# Thermodynamics & Statistical Physics

## Chapter 5. Thermodynamics of irreversible processes

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- $\Theta \geqslant 0$.

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- $\therefore \frac{\partial u}{\partial t} = - \bigtriangledown \cdot \vec{J}_q - \bigtriangledown \cdot (\mu \vec{J}_n)$.

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- Generally, $\Theta = \sum_k \vec{J}_k \cdot \vec{X}_k$.

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# 5.2 Linear and nonlinear processes, Onsager relation

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- Summation:
  $\vec{J} = L\vec{X}$, where $\vec{J}$ is the flux density, $\vec{X}$ is the "force".

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- $\Theta \geqslant 0$, requires, $L_{11} > 0$, $L_{11} L_{22} > L_{12}^2$.
  (Properties of of the coefficients $L$.)

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