Thermodynamics & Statistical Physics

Chapter 5. Thermodynamics of irreversible processes

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  $T\mathrm{d}S = \mathrm{d}U + p\mathrm{d}V - \sum_i \mu_i \mathrm{d}N_i$, where N_i is the number of molecules of component i, and μ_i is the chemical potential of one molecule of component i.

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- $\Theta \geqslant 0$.

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- Generally, $\Theta = \sum_k \vec{J}_k \cdot \vec{X}_k.$

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- Summation:
  $\vec{J} = L\vec{X}$, where $\vec{J}$ is the flux density, $\vec{X}$ is the "force".

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  $$= L_{11}X_1^2 + 2L_{12}X_1X_2 + L_{22}X_2^2.$$
- $\Theta \geqslant 0$, requires, $L_{11} > 0$, $L_{11}L_{22} > L_{12}^2$.
  (Properties of of the coefficients L.)

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