Models of Population Growth

1. Exponential or uninhibited growth:
If a population, P, grows at an annual rate of say 3% a year,continuously, that would mean that

$$\frac{dP}{dt} =$$

Note that in this case $\frac{dP/dt}{P}$ is constant. The ratio $\frac{dP/dt}{P}$ is called the productivity rate or the relative growth rate. If $P = P_0$ when $t = 0$, solve the initial value problem for P.

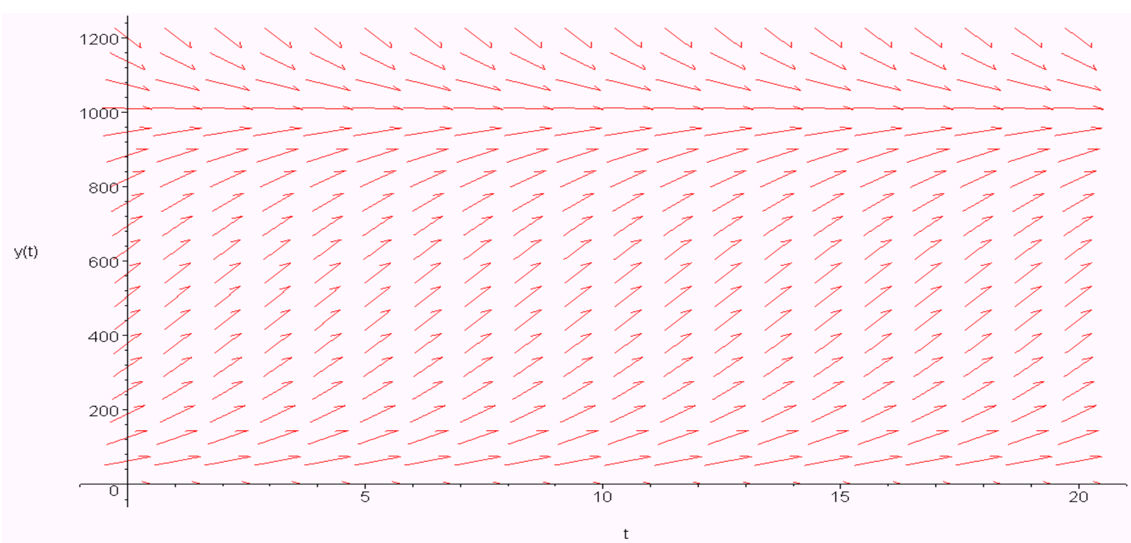
2. Logistic or inhibited growth:
More realistically, it seems appropriate to assume that a population will have a natural limit. A simple model that takes that into consideration assumes that the population grows at a rate proportional to the product of the population, P, and the difference between the limiting value of the population, M, and the population. Circle all the differential equations below that reflect these assumptions:

$$\frac{dP}{dt} = M - kP \qquad \frac{dP}{dt} = c(M - P) \qquad \frac{dP}{dt} = cP(M - P)$$

$$\frac{dP}{dt} = P(M - kP) \qquad \frac{dP}{dt} = kP(1 - \frac{P}{M})$$

Is the differential equation separable?

Below is a slope field of the differential equation where $M = 1000$ and $c = .0004$.
Sketch solutions when $P_0 = 100$ and $P_0 = 200$.



Using partial fractions one can solve $\frac{dP}{dt} = cP\,(M - P)$ or $k\,P\,(1 - \frac{P}{M})$ where $k = cM$ and $P((0) = P_0$. For example if $\frac{dP}{dt} = 0.4\,P\,(1 - \frac{P}{1000})$ and $P(0) = 100$ then

$\frac{dP}{dt} =$ __________ $P\,(1000 - P)$ and $\int \qquad\qquad\qquad = \int 0.0004\,dt$

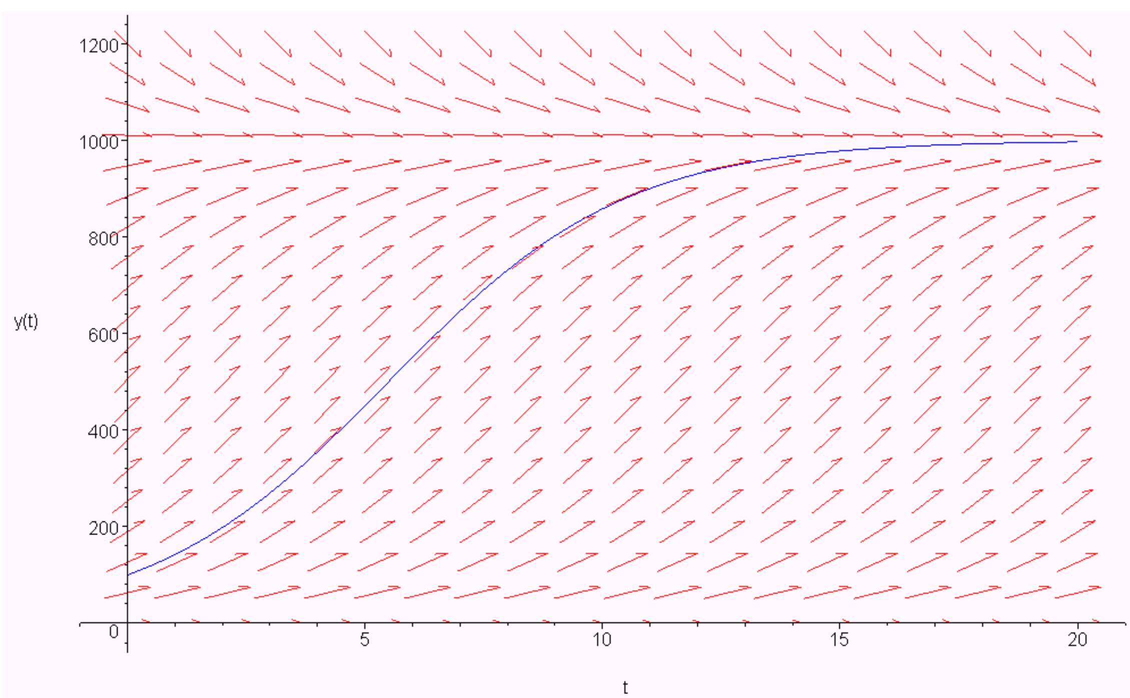
Since $\frac{1}{P\,(1000-P)} = (\frac{A}{P} + \frac{B}{(1000-P)}) = \frac{1}{1000}(\frac{\qquad}{P} + \frac{\qquad}{(1000-P)})$

then $\int \frac{1}{P\,(1000-P)}\,dP = \frac{1}{1000}(\ln P - \ln(1000 - P) = \frac{1}{1000}\ln \frac{P}{\qquad}$

Solving for P (do this!) one gets $P = \frac{1000}{1+9e^{-.4t}}$

How might you check the solution?

Below is a plot of $P = \frac{1000}{1+9e^{-.4t}}$ superimposed on the slope field. Compare this to your answer to 2 above.



3. A simple immigration model:
A population with uninhibited growth and a constant rate of immigration can be modeled by $\frac{dP}{dt} = kP + c$. What is the constant rate of immigration?

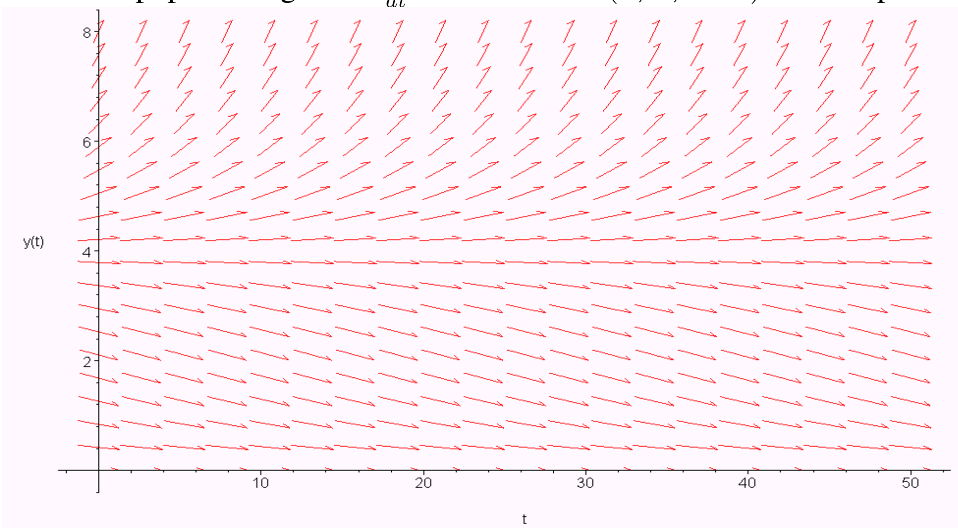
If $P = P_0$ when $t = 0$, solve the initial value problem for P.

4. Coalition model:
From http://www.math.duke.edu/education/ccp/materials/diffcalc/worldpop/world1.html:

In 1960 Heinz von Foerster, Patricia Mora, and Larry Amiot published a now-famous paper in Science (vol. 132, pp. 1291-1295). The authors argued that the growth pattern in the historic world's population data can be explained by improvements in technology and communication that have molded the human population into an effective coalition in a vast game against Nature -- reducing the effect of environmental hazards, improving living conditions, and extending the average life span. They proposed a coalition growth model for which the productivity rate (i.e. $\frac{dP/dt}{P}$) is not constant, but rather is an increasing function of P, namely, a function of the form kP^r, where the power r is positive and presumably small. (If r were 0, this would reduce to the uninhibited growth model)

Solve $\frac{dP}{dt} = kP^{r+1}$. Why is this called the "Doomsday model"?

5. Another model of population growth, assumes that $\frac{dP}{dt}$ = birth rate $-$ death rate. For organisms that need a partner for reproduction but rely on chance encounters for meeting a mate, the birth rate is proportional to the square of the population (actually the square of $\frac{1}{2}$ of the population, but that is the same thing). Explain what assumptions would lead to the following model for population growth $\frac{dP}{dt} = kP^2 - cP \ (k,\ c,\ > 0)$. The slope field is given below:



6. In the logistic growth model, show that $\frac{dP}{dt}$ is largest when $P = \frac{M}{2}$ Explain how this might be useful in estimating M.

7. Match the differential equation with the name of the population model:

$\frac{dP}{dt} = kP$	Exponential Growth
$\frac{dP}{dt} = A\, e^{kt}$	Logistic Growth
$\frac{dP}{dt} = kP(m - P)$	Immigration Model
$\frac{dP}{dt} = kP + c$	Coalition Model
$\frac{dP}{dt} = kP^{1+r}$	
$\frac{dP}{dt} = kP(1 - \frac{P}{M})$	
$\frac{dP}{dt} = kP + c$	
$\frac{dP}{dt} = kP(1 - \frac{P}{M})$	

8. Do p. 565/ 10a