Thermodynamics & Statistical Physics

Chapter 6. The most probable distribution of nearly independent particles

Yuan-Chuan Zou
zouyc@hust.edu.cn

School of Physics, Huazhong University of Science and Technology

December 30, 2013

Table of contents

6.1 Classical description of particle's movement

6.1 Classical description of particle's movement

- Macro quantities are statistical average of micro quantities.

6.1 Classical description of particle's movement

- Macro quantities are statistical average of micro quantities.
- To describe the state of motion, use generalized coordinates and generalized momentum (phase space): $(q_1, ..., q_r; p_1, ..., p_r)$, where r is the degree of freedom.

6.1 Classical description of particle's movement

- Macro quantities are statistical average of micro quantities.
- To describe the state of motion, use generalized coordinates and generalized momentum (phase space): $(q_1, ..., q_r; p_1, ..., p_r)$, where r is the degree of freedom.
- Change of the state is a curve in the phase space.

6.1 Classical description of particle's movement

- Macro quantities are statistical average of micro quantities.
- To describe the state of motion, use generalized coordinates and generalized momentum (phase space): $(q_1, ..., q_r; p_1, ..., p_r)$, where r is the degree of freedom.
- Change of the state is a curve in the phase space.
- Energy of a particle: $\varepsilon = \varepsilon(q_1, ..., q_r; p_1, ..., p_r)$.

6.1 Classical description of particle's movement

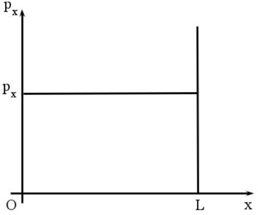
- Macro quantities are statistical average of micro quantities.
- To describe the state of motion, use generalized coordinates and generalized momentum (phase space): $(q_1, ..., q_r; p_1, ..., p_r)$, where r is the degree of freedom.
- Change of the state is a curve in the phase space.
- Energy of a particle: $\varepsilon = \varepsilon(q_1, ..., q_r; p_1, ..., p_r)$.

- Eg.1: Free particle:
  coordinate (x, y, z), momentum
  (p_x, p_y, p_z), trajectory$\rightarrow$

6.1 Classical description of particle's movement

- Macro quantities are statistical average of micro quantities.
- To describe the state of motion, use generalized coordinates and generalized momentum (phase space): $(q_1, ..., q_r; p_1, ..., p_r)$, where r is the degree of freedom.
- Change of the state is a curve in the phase space.
- Energy of a particle: $\varepsilon = \varepsilon(q_1, ..., q_r; p_1, ..., p_r)$.

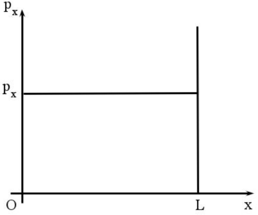
- Eg.1: Free particle:
  coordinate (x, y, z), momentum (p_x, p_y, p_z), trajectory$\rightarrow$



6.1 Classical description of particle's movement

- Macro quantities are statistical average of micro quantities.
- To describe the state of motion, use generalized coordinates and generalized momentum (phase space): $(q_1, ..., q_r; p_1, ..., p_r)$, where r is the degree of freedom.
- Change of the state is a curve in the phase space.
- Energy of a particle: $\varepsilon = \varepsilon(q_1, ..., q_r; p_1, ..., p_r)$.

- Eg.1: Free particle:
  coordinate (x, y, z), momentum (p_x, p_y, p_z), trajectory→
  energy: $\varepsilon = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$.



Classical description of particle's movement

- Eg.2. Linear oscillator:

Classical description of particle's movement

- Eg.2. Linear oscillator:
  $F = -Ax = m\ddot{x} \Rightarrow \omega = \sqrt{A/m}$,

Classical description of particle's movement

- Eg.2. Linear oscillator:
  $F = -Ax = m\ddot{x} \Rightarrow \omega = \sqrt{A/m}$,
  $x = C_1 \sin(\omega t + \phi_0)$,

Classical description of particle's movement

- Eg.2. Linear oscillator:
  $F = -Ax = m\ddot{x} \Rightarrow \omega = \sqrt{A/m}$,
  $x = C_1 \sin(\omega t + \phi_0)$,
  $p = m\dot{x} = m\omega C_1 \cos(\omega t + \phi_0)$,

Classical description of particle's movement

- Eg.2. Linear oscillator:
  $F = -Ax = m\ddot{x} \Rightarrow \omega = \sqrt{A/m}$,
  $x = C_1 \sin(\omega t + \phi_0)$,
  $p = m\dot{x} = m\omega C_1 \cos(\omega t + \phi_0)$,
  energy: $\varepsilon = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \mathrm{Const.}$

Classical description of particle's movement

- Eg.2. Linear oscillator:
  $F = -Ax = m\ddot{x} \Rightarrow \omega = \sqrt{A/m}$,
  $x = C_1 \sin(\omega t + \phi_0)$,
  $p = m\dot{x} = m\omega C_1 \cos(\omega t + \phi_0)$,
  energy: $\varepsilon = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \mathrm{Const.}$
  $\Rightarrow \frac{p^2}{2m\varepsilon} + \frac{x^2}{2\varepsilon/(m\omega^2)} = 1$ (trajectory in phase space).

Classical description of particle's movement

- Eg.2. Linear oscillator:
  $F = -Ax = m\ddot{x} \Rightarrow \omega = \sqrt{A/m}$,
  $x = C_1 \sin(\omega t + \phi_0)$,
  $p = m\dot{x} = m\omega C_1 \cos(\omega t + \phi_0)$,
  energy: $\varepsilon = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \text{Const.}$
  $\Rightarrow \frac{p^2}{2m\varepsilon} + \frac{x^2}{2\varepsilon/(m\omega^2)} = 1$ (trajectory in phase space).
- Eg.3. Rotator: Rotating with fixed radius:

Classical description of particle's movement

- Eg.2. Linear oscillator:
  $F = -Ax = m\ddot{x} \Rightarrow \omega = \sqrt{A/m}$,
  $x = C_1 \sin(\omega t + \phi_0)$,
  $p = m\dot{x} = m\omega C_1 \cos(\omega t + \phi_0)$,
  energy: $\varepsilon = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \mathrm{Const.}$
  $\Rightarrow \frac{p^2}{2m\varepsilon} + \frac{x^2}{2\varepsilon/(m\omega^2)} = 1$ (trajectory in phase space).
- Eg.3. Rotator: Rotating with fixed radius:
  Kinetic energy: $\varepsilon = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$.

Classical description of particle's movement

- Eg.2. Linear oscillator:
  $F = -Ax = m\ddot{x} \Rightarrow \omega = \sqrt{A/m}$,
  $x = C_1 \sin(\omega t + \phi_0)$,
  $p = m\dot{x} = m\omega C_1 \cos(\omega t + \phi_0)$,
  energy: $\varepsilon = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \mathrm{Const.}$
  $\Rightarrow \frac{p^2}{2m\varepsilon} + \frac{x^2}{2\varepsilon/(m\omega^2)} = 1$ (trajectory in phase space).
- Eg.3. Rotator: Rotating with fixed radius:
  Kinetic energy: $\varepsilon = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$.
  To polar coordinate: $x = r\sin\theta\cos\varphi$,
  $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$.

Classical description of particle's movement

- Eg.2. Linear oscillator:
  $F = -Ax = m\ddot{x} \Rightarrow \omega = \sqrt{A/m}$,
  $x = C_1 \sin(\omega t + \phi_0)$,
  $p = m\dot{x} = m\omega C_1 \cos(\omega t + \phi_0)$,
  energy: $\varepsilon = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \mathrm{Const.}$
  $\Rightarrow \frac{p^2}{2m\varepsilon} + \frac{x^2}{2\varepsilon/(m\omega^2)} = 1$ (trajectory in phase space).
- Eg.3. Rotator: Rotating with fixed radius:
  Kinetic energy: $\varepsilon = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$.
  To polar coordinate: $x = r\sin\theta\cos\varphi$,
  $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$.
  Kinetic energy: $\varepsilon = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2)$.

Classical description of particle's movement

- $\because \dot{r} = 0$, $\therefore \varepsilon = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2)$.

Classical description of particle's movement

- $\because \dot{r} = 0$, $\therefore \varepsilon = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2)$.
  Conjugate momentum of θ, φ: $p_\theta = mr^2\dot{\theta}$, $p_\varphi = mr^2 \sin^2\theta\dot{\varphi}$.

Classical description of particle's movement

- $\because \dot{r} = 0$, $\therefore \varepsilon = \frac{1}{2} m(r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2)$.
  Conjugate momentum of θ, φ: $p_\theta = mr^2\dot{\theta}$,
  $p_\varphi = mr^2 \sin^2\theta\dot{\varphi}$.
  $\therefore \varepsilon = \frac{1}{2}\left[\frac{(mr^2\dot{\theta})^2}{mr^2} + \frac{(mr^2\sin^2\theta\dot{\varphi})^2}{mr^2\sin^2\theta}\right]$

Classical description of particle's movement

- $\because \dot{r} = 0$, $\therefore \varepsilon = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2)$.
  Conjugate momentum of θ, φ: $p_\theta = mr^2\dot{\theta}$,
  $p_\varphi = mr^2\sin^2\theta\dot{\varphi}$.
  $$\begin{aligned}\therefore \varepsilon &= \frac{1}{2}\left[\frac{(mr^2\dot{\theta})^2}{mr^2} + \frac{(mr^2\sin^2\theta\dot{\varphi})^2}{mr^2\sin^2\theta}\right] \\ &= \frac{1}{2I}\left(p_\theta^2 + \frac{1}{\sin^2\theta}p_\varphi^2\right).\end{aligned}$$

Classical description of particle's movement

- $\because \dot{r} = 0$, $\therefore \varepsilon = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2)$.
  Conjugate momentum of θ, φ: $p_\theta = mr^2\dot{\theta}$,
  $p_\varphi = mr^2\sin^2\theta\dot{\varphi}$.
  $$\therefore \varepsilon = \frac{1}{2}\left[\frac{(mr^2\dot{\theta})^2}{mr^2} + \frac{(mr^2\sin^2\theta\dot{\varphi})^2}{mr^2\sin^2\theta}\right]$$
  $$= \frac{1}{2I}\left(p_\theta^2 + \frac{1}{\sin^2\theta}p_\varphi^2\right).$$
- Without external force, angular momentum conserved. Choose z-axis as the direction of the angular momentum, i.e., $\theta = \frac{\pi}{2} = \mathrm{const.}$,

Classical description of particle's movement

- $\because \dot{r} = 0$, $\therefore \varepsilon = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2)$.
  Conjugate momentum of θ, φ: $p_\theta = mr^2\dot{\theta}$, $p_\varphi = mr^2\sin^2\theta\dot{\varphi}$.
  $$\therefore \varepsilon = \frac{1}{2}\left[\frac{(mr^2\dot{\theta})^2}{mr^2} + \frac{(mr^2\sin^2\theta\dot{\varphi})^2}{mr^2\sin^2\theta}\right] = \frac{1}{2I}\left(p_\theta^2 + \frac{1}{\sin^2\theta}p_\varphi^2\right).$$
- Without external force, angular momentum conserved. Choose z-axis as the direction of the angular momentum, i.e., $\theta = \frac{\pi}{2} = \mathrm{const.}$, $p_\theta = 0$, $p_\varphi = mr^2\sin^2\theta\dot{\varphi} = mr^2\dot{\varphi} = mr^2\omega = M$,

Classical description of particle's movement

- $\because \dot{r} = 0$, $\therefore \varepsilon = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2)$.
  Conjugate momentum of θ, φ: $p_\theta = mr^2\dot{\theta}$,
  $p_\varphi = mr^2 \sin^2\theta\dot{\varphi}$.
  $$\therefore \varepsilon = \frac{1}{2}\left[\frac{(mr^2\dot{\theta})^2}{mr^2} + \frac{(mr^2\sin^2\theta\dot{\varphi})^2}{mr^2\sin^2\theta}\right]$$
  $$= \frac{1}{2I}(p_\theta^2 + \frac{1}{\sin^2\theta}p_\varphi^2).$$
- Without external force, angular momentum conserved. Choose z-axis as the direction of the angular momentum, i.e., $\theta = \frac{\pi}{2} = \text{const.}$,
  $p_\theta = 0$, $p_\varphi = mr^2\sin^2\theta\dot{\varphi} = mr^2\dot{\varphi} = mr^2\omega = M$,
  $\varepsilon = \frac{M^2}{2I}$.

Table of contents

6.2 Quantum description of particle's movement

6.2 Quantum description of particle's movement

- Wave-particle duality: $\varepsilon = h\nu$, $p = h/\lambda$, where $\lambda\nu = v_{\rm ph}$.

6.2 Quantum description of particle's movement

- Wave-particle duality: $\varepsilon = h\nu$, $p = h/\lambda$, where $\lambda\nu = v_{\rm ph}$.
- $\Delta q \Delta p \simeq h$, minimal "phase volume" in the phase space.

6.2 Quantum description of particle's movement

- Wave-particle duality: $\varepsilon = h\nu$, $p = h/\lambda$, where $\lambda\nu = v_{\rm ph}$.
- $\Delta q \Delta p \simeq h$, minimal "phase volume" in the phase space.
- Eg.1. Linear oscillator:

6.2 Quantum description of particle's movement

- Wave-particle duality: $\varepsilon = h\nu$, $p = h/\lambda$, where $\lambda\nu = v_{\mathrm{ph}}$.
- $\Delta q \Delta p \simeq h$, minimal "phase volume" in the phase space.
- Eg.1. Linear oscillator: $\varepsilon = h\nu(n + \frac{1}{2})$, $n = 0, 1, 2...$ discrete energy level.

6.2 Quantum description of particle's movement

- Wave-particle duality: $\varepsilon = h\nu$, $p = h/\lambda$, where $\lambda\nu = v_{\rm ph}$.
- $\Delta q \Delta p \simeq h$, minimal "phase volume" in the phase space.
- Eg.1. Linear oscillator: $\varepsilon = h\nu(n + \frac{1}{2})$, $n = 0, 1, 2...$ discrete energy level.
- Eg.2. Rotator:

6.2 Quantum description of particle's movement

- Wave-particle duality: $\varepsilon = h\nu$, $p = h/\lambda$, where $\lambda\nu = v_{\rm ph}$.
- $\Delta q \Delta p \simeq h$, minimal "phase volume" in the phase space.
- Eg.1. Linear oscillator: $\varepsilon = h\nu(n + \frac{1}{2})$, $n = 0, 1, 2...$ discrete energy level.
- Eg.2. Rotator: $\varepsilon = \frac{M^2}{2I}$, where $M^2 = l(l+1)\hbar^2$, $l = 0, 1, 2, ...$.

6.2 Quantum description of particle's movement

- Wave-particle duality: $\varepsilon = h\nu$, $p = h/\lambda$, where $\lambda\nu = v_{\rm ph}$.
- $\Delta q \Delta p \simeq h$, minimal "phase volume" in the phase space.
- Eg.1. Linear oscillator:
  $\varepsilon = h\nu(n + \frac{1}{2})$, $n = 0, 1, 2...$ discrete energy level.
- Eg.2. Rotator:
  $\varepsilon = \frac{M^2}{2I}$, where $M^2 = l(l+1)\hbar^2$, $l = 0, 1, 2, \ldots$.
  $M_z = m\hbar$, $m = 0, \pm 1, \pm 2, ..., \pm l$.

6.2 Quantum description of particle's movement

- Wave-particle duality: $\varepsilon = h\nu$, $p = h/\lambda$, where $\lambda\nu = v_{\rm ph}$.
- $\Delta q \Delta p \simeq h$, minimal "phase volume" in the phase space.
- Eg.1. Linear oscillator:
  $\varepsilon = h\nu(n + \frac{1}{2})$, $n = 0, 1, 2...$ discrete energy level.
- Eg.2. Rotator:
  $\varepsilon = \frac{M^2}{2I}$, where $M^2 = l(l+1)\hbar^2$, $l = 0, 1, 2, ...$
  $M_z = m\hbar$, $m = 0, \pm 1, \pm 2, ..., \pm l$.
  Degenerated.

Quantum description of particle's movement

- Eg.3. Spin angular momentum:

Quantum description of particle's movement

- Eg.3. Spin angular momentum:
  $\vec{s}^2 = s(s+1)\hbar^2$, where s is integer or half integer.

Quantum description of particle's movement

- Eg.3. Spin angular momentum:
  $\vec{s}^2 = s(s+1)\hbar^2$, where s is integer or half integer.
  $s_z = m_s\hbar$, $m_s = -s, ..., s$.

Quantum description of particle's movement

- Eg.3. Spin angular momentum:
  $\vec{s}^2 = s(s+1)\hbar^2$, where s is integer or half integer.
  $s_z = m_s\hbar$, $m_s = -s, ..., s$.
  Electron, $s = \frac{1}{2}$, $m_s = \pm\frac{1}{2}$.

Quantum description of particle's movement

- Eg.3. Spin angular momentum:
  $\vec{s}^2 = s(s+1)\hbar^2$, where s is integer or half integer.
  $s_z = m_s\hbar$, $m_s = -s, ..., s$.
  Electron, $s = \frac{1}{2}$, $m_s = \pm\frac{1}{2}$.
  Magnetic momentum: $\vec{\mu} = -\frac{e}{m}\vec{s}$.

Quantum description of particle's movement

- Eg.3. Spin angular momentum:
  $\vec{s}^2 = s(s+1)\hbar^2$, where s is integer or half integer.
  $s_z = m_s\hbar$, $m_s = -s, ..., s$.
  Electron, $s = \frac{1}{2}$, $m_s = \pm\frac{1}{2}$.
  Magnetic momentum: $\vec{\mu} = -\frac{e}{m}\vec{s}$. In the external $\vec{B}$, two choices of $m_z = \pm\frac{1}{2}$,

Quantum description of particle's movement

- Eg.3. Spin angular momentum:
  $\vec{s}^2 = s(s+1)\hbar^2$, where s is integer or half integer.
  $s_z = m_s\hbar$, $m_s = -s, ..., s$.
  Electron, $s = \frac{1}{2}$, $m_s = \pm\frac{1}{2}$.
  Magnetic momentum: $\vec{\mu} = -\frac{e}{m}\vec{s}$. In the external $\vec{B}$, two choices of $m_z = \pm\frac{1}{2}$, $\mu_z = \mp\frac{e\hbar}{2m}$.

Quantum description of particle's movement

- Eg.3. Spin angular momentum:
  $\vec{s}^2 = s(s+1)\hbar^2$, where s is integer or half integer.
  $s_z = m_s\hbar$, $m_s = -s, ..., s$.
  Electron, $s = \frac{1}{2}$, $m_s = \pm\frac{1}{2}$.
  Magnetic momentum: $\vec{\mu} = -\frac{e}{m}\vec{s}$. In the external $\vec{B}$, two choices of $m_z = \pm\frac{1}{2}$, $\mu_z = \mp\frac{e\hbar}{2m}$.
  Energy in the external field: $-\vec{\mu}\cdot\vec{B} = \pm\frac{e\hbar}{2m}B$.

Quantum description of particle's movement

- Eg.3. Spin angular momentum:
  $\vec{s}^2 = s(s+1)\hbar^2$, where s is integer or half integer.
  $s_z = m_s\hbar$, $m_s = -s, ..., s$.
  Electron, $s = \frac{1}{2}$, $m_s = \pm\frac{1}{2}$.
  Magnetic momentum: $\vec{\mu} = -\frac{e}{m}\vec{s}$. In the external $\vec{B}$, two choices of $m_z = \pm\frac{1}{2}$, $\mu_z = \mp\frac{e\hbar}{2m}$.
  Energy in the external field: $-\vec{\mu}\cdot\vec{B} = \pm\frac{e\hbar}{2m}B$.
- Eg.4. Free particle:

Quantum description of particle's movement

- Eg.3. Spin angular momentum:
  $\vec{s}^2 = s(s+1)\hbar^2$, where s is integer or half integer.
  $s_z = m_s\hbar$, $m_s = -s, ..., s$.
  Electron, $s = \frac{1}{2}$, $m_s = \pm\frac{1}{2}$.
  Magnetic momentum: $\vec{\mu} = -\frac{e}{m}\vec{s}$. In the external $\vec{B}$, two choices of $m_z = \pm\frac{1}{2}$, $\mu_z = \mp\frac{e\hbar}{2m}$.
  Energy in the external field: $-\vec{\mu}\cdot\vec{B} = \pm\frac{e\hbar}{2m}B$.
- Eg.4. Free particle:
  1-D, periodic border, $L = |n_x|\lambda$, $n_x = 0, \pm1, \pm2, ...$.

Quantum description of particle's movement

- Eg.3. Spin angular momentum:
  $\vec{s}^2 = s(s+1)\hbar^2$, where s is integer or half integer.
  $s_z = m_s\hbar$, $m_s = -s, ..., s$.
  Electron, $s = \frac{1}{2}$, $m_s = \pm\frac{1}{2}$.
  Magnetic momentum: $\vec{\mu} = -\frac{e}{m}\vec{s}$. In the external $\vec{B}$, two choices of $m_z = \pm\frac{1}{2}$, $\mu_z = \mp\frac{e\hbar}{2m}$.
  Energy in the external field: $-\vec{\mu}\cdot\vec{B} = \pm\frac{e\hbar}{2m}B$.
- Eg.4. Free particle:
  1-D, periodic border, $L = |n_x|\lambda$, $n_x = 0, \pm1, \pm2, ...$
  De Broglie relation: $p_x = h/\lambda = \frac{h}{L}n_x$,

Quantum description of particle's movement

- Eg.3. Spin angular momentum:
  $\vec{s}^2 = s(s+1)\hbar^2$, where s is integer or half integer.
  $s_z = m_s\hbar$, $m_s = -s, ..., s$.
  Electron, $s = \frac{1}{2}$, $m_s = \pm\frac{1}{2}$.
  Magnetic momentum: $\vec{\mu} = -\frac{e}{m}\vec{s}$. In the external $\vec{B}$, two choices of $m_z = \pm\frac{1}{2}$, $\mu_z = \mp\frac{e\hbar}{2m}$.
  Energy in the external field: $-\vec{\mu}\cdot\vec{B} = \pm\frac{e\hbar}{2m}B$.
- Eg.4. Free particle:
  1-D, periodic border, $L = |n_x|\lambda$, $n_x = 0, \pm1, \pm2, \ldots$.
  De Broglie relation: $p_x = h/\lambda = \frac{h}{L}n_x$,
  energy: $\varepsilon_{n_x} = \frac{p_x^2}{2m} = \frac{h^2}{2m}\frac{n_x^2}{L^2}$.

Quantum description of particle's movement

- 3-D, $p_x = \frac{h}{L} n_x$, $p_y = \frac{h}{L} n_y$, $p_z = \frac{h}{L} n_z$,

Quantum description of particle's movement

- 3-D, $p_x = \frac{h}{L} n_x$, $p_y = \frac{h}{L} n_y$, $p_z = \frac{h}{L} n_z$,
  $\varepsilon = \frac{p_x^2 + p_y^2 + p_z^2}{2m} = \frac{h^2}{2m} \frac{n_x^2 + n_y^2 + n_z^2}{L^2}$.

Quantum description of particle's movement

- 3-D, $p_x = \frac{h}{L} n_x$, $p_y = \frac{h}{L} n_y$, $p_z = \frac{h}{L} n_z$,
  $\varepsilon = \frac{p_x^2 + p_y^2 + p_z^2}{2m} = \frac{h^2}{2m} \frac{n_x^2 + n_y^2 + n_z^2}{L^2}$. Degenerated.

Quantum description of particle's movement

- 3-D, $p_x = \frac{h}{L} n_x$, $p_y = \frac{h}{L} n_y$, $p_z = \frac{h}{L} n_z$, $\varepsilon = \frac{p_x^2+p_y^2+p_z^2}{2m} = \frac{h^2}{2m}\frac{n_x^2+n_y^2+n_z^2}{L^2}$. Degenerated.
- $p_x = \frac{h}{L} n_x$, there is one state for each n_x.

Quantum description of particle's movement

- 3-D, $p_x = \frac{h}{L} n_x$, $p_y = \frac{h}{L} n_y$, $p_z = \frac{h}{L} n_z$, $\varepsilon = \frac{p_x^2+p_y^2+p_z^2}{2m} = \frac{h^2}{2m}\frac{n_x^2+n_y^2+n_z^2}{L^2}$. Degenerated.
- $p_x = \frac{h}{L} n_x$, there is one state for each n_x.
- The number of state $(p_x, p_x + \mathrm{d}p_x)$: $(n_x + \Delta n_x) - n_x$, where $p_x + \mathrm{d}p_x = h\frac{n_x+\Delta n_x}{L}$,

Quantum description of particle's movement

- 3-D, $p_x = \frac{h}{L}n_x$, $p_y = \frac{h}{L}n_y$, $p_z = \frac{h}{L}n_z$,
  $\varepsilon = \frac{p_x^2+p_y^2+p_z^2}{2m} = \frac{h^2}{2m}\frac{n_x^2+n_y^2+n_z^2}{L^2}$. Degenerated.
- $p_x = \frac{h}{L}n_x$, there is one state for each n_x.
- The number of state $(p_x, p_x + \mathrm{d}p_x)$: $(n_x + \Delta n_x) - n_x$,
  where $p_x + \mathrm{d}p_x = h\frac{n_x+\Delta n_x}{L}$,
  $\Delta n_x = \frac{L}{h}\mathrm{d}p_x$,

Quantum description of particle's movement

- 3-D, $p_x = \frac{h}{L}n_x$, $p_y = \frac{h}{L}n_y$, $p_z = \frac{h}{L}n_z$, $\varepsilon = \frac{p_x^2+p_y^2+p_z^2}{2m} = \frac{h^2}{2m}\frac{n_x^2+n_y^2+n_z^2}{L^2}$. Degenerated.
- $p_x = \frac{h}{L}n_x$, there is one state for each n_x.
- The number of state $(p_x, p_x + \mathrm{d}p_x)$: $(n_x + \Delta n_x) - n_x$, where $p_x + \mathrm{d}p_x = h\frac{n_x+\Delta n_x}{L}$, $\Delta n_x = \frac{L}{h}\mathrm{d}p_x$, also $\Delta n_y = \frac{L}{h}\mathrm{d}p_y$, $\Delta n_z = \frac{L}{h}\mathrm{d}p_z$.

Quantum description of particle's movement

- 3-D, $p_x = \frac{h}{L}n_x$, $p_y = \frac{h}{L}n_y$, $p_z = \frac{h}{L}n_z$, $\varepsilon = \frac{p_x^2+p_y^2+p_z^2}{2m} = \frac{h^2}{2m}\frac{n_x^2+n_y^2+n_z^2}{L^2}$. Degenerated.
- $p_x = \frac{h}{L}n_x$, there is one state for each n_x.
- The number of state $(p_x, p_x + \mathrm{d}p_x)$: $(n_x + \Delta n_x) - n_x$, where $p_x + \mathrm{d}p_x = h\frac{n_x+\Delta n_x}{L}$,
  $\Delta n_x = \frac{L}{h}\mathrm{d}p_x$, also $\Delta n_y = \frac{L}{h}\mathrm{d}p_y$, $\Delta n_z = \frac{L}{h}\mathrm{d}p_z$.
  Total number of state: $\Delta n_x \Delta n_y \Delta n_z = \frac{V}{h^3}\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z$.

Quantum description of particle's movement

- 3-D, $p_x = \frac{h}{L} n_x$, $p_y = \frac{h}{L} n_y$, $p_z = \frac{h}{L} n_z$,
  $\varepsilon = \frac{p_x^2+p_y^2+p_z^2}{2m} = \frac{h^2}{2m}\frac{n_x^2+n_y^2+n_z^2}{L^2}$. Degenerated.
- $p_x = \frac{h}{L} n_x$, there is one state for each n_x.
- The number of state $(p_x, p_x + \mathrm{d}p_x)$: $(n_x + \Delta n_x) - n_x$,
  where $p_x + \mathrm{d}p_x = h\frac{n_x+\Delta n_x}{L}$,
  $\Delta n_x = \frac{L}{h}\mathrm{d}p_x$, also $\Delta n_y = \frac{L}{h}\mathrm{d}p_y$, $\Delta n_z = \frac{L}{h}\mathrm{d}p_z$.
  Total number of state: $\Delta n_x \Delta n_y \Delta n_z = \frac{V}{h^3}\mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z$.
  h^3 acts as the elementary volume in phase space.

Quantum description of particle's movement

- 3-D, $p_x = \frac{h}{L}n_x$, $p_y = \frac{h}{L}n_y$, $p_z = \frac{h}{L}n_z$,
  $\varepsilon = \frac{p_x^2+p_y^2+p_z^2}{2m} = \frac{h^2}{2m}\frac{n_x^2+n_y^2+n_z^2}{L^2}$. Degenerated.
- $p_x = \frac{h}{L}n_x$, there is one state for each n_x.
- The number of state $(p_x, p_x + \mathrm{d}p_x)$: $(n_x + \Delta n_x) - n_x$,
  where $p_x + \mathrm{d}p_x = h\frac{n_x+\Delta n_x}{L}$,
  $\Delta n_x = \frac{L}{h}\mathrm{d}p_x$, also $\Delta n_y = \frac{L}{h}\mathrm{d}p_y$, $\Delta n_z = \frac{L}{h}\mathrm{d}p_z$.
  Total number of state: $\Delta n_x \Delta n_y \Delta n_z = \frac{V}{h^3}\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z$.
  h^3 acts as the elementary volume in phase space.
- Convert to polar coordinate: $p_x = p\sin\theta\cos\varphi$,
  $p_y = p\sin\theta\sin\varphi$, $p_z = p\cos\theta$,

Quantum description of particle's movement

- 3-D, $p_x = \frac{h}{L}n_x$, $p_y = \frac{h}{L}n_y$, $p_z = \frac{h}{L}n_z$, $\varepsilon = \frac{p_x^2+p_y^2+p_z^2}{2m} = \frac{h^2}{2m}\frac{n_x^2+n_y^2+n_z^2}{L^2}$. Degenerated.
- $p_x = \frac{h}{L}n_x$, there is one state for each n_x.
- The number of state $(p_x, p_x + \mathrm{d}p_x)$: $(n_x + \Delta n_x) - n_x$, where $p_x + \mathrm{d}p_x = h\frac{n_x+\Delta n_x}{L}$,
  $\Delta n_x = \frac{L}{h}\mathrm{d}p_x$, also $\Delta n_y = \frac{L}{h}\mathrm{d}p_y$, $\Delta n_z = \frac{L}{h}\mathrm{d}p_z$.
  Total number of state: $\Delta n_x \Delta n_y \Delta n_z = \frac{V}{h^3}\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z$.
  h^3 acts as the elementary volume in phase space.
- Convert to polar coordinate: $p_x = p\sin\theta\cos\varphi$,
  $p_y = p\sin\theta\sin\varphi$, $p_z = p\cos\theta$,
  $\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z = p^2\sin\theta\mathrm{d}p\mathrm{d}\theta\mathrm{d}\varphi$.

Quantum description of particle's movement

- Number of state:
  $\Delta n_x \Delta n_y \Delta n_z = \frac{V}{h^3} \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z = \frac{V}{h^3} p^2 \sin\theta \mathrm{d}p \mathrm{d}\theta \mathrm{d}\varphi$.

Quantum description of particle's movement

- Number of state:
  $\Delta n_x \Delta n_y \Delta n_z = \frac{V}{h^3} \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z = \frac{V}{h^3} p^2 \sin\theta \mathrm{d}p \mathrm{d}\theta \mathrm{d}\varphi$.
- For isotropic system, $\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta \mathrm{d}\theta = 4\pi$, the number of state: $\frac{4\pi V}{h^3} p^2 \mathrm{d}p$.

Quantum description of particle's movement

- Number of state:
  $\Delta n_x \Delta n_y \Delta n_z = \frac{V}{h^3}\mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z = \frac{V}{h^3}p^2 \sin\theta \mathrm{d}p\mathrm{d}\theta\mathrm{d}\varphi$.
- For isotropic system, $\int_0^{2\pi}\mathrm{d}\varphi \int_0^{\pi}\sin\theta\mathrm{d}\theta = 4\pi$, the number of state: $\frac{4\pi V}{h^3}p^2\mathrm{d}p$.
  Convert to energy $(\varepsilon = p^2/2m)$:
  $D(\varepsilon)\mathrm{d}\varepsilon = \frac{2\pi V}{h^3}(2m)^{3/2}\varepsilon^{1/2}\mathrm{d}\varepsilon$, where $D(\varepsilon)$ is the number density of state.

Quantum description of particle's movement

- Number of state:
  $\Delta n_x \Delta n_y \Delta n_z = \frac{V}{h^3} \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z = \frac{V}{h^3} p^2 \sin\theta \mathrm{d}p \mathrm{d}\theta \mathrm{d}\varphi$.
- For isotropic system, $\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta \mathrm{d}\theta = 4\pi$, the number of state: $\frac{4\pi V}{h^3} p^2 \mathrm{d}p$.
  Convert to energy ($\varepsilon = p^2/2m$):
  $D(\varepsilon)\mathrm{d}\varepsilon = \frac{2\pi V}{h^3}(2m)^{3/2} \varepsilon^{1/2} \mathrm{d}\varepsilon$, where $D(\varepsilon)$ is the number density of state.
- Also considering the spin s, the number of state enhances to $2s+1$ times.

Quantum description of particle's movement

- Number of state:
  $\Delta n_x \Delta n_y \Delta n_z = \frac{V}{h^3} \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z = \frac{V}{h^3} p^2 \sin\theta \mathrm{d}p \mathrm{d}\theta \mathrm{d}\varphi$.
- For isotropic system, $\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta \mathrm{d}\theta = 4\pi$, the number of state: $\frac{4\pi V}{h^3} p^2 \mathrm{d}p$.
  Convert to energy $(\varepsilon = p^2/2m)$:
  $D(\varepsilon)\mathrm{d}\varepsilon = \frac{2\pi V}{h^3}(2m)^{3/2}\varepsilon^{1/2}\mathrm{d}\varepsilon$, where $D(\varepsilon)$ is the number density of state.
- Also considering the spin s, the number of state enhances to $2s+1$ times.
  E.g., electrons, $s = 1/2$, degeneracy is $2s+1 = 2$.

Table of contents

6.3 Description of the system's microscopic arrangement

6.3 Description of the system's microscopic arrangement

- Object: system with nearly independent identical particles.

6.3 Description of the system's microscopic arrangement

- Object: system with nearly independent identical particles.
  nearly independent: very weak interaction, $E = \Sigma_i \varepsilon_i$,

6.3 Description of the system's microscopic arrangement

- Object: system with nearly independent identical particles.
  nearly independent: very weak interaction, $E = \Sigma_i \varepsilon_i$,
  identical: same kind of particle.

6.3 Description of the system's microscopic arrangement

- Object: system with nearly independent identical particles.
  nearly independent: very weak interaction, $E = \Sigma_i \varepsilon_i$,
  identical: same kind of particle.
- Classification: classical particles, Fermions, Bosons.

6.3 Description of the system's microscopic arrangement

- Object: system with nearly independent identical particles.
  nearly independent: very weak interaction, $E = \Sigma_i \varepsilon_i$,
  identical: same kind of particle.
- Classification: classical particles, Fermions, Bosons.
  Classical particles: distinguishable, can occupy the same state, like ideal gas particle;

6.3 Description of the system's microscopic arrangement

- Object: system with nearly independent identical particles.
  nearly independent: very weak interaction, $E = \Sigma_i \varepsilon_i$,
  identical: same kind of particle.
- Classification: classical particles, Fermions, Bosons.
  Classical particles: distinguishable, can occupy the same state, like ideal gas particle;
  Boson: indistinguishable, can occupy the same state, like photon, ^{4}He;

6.3 Description of the system's microscopic arrangement

- Object: system with nearly independent identical particles.
  nearly independent: very weak interaction, $E = \Sigma_i \varepsilon_i$,
  identical: same kind of particle.
- Classification: classical particles, Fermions, Bosons.
  Classical particles: distinguishable, can occupy the same state, like ideal gas particle;
  Boson: indistinguishable, can occupy the same state, like photon, ^{4}He;
  Fermion: indistinguishable, cannot occupy the same state, like electron, proton.

6.3 Description of the system's microscopic arrangement

- Object: system with nearly independent identical particles.
  nearly independent: very weak interaction, $E = \Sigma_i \varepsilon_i$,
  identical: same kind of particle.
- Classification: classical particles, Fermions, Bosons.
  Classical particles: distinguishable, can occupy the same state, like ideal gas particle;
  Boson: indistinguishable, can occupy the same state, like photon, ^{4}He;
  Fermion: indistinguishable, cannot occupy the same state, like electron, proton.
- Aim: find the number of microscopic arrangement.

Description of the system's microscopic arrangement

Classical particle system

State 1	State 2	State 3
A B		
	A B	
		A B
A	B	
B	A	
	A	B
	B	A
A		B
B		A

Description of the system's microscopic arrangement

Classical particle system

State 1	State 2	State 3
A B		
	A B	
		A B
A	B	
B	A	
	A	B
	B	A
A		B
B		A

Boson system

State 1	State 2	State 3
A A		
	A A	
		A A
A	A	
	A	A
A		A

Description of the system's microscopic arrangement

Classical particle system

State 1	State 2	State 3
A B		
	A B	
		A B
A	B	
B	A	
	A	B
	B	A
A		B
B		A

Boson system

State 1	State 2	State 3
A A		
	A A	
		A A
A	A	
	A	A
A		A

Fermion system

State 1	State 2	State 3
A	A	
	A	A
A		A

Table of contents

6.4 The principle of equal a priori probabilities

- Given an isolated system in equilibrium, it is found with equal probability in each of its accessible micro-states.

Table of contents

6.5 Distribution and microscopic state

6.5 Distribution and microscopic state

- Object: nearly independent identical particles.

6.5 Distribution and microscopic state

- Object: nearly independent identical particles.
- Constraints: N, E, V.

6.5 Distribution and microscopic state

- Object: nearly independent identical particles.
- Constraints: N, E, V.
- Aim: count the number of micro-states.

6.5 Distribution and microscopic state

- Object: nearly independent identical particles.
- Constraints: N, E, V.
- Aim: count the number of micro-states.
-

energy level	$\varepsilon_1, \varepsilon_2, ..., \varepsilon_l, ...$
degeneracy	$\omega_1, \omega_2, ..., \omega_l, ...$
particle number	$a_1, a_2, ..., a_l, ...$

6.5 Distribution and microscopic state

- Object: nearly independent identical particles.
- Constraints: N, E, V.
- Aim: count the number of micro-states.
-

energy level	$\varepsilon_1, \varepsilon_2, ..., \varepsilon_l, ...$
degeneracy	$\omega_1, \omega_2, ..., \omega_l, ...$
particle number	$a_1, a_2, ..., a_l, ...$

- $\{a_l\}$ represents one kind of distribution (contains multiple micro-state),

6.5 Distribution and microscopic state

- Object: nearly independent identical particles.
- Constraints: N, E, V.
- Aim: count the number of micro-states.
-

energy level	$\varepsilon_1, \varepsilon_2, ..., \varepsilon_l, ...$
degeneracy	$\omega_1, \omega_2, ..., \omega_l, ...$
particle number	$a_1, a_2, ..., a_l, ...$

- $\{a_l\}$ represents one kind of distribution (contains multiple micro-state), obeys: $\sum a_l = N$, $\sum a_l \varepsilon_l = E$.

6.5 Distribution and microscopic state

- Object: nearly independent identical particles.
- Constraints: N, E, V.
- Aim: count the number of micro-states.
-

energy level	$\varepsilon_1, \varepsilon_2, ..., \varepsilon_l, ...$
degeneracy	$\omega_1, \omega_2, ..., \omega_l, ...$
particle number	$a_1, a_2, ..., a_l, ...$

- $\{a_l\}$ represents one kind of distribution (contains multiple micro-state), obeys: $\sum a_l = N$, $\sum a_l \varepsilon_l = E$.
- For a distribution, the number of micro-states is NOT obvious.

Distribution and microscopic state for Boltzmann system

- Boltzmann system:

Distribution and microscopic state for Boltzmann system

- Boltzmann system:

1. Consider an energy level ε_l, there are ω_l quantum states, a_l particles.

Distribution and microscopic state for Boltzmann system

- Boltzmann system:

1. Consider an energy level ε_l, there are ω_l quantum states, a_l particles. For any particle, there are ω_l choices. (Each choice is a different micro-state.)

Distribution and microscopic state for Boltzmann system

- Boltzmann system:

1. Consider an energy level ε_l, there are ω_l quantum states, a_l particles. For any particle, there are ω_l choices. (Each choice is a different micro-state.) $\omega_l^{a_l}$ in total.

Distribution and microscopic state for Boltzmann system

- Boltzmann system:

1. Consider an energy level ε_l, there are ω_l quantum states, a_l particles. For any particle, there are ω_l choices. (Each choice is a different micro-state.) $\omega_l^{a_l}$ in total.
2. For another level $\varepsilon_{l'}$, number of micro-state: $\omega_{l'}^{a_{l'}}$.

Distribution and microscopic state for Boltzmann system

- Boltzmann system:

1. Consider an energy level ε_l, there are ω_l quantum states, a_l particles. For any particle, there are ω_l choices. (Each choice is a different micro-state.) $\omega_l^{a_l}$ in total.
2. For another level $\varepsilon_{l'}$, number of micro-state: $\omega_{l'}^{a_{l'}}$.
Total number of these two levels: $\omega_l^{a_l}\omega_{l'}^{a_{l'}}$. (not really)

Distribution and microscopic state for Boltzmann system

- Boltzmann system:

1. Consider an energy level ε_l, there are ω_l quantum states, a_l particles. For any particle, there are ω_l choices. (Each choice is a different micro-state.) $\omega_l^{a_l}$ in total.
2. For another level $\varepsilon_{l'}$, number of micro-state: $\omega_{l'}^{a_{l'}}$.

Total number of these two levels: $\omega_l^{a_l}\omega_{l'}^{a_{l'}}$. (not really)
Total number of all levels: $\prod \omega_l^{a_l}$.

Distribution and microscopic state for Boltzmann system

- Boltzmann system:

1. Consider an energy level ε_l, there are ω_l quantum states, a_l particles. For any particle, there are ω_l choices. (Each choice is a different micro-state.) $\omega_l^{a_l}$ in total.
2. For another level $\varepsilon_{l'}$, number of micro-state: $\omega_{l'}^{a_{l'}}$.
Total number of these two levels: $\omega_l^{a_l}\omega_{l'}^{a_{l'}}$. (not really)
Total number of all levels: $\prod \omega_l^{a_l}$.
3. Exchanging any two particle in different level produces a new micro-state: $N!/\prod a_l!$ (no new state for the same level).

Distribution and microscopic state for Boltzmann system

- Boltzmann system:

1. Consider an energy level ε_l, there are ω_l quantum states, a_l particles. For any particle, there are ω_l choices. (Each choice is a different micro-state.) $\omega_l^{a_l}$ in total.
2. For another level $\varepsilon_{l'}$, number of micro-state: $\omega_{l'}^{a_{l'}}$.
Total number of these two levels: $\omega_l^{a_l}\omega_{l'}^{a_{l'}}$. (not really)
Total number of all levels: $\prod \omega_l^{a_l}$.
3. Exchanging any two particle in different level produces a new micro-state: $N!/\prod a_l!$ (no new state for the same level).
4. Total number of micro-states for one distribution $\{a_l\}$:
$\boxed{\Omega_{\text{M.B.}} = \frac{N!}{\prod a_l!}\prod \omega_l^{a_l}}$.

Distribution and microscopic state for Bose system

- Bose system: indistinguishable

Distribution and microscopic state for Bose system

- Bose system: indistinguishable

Distribution and microscopic state for Bose system

- Bose system: indistinguishable

If they are distinguishable, the number is $\omega_l \cdot (\omega_l + a_l - 1)!$ (the first one must be a quantum state).

Distribution and microscopic state for Bose system

- Bose system: indistinguishable

If they are distinguishable, the number is $\omega_l \cdot (\omega_l + a_l - 1)!$ (the first one must be a quantum state).
But the particles are indistinguishable,

Distribution and microscopic state for Bose system

- Bose system: indistinguishable

If they are distinguishable, the number is $\omega_l \cdot (\omega_l + a_l - 1)!$ (the first one must be a quantum state).
But the particles are indistinguishable, so divided by $a_l!$.

Distribution and microscopic state for Bose system

- Bose system: indistinguishable

If they are distinguishable, the number is $\omega_l \cdot (\omega_l + a_l - 1)!$ (the first one must be a quantum state).
But the particles are indistinguishable, so divided by $a_l!$.
Exchange the position of two quantum states does not produce new arrangement,

Distribution and microscopic state for Bose system

- Bose system: indistinguishable

If they are distinguishable, the number is $\omega_l \cdot (\omega_l + a_l - 1)!$ (the first one must be a quantum state).
But the particles are indistinguishable, so divided by $a_l!$.
Exchange the position of two quantum states does not produce new arrangement, divided by $\omega_l!$.

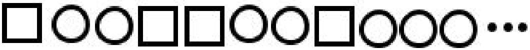
Distribution and microscopic state for Bose system

- Bose system: indistinguishable

If they are distinguishable, the number is $\omega_l \cdot (\omega_l + a_l - 1)!$ (the first one must be a quantum state).
But the particles are indistinguishable, so divided by $a_l!$.
Exchange the position of two quantum states does not produce new arrangement, divided by $\omega_l!$.
Then for one energy level ε_l, the number is $\frac{\omega_l \cdot (\omega_l + a_l - 1)!}{\omega_l! \cdot a_l!} = \frac{(\omega_l + a_l - 1)!}{a_l! (\omega_l - 1)!}$.

Distribution and microscopic state for Bose system

- Bose system: indistinguishable



If they are distinguishable, the number is $\omega_l \cdot (\omega_l + a_l - 1)!$ (the first one must be a quantum state).
But the particles are indistinguishable, so divided by $a_l!$.
Exchange the position of two quantum states does not produce new arrangement, divided by $\omega_l!$.
Then for one energy level ε_l, the number is $\frac{\omega_l \cdot (\omega_l + a_l - 1)!}{\omega_l! \cdot a_l!} = \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$.
The total number of micro-state for a given distribution $\{a_l\}$ is: $\boxed{\Omega_{\text{B.E.}} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}}$.

Distribution and microscopic state for Fermi system

- Fermi system: indistinguishable, repulsive.

Distribution and microscopic state for Fermi system

- Fermi system: indistinguishable, repulsive.

Each quantum state can only contain one particle.

Distribution and microscopic state for Fermi system

- Fermi system: indistinguishable, repulsive.

Each quantum state can only contain one particle.
To find the number of micro-states, equivalent to find the choices to pick up a_l quantum states in ω_l quantum states,

Distribution and microscopic state for Fermi system

- Fermi system: indistinguishable, repulsive.

Each quantum state can only contain one particle.
To find the number of micro-states, equivalent to find the choices to pick up a_l quantum states in ω_l quantum states, which is $\omega_l \cdot (\omega_l - 1) ... (\omega_l - a_l + 1) = \frac{\omega_l!}{(\omega_l - a_l)!}$.
(state is distinguishable)

Distribution and microscopic state for Fermi system

- Fermi system: indistinguishable, repulsive.

Each quantum state can only contain one particle.
To find the number of micro-states, equivalent to find the choices to pick up a_l quantum states in ω_l quantum states, which is $\omega_l \cdot (\omega_l - 1) \ldots (\omega_l - a_l + 1) = \frac{\omega_l!}{(\omega_l - a_l)!}$.
(state is distinguishable)
But the particles are indistinguishable, divided by $a_l!$.

Distribution and microscopic state for Fermi system

- Fermi system: indistinguishable, repulsive.

Each quantum state can only contain one particle.
To find the number of micro-states, equivalent to find the choices to pick up a_l quantum states in ω_l quantum states, which is $\omega_l \cdot (\omega_l - 1)...(\omega_l - a_l + 1) = \frac{\omega_l!}{(\omega_l - a_l)!}$.
(state is distinguishable)
But the particles are indistinguishable, divided by $a_l!$.
Then for one energy level ε_l, the number is $\frac{\omega_l!}{a_l!(\omega_l - a_l)!}$.

Distribution and microscopic state for Fermi system

- Fermi system: indistinguishable, repulsive.

Each quantum state can only contain one particle.
To find the number of micro-states, equivalent to find the choices to pick up a_l quantum states in ω_l quantum states, which is $\omega_l \cdot (\omega_l - 1)...(\omega_l - a_l + 1) = \frac{\omega_l!}{(\omega_l - a_l)!}$. (state is distinguishable)
But the particles are indistinguishable, divided by $a_l!$.
Then for one energy level ε_l, the number is $\frac{\omega_l!}{a_l!(\omega_l - a_l)!}$.
The total number of micro-state for a given distribution $\{a_l\}$ is: $\boxed{\Omega_{\text{F.D.}} = \prod \frac{\omega_l!}{a_l!(\omega_l - a_l)!}}$.

Classical limit $\boxed{\Omega_{\text{M.B.}} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l} = N! \prod \frac{\omega_l^{a_l}}{a_l!}}$

Classical limit $\boxed{\Omega_{\mathrm{M.B.}} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l} = N! \prod \frac{\omega_l^{a_l}}{a_l!}}$

- If $a_l \ll \omega_l$,

Classical limit $\boxed{\Omega_{\text{M.B.}} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l} = N! \prod \frac{\omega_l^{a_l}}{a_l!}}$

- If $a_l \ll \omega_l$,
- $\Omega_{\text{B.E.}} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$

Classical limit $\boxed{\Omega_{\text{M.B.}} = \frac{N!}{\prod a_l!}\prod \omega_l^{a_l} = N!\prod \frac{\omega_l^{a_l}}{a_l!}}$

- If $a_l \ll \omega_l$,
- $\Omega_{\text{B.E.}} = \prod \frac{(\omega_l+a_l-1)!}{a_l!(\omega_l-1)!} = \prod \frac{(\omega_l+a_l-1)(\omega_l+a_l-2)\ldots\omega_l}{a_l!}$

Classical limit $\boxed{\Omega_{\text{M.B.}} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l} = N! \prod \frac{\omega_l^{a_l}}{a_l!}}$

- If $a_l \ll \omega_l$,
- $\Omega_{\text{B.E.}} = \prod \frac{(\omega_l+a_l-1)!}{a_l!(\omega_l-1)!} = \prod \frac{(\omega_l+a_l-1)(\omega_l+a_l-2)\ldots\omega_l}{a_l!} \simeq \prod \frac{\omega_l^{a_l}}{a_l!}$

Classical limit $\boxed{\Omega_{\text{M.B.}} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l} = N! \prod \frac{\omega_l^{a_l}}{a_l!}}$

- If $a_l \ll \omega_l$,
- $\Omega_{\text{B.E.}} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!} = \prod \frac{(\omega_l + a_l - 1)(\omega_l + a_l - 2)\ldots\omega_l}{a_l!} \simeq \prod \frac{\omega_l^{a_l}}{a_l!}$ $= \frac{\Omega_{\text{M.B.}}}{N!}$.

Classical limit $\boxed{\Omega_{\text{M.B.}} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l} = N! \prod \frac{\omega_l^{a_l}}{a_l!}}$

- If $a_l \ll \omega_l$,
- $\Omega_{\text{B.E.}} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!} = \prod \frac{(\omega_l + a_l - 1)(\omega_l + a_l - 2)\ldots\omega_l}{a_l!} \simeq \prod \frac{\omega_l^{a_l}}{a_l!} = \frac{\Omega_{\text{M.B.}}}{N!}$.
- $\Omega_{\text{F.D.}} = \prod \frac{\omega_l!}{a_l!(\omega_l - a_l)!}$

Classical limit $\boxed{\Omega_{\text{M.B.}} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l} = N! \prod \frac{\omega_l^{a_l}}{a_l!}}$

- If $a_l \ll \omega_l$,
- $\Omega_{\text{B.E.}} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!} = \prod \frac{(\omega_l + a_l - 1)(\omega_l + a_l - 2)\ldots\omega_l}{a_l!} \simeq \prod \frac{\omega_l^{a_l}}{a_l!} = \frac{\Omega_{\text{M.B.}}}{N!}$.
- $\Omega_{\text{F.D.}} = \prod \frac{\omega_l!}{a_l!(\omega_l - a_l)!} = \prod \frac{\omega_l(\omega_l - 1)\ldots(\omega_l - a_l + 1)}{a_l!}$

Classical limit $\boxed{\Omega_{\text{M.B.}} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l} = N! \prod \frac{\omega_l^{a_l}}{a_l!}}$

- If $a_l \ll \omega_l$,
- $\Omega_{\text{B.E.}} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!} = \prod \frac{(\omega_l + a_l - 1)(\omega_l + a_l - 2)\ldots\omega_l}{a_l!} \simeq \prod \frac{\omega_l^{a_l}}{a_l!} = \frac{\Omega_{\text{M.B.}}}{N!}$.
- $\Omega_{\text{F.D.}} = \prod \frac{\omega_l!}{a_l!(\omega_l - a_l)!} = \prod \frac{\omega_l(\omega_l - 1)\ldots(\omega_l - a_l + 1)}{a_l!} \simeq \prod \frac{\omega_l^{a_l}}{a_l!}$

Classical limit $\boxed{\Omega_{\text{M.B.}} = \frac{N!}{\prod a_l!}\prod \omega_l^{a_l} = N!\prod \frac{\omega_l^{a_l}}{a_l!}}$

- If $a_l \ll \omega_l$,
- $\Omega_{\text{B.E.}} = \prod \frac{(\omega_l+a_l-1)!}{a_l!(\omega_l-1)!} = \prod \frac{(\omega_l+a_l-1)(\omega_l+a_l-2)\ldots\omega_l}{a_l!} \simeq \prod \frac{\omega_l^{a_l}}{a_l!} = \frac{\Omega_{\text{M.B.}}}{N!}$.
- $\Omega_{\text{F.D.}} = \prod \frac{\omega_l!}{a_l!(\omega_l-a_l)!} = \prod \frac{\omega_l(\omega_l-1)\ldots(\omega_l-a_l+1)}{a_l!} \simeq \prod \frac{\omega_l^{a_l}}{a_l!} = \frac{\Omega_{\text{M.B.}}}{N!}$.

Classical limit $\boxed{\Omega_{\text{M.B.}} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l} = N! \prod \frac{\omega_l^{a_l}}{a_l!}}$

- If $a_l \ll \omega_l$,
- $\Omega_{\text{B.E.}} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!} = \prod \frac{(\omega_l + a_l - 1)(\omega_l + a_l - 2)\ldots\omega_l}{a_l!} \simeq \prod \frac{\omega_l^{a_l}}{a_l!} = \frac{\Omega_{\text{M.B.}}}{N!}$.
- $\Omega_{\text{F.D.}} = \prod \frac{\omega_l!}{a_l!(\omega_l - a_l)!} = \prod \frac{\omega_l(\omega_l - 1)\ldots(\omega_l - a_l + 1)}{a_l!} \simeq \prod \frac{\omega_l^{a_l}}{a_l!} = \frac{\Omega_{\text{M.B.}}}{N!}$.
- In the classical limit, both F.D. and B.E. reach to the same number of micro-states $\frac{\Omega_{\text{M.B.}}}{N!}$.

Classical statistics

Classical statistics

State:$(q_1, q_2, ..., q_r; p_1, p_2, ..., p_r)$, continuous, uncountable.

Classical statistics

State:$(q_1, q_2, ..., q_r; p_1, p_2, ..., p_r)$, continuous, uncountable. Define unit "phase volume" (cell): $h_0 = \delta q_i \delta p_i$. Take the particles locate at the same unit "volume" as same state.

Classical statistics

State:$(q_1, q_2, ..., q_r; p_1, p_2, ..., p_r)$, continuous, uncountable. Define unit "phase volume" (cell): $h_0 = \delta q_i \delta p_i$. Take the particles locate at the same unit "volume" as same state. In the total phase volume $\Delta q \Delta p$, number of cells (states): $\frac{\Delta q \Delta p}{h_0}$.

Classical statistics

State:$(q_1, q_2, ..., q_r; p_1, p_2, ..., p_r)$, continuous, uncountable. Define unit "phase volume" (cell): $h_0 = \delta q_i \delta p_i$. Take the particles locate at the same unit "volume" as same state. In the total phase volume $\Delta q \Delta p$, number of cells (states): $\frac{\Delta q \Delta p}{h_0}$. For $r-$dimension, volume $\Delta w = \Delta q_1 ... \Delta q_r \Delta p_1 ... \Delta p_r$. Number of states: $\frac{\Delta w}{h_0^r}$.

Classical statistics

State:$(q_1, q_2, ..., q_r; p_1, p_2, ..., p_r)$, continuous, uncountable. Define unit "phase volume" (cell): $h_0 = \delta q_i \delta p_i$. Take the particles locate at the same unit "volume" as same state. In the total phase volume $\Delta q \Delta p$, number of cells (states): $\frac{\Delta q \Delta p}{h_0}$. For $r-$dimension, volume $\Delta w = \Delta q_1 ... \Delta q_r \Delta p_1 ... \Delta p_r$. Number of states: $\frac{\Delta w}{h_0^r}$.

energy	$\varepsilon_1, \varepsilon_2, ...\varepsilon_l, ...$
phase volume	$\Delta w_1, \Delta w_2, ...\Delta w_l, ...$
degeneracy	$\frac{\Delta w_1}{h_0^r}, \frac{\Delta w_2}{h_0^r}, ..., \frac{\Delta w_l}{h_0^r}, ...$
particle number	$a_1, a_2, ..., a_l, ...$

Classical statistics

State: $(q_1, q_2, ..., q_r; p_1, p_2, ..., p_r)$, continuous, uncountable. Define unit "phase volume" (cell): $h_0 = \delta q_i \delta p_i$. Take the particles locate at the same unit "volume" as same state. In the total phase volume $\Delta q \Delta p$, number of cells (states): $\frac{\Delta q \Delta p}{h_0}$. For $r-$dimension, volume $\Delta w = \Delta q_1 ... \Delta q_r \Delta p_1 ... \Delta p_r$. Number of states: $\frac{\Delta w}{h_0^r}$.

energy	$\varepsilon_1, \varepsilon_2, ... \varepsilon_l, ...$
phase volume	$\Delta w_1, \Delta w_2, ... \Delta w_l, ...$
degeneracy	$\frac{\Delta w_1}{h_0^r}, \frac{\Delta w_2}{h_0^r}, ..., \frac{\Delta w_l}{h_0^r}, ...$
particle number	$a_1, a_2, ..., a_l, ...$

- Number of micro-states: $\Omega_{\text{cl}} = \frac{N!}{\prod a_l!} \prod \left(\frac{\Delta w_l}{h_0^r} \right)^{a_l}$.

Table of contents

6.6 Boltzmann distribution

6.6 Boltzmann distribution

- For each distribution $\{a_l\}$, there are $\Omega(\{a_l\})$ micro-states.

6.6 Boltzmann distribution

- For each distribution $\{a_l\}$, there are $\Omega(\{a_l\})$ micro-states.
- "The principle of equal a priori probabilities" says each micro-state (including different distribution) appears in the same probability.

6.6 Boltzmann distribution

- For each distribution $\{a_l\}$, there are $\Omega(\{a_l\})$ micro-states.
- "The principle of equal a priori probabilities" says each micro-state (including different distribution) appears in the same probability. So, the distribution with more micro-states appears with more probability.

6.6 Boltzmann distribution

- For each distribution $\{a_l\}$, there are $\Omega(\{a_l\})$ micro-states.
- "The principle of equal a priori probabilities" says each micro-state (including different distribution) appears in the same probability. So, the distribution with more micro-states appears with more probability. The highest one is called the most probable distribution.

6.6 Boltzmann distribution

- For each distribution $\{a_l\}$, there are $\Omega(\{a_l\})$ micro-states.
- "The principle of equal a priori probabilities" says each micro-state (including different distribution) appears in the same probability. So, the distribution with more micro-states appears with more probability. The highest one is called the most probable distribution.
- Obeys $\delta\Omega = 0$, and $\delta^2\Omega < 0$.

6.6 Boltzmann distribution

- For each distribution $\{a_l\}$, there are $\Omega(\{a_l\})$ micro-states.
- "The principle of equal a priori probabilities" says each micro-state (including different distribution) appears in the same probability. So, the distribution with more micro-states appears with more probability. The highest one is called the most probable distribution.
- Obeys $\delta\Omega = 0$, and $\delta^2\Omega < 0$. Or equivalently, $\delta \ln \Omega = 0$, and $\delta^2 \ln \Omega < 0$.

6.6 Boltzmann distribution

- For each distribution $\{a_l\}$, there are $\Omega(\{a_l\})$ micro-states.
- "The principle of equal a priori probabilities" says each micro-state (including different distribution) appears in the same probability. So, the distribution with more micro-states appears with more probability. The highest one is called the most probable distribution.
- Obeys $\delta\Omega = 0$, and $\delta^2\Omega < 0$. Or equivalently, $\delta \ln \Omega = 0$, and $\delta^2 \ln \Omega < 0$.
- For Boltzmann system:

6.6 Boltzmann distribution

- For each distribution $\{a_l\}$, there are $\Omega(\{a_l\})$ micro-states.
- "The principle of equal a priori probabilities" says each micro-state (including different distribution) appears in the same probability. So, the distribution with more micro-states appears with more probability. The highest one is called the most probable distribution.
- Obeys $\delta\Omega = 0$, and $\delta^2\Omega < 0$. Or equivalently, $\delta \ln \Omega = 0$, and $\delta^2 \ln \Omega < 0$.
- For Boltzmann system: $\Omega = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l}$,

6.6 Boltzmann distribution

- For each distribution $\{a_l\}$, there are $\Omega(\{a_l\})$ micro-states.
- "The principle of equal a priori probabilities" says each micro-state (including different distribution) appears in the same probability. So, the distribution with more micro-states appears with more probability. The highest one is called the most probable distribution.
- Obeys $\delta\Omega = 0$, and $\delta^2\Omega < 0$. Or equivalently, $\delta \ln \Omega = 0$, and $\delta^2 \ln \Omega < 0$.
- For Boltzmann system: $\Omega = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l}$, $\ln \Omega = \ln N! - \sum \ln a_l! + \sum a_l \ln \omega_l$.

Boltzmann distribution

- Math: if $m \gg 1$, $\ln m! \simeq m(\ln m - 1)$.

Boltzmann distribution

- Math: if $m \gg 1$, $\ln m! \simeq m(\ln m - 1)$.
- $\ln \Omega = \ln N! - \sum \ln a_l! + \sum a_l \ln \omega_l$

Boltzmann distribution

- Math: if $m \gg 1$, $\ln m! \simeq m(\ln m - 1)$.
- $\ln \Omega = \ln N! - \sum \ln a_l! + \sum a_l \ln \omega_l$
  $\simeq N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l$.

Boltzmann distribution

- Math: if $m \gg 1$, $\ln m! \simeq m(\ln m - 1)$.
- $\ln \Omega = \ln N! - \sum \ln a_l! + \sum a_l \ln \omega_l$
  $\simeq N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l$.
- For different distribution, N and E does not change. Only $\{a_l\}$ changes.

Boltzmann distribution

- Math: if $m \gg 1$, $\ln m! \simeq m(\ln m - 1)$.
- $\ln \Omega = \ln N! - \sum \ln a_l! + \sum a_l \ln \omega_l$
  $\simeq N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l$.
- For different distribution, N and E does not change. Only $\{a_l\}$ changes. So:
  $\delta \ln \Omega \simeq \delta[N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l]$

Boltzmann distribution

- Math: if $m \gg 1$, $\ln m! \simeq m(\ln m - 1)$.
- $\ln \Omega = \ln N! - \sum \ln a_l! + \sum a_l \ln \omega_l$
  $\simeq N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l$.
- For different distribution, N and E does not change. Only $\{a_l\}$ changes. So:
  $$\begin{aligned}\delta \ln \Omega &\simeq \delta[N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l] \\ &= \delta[N \ln N - \sum a_l \ln a_l + \sum a_l \ln \omega_l]\end{aligned}$$

Boltzmann distribution

- Math: if $m \gg 1$, $\ln m! \simeq m(\ln m - 1)$.
- $\ln \Omega = \ln N! - \sum \ln a_l! + \sum a_l \ln \omega_l$
  $\simeq N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l$.
- For different distribution, N and E does not change. Only $\{a_l\}$ changes. So:

$$\begin{aligned}\delta \ln \Omega &\simeq \delta[N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l] \\ &= \delta[N \ln N - \sum a_l \ln a_l + \sum a_l \ln \omega_l] \\ &= 0 - [\sum a_l \tfrac{1}{a_l}\delta a_l + \sum \ln a_l \delta a_l] + \sum \ln \omega_l \delta a_l\end{aligned}$$

Boltzmann distribution

- Math: if $m \gg 1$, $\ln m! \simeq m(\ln m - 1)$.
- $\ln\Omega = \ln N! - \sum \ln a_l! + \sum a_l \ln \omega_l$
  $\simeq N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l$.
- For different distribution, N and E does not change. Only $\{a_l\}$ changes. So:

$$\begin{aligned}
\delta \ln \Omega &\simeq \delta[N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l] \\
&= \delta[N \ln N - \sum a_l \ln a_l + \sum a_l \ln \omega_l] \\
&= 0 - [\sum a_l \tfrac{1}{a_l}\delta a_l + \sum \ln a_l \delta a_l] + \sum \ln \omega_l \delta a_l \\
&= -\sum \delta a_l + \sum \ln \tfrac{\omega_l}{a_l} \delta a_l
\end{aligned}$$

Boltzmann distribution

- Math: if $m \gg 1$, $\ln m! \simeq m(\ln m - 1)$.
- $\ln \Omega = \ln N! - \sum \ln a_l! + \sum a_l \ln \omega_l$
  $\simeq N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l$.
- For different distribution, N and E does not change. Only $\{a_l\}$ changes. So:

$$\begin{aligned}\delta \ln \Omega &\simeq \delta[N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l] \\ &= \delta[N \ln N - \sum a_l \ln a_l + \sum a_l \ln \omega_l] \\ &= 0 - [\sum a_l \tfrac{1}{a_l} \delta a_l + \sum \ln a_l \delta a_l] + \sum \ln \omega_l \delta a_l \\ &= -\sum \delta a_l + \sum \ln \tfrac{\omega_l}{a_l} \delta a_l \\ &= 0 + \sum \ln \tfrac{\omega_l}{a_l} \delta a_l\end{aligned}$$

Boltzmann distribution

- Math: if $m \gg 1$, $\ln m! \simeq m(\ln m - 1)$.
- $\ln \Omega = \ln N! - \sum \ln a_l! + \sum a_l \ln \omega_l$
  $\simeq N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l$.
- For different distribution, N and E does not change. Only $\{a_l\}$ changes. So:

$$\begin{aligned}\delta \ln \Omega \simeq\ & \delta[N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l] \\ =\ & \delta[N \ln N - \sum a_l \ln a_l + \sum a_l \ln \omega_l] \\ =\ & 0 - [\sum a_l \tfrac{1}{a_l}\delta a_l + \sum \ln a_l \delta a_l] + \sum \ln \omega_l \delta a_l \\ =\ & -\sum \delta a_l + \sum \ln \tfrac{\omega_l}{a_l}\delta a_l \\ =\ & 0 + \sum \ln \tfrac{\omega_l}{a_l}\delta a_l\end{aligned}$$

$= 0$,

Boltzmann distribution

- Math: if $m \gg 1$, $\ln m! \simeq m(\ln m - 1)$.
- $\ln \Omega = \ln N! - \sum \ln a_l! + \sum a_l \ln \omega_l$
  $\simeq N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l$.
- For different distribution, N and E does not change. Only $\{a_l\}$ changes. So:
  $$\begin{aligned}\delta \ln \Omega \simeq\ & \delta[N(\ln N - 1) - \sum a_l(\ln a_l - 1) + \sum a_l \ln \omega_l] \\ =\ & \delta[N \ln N - \sum a_l \ln a_l + \sum a_l \ln \omega_l] \\ =\ & 0 - [\sum a_l \tfrac{1}{a_l}\delta a_l + \sum \ln a_l \delta a_l] + \sum \ln \omega_l \delta a_l \\ =\ & -\sum \delta a_l + \sum \ln \tfrac{\omega_l}{a_l}\delta a_l \\ =\ & 0 + \sum \ln \tfrac{\omega_l}{a_l}\delta a_l \end{aligned}$$
  $= 0$,
- $\Rightarrow \sum \ln \frac{\omega_l}{a_l}\delta a_l = 0$.

Boltzmann distribution

- $\delta \ln \Omega = \sum \ln \frac{\omega_l}{a_l} \delta a_l = 0$, a_l can not vary independently.

Boltzmann distribution

- $\delta \ln \Omega = \sum \ln \frac{\omega_l}{a_l} \delta a_l = 0$, a_l can not vary independently.
- Two constraints: $\delta N = \sum \delta a_l = 0$, $\delta E = \sum \varepsilon_l \delta a_l = 0$.

Boltzmann distribution

- $\delta \ln \Omega = \sum \ln \frac{\omega_l}{a_l} \delta a_l = 0$, a_l can not vary independently.
- Two constraints: $\delta N = \sum \delta a_l = 0$, $\delta E = \sum \varepsilon_l \delta a_l = 0$.
- Introduce two parameters α, β.

Boltzmann distribution

- $\delta \ln \Omega = \sum \ln \frac{\omega_l}{a_l} \delta a_l = 0$, a_l can not vary independently.
- Two constraints: $\delta N = \sum \delta a_l = 0$, $\delta E = \sum \varepsilon_l \delta a_l = 0$.
- Introduce two parameters α, β. $\delta \ln \Omega - \alpha \delta N - \beta \delta E = 0$,

Boltzmann distribution

- $\delta \ln \Omega = \sum \ln \frac{\omega_l}{a_l} \delta a_l = 0$, a_l can not vary independently.
- Two constraints: $\delta N = \sum \delta a_l = 0$, $\delta E = \sum \varepsilon_l \delta a_l = 0$.
- Introduce two parameters α, β. $\delta \ln \Omega - \alpha \delta N - \beta \delta E = 0$, or $-\sum [\ln \frac{a_l}{\omega_l} + \alpha + \beta \varepsilon_l] \delta a_l = 0$.

Boltzmann distribution

- $\delta \ln \Omega = \sum \ln \frac{\omega_l}{a_l} \delta a_l = 0$, a_l can not vary independently.
- Two constraints: $\delta N = \sum \delta a_l = 0$, $\delta E = \sum \varepsilon_l \delta a_l = 0$.
- Introduce two parameters α, β. $\delta \ln \Omega - \alpha \delta N - \beta \delta E = 0$, or $-\sum [\ln \frac{a_l}{\omega_l} + \alpha + \beta \varepsilon_l] \delta a_l = 0$.
- As two of $\{a_l\}$ are not independent, say a_1, a_2.

Boltzmann distribution

- $\delta \ln \Omega = \sum \ln \frac{\omega_l}{a_l} \delta a_l = 0$, a_l can not vary independently.
- Two constraints: $\delta N = \sum \delta a_l = 0$, $\delta E = \sum \varepsilon_l \delta a_l = 0$.
- Introduce two parameters α, β. $\delta \ln \Omega - \alpha \delta N - \beta \delta E = 0$, or $-\sum [\ln \frac{a_l}{\omega_l} + \alpha + \beta \varepsilon_l] \delta a_l = 0$.
- As two of $\{a_l\}$ are not independent, say a_1, a_2. Notice α, β are arbitrary.

Boltzmann distribution

- $\delta \ln \Omega = \sum \ln \frac{\omega_l}{a_l} \delta a_l = 0$, a_l can not vary independently.
- Two constraints: $\delta N = \sum \delta a_l = 0$,
  $\delta E = \sum \varepsilon_l \delta a_l = 0$.
- Introduce two parameters α, β.
  $\delta \ln \Omega - \alpha \delta N - \beta \delta E = 0$, or
  $-\sum [\ln \frac{a_l}{\omega_l} + \alpha + \beta \varepsilon_l] \delta a_l = 0$.
- As two of $\{a_l\}$ are not independent, say a_1, a_2. Notice α, β are arbitrary. Now set α, β by
  $\ln \frac{a_1}{\omega_1} + \alpha + \beta \varepsilon_1 = 0$,
  $\ln \frac{a_2}{\omega_2} + \alpha + \beta \varepsilon_2 = 0$.

Boltzmann distribution

- $\sum_{l=1}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l$

Boltzmann distribution

- $\sum\limits_{l=1}[\ln \frac{a_l}{\omega_l} + \alpha + \beta \varepsilon_l]\delta a_l = \sum\limits_{l=3}[\ln \frac{a_l}{\omega_l} + \alpha + \beta \varepsilon_l]\delta a_l$

Boltzmann distribution

- $\sum_{l=1}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = \sum_{l=3}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = 0.$

Boltzmann distribution

- $\sum_{l=1}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = \sum_{l=3}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = 0$.
  Now $\{a_l\}$ $(l = 3, 4, ...)$ are independent,

Boltzmann distribution

- $\sum_{l=1}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = \sum_{l=3}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = 0$.
  Now $\{a_l\}$ $(l = 3, 4, ...)$ are independent,
  $\therefore \ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l = 0$, $(l = 3, 4, ...)$.

Boltzmann distribution

- $\sum_{l=1}[\ln\frac{a_l}{\omega_l}+\alpha+\beta\varepsilon_l]\delta a_l=\sum_{l=3}[\ln\frac{a_l}{\omega_l}+\alpha+\beta\varepsilon_l]\delta a_l=0.$
  Now $\{a_l\}$ $(l=3,4,...)$ are independent,
  $\therefore \ln\frac{a_l}{\omega_l}+\alpha+\beta\varepsilon_l=0$, $(l=3,4,...)$.
- Combine, $\ln\frac{a_l}{\omega_l}+\alpha+\beta\varepsilon_l=0$, $(l=1,2,3,4,...)$,

Boltzmann distribution

- $\sum_{l=1}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = \sum_{l=3}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = 0$.
  Now $\{a_l\}$ $(l = 3, 4, ...)$ are independent,
  $\therefore \ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l = 0$, $(l = 3, 4, ...)$.
- Combine, $\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l = 0$, $(l = 1, 2, 3, 4, ...)$,
  or, $\boxed{a_l = \omega_l e^{-\alpha-\beta\varepsilon_l}}$.

Boltzmann distribution

- $\sum_{l=1}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = \sum_{l=3}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = 0.$
  Now $\{a_l\}$ $(l = 3, 4, ...)$ are independent,
  $\therefore \ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l = 0$, $(l = 3, 4, ...)$.
- Combine, $\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l = 0$, $(l = 1, 2, 3, 4, ...)$,
  or, $\boxed{a_l = \omega_l e^{-\alpha-\beta\varepsilon_l}}$.
  α, β are determined by the constraints:
  $N = \sum \omega_l e^{-\alpha-\beta\varepsilon_l}$, $E = \sum \omega_l e^{-\alpha-\beta\varepsilon_l}\varepsilon_l$.

Boltzmann distribution

- $\sum_{l=1}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = \sum_{l=3}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = 0.$
  Now $\{a_l\}$ $(l = 3, 4, ...)$ are independent,
  $\therefore \ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l = 0$, $(l = 3, 4, ...)$.
- Combine, $\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l = 0$, $(l = 1, 2, 3, 4, ...)$,
  or, $\boxed{a_l = \omega_l e^{-\alpha-\beta\varepsilon_l}}$.
  α, β are determined by the constraints:
  $N = \sum \omega_l e^{-\alpha-\beta\varepsilon_l}$, $E = \sum \omega_l e^{-\alpha-\beta\varepsilon_l}\varepsilon_l$.
- For any quantum state: the population is
  $f = \frac{a_l}{\omega_l} = e^{-\alpha-\beta\varepsilon_l}$.

Boltzmann distribution

- $\sum_{l=1}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = \sum_{l=3}[\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l]\delta a_l = 0.$
  Now $\{a_l\}$ $(l = 3, 4, ...)$ are independent,
  $\therefore \ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l = 0$, $(l = 3, 4, ...)$.
- Combine, $\ln \frac{a_l}{\omega_l} + \alpha + \beta\varepsilon_l = 0$, $(l = 1, 2, 3, 4, ...)$,
  or, $\boxed{a_l = \omega_l e^{-\alpha-\beta\varepsilon_l}}$.
  α, β are determined by the constraints:
  $N = \sum \omega_l e^{-\alpha-\beta\varepsilon_l}$, $E = \sum \omega_l e^{-\alpha-\beta\varepsilon_l}\varepsilon_l$.
- For any quantum state: the population is
  $f = \frac{a_l}{\omega_l} = e^{-\alpha-\beta\varepsilon_l}$.
- Only depends on the energy for the special quantum state. Suitable for any quantum state: $f_s = e^{-\alpha-\beta\varepsilon_s}$.

Boltzmann distribution

- Obeys: $N = \sum f_s = \sum e^{-\alpha-\beta\varepsilon_s}$,

Boltzmann distribution

- Obeys: $N = \sum f_s = \sum e^{-\alpha-\beta\varepsilon_s}$, $E = \sum f_s\varepsilon_s = \sum \varepsilon_s e^{-\alpha-\beta\varepsilon_s}$.

Boltzmann distribution

- Obeys: $N = \sum f_s = \sum e^{-\alpha-\beta\varepsilon_s}$,
  $E = \sum f_s\varepsilon_s = \sum \varepsilon_s e^{-\alpha-\beta\varepsilon_s}$.
- $\delta^2 \ln \Omega = \delta(-\sum \ln \frac{a_l}{\omega_l}\delta a_l)$

Boltzmann distribution

- Obeys: $N = \sum f_s = \sum e^{-\alpha-\beta\varepsilon_s}$,
  $E = \sum f_s \varepsilon_s = \sum \varepsilon_s e^{-\alpha-\beta\varepsilon_s}$.
- $\delta^2 \ln \Omega = \delta(-\sum \ln \frac{a_l}{\omega_l} \delta a_l)$
  $$= -\sum(\frac{\omega_l}{a_l}\frac{1}{\omega_l}\delta a_l)\delta a_l - \sum \ln \frac{a_l}{\omega_l}\delta^2 a_l$$

Boltzmann distribution

- Obeys: $N = \sum f_s = \sum e^{-\alpha-\beta\varepsilon_s}$,
  $E = \sum f_s \varepsilon_s = \sum \varepsilon_s e^{-\alpha-\beta\varepsilon_s}$.
- $\delta^2 \ln \Omega = \delta(-\sum \ln \frac{a_l}{\omega_l} \delta a_l)$
  $$= -\sum(\frac{\omega_l}{a_l}\frac{1}{\omega_l}\delta a_l)\delta a_l - \sum \ln \frac{a_l}{\omega_l}\delta^2 a_l$$
  $$= -\sum \frac{(\delta a_l)^2}{a_l} - \sum(-\alpha - \beta\varepsilon_l)\delta^2 a_l$$

Boltzmann distribution

- Obeys: $N = \sum f_s = \sum e^{-\alpha-\beta\varepsilon_s}$, $E = \sum f_s \varepsilon_s = \sum \varepsilon_s e^{-\alpha-\beta\varepsilon_s}$.
- $\delta^2 \ln \Omega = \delta(-\sum \ln \frac{a_l}{\omega_l} \delta a_l)$

$$
\begin{aligned}
&= -\sum (\frac{\omega_l}{a_l}\frac{1}{\omega_l}\delta a_l)\delta a_l - \sum \ln \frac{a_l}{\omega_l}\delta^2 a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} - \sum(-\alpha - \beta\varepsilon_l)\delta^2 a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} + \alpha \sum \delta^2 a_l + \beta \sum \delta^2 (\varepsilon_l a_l)
\end{aligned}
$$

Boltzmann distribution

- Obeys: $N = \sum f_s = \sum e^{-\alpha-\beta\varepsilon_s}$,
  $E = \sum f_s \varepsilon_s = \sum \varepsilon_s e^{-\alpha-\beta\varepsilon_s}$.
- $\delta^2 \ln \Omega = \delta(-\sum \ln \frac{a_l}{\omega_l} \delta a_l)$

$$
\begin{aligned}
&= -\sum (\tfrac{\omega_l}{a_l} \tfrac{1}{\omega_l} \delta a_l) \delta a_l - \sum \ln \tfrac{a_l}{\omega_l} \delta^2 a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} - \sum (-\alpha - \beta \varepsilon_l) \delta^2 a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} + \alpha \sum \delta^2 a_l + \beta \sum \delta^2 (\varepsilon_l a_l) \\
&= -\sum \frac{(\delta a_l)^2}{a_l} + \alpha \delta^2 \sum a_l + \beta \delta^2 \sum \varepsilon_l a_l
\end{aligned}
$$

Boltzmann distribution

- Obeys: $N = \sum f_s = \sum e^{-\alpha-\beta\varepsilon_s}$,
  $E = \sum f_s\varepsilon_s = \sum \varepsilon_s e^{-\alpha-\beta\varepsilon_s}$.
- $\delta^2 \ln\Omega = \delta(-\sum \ln\frac{a_l}{\omega_l}\delta a_l)$

$$
\begin{aligned}
&= -\sum(\frac{\omega_l}{a_l}\frac{1}{\omega_l}\delta a_l)\delta a_l - \sum \ln\frac{a_l}{\omega_l}\delta^2 a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} - \sum(-\alpha-\beta\varepsilon_l)\delta^2 a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} + \alpha\sum\delta^2 a_l + \beta\sum\delta^2(\varepsilon_l a_l) \\
&= -\sum \frac{(\delta a_l)^2}{a_l} + \alpha\delta^2\sum a_l + \beta\delta^2\sum\varepsilon_l a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} + \alpha\delta^2 N + \beta\delta^2 E
\end{aligned}
$$

Boltzmann distribution

- Obeys: $N = \sum f_s = \sum e^{-\alpha-\beta\varepsilon_s}$,
  $E = \sum f_s\varepsilon_s = \sum \varepsilon_s e^{-\alpha-\beta\varepsilon_s}$.
- $\delta^2 \ln\Omega = \delta(-\sum \ln \frac{a_l}{\omega_l}\delta a_l)$

$$
\begin{aligned}
&= -\sum(\frac{\omega_l}{a_l}\frac{1}{\omega_l}\delta a_l)\delta a_l - \sum \ln\frac{a_l}{\omega_l}\delta^2 a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} - \sum(-\alpha-\beta\varepsilon_l)\delta^2 a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} + \alpha\sum\delta^2 a_l + \beta\sum\delta^2(\varepsilon_l a_l) \\
&= -\sum \frac{(\delta a_l)^2}{a_l} + \alpha\delta^2\sum a_l + \beta\delta^2\sum\varepsilon_l a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} + \alpha\delta^2 N + \beta\delta^2 E \\
&= -\sum \frac{(\delta a_l)^2}{a_l}
\end{aligned}
$$

Boltzmann distribution

- Obeys: $N = \sum f_s = \sum e^{-\alpha-\beta\varepsilon_s}$,
  $E = \sum f_s \varepsilon_s = \sum \varepsilon_s e^{-\alpha-\beta\varepsilon_s}$.
- $\delta^2 \ln \Omega = \delta(-\sum \ln \frac{a_l}{\omega_l} \delta a_l)$

$$
\begin{aligned}
&= -\sum (\frac{\omega_l}{a_l} \frac{1}{\omega_l} \delta a_l) \delta a_l - \sum \ln \frac{a_l}{\omega_l} \delta^2 a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} - \sum (-\alpha - \beta \varepsilon_l) \delta^2 a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} + \alpha \sum \delta^2 a_l + \beta \sum \delta^2 (\varepsilon_l a_l) \\
&= -\sum \frac{(\delta a_l)^2}{a_l} + \alpha \delta^2 \sum a_l + \beta \delta^2 \sum \varepsilon_l a_l \\
&= -\sum \frac{(\delta a_l)^2}{a_l} + \alpha \delta^2 N + \beta \delta^2 E \\
&= -\sum \frac{(\delta a_l)^2}{a_l} \\
&< 0.
\end{aligned}
$$

Boltzmann distribution

- Obeys: $N = \sum f_s = \sum e^{-\alpha-\beta\varepsilon_s}$, $E = \sum f_s\varepsilon_s = \sum \varepsilon_s e^{-\alpha-\beta\varepsilon_s}$.
- $$\begin{aligned}\delta^2 \ln\Omega &= \delta(-\sum \ln\tfrac{a_l}{\omega_l}\delta a_l) \\ &= -\sum(\tfrac{\omega_l}{a_l}\tfrac{1}{\omega_l}\delta a_l)\delta a_l - \sum \ln\tfrac{a_l}{\omega_l}\delta^2 a_l \\ &= -\sum\frac{(\delta a_l)^2}{a_l} - \sum(-\alpha-\beta\varepsilon_l)\delta^2 a_l \\ &= -\sum\frac{(\delta a_l)^2}{a_l} + \alpha\sum\delta^2 a_l + \beta\sum\delta^2(\varepsilon_l a_l) \\ &= -\sum\frac{(\delta a_l)^2}{a_l} + \alpha\delta^2\sum a_l + \beta\delta^2\sum\varepsilon_l a_l \\ &= -\sum\frac{(\delta a_l)^2}{a_l} + \alpha\delta^2 N + \beta\delta^2 E \\ &= -\sum\frac{(\delta a_l)^2}{a_l} \\ &< 0.\end{aligned}$$
- This $\{a_l\}$ is the most probable distribution.

Deviation from the most probable distribution

Deviation from the most probable distribution

- Suppose a deviation from the most probable distribution $\{\Delta a_l\}$, which corresponds to the micro-states number $\Omega + \Delta\Omega = \Omega(\{a_l + \Delta a_l\})$.

Deviation from the most probable distribution

- Suppose a deviation from the most probable distribution $\{\Delta a_l\}$, which corresponds to the micro-states number $\Omega + \Delta\Omega = \Omega(\{a_l + \Delta a_l\})$.
- As $\Delta a_l \ll a_l$, expand $\Omega(\{a_l + \Delta a_l\})$ at $\Omega(\{a_l\})$:

Deviation from the most probable distribution

- Suppose a deviation from the most probable distribution $\{\Delta a_l\}$, which corresponds to the micro-states number $\Omega + \Delta\Omega = \Omega(\{a_l + \Delta a_l\})$.
- As $\Delta a_l \ll a_l$, expand $\Omega(\{a_l + \Delta a_l\})$ at $\Omega(\{a_l\})$:
  $\ln\Omega(\{a_l + \Delta a_l\}) = \ln\Omega(\{a_l\}) + \delta\ln\Omega + \frac{1}{2}\delta^2\ln\Omega + \ldots,$

Deviation from the most probable distribution

- Suppose a deviation from the most probable distribution $\{\Delta a_l\}$, which corresponds to the micro-states number $\Omega + \Delta\Omega = \Omega(\{a_l + \Delta a_l\})$.
- As $\Delta a_l \ll a_l$, expand $\Omega(\{a_l + \Delta a_l\})$ at $\Omega(\{a_l\})$:
  $\ln \Omega(\{a_l + \Delta a_l\}) = \ln \Omega(\{a_l\}) + \delta \ln \Omega + \frac{1}{2}\delta^2 \ln \Omega + \ldots,$
  $\Rightarrow \ln \frac{\Omega(\{a_l + \Delta a_l\})}{\Omega(\{a_l\})} = -\frac{1}{2} \sum \frac{(\Delta a_l)^2}{a_l}.$

Deviation from the most probable distribution

- Suppose a deviation from the most probable distribution $\{\Delta a_l\}$, which corresponds to the micro-states number $\Omega + \Delta\Omega = \Omega(\{a_l + \Delta a_l\})$.
- As $\Delta a_l \ll a_l$, expand $\Omega(\{a_l + \Delta a_l\})$ at $\Omega(\{a_l\})$:
  $\ln\Omega(\{a_l + \Delta a_l\}) = \ln\Omega(\{a_l\}) + \delta\ln\Omega + \frac{1}{2}\delta^2\ln\Omega + \ldots,$
  $\Rightarrow \ln\frac{\Omega(\{a_l+\Delta a_l\})}{\Omega(\{a_l\})} = -\frac{1}{2}\sum\frac{(\Delta a_l)^2}{a_l}.$
- For a small deviation, $\frac{\Delta a_l}{a_l} = 10^{-5}$,

Deviation from the most probable distribution

- Suppose a deviation from the most probable distribution $\{\Delta a_l\}$, which corresponds to the micro-states number $\Omega + \Delta\Omega = \Omega(\{a_l + \Delta a_l\})$.
- As $\Delta a_l \ll a_l$, expand $\Omega(\{a_l + \Delta a_l\})$ at $\Omega(\{a_l\})$:
  $\ln\Omega(\{a_l + \Delta a_l\}) = \ln\Omega(\{a_l\}) + \delta\ln\Omega + \frac{1}{2}\delta^2\ln\Omega + \ldots,$
  $\Rightarrow \ln\frac{\Omega(\{a_l+\Delta a_l\})}{\Omega(\{a_l\})} = -\frac{1}{2}\sum\frac{(\Delta a_l)^2}{a_l}.$
- For a small deviation, $\frac{\Delta a_l}{a_l} = 10^{-5}$,
  $\ln\frac{\Omega+\Delta\Omega}{\Omega} \simeq -\frac{1}{2}\sum(\frac{\Delta a_l}{a_l})^2 a_l$

Deviation from the most probable distribution

- Suppose a deviation from the most probable distribution $\{\Delta a_l\}$, which corresponds to the micro-states number $\Omega + \Delta\Omega = \Omega(\{a_l + \Delta a_l\})$.
- As $\Delta a_l \ll a_l$, expand $\Omega(\{a_l + \Delta a_l\})$ at $\Omega(\{a_l\})$:
  $\ln \Omega(\{a_l + \Delta a_l\}) = \ln \Omega(\{a_l\}) + \delta \ln \Omega + \frac{1}{2}\delta^2 \ln \Omega + \ldots,$
  $\Rightarrow \ln \frac{\Omega(\{a_l+\Delta a_l\})}{\Omega(\{a_l\})} = -\frac{1}{2}\sum \frac{(\Delta a_l)^2}{a_l}.$
- For a small deviation, $\frac{\Delta a_l}{a_l} = 10^{-5}$,
  $\ln \frac{\Omega+\Delta\Omega}{\Omega} \simeq -\frac{1}{2}\sum(\frac{\Delta a_l}{a_l})^2 a_l = -\frac{1}{2} \times 10^{-10} \sum a_l$

Deviation from the most probable distribution

- Suppose a deviation from the most probable distribution $\{\Delta a_l\}$, which corresponds to the micro-states number $\Omega + \Delta\Omega = \Omega(\{a_l + \Delta a_l\})$.
- As $\Delta a_l \ll a_l$, expand $\Omega(\{a_l + \Delta a_l\})$ at $\Omega(\{a_l\})$:
$\ln\Omega(\{a_l + \Delta a_l\}) = \ln\Omega(\{a_l\}) + \delta\ln\Omega + \frac{1}{2}\delta^2\ln\Omega + \ldots,$
$\Rightarrow \ln\frac{\Omega(\{a_l+\Delta a_l\})}{\Omega(\{a_l\})} = -\frac{1}{2}\sum\frac{(\Delta a_l)^2}{a_l}.$
- For a small deviation, $\frac{\Delta a_l}{a_l} = 10^{-5}$,
$\ln\frac{\Omega+\Delta\Omega}{\Omega} \simeq -\frac{1}{2}\sum(\frac{\Delta a_l}{a_l})^2 a_l = -\frac{1}{2}\times 10^{-10}\sum a_l$
$\sim -3\times 10^{13}\ (N \sim 1\mathrm{mol} = 6.02\times 10^{23}).$

Deviation from the most probable distribution

- Suppose a deviation from the most probable distribution $\{\Delta a_l\}$, which corresponds to the micro-states number $\Omega + \Delta\Omega = \Omega(\{a_l + \Delta a_l\})$.
- As $\Delta a_l \ll a_l$, expand $\Omega(\{a_l + \Delta a_l\})$ at $\Omega(\{a_l\})$:
$\ln\Omega(\{a_l + \Delta a_l\}) = \ln\Omega(\{a_l\}) + \delta\ln\Omega + \frac{1}{2}\delta^2\ln\Omega + \ldots,$
$\Rightarrow \ln\frac{\Omega(\{a_l+\Delta a_l\})}{\Omega(\{a_l\})} = -\frac{1}{2}\sum\frac{(\Delta a_l)^2}{a_l}.$
- For a small deviation, $\frac{\Delta a_l}{a_l} = 10^{-5}$,
$\ln\frac{\Omega+\Delta\Omega}{\Omega} \simeq -\frac{1}{2}\sum(\frac{\Delta a_l}{a_l})^2 a_l = -\frac{1}{2}\times 10^{-10}\sum a_l$
$\sim -3\times 10^{13}$ $(N \sim 1\mathrm{mol} = 6.02\times 10^{23})$.
- $\frac{\Omega+\Delta\Omega}{\Omega} \sim e^{-3\times 10^{13}} \to 0.$

Deviation from the most probable distribution

- Suppose a deviation from the most probable distribution $\{\Delta a_l\}$, which corresponds to the micro-states number $\Omega + \Delta\Omega = \Omega(\{a_l + \Delta a_l\})$.
- As $\Delta a_l \ll a_l$, expand $\Omega(\{a_l + \Delta a_l\})$ at $\Omega(\{a_l\})$: $\ln \Omega(\{a_l + \Delta a_l\}) = \ln \Omega(\{a_l\}) + \delta \ln \Omega + \frac{1}{2}\delta^2 \ln \Omega + \ldots,$ $\Rightarrow \ln \frac{\Omega(\{a_l+\Delta a_l\})}{\Omega(\{a_l\})} = -\frac{1}{2}\sum \frac{(\Delta a_l)^2}{a_l}.$
- For a small deviation, $\frac{\Delta a_l}{a_l} = 10^{-5}$, $\ln \frac{\Omega+\Delta\Omega}{\Omega} \simeq -\frac{1}{2}\sum(\frac{\Delta a_l}{a_l})^2 a_l = -\frac{1}{2} \times 10^{-10} \sum a_l$ $\sim -3 \times 10^{13}$ $(N \sim 1\mathrm{mol} = 6.02 \times 10^{23})$.
- $\frac{\Omega+\Delta\Omega}{\Omega} \sim e^{-3\times 10^{13}} \to 0.$
- The most probable distribution is very close to the whole distribution.

Boltzmann statistics on classical particles

Boltzmann statistics on classical particles

- $\omega_l \rightarrow \frac{\Delta\omega_l}{h_0^r}$.

Boltzmann statistics on classical particles

- $\omega_l \to \frac{\Delta\omega_l}{h_0^r}$.
- $\therefore a_l = e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$,

Boltzmann statistics on classical particles

- $\omega_l \to \frac{\Delta\omega_l}{h_0^r}$.
- $\therefore a_l = e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$,
  α and β obeys:
  $N = \sum a_l = \sum e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$,
  $E = \sum a_l\varepsilon_l = \sum \varepsilon_l e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$.

Boltzmann statistics on classical particles

- $\omega_l \to \frac{\Delta\omega_l}{h_0^r}$.
- $\therefore a_l = e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$,
  α and β obeys:
  $N = \sum a_l = \sum e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$,
  $E = \sum a_l\varepsilon_l = \sum \varepsilon_l e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$.
- Or if the (q, p) are continuous,

Boltzmann statistics on classical particles

- $\omega_l \to \frac{\Delta\omega_l}{h_0^r}$.
- $\therefore a_l = e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$,
  α and β obeys:
  $N = \sum a_l = \sum e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$,
  $E = \sum a_l\varepsilon_l = \sum \varepsilon_l e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$.
- Or if the (q, p) are continuous,
  $a_l = e^{-\alpha-\beta\varepsilon(\{q_i,p_i\})}\frac{\mathrm{d}q_1\ldots\mathrm{d}q_r\mathrm{d}p_1\ldots\mathrm{d}p_r}{h_0^r}$.

Boltzmann statistics on classical particles

- $\omega_l \rightarrow \frac{\Delta\omega_l}{h_0^r}$.
- $\therefore a_l = e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$,
  α and β obeys:
  $N = \sum a_l = \sum e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$,
  $E = \sum a_l\varepsilon_l = \sum \varepsilon_l e^{-\alpha-\beta\varepsilon_l}\frac{\Delta\omega_l}{h_0^r}$.
- Or if the (q,p) are continuous,
  $a_l = e^{-\alpha-\beta\varepsilon(\{q_i,p_i\})}\frac{\mathrm{d}q_1\ldots\mathrm{d}q_r\mathrm{d}p_1\ldots\mathrm{d}p_r}{h_0^r}$.
  Obeys:
  $\int\cdots\int e^{-\alpha-\beta\varepsilon(\{q_i,p_i\})}\frac{\mathrm{d}q_1\ldots\mathrm{d}q_r\mathrm{d}p_1\ldots\mathrm{d}p_r}{h_0^r} = N$,
  $\int\ldots\int \varepsilon(\{q_i,p_i\})e^{-\alpha-\beta\varepsilon(\{q_i,p_i\})}\frac{\mathrm{d}q_1\ldots\mathrm{d}q_r\mathrm{d}p_1\ldots\mathrm{d}p_r}{h_0^r} = E$.

Table of contents

6.7 Bose distribution and Fermi distribution

6.7 Bose distribution and Fermi distribution

- N, V, E conserved, require $\{a_l\}$.

6.7 Bose distribution and Fermi distribution

- N, V, E conserved, require $\{a_l\}$.
- Obeys $\sum a_l = N$, $\sum a_l \varepsilon_l = E$.

6.7 Bose distribution and Fermi distribution

- N, V, E conserved, require $\{a_l\}$.
- Obeys $\sum a_l = N$, $\sum a_l \varepsilon_l = E$.
- Bose system: $\Omega_{\mathrm{B}} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$.

6.7 Bose distribution and Fermi distribution

- N, V, E conserved, require $\{a_l\}$.
- Obeys $\sum a_l = N$, $\sum a_l \varepsilon_l = E$.
- Bose system: $\Omega_{\mathrm{B}} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$.
- $\ln \Omega_{\mathrm{B}} = \sum [\ln(\omega_l + a_l - 1)! - \ln a_l! - \ln(\omega_l - 1)!]$

6.7 Bose distribution and Fermi distribution

- N, V, E conserved, require $\{a_l\}$.
- Obeys $\sum a_l = N$, $\sum a_l \varepsilon_l = E$.
- Bose system: $\Omega_\mathrm{B} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$.
- $\ln \Omega_\mathrm{B} = \sum [\ln(\omega_l + a_l - 1)! - \ln a_l! - \ln(\omega_l - 1)!]$
  $\simeq \sum [\ln(\omega_l + a_l)! - \ln a_l! - \ln \omega_l!]$

6.7 Bose distribution and Fermi distribution

- N, V, E conserved, require $\{a_l\}$.
- Obeys $\sum a_l = N$, $\sum a_l \varepsilon_l = E$.
- Bose system: $\Omega_{\rm B} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$.
- $$\begin{aligned} \ln \Omega_{\rm B} &= \sum [\ln(\omega_l + a_l - 1)! - \ln a_l! - \ln(\omega_l - 1)!] \\ &\simeq \sum [\ln(\omega_l + a_l)! - \ln a_l! - \ln \omega_l!] \\ &\simeq \sum \{(\omega_l + a_l)[\ln(\omega_l + a_l) - 1] - a_l(\ln a_l - 1) \\ &\quad -\omega_l(\ln \omega_l - 1)\} \end{aligned}$$

6.7 Bose distribution and Fermi distribution

- N, V, E conserved, require $\{a_l\}$.
- Obeys $\sum a_l = N$, $\sum a_l \varepsilon_l = E$.
- Bose system: $\Omega_{\mathrm{B}} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$.
- $$\begin{aligned} \ln \Omega_{\mathrm{B}} &= \sum [\ln(\omega_l + a_l - 1)! - \ln a_l! - \ln(\omega_l - 1)!] \\ &\simeq \sum [\ln(\omega_l + a_l)! - \ln a_l! - \ln \omega_l!] \\ &\simeq \sum \{(\omega_l + a_l)[\ln(\omega_l + a_l) - 1] - a_l(\ln a_l - 1) \\ &\quad - \omega_l(\ln \omega_l - 1)\} \\ &= \sum [(\omega_l + a_l) \ln(\omega_l + a_l) - a_l \ln a_l - \omega_l \ln \omega_l]. \end{aligned}$$

6.7 Bose distribution and Fermi distribution

- N, V, E conserved, require $\{a_l\}$.
- Obeys $\sum a_l = N$, $\sum a_l \varepsilon_l = E$.
- Bose system: $\Omega_{\rm B} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$.
- $$\begin{aligned}\ln \Omega_{\rm B} &= \sum [\ln(\omega_l + a_l - 1)! - \ln a_l! - \ln(\omega_l - 1)!] \\ &\simeq \sum [\ln(\omega_l + a_l)! - \ln a_l! - \ln \omega_l!] \\ &\simeq \sum \{(\omega_l + a_l)[\ln(\omega_l + a_l) - 1] - a_l(\ln a_l - 1) \\ &\quad -\omega_l(\ln \omega_l - 1)\} \\ &= \sum [(\omega_l + a_l)\ln(\omega_l + a_l) - a_l \ln a_l - \omega_l \ln \omega_l].\end{aligned}$$
- $\delta \ln \Omega_{\rm B} = \sum [\delta a_l \ln(\omega_l + a_l) + (\omega_l + a_l)\frac{1}{\omega_l + a_l}\delta a_l - (\delta a_l \ln a_l + a_l \frac{1}{a_l}\delta a_l) - 0]$

6.7 Bose distribution and Fermi distribution

- N, V, E conserved, require $\{a_l\}$.
- Obeys $\sum a_l = N$, $\sum a_l \varepsilon_l = E$.
- Bose system: $\Omega_{\rm B} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$.
- $\ln \Omega_{\rm B} = \sum[\ln(\omega_l + a_l - 1)! - \ln a_l! - \ln(\omega_l - 1)!]$
  $\simeq \sum[\ln(\omega_l + a_l)! - \ln a_l! - \ln \omega_l!]$
  $\simeq \sum\{(\omega_l + a_l)[\ln(\omega_l + a_l) - 1] - a_l(\ln a_l - 1)$
  $-\omega_l(\ln \omega_l - 1)\}$
  $= \sum[(\omega_l + a_l)\ln(\omega_l + a_l) - a_l \ln a_l - \omega_l \ln \omega_l]$.
- $\delta \ln \Omega_{\rm B} = \sum[\delta a_l \ln(\omega_l + a_l) + (\omega_l + a_l)\frac{1}{\omega_l + a_l}\delta a_l -$
  $(\delta a_l \ln a_l + a_l \frac{1}{a_l}\delta a_l) - 0]$
  $= \sum \ln \frac{\omega_l + a_l}{a_l}\delta a_l = 0.$

Bose distribution and Fermi distribution

- Introduce the α, β:

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l]\delta a_l=0$

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln \frac{\omega_l+a_l}{a_l} - \alpha - \beta\varepsilon_l]\delta a_l = 0$
  $\Rightarrow \ln \frac{\omega_l+a_l}{a_l} - \alpha - \beta\varepsilon_l = 0$,

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln \frac{\omega_l + a_l}{a_l} - \alpha - \beta\varepsilon_l]\delta a_l = 0$
  $\Rightarrow \ln \frac{\omega_l + a_l}{a_l} - \alpha - \beta\varepsilon_l = 0$, or $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l} - 1}$,

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l]\delta a_l=0$
  $\Rightarrow \ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l=0$, or $a_l=\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$,
  where α,β obeys: $\sum\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}=N$, $\sum\frac{\omega_l\varepsilon_l}{e^{\alpha+\beta\varepsilon_l}-1}=E$.

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l]\delta a_l=0$
  $\Rightarrow \ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l=0$, or $a_l=\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$,
  where α,β obeys: $\sum\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}=N$, $\sum\frac{\omega_l\varepsilon_l}{e^{\alpha+\beta\varepsilon_l}-1}=E$.
- Fermi system: $\Omega_{\mathrm{F}}=\prod\frac{\omega_l!}{a_l!(\omega_l-a_l)!}$.

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l]\delta a_l=0$
  $\Rightarrow \ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l=0$, or $a_l=\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l-1}}$,
  where α,β obeys: $\sum\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l-1}}=N$, $\sum\frac{\omega_l\varepsilon_l}{e^{\alpha+\beta\varepsilon_l-1}}=E$.
- Fermi system: $\Omega_\mathrm{F}=\prod\frac{\omega_l!}{a_l!(\omega_l-a_l)!}$.
- $\ln\Omega_\mathrm{F}=\sum[\ln\omega_l!-\ln a_l!-\ln(\omega_l-a_l)!]$

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l]\delta a_l=0$
  $\Rightarrow \ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l=0$, or $a_l=\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$,
  where α,β obeys: $\sum\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}=N$, $\sum\frac{\omega_l\varepsilon_l}{e^{\alpha+\beta\varepsilon_l}-1}=E$.
- Fermi system: $\Omega_{\mathrm{F}}=\prod\frac{\omega_l!}{a_l!(\omega_l-a_l)!}$.
- $\ln\Omega_{\mathrm{F}}=\sum[\ln\omega_l!-\ln a_l!-\ln(\omega_l-a_l)!]$
  $\simeq\sum\{\omega_l(\ln\omega_l-1)-a_l(\ln a_l-1)$
  $-(\omega_l-a_l)[\ln(\omega_l-a_l)-1]\}$

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l]\delta a_l=0$
  $\Rightarrow \ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l=0$, or $a_l=\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$,
  where α,β obeys: $\sum\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}=N$, $\sum\frac{\omega_l\varepsilon_l}{e^{\alpha+\beta\varepsilon_l}-1}=E$.
- Fermi system: $\Omega_{\rm F}=\prod\frac{\omega_l!}{a_l!(\omega_l-a_l)!}$.
- $\ln\Omega_{\rm F}=\sum[\ln\omega_l!-\ln a_l!-\ln(\omega_l-a_l)!]$
  $\simeq\sum\{\omega_l(\ln\omega_l-1)-a_l(\ln a_l-1)$
  $-(\omega_l-a_l)[\ln(\omega_l-a_l)-1]\}$
  $=\sum[\omega_l\ln\omega_l-a_l\ln a_l-(\omega_l-a_l)\ln(\omega_l-a_l)]$.

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l]\delta a_l=0$
  $\Rightarrow \ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l=0$, or $a_l=\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$,
  where α,β obeys: $\sum\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}=N$, $\sum\frac{\omega_l\varepsilon_l}{e^{\alpha+\beta\varepsilon_l}-1}=E$.
- Fermi system: $\Omega_{\mathrm{F}}=\prod\frac{\omega_l!}{a_l!(\omega_l-a_l)!}$.
- $\ln\Omega_{\mathrm{F}}=\sum[\ln\omega_l!-\ln a_l!-\ln(\omega_l-a_l)!]$
  $\simeq\sum\{\omega_l(\ln\omega_l-1)-a_l(\ln a_l-1)$
  $-(\omega_l-a_l)[\ln(\omega_l-a_l)-1]\}$
  $=\sum[\omega_l\ln\omega_l-a_l\ln a_l-(\omega_l-a_l)\ln(\omega_l-a_l)]$.
- $\delta\ln\Omega_{\mathrm{F}}=\sum\{0-[\delta a_l\ln a_l+a_l\frac{1}{a_l}\delta a_l]-[-\delta a_l\ln(\omega_l-a_l)+(\omega_l-a_l)\frac{-\delta a_l}{\omega_l-a_l}]\}$

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l]\delta a_l=0$
  $\Rightarrow \ln\frac{\omega_l+a_l}{a_l}-\alpha-\beta\varepsilon_l=0$, or $a_l=\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$,
  where α,β obeys: $\sum\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}=N$, $\sum\frac{\omega_l\varepsilon_l}{e^{\alpha+\beta\varepsilon_l}-1}=E$.
- Fermi system: $\Omega_{\mathrm{F}}=\prod\frac{\omega_l!}{a_l!(\omega_l-a_l)!}$.
- $\ln\Omega_{\mathrm{F}}=\sum[\ln\omega_l!-\ln a_l!-\ln(\omega_l-a_l)!]$
  $\simeq\sum\{\omega_l(\ln\omega_l-1)-a_l(\ln a_l-1)$
  $-(\omega_l-a_l)[\ln(\omega_l-a_l)-1]\}$
  $=\sum[\omega_l\ln\omega_l-a_l\ln a_l-(\omega_l-a_l)\ln(\omega_l-a_l)]$.
- $\delta\ln\Omega_{\mathrm{F}}=\sum\{0-[\delta a_l\ln a_l+a_l\frac{1}{a_l}\delta a_l]-[-\delta a_l\ln(\omega_l-a_l)+(\omega_l-a_l)\frac{-\delta a_l}{\omega_l-a_l}]\}$
  $=\sum[\ln(\omega_l-a_l)-\ln a_l]\delta a_l=0$.

Bose distribution and Fermi distribution

- Introduce the α, β:

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln \frac{\omega_l - a_l}{a_l} - \alpha - \beta\varepsilon_l]\delta a_l = 0$

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln \frac{\omega_l - a_l}{a_l} - \alpha - \beta\varepsilon_l]\delta a_l = 0$
  $\Rightarrow \ln \frac{\omega_l - a_l}{a_l} - \alpha - \beta\varepsilon_l = 0$,

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln \frac{\omega_l - a_l}{a_l} - \alpha - \beta\varepsilon_l]\delta a_l = 0$
  $\Rightarrow \ln \frac{\omega_l - a_l}{a_l} - \alpha - \beta\varepsilon_l = 0$, or $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1}$,

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln\frac{\omega_l - a_l}{a_l} - \alpha - \beta\varepsilon_l]\delta a_l = 0$
  $\Rightarrow \ln\frac{\omega_l - a_l}{a_l} - \alpha - \beta\varepsilon_l = 0$, or $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1}$,
  where α, β similarly obeys:
  $\sum\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1} = N$, $\sum\frac{\omega_l\varepsilon_l}{e^{\alpha+\beta\varepsilon_l}+1} = E$.

Bose distribution and Fermi distribution

- Introduce the α, β: $\sum[\ln\frac{\omega_l - a_l}{a_l} - \alpha - \beta\varepsilon_l]\delta a_l = 0$
  $\Rightarrow \ln\frac{\omega_l - a_l}{a_l} - \alpha - \beta\varepsilon_l = 0$, or $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1}$,
  where α, β similarly obeys:
  $\sum\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1} = N$, $\sum\frac{\omega_l\varepsilon_l}{e^{\alpha+\beta\varepsilon_l}+1} = E$.
- Similarly, for any individual quantum state s, the average number of particles: $f_s = \frac{1}{e^{\alpha+\beta\varepsilon_s}\pm 1}$,
  and $\sum f_s = N$, $\sum \varepsilon_s f_s = E$.

Table of contents

6.8 Relations between Boltzmann, Bose and Fermi distributions

6.8 Relations between Boltzmann, Bose and Fermi distributions

- Boltzmann distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}}$.

6.8 Relations between Boltzmann, Bose and Fermi distributions

- Boltzmann distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}}$.
- Bose distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l-1}}$.

6.8 Relations between Boltzmann, Bose and Fermi distributions

- Boltzmann distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}}$.
- Bose distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$.
- Fermi distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1}$.

6.8 Relations between Boltzmann, Bose and Fermi distributions

- Boltzmann distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}}$.
- Bose distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$.
- Fermi distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1}$.
- If $\alpha + \beta\varepsilon_l \gg 1$, i.e., $\frac{a_l}{\omega_l} \ll 1$, three distributions merge. Called non-degenerate condition.

6.8 Relations between Boltzmann, Bose and Fermi distributions

- Boltzmann distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}}$.
- Bose distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$.
- Fermi distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1}$.
- If $\alpha + \beta\varepsilon_l \gg 1$, i.e., $\frac{a_l}{\omega_l} \ll 1$, three distributions merge. Called non-degenerate condition.
- Remind: $\Omega_\mathrm{M} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l}$, $\Omega_\mathrm{B} = \prod \frac{(\omega_l+a_l-1)!}{a_l!(\omega_l-1)!}$, $\Omega_\mathrm{F} = \prod \frac{\omega_l!}{a_l!(\omega_l-a_l)!}$.

6.8 Relations between Boltzmann, Bose and Fermi distributions

- Boltzmann distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}}$.
- Bose distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$.
- Fermi distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1}$.
- If $\alpha+\beta\varepsilon_l \gg 1$, i.e., $\frac{a_l}{\omega_l} \ll 1$, three distributions merge. Called non-degenerate condition.
- Remind: $\Omega_{\mathrm{M}} = \frac{N!}{\prod a_l!}\prod \omega_l^{a_l}$, $\Omega_{\mathrm{B}} = \prod \frac{(\omega_l+a_l-1)!}{a_l!(\omega_l-1)!}$, $\Omega_{\mathrm{F}} = \prod \frac{\omega_l!}{a_l!(\omega_l-a_l)!}$.

  In the case $\frac{a_l}{\omega_l} \ll 1$, $\Omega_{\mathrm{B}} \simeq \frac{\omega_l^{a_l}}{\prod a_l!}$, $\Omega_{\mathrm{F}} \simeq \frac{\omega_l^{a_l}}{\prod a_l!}$.

6.8 Relations between Boltzmann, Bose and Fermi distributions

- Boltzmann distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}}$.
- Bose distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$.
- Fermi distribution: $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1}$.
- If $\alpha + \beta\varepsilon_l \gg 1$, i.e., $\frac{a_l}{\omega_l} \ll 1$, three distributions merge. Called non-degenerate condition.
- Remind: $\Omega_{\mathrm{M}} = \frac{N!}{\prod a_l!} \prod \omega_l^{a_l}$, $\Omega_{\mathrm{B}} = \prod \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$, $\Omega_{\mathrm{F}} = \prod \frac{\omega_l!}{a_l!(\omega_l - a_l)!}$.

  In the case $\frac{a_l}{\omega_l} \ll 1$, $\Omega_{\mathrm{B}} \simeq \frac{\omega_l^{a_l}}{\prod a_l!}$, $\Omega_{\mathrm{F}} \simeq \frac{\omega_l^{a_l}}{\prod a_l!}$.

  $\therefore \Omega_{\mathrm{B}} \simeq \Omega_{\mathrm{F}} \simeq \frac{\Omega_{\mathrm{M}}}{N!}$.

Table of contents