U.S. Population Models*

Year	U.S. Population
1790	3,929,214
1800	5,308,438
1810	7,239,881
1820	9,638,453
1830	12,866,020
1840	17,069,453
1850	23,191,876
1860	31,443,321
1870	38,558,371
1880	50,189,209
1890	62,979,776
1900	76,212,168
1910	92,228,496
1920	106,021,537
1930	123,202,624
1940	132,164,569
1950	151,325,798
1960	179,323,175
1970	203,302,031
1980	226,542,199
1990	248,709,873
2000	281,421,906
2010	308,745,538

On the left, is U.S. Population data gathered from the U.S. Census website. In this application, you will investigate various models of the population and compare them to the actual growth seen in U.S. History. The first model, you have probably seen in class:

$$\frac{dP}{dt} = kP.$$

This model claims that the rate of change of growth is directly proportional to the population itself. While this idea may seem straightforward, how well does it work in practice? There are many other possible forces that might influence

1. The general solution to the differential equation $\frac{dP}{dt} = kP$ is Ae^{kt}. Using this fact it is possible choose the parameter k so that it is in some sense "best". In particular it suggests that $\ln(P) = \ln(A) + kt$, so if we plot the log of the population we expect the slope of the resulting line to be k. However, a plot of the U.S. population data does not give a precisely straight line. Picking a line where through the data is "most linear" suggests $k \approx 0.027$.

$$\begin{cases} \frac{dP}{dt} = .027P \\ P(0) = 3,929,214 \end{cases}$$

What does the solution to this differential equation have to do with the U.S. population data (i.e. what does P represent, what are its units? What does t represent, what are its units?)

2. Another model for population is

$$\frac{dP}{dt} = kP + c$$

For now, we'll call this model #2. What is a possible interpretation for c? Explain your interpretation in terms of a positive value of c and negative value of c.

3. Draw the phase diagram for the model $\frac{dP}{dt} = kP + c$. Based on the phase diagram alone, do you believe this model could be sufficient to capture the long run behavior of the U.S. Population seen in the table?

4. Verify that $\frac{c}{k}\left(e^{kt} - 1\right) + P_0 e^{kt}$ is a solution to the initial value problem

$$\begin{cases} \frac{dP}{dt} = kP + c \\ P(0) = P_0 \end{cases}$$

*Thursday, March 30 (TTh sections) and Friday, March 31 (MWF sections)

5. Given the actual U.S. population data, the parameters that minimize net distance from the model and the data are $k = 0.011$ and $c = 296,684$. Do these numbers seem reasonable to you? Why is it that the relative growth rate (k) for this model seems so much lower than the relative growth rate in the exponential model?

6. Another model you may have seen in class is the logistic model. This model encodes the idea that there is a maximum possible population would limit towards. Your textbook presents this as

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Given the U.S. Population data, the choices $k = 0.027$ and $M = 369,557,000$. That M represents the maximum population can readily be seen by examining the phase diagram. Why is it that for this model, the relative growth rate (k) seems so close to what it had been for the exponential model.

7. Below are graphs of the three models for population together with actual US data. Which model do you think is "best"? Be sure to explain why you think your choice is the most realistic.