar)^3} + \beta \frac{\pi^2 k^4}{15c^3 \hbar^3} T^4 V \right] \\ &= k \left(\frac{\pi^2 V}{45c^3} \frac{k^3 T^3}{\hbar^3} + \frac{\pi^2 V}{15c^3} \frac{k^3 T^3}{\hbar^3} \right) \end{aligned}$$

Photon gas

$$U = \frac{\pi^2 k^4}{15c^3 \hbar^3} T^4 V, \quad \ln \mathcal{Z} = \frac{\pi^2 V}{45c^3} \frac{1}{(\beta \hbar)^3}.$$

- Internal energy:

$$U = -\frac{\partial}{\partial \beta} \ln \mathcal{Z} = -\frac{\pi^2 V}{45c^3} \frac{1}{\hbar^3} \frac{-3}{\beta^4} = \frac{\pi^2 k^4}{15c^3 \hbar^3} T^4 V.$$

- Pressure:

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- Entropy:

$$\begin{aligned} S &= k \left(\ln \mathcal{Z} - \alpha \frac{\partial \ln \mathcal{Z}}{\partial \alpha} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta} \right) = k \left(\ln \mathcal{Z} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta} \right) \\ &= k \left[\frac{\pi^2 V}{45c^3} \frac{1}{(\beta \hbar)^3} + \beta \frac{\pi^2 k^4}{15c^3 \hbar^3} T^4 V \right] \\ &= k \left(\frac{\pi^2 V}{45c^3} \frac{k^3 T^3}{\hbar^3} + \frac{\pi^2 V}{15c^3} \frac{k^3 T^3}{\hbar^3} \right) \\ &= \frac{4}{45} \frac{\pi^2 k^4 V}{c^3 \hbar^3} T^3. \quad (\text{same as (2.6.4)}) \end{aligned}$$

Table of contents

1 §8. Bose statistics and Fermi statistics

- 8.1 Thermal parameters in statistics
- 8.2 U of weak degenerated ideal Fermion(Boson) gas
- 8.3 Bose-Einstein Condensate (BEC)
- 8.4 Photon gas
- 8.5 Free electron gas in metal

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$$p_{\text{Cu}} = \frac{2}{5} \cdot 8.5 \times 10^{28} \text{m}^{-3} \cdot 1.38 \times 10^{-23} \text{J} \cdot \text{K}^{-1} \cdot 8.2 \times 10^4 \text{K}$$

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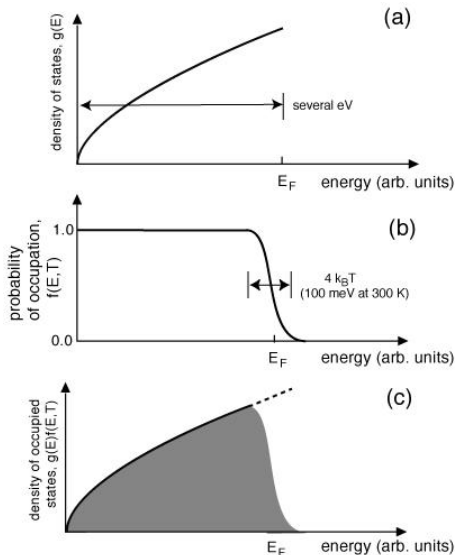
$$f \begin{cases} > \frac{1}{2}, & \varepsilon < \mu; \\ = \frac{1}{2}, & \varepsilon = \mu; \\ < \frac{1}{2}, & \varepsilon > \mu. \end{cases}$$

- Number of states:

$$\frac{4\pi V}{h^3} (2m)^{\frac{3}{2}} \varepsilon^{\frac{1}{2}} d\varepsilon.$$

- Energy density:

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$$E = \frac{2}{5} C \mu^{\frac{5}{2}} \left[1 + \frac{5\pi^2}{8} \left(\frac{kT}{\mu} \right)^2 \right], \text{ where } C = \frac{4\pi V}{h^3} (2m)^{\frac{3}{2}}.$$

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- $\therefore C_V = (\frac{\partial U}{\partial T})_V = Nk \frac{\pi^2 k}{2\mu(0)} T = \frac{\pi^2}{2} Nk \frac{T}{T_F} \propto T.$

Table of contents

1 §8. Bose statistics and Fermi statistics

- 8.1 Thermal parameters in statistics
- 8.2 U of weak degenerated ideal Fermion(Boson) gas
- 8.3 Bose-Einstein Condensate (BEC)
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- 8.5 Free electron gas in metal