NEURAL NETS IN THE GUIDING TERM

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Consider the SDE $\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t$. We wish to condition this SDE. We follows the approach of **?**. The guided process X° has drift $(t, x) \mapsto b(t, x) + \gamma(t, x)$. One can define $\gamma(t, x) = \sigma(t, x)g(t, x)$, where

$$g(t, x) = \sigma'(t, x)\nabla_x \log \tilde{p}(t, x; T, x_T) + v_\varphi(t, x), \tag{1}$$

with v_φ a bounded function (this would be the neural net) Denote the law of the process with this guiding term superimposed by $\mathbb{Q}_\varphi$. We learn φ from stochastic gradient descent on

$$\varphi \mapsto KL(\mathbb{P}^\star \mid \mathbb{Q}_\varphi) = \mathbb{E}\,\xi\left(\frac{d\mathbb{P}^\star}{d\mathbb{Q}_\varphi}(\mathcal{G}P_\varphi(W))\right)$$

with $\xi(x) = x\log x$. Here, W is the driving Brownian motion and $\mathcal{G}P_\varphi(W) \sim \mathbb{Q}_\varphi$ is the strong solution to the SDE

$$\mathrm{d}X_t^\circ = b(t, X_t^\circ)\,\mathrm{d}t + \sigma(t, X_t^\circ)g(t, X^\circ)\,\mathrm{d}t + \sigma(t, X_t^\circ)\,\mathrm{d}W_t$$

In control theory, one would call g the applied control. In **?** the likelihood ratio can be found for the case where $v_\varphi \equiv 0$. Using Girsanov we can see how this ratio changes by shifting the the drift of the process by σv_φ.

Note that if $v_\varphi(t, x)$ were to equal

$$\sigma'(t, x)\nabla_x \log \frac{p(t, x; T, x_T)}{\tilde{p}(t, x; T, x_T)}$$

then $KL(\mathbb{P}^\star \mid \mathbb{Q}_\varphi) = 0$. This motivates to minimise

$$\varphi \mapsto \mathbb{E}_{\mathbb{Q}_0}\int_0^T \left\|\sigma'(t, X_t)\nabla_x \log \frac{p(t, X_t; T, x_T)}{\tilde{p}(t, X_t; T, x_T)} - \sigma'(t, X_t)v_\varphi(t, X_t)\right\|^2 \mathrm{d}t$$

Now approximate p by Euler discretisation on intervals (t_{m-1}, t_m). As $\tilde{p}$ is Gaussian, the RHS is approximately

$$\sum_{m=1}^{M}\int_{t_{m-1}}^{t_m} \mathbb{E}_{\mathbb{Q}_0}\left\|\sigma'(t, X_t)g(t_{m-1}, X_{t_{m-1}}; t_m, X_{t_m}) - \sigma'(t, X_t)v_\varphi(t, X_t)\right\|^2 \mathrm{d}t$$

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1.

Define $X^\diamond$ with drift

$$b(x) + a(s,x)\nabla \log \tilde{p}(t,x,T,v) + \sigma f_\varphi(t,x)$$

and try to optimise

$$KL(P^\star \mid P^\diamond) = E \log(\frac{dP^\star}{dP^\diamond}(Z^\diamond))\frac{dP^\star}{dP^\diamond}(Z^\diamond)$$

or

$$KL(P^\diamond \mid P^\star) = -E \log(\frac{dP^\star}{dP^\diamond}(Z^\diamond))$$

The underlying idea is that f models $\sigma'(t,x)\nabla \log \frac{p(t,x;T,v)}{\tilde{p}(t,x;T,v)}$

Then use Girsanov to get

$$\frac{dP^\star}{dP^\diamond}(X_{[0,T]}) = \frac{dP^\star}{dP^\circ}(X_{[0,T]}) \cdot \Gamma(X_{[0,T]})$$

with $X_s = [GP_\varphi(W)]_s$ (so this is sampled under $P^\diamond$) and

$$\Gamma(X_{[0,T]}) = \exp\left(\int_0^T f_\varphi(s,X_s)dW_s - \frac{1}{2}\int_0^T \|f_\varphi(s,X_s)\|^2 ds\right).$$

The second term on the RHS is just the "ordinary" likelihood for guided proposals (so without the f_φ-term), but evaluated in the process which has f_φ in the drift.