Introduction to Differential Equations

Monday, February 16, 2015

---

1. (a) Is the point $x = 1$, $y = 3$ (i.e. the point $(1, 3)$) a solution to the equation $2y+x = y+4$? Why?

   (b) Is the point $(2, 3)$ a solution to the equation $y^2 + x = y + 8$? Why?

   (c) Is the point $(1, 1)$ a solution to the equation $x^2 + y^2 = 1$? Why?

   (d) What does it mean that a point (a, b) is a solution to an equation?

2. (a) Is the function $y = x^2$ a solution to the equation $\frac{dy}{dx} = 2x$? What about $y = x^2 + 5$? Why?

   (b) Is the function $y = -\frac{1}{x}$ a solution to the equation $\frac{dy}{dx} = y^2$?

   If $y = -\frac{1}{x}$, then $\frac{dy}{dx} =$ ______________

   If $y = -\frac{1}{x}$, then $y^2 =$ ______________

   If $y = -\frac{1}{x}$, is $\frac{dy}{dx} = y^2$?

   (c) Is the function $y = \frac{x^2}{2}$ a solution to the equation $\frac{dy}{dx} = y + x$?

   If $y = \frac{x^2}{2}$, then $\frac{dy}{dx} =$ ______________

   If $y = \frac{x^2}{2}$, then $y + x =$ ______________

   If $y = \frac{x^2}{2}$, is $\frac{dy}{dx} = y + x$?

3. (a) Is the function $y = e^{3x}$ a solution to the equation $\frac{dy}{dx} = 3y$?

If $y = e^{3x}$, then $\frac{dy}{dx} =$ ______________

If $y = e^{3x}$, then $3y =$ ______________

(b) Is the function $y = e^{3x}$ a solution to the equation $\frac{dy}{dx} = 3y$?

If $y = e^{3x} + 1$, then $\frac{dy}{dx} =$ ______________

If $y = e^{3x} + 1$, then $3y =$ ______________

(c) Is the function $y = e^{3x}$ a solution to the equation $\frac{dy}{dx} = 3y$?

If $y = 2e^{3x}$, then $\frac{dy}{dx} =$ ______________

If $y = 2e^{3x}$, then $3y =$ ______________

4. Find a function that satisfies the differential equation $\frac{dy}{dx} = 3x$.

Can you name another such function ?

5. (a) Express the following sentence as a differential equation (symbols instead of words): "the rate of change of population is directly proportional to the size of the population."

   (b) Can you guess a function that satisfies the differential equation you wrote down in (a)?

   (c) Can you find another solution?

6. Which of the following could be a solution to the simple immigration model with a growth rate of 2% and an immigration rate of one hundred thousand people per year? Of those which are mathematical solutions, is there any reason to rule out such solutions? ($P(t)$ is the population in millions)

   (a) $P(t) = -5 + 4e^{0.02t}$

   (b) $P(t) = 5 + 10e^{0.02t}$

   (c) $P(t) = -5 + 10e^{0.02t}$

   (d) $P(t) = 10e^{0.02t} - 5t$

   (e) $P(t) = 5e^{0.02t}$

7. Both of the previous models allowed for the population to grow indefinitely, which may not be the most realistic idea. Yet the exponential model does a pretty decent job particularly when the population is reasonably small. For the next model, let's introduce the idea of the carrying capacity, M–a maximum allowed population Here we're going to try and write down a model which looks like the exponential model when M is very large, limits growth of the population when M is small.

   (a) Write a polynomial $f(x)$ such that $f(0) = 0$ and $f(M) = 0$.

   (b) Write a differential equation for the population where the rate of change is 0 when the population is M (and when the population is 0).

8. Below is a slope field for a differential equation very much like the one in the previous question. Sketch several solution curves on the graph below.